

A PRECISE PROOF OF THE n -VARIABLE BEKIĆ PRINICIPLE

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ABSTRACT. We provide a proof of the n -ary Bekić principle, which states that a vectorial fixpoint of size n can be written in terms of nested fixpoints in each coordinate according to lexicographic order. The proof is inductive.

The n -ary Bekić principle for $n > 3$ has been noticeably missing from mathematics literature, yet it should be a direct generalisation of the $n = 2$ case. We first state the $n = 2$ version for least upper bounds.

Theorem 1 (Bekić). *Let E, F be a complete lattices, and let $f : E \times F \rightarrow E \times F$ be a monotone function. Let f_1, f_2 be the projection of f onto the respective coordinates. Then the following equality holds:*

$$\mu \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = \begin{bmatrix} \mu x. f_1(x, \mu y. f_2(x, y)) \\ \mu y. f_2(\mu x. f_1(x, y), y) \end{bmatrix}$$

Proof. Refer to a textbook on fixpoints. □

The equality only asserts that RHS is one way to express LHS. It does not assert that it is the only way. In fact, the question of whether there are shorter ways to express LHS (less than exponentially long) for $n > 2$ is open.

One might venture a guess for what higher dimensions look like. For example, for $n = 3$,

$$(1) \quad \mu \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} f_1(x, y, z) \\ f_2(x, y, z) \\ f_3(x, y, z) \end{bmatrix} = \begin{bmatrix} \mu x. f_1(x, \mu y. f_2(x, y, \mu z. f_3(x, y, z)), \mu z. f_3(x, \mu y. f_2(x, y, z), z)) \\ \mu y. f_2(\mu x. f_1(x, y, \mu z. f_3(x, y, z)), y, \mu z. f_3(\mu x. f_1(x, y, z), y, z)) \\ \mu z. f_3(\mu x. f_1(x, y, z), \mu y. f_2(x, y, z), z), \mu y. f_2(\mu x. f_1(x, y, z), y, z), z \end{bmatrix}$$

Observing the nesting structure of the above, one can naturally organise the fixpoints into a tree. Such tree corresponds to strings of length n over the alphabet $\{1, \dots, n\}$, without repetition, according to lexicographic order. It turns out this is the correct intuition; next we make it precise and state the n -variable Bekić principle.

For general n , we use a recursive definition for nested fixpoints. We fix some notations. Write *nested* as the recursive, nested fixpoint we want to define. Write $\lambda j.[V(j)]$ as a vector in $L_1 \times \dots \times L_n$ such that the j -th coordinate is given by $V(j) \in L_j$. Given a binary predicate b and values $G, H \in L_i$, write $?b : G; H$ as the value in L_i given by if-then-else: if b evaluates to True, then $?b : G; H = G$; else $?b : G; H = H$. In particular, we write $v \neq \text{undef}$ as the predicate saying that v is NOT undefined. We use a special symbol *undef* to indicate an undefined value. For a map $B : \{1 \dots n\} \rightarrow (\bigsqcup L_i) \sqcup \{\text{undef}\}$, we write $B(j := v)$ as the modified map that sends j to v and i to $B(i)$ for $i \neq j$.

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Definition 2. Let L_1, \dots, L_n be complete lattices. Let $f : L_1 \times \dots \times L_n \rightarrow L_1 \times \dots \times L_n$ be a monotone function. Let f_i be the projection of f to the i -th coordinate. Fix a map $B : \{1 \dots n\} \rightarrow (\bigsqcup L_i) \sqcup \{\text{undef}\}$ mapping coordinate positions i to L_i . We define

$$(2) \quad \text{nested}(n, i, B, f) = \mu x_i. f_i(\lambda j. [?j = i : x_i; ?(B(j) \neq \text{undef}) : B(j); \text{nested}(n, j, B(i := x_i), f)])$$

A nested fixpoint for $f : L_1 \times \dots \times L_n \rightarrow L_1 \times \dots \times L_n$, of size n , at coordinate i , is given by

$$\text{nested}(n, i, B_0, f)$$

where $B_0(i) = \text{undef}$ for each $i \in \{1 \dots n\}$.

To see that the above procedure (2) terminates, consider induction on the unbound coordinates of B . If B binds all coordinates, then the definition has no recursion thus terminates. If B has x_1, \dots, x_k unbound, then the if-then-else clause in the definition tests for an unbound j -th coordinate for B . It goes into recursion only when B does not bound the j -th coordinate, and the recursive *nested* clause uses $B(j := x_j)$ which has one fewer unbound coordinate than B . Hence the recursive clause terminates by induction, hence the whole definition eventually terminates.

Intuitively, the recursive case produces f_i applied to a vector that is bound to x_i in the i -th coordinate and recursively go down one level, with i -th coordinate bound to x_i thereafter. The information for bound values is recorded in B .

Example 3. We expand $\text{nested}(3, B_0, f)$ for illustration. Firstly we reduce to:

$$\mu x_1. f_1(\lambda j. [?j = 1 : x_1; ?(B_0(j) \neq \text{undef}) : B_0(j); \text{nested}(3, B_0(1 := x_1), f)])$$

The vector $\lambda j. [?j = 1 : x_1; ?(B_0(j) \neq \text{undef}) : B_0(j); \text{nested}(3, j, B_0(1 := x_1), f)]$ can be written as

$$(x_1, \text{nested}(3, 2, B_0(1 := x_1), f), \text{nested}(3, 3, B_0(1 := x_1), f))$$

We expand the depth 2 nested fixed points:

$$\begin{aligned} \text{nested}(3, 2, B_0(1 := x_1), f) &= \mu x_2. f_2(x_1, x_2, \text{nested}(3, 3, B_0(1 := x_1, 2 := x_2), f)) \\ \text{nested}(3, 3, B_0(1 := x_1), f) &= \mu x_3. f_3(x_1, \text{nested}(3, 2, B_0(1 := x_1, 3 := x_3), f), x_3) \end{aligned}$$

We expand the depth 3 nested fixed points. Note we are in the base case:

$$\begin{aligned} \text{nested}(3, 3, B_0(1 := x_1, 2 := x_2), f) &= \mu x_3. f_3(x_1, x_2, x_3) \\ \text{nested}(3, 2, B_0(1 := x_1, 3 := x_3), f) &= \mu x_2. f_2(x_1, x_2, x_3) \end{aligned}$$

Substituting into the original fixpoint, we get that $\text{nested}(3, 1, B_0(1 := x_1), f)$ equals:

$$\mu x_1. f_1(x_1, \mu x_2. f_2(x_1, x_2, \mu x_3. f_3(x_1, x_2, x_3)), \mu x_3. f_3(x_1, \mu x_2. f_2(x_1, x_2, x_3), x_3))$$

This is exactly the first coordinate of equation 1.

Now we are ready to state the n -variable Bekić principle. Write $\mu \vec{x}. f(\vec{x})$ for

$$\mu[x_1, \dots, x_n]^T \cdot [f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)]^T$$

for some n that is implicitly understood.

Theorem 4 (n -ary Bekič). *For any $i = 1, \dots, n$, the i -th coordinate of $\mu\vec{x}.f$ is equal to*

$$\text{nested}(n, i, B_0, f)$$

where $B_0(j) = \text{undef}$ for all $j = 1, \dots, n$.

To prove the theorem we need the notion of specializing a function to a value. Given a monotone function $f : L_1 \times \dots \times L_n \rightarrow L_1, \dots, L_n$ and some $a_i \in L_i$, we call the function

$$(x_1, \dots, \widehat{x_i}, \dots, x_n) \mapsto (f_1(x_1, \dots, a_i, \dots, x_n), \dots, \widehat{f_i(x_1, \dots, a_i, \dots, x_n)}, \dots, f_n(x_1, \dots, a_i, \dots, x_n))$$

the **specialization** of f at a_i , written as $Sp_{i,a_i}f$. Note that $Sp_{i,a_i}f$ has domain and codomain $L_1 \times \dots \times \widehat{L_i} \times \dots \times L_n$. For (y_1, \dots, y_{n-1}) , we define the (i, a_i) -shifted vector to be

$$(3) \quad E_{i,a_i}\vec{y} = \lambda j \in \{1 \dots n\}. ?j < i; y_j; j = i : a_i; y_{j-1}$$

or more intuitively $(y_1, \dots, a_i, y_i, \dots, y_{n-1})$.

Note there is a coordinate vector shift in the projections of specialized function. Let $j \in \{1, \dots, n-1\}$. Then:

$$(4) \quad (Sp_{i,a_i}f)_j(y_1, \dots, y_{n-1}) = \begin{cases} f_j(E_{i,a_i}\vec{y}) & \text{if } 1 \leq j < i \\ f_{j+1}(E_{i,a_i}\vec{y}) & \text{if } i \leq j \leq n-1 \end{cases}$$

There is a corresponding shift in B . Given $j \in \{1, \dots, n-1\}$:

$$(5) \quad (Sp_i B)(j) = \begin{cases} B(j) & \text{if } 1 \leq j < i \\ B(j+1) & \text{if } i \leq j \leq n-1 \end{cases}$$

What variables do $Sp_{i,a_i}f$ use? When we construct a nested fixpoint $\text{nested}(n-1, k, B, Sp_{i,a_i}f)$, the algorithm is going to create a clause $\mu x_k.(Sp_{i,a_i}f)_k$, which binds the x_k position of $Sp_{i,a_i}f$ and projects to coordinate k . The ambiguity with this expression is that often we would make $\mu x_k.f_k(\vec{v})$ with some special vector \vec{v} that fixes one coordinate of f_k , making it equal to some $Sp_{j,a_j}f$. Then $k \in \{1, \dots, n\}$ has to be put to the appropriate coordinate in the $(n-1)$ -ary input vector, and $(Sp_{i,a_i}f)(x_1, \dots, \widehat{x_i}, \dots, x_n)$ has input vector y_1, \dots, y_{n-1} , so there is a skip of i when mapping to f 's coordinates. We keep to the convention that any n -ary function has input vector coordinates numbered from 1 to n , and track any transformation between $(n-1)$ -ary and n -ary input vectors with a map, similar to the simplicial category.

Lemma 5. *Let $i, j \in \{1, \dots, n\}$, where $n > 1$. For $i < j$, we have*

$$\text{nested}(n, i, B(j := a_j), f) = \text{nested}(n-1, i, Sp_j B, Sp_{j,a_j}f)$$

For $i > j$, we have

$$\text{nested}(n, i, B(j := a_j), f) = \text{nested}(n-1, i-1, Sp_j B, Sp_{j,a_j}f)$$

Proof. By induction on the number of unbound coordinate in B . We assume B does not bind j so that $B(j := a_j)$ is a genuine modification. Hence B has at least 1 unbound coordinate. The base case says B only has 1 unbound coordinate and that has to be j . Then LHS reduces to $\mu x_i.f_i(B(1), \dots, a_j, \dots, B(n)) = f_i(B(1), \dots, a_j, \dots, B(n))$. Meanwhile RHS has B binding every coordinate $1, \dots, \hat{j}, \dots, n$, as $Sp_{j,a_j}f$ acts on $L_1 \times \dots \times \widehat{L_j} \times \dots \times L_n$. When $i < j$, RHS reduces to

$$\mu x_i.(Sp_{j,a_j}f)_i(B(1), \dots, \widehat{B(j)}, \dots, B(n)) = \mu x_i.f_i(B(1), \dots, a_j, \dots, B(n)) = f_i(B(1), \dots, a_j, \dots, B(n)).$$

When $i > j$, RHS takes the value $\mu y_{i-1}.(Sp_{j,a_j}f)_{i-1}(B(1), \dots, \widehat{B(j)}, \dots, B(n)) = \mu y_{i-1}.f_i(B(1), \dots, a_j, \dots, B(n)) = f_i(B(1), \dots, a_j, \dots, B(n))$. So the base case is proven.

For the inductive case, consider $i < j$. Unwrap the definition of LHS to get

$$\mu x_i.f_i(\lambda k.?k = i : x_i; ?k = j : a_j; ?B(k) \neq \text{undef} : B(k); \text{nested}(n, k, B(j := a_j, i := x_i), f))$$

Since $B(i := x_i)$ has 1 fewer variable bound than B , IH can be applied to yield

$$\text{nested}(n, k, B(j := a_j, i := x_i), f) = \begin{cases} \text{nested}(n - 1, k, Sp_j(B(i := x_i)), Sp_{j,a_j}f) & \text{if } k < j \\ \text{nested}(n - 1, k - 1, Sp_j(B(i := x_i)), Sp_{j,a_j}f) & \text{if } k > j \end{cases}$$

Note that $Sp_j(B(i := x_i)) = Sp_j B(i := x_i)$ when $i < j$. Hence the expression

$$f_i(\lambda k.?k = i : x_i; ?k = j : a_j; ?B(k) \neq \text{undef} : B(k); \text{nested}(n, k, B(j := a_j, i := x_i), f))$$

with a map of coordinates $h \in \{1, \dots, n - 1\}$ to $k \in \{1, \dots, n\}$ skipping j , evaluates to

$$\begin{aligned} (Sp_{j,a_j}f)_i(\lambda h \in \{1 \dots n - 1\}.?h = i : x_i; ?Sp_j B(h) \neq \text{undef} : Sp_j B(h); \\ ?h < j : \text{nested}(n - 1, h, Sp_j B(i := x_i), Sp_{j,a_j}f); \\ \text{nested}(n - 1, h + 1 - 1, Sp_j B(i := x_i), Sp_{j,a_j}f) \end{aligned}$$

which is precisely

$$\begin{aligned} (Sp_{j,a_j}f)_i(\lambda h \in \{1 \dots n - 1\}.?h = i : x_i; ?Sp_j B(h) \neq \text{undef} : Sp_j B(h); \\ \text{nested}(n - 1, h, Sp_j B(i := x_i), Sp_{j,a_j}f)) \end{aligned}$$

Adding μx_i at the front we get equality to RHS.

When $i > j$, unwrap the LHS to get the same expression. We have a symmetric IH, so it can be applied in the same manner to yield

$$\text{nested}(n, k, B(j := a_j, i := x_i), f) = \begin{cases} \text{nested}(n - 1, k, Sp_j(B(i := x_i)), Sp_{j,a_j}f) & \text{if } k < j \\ \text{nested}(n - 1, k - 1, Sp_j(B(i := x_i)), Sp_{j,a_j}f) & \text{if } k > j \end{cases}$$

However, this time $Sp_j(B(i := x_i)) = (Sp_j B)(i - 1 := x_i)$. We evaluate to

$$\begin{aligned} (Sp_{j,a_j}f)_{i-1}(\lambda h \in \{1 \dots n - 1\}.?h = i - 1 : x_i; ?Sp_j B(h) \neq \text{undef} : Sp_j B(h); \\ ?h < j : \text{nested}(n - 1, h, Sp_j B(i - 1 := x_i), Sp_{j,a_j}f); \\ \text{nested}(n - 1, h + 1 - 1, Sp_j B(i - 1 := x_i), Sp_{j,a_j}f) \end{aligned}$$

which is

$$\begin{aligned} (Sp_{j,a_j}f)_{i-1}(\lambda h \in \{1 \dots n - 1\}.?h = i - 1 : x_i; ?Sp_j B(h) \neq \text{undef} : Sp_j B(h); \\ \text{nested}(n - 1, h, Sp_j B(i - 1 := x_i), Sp_{j,a_j}f)) \end{aligned}$$

We observe that the μx_i in front actually corresponds to the $i - 1$ -th coordinate of the input vector of $Sp_{j,a_j}f$. Hence we get precisely

$$\text{nested}(n - 1, i - 1, Sp_j B, Sp_{j,a_j}f)$$

which is RHS. \square

We are ready to prove the n -ary Bekić principle.

0.1. Proof of theorem 4. Write the vectorial fixpoint as

$$\mu \vec{x}.f = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix}$$

We prove two inequalities separately. First we make a claim about a_i being larger. We use $[i_1 := a_{i_1}, \dots, i_k := a_{i_k}]$ to denote the map $B_0(i := a_i, \dots, i_k := a_{i_k})$, where B_0 is our favorite empty map undefined on every coordinate.

Claim 6. Let a_1, \dots, a_n be components of the vectorial fixpoint of a monotone function f . Then $a_i \geq f_i(\lambda j.?j = i : a_i; \text{nested}(n, j, [i := a_i], f))$.

Proof. By induction on n . Let $n = 1$. Then $f : L \rightarrow L$ has least fixpoint $a_1 = \mu x.f(x)$. Then $a_1 = f(a_1)$.

Suppose $n > 1$. We fix a general i such that $1 < i < n$. The cases $i = 1$ and $i = n$ are simplifications of the general case. First for each $j \neq i$, observe that a_j is part of a vectorial fixpoint for f specialised on a_i :

$$Sp_{i,a_i}f(a_1, \dots, \widehat{a_i}, \dots, a_n) = (a_1, \dots, \widehat{a_i}, \dots, a_n)$$

The above input and output vectors are of length $n - 1$, and $Sp_{i,a_i}f$ is obviously monotone. Note that when $j > i$, a_j is at the $(j - 1)$ -th coordinate of $Sp_{i,a_i}f$. By IH we have for each $j \neq i$

$$a_j \geq \begin{cases} (Sp_{i,a_i}f)_j(\lambda k.?k = j : a_j; \text{nested}(n - 1, k, [j := a_j], Sp_{i,a_i}f)) & \text{if } j < i \\ (Sp_{i,a_i}f)_{j-1}(\lambda k.?k = j - 1 : a_j; \text{nested}(n - 1, k, [j - 1 := a_j], Sp_{i,a_i}f)) & \text{if } j > i \end{cases}$$

When $j < i$ we have by lemma 5

$$\text{nested}(n - 1, k, [j := a_j], Sp_{i,a_i}f) = \begin{cases} \text{nested}(n, k, [j := a_j, i := a_i], f) & \text{if } k < i \\ \text{nested}(n, k + 1, [j := a_j, i := a_i], f) & \text{if } k + 1 > i \end{cases}$$

When $j > i$ we have again by lemma 5

$$\text{nested}(n - 1, k, [j - 1 := a_j], Sp_{i,a_i}f) = \begin{cases} \text{nested}(n, k, [j := a_j, i := a_i], f) & \text{if } k < i \\ \text{nested}(n, k + 1, [j := a_j, i := a_i], f) & \text{if } k + 1 > i \end{cases}$$

Let us fix some $j > i$. Then

$$\begin{aligned} a_j &\geq (Sp_{i,a_i}f)_{j-1}(\lambda k.?k = j - 1 : a_j; \text{nested}(n - 1, k, [j - 1 := a_j], Sp_{i,a_i}f)) \\ &= f_j(\lambda h \in \{1, \dots, n\}.?h = i : a_i; ?h = j : a_j; \\ &\quad ?h < i : \text{nested}(n, h, [j := a_j, i := a_i], f); \\ &\quad \text{nested}(n, h - 1 + 1, [j := a_j, i := a_i], f)) \\ &= f_j(\lambda h.?h = i : a_i; ?h = j : a_j; \text{nested}(n, h, [j := a_j, i := a_i], f)) \end{aligned}$$

By definition of least fixpoint we get

$$\begin{aligned} a_j &\geq \mu x_j.f_j(\lambda h.?h = i : a_i; ?h = j : x_j; \text{nested}(n, h, [j := x_j, i := a_i], f)) \\ &= \text{nested}(n, j, [i := a_i], f) \end{aligned}$$

Now fix some $j < i$. Then we have similarly

$$\begin{aligned} a_j &\geq (Sp_{i,a_i}f)_j(\lambda k.?k = j : a_j; nested(n - 1, k, [j := a_j], Sp_{i,a_i}f)) \\ &= f_j(\lambda h \in \{1, \dots, n\}.?h = i : a_i; ?h = j : a_j; \\ &\quad ?h < i : nested(n, h, [j := a_j, i := a_i], f); \\ &\quad nested(n, h - 1 + 1, [j := a_j, i := a_i], f) \\ &= f_j(\lambda h.?h = i : a_i; ?h = j : a_j; nested(n, h, [j := a_j, i := a_i], f)) \end{aligned}$$

and we get the same inequality

$$a_j \geq nested(n, j, [i := a_i], f)$$

We substitute this inequality into the equation

$$a_i = f_i(\lambda j.j = i : a_i; a_j)$$

and we get

$$a_i \geq f_i(\lambda j.?j = i : a_i; nested(n, j, [i := a_i], f))$$

which is exactly what is claimed. \square

With Claim 6 and definition of least fixpoint we immediately get half of what we are supposed to show:

$$a_i \geq \mu x_i.f_i(\lambda j.?j = i : x_i; nested(n, j, [i := x_i], f)) = nested(n, i, [], f)$$

For the other half, we write $a'_i = nested(n, i, [], f)$. Then we have

$$a'_i = f_i(\lambda j.?j = i : a'_i; nested(n, j, [i := a'_i], f))$$

$$Sp_{1,a'_1}f(x_1, \dots, x_{n-1}) = (f_2(a'_1, x_1, \dots, x_{n-1}), \dots, f_n(a'_1, x_1, \dots, x_{n-1}))$$

By IH, we have

$$\mu \begin{bmatrix} x_1 \\ \dots \\ x_{n-1} \end{bmatrix} \cdot \begin{bmatrix} (Sp_{1,a'_1}f)_1 \\ \dots \\ (Sp_{1,a'_1}f)_{n-1} \end{bmatrix} = \begin{bmatrix} nested(n-1, 1, [], Sp_{1,a'_1}f) \\ \dots \\ nested(n-1, n-1, [], Sp_{1,a'_1}f) \end{bmatrix}$$

By lemma 5, the RHS equals

$$\begin{bmatrix} nested(n, 2, [1 := a'_1], f) \\ \dots \\ nested(n, n, [1 := a'_1], f) \end{bmatrix}$$

Write $a''_i = nested(n, i, [1 := a'_1], f)$. Hence from the vectorial fixpoint for $Sp_{1,a'_1}f$ we have equations

$$\begin{cases} f_2(a'_1, a''_2, \dots, a''_n) = a'_1 \\ f_3(a'_1, a''_2, \dots, a''_n) = a''_2 \\ \dots \\ f_n(a'_1, a''_2, \dots, a''_n) = a''_n \end{cases}$$

Moreover since $a'_1 = nested(n, 1, [], f) = \mu x_1.f_1(\lambda j.j = 1 : x_1; a''_j)$ we have

$$f_1(a'_1, a''_2, \dots, a''_n) = a'_1$$

This implies that the vector $(a'_1, a''_2, \dots, a''_n)$ is a fixpoint of f . Since (a_1, \dots, a_n) is the least fixpoint, we get $a_1 \leq a'_1$. By a permutation argument we also get $a_i \leq a'_i$ for all $i > 1$. This finishes the proof of theorem 4.

0.2. Concluding remarks. We have presented an inductive proof of the n -ary Bekič principle. Our principal motivation for going into such technical detail is the need for this proof to be formalized in the proof assistant **Isabelle**. Hence, one cannot merely present an intuitive argument that relies on exhibiting the explicit form of nested fixpoint for a number n that is not fixed. Such an argument would require human imagination and the size of the argument grows in the order of $\text{EXP}(n)$. In this regard, induction is a higher-order reasoning mechanism that allows for a proof of fixed length to work for all numbers n . This is of course reflected in our proof as well.

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