Tidbits on

Physics

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These notes and much more are on my website: will-s-h.github.io. These notes were mainly written at the time I was trying to make my 2nd IPhO team (sadly, I didn't make it). For context, I had taken a half-year hiatus from physics, and given my terrible memory I had forgotten many of the concepts I had learned, besides the basics. Here, I note down some of the things that were surprising or surprisingly helpful in solving and understanding physics olympiad problems. My main sources were:

- ⋆⋆⋆ *Kevin Zhou's Handouts.* An extremely well compiled list of problems and concepts for physics olympiads that you can use for free at `https://knzhou.github.io/#handouts`.
- ⋆ *An Introduction to Classical Mechanics* by David Morin.
- ⋆ *Physics* by Halliday, Resnick and Krane.
- I also reference several physics papers along the way, which I mainly found through the above resources.

Important Disclaimer: do not expect to find every problem solving technique or every physical concept in these notes. These notes are intended to be used only as a supplement to your learning in physics.

Contents

0 Mathematics

0.1 Hyperbolic Functions

Definition: Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2} \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} \quad \operatorname{sech}(x) = \frac{1}{\cosh(x)} \quad \coth(x) = \frac{1}{\tanh(x)}$$

The following hyperbolic identities can be verified using their definitions:

- $\cosh^2(x) - \sinh^2(x) = 1$
- $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

The Taylor series of $\sinh(x)$ and $\cosh(x)$ are similar to that of $\sin(x)$ and $\cos(x)$:

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

From the Taylor series, the following are clear:

- $\sinh(ix) = i\sin x$, $\sinh x = -i\sin(ix)$
- $\cosh(ix) = \cos x$, $\cosh x = \cos(ix)$
- $\tanh(x) = -i\tan(ix)$

The following derivatives and integrals are relevant for hyperbolic functions:

- $\frac{d}{dx}\sinh x = \cosh x$, $\frac{d}{dx}\cosh x = \sinh x$, $\frac{d}{dx}\tanh x = \operatorname{sech}^2(x)$
- $\int \sinh x = \cosh x$, $\int \cosh x = \sinh x$
- Note the following integrals for inverse hyperbolic functions:

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x)$$

$$\int \frac{1}{x\sqrt{1-x^2}} = -\operatorname{sech}^{-1}(x)$$

0.2 Iteration

Suppose you have an equation of the form $x = f(x)$. One way to numerically solve it is to guess a value x_0, then repeatedly apply the function f until the value no longer changes. Consider the visual that happens when you do such a thing for $f(x) = \cos(x)$, located in Figure 1.

Suppose one chooses an x_0 to be close to the Dottie number, or the solution to $x = f(x)$. Let us call the Dottie number x_f. If $x_0 = x_f + dx$, where $dx \ll x_f$, then

$$x_1 = f(x_0) = f(x_f + dx) \approx f(x_f) + f'(x_f)dx = x_f + f'(x_f)dx$$

If $|f'(x_f)| < 1$, dx will continue to decrease until $x \to x_f$.

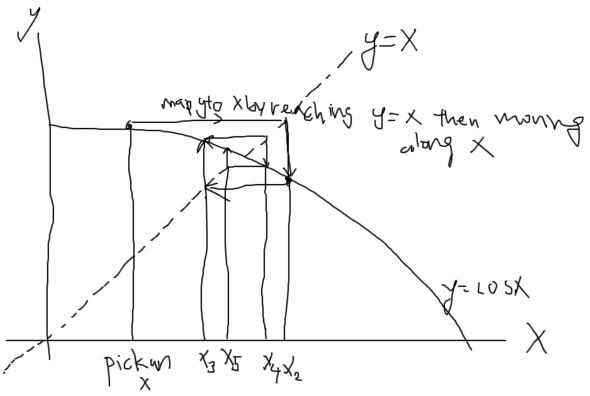

Figure 1: Iteration on $x = \cos x$

1 Mechanics I: Force and Energy

1.1 Force

A force is often defined, in Newton's second law, as $\mathbf{F} = m\mathbf{a}$. It's one of physics's most recognizable equations. Except, it's wrong. Force is actually more accurately:

Definition: Accurate Force

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

The difference is clear when we write out the derivative on $\mathbf{p} = m\mathbf{v}$. By the product rule:

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} + \frac{dm}{dt}\mathbf{v}$$

Therefore, $\mathbf{F} = m\mathbf{a}$ is only true when $dm/dt = 0$ or $\mathbf{v} = 0$.

1.2 Ropes and Strings

A rope (or string) in physics (**to be filled in**). (fill in that principle where ropes/strings under certain conditions follow the shortest path distance between two points or something of that sort).

(Should also add the fact that their potential energy is like $T \cdot L$, where T is tension and L is length).

1.3 Work

Work is defined as $\int F dx$, but its definition is much more subtle than it might seem. Consider the following examples:

Example: Confusing Work

1. You are running at full speed when you crash into the wall, losing all of your kinetic energy. How much work does the wall do on you?
2. A person sliding their hand across a stationary table, causing frictional heat. How much work does the hand's friction do on the table?

The key confusion that might occur is what dx means in $\int F dx$. The definition of dx is the motion of the **point of application** of the force. Importantly, the point of application is not an abstract point; it is the physical portion of a system that a force is acting on. This definition allows us to explain both examples:

1. The normal force from the wall is always applied at the portion of your body that is touching the wall, which are stationary. Therefore, the work is $\boxed{0}$.
2. In this case, *the work is also* $\boxed{0}$. The reason is the point of application does not refer to the hand, which is always in motion; it refers to each individual piece of the table on which friction acts, which is always at rest. Note that the location of the point of application does move, but the point of application itself does not move—each new location is independent of the last.

1.4 Potential Energy

You might have seen the definition of potential energy as $\Delta U = -W$ (such as HRK's equation 12-4), but note some subtleties regarding the equation:

Idea: Potential Energy Facts

- Potential energy is only for *conservative forces*. Other forces do not have potential energies associated with a particular configuration.
- Potential energy is for *systems* of objects. There is a potential energy associated with two objects A and B that are under mutual gravitational influence; there is no gravitational potential energy associated with A alone.
- What exactly is the work W? It is the work that one part of the system has done on the other. For example, in the case of two massive bodies A and B, W is the negative work that A does on B while B is moved away from A. In this process, the gravitational potential energy increases. Note that the work on this equation is exactly the opposite of the amount of work that would be done if B were to move back to its original position.
- $\mathbf{F} = -\nabla U$

1.5 Springs

Springs are things that stretch and contract, and based on amount of stretch and contraction they exert different amounts of force, given by Hooke's law:

$$F = -kx$$

Note x is relative to a spring's rest length. The potential energy of a spring either compressed or stretched by a length A is

$$U = -\int_0^A -kx dx = \frac{1}{2}kA^2$$

Proof: Springs in Parallel/Series

Parallel: Since the effective force is the sum of the two individual forces, we can replace the spring with an effective spring with effective spring constant $k = k_1 + k_2$.

Series: The force they transmit must be equal throughout all the springs: $F = k_1 x_1 = k_2 x_2$. Thus, the effective spring constant has the following expression:

$$\frac{1}{k} = \frac{x_1 + x_2}{F} = \frac{1}{k_1} + \frac{1}{k_2}$$

Note that springs attached in series also gives us insight into subsets of uniform springs. If a uniform spring is split into two equivalent springs, we have just derived that each of their spring constants is $2k$. If we were to have an infinitesimal piece dx of a spring of length L, it must have an extremely high spring constant $k' = kL/dx$.

Idea: Linearity of Hooke's Law

Because Hooke's law is linear, we can replace any stretched or compressed by a constant amount with an equivalent spring with equal spring constant but different rest length.

Example: Application of Linearity of Hooke's Law

Consider a mass m hanging from a vertical spring with spring constant k. After reaching equilibrium, we can replace the spring that is stretched by $x = mg/k$ with an equivalent spring that takes into account both the gravitational force and the spring force–from there, it is clear that the oscillation frequency is unchanged by gravity and is still $\omega = \sqrt{k/m}$.

With all this said, springs are *still* not as easy as they might seem. For example:

Example: Unequal Forces

If pull on a uniform spring (of constant k) with unequal forces F_a and F_b on opposite sides, how much does it elongate?

It is commonly thought to be $\min(F_a, F_b)/k$, but the answer to this question is actually $(F_a + F_b)/2k$. Let us assume without loss of generality that $F_a > F_b$. The first thing we should notice is that there must be a net acceleration of the spring. In order for this acceleration not to be infinite, the spring must have mass. Since Hooke's law definitely applies in the case of ideal, uniform, massless springs, but we're unsure of how it might apply to the general case, we will split the spring into N equal mass pieces (where $N \to \infty$),

where each piece has mass m/N and spring constant Nk.

In the accelerating reference frame of the center of mass of the spring, the steady state of the spring is clearly at rest (think of a vertical spring of mass m sitting on a table in a uniform gravitational field; the force applied at the top is F_b, the force applied by the table is F_a, and the gravitational field is akin to a). Therefore, each piece of the spring has the same acceleration as the center of mass, $a = (F_a - F_b)/m$, and the tension increases along the spring in the $F_b \to F_a$ direction linearly in mass, as $dF = a dm$. For the ith piece, the force is

$$F - F_b = a\frac{i}{N}m$$

$$F = F_b + \frac{i}{N}(F_a - F_b)$$

Therefore, the elongation of the ith piece is

$$\Delta x_i = \frac{F}{Nk} = \frac{F_b}{Nk} + \frac{i}{N^2 k}(F_a - F_b)$$

And the total elongation is:

$$\Delta L = \lim_{N\to\infty} \sum_{i=1}^{N} \Delta x_i = \int_0^1 \left(\frac{F_b}{k} + f\frac{F_a - F_b}{k}\right) df$$

where $f = i/N$ is the fraction of mass traversed. The integral evaluates to

$$\Delta L = \frac{F_b}{k} + \frac{F_a - F_b}{2k} = \boxed{\frac{F_a + F_b}{2k}}$$

1.6 Internal Energy

(From HRK 13-2) Consider an ice skater as she pushes herself away from a railing at the edge of a skating rink. In this situation:

$$\Delta U = 0 = W_{\text{ext}} = 0$$

$$\Delta K \neq 0$$

Where did the kinetic energy come from? It comes from the *internal energy* of the skater.

Definition: Internal Energy

E_{int} includes the microscopic unordered kinetic energies that is encapsulated in thermal energy, as well as rotational energy, chemical energy, and more.

The concept of internal energy may seem stupid (it's just every other form of energy!) but sometimes its easy to forget that internal energy exists. In the first example in Section 1.2, for example, your kinetic energy doesn't go into work exerted on the wall; it is converted into internal energy.

1.7 Collisions

Consider a 1D elastic collision between masses m_1 and m_2 such that their initial velocities are v_1 and v_2. The typical way of finding the final velocities is by writing conservation of kinetic energy and conservation of momentum:

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$$
$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$$

How would we brute force solve this set of equations? One of the easier ways is by taking the energy conservation equation and separating the m_1 and m_2 terms:

$$\frac{1}{2}m_1(v_1^2 - u_1^2) = \frac{1}{2}m_2(u_2^2 - v_2^2)$$
$$m_1(v_1 + u_1)(v_1 - u_1) = m_2(u_2 + v_2)(u_2 - v_2)$$

By momentum conservation, $m_1(v_1 - u_1) = m_2(u_2 - v_2)$, leaving us with a simplified form of energy conservation:

$$v_1 + u_1 = v_2 + u_2$$

Is there a deeper reasoning for why the final equation for energy conservation is much simpler than the quadratic terms we started with? In fact, there is.

Idea: Center of Mass Reference Frame

Consider the same 1D elastic collision, except in the center of mass reference frame with velocity

$$v_{\text{CM}} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$

In the center of mass reference frame, the total momentum of the system is 0, because each of the masses provide an equal and opposite momentum. In order to keep the total momentum 0 and preserve kinetic energy, the only possible final velocities are simply the opposite of the initial velocities:

$$u_1' = -v_1' \qquad u_2' = -v_2'$$
$$v_1' + u_1' = v_2' + u_2' = 0$$

Since each velocity increases by a quantity v_{CM} upon moving back to the initial frame, the sums of velocities remain equal, which proves our simplified form of energy conservation.

This center of mass reference frame technique is also a useful tool in inelastic collisions. Completely inelastic collisions are collisions in which bodies stick together—one can prove this type of collision consumes the most kinetic energy by considering the center of mass reference frames (note that ΔK due to the collision is the same in all reference frames, as proven in the later "Changes in Energy" section). In the center of mass reference frame, the collision that results in the greatest kinetic energy decrease while satisfying momentum conservation is the collision where both masses come to rest: in essence, the completely inelastic collision.

1.8 Friction

There are many subtleties with friction that cannot be covered in these notes. One brief subtlety I will mention is the direction of frictional motion. For kinetic friction, the frictional force opposes the direction of relative motion, and for static friction, the frictional force opposes the direction of relative acceleration. The direction of frictional force is more complicated than it might seem: the Painleve paradox, for example, is briefly discussed in Kevin Zhou's M2 handout—therefore I will refrain from discussing them here.

Another subtlety, turns out, is that the work due to friction is typically not well-defined because it depends on the details of its microscopic origin! How is that possible? Well, consider one model of friction in which two surfaces have "teeth" that weld together due to high pressures, then at a later time break, resulting in vibrations of thermal energy:

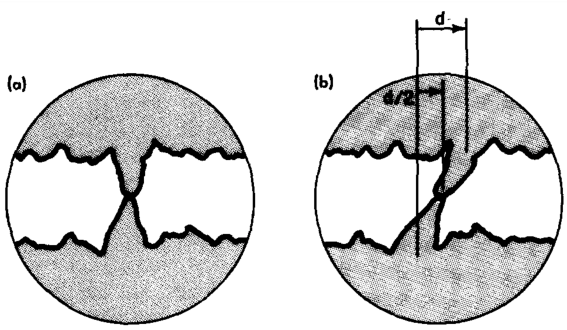

Figure 2: One model of Sliding Friction (Sherwood & Bernard, 1984).

In this particular model, the point of contact moves a distance $d/2$ for every distance d that the top surface moves. Therefore, frictional work in this model on the top surface is $W = -fd/2$, not $W = -fd$ as we might expect from the macroscopic view. This model shows how frictional work depends on how friction acts microscopically.

Despite this impediment, we can still determine quite a few general things regarding friction.

Example: Friction Work

Consider a block pulled along a horizontal table by tension T such that it is at constant speed, due an equal an opposite friction force f. By conservation of energy on the block, we have

$$\Delta K + \Delta E_{\text{int, block}} = W_T + W_f$$

$$W_f = -W_T + \Delta E_{\text{int, block}} = -Ts + \Delta E_{\text{int, block}} = -fs + \Delta E_{\text{int, block}}$$

Where s is the displacement of the block. Since internal energy in this case is only heat, by the second law of thermodynamics, $\Delta E_{\text{int, block}} > 0$. Therefore $|W_f| < fs$, unlike the naive expression $W_f = -fs$. If we instead choose the table as our system, we obtain

$$\Delta E_{\text{int, table}} = W_f' = -W_f$$

And since $\Delta E_{\text{int, table}} > 0$ as well by similar thermodynamic arguments, we have

$$-fs < W_f < 0$$

Finally, applying conservation of energy to the entire block + table system

$$\Delta E_{\text{int, block}} + \Delta E_{\text{int, table}} = W_T = fs$$
$$\boxed{\Delta E_{\text{int, total}} = fs}$$

The result we obtain is that the total increase in internal energy of the system is equal to magnitude of the "naive work" that friction exerts on the block. We cannot find the distribution of this internal energy unless we know the microscopic details OR the situation is highly symmetrical, in which case each object receives $+fs/2$ internal energy, independent of the frictional model.

We can generalize this result to more situations using an idea from the next section, the Center of Mass Energy. If you're unaware of Center of Mass Energy, you can skip this section for now and come back to it.

Proof: Generalization of Friction

Consider an example: block 1 is on top of block 2 such that block 1 moves to the right and block 2 also moves to the right, but at a lower speed. They have masses m_1 and m_2, respectively. The pair of surfaces has a coefficient of friction μ. Additionally, consider general forces F_1 acting on block 1 and F_2 acting on block 2, both positive and to the right. Using center of mass energy on each block yields:

$$(F_1 - f)s_1 = \Delta K_1$$
$$(F_2 + f)s_2 = \Delta K_2$$

Summing the two equations yields:

$$F_1 s_1 + F_2 s_2 - f(s_1 - s_2) = W_{\text{ext}} - f s_{\text{rel}} = \Delta K_{\text{total}}$$

Since any energy that is not going into kinetic energy is going into internal energy, we have

$$\Delta E_{\text{int, total}} = f s_{\text{rel}}$$

where s_{rel} is the relative displacement between the two blocks.

You may be wondering, why is it that this nuance is so rarely discussed, and even when you are not aware of this nuance most friction problems are solved correctly? That is because most often, one is actually mistakenly using center-of-mass energy, the topic of the next section.

1.9 Center of Mass Energy

In cases where internal energy is involved (such as when trying to determine frictional work), it occasionally makes normal conservation of energy useless. A new kind of work called *center of mass work* can be defined to work around such problems.

Definition: Center of Mass Work

$$W_{\text{cm}} = \int F dx_{\text{cm}}$$

Using a similar proof to the regular work-energy theorem, we have:

Proof: Center of Mass Work-Energy Theorem

$$W_{\text{cm}} = \int M \frac{dv_{\text{cm}}}{dt} v_{\text{cm}} dt = \int M v_{\text{cm}} dv_{\text{cm}} = \Delta\left(\frac{1}{2} M v_{\text{cm}}^2\right) = \Delta K_{\text{cm}}$$

This expression looks nearly identical to the regular work-energy theorem, except with cm subscripts. The key difference is how work is defined now: it no longer depends on the point of application; only the motion of the center of mass matters in this type of work. This can be very powerful—I highly recommend you read the four examples given in HRK 13-5.

For a very challenging problem on this topic, see USAPhO 2013 B1.

1.10 Changes in Energy: Reference Frames and Transfers

In this section we discuss both "changes in energy" due to change of reference frame, as well as actual changes in energy. Does conservation of kinetic energy hold in different reference frames? The answer is an obvious yes, but for completeness consider the following proof:

Proof: Kinetic Energy in Different Frames

Following Morin 5.6.2, suppose that in the CM frame of N objects, each object i has a velocity $\mathbf{v}_i'$. In a different inertial reference frame, each object has a velocity $\mathbf{v}_i = \mathbf{v}_i' + \mathbf{u}$. In the center of mass reference frame, the kinetic energy is

$$K_{\text{CM}} = \frac{1}{2} \sum m_i |\mathbf{v}_i'|^2$$

In the different reference frame, the kinetic energy is

$$K = \frac{1}{2} \sum m_i |\mathbf{v}_i' + \mathbf{u}|^2$$

$$= \frac{1}{2} \sum m_i (\mathbf{v}_i' \cdot \mathbf{v}_i' + 2\mathbf{v}_i' \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u})$$

$$= \frac{1}{2} \sum m_i |\mathbf{v}_i'|^2 + \mathbf{u} \cdot \left(\sum m_i \mathbf{v}_i'\right) + \frac{1}{2} |\mathbf{u}|^2 \sum m_i$$

Since $\sum m_i \mathbf{v}_i' = 0$ in the center of mass frame, the kinetic energy is simply:

$$K = K_{\text{CM}} + \frac{1}{2} M u^2$$

This is a useful theorem in and of itself, but it also easily proves the claim we are considering. Clearly, any frame in which K is conserved implies K is conserved in any other frame, since u is always constant for a given frame. ■

Note some of the results of this proof:

Idea: Corollaries of Kinetic Energy in Different Frames

- Though energy within one frame is conserved, the amount of energy that is conserved is different in different reference frames
- Momentum conservation is crucial to energy conservation in different frames. For example, suppose an otherwise unbothered system is given +20 J of energy in one frame. Depending on the Δp due to that same action, in a different frame, the system does not necessarily gain +20 J. This proof also helps us understand scenarios in which kinetic energy is not conserved. Consider P22 of the 2020 $F = ma$ A. Since $\Delta p = 0$, ΔK only depends on ΔK in the initial frame and not w; therefore, ΔK is constant.

Moving on to actual transfers of energy. Consider again P22 of the 2020 $F = ma$ A, and consider an inelastic collision between two masses. If there are no other energy sources, the lost kinetic energy must go into the heating of the two masses. Thus:

Idea: "Invariance of Heat Transfer"

Since heat capacity and temperature are intrinsic and frame-invariant, the amount of energy going into heating the masses (and thus the amount of kinetic energy lost) is constant.

While this seems like cop-out reasoning, this is completely valid reasoning.

Another instance of "invariance of amount of heat transfer" is in the following example:

Example: Another "Invariance of Heat Transfer" Problem

Sand falls slowly at a constant rate dm/dt onto a horizontal belt at constant v. Why is the power P needed to drive the belt two times the rate of increase of kinetic energy? Two solutions:

- One way we can arrive at this result is by calculating both quantities. The power needed to maintain the belt is $P = Fv = (dp/dt)v = (dm/dt)v^2$. The rate of increase of kinetic energy is $(1/2)(dm/dt)v^2 = P/2$. We have reached the desired result, albeit a little unsatisfactorily because we didn't uncover the deeper reasoning. Where is the energy lost?
- The way we can find the energy loss is by going into the belt's reference frame. There, the kinetic energy of each piece of sand dm is initially $(1/2)dmv^2$, and over a time dt the kinetic energy of that piece goes to 0, becoming heat. Since the quantity of heat generated is the same in all frames, the power going into heat loss is $(1/2)(dm/dt)v^2$, exactly the amount of energy that is lost.

1.11 Virtual Work

(to be completed: I should look at some handouts/resources before commenting)

2 Mechanics II: Multi-body Physics (Rotation, Fluids, etc.)

2.1 Rotation

Ah, rotation. It looks hard but is actually just as simple as translational motion because every rotational concept has a translational analogy . . . right? Try answering all parts of problem 1 of M5 and you'll see that this idea is totally wrong—rotation is complex in its own right.

Idea: Unique Ideas

- Parallel transport of a vector. Because of this, a rotation can NOT be expressed in terms of a vector.
- Net rotation without angular momentum is possible.

2.2 Torque and Angular Momentum, Basic Definitions

Definition: Torque

Torque about the origin is defined as

$$\boldsymbol{\tau} = \sum_i \mathbf{r}_i \times \mathbf{F}_i$$

Definition: Angular Momentum

Angular momentum about the origin is defined as

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

Proof: $\boldsymbol{\tau} = d\mathbf{L}/dt$

Taking a time derivative of angular momentum yields

$$\frac{d\mathbf{L}}{dt} = \sum_i \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i + \sum_i \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt}$$

Since $d\mathbf{r}_i/dt \parallel \mathbf{p}_i$, the first summation is equal to 0 and thus

$$\boxed{\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}}$$

Note that this expression is only true if torques are taken about the CM or about a fixed point (plus some edge cases, see later sections). This expression can be further simplified if we assume for internal forces that the force between two particles acts on the line between them (this is true in most cases, a notable exception being magnetic forces). In that case, $\boldsymbol{\tau}_{\text{int}} = \mathbf{0}$, so the above simplifies to

$$\boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}}{dt}$$

Note that for magnetic forces, one way to deal with this conundrum is to consider the field itself to have momentum and angular momentum.

Common results in physics textbooks include:

- Moment of inertia: $I = \int r^2 dm$
- Decomposition of $\mathbf{L}$ as $I\boldsymbol{\omega}$ or $I_{\text{CM}}\boldsymbol{\omega} + M\mathbf{r}_{\text{CM}} \times \mathbf{v}_{\text{CM}}$
- Rotational Kinetic Energy as $\frac{1}{2}I_p\omega^2$ or $\frac{1}{2}I_{\text{CM}}\omega^2 + \frac{1}{2}Mv_{\text{CM}}^2$

An important note to remember: these results all assume a fixed rotation axis, as well as some degree of symmetry for $\boldsymbol{\omega} \parallel \mathbf{L}$ to be true—check the next section for more details.

2.3 3D Rotation

Now this is where sh⋆t hits the fan (we now follow Morin Chapter 9).

Proof: Moment of Inertia Tensor

Suppose a rigid body is rotating with some angular velocity $\boldsymbol{\omega}$ about the origin. Omitting the dm, we expand the expression for $\mathbf{L}$ in components:

$$\mathbf{L} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = \int \langle x, y, z\rangle \times \langle \omega_2 x - \omega_3 y, \omega_3 x - \omega_1 z, \omega_1 y - \omega_2 x\rangle$$

$$\begin{aligned}\mathbf{L} = \int \langle &\omega_1(y^2+z^2) - \omega_2 xy - \omega_3 zx,\\ &\omega_2(z^2+x^2) - \omega_3 yz - \omega_1 xy,\\ &\omega_3(x^2+y^2) - \omega_1 zx - \omega_2 yz\rangle\end{aligned}$$

The angular momentum can therefore be written in the concise matrix form:

$$\begin{bmatrix}L_1\\L_2\\L_3\end{bmatrix} = \begin{bmatrix}\int y^2+z^2 & -\int xy & -\int zx\\ -\int xy & \int z^2+x^2 & -\int yz\\ -\int zx & -\int yz & \int x^2+y^2\end{bmatrix}\begin{bmatrix}\omega_1\\\omega_2\\\omega_3\end{bmatrix}$$

$$\equiv \begin{bmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{bmatrix}\begin{bmatrix}\omega_1\\\omega_2\\\omega_3\end{bmatrix}$$

$$\boxed{\mathbf{L} = \mathbf{I}\boldsymbol{\omega}}$$

where $I_{xz} = I_{zx}$, $I_{xy} = I_{yx}$, and $I_{yz} = I_{zy}$, i.e. the matrix is symmetric. Note the very unintuitive fact that $\boldsymbol{\omega}$ and $\mathbf{L}$ are NOT parallel in general.

Proof: Kinetic Energy

The kinetic energy of such motion is

$$K = \frac{1}{2}\int dm|\boldsymbol{\omega}\times\mathbf{r}|^2 = \frac{1}{2}\int dm\left((\omega_2 z-\omega_3 y)^2+(\omega_3 x-\omega_1 z)^2+(\omega_1 y-\omega_2 x)^2\right)$$

which upon expanding, one can see is equivalent to

$$K = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\boldsymbol{\omega} = \boxed{\frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}}$$

Now, we calculate the angular momentum for the general motion of a rigid body (which can be described as rotation + translation about the CM, by Chasles's theorem).

Proof: Decomposition of L

Using the position and velocity of the CM relative to the origin to be $\mathbf{R}$ and $\mathbf{V}$ respectively, and letting the position and velocity relative to the center of mass be $\mathbf{r}'$ and $\mathbf{v}'$ respectively, we have:

$$\mathbf{L} = \int \mathbf{r}\times\mathbf{v}dm = \int(\mathbf{R}+\mathbf{r}')\times(\mathbf{V}+\mathbf{v}')dm$$

$$= \int(\mathbf{R}\times\mathbf{V})dm + \mathbf{R}\times\left(\int\mathbf{v}'dm\right)+\left(\int\mathbf{r}'dm\right)\times\mathbf{V}+\int\mathbf{r}'\times\mathbf{v}'dm$$

Since $\int\mathbf{r}'dm = \int\mathbf{v}'dm = 0$, the second and third terms equal 0, so the result is

$$\mathbf{L} = M(\mathbf{R}\times\mathbf{V})+\mathbf{L}_{\text{CM}}$$

Examining the kinetic energy yields

Proof: Decomposition of K

$$K = \int\frac{1}{2}v^2dm = \int\frac{1}{2}|\mathbf{V}+\mathbf{v}'|^2dm$$

$$= \int\frac{1}{2}V^2dm+\int\frac{1}{2}v'^2dm \quad \text{(cross terms cancel again)}$$

$$K = \boxed{\frac{1}{2}MV^2+\frac{1}{2}\boldsymbol{\omega}'\cdot\mathbf{L}_{\text{CM}}}$$

One can also derive the general form of the parallel axis theorem.

Proof: General Parallel Axis Theorem

Suppose a rigid body rotates about the origin, but its center of mass is located at $\mathbf{R} = \langle X, Y, Z\rangle$. Using the fact that $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$, and splitting $x = X + x'$, etc. for all coordinates (and again finding finding cross terms cancel), we find

$$\mathbf{L} = M\begin{bmatrix} Y^2+Z^2 & -XY & -ZX \\ -XY & Z^2+X^2 & -YZ \\ -ZX & -YZ & X^2+Y^2 \end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}$$
$$+\begin{bmatrix} \int y'^2+z'^2 & -\int x'y' & -\int z'x' \\ -\int x'y' & \int z'^2+x'^2 & -\int y'z' \\ -\int z'x' & -\int y'z' & \int x'^2+y'^2 \end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}$$
$$\equiv (\mathbf{I}_{\mathrm{R}}+\mathbf{I}_{\mathrm{CM}})\boldsymbol{\omega}$$

For kinetic energy in the general parallel axis theorem, if $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ are equal, so that $\mathbf{V} = \boldsymbol{\omega}' \times \mathbf{R}$ (e.g. a rigid object rotating about the origin), then the kinetic energy is

$$K = \frac{1}{2}M|\boldsymbol{\omega}\times\mathbf{R}|^2 + \int \frac{1}{2}|\boldsymbol{\omega}\times\mathbf{r}'|^2 dm$$

which similarly simplifies to

$$K = \frac{1}{2}\boldsymbol{\omega}\cdot(\mathbf{I}_{\mathrm{R}}+\mathbf{I}_{\mathrm{CM}})\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}$$

Dealing with the general moment of inertia tensor all the time is painful. In fact, there is a way to avoid this pain:

Definition: Principal Axes

Every tensor has a special set of basis vectors, called the *principal axes*, such that $\mathbf{I}$ comes in the nice form

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Alternatively, principal axes can be defined as the axis $\hat{\boldsymbol{\omega}}$ such that $\mathbf{I}\hat{\boldsymbol{\omega}} = I\hat{\boldsymbol{\omega}}$ (i.e. $\boldsymbol{\omega} \parallel \mathbf{L}$ if rotating *only* about a principal axis), or the axes about which an object can rotate with constant angular speed when 0 torque is applied to it.

It's an amazing fact that for any $\mathbf{I}$, there exists a set of principal axes, as well as a general method of finding them–you can check Morin Appendix D for the full proof.

Idea: $\mathbf{L}$ and K for Principal Axes

If principal axes do exist, angular momentum and kinetic energy take on the following nice forms:

$$\mathbf{L} = \langle I_1\omega_1, I_2\omega_2, I_3\omega_3\rangle$$
$$K = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

In these notes I'll ignore the general way to find principal axes and focus on how practically this might come up in problems. Usually there are axes or planes of symmetry in a problem to make the calculations easier:

Idea: Special Circumstances for Principal Axes

In these special circumstances, it is possible to determine the principal axes. For a given origin:

- If the object has a plane of symmetry about the origin (ex. $x = 0$), then that plane of symmetry is a principal axis (since $\int xz = \int xy = 0$). As a corollary, if the object has two orthogonal planes of symmetry, the principal axes are completely defined.
- If the object is an axisymmetric body of revolution such that the axis passes through the origin, the axis of symmetry is a principal axis. The remaining two axes are any two orthogonal axes in the plane perpendicular to the axis of symmetry passing through the origin. In this case, since $I_2 = I_3$, we can use the idea in M8 of thinking of the rotation as $\mathbf{L} = I_{\parallel}\boldsymbol{\omega}_{\parallel} + I_{\perp}\boldsymbol{\omega}_{\perp}$

2.4 Euler's Equations

Although the principal axes are nice, they also introduce some new challenges, such as the fact that the principal axes themselves move in space. Therefore, $d\mathbf{L}/dt$ no longer equals $\boldsymbol{\tau}$ in the rotating frame of the principal axes, since $\mathbf{L}$ can change even without $\boldsymbol{\tau}$. The Euler equations resolve this:

Idea: Euler's Equations

The following equations are true if we take $\boldsymbol{\tau}$ about a fixed point or the CM, as usual:

$$\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2$$

$$\tau_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3$$

$$\tau_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1$$

Note that you only need to remember one of them, because the other two can be obtained by cyclic permutation of the indices.

Proof: Euler's Equations

In the lab frame, we have $\boldsymbol{\tau} = d\mathbf{L}/dt$, and we want to translate this equation to the body frame (i.e. the frame of the principal axes). Suppose $\mathbf{L}_0$ is the vector $\mathbf{L}$ at any given instant, such that it is fixed to the rigid body. Then

$$\frac{d\mathbf{L}}{dt} = \frac{d(\mathbf{L} - \mathbf{L}_0)}{dt} + \frac{d\mathbf{L}_0}{dt}$$

Or in other words, the rate of change of $\mathbf{L}$ is the rate of change of $\mathbf{L}$ relative to the body plus the rate of change of a fixed vector in the lab frame due to simply the rotation of the body. We can write this equation as

$$\frac{d\mathbf{L}_{\text{lab}}}{dt} = \frac{d\mathbf{L}_{\text{body}}}{dt} + \boldsymbol{\omega} \times \mathbf{L}_{\text{lab}}$$

The left hand side is simply the torque $\boldsymbol{\tau}$, whereas the right hand side yields:

$$\boldsymbol{\tau} = \frac{d}{dt}\langle I_1\omega_1, I_2\omega_2, I_3\omega_3\rangle + \langle \omega_1, \omega_2, \omega_3\rangle \times \langle I_1\omega_1, I_2\omega_2, I_3\omega_3\rangle$$

which upon expanding yields the desired result.

2.5 In-Depth Torque and Angular Momentum Definitions

(Illarramendi and Gaztelurrutia 1995, Moments to be Cautious of—Relative versus Absolute Angular Momentum) There's actually a subtlety regarding angular momentum that hasn't been addressed yet. Suppose we choose a point P that can move arbitrarily. There are two angular momentums we could calculate:

Definition: Absolute Angular Momentum

The *absolute angular momentum* is $\mathbf{L}$ calculated from an external inertial frame of reference:

$$\mathbf{L}^A = \sum_i m_i(\mathbf{r}_i - \mathbf{r}_p) \times \dot{\mathbf{r}}_i$$

Definition: Relative Angular Momentum

Relative angular momentum is taken in the frame of an observer instantaneously moving with the same velocity as P:

$$\mathbf{L}^R = \sum_i m_i(\mathbf{r}_i - \mathbf{r}_p) \times (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_p)$$

Since $\sum m_i\mathbf{r}_i = M\mathbf{r}_{\text{CM}}$, we can show that the two definitions of angular momentum are related by the following

$$\mathbf{L}^A = \mathbf{L}^R + M(\mathbf{r}_{\text{CM}} - \mathbf{r}_p) \times \dot{\mathbf{r}}_p$$

Notice that if P is chosen to be the center of mass (or the motion of P is parallel to its displacement from the center of mass), the relative and absolute angular momentums are equal. The reason we differentiated between these two angular momentums is that it turns out $\tau = d\mathbf{L}/dt$ is only valid in certain conditions, and those conditions are different for the two angular momentums!

Proof: Conditions of Validity for $\tau = d\mathbf{L}/dt$

Let us now differentiate (with respect to time) the absolute and relative angular momentum:

$$\frac{d\mathbf{L}^A}{dt} = \sum_i m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_p) \times \dot{\mathbf{r}}_i + \sum_i m_i(\mathbf{r}_i - \mathbf{r}_p) \times \ddot{\mathbf{r}}_i = -M(\dot{\mathbf{r}}_p \times \mathbf{r}_{\text{cm}}) + \tau$$

$$\frac{d\mathbf{L}^R}{dt} = \sum_i m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_p) \times (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_p) + \sum_i m_i(\mathbf{r}_i - \mathbf{r}_p) \times (\ddot{\mathbf{r}}_i - \ddot{\mathbf{r}}_p) = 0 + \tau - M(\mathbf{r}_{\text{CM}} - \mathbf{r}_p) \times \ddot{\mathbf{r}}_p$$

Or alternatively:

$$\tau = \frac{d\mathbf{L}^A}{dt} + M(\dot{\mathbf{r}}_p \times \dot{\mathbf{r}}_{\text{CM}})$$

$$\tau = \frac{d\mathbf{L}^R}{dt} + M(\mathbf{r}_{\text{CM}} - \mathbf{r}_p) \times \ddot{\mathbf{r}}_p$$

Note that this means $\tau = dL/dt$ applies when

- P is a fixed point or is the CM. This works for both definitions.
- $\dot{\mathbf{r}}_P \parallel \dot{\mathbf{r}}_{\text{CM}}$ or $\dot{\mathbf{r}}_{\text{CM}} = 0$, which works for absolute angular momentum
- P moves with constant velocity or $\mathbf{r_{CM}} - \mathbf{r}_p \parallel \ddot{\mathbf{r}}_p$. Works for relative angular momentum.
- Any P, as long as you account for the fictitious torque due to the frame of P accelerating. This works for relative angular momentum. Importantly, this means that you can use *accelerating but non-rotating* reference frames and still use $\tau = dL/dt$ provided you include fictitious torque.

To give you an idea of the ramifications, here is an example:

Example: Angular Momentum Subtlety

Let us use Morin 2.17 (M2 P12) as an example. Consider the case that $\theta = 0$. While it is rolling without slipping, straightforward analysis on the center of mass yields:

$$fR - Tr = I\alpha$$

$$T - f = ma$$

The non-slip condition implies $\alpha = a/R$, which results in

$$a = \frac{T(1 - r/R)}{m + I/R^2} > 0$$

A more slick way of approaching this is by considering torque about the spool's contact point on the ground. Naively, this solves everything completely: regardless of the value of f the torque is applied in the clockwise direction, so the spool will begin rotating clockwise. But what exactly does "the point on the ground" mean? Let us consider multiple definitions of this point, which we'll call P.

- P is a fixed point; it is only the spool's contact point with the ground initially. In this case, the naive $\tau = d\mathbf{L}/dt$ works completely; however, this angular momentum includes both spin angular momentum (angular momentum due to rotation about the center of mass) and orbital angular momentum (angular momentum due to the motion of the center of mass). In this case, we are only interested in spin angular momentum–although

we can use an extra step to figure out that rolling without slipping implies that the spin and orbital angular momentums are proportional, this $\tau = d\mathbf{L}/dt$ does not give us the most general information about which direction the spool would rotate.

- P is an accelerating point: it is always the spool's contact point with the ground. **(to be completed)**

2.6 Orbital Mechanics

2.7 Fluids

3 Electricity and Magnetism I: Basics

3.1 Field Lines

Field lines are curves whose tangent, at any point, lies in the direction of the field at that point (definition from Purcell 1.7). Practically, we often consider the following additional properties of field lines, due to a combination of electrostatic principles and just convention:

Idea: Field Line Principles/Convention

- Field lines must begin at positive charges (or ∞) and end at negative charges (or ∞).
- The density of field lines is proportional to the strength of the field at the location.
- Larger charges have more field lines starting from or ending on them.
- Field lines never cross, since $\mathbf{E}$ is a function and can't have multiple values at a single point.

Note that there are additional useful facts about field lines that come up:

Idea: Additional Useful Facts

- Number of field lines is proportional to flux.
- Electrostatic potential decreases monotonically along a field line.

3.2 Conductors

There are more to conductors than what meets the eye (literally). Conductors are rigid metal lattices of positive ions plus a sea of mobile electrons. Even when no net charge is placed on a conductor, electrons can move about, which is why the following phenomena can happen:

- A neutral conductor in a gravitational field has a slight electric field in its interior, exactly enough to counteract the force of gravity on an electron. (**E4** P4).
- A coil of wire, if spinned fast enough, can produce a current. The electrons are essentially "left-behind" while the positive lattice is rotated. (**ERev** P16).

Note that in the second example, the electrons are not completely stationary in the inertial reference frame. If electrons were simply left behind when the lattice of positive nuclei were rotated, that would mean any motion of conductors would result in insanely high amounts of current, which obviously does not happen. In most motions, the electrons are dragged along with the metal lattice, but with a slight lag in speed from the positive lattice. Why electrons are dragged along can be explained using *Drude theory*, which proposes a model for resistance that is dependent on electrons slowing down due to collisions with the lattice. Essentially, it is a metal's resistance that makes the lattice drag the electrons along for the ride.

3.3 Image Charges

The point of image charges is to be able to easily simulate the effect of surface charges on a conductor. An important thing to note: *image charges must not be in the same "region" as the region you would like to find electric fields in.* What exactly does that mean?

Proof: "Not In Same Region" Statement

Suppose you tried to simulate the effects of surface charges using an image charge that was in the same "region" that you were trying to calculate, like P5(b) of **E2**. Then you could make an arbitrary Gaussian surface surrounding that image charge. According to the image charge, there should be a net flux through the Gaussian surface, but there is no true charge within the surface, yielding a contradiction of Gauss's law. In other words, there is no way for surface charges to orient themselves so that they simulate the field of a point charge that is in the region that you are trying to calculate the electric field.

3.4 Accounting for All Energies in E&M

Example: Hidden Energy

Consider problem 13b in **E2**. In order for $F = -dU/dx$, U has to reflect the entire energy of the system, which includes the work done by the battery. Since $U_{\text{bat}} = -W_{\text{bat}}$, we have that

$$dU_{\text{bat}} = -\phi dQ = -\phi^2 dC$$

Since the work done by a battery on a charge dQ is ϕdQ, where ϕ is the potential difference across the battery. Therefore, the force that the plate feels is

$$F = -\frac{dU_{\text{total}}}{dx} = -\frac{dU_{cap}}{dx} - \frac{dU_{\text{bat}}}{dx} = -\frac{1}{2}\phi^2\frac{dC}{dx} + \phi^2\frac{dC}{dx} = \boxed{\frac{1}{2}\phi^2\frac{dC}{dx}}$$

The energy expended by the battery is what I call a "hidden energy." (There's probably something that's more official that is actually called hidden energy, similar to how "hidden momentum" is an actual term in relativity and E&M, but a quick Google search didn't pull up anything so...) These come up a lot, especially in E&M, so it's important to keep track of every possible source of energy when using energy methods.

Example: Hidden Energy, Pt. 2

What is the change in length of the spiral spring/wire, which has N turns, radius R, length x_0, and spring constant k, when a small current I is made to flow through it by an ideal current source?

(Note that the version with a superconducting wire is in Example 8, **E5**, and that the answers should be the same since investigating forces reveals that the parallel currents in each loop attract by the same amount when the current is passed through the spring/wire, regardless of its source.)

We can consider an ideal current source to be a battery of variable voltage $V = d\Phi/dt$, whose power is $P = IV = Id\Phi/dt$. Therefore, the work dW done in a time dt is $Id\Phi = I^2 dL$, and the equivalent "potential energy" change is $-I^2 dL$. Therefore, for a virtual displacement of dx, the total changes in potential energy look like the following:

$$dU = dU_{\text{spring}} + dU_{\text{mag}} + dU_{\text{current}}$$

$$0 = k(x - x_0)dx + \frac{1}{2}I^2 dL - I^2 dL$$

$$x - x_0 = \frac{I^2}{2k}\frac{dL}{dx} = \frac{I^2}{2k}\frac{d}{dx}\left(\frac{N(\mu_0 \frac{N}{x} I)\pi R^2}{I}\right) = -\frac{I^2}{2k}\frac{\mu_0 N^2 \pi R^2}{x^2}$$

Since I is small, $x \approx x_0$ on the right hand side, so

$$\boxed{x \approx x_0 - \frac{I^2}{2k}\frac{\mu_0 N^2 \pi R^2}{x_0^2}}$$

There are a couple similarities in the two examples that I would like to point out:

1. Hidden energies are sneaky because not accounting for them results in a lost negative sign, which is insidious because it gives a physically wrong result but seems to stem from an error. For the second example in this section, one may infer that parallel currents attract, but not accounting for the hidden energy might result in the conclusion that it repels.

2. There is usually a related problem, which through a force analysis can be shown to be equivalent, but avoids the hidden energy problem altogether. For the first example, it's the case where the charge is held constant and there is no battery. For the second example, it's the case where a superconducting wire is used and there is no ideal current source.

3.5 Related Problems/Analogies

In E&M it is a common technique to think of analogies that make something easier to visualize or easier to calculate. Below is an example of an E&M to mechanics analogy:

Example: AC Circuits $\leftrightarrow$ Driven Damped Harmonic Oscillators

Components in AC Circuits correspond to the following components in driven damped oscillators:

$$Q \leftrightarrow x, \quad I \leftrightarrow v, \quad \dot{I} \leftrightarrow a, \quad L \leftrightarrow m, \quad R \leftrightarrow b, \quad C \leftrightarrow 1/k, \quad V_0 \leftrightarrow F_0.$$

Next, we consider two different circuits with similar properties:

Example: RC Circuit Time Constant

For problem 13 in **E3** (Kalda), the time constant for the circuit will come down to the following equation:

$$\mathcal{E} = (\cdots)I + (\cdots)Q$$

If you were to consider the same circuit, except excluding the battery, the circuit will have essentially the same equation, except:

$$0 = (\cdots)I + (\cdots)Q$$

Both have the same time constants but in one of the circuits, the equivalent resistance is suddenly very easy to calculate.

Finally, we consider an analogy between electricity and magnetism:

Example: Solving for $\mathbf{E}$ using Biot-Savart!

For a system with no net charges, the following relations about electricity are true:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{E} = 0$$

This is analagous to the equations regarding magnetism:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0$$

The neat thing is, we already know the solution to the latter set of differential equations: Biot-Savart's law! Therefore, if we replace every instance of $\mu_0\mathbf{J}$ with $-\dot{\mathbf{B}}$ (equivalently, replacing $\mu_0 I$ with $-\dot{\Phi}$, assuming the boundary through which we calculate Φ is stationary), we can find the electric field:

$$\mathbf{E} = -\frac{\dot{\Phi}}{4\pi} \int \frac{d\mathbf{s} \times \hat{\boldsymbol{r}}}{r^2}$$

See **E5** problem 16 to see more.

3.6 Circuits Intuition

How exactly do DC circuits work? One calculation you might do at some point is find the drift velocity (average velocity) of electrons in a conductor:

Example: Drift Velocity

What is the drift velocity of a copper wire with radius $r \approx 1$ mm and current 1 A? The density of free electrons in copper is 8.4×10^{28} m^{-3}.

The current can be found in terms of the drift velocity in the following expression:

$$I = neAv_d$$

Therefore

$$v_d = \frac{I}{neA} = \frac{1 \text{ A}}{(8.4 \times 10^{28} \text{ m}^{-3})(1.6 \times 10^{-19} \text{ C})(\pi \cdot 10^{-6} \text{ m}^2)} \approx 0.02 \text{ mm/s}$$

The numbers differ from example to example, but typically drift velocities are actually rather slow, on the order of mm/s or less. Then how is it possible for circuits, such as light bulbs, to begin working almost instantaneously, if the electrons are that slow? There are two things to note here:

1. The speed of electrons is actually closer to the *Fermi velocity*, which is so fast that it approaches the speed of light. The only thing is, electrons continually collide with the positive lattice and change directions, so that their average velocity over a period of time is the drift velocity. But this effect does not explain how circuits begin working almost simultaneously.
2. The real reason is because individual electrons don't need to travel from the battery through all the circuit components and to the other end. As the drift velocity suggests, the average electron doesn't actually make it that far. But since there are free electrons *throughout the circuit*, in the wires and in the electrical components that make up the circuit, what matters is the propagation of the electric field; as long as there is electric field nearby, the nearby free electrons will cause the current to flow.

The second point brings up another conceptual question: how does the electric field propagate and get setup so it points exclusively parallel to a wire? The answer is **surface charges:**

Idea: Surface Charges

(Müller 2012, A semiquantitative treatment of surface charges in DC circuits). There are three main roles of surface charges:

1. They maintain potential around the circuit
2. They provide the electric field in the space outside the conductor
3. They ensure the confined flow of current by generating electric field parallel to the wire

There are two types of surface charges:

1. *Type-I surface charges* occur at the boundary of two conductors of different resistivities. Note they exist because in order for current to be constant, the electric field must be higher in materials of higher resistivity ($\mathbf{E} = \rho\mathbf{J}$).
2. *Type-II surface charges* reside at the surface of conductors. These accumulate on the sides and corners of wires to direct other electrons.

How do the surface charges get to where they are supposed to be? Initially, they aren't present, so the electrons that reach a conductor-conductor interface or a corner simply stay there. Those hundreds or so electrons (it doesn't take too many to direct the rest of the current) become surface charges that allow the rest of the electrons to flow properly.

There's some more fascinating stuff in the paper, which I'll leave to you to check out.

3.7 Kirchoff's Laws and Kinetic Energy

(the end remark of **E5**).

3.8 Failure of Ideal Circuits

There are several situations in which ideal circuits fail to give sensible results:

Example: Ideal Batteries/Current Sources

Two batteries in parallel will try to set the voltage difference to different amounts, and two ideal current sources in series will try to set the current to different amounts, leading to a contradiction. Practical batteries can be considered as a battery with an internal resistance in series, whereas practical current sources can be considered as ideal current sources with an internal resistance in parallel, which helps resolve the contradiction.

Next, we consider another case in which ideal circuits fail to give us a satisfying explanation:

Example: Ideal Capacitor/Inductor

Consider an ideal capacitor connected to a battery (or similarly, an ideal inductor connected to an ideal current source). If everything is ideal and there are no energy losses, how can we explain the fact that the battery does a work $W = \mathcal{E}Q$ but the capacitor only has energy $U = \mathcal{E}Q/2$?

The problem is there is no such thing as "everything is ideal." (Using the analysis presented in E6, example 2). If the resistance of the circuit dominates (overdamping), the capacitor charges monotonically, and if the self-inductance of the circuit dominantes (underdamping), the capacitor voltage oscillates around $\mathcal{E}$, eventually settling down due to resistance. In total, as the energies above show, half the energy is lost regardless of how "ideal" the circuit is!

Jumping ahead and presenting an analogy with thermodynamics, attaching a battery of fixed voltage is kind of like instantly dropping a piston, and letting it oscillate until it comes to a stop. The non-adiabatic process results in an increase in entropy; we could avoid this using an adiabatic process. In thermodynamics, this would mean adiabatically compressing the piston; in this circuit, it would mean adjusting the voltage in infinitesimal increments, so that the circuit is always in equilibrium.

3.9 Capacitor and Inductor Intuition

I'm sure you've heard this one before, but here's one way to think about capacitors and inductors in DC circuits:

Idea: Early/Late Time Approximations

When a DC circuit has reached its steady state, all quantities (charges, currents, voltages) are constant. Therefore:

- Capacitors are essentially breaks in a circuit, because Q is constant and therefore $I = 0$.
- Inductors are essentially zero-resistance wires since $dI/dt = 0$.

When a DC circuit has just been closed, it is actually a little more complicated:

- A capacitor is instantaneously a battery of emf Q/C, where Q is what it is before the circuit is closed since $\Delta Q = 0$ in infinitesimal times unless $I = \infty$. If $Q = 0$, it is like a zero-resistance wire.
- An inductor is instantaneously an ideal current source of current I, where I is what it is before the circuit is closed since $\Delta I = 0$ for infinitesimal times unless $\mathcal{E} = \infty$. If $I = 0$, it is like a circuit break.

3.10 Maxwell's Equations

The following are the four equations that make up Maxwell's Equations:

Definition: Gauss's Law for Electricity

$$\iint_\Sigma \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

where Σ is a closed 2D surface, i.e. there is volume enclosed in it.

Definition: Gauss's Law for Magnetism

$$\iint_\Sigma \mathbf{B} \cdot d\mathbf{A} = 0$$

Σ is again a closed 2D surface.

Definition: Faraday's Law

$$\oint_{\partial\Sigma} \mathbf{E} \cdot d\mathbf{s} = -\iint_\Sigma \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

where Σ is this time an *open* 2D surface (one that has no volume enclosed), and $\partial\Sigma$ is the boundary of that surface.

Definition: Ampere-Maxwell Equation

$$\oint_{\partial\Sigma} \mathbf{B} \cdot d\mathbf{s} = \mu_0 \iint_{\Sigma} \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{A}$$

where Σ and $\partial\Sigma$ are again an open 2D surface and its boundary, respectively.

We'll discuss some subtleties with Faraday's law and the Ampere-Maxwell equation in a later section. For now, note the similarities between the Ampere-Maxwell equation and Faraday's law. The only asymmetry is the missing $\mathbf{J}_m$ in Faraday's law, and that is since there is no need to consider magnetic currents.

Definition: Differential Form of Maxwell's Equations

The four equations listed above are listed again in the same order, but in differential form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

There is yet another form of Maxwell's Equations that is used in terms of polarization, magnetization, and all that good stuff, but we'll leave that for its own section.

3.11 Induction

The work done per unit charge (often considered while travelling through a loop) is unintuitively called the *electromotive force*, even though it is not a force. It can be calculated as follows:

Definition: Electromotive Force (emf)

$$\mathcal{E} = \frac{1}{q} \oint_C \mathbf{F} \cdot d\mathbf{s}$$

It's important to keep in mind this definition and not depend on the results we derive below, for a number of reasons. Firstly, often there is no "loop" to use for the universal law of induction we will derive below. Further, the electromotive force need not even be due to electromagnetic fields (see, for instance, **ERev** P16).

But suppose the emf were due to just electromagnetic fields. Then the following important result is true:

Idea: Universal Law of Induction

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

As long as only electricity and magnetism are involved, this statement is always true!

Proof: Universal Law of Induction

We'll consider two separate cases: a time-independent magnetic field (but changing loop) and a stationary loop with a changing magnetic field. The general case is proven by considering both of these effects at the same time.

First, a time-independent magnetic field. Note that since $\partial\mathbf{B}/\partial t = 0$ in this case, Faraday's law says that there are no nonconservative loops of $\mathbf{E}$, and thus $\mathbf{E}$ contributes nothing to the emf. Therefore

$$\mathcal{E} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s}$$

It remains to be shown that the expression on the right is $-d\Phi_B/dt$. Consider $d\Phi_B$ over an infinitesimal increment in time dt. Each portion $d\mathbf{s}$ of the loop moves a distance $\mathbf{v}dt$, which results in an additional flux $\mathbf{B} \cdot (\mathbf{v}dt \times d\mathbf{s})$. Therefore:

$$d\Phi_B = \oint_C \mathbf{B} \cdot (\mathbf{v}dt \times d\mathbf{s})$$

$$\frac{d\Phi_B}{dt} = \oint_C \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{s})$$

Using the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$, the result follows:

$$\frac{d\Phi_B}{dt} = -\oint_C d\mathbf{s} \cdot (\mathbf{v} \times \mathbf{B}) \implies \mathcal{E} = -\frac{d\Phi_B}{dt}$$

Next, we consider a stationary loop. Since $\mathbf{v} = 0$, emf is only due to the presence of nonconservative $\mathbf{E}$. In this case, we can directly apply Faraday's law:

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{s} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

Since the surface S is fixed, we can move the time derivative outside, which yields the desired result:

$$\mathcal{E} = -\frac{d}{dt}\iint_S \mathbf{B} \cdot d\mathbf{A} = -\frac{d\Phi_B}{dt}$$

In this proof, we already navigated several subtleties regarding Faraday's law:

- The electric field induced by a changing magnetic field is nonconservative, unlike electric fields due to charges.
- In the form of Faraday's law we have in these notes, the right hand side is NOT the time derivative of the (flux of the magnetic field), but rather, it is the flux of (the time derivative of

the magnetic field). Both integrals are equal if the surface is stationary, however.

- $\mathcal{E} \neq \oint_C \mathbf{E} \cdot d\mathbf{s}$, although the equality is true when the loop is stationary and no other forces are involved.

3.12 Mutual Inductance

For two inductors L_1 and L_2 with mutual inductance M, the differential work to change the currents is:

$$dU = (L_1 I_1 + M I_2) dI_1 + (L_2 I_2 + M I_1) dI_2$$
$$dU = L_1 I_1 dI_1 + L_2 I_2 dI_2 + M d(I_1 I_2)$$

Therefore, the total work is

$$U = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$$

Physically, the work done must be nonnegative for all values of I_1 and I_2. Taking $x = I_2/I_1$, we have that

$$U \propto L_2 x^2 + 2Mx + L_1 = L_2 \left(x + \frac{M}{L_2} \right)^2 + L_1 - \frac{M^2}{L_2^2}$$

Clearly, $U \geq 0$ for all x if

$$|M| \leq \sqrt{L_1 L_2}$$

4 Electricity of Magnetism II: Advanced Topics

4.1 Electric Dipoles

I think the definition of an electric dipole moment given in **E1** is a little misleading, since the dimensions don't seem exactly right when you first look at the integral. It is better written as

$$\mathbf{p} = \int \rho(\mathbf{r}) \mathbf{r} dV$$

4.2 Magnetic Dipoles

4.3 Quasistatic Regime

When we apply Faraday's law, we often use Ampere's law (without the extra displacement current term) to calculate the magnetic field. This is not generally valid, but works if the currents are in the slowly changing "quasistatic" regime, which means radiation effects are negligible.

4.4 Displacement Current

Proof: Ignoring Displacement Current

Suppose $\mathbf{J}$ is constant, regardless of whether ρ is constant. If we look at displacement currents in this scenario:

$$\nabla \times \mathbf{J}_d = \varepsilon_0 \nabla \times \frac{\partial \mathbf{E}}{\partial t} = \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

If currents are constant, then $\mathbf{B}$ is constant so $\nabla \times \mathbf{J}_d = 0$. However, this means that $\mathbf{J}_d$ can be described as a superposition of radial, spherically symmetric currents, which produce no magnetic field because

1. Each spherically symmetric current distribution's magnetic field would have to be radial by symmetry.
2. However, any non-zero radial field would violate Gauss's law, so it is not only radial; the magnetic field must also be 0.

Therefore, $\mathbf{J} = \text{const.} \implies$ one can ignore displacment currents' effect on the magnetic field. Note too that even if $\mathbf{J}$ is changing, as long as it changes slowly, $\partial^2 \mathbf{B}/\partial t^2$ is small enough that this approximation still applies.

4.5 Electromagnetic Radiation

5 Thermodynamics

5.1 Statistical Mechanics I: Boltzmann Factor

Definition: Macrostates and Microstates

To describe the state of a system, you can use microstates and macrostates:

- Microstates usually entail microscopic configurations (e.g. position and velocity), and each microstate is usually equally likely.
- Macrostates are described using macroscopic quantities (such as pressure, volume, or total energy), and each macrostate has different probability, since each macrostate corresponds to a different number of microstates.

Armed with this knowledge, we derive the probability distribution for a particle, which explicity has just one microstate per macrostate of total energy:

Idea: Boltzmann Factor/Canonical Factor

The probability distribution for states of a particle in a system of temperature T is proportional to $e^{-E/k_B T}$.

Proof: Boltzmann Factor

(Blundell and Blundell Chapter 4). First, we introduce some assumptions:

1. Each one of the possible microstates of a system is equally likely to occur
2. The system's internal dynamics are such that the microstates of the system are continually changing
3. Given enough time, the system will explore all possible microstates and spend an equal time in each of them (called the **ergodic hypothesis**.)

Given these assumptions, we find that *a system will appear to choose macroscopic configurations that maximizes the number of microstates.* This is because practically, the macrostate of with the most microstates has an extremely large number of microstates; therefore, the system will pretty much exclusively spend its time in this macrostate.

Let us consider then a **canonical system**, which is composed of a particle of energy ϵ and a reservoir of energy $E - \epsilon$ such that $\epsilon \ll E$ and E is the total energy of the particle and its environment, in effect providing a constant temperature environment. Since the number of microstates for the particle is always 1, the number of total microstates is $1 \times \Omega(E - \epsilon)$, where Ω is an arbitrary function describing the number of microstates of the environment. Then, using a Taylor approximation to first order about E:

$$\ln \Omega(E - \epsilon) \approx \ln \Omega(E) - \frac{d \ln \Omega}{dE} \epsilon$$

For a formal justification on why this first order approximation works, see Exercise 4.4 of Blundell. But for intuition, it suffices to say that Ω is so large that one must take the log of it for second and greater order terms to be sufficiently small. If we define temperature of a system to be

$$\frac{1}{k_B T} \equiv \frac{d \ln \Omega}{dE}$$

(The factor of $1/k_B T$ is also sometimes parameterized as β.) Then we have

$$\ln \Omega(E - \epsilon) = \ln \Omega(E) - \frac{\epsilon}{k_B T}$$

$$\Omega(E - \epsilon) = \Omega(E) e^{-\epsilon / k_B T}$$

$$\boxed{p(\epsilon) \propto e^{-\epsilon / k_B T}}$$

5.2 Statistical Mechanics II: Equipartition Theorem

Idea: Equipartition Theorem

Suppose some degree of freedom in phase space contributes to the energy by a power law,

$$E \propto p^n \quad \text{or} \quad H \propto x^n$$

Then in thermal equilibrium, there is an average energy of $k_B T/n$ in this degree of freedom.

Proof: Equipartition Theorem

Let the energy E of a particular system be αx^n. Then the probability of the system being at that energy level is $\propto e^{-\beta E}$:

$$p(x) = \frac{e^{-\beta \alpha x^n}}{\int_{-\infty}^{\infty} e^{-\beta \alpha x^n} dx}$$

The mean energy is

$$\langle E \rangle = \int_{-\infty}^{\infty} E p(x) dx = \frac{\int_{-\infty}^{\infty} \alpha x^n e^{-\beta \alpha x^n} dx}{\int_{-\infty}^{\infty} e^{-\beta \alpha x^n} dx}$$

For even integers, one can find the desired result using integration by parts directly. To find the result for general n, we can first begin by nondimensionalizing the integral:

$$\langle E \rangle = \frac{1}{\beta} \frac{\int_{-\infty}^{\infty} u^{1/n} e^{-u} du}{\int_{-\infty}^{\infty} u^{-1+1/n} e^{-u} du}$$

To avoid the dumb shit that happens when $u < 0$, let's set the limits to be from 0 to ∞. Upon integration by parts of the top integral:

$$= \frac{1}{\beta} \frac{u^{1/n} e^{-u}|_0^{\infty} - \int_0^{\infty} -\frac{1}{n} u^{-1+1/n} e^{-u} du}{\int_0^{\infty} u^{-1+1/n} e^{-u} du} = \frac{1}{n\beta} = \boxed{\frac{k_B T}{n}}$$

Note that what we did here was simply use the properties of the Gamma function:

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx$$

where $\Gamma(n) = (n-1)!$ for positive integers n.

Proof: Slicker Way to Do the Integral

Now we know that

$$\frac{\int_0^{\infty} \alpha x^n e^{-\beta \alpha x^n} dx}{\int_0^{\infty} e^{-\beta \alpha x^n} dx} = \frac{1}{n\beta}$$

But we could have proved this an even slicker way than before, albeit less directly. In order for this integral to be true, the following must also be true:

$$\int_0^\infty \beta\alpha x^n e^{-\beta\alpha x^n}\,dx = \frac{1}{n}\int_0^\infty e^{-\beta\alpha x^n}\,dx$$
$$\int_0^\infty \left(\beta\alpha x^n - \frac{1}{n}\right) e^{-\beta\alpha x^n}\,dx = 0$$
$$\int_0^\infty \left(-\frac{1}{n}xe^{-\beta\alpha x^n}\right)'\,dx = 0$$
$$-\frac{1}{n}xe^{-\beta\alpha x^n}\Big|_0^\infty = 0$$

The last statement is clearly true, so the integral has the desired value. ■

This is an entirely classical theorem. When the temperature gets too low, the states can no longer be approximated as continuous, and the quantized nature of energy states comes into play. At low temperatures, some degrees of freedom are said to "freeze out": they don't contribute to any heat capacities. In fact, most molecules are not vibrationally excited even at room temperatures.

5.3 Pressure

What is pressure? It is often defined as F/A, but at the same time we often ascribe it to random points in space where a gas or some other thing isn't necessarily exerting a force on a surface. A surface also doesn't necessarily feel the same amount of force that "pressure" would indicate, as the following example illustrates. Therefore, it is better that we think of pressure as the rate at which momentum passes through a surface, per unit area.

Example: Pressure Thought Experiment

A semi-infinite tube of ideal gas at pressure p and temperature T has a piston on one end. If the piston has a cross-sectional area A and is moving at a velocity v away from the gas, what is the force on the piston?

Clearly, if $v = 0$, $F = pA$. However, for $v > 0$, the average relative velocity of particles elastically bouncing off of the piston decreases, which should result in a lower force. Calculating the exact force would be a pain and I don't want to do it.

The general force due to pressure can be simplified using the following trick:

Example: Hemisphere Trick

A sphere of radius R contains a gas with uniform pressure P. Find the total force exerted by the gas on one hemisphere.

In theory, one can perform a surface integral to integrate the forces. But consider a hemispherical container with a gas of the same pressure P. In total, the force from the gas onto the container must be 0. Therefore, force on the flat portion must equal the force on the curved portion. The force on the flat portion is easy to calculate as $\pi R^2 P$, which is our answer.

This method works because of the following, more general property:

Idea: Vector Integral Trick

The integral $\int d\mathbf{S}$ does not depend on the general properties of the surface; it only depends on the boundary of said surface.

5.4 Surface Tension

There are several useful ways to think about surface tension:

Idea: Surface Tension Ideas

- *Forces:* if you cut a surface into two, the tension force of one piece on the other is

$$d\mathbf{F} = \gamma d\mathbf{s} \times \hat{\mathbf{n}}$$

- *Energy:* the energy of an area element dA is γdA for each exposed side of the area element.

5.5 Second Law of Thermodynamics

The second law of thermodynamics has an excellent introduction in Blundell & Blundell Ch. 13.

Proof: Clausius's Theorem

For a Carnot cycle, $Q_h/T_h = Q_c/T_c$. If we define $đQ$ as the amount of heat entering the system and T to be the temperature of the system, we have

$$\oint \frac{đQ_{\text{rev}}}{T} = \frac{Q_h}{T_h} + \frac{-Q_c}{T_c} = 0$$

Note that above we have the subscript $_{\text{rev}}$ since heat is transferred reversibly, since it is a Carnot cycle. For general cycles, we start by noting the total work outputted, by the first law of thermo, is

$$đW_{\text{out}} = -đW = đQ - dU$$
$$W_{\text{out}} = \oint đQ - dU = \oint đQ$$

Now suppose that at each infinitesimal step of the cycle, the infinitesimal heat $đQ_i$ is supplied to the system in the following manner:

1. There is the environment, a large reservoir of temperature T_{surr}, used for every infinitesimal step.
2. A Carnot engine takes in $đW_c + đQ$ heat, and outputs $đQ$ heat to a reservoir of temperature T (where T is the temperature of the system during this infinitesimal step), while outputting a work $đW_c$.
3. The reservoir of temperature T directly transfers $đQ$ to the system, with no work extracted.

This specific sequence of steps represents the most efficient way to conduct a general cycle with one reservoir of temperature T_{surr}. Using the properties of the Carnot engine in step 2, we have:

$$\frac{đQ}{T} = \frac{đQ + đW_c}{T_{\text{surr}}}$$

$$đW_c = đQ\left(\frac{T_{\text{surr}}}{T} - 1\right)$$

Now, considering the process of the cycle + carnot engines, the Kelvin statement of the second law says that it is impossible for a process to have the net effect of taking heat from the large reservoir of temperature T_{surr} and convert into a positive work $W_{\text{out}} + \oint đW_c$. Therefore,

$$W_{\text{out}} + \oint đW_c \leq 0$$

$$\oint đQ + \oint đQ\left(\frac{T_{\text{surr}}}{T} - 1\right) \leq 0$$

$$T_{\text{surr}} \oint \frac{đQ}{T} \leq 0$$

$$\boxed{\oint \frac{đQ}{T} \leq 0}$$

where as proven above, equality holds when the process is reversible.

Definition: Entropy

We define entropy as

$$dS = \frac{đQ_{\text{rev}}}{T}$$

Note that this definition only works for reversible processes. For irreversible processes, we can get around this definition in the following manner:

- Consider a reversible process that brings the system from the same initial to the same final state as the irreversible process. (The reversible process differs in what it does to the surroundings).
- The entropy change of the *system* is the same in both processes, since entropy is a state function.
- Then, one can calculate entropy using $dS = đQ/T$.

Proof: Another Form of the Clausius Inequality

Consider a cycle from $A \to B \to A$ where from $A \to B$ we consider a general path and from $B \to A$ we consider the reversible path. Then:

$$\oint \frac{đQ}{T} \leq 0$$

$$\int_A^B \frac{đQ}{T} + \int_B^A \frac{đQ_{\text{rev}}}{T} \leq 0$$

$$\int_A^B \frac{đQ}{T} \leq \int_A^B \frac{đQ_{\text{rev}}}{T}$$

This is true no matter how close A and B get, so it is true for the differential as well:

$$\frac{đQ_{\text{rev}}}{T} = \boxed{dS \geq \frac{đQ}{T}}$$

This implies $dS \geq 0$ for any isolated system.

6 Waves

6.1 Wave Equation

Idea: Wave Equation

A wavefunction $y(x,t)$ must satisfy the following relation:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

Factoring the equation yields:

$$(\partial_t^2 - v^2\partial_x^2)y = (\partial_t + v\partial_x)(\partial_t - v\partial_x)y = 0$$

Therefore, equations that satisfy $(\partial_t \pm v\partial_x)y = 0$ satisfy the wave equation. From there, it is clear that equations can be of the form

$$y = f(x \pm vt)$$

or in general, since the wave equation is a linear equation,

$$y = f(x - vt) + g(x + vt)$$

Proof: For Waves on Strings

We prove the wave equation for a wave on a string. Assume the wave is shallow, i.e. $\partial y/\partial x \ll 1$. Tension at a given point provides forces, to first order in y', of

$$T_x = \frac{T}{\sqrt{1+y'^2}} \approx T$$

$$T_y = \frac{Ty'}{\sqrt{1+y'^2}} \approx Ty'$$

Therefore, on a piece of string of length dx, the x forces approximately cancel while the y forces satisfy

$$F_{y,\text{net}} = dT_y = Ty''dx = \ddot{y}(\mu dx)$$
$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}$$

where the desired result is proven if $v^2 = T/\mu$.

6.2 Some Definitions

Definition: Wave Terms

- Standing Waves: in general, wavefunctions of the form

$$\psi = f(x)g(y)h(z)\cos(\omega t)$$

- Phase velocity: ω/k. Intuitively, the velocity of something at a constant phase. Since not the velocity of an actual object, can exceed c. Note that $v_p = \omega/k$ since $\phi = kx - \omega t =$ const. with respect to time, or $\partial\phi/\partial t = 0$.
- Group velocity: $d\omega/dk$. Intuitively, the speed of the peak of a finite wavepacket. Note a peak of a wavepacket is located where the phases for waves of every wave number line up, or $\phi = kx - \omega t =$ const. with respect to k or $\partial\phi/\partial k = 0$. This yields $x - (d\omega/dk)t = 0$ or $v_g = x/t = d\omega/dk$.

6.3 Transmission and Reflection

Idea: Transmission and Reflection Coefficients

The original, reflected, and transmitted waves that occur at the boundary of two different media can be described, respectively, as

$$y = e^{i(kx-\omega t)} \qquad y_r = re^{i(-kx-\omega t)} \qquad y_t = te^{i(k'x-\omega t)}$$

The shared $e^{-i\omega t}$ term is because most often, something at the boundary is constant in time, which is only possible if every wave has the same frequency. The values of r and t are determined using the boundary condition.

6.4 Energy

Idea: Energy of a Wave

In general, the energy within a wave is proportional to A^2 *and* the properties of the medium.

Example: Kinetic = Potential Energy

Sometimes, $K = U$ exactly! For example, for a wave on a string that is purely moving in one direction (i.e. it can be described as a single function $f(x - vt)$), the kinetic and potential

energy densities are equal at every point:

$$\frac{dK}{dx} = \frac{1}{2}\mu\dot{y}^2$$

$$\frac{dU}{dx} = T\frac{\sqrt{dy^2 + dx^2} - dx}{dx} \approx \frac{1}{2}Ty'^2$$

Due to the nature of the function $f(x - vt)$, we can easily check that $\dot{y} = vy'$. Thus:

$$\frac{dK}{dx} = \frac{1}{2}\mu\dot{y}^2 = \frac{1}{2}\mu v^2 y'^2 = \frac{1}{2}Ty'^2 = \frac{dU}{dx}$$

Because the densities are the same at every point, integration reveals $K = U$ exactly. If there are waves going in opposite directions, however, this need not be true; a quick counterexample is a standing wave. However, due to the virial theorem, the time averages of U and K are still equal.

Example: Energy Conservation

If the two media at a boundary are the same, just with certain boundary conditions, one can write down

$$1 = |r|^2 + |t|^2$$

because the energy is proportional to A^2 and all other terms cancel. However, in the case of different media, this is not necessarily true. For example, consider the case where two strings of linear mass densities μ_1 and μ_2 and tensions T_1 and T_2, connected by a massless ring which slides on a vertical frictionless pole.

If we consider a finite wave, then the incoming energy must equal the energies of the reflected and transmitted waves. At some point the incoming wave runs out and no longer interferes with the reflected wave, which means in both strings the conditions necessary for the above example to apply are satisfied, and $K = U$ for each wave! Therefore, we investigate just the U of each wave in the following reasoning:

- $dU/dx \propto Ty'^2 \propto T(Ak)^2$, where A is the amplitude and k is the wavenumber. This can be verified using $e^{i(kx-\omega t)}$.
- In the above example, ω is constant, so $k \propto 1/v$.
- The duration of each wave τ is constant, so the length $L = v\tau \propto v$.
- Therefore:

$$U \propto L\frac{dU}{dx} \propto vT\left(A\frac{1}{v}\right)^2 \propto \frac{T}{v}A^2 \propto \sqrt{\mu T}A^2$$

$Z \equiv \sqrt{\mu T}$ is called the impedance of the string, and it depends on the properties of the medium. Using this information, the statement of conservation of energy becomes

$$Z_1 = Z_1|r|^2 + Z_2|t|^2$$

6.5 Optics

Idea: Coherence

Coherence between light sources means they have a constant phase difference; in other words, light from each source exhibits only one phase at a particular location. This is why two light sources of the same color often cannot produce interference patterns; though they are monochromatic, they are still incoherent.

Idea: Double Slit

For two slits separated by a distance d, the interference pattern as a function of angle θ from the normal has the following intensity:

$$I(\theta) \propto \cos^2\left(\frac{kd\sin\theta}{2}\right)$$

Proof: Double Slit

The amplitude at a particular location on the screen is given by

$$A(\theta) \propto e^{i(kr_1-\omega t)} + e^{i(kr_2-\omega t)}$$

We're looking for the relative magnitudes, so we can take out a phase difference as well as the time dependence to find that

$$A(\theta) \propto e^{-ik\Delta r/2} + e^{ik\Delta r/2}$$

where $\Delta r \equiv r_2 - r_1$. The intensity is therefore proportional to:

$$I \propto A^2 \propto |e^{-ik\Delta r/2} + e^{ik\Delta r/2}|^2 \propto |2\cos k(\Delta r/2)|^2 \propto \cos^2(k\Delta r/2)$$

In the far-field approximation (where the distance between the slits and the screen $D \gg d$), $\Delta r \approx d\sin\theta$, so the result is proven.

Idea: Diffraction Grating

Consider a diffraction grating with N identical slits, each separated by a distance d. The intensity is then:

$$I(\theta) \propto \frac{\sin^2(Nk\Delta r/2)}{\sin^2(k\Delta r/2)}$$

Intuitively, the pattern looks like an evenly spaced row of spikes, corresponding to the slow evolution of the denominator, as well as rapid oscillations that are barely visible around those spikes, corresponding to the numerator.

Proof: Diffraction Grating

Defining $\Delta r \equiv d \sin\theta$ as before, we have

$$A \propto 1 + e^{ik\Delta r} + e^{2ik\Delta r} + \cdots + e^{(N-1)ik\Delta r} = \frac{e^{ikN\Delta r} - 1}{e^{ik\Delta r} - 1}$$

Factoring out a common phase in both the numerator and the denominator yields

$$A \propto \frac{e^{ikN\Delta r/2} - e^{-ikN\Delta r/2}}{e^{ik\Delta r/2} - e^{-ik\Delta r/2}} \propto \frac{\sin(kN\Delta r/2)}{\sin(k\Delta r/2)}$$

So the desired result follows.

Idea: Single Slit

A single slit of width a has the following intensity pattern:

$$I(\theta) \propto \frac{\sin^2(ka\sin\theta/2)}{(ka\sin\theta/2)^2}, \quad I \propto \frac{\sin^2\beta}{\beta^2} \quad (\beta \equiv ka\sin\theta/2)$$

Proof: Single Slit

We can consider the single slit as a diffraction grating in the case that $d \to 0$, $N \to \infty$, $Nd \to a$, and keeping the total energy constant. In particular, that last condition requires us to put a $1/N$ factor on A in the proof above,

Example: Amplitude versus Area

Suppose you shine a coherent light source on a pair of slits, resulting in a double slit pattern. If you double one slit's width, what is the resulting interference pattern?

It turns out, the *amplitude* of the slit that is twice as wide is doubled, *not its intensity!* Therefore,

$$A \propto 2 + e^{ik\Delta r}$$

and the intensity will vary from I_0 to $9I_0$ instead of the original 0 to $4I_0$. Why is it that area is proportional to amplitude, and not intensity? We'll come back to answer that question, but first:

Consider the simpler case of just one slit, which has a single slit pattern. If the slit width is doubled, the energy passing through it doubles *and* the total amplitude can be considered to be doubled. How is this possible?

- Consider the final amount of energy. The amplitude doubling means the intensity on the screen quadruples, but overall the intensity pattern shrinks by half due to the widening of the slit! The total energy is therefore only doubled.
- What about the initial amount of energy? Well, in both cases the incoming intensity is

the same, but the area of the slit doubles, so the incoming energy is doubled. Energy is therefore conserved.

- What about amplitude? In this case, amplitude *cannot* just be said to be 2 like in the case above. Strictly speaking, in the case with two slits, the amplitude due to the wider slit is not just 2, but because of the the interference between the two slits is assumed to dominate the interference pattern given how thin the slits are, which result in the single slit pattern to be roughly $A \propto (\sin\beta)/\beta \approx$ const. throughout. *Since there is no other interference,* the single slit diffraction is non-negligible no matter how thin the slit is.

The difference between this case and the first... **(incomplete, still more thought required)**