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On Lynden-Bell's method for the determination of the luminosity function

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Summary. A new, simple argument leading to Lynden-Bell's C method is presented. This argument leads to an extended version of the C method which gives the properly normalized luminosity function.

1 Introduction

The problem of the determination of the luminosity function from a magnitude-limited sample has been solved by Lynden-Bell (1971) by introducing the so called C method. I shall present in this paper a new argument leading to the C method. This argument has two advantages in comparison with Lynden-Bell's original paper: it seems to be simpler and it leads to a properly normalized luminosity function, while the original C method gives only its shape (in a cumulative form).

The C method has been used only for quasars (Lynden-Bell 1971; Jackson 1974; Marshall *et al.* 1983) but it can be used also for other astronomical objects. The method should be especially useful for galaxies, since several new magnitude-limited complete (or nearly complete) samples of galaxies with known redshifts (e.g. Kirshner, Oemler & Schechter 1978; Kirshner *et al.* 1983; Huchra *et al.* 1983; Merighi *et al.* 1986; Parker *et al.* 1986; Peterson *et al.* 1986; Loh & Spillar 1986) have been recently obtained.

To make the bibliography of the C method complete we should also mention the papers of Carswell (1973), Felten (1976) and Nicoll & Segal (1983) where some aspects of the method have been discussed.

2 The statement of the problem

Let us denote M as absolute brightness, μ a distance modulus, m an observed brightness, and r a distance in Mpc, where:

$$m = M + \mu \tag{1}$$

$$r = 10^{0.2(\mu - 25)}. \tag{2}$$

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The same notation has been used in Chołoniewski (1986, hereafter Paper I). Let us consider galaxies on the (M, μ)-plane such that:

$$M_{\min} \leqq M \leqq M_{\max},$$

$$\mu_{\min} \leqq \mu \leqq \mu_{\max}. \tag{3}$$

The observed distribution of galaxies in the above region $N(M, \mu)$ can be expressed as:

$$N(M,\mu) = \phi(M)\, D(\mu)\, \Theta[m_{\lim} - (M+\mu)], \tag{4}$$

where $\phi(M)$ and $D(\mu)$ are distribution functions of their arguments, and $\Theta(m_{\lim} - m)$ is the Heaviside function which describes here the magnitude cut-off: $m \leqq m_{\lim}$ (see Fig. 1). Our task is to obtain the functions $\phi(M)$ and $D(\mu)$.

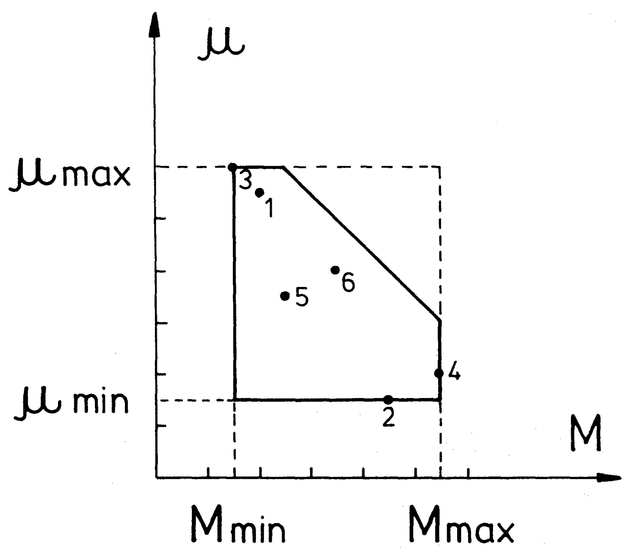

Figure 1. An instantaneous distribution of galaxies (quasars, stars) in the (M, μ)-plane. The tilted line represents the selection line: $M + \mu = m_{\lim}$. No galaxies are observed over this line due to the magnitude cut-off. The data presented here have been used in computations in Section 6.

3 The solution

The function $N(M, \mu)$ can be expressed as:

$$N(M, \mu) = \sum_{k=1}^{N} \delta(M - M_k, \mu - \mu_k), \tag{5}$$

where the sum is over the objects in the sample (which are observed in M_k, μ_k points; $k = 1, \ldots, N$). Substitution of the above formula into equation (4) prompts us to assume $\phi(M)$ and $D(\mu)$ to be in the following form:

$$\phi(M) = \sum_{i=1}^{N} \psi_i \delta(M - M_i), \tag{6}$$

$$D(\mu) = \sum_{j=1}^{N} d_j \delta(\mu - \mu_j). \tag{7}$$

Now, equation (4) becomes:

$$\sum_{k=1}^{N} \delta(M-M_k,\ \mu-\mu_k)=\sum_{i:}^{M_i+\mu_j\leq m_{\rm lim}}\sum_{j:} \psi_i d_j \,\delta(M-M_i)\,\delta(\mu-\mu_j). \tag{8}$$

Integration of equation (8) over M and μ, gives, respectively:

$$1=d_j \sum_{i:}^{M_i+\mu_j\leq m_{\rm lim}} \psi_i \qquad j=1,\ldots,N \tag{9}$$

$$1=\psi_i \sum_{j:}^{M_i+\mu_j\leq m_{\rm lim}} d_j \qquad i=1,\ldots,N. \tag{10}$$

Equations (9) and (10) can be easily solved with respect to $\psi_i(i=1,\ldots,N)$ and $d_j(j=1,\ldots,N)$ – see Section 6 for an example. These quantities give us, via equations (6) and (7), the solution of our problem, namely: the determination of $\phi(M)$ and $D(\mu)$.

4 The construction of the final results

Knowing the functions $\phi(M)$ and $D(\mu)$ we can reconstruct the distribution of galaxies in the whole considered (see equation 3) region, in particular above the selection line (see Fig. 1):

$$N_{\rm t}(M,\mu)=\phi(M)\,D(\mu). \tag{11}$$

The distribution $N_{\rm t}(M,\ \mu)$ is all we need for computing: the properly normalized luminosity function – $\Phi(M)$, the solid angle averaged number density – $\varrho(\mu)$, and the total number of galaxies – $N_{\rm t}$:

$$\Phi(M)=\int_{\mu_{\rm min}}^{\mu_{\rm max}} N_{\rm t}(M,\ \mu)\,d\mu\ \frac{1}{V_{\rm t}}, \tag{12}$$

$$\varrho(\mu)=\int_{M_{\rm min}}^{M_{\rm max}} N_{\rm t}(M,\ \mu)\,dM\ \frac{d\mu}{dV'}, \tag{13}$$

$$N_{\rm t}=\int_{M_{\rm min}}^{M_{\rm max}}\int_{\mu_{\rm min}}^{\mu_{\rm max}} N_{\rm t}(M,\ \mu)\,dM\,d\mu, \tag{14}$$

where $V_{\rm t}$ denotes the total volume of the sample, dV' denotes a solid angle-integrated differential volume element:

$$V_{\rm t}=\int_{\mu_{\rm min}}^{\mu_{\rm max}}\frac{dV'}{d\mu}\,d\mu=\frac{\Omega}{3}\left[10^{0.6(\mu_{\rm max}-25)}-10^{0.6(\mu_{\rm min}-25)}\right] \tag{15}$$

$$dV'\equiv\Omega r^2\,dr \tag{16}$$

and Ω denotes the solid angle of the sample. Using equations (6), (7) and (11) we have:

$$\Phi(M)=\sum_{i=1}^{N}\psi_i\,\delta(M-M_i)\sum_{j=1}^{N} d_j\,\frac{1}{V_{\rm t}}, \tag{17}$$

$$\varrho(\mu)=\sum_{j=1}^{N} d_j\,\delta(\mu-\mu_j)\sum_{i=1}^{N}\psi_i\frac{d\mu}{dV'}, \tag{18}$$

$$N_t = \sum_{i=1}^{N} \psi_i \sum_{j=1}^{N} d_j. \tag{19}$$

The value N_t can be used for computation of the total average number density – n:

$$n \equiv \frac{N_t}{V_t}. \tag{20}$$

It should be noted here that the functions $\Phi(M)$ and $\varrho(\mu)$ as expressed in equations (12), (13) or (15), (16) fulfil automatically the well-known fundamental equations:

$$\int_{M_{\min}}^{M_{\max}} \Phi(M)\, dM = n, \tag{21}$$

$$\iiint_{V_t} \varrho(x, y, z)\, dx\, dy\, dz = \int_{\mu_{\min}}^{\mu_{\max}} \varrho(\mu)\, \frac{dV'}{d\mu}\, d\mu = N_t, \tag{22}$$

(for comparison see equations 20 and 21 in Paper I).

Our estimators of the functions $\Phi(M)$ and $\varrho(\mu)$ are equal to the weighted sum of Dirac delta functions (see equations 17 and 18) while we expect that they should be smooth and continuous. Hence, it is reasonable to take as the final results not $\Phi(M)$ and $\varrho(\mu)$ directly but their average values inside appropriate intervals:

$$\langle \Phi(M) \rangle \equiv \int_{M}^{M+\Delta M} \Phi(M)\, dM \Big/ \int_{M}^{M+\Delta M} dM, \tag{23}$$

$$\langle \varrho(\mu) \rangle \equiv \int_{\mu}^{\mu+\Delta\mu} \varrho(\mu)\, \frac{dV'}{d\mu}\, d\mu \Big/ \int_{\mu}^{\mu+\Delta\mu} \frac{dV'}{d\mu}\, d\mu, \tag{24}$$

which gives:

$$\langle \Phi(M) \rangle = \frac{\sum\limits_{i:}^{M_i \varepsilon [M,\, M+\Delta M]} \psi_i \sum\limits_{j=1}^{N} d_j}{V_t \Delta M}, \tag{25}$$

$$\langle \varrho(\mu) \rangle = \frac{\sum\limits_{j:}^{\mu_j \varepsilon [\mu,\, \mu+\Delta\mu]} d_j \sum\limits_{i=1}^{N} \psi_i}{(\Omega/3)[10^{0.6(\mu+\Delta\mu-25)} - 10^{0.6(\mu-25)}]}, \tag{26}$$

(for comparison see equations 14 and 15 in Paper I).

5 Lynden-Bell's solution

In this section we assume that our data $(M_k, \mu_k)_{k=1,N}$ are ordered in such a way that $M_{k+1} \geqq M_k$.

Let us define, after Lynden-Bell (1971), a so-called $C^-(M')$ function which is equal to the number of galaxies inside the following region:

$$M_{\min} \leqq M < M'$$

$$\mu_{\min} \leqq \mu \leqq m_{\lim} - M' \tag{27}$$

(see Fig. 2). Let us introduce now the numbers C_k defined as:

$$C_k \equiv C^-(M_k), \qquad k = 1, \ldots, N \tag{28}$$

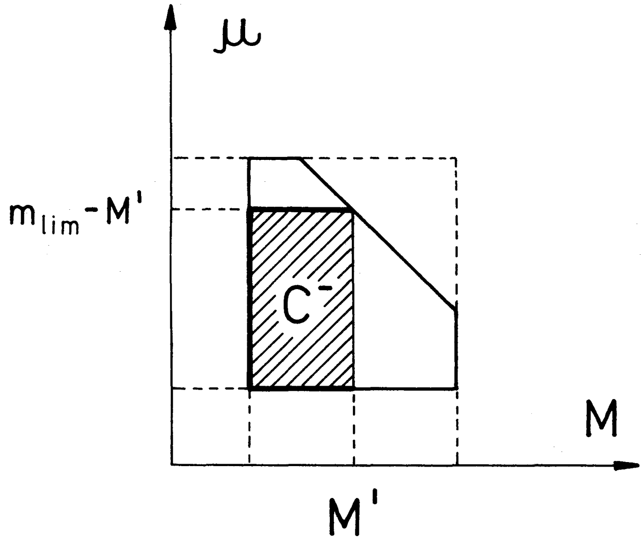

Figure 2. Illustration of the definition of the $C^-(M')$ function. Its value is equal to the observed number of galaxies in the shaded box.

(note, that $C_1=0$ for any sample). It can be shown, basing on equations (9) and (10) that:

$$C_k+1=\sum_{i=1}^{k}\psi_i \sum_{j:}^{M_k+\mu_j\leq m_{\rm lim}} d_j. \tag{29}$$

Equations (10) and (29) give:

$$C_k+1=\sum_{i=1}^{k}\psi_i/\psi_k, \tag{30}$$

which implies a simple recursion relation:

$$\psi_{k+1}=\psi_k\frac{C_k+1}{C_{k+1}} \qquad k=1,\ldots,N \tag{31}$$

and finally:

$$\psi_2=\psi_1\frac{C_1+1}{C_2},$$

$$\psi_3=\psi_1\frac{C_1+1}{C_2}\,\frac{C_2+1}{C_3},$$

$$\psi_4=\psi_1\frac{C_1+1}{C_2}\,\frac{C_2+1}{C_3}\,\frac{C_3+1}{C_4}, \qquad \text{etc.} \tag{32}$$

Equation (31), which gives the explicit solution for $\psi_k(k=1,\ldots,N)$, can be recommended for practical application. Analogous formulae can also be easily derived for $d_k(k=1,\ldots,N)$.

Let us express now the function $\phi(M)$ [which is proportional to the luminosity function $\phi(M)$] in the cumulative form:

$$\int_{M_{\rm min}}^{M}\phi(M)\,dM=\sum_{k:}^{M_k<M}\psi_k. \tag{33}$$

Taking into account equation (32) we have:

$$\int_{M_{\min}}^{M} \phi(M)\, dM = \psi_1 \prod_{k:}^{M_k<M} \frac{C_k+1}{C_k}, \tag{34}$$

where the factor $(C_1+1)/C_1$ should be replaced by unity. Equation (34) has a form identical to the appropriate expression in Lynden-Bell (1971).

The sequence $C_k(k=1, \ldots, N)$ can be used to express in a simple and elegant way the formula for the total number of galaxies N_t:

$$N_t = \prod_{k=2}^{N} \frac{C_k+1}{C_k} \tag{35}$$

which comes from equations (10) and (34).

6 An example

Let us consider a sample such that:

$M_1=4,\quad \mu_1=11,$
$M_2=9,\quad \mu_2=3,$
$M_3=3,\quad \mu_3=12,$
$M_4=11,\quad \mu_4=4,$
$M_5=5,\quad \mu_5=7,$
$M_6=7,\quad \mu_6=8,$
$m_{\lim}=17,\quad N=6.$

The sample is presented in Fig. 1. Equations (9) and (10) have the form:

$$\begin{aligned}
1&=d_1(\psi_1+\psi_3+\psi_5), & 1&=\psi_1(d_1+d_2+d_3+d_4+d_5+d_6),\\
1&=d_2(\psi_1+\psi_2+\psi_3+\psi_4+\psi_5+\psi_6), & 1&=\psi_2(d_2+d_4+d_5+d_6),\\
1&=d_3(\psi_1+\psi_3+\psi_5), & 1&=\psi_3(d_1+d_2+d_3+d_4+d_5+d_6),\\
1&=d_4(\psi_1+\psi_2+\psi_3+\psi_4+\psi_5+\psi_6), & 1&=\psi_4(d_2+d_4),\\
1&=d_5(\psi_1+\psi_2+\psi_3+\psi_5+\psi_6), & 1&=\psi_5(d_1+d_2+d_3+d_4+d_5+d_6),\\
1&=d_6(\psi_1+\psi_2+\psi_3+\psi_5+\psi_6), & 1&=\psi_6(d_2+d_4+d_5+d_6).
\end{aligned}$$

Let us assume now that $\psi_1=1$. This makes the solution of the above set of equation a unique one. The solution is:

$\psi_1=1,\quad d_1=1/3,$
$\psi_2=3,\quad d_2=1/18,$
$\psi_3=1,\quad d_3=1/3,$
$\psi_4=9,\quad d_4=1/18,$
$\psi_5=1,\quad d_5=1/9,$
$\psi_6=3,\quad d_6=1/9,$

which gives, according to equation (19):

$$N_t=(1+3+1+9+1+3)\times(1/3+1/18+1/3+1/18+1/9+1/9)=18$$

and:

$$\phi(M)=1\delta(M-4)+3\delta(M-9)+1\delta(M-3)+9\delta(M-11)+1\delta(M-5)+3\delta(M-7),$$
$$D(\mu)=1/3\delta(\mu-11)+1/18\delta(\mu-3)+1/3\delta(\mu-12)+1/18\delta(\mu-4)+1/9\delta(\mu-7)+1/9\delta(\mu-8).$$

The solution for ψ_k can be also obtained by using equation (31). The values C_k (see Section 5) which are needed for this purpose are:

$$C_1=0,$$
$$C_2=1,$$
$$C_3=2,$$
$$C_4=1,$$
$$C_5=2,$$
$$C_6=1.$$

The d_k can be computed by an analogous procedure.

Based on the known functions $\phi(M)$ and $D(\mu)$, the final results for $\Phi(M), \varrho(\mu)$ and n can easily be constructed (see Section 4).

7 Likelihood analysis

The likelihood for our problem is (according to Marshall *et al.* 1983):

$$L=\sum_{k=1}^{N}\ln\phi(M_k)\,D(\mu_k)-\int_{M_{\min}}^{M_{\max}}\int_{\mu_{\min}}^{\mu_{\max}}\phi(M)\,D(\mu)\,\theta[m_{\lim}-(M+\mu)]\,dM\,d\mu. \quad (36)$$

Substitution of equations (6) and (7) to equation (36) gives:

$$L=\sum_{k=1}^{N}\ln\,\psi_k\,d_k-\sum_{i:}^{M_i+\mu_j\leq m_{\lim}}\sum_{j:}\psi_i d_j. \quad (37)$$

Differentiation of L with respect to ψ_i and d_j again gives the same result as developed earlier (see Section 3), namely: equations (9) and (10).

8 Conclusions

I show in this paper that Lynden-Bell's method for the determination of the luminosity function (in the extended version presented in this paper) has the following features:

(i) It is based on a very simple idea.
(ii) It gives results after elementary, one-step computations.
(iii) It leads in a natural manner to a luminosity function which is properly normalized.

I argue that the above features of the method should make it more popular.

Is Lynden-Bell's method the best one? The answer to this important question needs a much broader context and will be discussed in a later paper.

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