Multivariable Calculus

Based on *Calculus: Early Transcendentals*, by Briggs, Cochran, Gillett, and Schultz, Third Edition. Note that in general, I skipped stuff that I believe I already know pretty well.

1 Vectors

1.1 Cross Product

Definition 1: Cross Product

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

Properties:

- Anticommutative: $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$
- Associative: $(a\mathbf{u})\times(b\mathbf{v}) = ab(\mathbf{u}\times\mathbf{v})$
- Distributive: $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{w} + \mathbf{v}\times\mathbf{w}$

One application: the volume of a parallelopiped is $|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|$

1.2 Lines and Planes in Space

Definition 2: Line in Space

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

The symmetric equation form of a line in space is:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Theorem 1: Distance Between Point and Line

To find the distance between a point Q and a line with direction vector $\mathbf{v}$, pick any point P on the line and do the following:

$$d = \frac{|\mathbf{v}\times\vec{PQ}|}{|\mathbf{v}|}$$

The equation of a plane can be found as follows. Suppose a plane passes through a point $P_0 = (x_0, y_0, z_0)$ and has a normal vector $\mathbf{n} = \langle a, b, c\rangle$. Then for any point P in the plane:

$$\mathbf{n}\cdot\vec{P_0P} = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Definition 3: Plane

A plane can be expressed in the following form:

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$

Theorem 2: Distance Between Point and Plane

For the point of interest Q, any point on the plane P, and the plane's unit normal vector $\hat{\mathbf{n}}$:

$$d = |\hat{\mathbf{n}} \cdot \vec{PQ}|$$

Procedure to check if lines intersect:

1. Check if they are parallel: if $\mathbf{v}_1 = c\mathbf{v}_2$, then the two lines are either parallel or the same line.

2. Check if skew:

$$x_1(t) = x_2(s)$$
$$y_1(t) = y_2(s)$$
$$z_1(t) = z_2(s)$$

   Since there are 3 equations and 2 unknowns (s and t), either one of the equations is redundant (in which case there is an intersection point) or one of the equations is contradictory (in which case there is no intersection and the lines are skew).

Procedure to check if planes intersect:

1. Check if they are parallel: if $\mathbf{n}_1 = c\mathbf{n}_2$, then either parallel or the same plane.

2. Otherwise, the vector of the line is:

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$$

1.3 Cylinders and Quadric Surfaces

Definition 4: Cylinder

In the context of three-dimensional surfaces, the term cylinder refers to a surface that is parallel to a line. If one of the coordinates (x, y, z) are missing in an equation, the equation denotes a cylinder parallel to the missing axis (e.g. $y = x^2$ is parallel to the z axis).

Definition 5: Trace

The trace is the set of points at which a surface intersects a plane (e.g. xy trace is the intersection of a surface and the xy plane, and to find it you would set $z = 0$).

Definition 6: Intercept

The intercepts are where a surface intersects the coordinate axis (e.g. the x-intercepts are found when setting $y = z = 0$).

Definition 7: Quadric Surface

A quadric surface only includes terms up to the 2nd degree. The general quadric equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where at least one of $A \to F$ must be nonzero.

Shape	Equation	Features
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	all traces are ellipses
Elliptic paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	xy traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Traces of $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are hyperbolas.
Hyperboloid of two sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Traces with $z = z_0$, $\lvert z_0\rvert > \lvert c\rvert$ are ellipses. Traces with $x = x_0$, $y = y_0$ are hyperbolas.
Elliptic Cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$	Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.
Hyperbolic Paraboloid	$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$	Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ and $y = y_0$ are parabolas.

Table 3: Types of Quadric Surfaces

2 Vector-valued Functions

2.1 Vector-valued Functions

Vector valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. **Domain:** largest set of values of t for which x, y, z are defined.

The natural direction or **orientation** of a curve is the direction in which the curve is generated as the parameter is increased. Orientation only exists in parameterized curves.

Definition 8: Limit of Vector Valued Function

A vector valued function $\mathbf{r}$ approaches $\mathbf{L}$ as t approaches a provided $\lim_{t\to a} \lvert\mathbf{r} - \mathbf{L}\rvert = 0$. In other words:

$$\lim_{t\to a} \lvert\mathbf{r} - \mathbf{L}\rvert = 0 \implies \lim_{t\to a} \mathbf{r} = \mathbf{L}$$

Definition 9: Continuity

A function $\mathbf{r}(t)$ is continuous at a provided

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$$

A function is continuous on an interval I if it is continuous for all t in I.

2.2 Calculus of Vector-Valued Functions

Definition 10: Vector Function Derivative

$$\frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$
$$= \langle f'(t), g'(t), h'(t) \rangle$$

Provided $\mathbf{r}'(t) \neq 0$, $\mathbf{r}'(t)$ is the tangent vector at the point $\mathbf{r}(t)$. Note: if $\mathbf{r}'(t) = 0$, a curve may charge direction of velocity abruptly even with finite acceleration. $\mathbf{r}(t)$ is said to be **smooth** if f, g, h are differentiable, and $\mathbf{r}'(t) \neq 0$ on the interval.

Definition 11: Unit Tangent Vector

The unit tangent vector $\mathbf{T}$ is

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Theorem 4: Derivative Rules

- Constant:
$$\frac{d\mathbf{C}}{dt} = 0$$
- Sum:
$$\frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \mathbf{u}' + \mathbf{v}'$$
- Product:
$$\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$
- Chain Rule:
$$\frac{d}{dt}(\mathbf{u}(f(t)) = \mathbf{u}'(f(t)) \cdot f'(t)$$
- Dot Product Rule:
$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$$
- Cross Product Rule:
$$\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$$

Definition 12: Integrals of Vector-valued Function

For $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, let $\mathbf{R}(t) = \langle F(t), G(t), H(t) \rangle$ where F, G, H are the antiderivatives of f, g, h respectively. Then the indefinite integral is:

$$\int \mathbf{r}(t)dt = \mathbf{R}(t) + \mathbf{C}$$

The definite integral is:

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(b) - \mathbf{R}(a)$$

2.3 Motion in Space

Definition 13: Motion Terms

Velocity:

$$\mathbf{v} = \mathbf{r'}(t)$$

Speed:

$$|\mathbf{v}| = |\mathbf{r'}(t)|$$

Acceleration:

$$\mathbf{a} = \mathbf{r''}(t)$$

Theorem 5: Circular Motion

If $|\mathbf{r}|$ is constant (i.e. if something is in circular motion), then $\mathbf{r} \cdot \mathbf{v} = 0$.

2.4 Length of Curves

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt = \int_a^b |\mathbf{r'}(t)| dt$$

Arc length function $\mathbf{s}$:

$$\mathbf{s}(t) = \int_a^t |\mathbf{v}(u)| du$$

where $ds/dt = |\mathbf{v}|$ by fundamental theorem of calculus.

Theorem 6: Arc Length as a Parameter

If $|\mathbf{v}| = 1$ for all $t \geq a$, then t represents arc length.

2.5 Curvature, Normal Vectors

Given the unit tangent vector $\mathbf{T} = \mathbf{r'}/|\mathbf{r'}|$, we can define a new concept called **curvature** describing how fast something changes direction.

Definition 14: Curvature

If s is arclength and $\mathbf{T}$ is the unit tangent vector, curvature is

$$\kappa(s) = \left| \frac{d\mathbf{T}}{ds} \right|$$

Using the chain rule, we can also express κ as follows:

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|d\mathbf{T}/ds|}{|ds/dt|} = \boxed{\frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right|}$$

Alternatively, you can do a lot of ugly vector algebra to get the following:

Theorem 7: Alternate Curvature Formulas

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

If we wanted to find the curvature a function $y = f(x)$, we can use the substitution $\mathbf{r} = \langle x, f(x) \rangle$ to find:

$$\kappa = \frac{f''(x)}{(1+[f'(x)]^2)^{3/2}}$$

The curvature answers the question of how fast a curve turns. The principal unit normal vector determines the direction in which a curve turns.

Definition 15: Principal Unit Normal Vector

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$$

Alternatively, using the chain rule:

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

Theorem 8: Properties of Principal Unit Normal Vector

- $\mathbf{T}$ and $\mathbf{N}$ are orthogonal at all points on a curve
- The principal unit vector points to the inside of a curve, in the direction where the curve is turning.

Components of acceleration:

$$\begin{aligned}\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) \\ &= \frac{d\mathbf{T}}{dt}\frac{ds}{dt} + \mathbf{T}\frac{d^2s}{dt^2} \\ &= \frac{d\mathbf{T}}{ds}\left(\frac{ds}{dt}\right)^2 + \mathbf{T}\frac{d^2s}{dt^2}\end{aligned}$$

Therefore, the following theorem holds:

Theorem 9: Components of Acceleration

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} = \kappa|\mathbf{v}|^2\mathbf{N} + \frac{d^2s}{dt^2}\mathbf{T}$$

The vectors $\mathbf{T}$ and $\mathbf{N}$ form the **oscullating plane**, the plane that the curve is currently curving within. How quickly does a curve move out of this plane? This is what the unit binormal vector $\mathbf{B}$ tells us:

Definition 16: Unit Binormal Vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

The rate at which a curve twists out of the oscullating plane is given by the rate at which $\mathbf{B}$ changes as you move along the curve, which is $d\mathbf{B}/ds$.

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = \frac{d\mathbf{T}}{ds}\times\mathbf{N} + \mathbf{T}\times\frac{d\mathbf{N}}{ds}$$

Since $d\mathbf{T}/ds$ and $\mathbf{N}$ are parallel, their cross product is $\mathbf{0}$. Therefore:

$$\frac{d\mathbf{B}}{ds} = \mathbf{T}\times\frac{d\mathbf{N}}{ds}$$

Note that since the following three facts are true:

1. $d\mathbf{B}/ds$ is orthogonal to $\mathbf{T}$
2. $d\mathbf{B}/ds$ is orthogonal to $\mathbf{B}$ (since $\mathbf{B}$ is a unit vector, and the derivative of a constant length vector always satisfies $\mathbf{r} \cdot \mathbf{v} = 0$).
3. $\mathbf{B}$ and $\mathbf{T}$ are orthogonal to $\mathbf{N}$

Therefore $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$. Because of this proportionality, we define the constant **torsion**, denoted by τ.

Definition 17: Torsion

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

Taking the dot product with $\mathbf{N}$ on both sides yields:

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Theorem 10: Alternate Formulas for $\mathbf{B}$ and τ

Defining $\mathbf{r'} = \mathbf{v}$, $\mathbf{r''} = \mathbf{a}$, and $\mathbf{r'''} = \mathbf{a'}$, the following are true:

$$\mathbf{B} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$$

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a'}}{|\mathbf{v} \times \mathbf{a}|^2}$$

3 Functions of Several Variables

3.1 Limits and Continuity

Definition 18: Limit Definition

A function f has a limit L as $P = (x, y)$ approaches $P_0 = (a, b)$, written as

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L$$

if, for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that (note that $\implies$ means 'implies')

$$|PP_0| < \delta \implies |f(x, y) - L| < \varepsilon$$

Note that this limit definition means the value L must be approached along all possible paths.

Definition 19: Boundary Points

Let R be a region in $\mathbb{R}^2$.

- an **interior point** P is entirely within R, meaning it is possible to find a disk centered at P that contains only points within R.
- a **boundary point** Q is on an edge of R, meaning every disk centered at Q contains at least one point in R and at least one point not in R.
- an **open region** consists only of interior points

- a **closed region** contains all boundary points of a region

Note that if P_0 is a boundary point of the domain, even if P_0 is not within the domain, $\lim_{(x,y)\to P_0} f(x,y)$ exists provided $f(x,y)$ approaches the same value along all paths within the domain.

Procedure: if $f(x,y)$ approaches two different values as (x,y) approaches (a,b) along two different paths in the domain of f, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Definition 20: Continuity

$f(x,y)$ is continuous at (a,b) provided

- f is defined at (a,b)
- $\lim_{(x,y)\to(a,b)} f(x,y)$ exists
- $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

3.2 Partial Derivatives

Definition 21: Partial Derivative

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

provided the limit exists. The following notations are equivalent:

$$\frac{\partial f}{\partial x}(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b)$$

To calculate a partial derivative, simply take the normal derivative while treating other variables as constants.

Example 1: Partial Derivatives

Example: Given

$$f(x,y,z) = \frac{x^2+xy}{\sin z}$$

find $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$.

$$\frac{\partial f}{\partial x} = \frac{2x+y}{\sin z}$$
$$\frac{\partial f}{\partial y} = \frac{x}{\sin z}$$
$$\frac{\partial f}{\partial z} = -\frac{x^2+xy}{\sin^2 z}\cos z$$

Higher Order Partial Derivatives. Note the following notation:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx} \text{ (mixed partial derivative)}$$

Theorem 11: Clairaut Equality of Mixed Partial Derivatives

If f is defined on an open set D of $\mathbb{R}^2$, and f_{xy} and f_{yx} are continuous throughout D, then $f_{xy} = f_{yx}$ at all points of D.

Assuming sufficient continuity, higher derivatives work as well:

$$f_{xyx} = f_{xxy} = f_{yxx}$$

Differentiability. Recall for a one variable function f, it is differentiable at $x = a$ provided the following limit exists:

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

This would imply:

$$\varepsilon \equiv \frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a)$$

$$\lim_{\Delta x \to 0} \varepsilon = 0$$

$$\varepsilon \Delta x = (f(a + \Delta x) - f(a)) - f'(a)\Delta x$$

$$\boxed{\delta y = f'(a)\Delta x + \varepsilon \Delta x}$$

Analogously for multiple variables:

Definition 22: Differentiability

$z = f(x, y)$ is differentiable at (a, b) provided $f_x(a, b)$ and $f_y(a, b)$ exist, and

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where for fixed a and b, ε_1 and ε_2 only depend on Δx and Δy respectively, with $(\varepsilon_1, \varepsilon_2) \to (0, 0)$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem 12: Determining Differentiability

If f_x and f_y are defined on an open set containing (a, b), with f_x and f_y continuous at (a, b), then f is differentiable at (a, b).

Theorem 13: Differentiability Implies Continuity

If f is differentiable at (a, b), then f is also continuous at (a, b). Contrapositive: if not continuous, then not differentiable.

3.3 Chain Rule

Theorem 14: Chain Rule

For one independent variable (e.g. x, y are functions of t):

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

For two independent variables (e.g. x, y are functions of both s and t):

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example 2: Chain Rule

Suppose $f(x, y) = xy$, where $x = s + t$ and $y = t^2$, and one wanted to find $\partial f/\partial s$. In this case, there are two ways to do this:

1. One can substitute x and y with s and t to calculate partial derivatives directly:

$$f(x, y) = (s + t)t^2 = st^2 + t^3$$
$$\frac{\partial f}{\partial s} = t^2$$

2. One can use the chain rule, which says that

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = (y)(1) + (x)(0) = y = t^2$$

Theorem 15: Implicit Differentiation

For $F(x, y) = 0$:

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y}$$

given $\partial F/\partial y \neq 0$.

3.4 Directional Derivative and Gradient

Definition 23: Directional Derivative

The net rate of change in $f(x, y)$ in the direction of $\hat{\mathbf{u}}$ is

$$D_{\hat{\mathbf{u}}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_x, b + hu_y) - f(a, b)}{h}$$

Evaluating the directional derivative without limits:

$$g(s) \equiv f(a + su_1, b + su_2) = f(x, y)$$
$$D_{\hat{\mathbf{u}}} f(a, b) = \frac{dg}{ds}(0) = \left(\frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}\right)\bigg|_{s=0}$$
$$D_{\hat{\mathbf{u}}} f(a, b) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \bigg|_{(a,b)} \cdot \hat{\mathbf{u}}$$

Definition 24: Gradient Vector

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \bigg|_{(x,y)}$$

Therefore,

$$\boxed{D_{\hat{\mathbf{u}}} f(a,b) = \nabla f(a,b) \cdot \hat{\mathbf{u}}}$$

Example 3: Directional Derivative

What is the derivative of $f(x,y) = x^2 + y^2$ in the direction of $\langle 2, 1\rangle$ at $(2,2)$? The unit vector is

$$\hat{u} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$\nabla f(2,2)$ is

$$\nabla f(2,2) = \left\langle 2x, 2y \right\rangle\Big|_{(2,2)} = \langle 4, 4\rangle$$

Therefore, $D_{\hat{\mathbf{u}}} f(2,2) = \boxed{\langle 8/\sqrt{5}, 4/\sqrt{5}\rangle}$

Theorem 16: Properties of Gradient

For scalar fields ψ and ϕ:

$$\nabla(\psi + \phi) = \nabla\psi + \nabla\phi$$
$$\nabla(\psi\phi) = \psi\nabla\phi + \phi\nabla\psi$$
$$\nabla\left(\frac{\psi}{\phi}\right) = \frac{\phi\nabla\psi - \psi\nabla\phi}{\phi^2}$$

Interpretation of gradient:

$$\begin{aligned} D_{\hat{\mathbf{u}}} f(a,b) &= \nabla f(a,b) \cdot \hat{\mathbf{u}} \\ &= |\nabla f(a,b)| \cos\theta \end{aligned}$$

where θ is the angle between $\nabla f(a,b)$ and $\hat{\mathbf{u}}$. This implies the following:

Theorem 17: Directions of change

Let f be differentiable at (a,b) with $\nabla f(a,b) \neq \mathbf{0}$.

- f has maximum rate of increase at (a,b) in the direction of gradient $\nabla f(a,b)$ with the rate of change $|\nabla f(a,b)|$.
- f has maximum rate of decrease at (a,b) in the direction of gradient $-\nabla f(a,b)$ with the rate of change $-|\nabla f(a,b)|$.
- The directional derivative is 0 in directions orthogonal to $\nabla f(a,b)$.

Theorem 18: Gradient and Level Curves

Given f is differentiable at (a,b), the line tangent to the level curve of f at (a,b) is orthogonal to the gradient $\nabla f(a,b)$, provided $\nabla f(a,b) \neq \mathbf{0}$.

This implies an alternate form of the tangent line to a level curve:

$$\nabla f(x,y) \cdot \mathbf{r}'(t) = 0$$

$$\frac{\partial f}{\partial x}\bigg|_{(a,b)} (x-a) + \frac{\partial f}{\partial y}\bigg|_{(a,b)} (y-b) = 0$$

For 3D gradients, level curves simply become level surfaces such that $f(x,y,z) = \text{const.}$, and all the above definitions and theorems can be extended straightforwardly.

3.5 Tangent Planes and Linear Approximation

For an implicitly defined surface $F(x,y,z) = 0$, we have the following:

$$\begin{aligned}\frac{dF}{dt} &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} \\ &= \nabla F \cdot \mathbf{r'}(t)\end{aligned}$$

Therefore, $\nabla F \cdot \mathbf{r'}(t) = 0$ at any point on the curve. Thus, for a particular point $P = (a,b,c)$, any curve that passes through P that is within $F(x,y,z) = 0$ is tangent to ∇F. Therefore, the tangent plane is

$$\boxed{\nabla F \cdot \langle x-a, y-b, z-c\rangle = 0}$$

For an explicitly defined surface $z = f(x,y)$, we can do the following:

$$\begin{gathered}\nabla F = \nabla(z - f(x,y)) = \langle -f_x, -f_y, 1\rangle \\ -f_x(x-a) - f_y(y-b) + z - f(a,b) = 0 \\ \boxed{z = f_x(x-a) + f_y(y-b) + f(a,b)}\end{gathered}$$

Definition 25: Linear Approximation

For a function $w = f(x,y,z)$, the linear approximation at the point $(a,b,c,f(a,b,c)$ is:

$$L(x,y,z) = f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) + f(a,b,c)$$

Similarly with differentials:

$$dw = f_x(a,b,c)dx + f_y(a,b,c)dy + f_z(a,b,c)dz$$

3.6 Maximums/Minimums

Definition 26: Local Maximum/Minimum Values

Suppose (a,b) is a point in a region R on which f is defined. If $f(x,y) \leq (a,b)$ for all (x,y) in the domain of f and in some open disk centered at (a,b), then $f(a,b)$ is a **local maximum** value of f. (similar definition for min). Local maximum and local minimum values are also called **local extreme values** or **local extrema**.

Theorem 19: Derivatives and Local Extrema

If f has a local maximum or minimum at (a,b), and if f_x and f_y exist at (a,b), then $f_x(a,b) = f_y(a,b) = 0$

Using this theorem, we can narrow down the candidates for local extrema to a set of points called **critical points**.

Definition 27: Critical Points

An interior point (a,b) in the domain of f is a critical point if either

- $f_x(a,b) = f_y(a,b) = 0$ or

- at least one of f_x or f_y does not exist at (a, b)

Some critical points are neither minima nor maxima. In this case, they are **saddle points**.

Definition 28: Saddle Point

Consider a function f that is differentiable at a critical point (a, b). Then f has a saddle point at (a, b) if, in every open disk centered at (a, b), there are points (x, y) for which $f(x, y) < f(a, b)$ and points for which $f(x, y) > f(a, b)$.

Theorem 20: Second Derivative Test

Suppose $f_x(a, b) = f_y(a, b) = 0$. Using the quantity $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b).
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b)
3. If $D(a, b) < 0$, then f has a saddle point at (a, b)
4. If $D(a, b) = 0$, then the test is inconclusive.

The second derivative depends on $D(x, y)$, called the **discriminant** of f. It can be remembered as the determinant of the 2×2 **Hessian** matrix:

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Note that analagous to the Extreme Value Theorem in one variable, higher dimensional absolute max/min values occur either at local min/max or at the boundary of the region of interest.

The following presents the intuition behind the Second Derivative Test. Suppose you have $f_x(a, b) = f_y(a, b) = 0$, and you wanted to create a quadratic estimate of the function f at (a, b). In order to do that, we would try to create a function with equal second derivatives f_{xx}, f_{xy}, and f_{yy}. To do this, we would construct the function

$$g(x, y) = f(a, b) + \frac{1}{2}f_{xx}(a, b)(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{1}{2}f_{yy}(a, b)(y - b)^2$$

For simplicity of notation, let us consider let us consider $a = b = f(a, b) = 0$, and drop all instances of (a, b). In this case:

$$g(x, y) = \frac{1}{2}f_{xx}x^2 + f_{xy}xy + \frac{1}{2}f_{yy}y^2$$

It remains to determine when $g(x, y)$ is a minimum, maximum, or saddle. Suppose we fix $y = y_0$. The roots of the resulting one dimensional parabola are:

$$x_{\text{root}} = \frac{-f_{xy}y_0 \pm \sqrt{(f_{xy}y_0)^2 - f_{xx}f_{yy}y_0^2}}{f_{xx}} = y_0 \left(\frac{-f_{xy} \pm \sqrt{f_{xy}^2 - f_{xx}f_{xy}}}{f_{xx}} \right)$$

If there is more than one root, this means that along y_0, $g(x, y)$ has both positive and negative values, implying that $g(x, y)$ is a saddle. Therefore:

$$f_{xy}^2 - f_{xx}f_{xy} > 0 \implies \text{saddle}$$

If $f_{xy}^2 - f_{xx}f_{xy} < 0$, this means there are no places where $g(x, y) = 0$ other than $(0, 0)$, which means that $(0, 0)$ is either a maximum or minimum depending on the sign of f_{xx}.

Finally, if $f_{xy}^2 - f_{xx}f_{xy} = 0$, there is a string of roots along the line $x = -y(f_{xy}/f_{xx})$, which means that higher orders of derivatives along that line may influence whether the point is a maximum, minimum, or saddle. Therefore, the second derivative test is inconclusive in this case.

3.7 Lagrange Multipliers

Theorem 21: Lagrange Multipliers

Suppose we seek the extrema of an **objective function** f, with the restriction that (x, y) must lie on a **constraint curve** C given by $g(x, y) = 0$. Assuming $\nabla g(a, b) \neq \mathbf{0}$, the extreme points satisfy the following relation:

$$\nabla f(a, b) = \lambda \nabla g(a, b)$$

where λ is a real number called the **Lagrange multiplier**.

The reason $\nabla f(a, b)$ and $\nabla g(a, b)$ are parallel is simple:

1. if f is at an extremum, the path of $g(x, y) = 0$ must be perpendicular to $\nabla f(a, b)$.
2. since $g(x, y) = 0$ is a level curve of $z = g(x, y)$, $\nabla g(a, b)$ is perpendicualr to the path of $g(x, y) = 0$
3. since $\nabla g(a, b)$ and $\nabla f(a, b)$ are two dimensional vectors, and both are perpendicular to the same thing, these two vectors must be parallel.

Similarly, in $\mathbb{R}^3$ the following equation must be met:

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

4 Multiple Integration

4.1 Multiple Integrals

Definition 29: Double Integral

A function f defined on a rectangular region R in the xy-plane is integrable on R if

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k$$

exists for all partitions of R and all choices of (x_k^*, y_k^*) within those partitions. The limit is the **double integral of f over R**.

$$\iint_R f(x, y) dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k$$

Iterated integrals are how double integrals are evaluated in practice. The following two expressions are equivalent:

$$\int_0^1 \int_0^2 (6 - 2x - y) dy dx = \int_0^1 \left(\int_0^2 (6 - 2x - y) dy \right) dx$$

The inner integral would be evaluated as a single integral with x held constant. Then, the outer integral would be calculated with respect to x.

Theorem 22: Fubini's Theorem over Rectangular Regions

The area integral of a rectangular region can be evaluated as an interated integral, where the order of integration does not matter, i.e. for the region $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Definition 30: Average Value

$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x,y)dA$$

General integrals can have functions of a different variable in their limits, e.g.

$$\int_0^3 \int_{x+1}^{x^2+3} ydydx$$

In polar coordinates, a double integral looks like

$$\iint_R f(x,y)dA = \iint_R f(x,y)rdrd\theta$$

Triple integrals are much of the same:

$$\iiint_D f(x,y,z)dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,y,z)dzdydx$$

And average values divide by volume of D rather than area of R.

4.2 Triple Integrals in Other Coordinates

- Cylindrical Coordinates:

$$\iiint_D f(x,y,z)dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)dzrdrd\theta$$

- Spherical Coordinates:

$$\iiint_D f(x,y,z)dV = \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2 \sin\varphi d\rho d\varphi d\theta$$

4.3 Mass Integrals

Definition 32: Center of Mass in One Dimension

$$\bar{x} = \frac{M}{m}$$

where M is the total moment and m is the mass, each given by

$$M = \int_a^b x\rho(x)dx$$

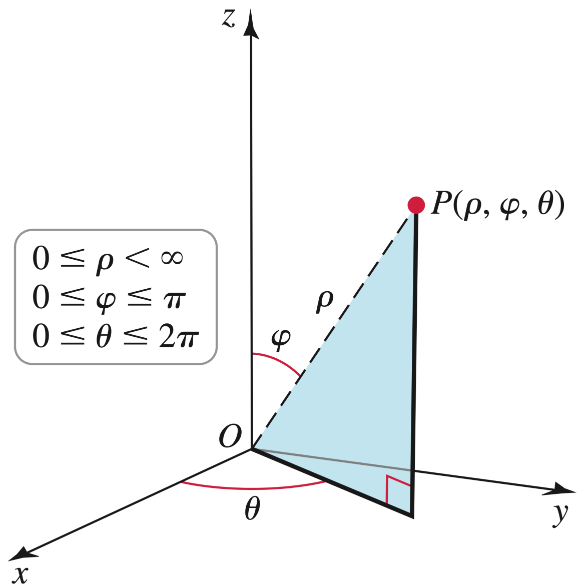

Figure 31: Spherical Coordinates

$$m = \int_a^b \rho(x)dx$$

Definition 33: Center of Mass in Two Dimensions

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho dA$$
$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho dA$$

where $m = \iint_R \rho dA$ and M_x and M_y are the moments with respect to the x and y axes, respectively. If ρ is constant, the center of mass is called the **centroid** and is independent of density.

Definition 34: Center of Mass in Three Dimensions

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint_D x\rho dV$$
$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint_D y\rho dV$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint_D z\rho dV$$

where $m = \iiint_D \rho dV$ and M_{yz}, M_{xz}, and M_{xy} are the moments with respect to the coordinate planes.

4.4 Change of Coordinates in Multiple Integrals

Definition 35: Terms Related to Transformations

A change of variables of $(u, v) \to (x, y)$ is written compactly as $(x, y) = T(u, v)$. T has two components:

$$x = g(u, v)$$

$$y = h(u, v)$$

Geometrically, T maps the region S in uv space to the region R in xy space. We write the outcome of this process as $R = T(S)$ and call R the **image** of S under T. A transformation T is **one-to-one** if $T(P) = T(Q)$ only when $P = Q$, where P and Q are points in S.

We define a new concept, the determinant of the Jacobian matrix:

Definition 36: Jacobian Determinant

For two variables, given a transformation $(x, y) = T(u, v)$, the Jacobian determinant (or just Jacobian) of T is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

For three variables, given a transformation $(x, y, z) = T(u, v, w)$, the Jacobian of T is:

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Now, we are ready to change variables in multiple integrals:

Theorem 23: Change of Variables

For two variables:

$$\iint_R f(x, y)dA = \iint_S f(g(u, v), h(u, v))|J(u, v)|dA$$

For three variables:

$$\iiint_D f(x, y, z)dV = \iiint_S f(g(u, v, w), h(u, v, w), p(u, v, w))|J(u, v, w)|dV$$

The reasoning behind the Jacobian determinant factor is explained in the following. In the two dimensional case, the vector change due to a small change du in the xy plane is

$$d\mathbf{u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle du$$

Similarly, for dv it is:

$$d\mathbf{v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle dv$$

Thus, an area of $dA = dudv$ in the uv plane has an equivalent area in the xy plane of

$$|d\mathbf{u} \times d\mathbf{v}|$$

which if you ignore the scuffed-ness of 2D cross products is equivalent to the Jacobian multiplied by $dudv$.

Strategies for Choosing New Variables:

- aim for simple regions of integration in the uv plane
- let the integrand suggest new variables
- let the region suggest new variables

5 Vector Calculus

5.1 Vector Fields

Definition 37: Vector Field Definitions

$$\mathbf{F}(x,y) = \langle f(x,y), g(x,y)\rangle$$

A radial vector field is

$$\mathbf{F} \propto \mathbf{r} = \langle x, y\rangle$$

A gradient vector field is one that can be defined as

$$\mathbf{F} = \nabla\varphi$$

5.2 Line Integrals

Definition 38: Line Integral

$$\int_C f ds = \int f(x(t), y(t))|\mathbf{r'}(t)|dt$$

For a line integral of a vector field

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_C \mathbf{F}\cdot\mathbf{r'}(t)dt$$

In practice line integrals are evaluated using parameterization. Using the notation that $-C$ is the same curve as C but oriented in the opposite direction, note the intuitive fact that

$$\int_{-C}\mathbf{F}\cdot\mathbf{T}ds = -\int_C \mathbf{F}\cdot\mathbf{T}ds$$

Example 4: Line Integral

Find the line integral of $f(x,y) = xy$ over the curve $x^2 + y^2 = 9$ in the first quadrant oriented from $(3,0)$ to $(0,3)$.

To evaluate this integral, we use the parameterization of $\mathbf{r}(t) = \langle 3\cos t, 3\sin t\rangle$ on the time range $(0, \pi/2)$:

$$\int_C f ds = \int_0^{\pi/2} xy\cdot 3dt = \int_0^{\pi/2} 27\cos t\sin t dt = \frac{27}{2}$$

Definition 39: Circulation

On a closed, oriented curve, the circulation of $\mathbf{F}$ on C is

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_C \mathbf{F}\cdot\frac{\mathbf{r'}(t)}{|\mathbf{r'}(t)|}|\mathbf{r'}(t)|dt = \int_C \mathbf{F}\cdot\mathbf{r'}(t)dt = \int_C \mathbf{F}\cdot d\mathbf{r}$$

Note that circulation is positive for vectors generally pointing in the counterclockwise direction.

Definition 40: Flux (for curve in $\mathbb{R}^2$

The flux is

$$\int_C \mathbf{F}\cdot\hat{\mathbf{n}}ds = \int_a^b (f(t)y'(t) - g(t)x'(t))dt$$

5.3 Conservative Vector Fields

Definition 41: Conservative Vector Field

A vector field $\mathbf{F}$ is conservative if there exists a scalar field φ such that $\mathbf{F} = \nabla\varphi$.

A consequence of this definition is that $f = \varphi_x$, $g = \varphi_y$, and $h = \varphi_z$. Therefore, the statement of Clairaut's theorem gives us some constraints on $\mathbf{F}$:

$$\varphi_{xy} = \varphi_{yx} \implies f_y = g_x$$
$$\varphi_{xz} = \varphi_{zx} \implies f_z = h_x$$
$$\varphi_{yz} = \varphi_{zy} \implies g_z = h_y$$

Theorem 24: Test for Conservative Vector Fields

$\mathbf{F}$ is conservative iff

$$\mathbb{R}^3 : f_y = g_x, f_z = h_x, \text{and } g_z = h_y$$
$$\mathbb{R}^2 : f_y = g_x$$

The procedure for finding potential functions for conservative vector fields in $\mathbb{R}^3$ is as follows:

1. integrate $\varphi_x = f$ with respect to x to obtain φ, which includes the arbitrary function $C(y, z)$ (a generalization of an arbitrary constant).
2. Compute φ_y and equate it to g to obtain $C_y(y, z)$.
3. Integrate $C_y(y, z)$ to find $C(y, z)$ in terms of an arbitrary function $d(z)$
4. Repeat for z component to find φ.

Theorem 25: Fundamental Theorem for Line Integrals

If $\mathbf{F} = \nabla\varphi$:

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

where A is the beginning point and B is the endpoint. This theorem implies for a closed loop C:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \varphi(A) - \varphi(A) = 0$$

This theorem also implies that the integral is path independent, as long as the endpoints are the same:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

Note that one can prove that

$$\text{path independence} \iff \mathbf{F} \text{ is conservative} \iff \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

5.4 Green's Theorem

Theorem 26: Green's Theorem: Circulation Form

Let C be a simple closed piecewise curve, oriented counterclockwise, that encloses a simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C f dx + g dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

Note that the left two expressions are the circulation, and the right side is the surface integral of the two-dimensional curl: $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$.

In physics terms, this intuitively makes sense as follows. Consider a magnet with constant magnetization vector so that every little region has the same dipole moment vector per unit volume. The right side would be equivalent to summing each individual region's dipole moment, whereas the left side would be equivalent to finding the dipole moment by integrating the current around the outer edge. These two sides are equal since each little piece's neighbors' currents cancel with their own currents, leaving only the very boundary in the integral.

Example 5: Green's Theorem to Calculate Area

For vector field $\mathbf{F} = \langle 0, x \rangle$, Green's theorem becomes:

$$\oint_C x dy = \iint_R dA = \text{area of } R$$

Similarly for $\mathbf{F} = \langle y, 0 \rangle$:

$$\oint_C y dx = -\iint_R dA = -\text{area of } R$$

Thus:

$$\boxed{\text{area of } R = \frac{1}{2} \oint_C (x dy - y dx)}$$

Due to the symmetry of this relation, this may simplify the calculation of area. For example, in calculating the area of an ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

one can use the parametric relations $\mathbf{r}(t) = \langle a \cos t, b \sin t \rangle$ to find that

$$x dy - y dx = (a \cos t)(b \cos t dt) - (b \sin t)(-a \sin t dt) = ab dt$$

$$A = \frac{1}{2} \int_0^{2\pi} ab dt = \pi ab$$

Theorem 27: Green's Theorem: Flux Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C f dy - g dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA$$

The left two expressions are the outward flux, and the right side is the surface integral of the two-dimensional divergence.

Example 6: Green's Theorem on Complicated Regions

Find the outward flux of the vector field $\mathbf{F} = \langle xy^2, x^2y \rangle$ across the boundary of the annulus $R = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$.

Because R is not simply connected, Green's Theorem in its circulation form does not apply as stated. However, consider paths L_1 and L_2 along with the two paths along the circles of the annulus, as shown in the figure below. L_1 and L_2 would make the region a simply connected region, and as L_1 approaches L_2 their contributions to the circulation cancel. In other words, we can take the effective circulation of R (as if it were a simply connected region) to simply be the sum of the counterclockwise circulation of the outer circle and the clockwise circulation of the inner circle.

Using Green's Theorem in polar coordinates:

$$\begin{aligned} \text{circulation} &= \iint_R (y^2 + x^2) dA \\ &= 2\pi \int_1^2 r^2 (r dr) \\ &= \boxed{\frac{15\pi}{2}} \end{aligned}$$

Definition 43: Stream Function for $\mathbb{R}^2$ Vector Fields

A stream function for a vector field $\mathbf{F} = \langle f, g \rangle$ (if it exists) satisfies the following properties:

$$\frac{\partial \psi}{\partial y} = f$$

$$\frac{\partial \psi}{\partial x} = -g$$

The 2D divergence of a vector field with a stream function is:

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \psi_{yx} - \psi_{xy} = 0$$

For vector fields $\mathbf{F}$ that are both conservative and source free (i.e. there exists a stream function for the vector field), they have both zero curl and zero divergence. This means the following:

$$\mathbf{F} = \nabla \varphi = \left\langle \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right\rangle$$

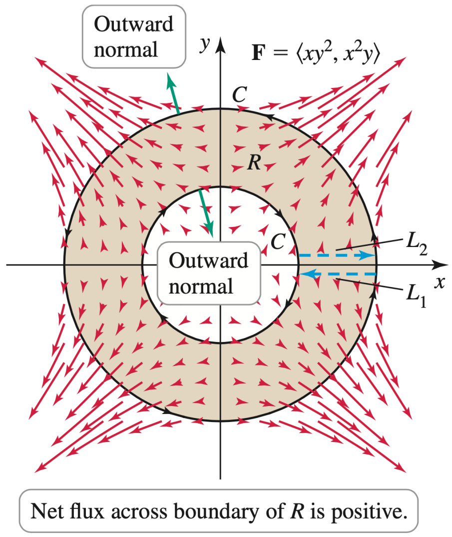

Figure 42: Green's Theorem Example

(2D divergence):

$$\text{div } \mathbf{F} = f_x + g_y = \varphi_{xx} + \varphi_{yy} = 0$$

(2D curl):

$$\text{curl } \mathbf{F} = g_x - f_y = -\psi_{xx} - \psi_{yy} = 0$$

Thus, both functions satisfy an important equation known as **Laplace's equation**:

$$\varphi_{xx} + \varphi_{yy} = 0$$
$$\psi_{xx} + \psi_{yy} = 0$$

5.5 Divergence and Curl

Definition 44: Divergence

The divergence of a vector field $\mathbf{F}$ that is differentiable in a region of $\mathbb{R}^3$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Theorem 28: Divergence of Radial Vector Fields

For a vector field: $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^p$, its divergence is

$$\nabla \cdot \mathbf{F} = \frac{3-p}{|\mathbf{r}|^p}$$

Definition 45: Curl

For a vector field $\mathbf{F} = \langle f, g, h \rangle$, the curl is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

The $\mathbf{i}$th component of $\nabla \times \mathbf{F}$ gives the rotation at a particular axis about the $\mathbf{i}$ axis, where the positive rotation direction is given by the right hand rule. The other components follow similarly.

Some terminology:

- If $\nabla \cdot \mathbf{F} = 0$, a vector field is **source-free/incompressible**.
- If $\nabla \times \mathbf{F} = \mathbf{0}$, a vector field is **irrotational**.

For a general rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a}$ is a constant vector, brute calculation tells us that

$$\nabla \cdot \mathbf{F} = 0$$

$$\nabla \times \mathbf{F} = 2\mathbf{a}$$

If $\mathbf{F}$ is a velocity field, it can be shown that the angular velocity $\omega = |\mathbf{a}|$. Therefore:

$$\omega = \frac{1}{2}|\nabla \times \mathbf{F}|$$

And just as $\nabla f \cdot \mathbf{n}$ is the directional derivative in the direction of $\mathbf{n}$, $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ is the directional spin in the direction of $\mathbf{n}$, which is maximized when the curl aligns with $\mathbf{n}$.

Divergence and Curl Properties

- $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$
- $\nabla \cdot (c\mathbf{F}) = c\nabla \cdot \mathbf{F}$
- $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$
- $\nabla \times (c\mathbf{F}) = c\nabla \times \mathbf{F}$
- The curl of a conservative vector field is $\mathbf{0}$, i.e. $\nabla \times (\nabla \varphi) = \mathbf{0}$
- $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

Definition 46: Laplacian

The Laplacian of a scalar field u is:

$$\nabla^2 u = \nabla \cdot \nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Theorem 29: Product Rule for Divergence

Let u be a scalar field that is differentiable on a region D and let $\mathbf{F}$ be a vector field that is differentiable on D. Then:

$$\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u(\nabla \cdot \mathbf{F})$$

Summary of Conservative Field Properties:

- (definition) There exists a potential function such that $\mathbf{F} = \nabla\varphi$
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points and regardless of path
- $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all curves
- $\nabla \times \mathbf{F} = \mathbf{0}$ at all points

5.6 Surface Integrals

Parameterized surfaces in $\mathbb{R}^3$ take on the following form:

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$$

Some examples:

- Cylinder of radius a and height h: $\langle a\cos u, a\sin u, v\rangle$ for $0 \le u \le 2\pi, 0 \le v \le h$
- Cone of radius a and height h: $\langle (av/h)\cos u, (av/h)\sin u, v\rangle$ for $0 \le u \le 2\pi, 0 \le v \le h$
- Sphere of radius a: $\langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$ for $0 \le u \le \pi, 0 \le v \le 2\pi$

Definition 47: Surface Integral of Scalar Field

Assume tangent vectors $\mathbf{t}_u$ and $\mathbf{t}_v$ have the following definitions:

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$
$$\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

Then the surface integral of the scalar function f over the smooth surface S is

$$\iint_S f(x, y, z)dS = \iint_R f(x(u, v), y(u, v), z(u, v))|\mathbf{t}_u \times \mathbf{t}_v|dA$$

Essentially, $dS = |\mathbf{t}_u \times \mathbf{t}_v|dA$ converts between area in coordinate space and the actual area element of the surface.

For an explicitly defined surface $z = g(x, y)$, using x and y as our parameters, we find that

$$\mathbf{t}_x = \frac{\partial \mathbf{r}}{\partial x} = \left\langle 1, 0, \frac{\partial z}{\partial x} \right\rangle$$

$$\mathbf{t}_y = \frac{\partial \mathbf{r}}{\partial y} = \left\langle 0, 1, \frac{\partial z}{\partial y} \right\rangle$$

$$|\mathbf{t}_x \times \mathbf{t}_y| = \left| \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle \right| = \sqrt{1 + z_x^2 + z_y^2}$$

Therefore:

$$\iint_S f(x,y,z)dS = \iint_R f(x,y,g(x,y))\sqrt{1+z_x^2+z_y^2}dA$$

Definition 48: Surface Integral of Vector Field

Note: the surface must be **orientable** (**two-sided**), meaning at every point there is a non-ambiguous normal vector (e.g. can't be Mobius strip, which has only one side). For a vector field $\mathbf{F} = \langle f, g, h \rangle$:

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) dA = \iint_R (-f z_x - g z_y + h) dA$$

5.7 Stokes' Theorem

Stokes' theorem is the $\mathbb{R}^3$ generalization of Green's theorem in circulation form.

Theorem 30: Stokes' Theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

Note that $\mathbf{n}$ and the orientation of C satisfy the right hand rule.

Stokes' theorem allows us to interpret curl in terms of **average circulation**. Define average circulation of $\mathbf{F}$ over S to be

$$\frac{1}{\text{area of } S} \oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{\text{area of } S} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

For a general rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, we know that $\nabla \times \mathbf{F} = 2\mathbf{a}$ and therefore the average circulation equals: $2\mathbf{a} \cdot \mathbf{n}$. Thus:

- The scalar component of $\nabla \times \mathbf{F}$ at a point in the direction of $\mathbf{n}$ is the average circulation of $\mathbf{F}$.
- The direction of $\nabla \times \mathbf{F}$ is the direction that maximizes the average circulation at a point.

Final notes about Stokes' Theorem:

- For any two surfaces with consistent boundary curve and orientation C:

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

- A closed surface can be split into two surfaces with the same boundary curve but different orientation. Therefore,

$$\iint_{S_{\text{closed}}} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = 0$$

5.8 Divergence Theorem

The divergence theorem is the $\mathbb{R}^3$ version of Green's theorem (flux version).

Theorem 31: Divergence Theorem

Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by an oriented surface S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_D (\nabla \cdot \mathbf{F}) \, dV$$

Theorem 32: Divergence Theorem for Hollow Regions

Suppose a region is bounded by S_1 and S_2, where S_1 lies within S_2, and let n_1 and n_2 be the outward normal vectors. Then:

$$\iiint_D \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_2} \mathbf{F} \cdot \mathbf{n_2} \, dS - \iint_{S_1} \mathbf{F} \cdot \mathbf{n_1} \, dS$$