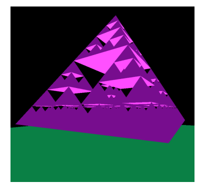
1 Introduction

This document will explore the volume of the 3D Sierpinski Triangle as the number of recursive subdivisions tends towards infinity. We will ultimately show that the limit of the volume is zero.

We will look at a sequence of Sierpinski triangles $(v_n)_{n\in\mathbb{N}}$, where v_1, the base case is a square pyramid. The figure below shows the pictorial representation of the first four terms of the sequence.

v1:  v2:  v3:  v4: 

2 Base Case (v_1)

For a square base pyramid we have Volume $V = \frac{1}{3}bh$, where b is the area of the base and h is the height. Let s be the side length. Since $s = h$ and $b = s^2$ we have

$$V = \frac{1}{3}(s^2)s = \frac{1}{3}s^3$$

.

So the volume of the initial pyramid v_1 is $\frac{1}{3}s^3$, where s is some arbitrary real number scalar.

3 Recursive Relationship

Now we define two recursive relationships for this shape, one for size and one for volume. We will denote the size and volume of the nth term as s_n and v_n respectively.

3.1 Size Relation

The first terms has the trivial size of s, that is $s_1 = s$. Now for each subsequent term the height of the pyramid is halved, so we have that $s_{n+1} = \frac{1}{2}s_n$. So the size of the nth pyramid or s_n is defined as:

$$\mathrm{s}_n = \begin{cases} s & n = 1 \\ \frac{1}{2^{n-1}}s & n \geq 2 \end{cases}$$

3.2 Volume Relation

We need a formula for the volume of the nth pyramid (v_n) in terms of its size s_n. For the sake of simplicity let $s = 1$.

As we did for the size relation, lets define the volume of v_1. We know that $v_1 = \frac{1}{3}(s_1)^3$. In other words, the volume of the first term is simply a square base pyramid with an equal height and side length.

For v_n, we need to calculate the number of pyramids, and multiply it by the volume of each pyramid. A simple inductive argument will show that v_n contains 5^{n-1} pyramids. Since each of these 5^{n-1} pyramids is a square pyramid (with height = side length), we know the volume each pyramid is $\frac{1}{3}(s_n)^3$. So we have

$$v_n = 5^{n-1}\frac{1}{3}(s_n)^3$$

Recall that $\forall n \geq 1,\ s_n = \frac{1}{2^{n-1}}$

Substituting this for s_n, we get v_n completely in terms of n

$$v_n = 5^{n-1}\frac{1}{3}(\frac{1}{2^{n-1}})^3$$

$$v_n = \frac{1}{3}5^{n-1}\frac{1}{2^{3(n-1)}}$$

$$v_n = \frac{1}{3}\frac{5^{n-1}}{8^{n-1}}$$

$$v_n = \frac{1}{3}\left(\frac{5}{8}\right)^{n-1}, \forall n \geq 1$$

4 Limit of Volume

An introductory calculus course would be sufficient to convince you that the limit of the sequence (v_n) is zero.

So we have our result, the limit of the volume of this shape. I think it illustrates a very nice point. This result provides a geometric intuition for why the limit of some sequence need not be a term in that sequence.

We are permitted to create shapes that have volumes which are arbitrarily close, but never equal to zero. In this way we get to look at the beauty of convergence while at the same time respecting the fact that a 3D shape must have non-zero volume.