Cosmology Notes

Going over basic cosmology needed for IOAA without going into any Robertson-Walker metrics, spacetime curvature, or general relativity.

1 Definitions

Redshift. Denoted as z, defined as:

$$z = \frac{\lambda - \lambda_0}{\lambda_0}$$

Some properties:

1. $z > 0$ for a shift towards red, $z < 0$ for blueshift
2. Can be created due to the Doppler effect or cosmological expansion.
3. Because of its relationship with cosmological expansion, a higher z also denotes a farther distance and an earlier time.

Important to note: sometimes there is some ambiguity in what redshift refers to, because for complete information, the two reference points in time that λ and λ_0 are measured in must be mentioned. In any situation in which there may be ambiguity, these notes use $z_{a\to b}$ to refer to a redshift between times t_a and t_b. If there is no subscript, it is most likely that z refers to $z_{em\to 0}$, where $_{em}$ refers to the time of emission and 0 refers to the present.

Scale Factor. Denoted as a, this represents the relative linear size of the universe. Typically in the present, $a \equiv 1$. As an example of how to use the scale factor, a photon of wavelength λ at time t_1 has a wavelength $\lambda' = a(t_2)\lambda/a(t_1)$. In fact, the above equation reveals an important relation between redshift and scale factor:

$$\frac{a(t_b)}{a(t_a)} = 1 + z_{a\to b}$$

$$a_{em} = \frac{a(t)}{1 + z_{em\to t}}$$

If t is the present time, $a(t) = 1$ and the relation becomes the familiar

$$a_{em} = \frac{1}{1 + z_{em\to 0}}$$

Note that if there is no subscript, it is most likely that a refers to a_{em}, but one should always check the context.

Hubble Constant. Denoted as H or H_0, defined as $\dot{a}/a$, where $\dot{a} \equiv da/dt$. Typically, $H(t)$ or H is used to define the Hubble "parameter" that changes with time and with the scale factor, while H_0 refers to the Hubble "constant", i.e. the value of the Hubble "parameter" at the current time.

Hubble's Law. Hubble's law states that at cosmological distances, objects recede from us at a speed of $v = H_0 d$, where d is the object's distance from us (it will turn out to be the proper distance, which will be defined later).

Besides these basic definitions, the more involved definitions will be introduced later in these notes.

2 Fundamental Equations of Cosmology

We follow these notes and show a Newtonian derivation of the fundamental equations of cosmology. Speaking of which, the fundamental equations of cosmology (they're not actually called these by the way; this is just my opinion) are:

1. Friedmann equation
2. Fluid equation
3. Acceleration equation

Note that these equations are not independent; any two of the above equations can be used to derive the third.

2.1 Friedmann Equation

Suppose an isolated sphere of mass M and radius R collapses under its own gravity. We consider the motion of the outermost radius, $R(t)$. The sphere will stay uniform, and as such it will be useful to describe the motion in terms of the scale factor, a:

$$R = ar_C$$

where r_C is the comoving distance of the boundary, which is constant. By conservation of energy (per unit mass at the boundary), we have:

$$\frac{1}{2}R^2 - \frac{GM}{R} = U = \text{const.}$$

Substituting $R = ar_C$ and $M = (4\pi/3)\rho R^3$, we obtain:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2U}{r_C^2 a^2}$$

Now, without proof, we state that in General Relativity, a similar equation is formed when relating the size of spacetime itself (rather than the size of a an isolated sphere) to energy density:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\epsilon}{c^2} - \frac{kc^2}{R_0^2 a^2}}$$

R_0 is the present value of the curvature radius. The sign of k depends on curvature. If $k < 0$, then the universe has negative curvature, like a saddle. In this universe, parallel lines eventually diverge. If $k = 0$, the universe is flat—such a universe obeys Euclidean geometry, and parallel lines neither diverge nor converge. If $k > 0$, the universe has positive curvature, like a sphere—parallel lines converge in such a universe.

Note that for $k \leq 0$, an expanding universe continues indefinitely, because the right hand side would always be positive, and $\dot{a}$ could never reach 0 and reverse signs.

2.2 Easier to use form of Friedmann Equation

We can simplify this expression using the critical density, which is defined as the density required for $k = 0$. The critical energy density is thus:

$$\epsilon_c = \frac{3H^2(t)}{8\pi G}$$

Note that $H^2(t)$ will change with time. With this definition, we can define a density parameter Ω:

$$\Omega \equiv \frac{\epsilon}{\epsilon_c}$$

Thus:

$$H^2 = H^2 \left(\frac{\epsilon}{\epsilon_c} - \frac{kc^2}{H^2 R_0^2 a^2} \right)$$

It is important to remember that while $\epsilon \propto a^{-3}$ for matter, as an example, this does not mean $\Omega_m \propto a^{-3}$ because the critical density changes with a as well. In fact, this has a pretty important consequence. If we consider Ω, the overall density parameter, the Friedmann equation becomes:

$$1 - \Omega = -\frac{kc^2}{H^2 a^2 R_0^2}$$

If $\Omega = 1$, then $k = 0$, meaning no matter how a evolves Ω will stay at 1. Similarly, no matter the value of k, the sign of the right side of the equation will remain the same, so any $\Omega > 1$ will stay > 1 and any $\Omega < 1$ will stay < 1.

We move on with our Friedmann equation. If we divide the energy density into four components: matter, radiation, dark energy, and a "curvature" component (with $\epsilon = -3kc^2/(8\pi G R_0^2 a^2) \propto a^{-2}$), we can change the expression to become:

$$H^2 = H^2 \left(\frac{\epsilon_{m,0} a^{-3} + \epsilon_{r,0} a^{-4} + \epsilon_{\Lambda,0} + \epsilon_{k,0} a^{-2}}{\epsilon_c} \right)$$

An important note is that H^2 is both outside and in ϵ_c, cancelling out. Thus, we can replace both with H_0, which leads us to the important breakthrough:

$$\boxed{H^2 = H_0^2 \left(\Omega_{m,0} a^{-3} + \Omega_{r,0} a^{-4} + \Omega_{\Lambda,0} + \Omega_{k,0} a^{-2} \right)}$$

2.3 Fluid Equation

The first law of thermodynamics:

$$dE = dW + dQ = -PdV + dQ$$

Expansion of the universe is adiabatic, since there is no external heat and most processes are reversible (exceptions include positron/electron annihilations). Thus, we can rewrite the equation as follows:

$$dE + PdV = 0 \implies \dot{E} + P\dot{V} = 0$$

First of all, we can relate $\dot{V}$ with $\dot{a}$ by noting that $\dot{V}/V = 3\dot{a}/a$. Next, replacing the total energy with the energy density $E = \epsilon V$ yields:

$$\dot{E} = V\dot{\epsilon} + \dot{V}\epsilon = V\left(\dot{\epsilon} + (3\dot{a}/a)\epsilon\right)$$

Plugging the above equation back into our expression for the first law of thermodynamics yields:

$$\boxed{\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0}$$

Most components of the universe (e.g. matter, radiation) satisfy, at least approximately, the following equation of state:

$$P = w\epsilon$$

where w is a dimensionless constant that is typically independent of time. The solution for $\epsilon(a)$ given w is thus:

$$\boxed{\epsilon \propto a^{-3(1+w)}}$$

Notice that $w \approx 0$ for matter (since in general, mass energy density is much larger than the pressure), $w = 1/3$ for radiation (note that you can derive $P = U/(3V)$ for a photon gas), and $w = -1$ for dark energy (from the definition of dark energy).

2.4 Acceleration Equation

Thus far, we have derived the Friedmann equation and fluid equation independently. We can use these two equations to derive the acceleration equation. First, we multiply the Friedmann equation by a^2:

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\epsilon a^2 - \frac{kc^2}{R_0^2}$$

Then we take the time derivative:

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}(\dot{\epsilon}a^2 + 2\epsilon a\dot{a}^2)$$

We substitute $\dot{\epsilon}(a/\dot{a}) = -3(\epsilon + P)$ and divide by $2a\dot{a}$:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P)}$$

If ϵ and P are positive, the expansion of the universe decelerates.

3 Cosmological Distance Measures

In non-cosmological settings, all of the following distance measures you will see are equivalent, which is why these may be new concepts to you. However, since the universe is expanding, this results in different distances being used in different scenarios.

3.1 Proper Distance

The proper distance d_P is the physical distance between two objects, i.e. the number that would be measured if you had a very long measuring stick. The proper distance scales linearly with a, so if the proper distance between two objects was $d_{P,0}$ when $a(t) = a_0$, then the new proper distance at any time a is

$$d_P = d_{P,0}\frac{a(t)}{a_0}$$

3.2 Comoving Distance

Since proper distance (the distance that is likely most intuitive in non-cosmological settings) changes with time, it is helpful to use a distance metric that is invariant with the expansion of space. This is the comoving distance (denoted d_C or χ): it is often defined as the proper distance at a time when $a = 1$. If objects are not moving relative to the expansion of space, their comoving distance remains constant by definition. With this definition, the relationship between comoving distance and proper distance is as follows:

$$\boxed{d_P = a(t)d_C}$$

3.3 Luminosity Distance

Luminosity distance (d_L) is defined as the distance in the following equation:

$$F \equiv \frac{L}{4\pi d_L^2}$$

Consider a set of photons emitted isotropically from a point source with intrinsic luminosity L. The amount of energy emitted in time dt will thus be Ldt. There are two reasons why the flux will not be $F = L/(4\pi d_P^2)$:

1. Photons will be redshifted due to cosmological expansion. Thus, the amount of power per unit area (i.e. flux) will be decreased by a factor of $a(t_{em})/a(t_0) = a(t_{em})$, where t_{em} is the time of emission and t_0 is the current time, and $a(t_0) \equiv 1$.

2. The amount of time dt between the first and last photons will be stretched due to cosmological expansion, by the same factor as before. As such, the power (which is energy per unit time) will decrease by the same factor.

Due to these two factors:

$$\frac{La_{em}^2}{4\pi d_P^2} = \frac{L}{4\pi d_L^2}$$

$$d_L = d_P/a_{em} = \boxed{d_P(1+z_{em\to 0})}$$

where 0 in the subscript refers to the present.

3.4 Angular Distance

Angular distance is defined as:

$$d_A \equiv \frac{D}{\theta}$$

where D is a diameter, and θ is an object's angular size. In order to derive the relation between d_A and d_P, notice that the direction of any given photon will not change with expansion (in a flat universe). Thus, the angle θ between any two photons' paths is constant. By geometry at the time of emission, the following relation is true:

$$d_P(t_{em}) = \frac{D}{\theta}$$

Using the definition of angular distance, we discover the following relation:

$$d_A = d_P a_{em} = \boxed{\frac{d_P}{1+z_{em\to 0}}}$$

where again we have dropped the subscript $_{em}$ for convenience.

4 Common Confusions

Evolution of Density Parameter. Take the evolution of matter energy density, for example. While $\epsilon_m \propto a^{-3}$, Ω_m is NOT $\propto a^{-3}$ because the critical density evolves with a as well. Because of this, the form of the Friedmann equation present in NAO 2019 P8 is incorrect:

$$H^2 = H_0^2(\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k)$$

The correct version is actually:

$$1 = \Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k$$

Overall Density Parameter. Typically, when Ω is used, this does not include Ω_k. Therefore, the below equation is not necessarily correct:

$$\Omega = 1$$

whereas the below equation is correct:

$$1 - \Omega = \Omega_k$$

However, in almost all cases you encounter in IOAA, this distinction will not matter, because $\Omega_k = 0$ for flat universes.

Motion of a Photon. It will often be helpful to consider the motion of a photon, e.g. when considering different distance metrics. The proper speed of a photon is always c, but the proper distance it has already travelled will always be expanding, so in order to get a grip on the distance a photon travels, we will use the comoving distance χ:

$$cdt = ad\chi$$
$$\chi = \int \frac{cdt}{a(t)}$$

where $a(t)$ would be determined from integrating the Friedmann equation. Using $dt = da/\dot{a}$, we have:

$$\chi = \int_{a_{em}}^{1} \frac{cda}{Ha^2}$$

or alternatively, using z:

$$\chi = \int_0^z \frac{cdz}{H}$$

Only after we have found the comoving distance can we determine other distance measures, e.g. the proper distance:

$$d_p = a(t)\chi$$

Notice that the two instances of $a(t)$ do not cancel, because one is within the integral and changes with dt, while the other is outside the integral and is only the value at t. Additionally, notice the close parallel to the above formulas with Hubble's law, $v = cz = Hd_P$.

Curvature Issues. Perhaps you have seen formulas such as

$$D_M = D_H \frac{1}{\sqrt{\Omega_k}} \sinh\left(\sqrt{\Omega_k} D_C / D_H\right)$$

In general, anything that involves Ω_k is beyond the scope of these notes, because $\Omega_k \neq 0$ involves non-Euclidean geometry and in general complicates things a lot. You should therefore be very careful when trying to generalize the results of these notes, especially the cosmological distances section, to general cosmological models.

Hubble's Law Subtleties. You may have seen Hubble's law before:

$$v = H_0 d_P$$

Suppose photons travelled infinitely fast, i.e. we can see how every point in the universe is moving at this very instant. Then Hubble's law would be true, because:

$$v = \frac{d(d_P)}{dt} = \dot{a}\chi = \frac{\dot{a}}{a} d_P$$

However, note that Hubble's law would not describe the motion of galaxies as we see right now, because the photons we receive from very distant galaxies is from the past, a time with a different Hubble constant $H(t)$.

Cosmological Redshift versus Doppler Redshift. Doppler redshift due to the motion of objects relative to each other and cosmological redshift due to the expansion of the universe are two different things. Case in point: two points in an expanding universe may be moving apart faster than the speed of light—this does not violate relativity because space itself is expanding, and the objects themselves may not be moving at all relative to space.

Now, this brings up an excellent point: there is now a "special" frame in which things like the CMB are at rest. Does this mean cosmology and special relativity are incompatible? No, we didn't violate any well-respected theories today; see this link for more. The gist is that although there is a special reference frame, the laws of physics remain the same in all inertial reference frames, which is all that matters for special relativity.

Now, we tackle a common practice in the earlier USAAAO rounds. To determine distance from redshift, a common practice is used:

$$v = cz = H_0 d$$
$$d = \frac{cz}{H_0}$$

This is bad practice and is only applicable for $z \ll 1$. Even if we use the relativistic Doppler formula:

$$z = \sqrt{\frac{1 + v/c}{1 - v/c}}$$

The expression for d still does not hold for large redshifts, because for one, we are implicitly assuming doppler redshift and cosmological redshift give the same results, and two, we are ignoring the subtlety with Hubble's law mentioned above. In fact, it's a miracle that this holds even for $z \ll 1$, but it does almost by sheer chance. From Ryden 6.1, we can approximate the proper distance d_P starting from the exact expression:

$$d_P = c \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

We can Taylor approximate a using Hubble's constant H_0:

$$\frac{a(t)}{a(t_0)} \approx 1 + \left.\frac{\dot{a}}{a}\right|_{t=t_0} (t - t_0)$$

Using $a(t_0) = 1$ and $H_0 = (\dot{a}/a)|_{t=t_0}$, we have:

$$a(t) \approx 1 + H_0(t - t_0)$$

Using $(1 + x)^\alpha \approx 1 + \alpha x$, we have:

$$\frac{1}{a(t)} \approx 1 - H_0(t - t_0)$$

Thus, an approximate expression for redshift z would be

$$z \approx H_0(t_0 - t)$$

If we integrate the expression for d_P using our approximate expression for $a(t)$:

$$d_P \approx c(t_0 - t_e) - \frac{1}{2} c H_0 (t_0 - t_e)^2$$

To first order, we have:

$$d_P \approx \frac{cz}{H_0}$$

Identical to our original expression.

5 Lightweight Questions

Problems:

1. The derivation of the Newtonian version of the Friedmann equation presented implicitly assumes that nothing changes as $R \to \infty$, but in an infinite homogenous isotropic universe, there is no longer a preferred direction. Therefore, how do we know that an infinite universe of matter would collapse at all? If it does collapse, in which direction does the matter go?

2. The curvature constant k found in the Friedmann equation can have values of $+1$, 0, and -1. For each value, answer the following questions:
   (a) What is the sign of the Newtonian energy per unit mass, U?
   (b) What are the possible values of Ω, and how does Ω evolve with time?
3. Select all of the following that necessarily indicates a universe with positive curvature:
   (a) $\Omega > 1$
   (b) $k > 1$
   (c) $\Omega_k > 1$
4. In the equation of state $P = w\epsilon$, what value(s) of w result in
   (a) no pressure?
   (b) ϵ independent of a?
   (c) no net acceleration of the universe?
5. The redshift and luminosity distance to the Starkiller base are $z_0 = 1$ and $d_L = 5$ Gpc. Fast forward billions of years, and now the redshift to the Starkiller base is observed to be $z = 3$. What is
   (a) the scale factor of the universe at this later time?
   (b) the new luminosity distance?
6. (Adapted from USAAAO Training 2020) For a flat ($k = 0$), one-component universe of pressure-less dust, show that the angular diameter of an extended object with linear diameter D at redshift z is

$$\theta = \frac{H_0 D}{2c} \frac{(1+z)^{3/2}}{\sqrt{1+z} - 1}$$

6 Solutions to Lightweight Questions

Solutions:

1. An infinite universe *does collapse*, despite there being no preferred direction. Collapse is possible without a preferred direction because although any one observer within this universe would observe the universe collapsing towards themselves, any other observer would also observe the universe to collapse about a different point, namely towards themselves.

   This explains why a collapse can occur without violating isotropy and homogeneity but does not explain why a collapse would occur at all. Indeed, you could argue that if you integrate the gravitational force about any point in concentric shells to infinity, each shell exerts no gravitational force and thus there is no gravitational force at all. **(expand on this later but tl;dr: concentric shells is just a limiter, force is undefined)**
2. (a) From the derivation of the Friedmann equation, we see that $2U/r_C^2 = -kc^2/R_0^2$. Therefore, U always has the opposite sign of k, and $U = 0$ when $k = 0$.

   (b) As stated earlier, if $k = 0$, $\Omega = 1$ at all times. Additionally, no matter the sign of k, due to the following equation:

$$1 - \Omega = -\frac{kc^2}{H^2 a^2 R_0^2}$$

   no matter a evolves, Ω will stay either above 1 or below 1. With that stated, clearly $\Omega > 1$ at all times for $k = +1$ and $\Omega < 1$ at all times for $k = -1$.

It's impossible to pin down what exactly happens to Ω for $k = \pm 1$. Suppose, for example, a universe with only dark energy and with $k = -1$. In this case, the Friedmann equation becomes:

$$H^2 = \frac{8\pi G}{3}\frac{\epsilon_\Lambda}{c^2} + \frac{c^2}{R_0^2 a^2} = C_1 + \frac{C_2}{a^2}$$

$$\dot{a}^2 = C_1 a^2 + C_2$$

Since $\Omega = 1 + (kc^2)/(H^2 a^2 R_0^2)$, and $H \to \sqrt{C_1}$ while $a \to \infty$, Ω starts from a value below 1 and approaches 1 as time goes on.

But suppose a universe with a curious type of energy such that $\epsilon \propto a^{-2}$. In this case:

$$H^2 = \frac{C}{a^2}$$

Which means $H^2 a^2$ is constant and therefore Ω is constant. Thus, different behaviors can occur with Ω, but its relative position to 1 will remain the same.

3. By definition, $k > 0$ means positive curvature, therefore B is correct. Further, by the equation

$$1 - \Omega = -\frac{kc^2}{H^2 a^2 R_0^2}$$

   $\Omega > 0$ indicates $k > 0$, therefore A is correct as well. Finally,

$$\Omega_k \propto \epsilon_k \propto -k$$

   so C is incorrect.

4. (a) By definition, $w = 0$.
   (b) Using $\epsilon \propto a^{-3(1+w)}$, we have that $w = -1$.
   (c) Using $a \propto \epsilon + 3P \propto 1 + 3w$, we have that $w = -1/3$.

5. This problem doesn't seem challenging but actually opens up a lot of potholes, if you're not careful about z and a.
   (a) The main pitfall here is to directly use the formula $a = (1+z)^{-1}$: this would give a value of $a = 1/4$. Why is this? Remember that the derivation is from the fact that $\lambda_{em}/a_{em} = \lambda(t)/a(t)$, and therefore

$$a_{em} = \frac{a(t)}{\lambda(t)/\lambda_{em}} = \frac{a(t)}{1 + z_{em\to t}}$$

   where $a(t)$ is typically taken to be $= 1$. Therefore, to apply the formula in this manner would be equivalent to getting the scale factor of the universe at the time of emission for an object whose redshift is $z = 3$ at the present day, whereas what you want is the scale factor in the future, when an object at $z = 1$ in the present day has reached $z = 3$.

   Thus, we can apply this new formula we have derived, using the fact that $a_{em} = 1/2$ and $z_{em\to t} = 3$, to find that $a(t)$ in the future is $\boxed{2}$.
   (b) There are several pitfalls here, but they are mainly confusions with a_{em} and $a(t)$, $z_{em\to 0}$ and $z_{em\to t}$ as before, so they will not be addressed here. To do this question correctly, note that

$$d_L = d_P(1 + z_{em\to 0}) = a(t) d_c (1 + z_{em\to 0}) \propto a(t)$$

   so $d_L' = \boxed{10 \text{ Gpc}}$.

6. A photon travelling from the object (at the time of emission) to the observer would traverse a comoving distance

$$\chi = \int_0^z \frac{cdz}{H} = \int_0^z \frac{cdz}{H_0\sqrt{\Omega_{m,0}(1+z)^3}} = \frac{c}{H_0\sqrt{1}} \times -2\left(\frac{1}{\sqrt{1+z}} - 1\right)$$

The angular distance d_A is:

$$d_A = \frac{\chi}{1+z} = \frac{2c}{H_0}\frac{\sqrt{1+z}-1}{(1+z)^{3/2}}$$

Thus, $\theta = D/d_A$ and the intended result follows.