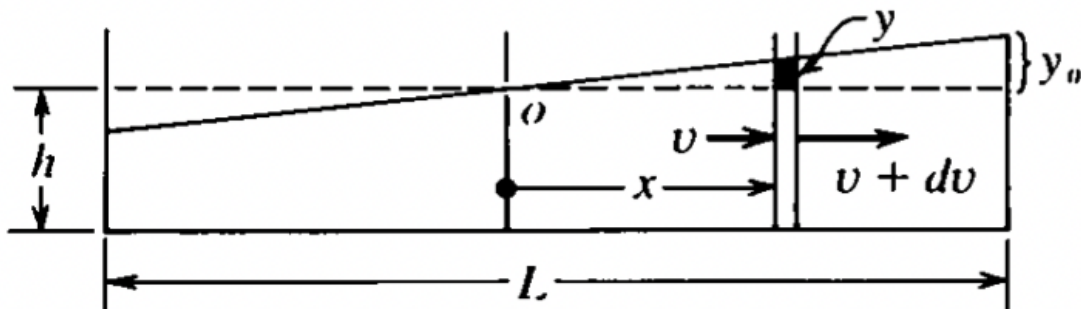# USAPhO Mock

Yep. This is probably going to be a "super-USAPhO", so don't be down if you get like a 2/6. Sources for the problems are on the last page, plus where you can find solutions to them. Among us picture for good luck.

1 Oscillating Lake

The simplest approximation to water sloshing around in a closed container is that the water's surface *tilts* while still retaining its flat surface. Imagine a lake of rectangular cross section, as shown, of length L and with water depth $h \ll L$. The problem resembles that of the simple pendulum, in that the kinetic energy is almost entirely due to horizontal flow of the water, whereas the potential energy depends on the very small change of vertical level.

(a) Imagine at some instant the water level at the extreme ends is at $\pm y_0$ with respect to the normal water level. Show that the increased gravitational potential energy of the whole mass of the water is given by

$$U = \frac{1}{6} b \rho g L y_0^2,$$

where b is the width of the lake.

(b) Assuming the water flow is purely horizontal, its speed v must vary with x, being greatest at $x = 0$ and zero at $x = \pm L/2$. Show that the speed of the water as a function of x is given by,

$$v(x) = v(0) - \frac{x^2}{hL} \frac{dy_0}{dt}.$$

where,

$$v(0) = \frac{L}{4h} \frac{dy_0}{dt}.$$

You may assume the lake water is incompressible, and thus obeys the continuity equation.

(c) Hence show that at any given instant, the total kinetic energy of the lake associated with the horizontal motion of the water is given by,

$$K = \frac{1}{60} \frac{b \rho L^3}{h} \left(\frac{dy_0}{dt} \right)^2.$$

(d) From this, find the period of oscillation of the lake. **Hint:** consider the sum of K and U. Does this value change with time? You may make reasonable approximations if needed.

2   (Not An) Inverse Square

Imagine that new and extraordinarily precise measurements have revealed an error in Coulomb's law. The *actual* force of interaction between two point charges is found to be

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \left(1 + \frac{|\mathbf{r} - \mathbf{r}'|}{\lambda}\right) e^{(|\mathbf{r} - \mathbf{r}'|)/\lambda},$$

where λ is a new constant of nature called the PhODs constant. You, the test taker, are now charged with the task of reformulating electrostatics to accomodate for the new discovery. Assume superposition still holds.

(a) What is the electric field a distance $\mathbf{r}$ from a uniform charged spherical shell of radius $\mathbf{a}$, with charge density σ?

(b) Does this version of electrostatics permit a scalar potential? Why or why not? (You need only give a convincing argument; a proof is not required).

(c) Show that under this formulation of electrostatics, not *all* excess charge on a three-dimensional conducting object goes to its boundary, only some.

(d) Now suppose we have found *another* error in our measurements, and it turns out the new adjustment is (completely ignore the previous "actual" force for this sub-problem).

$$\mathbf{F} \propto r^{-2+\delta}.$$

You are going to follow in Maxwell's footsteps, and experimentally measure the value of δ. Based on the work of Cavendish, Maxwell used the following experiment to test the value of δ: Place two conducting, concentric, thin, spherical shells with radii a and b ($a > b$) down, connect a thin wire between them. After charging the outer spherical shell to potential U, remove the power supply, then remove the thin wire connecting the two spherical shells, and then ground the outer spherical shell. At this time, the measured potential of the inner spherical shell is not greater than U. From this, estimate the value of δ. It is known that $\delta \ll 1$, as it has gone unnoticed for so long.

3   The Photoelectric Effect

A zinc ball of radius $R = 1cm$ is located in vacuum, far away from any other charged bodies, and is charged to a potential of $V = -0.5V$ (assume $V = 0$ at infinity). The ball is being illuminated by a monochromatic ultraviolet light with a wavelength of $\lambda = 290nm$.

(a) What is the maximum velocity, $v_{\max}$ of the photoelectrons flying out of the ball?

(b) What is the maximum velocity v_2 of a single photoelectron very far away from the ball?

(c) Determine the ball's potential after a prolonged exposure to the UV light.

(d) Determine the net number, N, of photoelectrons escaped from the ball after the prolonged exposure to the UV light.
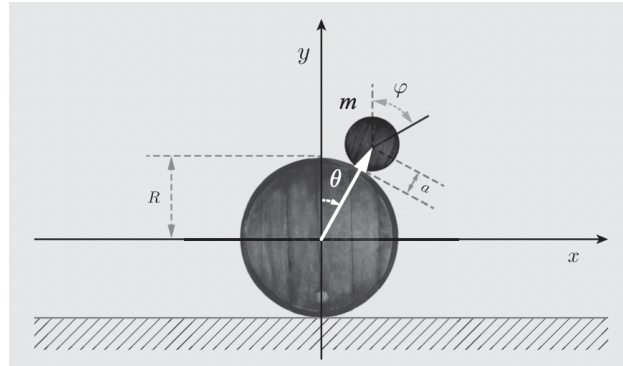
Some useful numbers:

- Photoelectric threshold for zinc: $\lambda_0 = 332nm$.

- $c = 3 \cdot 10^8 m/s$, $h = 6.626 \cdot 10^{-34} J/s$, $\epsilon_0 = 8.85 \cdot 10^{-12} F/m$, $e = -1.6 \cdot 10^{-19}$, and the electron mass is $m_e = 9.1 \cdot 10^{-31} kg$.
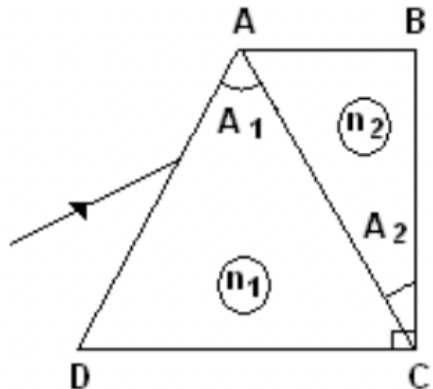
Intermission

4   Barrel Stacking

Consider the problem of two cylindrical barrels, one on top of the other, as shown in Figure 6.3. The bottom barrel is fixed in position and orientation, but the top one, of mass m, is free to move. It starts rolling down from its initial position at the top, rolling without slipping due to friction between the barrels.

The bottom barrel has radius R, and the top barrel has radius a. θ is measured from the positive vertical. The bottom barrel is unable to move, and the top barrel rolls without slipping on the bottom barrel.

(a) Find the height when the top barrel loses contact with the bottom barrel.

(b) Find the angle θ from the vertical when the top barrel falls off of the bottom barrel.

(c) Find the distance elapsed on the top barrel when the bottom barrel falls off (as in, how much circumference of the bottom barrel has the top barrel rolled across).

(d) [***] Find the normal force as a function of θ (maybe skip this part; I'm pretty sure it's harder than all of the other ones combined).

5   Prism Stacking

Two dispersive prisms having apex angles of $A_1 = 60°$ and $A_2 = 30°$ are glued into the shape above. The refractive indices of the prisms, dependent on the wavelength, are respectively

$$n_1(\lambda) = a_1 + \frac{b_1}{\lambda}$$

$$n_2(\lambda) = a_2 + \frac{b_2}{\lambda}.$$

Where $a_1 = 1.1, a_2 = 1.3, b_1 = 10^5 nm^2, b_2 = 5 \cdot 10^4 nm^2$.

(a) Determine the wavelength λ_0 of the incident radiation that passes through the prisms without refraction on the AC face at any incident angle; determine the corresponding refractional indices of the prisms.

(b) Draw the ray path in the system of prisms for three different radiations, $\lambda_{\text{red}}, \lambda_0, \lambda_{\text{violet}}$, which are incident on the system at the same angle.

(c) Determine the minimum deviation angle for a ray of wavelength λ_0.

(d) Calculate the wavelength of the ray that penetrates and then exits the system parallel to DC.

6   Traveling Faster Than Light

Can a body move faster than the speed of light? The answer is "No" if the object is moving in the vacuum. But the answer can be "Yes", if we deal with the phase speed of light in an optically dense medium with refractive index of n ($n = c/u$, where u is the speed of light in the medium).

We say a body is superluminal if $u < v < c$, where v is the velocity of the body. Throughout the problem, we will be dealing with a superluminal body of constant velocity v in an optical medium without dispersion, with u begin the velocity of light in the medium.

Note that we define $\gamma \equiv 1/\sqrt{1 - (v/c)^2}$, and $\tan \theta = \sqrt{v^2/u^2 - 1}$.
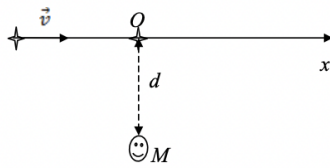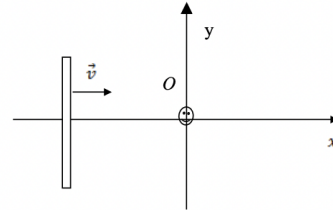

Figure 1

Figure 3

Note: there is no figure 2. I had to trim this question to make it USAPhO-length, so I got rid of the figure 2 part.

6.0.1   Figure 1

For all of these sub-questions, refer to figure 1. For this figure, a radiating particle is moving along the x-axis with a constant velocity v, with $v > u$. An observer M is located at a distance d from the x-axis. We choose the point closest to the observer on the x-axis to be the origin. The time when the particle passes $x = 0$ is $t = 0$.

(a) At time $t = t_0$, the observer first sees the particle at position x_0'. Find the apparent position x_0' and the observed time t_0 for this first appearance in terms of d, v, and θ.

(b) Find the apparent position $x'(t)$ at all times t. Write your answer in terms of v, θ, t, and t_0.

(c) From this, find the apparent velocity, $v'(t)$, at all times t. Write your answer in terms of v, θ, t, and t_0.

(d) Can an apparent velocity be greater than c? Why or why not?

6.0.2   Figure 3

You should refer to figure 3 for all parts of this question. Assume that the linear radiating object moves perpendicularly along the x-axis. Let the observer be located at the origin, which we define to be at $x = 0$. The object is symmetric with respect to the x-axis.

(a) Show that for a given time t, the apparent form of this object is an ellipse, or part(s) of an ellipse.

(b) Find the position of the center of symmetry for the ellipse at a given time t, in terms of v, θ, and t.

(c) Determine the lengths of the semi-major and semi-minor axes of the ellipse for a given time t, in terms of v, θ, and t.

Sources

These problems were inspired/taken from:

1. French's *Vibrations and Waves*, problem 3-18

2. Griffiths EM, problem 2.54, and Guopei, problem idk.

3. Kiselev and Slobodyanin, page 8, problem 10.

4. Helliwell and Sahakian's *Modern Classical Mechanics* example 6.2. They do it with Lagrangians, but you can do it with Newtonian; it's just harder.

5. IPhO 1983 problem 3

6. (Some of) APhO 2008 problem 3 (If I included all of it, it would have been way too long).

You can get solutions to these problems from these sources, which you can get by Googling or by using the resources on the server. For the Kiselev and Slobodyanin book, I'll post it in the server or something, along with Guopei. For the IPhO and APhO problems, see this problem archive.

Solutions

- See here.

- See here.

- See in Discord resources. Ping me and I'll post them.

- You can find the book on Libgen or Anna's Archive. Not that I'm recommending you do that or anything. . .

- See this problem archive for the 5th and 6th problem.