Spherical Trig, Time Notes
v 1.3.1

Spherical trig notes and time notes: basic formulas, tips and tricks, maybe some problems.

1 Spherical Trig

Most formulas here will be presented without proofs, unless the proof is important to know. Proofs of many of these equations are in the appendix.

Great Circle. The circle that is formed through the intersection of a plane and the surface of a sphere, such that the plane passes through the center of the sphere.

Small Circle. Any circle on the surface of a sphere that is not a great circle.

Spherical Polygon. A closed geometric figure on the surface of a sphere which is formed by the arcs of great circles. The most common is a spherical triangle, but lunes/diangles (polygons with only two sides) exist as well.

Angles and Side lengths. Typically, both angles and side lengths of spherical polygons will be measured in angular units such as degrees or radians. Like in Euclidean geometry, angles are typically denoted with capital letters (A), while side lengths are denoted with lowercase letters (a). Points are usually indicated with the same capital letter as the angle there. We use this notation in the rest of these notes unless otherwise specified.

Uniqueness. There is only one great circle that passes through two arbitrary points A and B, unless A and B are antipodal or the same point, in which case there are infinitely many great circles. There are, however, infinitely many small circles that pass through two arbitrary points. Important thing to note: the shortest path between two points is always a great circle!

Absence of Parallel Lines. Sometimes, great circle arcs are called "lines" because they are the equivalent of lines in Euclidean geometry. However, unlike lines in Euclidean geometry, there are no parallel "lines": all great circle intersect.

The following formulas apply to spherical triangles made of three great circle arcs:

1. Spherical Law of Sines:
$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$
2. Spherical law of Cosines:
$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$
3. Four-part formula:
$$\cos i \cos I = \sin i \cot o - \sin I \cot O$$
where i, o, I, and O represent the inner side, outer side, inner angle, and outer angle respectively.

The proofs of these equations are not important to remember, except for the fact that *all three equations are dependent!*

Congruent Triangles. One interesting thing is that in spherical geometry there are no "similar" triangles in the sense that they are scaled up or down versions of the same triangle—the only similarity allowed is exact congruence. SSS, SAS, ASA, and AAA are all enough to define a unique triangle.

Girard's Theorem. The area of a spherical triangle on a unit sphere is

$$\text{area} = A + B + C - \pi$$

Corollary. For a spherical polygon with n sides and interior angles $A_1, ..., A_n$ on a unit sphere, the area is

$$\text{area} = A_1 + ... + A_n - (n-2)\pi$$

which can easily be proved from Girard's Theorem by splitting a polygon into $n-2$ triangles.

Cartesian Coordinate Bash. Sometimes, a question lends itself better to a cartesian coordinate bash. The main techniques here are:

1. Convert any angular coordinates into cartesian coordinates. As an example of alt-az $\rightarrow$ xyz coordinates (alt-az will be defined in the next section, if you're unfamiliar), we use x as East, y as North, and z as the zenith direction:

$$\mathbf{r} = (x, y, z) = (\cos a \sin A, \cos a \cos A, \sin a)$$

2. A great circle is defined by a plane. The plane must pass through the center of a sphere, so we can define the plane by its perpendicular/normal vector.
3. A plane's normal vector can be found using any two points along the great circle. The normal vector is then:

$$\mathbf{n} = \mathbf{r_1} \times \mathbf{r_2}$$

4. An arbitrary point is within the great circle if:

$$\mathbf{n} \cdot \mathbf{r} = 0$$

5. The intersection points of two great circles are:

$$\mathbf{r} = \pm(\mathbf{n_1} \times \mathbf{n_2})$$

6. Typically, small circles will be defined such that they are easy to express in cartesian coordinates. For example, using the example coordinates in part one, a small circle defined by $a = 30°$ would be equivalent to $z = 0.5$ in cartesian coordinates.

2 Coordinate Definitions

Easier Definitions.

1. **Celestial Sphere**
2. **Celestial Equator**
3. **Ecliptic**
4. **Horizon**
5. **Declination**
6. **Altitude**

Tricky Definitions.

1. **Azimuth** A: clockwise from South, along the local horizon. Definition holds regardless of hemisphere. (Definitions may vary; this seems to be the most commonly used. However, in free response questions, always state your definition of azimuth to avoid confusion.)

2. **Right Ascension** α: eastward from the vernal equinox on celestial equator, CCW as viewed from North Pole. OR number of hours behind the Sun on the vernal equinox.

3. **Vernal Equinox**: intersection of ecliptic and celestial equator when travelling eastward on the ecliptic (the ascending node). $\alpha = 0$.

4. **Hour angle** H: angle from meridian in equatorial frame. Increases in the direction of rotation of the sky.

5. **The fictitious "mean sun"**: fictitious body that moves in a circular orbit with constant angular velocity around the celestial equator, rather than on ecliptic.

Location of the Sun. In (α, δ):

1. Vernal Equinox (0^h, $0°$)

2. Summer Solstice (6^h, $+23.45°$)

3. Autumnal Equinox (12^h, $0°$)

4. Winter Solstice (18^h, $-23.45°$)

5. Approximate RA of the Sun at all times, where Δt is time after vernal equinox:

$$\alpha_\odot \approx \frac{\Delta t}{1 \text{ year}} 24^h$$

Note that this approximate expression is the exact expression for the RA of the mean Sun.

3 Configurations to Remember

Solving Oblique Triangles (oblique = without a right angle)

In general, using an inverse sine is not recommended, because of the ambiguity between the angle and its supplement. But, as you will see indicated by a *, in some cases not enough information is provided to avoid ambiguity, and the ambiguity represents the two possible triangles that fit the given description.

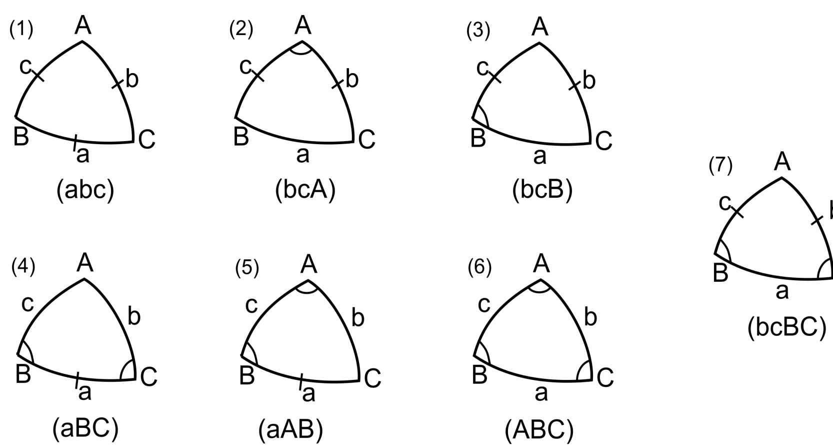

Figure 1: Types of Triangles (Source: Wikipedia)

1. **SSS.** Use law of cosines to find angles.

2. **SAS.** Use law of cosines to find a, then back to case 1.

3. **SSA.*** Use law of sines to find C (if $b < c$, there are two solutions, C and $180° - C$), then go to case 7.

4. **ASA.** Use 4-part formula to find c and b, then law of cosines to find A.
5. **AAS.*** Use law of sines to find b (if $A > B$ and $a < 90°$, there are two solutions, b and $180° - b$), then go to case 7.
6. **AAA.** Use supplemental cosine rules to find sides.
7. **SSAA.** Split into two right triangles such that A is split into two angles. The two portions of a, defined as a_1 and a_2, can be found using the 4-part formula on the right triangles. $a = a_1 + a_2$ and A can be found from law of sines.

Solving Right Triangles

Whenever there is a right angle, any two other components will completely determine the triangle. Many of the formulas simplify dramatically upon using a $90°$ angle.

4 Time

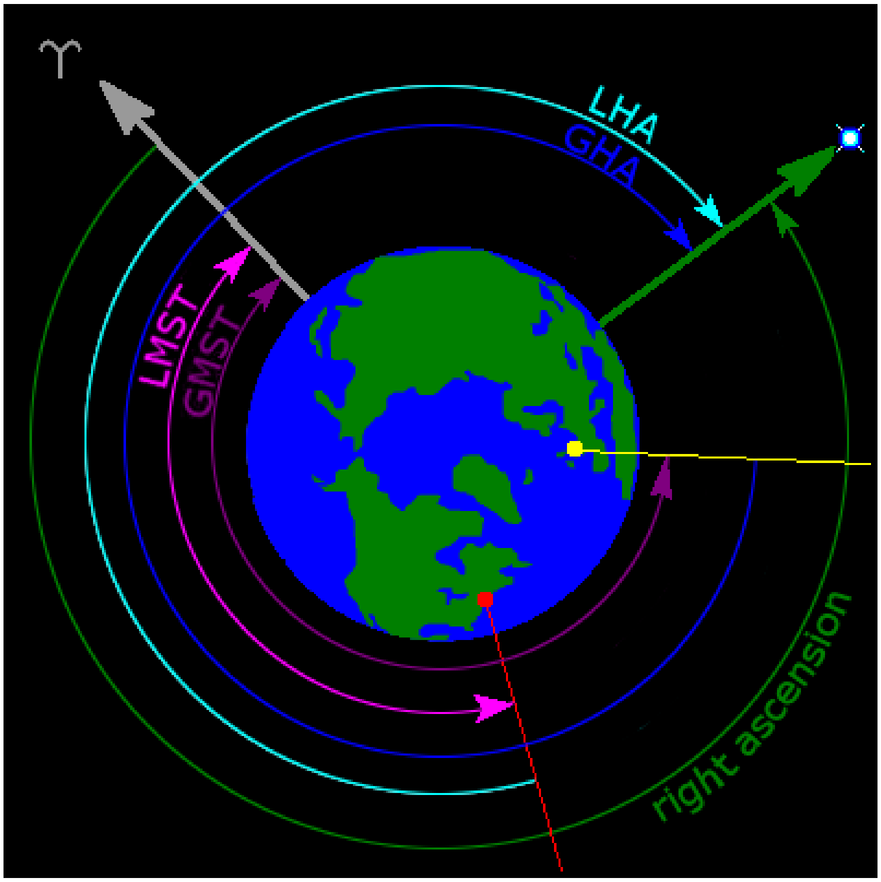

Figure 2: Several Time Measurements (Source: Wikipedia)

In the above figure, the gray line represents the vernal equinox, the red line represents the local meridian, the yellow line represents the Greenwich meridian, and the green line represents the location of a particular star. Pay attention to:

1. **Right Ascension (α).** eastward from the vernal equinox on celestial equator, CCW as viewed from North Pole. OR number of hours behind the Sun on the vernal equinox.
2. **Local Hour Angle (LHA = H).** The local hour angle is the angle between the local meridian and the star, measured clockwise.
3. **Local Mean Sidereal Time (LMST).** The hour angle of the vernal equinox. The "mean" portion is sometimes omitted from the name.
4. **Greenwich Mean Sidereal Time (GMST).** The hour angle, from Greenwich, of the vernal equinox. The "mean" portion is sometimes omitted from the name.

From the above definitions, one finds a a very useful equation:

$$H + \alpha = \mathrm{LST}$$

Conversion Between Greenwich Solar Time and Greenwich Sidereal Time.

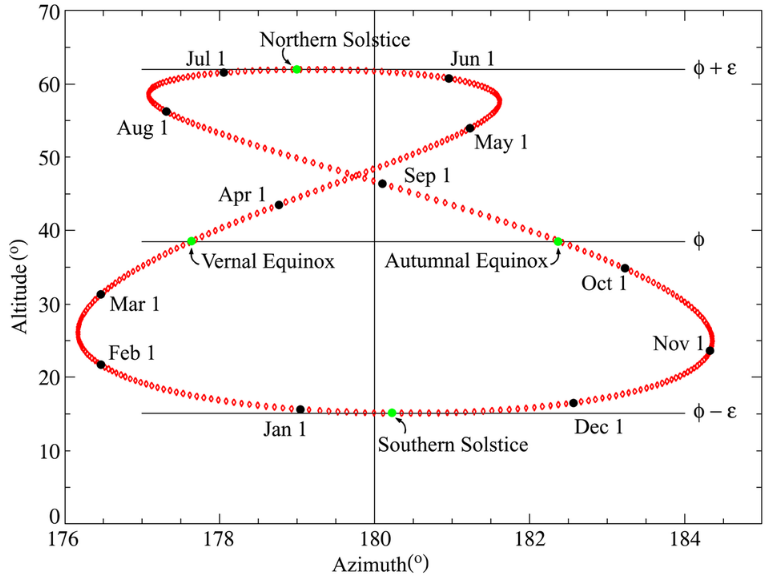

Figure 3: An Example Analemma

Greenwich solar time is indicated by GMT+0 or UTC+0. On the day of the vernal equinox, the following relation between solar time and sidereal time is true:

$$0^h \text{ (sidereal time)} = 12^h \text{ (solar time)}$$

At all other times, we can use the fact that (mean) sidereal time marches forward at a slightly faster rate than solar time, eventually exceeding solar time by exactly 24^h in one tropical year:

$$\Delta(\mathrm{LST}) = \frac{\Delta T_{\text{solar}}}{23.9344^h} 24^h + \frac{\Delta t \text{ (in solar days)}}{365.2422 \text{ days}} 24^h$$

Conversion from GST to LST. To convert the Greenwich Sidereal Time to Local Sidereal Time:

$$\mathrm{LST} = \mathrm{GST} + \lambda$$

where $\lambda > 0$ is longitude in the eastward direction.

Analemma. As defined from Wikipedia: an analemma is a diagram showing the position of the Sun in the sky as seen from a fixed location on Earth at the same mean solar time, as that position varies over the course of a year.

Why is it not in the same location throughout the year? For starters, the obliquity of the Earth causes the declination of the true Sun to vary between $+23.45°$ and $-23.45°$ Additionally, because the mean solar time is fixed, the true solar time is not, due to eccentricity and obliquity concerns causing the true sun to increase in right ascension at varying rates.

The x-axis of an analemma is the equation of time (ET):

$$\mathrm{ET} = T_{\text{true}} - T_{\text{mean}} = H_{\text{true}} - H_{\text{mean}} = \boxed{\alpha_{\text{mean}} - \alpha_{\text{true}}}$$

5 Worked Examples

First, we work through NAO 2019 P9. The question is as follows:

(a) Find the shortest distance from Boston (42.36010 N, 71.05890 W) to Beijing (39.90420 N, 116.40740 E) traveling along the Earth's surface. Assume that the Earth is a uniform sphere of radius 6371 km.

(b) What fraction of the path lies within the Arctic circle (north of 66.56080 N)?

Part (a) is a classic, straightforward question. If you do not see how to immediately do this question, review the configurations to remember section as well as the official solution. Part (b) is a little tricky, because we have to deal with both small circles and great circles—see the official solution for the problem writers' way to do it using cartesian coordinates. Personally, I find this way very ugly and prone to mistakes, so I present a way using spherical triangles.

Consider the following diagram:

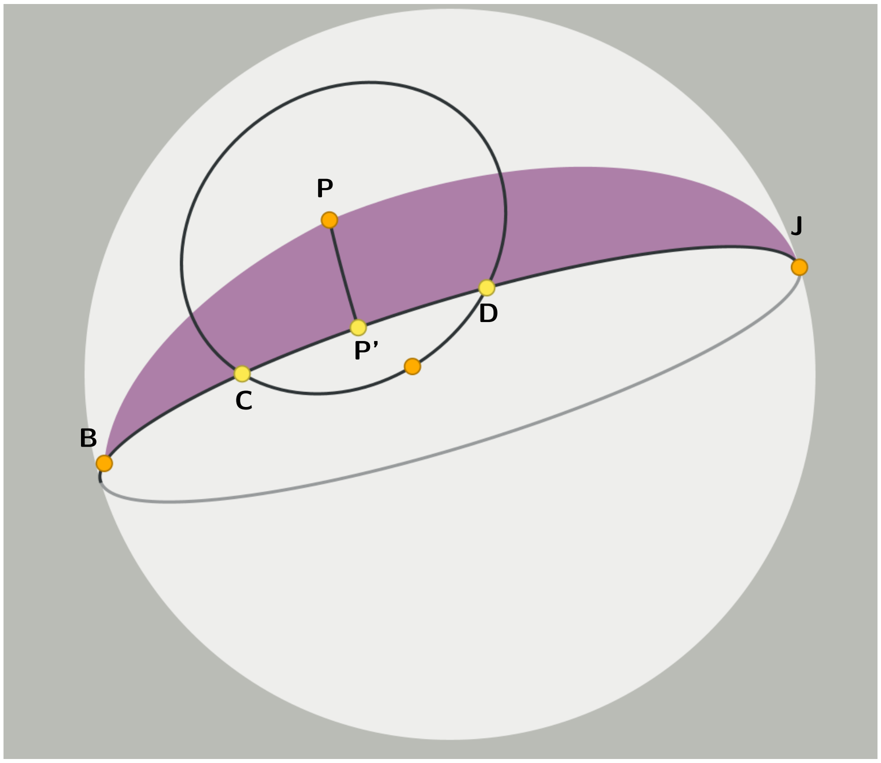

Figure 4: NAO 2019 P9. Not to scale.

For part (a), we formed a spherical triangle with Boston, Beijing, and the North Pole. We label Boston with B, Beijing with J, the North Pole P, and the intersection of the arctic circle with BJ as C and D. Now, we solve this by introducing a new great circle segment PP', such that $PP' \perp BJ$.

Part (a) is simply finding the length of BJ, which is $97.458°$. Since we know $\angle BPJ = 172.5337°$, $PB = 47.6399°$, and $PJ = 50.0958°$, we can use the law of sines to determine the other angles in the triangle. We will only need to know one of them, in this example $\angle PBJ$:

$$\frac{\sin \angle PBJ}{\sin PJ} = \frac{\sin \angle BPJ}{\sin PJ}$$
$$\angle PBJ = 7.4663°$$

We can then perform another law of sines treatment, this time on $\triangle PBP'$:

$$\frac{\sin \angle BP'P}{\sin PB} = \frac{\sin \angle PBJ}{\sin PP'}$$

As $\angle BP'P = 90°$ by definition, $PP' = 5.5099°$. Again using that $90°$ angle, and now $PC = 23.4392°$ we can find $P'C$ by law of cosines:

$$\cos PC = \cos P'C \cos PP'$$

$$P'C = 22.8180°$$

By symmetry, $CD = 2 \times P'C$. Thus, our final result is:

$$\frac{2 \times P'C}{BJ} = \boxed{0.468}$$

Next, we work through 7a of the GeCAA:

A stargazer in Chiayi, Chinese Taipei (23.5°N, 120.4°E, GMT+8) saw two meteors streaking through the sky at 21:00 (Chinese Taipei time) on 25th September 2020. The Greenwich Sidereal Time (GST) at 00:00 UT on 1st January 2020 is $6^h 40^m 30^s$. What is the Local Sidereal Time (LST) at the time of observation?

We can very quickly find GMT by subtracting 8^h from 21^h, giving us that $\mathrm{GMT} = 13^h$. This is the solar time at Greenwich. Then, we can use the solar time $\to$ sidereal time converter, by choosing 00:00 UT on 01/01/2020 as our starting day, and the observation time and day as our ending day:

$$\mathrm{LST} - 6^h 40^m 30^s = (13^h - 0^h)\frac{24}{23.9344} + \frac{268 \times 24^h}{365.2422}$$

where 268 is the number of days between 09/25 and 01/01. Thus, we have:

$$\mathrm{LST} = 37^h 19^m 14^s = \boxed{13^h 19^m 14^s}$$

Appendix

A Derivation of Three Fundamental Formulas

Spherical Law of Cosines. To be added.

Spherical Law of Sines. To be added.

Four part formula. To be added.

B Polar Triangles

Polar Triangle. Given a spherical triangle $\triangle ABC$, the polar triangle $\triangle A'B'C'$ is the triangle with A a pole of $B'C'$ on the same side as A', B a pole of $A'C'$ on the same side as B', and C a pole of $A'B'$ on the same side as C'.

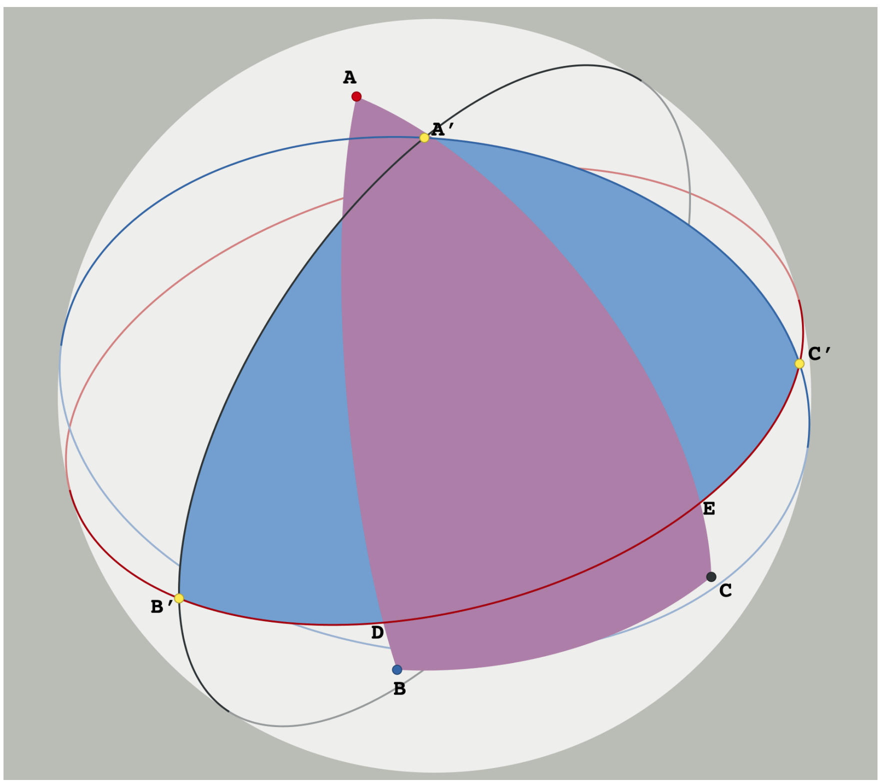

Figure 5: Example of polar triangles

Polar Triangles are subject to the following theorems:

Theorem 1. If $\triangle A'B'C'$ is the polar triangle to $\triangle ABC$, then $\triangle ABC$ is the polar triangle to $\triangle A'B'C'$.

Proof: By assumption, B is the pole of $A'C'$. Therefore $A'B = \pi/2$. Similarly, as C is the pole of $A'B'$, $A'C = \pi/2$. Now consider the relationship between A' and BC. A' is a distance $\pi/2$ from both B and C, which means that OA' is perpendicular to the plane defined by O, B, and C, where O is the center of the sphere. Thus, A' must be a pole of BC. Similarly B' and C' are poles of AC and AB on the appropriate sides.

Theorem 2. If $\triangle A'B'C'$ is the polar triangle to $\triangle ABC$, then $\angle A + B'C' = \pi$

Proof: Extend the lines AB and AC so that they meet $B'C'$ in points D and E. Then $B'E = C'D = \pi/2$, since B' and C' are the poles of AC and AB, respectively. Since DE is on the 'equator' in which A is the pole, we have $\angle A = DE$. Thus:

$$B'C' + \angle A = \pi$$

Proof of Supplemental Rules. Using the existence of polar triangles and theorem 2, we can simply replace each instance of a with $180° - A$, etc. and A with $180° - a$, etc. into the Law of Sines and Law of Cosines to derive the supplemental rules.