Scalar Curvature Rigidity of Subsets of Spheres

Master's Thesis
of

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Chapter 1

Introduction

This thesis is an exposition of the recent work of [BBHW24], which deals with scalar curvature rigidity of subsets of spheres. This line of research began with Llarull's result [Lla88], which states that there can be no metric on a sphere that is greater than the standard metric and has greater scalar curvature at the same time. The intuition for this is "the higher the metric the smaller the curvature" - indeed if one looks at S^2 equipped with a large radius $r^2 g_{S^2}$, the scalar curvature decreases by a factor of $\frac{1}{r}$. However, the proof relies on the existence of harmonic (and indeed parallel) spinor fields on the sphere which puts an upper bound on the given metric via the Weitzenböck formula with an appropriately twisted spinor bundle. The Atiyah-Singer index theorem is used in a crucial place to summon such a harmonic spinor field. In [BBHW24], the same question is asked about different subsets of the sphere, namely 1) the sphere with open disks removed at the top and bottom and 2) the sphere with antipodal points removed, which is considered as a "limit" of the case of open disks. For the proof, the line of thought is very similar to [Lla88], but since one needs to consider manifolds with nonempty boundary the involvement with spinor bundles and Dirac operators *of* the boundary is much subtle. In particular, one needs to pass the entire index-theoretic argument to the boundary, which is assumed to be closed. Additionally, the calculations quickly become more technical.

The structure of this thesis is as follows. Chapter 2 provides the preparatory materials, with [BB11],[LM89],[GHL04],[Bä11] and [Pet16] as go-to sources. Chapter 3 contains the major part of this thesis which deals with the main results of [Lla88], and the open disk case of [BBHW24]. The case of antipodal points is untouched because of time constraints.

Chapter 2

Preliminaries

2.1 Manifolds, vector bundles and differential operators

The main stage for our considerations in this thesis is smooth manifolds with boundary. We recall a few essential definitions. A manifold with boundary is a topological space locally diffeomorphic to either an open subset of Euclidean space, or an open subset of the upper half plane of Euclidean space. Such a diffeomorphism is called a **coordinate chart**. A vector bundle (over some base manifold) is a manifold itself, but locally it is diffeomorphic (such a diffeomorphism is called **trivialisation**) to the product of an open neighbourhood of the base manifold and a vector space of some fixed dimension. This open neighbourhood comes from a coordinate chart of the base manifold, and again can be diffeomorphic to possibly an open subset of upper half plane. A vector bundle comes with a projection to the base manifold. A **fiber** of the vector bundle, considered above some point of the base manifold, is the preimage of this projection of the said point. It is equipped with a finite dimensional vector space structure, hence also a canonical topology. A **section** of a vector bundle is a continuous map from the base manifold to the bundle such that the fiber is preserved, *i.e.* the point p is mapped to the fiber above p. A section can be smooth, of different degrees, when we locally view it as a vector valued function with the vector bundle trivialised and on a coordinate chart of the base. This notion is well-defined regardless of choice of charts, so we have the notion of local $C^1, C^2, \ldots, C^\infty$ smooth sections following the usual terminology of multivariable calculus. Sometimes a scalar section is, locally, not very smooth but is continuous and measurable. We can therefore integrate it in L^2 and take its L^2-norm (which is not guaranteed to be $< \infty$).

With the choice of a partition of unity (with some additional assumptions such as locally finite, which is achievable for a *paracompact* base manifold), we can define global notions of smoothness and L^2-integrability.

Suppose we have two vector bundles E, F over the same base manifold. A **differential operator** of order at most k is a linear map (usually over $\mathbb{R}$) $P : C^\infty(M, E) \to C^\infty(M, F)$ that locally (say over a coordinate chart U) has the form

$$\sum_{|\alpha| \leq k} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

where $A^\alpha(x)$ is a smooth section of the bundle of homomorphisms $\mathrm{Hom}(E, F)$ over $x \in U$. In other words, a differential operator is locally a differential operator defined for Euclidean spaces.

For a smooth section s of E, P acts on $s|_U$ by differentiating with respect to the coordinate variables x^α and applying the homomorphism A^α, and summing across all α. However, across charts P satisfies some compatibility conditions such that it is globally well-defined. Note if $x \in U$ is a point on the boundary, one needs to extend $s|_U$ to an open subset of the Euclidean space and show that different extensions do not alter the outcome of the differential. This should be a generalisation of the fact from analysis that upper and lower limits are equal whenever a limit exists.

2.2 Metrics, connections, Laplace and Dirac-type operators

A Riemannian metric g on a manifold is a smooth section of the bundle of symmetric 2-tensors on the manifold, i.e. an element of $C^\infty(T^*M \odot T^*M)$, which is pointwise symmetric, nondegenerate and positive definite. In other words, on each point p of the manifold the metric gives a positive definite inner product on the tangent space T_pM. The metric gives a canonical isomorphism between T_pM and its dual space T_p^*M, given by

$$v \mapsto g(v, \cdot)$$

Hence there is an induced Riemannian metric on the cotangent bundle T^*M.

For a map between Riemannian manifolds $f : (M, g) \to (N, h)$, we can pull back the Riemannian metric $f^*h(X, Y) = h(f_*X, f_*Y)$. There are two different notions of comparison.

Definition 2.2.1. Let f be as above. f is called **ε-contracting** if for all $X \in TM$

$$f^*h(X, X) \leq \varepsilon g(X, X)$$

f is called **C-Lipschitz** for some $C \in \mathbb{R}_{>0}$ if for any $X, Y \in TM$

$$f^*h(X, Y) \leq Cg(X, Y)$$

Remark. Lipschitz is stronger than contracting. The names C and ε are just conventional choices. Intuitively, contracting only considers the effect in a "radial" direction, whereas Lipschitz additionally considers interacting tangent vectors in different directions.

For a vector bundle E, we can similarly define a Riemannian metric on E to be a smooth, pointwise symmetric inner product on E. For smooth sections ϕ, ψ on E, we can take its pointwise L^2-inner product $\langle \phi, \psi \rangle$ which is a smooth function. Then we can integrate this function over M and define the L^2 inner product on $C^\infty(M, E)$:

$$(\phi, \psi)_{L^2} := \int_M \langle \phi, \psi \rangle \, dV$$

where V is the Riemannian volume form.

This inner product induces a norm, denoted $||\cdot||_{L^2}$, on $C^\infty(M, E)$ which is usually not complete. The completion with respect to this norm is denoted $L^2(M, E)$, and it is a fact that this space corresponds to the space of L^2-integrable sections, i.e. continuous sections ϕ such that

$$\int_M |\phi|^2 \, dV < \infty$$

A **connection** on a vector bundle E is a map $\nabla : C^\infty(E) \times C^\infty(TM) \to C^\infty(E)$. It is denoted by $\nabla_X s \coloneqq \nabla(s, X)$. It is required to be

(1) tensorial in the second component, i.e. $\nabla_{fX} s = f \nabla_X s$ for any smooth function f.
(2) Leibniz in the first component, i.e. $\nabla_X(fs) = df(X)s + f\nabla_X s$ for any smooth function f.

Moreover if a connection is required to be **metric** and **torsion-free**, it can be uniquely determined. Such a connection is called the **Levi-Civita** connection, and acts on $E = TM$. We write ∇^{LC} for this.

The **principal symbol** of a differential operator P is a linear map $\sigma_k(P, \xi) : E_x \to F_x$ for a point $x \in M$ and a linear map $\xi \in T_x^* M$. Choose a smooth function f such that $df|_x = \xi$ and $f(x) = 0$, and extend e locally to a vector field $\tilde{e}$. Then the principal symbol is defined as

$$\sigma_k(P, \xi)(e) \coloneqq \frac{1}{k!} P(f^k \tilde{e})|_x$$

One can verify by direct computation, that upon writing $\xi_i = \xi(\frac{\partial}{\partial x^i}) = \frac{\partial f}{\partial x^i}$,

$$\sigma_k(P, \xi)(e) = \sum_{|\alpha|=k} A^\alpha(x) \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n} \cdot e$$

which contains information about the top-degree coefficients A^α.

An operator of degree 2 from E to E is of **Laplace-type** if the principal symbol is given by scalar multiplication:

$$\sigma_2(P, \xi)e = -|\xi|^2 e$$

where $|\cdot|$ is the induced Riemannian metric on T^*M.

For an operator $P : C^\infty(M, E) \to C^\infty(M, F)$, it is a fact that there always exists a **formal adjoint** of P, denoted $P^* : C^\infty(M, F) \to C^\infty(M, E)$, such that the following is satisfied:

$$(P\phi, \psi)_{L^2} = (\phi, P^*\psi)_{L^2}$$

An operator D of degree 1 from E to E is of **Dirac-type** if D^*D and DD^* are of Laplace-type.

There are several operators involving second-order derivatives on sections of vector bundles. We introduce them here. Fix a vector bundle E over a manifold M. Let ∇ be a connection on E. We denote the Levi-Civita connection by ∇^{LC}.

Definition 2.2.2. Let V, W be fixed vector fields on M. The **invariant second derivative** $\nabla^2_{V,W} : \Gamma(E) \to \Gamma(E)$ is given by $\nabla^2_{V,W} s = \nabla_V \nabla_W s - \nabla_{\nabla^{LC}_V W} s$.

Definition 2.2.3. The **connection Laplacian** of a vector bundle E, denoted by $\nabla^*\nabla : \Gamma(E) \to \Gamma(E)$, is a given by the negative of trace of the invariant second derivative, i.e. $\nabla^*\nabla s = -\operatorname{tr} \nabla^2_{(-,-)} s$.

Remark. In a local orthonormal frame $e_1, \dots, e_n$, we have

$$\nabla^*\nabla s = -\sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} s - \nabla_{\nabla^{LC}_{e_i} e_i} s \tag{2.1}$$

We will need a definition of index of linear operators between Hilbert spaces.

Definition 2.2.4. Let V, W be Hilbert spaces. Let $P : V \to W$ be a linear operator that has closed range, finite-dimensional kernel and finite-dimensional cokernel.

Then the **index** of P is given by

$$\operatorname{ind} P = \dim \ker P - \dim \operatorname{coker} P$$

Alternatively, the adjoint $P^* : W \to V$ exists and

$$\operatorname{ind} P = \dim \ker P - \dim \ker P^*$$

Remark. It is a standard exercise in functional analysis to check that $\ker P^* = (\operatorname{Im} P)^\perp$. Since $(\operatorname{Im} P)^\perp$ is a complement of $\operatorname{Im} P$, its dimension equals $\dim \operatorname{coker} P$. So the two definitions of index agree.

2.3 Clifford Algebras, Spin groups, spinor spaces, spinor representations

A Clifford algebra is an algebra that realises the **Clifford relation**. Fix (V, β) where V is a k-vector space and β a symmetric bilinear form on V. Let A be a unital algebra over k. A k-linear map $\iota : V \to A$ satisfies the Clifford relation if

$$\iota(v)^2 = -\beta(v, v) \cdot 1$$

A **Clifford algebra** of (V, β) is a unital algebra universal with respect to the Clifford relation. It is denoted by $Cl(V, \beta)$. In words, if B is a unital algebra with a k-linear map $f : V \to B$ such that $f(v)^2 = -\beta(v, v) \cdot 1$, then there is a unique algebra homomorphism $\tilde{f} : Cl(V, \beta) \to B$ such that $\tilde{f} \circ \iota = f$. One can check that $Cl(V, \beta)$ is uniquely determined up to unique isomorphism.

As is true with all objects governed by universal properties, there is a definition from the ground up. Here is the concrete construction of Clifford algebra for (V, β).

Definition 2.3.1. Given a vector space V equipped with a symmetric bilinear form $\beta : V \times V \to V$, the **Clifford algebra** of (V, β), denoted by $Cl(V, \beta)$ is the quotient of the tensor algebra:

$$\left(\bigoplus_{i=0}^{\infty} \mathcal{T}^i(V) \right) / \mathcal{I} \tag{2.2}$$

with $\mathcal{I} = \langle \{ a \otimes b + b \otimes a + 2\beta(a, b) \cdot 1 : a, b \in V \} \rangle$, where $\langle . \rangle$ denotes ideal generation.

The algebra multiplication on $Cl(V, \beta)$ is given by concatenation of elementary tensors. The tensor algebra, before quotienting by $\mathcal{I}$, is obviously $\mathbb{Z}$-graded. But $\mathcal{I}$ is not a homogeneous ideal so the grading does not descends to a $\mathbb{Z}$-grading on the quotient. However, $\mathcal{I}$ is generated by elements in the even part of the tensor algebra, so we have a $\mathbb{Z}_2$ grading on $Cl(V, \beta)$.

Lemma 2.3.2. There is a $\mathbb{Z}_2$ grading on $Cl(V, \beta)$, i.e. there is a splitting of additive abelian groups $Cl(V,\beta) = Cl(V,\beta)^0 \oplus Cl(V,\beta)^1$, where we have spans in the quotient vector space

$$Cl(V,\beta)^0 = \operatorname{span}\{\overline{v_{i_1} \otimes \cdots \otimes v_{i_k}} : k \in 2\mathbb{N}_0\}$$
$$Cl(V,\beta)^1 = \operatorname{span}\{\overline{v_{i_1} \otimes \cdots \otimes v_{i_k}} : k \in 2\mathbb{N}_0 + 1\}$$

Note that the $\mathbb{Z}_2$-grading is not a direct sum of algebras. The multiplication of two elements in the odd part $Cl(V,\beta)^1$ lands in the even part $Cl^0(V,\beta)$.

Example 1. The most common Clifford algebra is $Cl(\mathbb{R}^n, \cdot)$, where $\cdot$ is the usual dot product on $\mathbb{R}^n$. We abbreviate it to Cl_n. Its dimension is 2^n.

Example 2. Clifford algebra is well-behaved under field extensions. Given a field extension $k \subset K$, a vector space V over k can be extended by $V \otimes K$ to a vector space over K. If V is equipped with a symmetric k-bilinear form β, one can check that $V \otimes K$ is equipped with an induced K-bilinear form β' in a canonical way. Then one checks that $Cl(V,\beta) \otimes K = Cl(V \otimes K, \beta')$ by verifying the universal property.

As a use case, we can define the **complexified Clifford algebra** by taking tensor product with $\mathbb{C}$. Write $\mathbb{C}l_n = Cl_n \otimes \mathbb{C}$. Its real dimension is 2^{n+1}.

There is an inner product on Cl_n. It is such that the basis

$$\{e_I = e_{i_1} \dots e_{i_k} | I = (i_1, \dots, i_k) \text{ is a multi-index of cardinality } n\}$$

is declared orthonormal. On $\mathbb{C}l_n$ we extend complex linearly on the second coordinate, and complex anti-linearly on the first. We have

Lemma 2.3.3. Let v, w be elements of a certain Clifford algebra $Cl(V)$. Then for any $X \in V$,

$$\langle X \cdot v, w\rangle = -\langle v, X \cdot w\rangle$$

Proof. See [Bä11, Chapter 2]. □

Lemma 2.3.4. Let $|.|$ denote the norm induced by the inner product on a Clifford algebra. Let $v, w \in Cl(V)$ be elements of a Clifford algebra, where $v \in V$. Then

$$|\langle v \cdot w, w\rangle| \leq |v||w|^2 \tag{2.3}$$

Proof. $|\langle v \cdot w, w\rangle| \leq |v \cdot w||w|$ by Cauchy-Schwartz inequality. Then $|v \cdot w| = \sqrt{\langle v \cdot w, v \cdot w\rangle} = \sqrt{-\langle v \cdot v \cdot w, w\rangle} = \sqrt{-\langle -|v|^2 w, w\rangle} = |v||w|$. The inequality follows. □

Definition 2.3.5 (Pin and Spin groups)**.** The **Pin** group is the following multiplicative subgroup of Cl_n:

$$\operatorname{Pin}(n) = \{v_1 \cdot ... \cdot v_m \in Cl_n : v_i \in S^{n-1} \subset \mathbb{R}^n \subset Cl_n,\ m \in \mathbb{N}_0\}$$

The **SPin** group is the following subgroup of $\operatorname{Pin}(n)$:

$$\operatorname{Spin}(n) = \{v_1 \cdot ... \cdot v_m \in Cl_n : v_i \in S^{n-1} \subset \mathbb{R}^n,\ m \in 2\mathbb{N}_0\}$$

Let $n = 2m$ be an even number. Let $e_1, \dots, e_{2m}$ be the standard basis of $\mathbb{R}^n$. Then

$$\{e_{i_1} \dots e_{i_k} : 1 \leq i_1 < \dots < i_k \leq n\}$$

forms a basis of Cl_n. This set together with its multiplication by i forms a basis of $\mathbb{C}l_n$. For $j = 1, \dots, m$ define

$$z_j = \frac{1}{2}(e_{2j-1} - ie_{2j}) \quad \bar{z}_j = \frac{1}{2}(e_{2j-1} + ie_{2j})$$

The following set forms a basis of $\mathbb{C}l_n$:

$$\{z_{j_1} \dots z_{j_k} \bar{z}_{i_1} \dots \bar{z}_{i_l} : 1 \leq j_1 < \dots < j_k \leq n, \ 1 \leq i_1 < \dots < i_l \leq n\}$$

Let

$$z(j_1, \dots, j_k) \coloneqq z_{j_1} \dots z_{j_k} \bar{z}_1 \dots \bar{z}_m$$

Definition 2.3.6. The **spinor space** in dimension n, denoted Σ_n, is

$$\Sigma_n \coloneqq \mathrm{span}_{\mathbb{C}}\{z(j_1, \dots, j_k) : 0 \leq k \leq m, \ 1 \leq j_1 < \dots < j_k \leq m\} \subset \mathbb{C}l_n$$

An element of Σ_n is called a **spinor**.

The **volume element** of $\mathbb{C}l_n$ is given by

$$\omega = e_1 \dots e_n \in Cl_n \subset \mathbb{C}l_n$$

One checks that $\dim_{\mathbb{C}}(\Sigma_n) = 2^m = 2^{\frac{n}{2}}$.

Lemma 2.3.7.

$$i^m \omega \cdot z(j_1, \dots, j_k) = (-1)^k z(j_1, \dots, j_k)$$

Proof. Direct computation. $\square$

We can define the ± 1 eigenspace of multiplication by $i^m \omega$.

Definition 2.3.8.

$$\Sigma_n^{\pm} = \mathrm{span}_{\mathbb{C}}\{z(j_1, \dots, j_k) : i^m \omega \cdot z(j_1, \dots, j_k) = \pm z(j_1, \dots, j_k)\}$$

We have $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$. One can also characterise $\Sigma_n^{\pm}$ directly using the parity of k in $z(j_1, \dots, j_k)$.

Now let n be an odd number. We define

$$\Sigma_n = \Sigma_{n+1}^+$$

The positive part of the spinor space of 1 dimension higher.

Some basic computational results we need are as follows.

Lemma 2.3.9. Let n be an even number. Let $z(j_1, \dots, j_k)$ be a spinor. Let e_{2l} and e_{2l-1} be elements of $\mathbb{R}^n \subset \mathbb{C}l_n$. In the multiplicands of $z(j_1, \dots, j_k)$, if e_{2l} or e_{2l-1} are not contained as a term in $z(j_1, \dots, j_k)$, then there is ν such that $j_\nu < l < j_{\nu+1}$ and $z(j_1, \dots, j_k) = z(j_1, \dots, j_\nu, j_{\nu+1}, \dots, j_k)$. Else, $z(j_1, \dots, j_k)$ is of the form $z(j_1, \dots, j_\nu, l, j_{\nu+1}, \dots, j_k)$. There are 4 cases:

(1) e_{2l} is NOT contained in $z(j_1, \dots, j_k)$. Then

$$e_{2l} \cdot z(j_1, \dots, j_k) = i(-1)^\nu z(j_1, \dots, j_\nu, l, j_{\nu+1}, \dots, j_k)$$

(2) e_{2l-1} is NOT contained in $z(j_1, \dots, j_k)$. Then

$$e_{2l-1} \cdot z(j_1, \dots, j_k) = (-1)^\nu z(j_1, \dots, j_\nu, l, j_{\nu+1}, \dots, j_k)$$

(3) e_{2l} is contained in $z(j_1, \dots, j_k)$. Then

$$e_{2l} \cdot z(j_1, \dots, j_\nu, l, j_{\nu+1}, j_k) = (-1)^\nu z(j_1, \dots, j_\nu, j_{\nu+1}, \dots, j_k)$$

(4) e_{2l-1} is contained in $z(j_1, \dots, j_k)$. Then

$$e_{2l-1} \cdot z(j_1, \dots, j_\nu, l, j_{\nu+1}, j_k) = (-1)^{\nu+1} z(j_1, \dots, j_\nu, j_{\nu+1}, \dots, j_k)$$

Proof. Direct computation. □

From the above lemma, we know Σ_n is a $\mathbb{C}l_n$-module by Clifford multiplication. Moreover, the action satisfies $\mathbb{C}l_n^0 \cdot \Sigma_n^\pm \subset \Sigma_n^\pm$ and $\mathbb{C}l_n^1 \cdot \Sigma_n^\pm \subset \Sigma_n^\mp$. In particular, $\mathrm{Spin}(n) \subset \mathbb{C}l_n^0$ and it preserves the parity of spinor spaces. Hence the action descends to an action on both Σ_n^+ and Σ_n^-.

Now let n be an odd number. We would like to have an analogous action. We need the following lemma:

Lemma 2.3.10. Let $n \in \mathbb{N}$. Let $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ be inclusion into the first n coordinates. The map $\alpha : \mathbb{R}^n \to \mathbb{C}l_{n+1}^0$ given by

$$\alpha(e_i) = e_i \cdot e_{n+1}$$

satisfies the Clifford relations and lifts to a map $\alpha : \mathbb{C}l_n \to \mathbb{C}l_{n+1}^0$. In fact α is an algebra isomorphism.

Proof. See [Bä11, Lemma 2.3.12]. □

The action of $\mathbb{C}l_n$ on $\Sigma_n = \Sigma_{n+1}^0$ is defined by

$$a \cdot x := \alpha(a) \cdot x \in \Sigma_{n+1}^0$$

Since $n+1$ is even we have that $\alpha(a)$ preserves parity of Σ_{n+1} and the above action is well-defined.

Definition 2.3.11. Let n be an even number. The map $\sigma_n : \mathrm{Spin}(n) \to \mathrm{End}(\Sigma_n)$ given by

$$\sigma_n(v)(a) = v \cdot a \in \Sigma_n$$

is called the **spinor representation**. The respective maps $\sigma_n^\pm : \mathrm{Spin}(n) \to \mathrm{End}(\Sigma_n^\pm)$ given by $\sigma_n^\pm(v) = \sigma_n(v)|_{\Sigma_n^\pm}$ are called **positive** and **negative** spinor representations.

Let n be an odd number. We similarly have spinor representation $\sigma_n : \mathrm{Spin}(n) \to \mathrm{End}(\Sigma_n) = \mathrm{End}(\Sigma^0_{n+1})$ given by

$$\sigma_n(v)(a) = \alpha(v) \cdot a \in \Sigma^0_{n+1}$$

A Spinor representation is a special case of an algebra representation. We will introduce a more general concept of group representation in the next section.

2.4 Principal G-bundles, G representations and associated vector bundles

The notion of fibre bundles generalises that of vector bundles. One replaces the fiber with some fixed manifold, or possibly a Lie group. When we write G in this section, it should be thought of as a Lie group.

Definition 2.4.1. Let M be a smooth manifold. A (smooth) fibre bundle is a smooth manifold P with a smooth surjective map $\pi : P \to M$, with another smooth manifold F, such that for each point $x \in M$, there is a open neighbourhood U of x and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times F$ such that $\pi_1 \circ \phi = \pi$.

We say P is a fibre bundle over M with fibre F.

The trivialisation of a fiber bundle over a manifold with boundary is the same as that of a vector bundle. In fact we reduce to the definition of a vector bundle when F is a finite dimensional vector space.

Upcoming is the notion of principal G-bundle. It is merely a fiber bundle with an extra action from a Lie group G, that satisfies some compatibility conditions. This action defines (non-canonical) diffeomorphisms from the fiber to G. Then the local trivialisations fix for each point on M such a diffeomorphism.

Definition 2.4.2. Let G be a Lie group and $P \xrightarrow{\pi} M$ a fiber bundle. We say P is a **principal G-bundle** if:

(1) there is a free smooth *right* action of G on P that preserves fiber In other words, there is a smooth map $P \times G \to P$ denoted by $(p, g) \mapsto p \cdot g$ such that $\pi(p \cdot g) = \pi(p)$. Note that for any $x \in M$, the fiber P_x is diffeomorphic to G via this action.

(2) With the identification $P_x \cong G$, any bundle trivialisation map is **G-equivariant** in the following sense. Fix a trivialisation map ϕ_α, consider the following trivialisation

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi_\alpha"] \arrow[dr, "\pi"'] & U \times G \arrow[d, "\pi_1"] \\ & M \end{tikzcd}$$

For any $a \in \pi^{-1}(U)$ we have $\phi_\alpha(a \cdot g) = \phi_\alpha(a) \cdot g$ for any $g \in G$, where $(\cdot)$ on the RHS denotes the canonical *right* action $(p, g_0) \cdot g = (p, g_0 g)$.

There is a correspondence between principal bundles and vector bundles given by orientations.

Example 3. Suppose M is orientable. A **spin structure** on M is a principal $\mathrm{Spin}(n)$-bundle on M that is also a double cover of the frame bundle $P^{SO}(M)$ such that the covering map, is compatible with the right action of $\mathrm{Spin}(n)$ and $SO(n)$ on the bundles; see definition 2.5.1.

In what follows, let $k \subset K$ be an inclusion of fields. The notion of group representation is nothing but that of a G-action on a vector space by endomorphisms.

Definition 2.4.3 (Group and algebra representations). Let G be a group and let M be a vector space. A **representation** of G in M is a group homomorphism

$$\rho : G \to \mathrm{End}(M)$$

Let A be an algebra over k, M be a vector space over K. A **representation** of A in M is a k-algebra homomorphism

$$\rho : A \to \mathrm{End}_K(M)$$

Two representations ρ, ρ' of A in M, M' respectively are **equivalent** if there exists a vector space isomorphism $f : M \to M'$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[dr, "\rho'"'] & End(M) \\ & End(M') \arrow[u, "(-)^f"'] \end{tikzcd}$$

where $g^f = f^{-1} \circ g \circ f$ denotes conjugation by f. In words, $\forall a \in A$, we have $f^{-1} \circ \rho'(a) \circ f = \rho(a)$. For two representations of a group, the definition of equivalence is exactly the same.

A representation $\rho : G \to W$ is **decomposable** if there are W_1, W_2 such that $W = W_1 \oplus W_2$ and $\rho = \rho_1 \oplus \rho_2 : G \to \mathrm{End}(W_1) \oplus \mathrm{End}(W_2)$, where ρ_i is a representation in W_i. A representation is **indecomposable** if it is not decomposable.

Example 4. The spinor representation introduced in definition 2.3.11 is a representation of the group $\mathrm{Spin}(n)$.

Definition 2.4.4 (Associated fiber bundle). Let P be a principal G-bundle, and let F be a topological space. Suppose $\rho : G \to Homeo(F)$ is a continuous group homomorphism. Then the space

$$P \times_\rho F = \frac{P \times F}{(p, a) \sim (pg^{-1}, \rho(g)a)}$$

is the **fiber bundle associated to P** by ρ.

When a fiber bundle E is isomorphic as a fiber bundle to an associated bundle to P, we say that E has an associated bundle structure with respect to P.

An **associated vector bundle** is nothing but an associated fiber bundle where the fiber is a finite dimensional vector space V, and $Homeo(F)$ is replaced with $GL(V)$. One should check that $P \times_\rho F$ has a well-defined manifold structure and a fiber bundle structure over M.

Example 5 (Clifford algebra bundle). Suppose $\dim M = n$ and M is oriented. Let $F = \mathbb{C}l_n$ the complexified Clifford algebra with the standard inner product. Let $P^{SO}(M)$ be the oriented orthonormal frame bundle of M. The canonical homomorphism $\rho : SO(n) \to Aut(\mathbb{R}^n)$

induces for each $A \in SO(n)$ an automorphism from $\mathbb{R}^n$ to itself which preserves the standard inner product. Consider the target $\mathbb{R}^n \subset \mathbb{C}l_n$, the automorphism satisfies the Clifford relation $(Av)^2 = -\langle Av, Av\rangle \cdot 1 = -\langle v, v\rangle \cdot 1$ so it induces an automorphism on $\mathbb{C}l_n$. Hence we get a representation $SO(n) \xrightarrow{\rho} Aut(\mathbb{C}l_n)$. One can construct the bundle $\mathbb{C}l(M) := P^{SO}(M) \times_\rho \mathbb{C}l_n$, called the Clifford algebra bundle of M.

If M is spin, One can show further that $\mathbb{C}l(M)$ has an associated bundle structure with respect to $P^{\mathrm{Spin}}(M)$. Define the adjoint representation $Ad : \mathrm{Spin}(n) \to \mathbb{C}l_n$ given by $Ad(g)(\phi) = g\phi g^{-1}$, where the product is Clifford multiplication. Take the covering map $f : \mathrm{Spin}(n) \to SO(n)$ and consider the exact sequence

$$1 \to \{\pm 1\} \hookrightarrow \mathrm{Spin}(n) \xrightarrow{f} SO(n) \to 1$$

Given $A = f(\pm a) \in SO(n)$, define $Ad'(A) := Ad(a)$. Then $Ad(-a) = Ad(-1) \circ Ad(a) = Ad(a)$ which shows Ad' is well-defined. Now for any $v \in \mathbb{R}^n$, we have $Ad(a)(v) = ava^{-1}$. Write $a = v_1 \dots v_{2m}$, then $\rho(f(v_1 \dots v_{2m}))$ is by definition of f the composition of hyperplane reflections with respect to $v_{2m}, \dots, v_1$ successively. But this is exactly ava^{-1}. Since $Ad'(A)$ and $\rho(A)$ are both algebra automorphisms and are the same on $\mathbb{R}^n$, they are the same on $\mathbb{C}l_n$.

Form the associated fiber bundle $P^{\mathrm{Spin}}(M) \times_{Ad} \mathbb{C}l_n$. By [LM89, Example 3.7], this bundle is isomorphic to $\mathbb{C}l(M)$ as fiber bundles.

Remark. Let us calculate the trivialisation map of an associated fiber bundle. Given a G-principal bundle P, and a representation $\rho : G \to Homeo(F)$, take $E = P \times_\rho F$. For $p \in M$, there exists a trivialising neighbourhood U of P and a trivialising map $\varphi_\alpha : P|_U \to U \times G$. Then the trivialising map of $\phi_\alpha : E|U \to U \times F$ is given by

$$\begin{aligned} &\phi_\alpha : E|_U \to U \times F \\ &\phi_\alpha : [\![a|_p, v]\!] \mapsto (p, \rho(\pi_2(\varphi_\alpha(a))) \cdot v) \end{aligned} \tag{2.4}$$

Notice the implicit dependence on φ_α, where p on RHS is in fact $\pi_1\varphi_\alpha(a)$. Also by the definition of principal bundle, the isomorphic property $P_x \cong G$ is obtained via the free action on P; but a specific *isomorphism* map is given by trivialisation φ_α.

Let us calculate the transition map of an associated fiber bundle.

We prove a "reduction of structure group" lemma characterising isomorphism types of associated fiber bundles.

Lemma 2.4.5. Let G be a Lie group and $i : H \leq G$ be a Lie subgroup. Let P be a principal G-bundle and Q be a principal H-bundle, and let $\iota : Q \subset P$ be a smooth embedding compatible with group actions, i.e. $\iota(a \cdot v) = \iota(a) \cdot i(v)$. If $\rho : G \to F$, $\rho' : H \to F$ are representations of G, H in F such that $\rho' = \rho \circ i$, then there is an isomorphism of fiber bundles $P \times_\rho F \cong Q \times_{\rho'} F$.

Proof. Let ϕ_U be a trivialisation of the Q, ψ_U be a trivialisation of P. The locally defined map $f : Q \times_{\rho'} F \to P \times_\rho F$ given by $[\![a|_p, v]\!] \mapsto [\![\psi_U^{-1}(i(\phi_U(a)))|_p, v]\!]$ trivialises to the identity map for a choice of $a \in Q$. But $\psi_U^{-1}(i(\phi_U(a)))|_p = \iota(a)|_p$ is well-defined globally. $\square$

Next we define a bundle of modules over a bundle of algebras.

Definition 2.4.6. A vector bundle A is a **bundle of algebras** if there is an algebra structure on the fiber of A.

A vector bundle E is a **bundle of modules** over a bundle of algebras A if there is a smooth bilinear bundle homomorphism $\mu : A \times E \to E$, called the **action of A on E**, satisfying the following commutativity:

$$\begin{tikzcd} A \times A \times E \arrow[r, "id_A \times \mu"] \arrow[d, "(\cdot)\times id_E"] & A \times E \arrow[d, "\mu"] \\ A \times E \arrow[r, "\mu"] & E \end{tikzcd}$$

where $(\cdot)$ denotes the natural multiplication function on the algebra structure of A.

Remark. The above definition generalises the usual definition of a module over an algebra. However, the action does not necessarily come from an action on a single fiber. Instead, on any trivialising neighbourhood $U \subset M$, there is a smooth family of actions of the fiber of the algebra bundle on the fiber of the module bundle. To make the action come from a fixed action on fibers, we can use the associated fiber bundle construction. In particular, we would like to consider a bundle of modules over a bundle of algebras where both are associated to the same principal G bundle.

Lemma 2.4.7. Fix a principal G-bundle P and two associated vector bundles $P \times_\rho A$ and $P \times_\sigma W$. Suppose there is an algebra structure on A and there is a representation of A in W, denoted $\mu : A \times W \to W$ (over some field). If μ satisfies the following commutativity:

$$\begin{tikzcd} P \times A \times W \arrow[r, "\mu"] \arrow[d, "g"] & P \times W \arrow[d, "g"] \\ P \times A \times W \arrow[r, "\mu"] & P \times W \end{tikzcd}$$

where $\mu(p, a, w) = (p, \mu(a)(w))$, and $g(p, a, w) = (pg^{-1}, \rho(g)a, \sigma(g)w)$. Then $\tilde{\mu}$ given by

$$\begin{aligned} \tilde{\mu} : (P \times_\rho A) \times (P \times_\sigma E) &\to (P \times_\sigma E) \\ \tilde{\mu}([\![a, v]\!], [\![a, w]\!]) &= [\![a, \mu(v, w)]\!] \end{aligned} \tag{2.5}$$

is a smooth bilinear bundle homomorphism with any choice of $a \in P_x$.

Remark. In $\tilde{\mu}$ one can choose any element $a \in P_x \cong G$ as long as it stays the same across the map. Moreover, one can always choose to define the map on just one choice of a, say e_G, then require the result varies consistently with the action from G.

Proof of lemma 2.4.7. The condition on commutativity translates to

$$g\mu(a, b) = \mu(\rho(g)a, \sigma(g)(b)) \tag{2.6}$$

which is a diagonal commutative condition on the interaction between g and μ. Using this condition one can check that (2.5) is a well-defined map of sets.

A technical verification using trivialisations gives smoothness. $\square$

Remark. If μ satisfies the criterion of lemma 2.4.7, we say it is a **pointwise action** of the associated bundle. This allows us to extend the G-representation to an A-representation, via the representation of G in A. If two equivalent pointwise actions coming from the same principal bundle are used to induce global actions, then the global actions are also equivalent, in an appropriate sense. This is formulated in the following lemma.

Lemma 2.4.8. Let μ, μ' be two equivalent actions of an algebra A in W, W' respectively. Let $Ad : G \to GL(A)$ be a representation of G in A. Let $\rho_W, \rho_{W'}$ be equivalent G-representations in W, W' respectively. Suppose these representations satisfy

$$\begin{aligned}\rho_W(g)(\mu(a)(w)) &= \mu(Ad_g(a))(\rho_W(w))\\ \rho_{W'}(g)(\mu'(a)(w')) &= \mu'(Ad_g(a))(\rho_{W'}(w'))\end{aligned}$$

Then $\tilde{\mu}$ and $\tilde{\mu'}$ are well-defined global actions by lemma 2.4.7.

Suppose P is a principal G-bundle. If μ and μ' are equivalent, then there is a isomorphism of fiber bundles $f : P \times_{\rho_W} W \to P \times_{\rho_{W'}} W'$ such that the following is commutative:

$$\begin{tikzcd}(P \times_{Ad} A) \times (P \times_{\rho_W} W) \arrow[r, "\mu"] \arrow[d, "id\times f"] & P \times_{\rho_W} W \arrow[d, "f"] \\ (P \times_{Ad} A) \times (P \times_{\rho_{W'}} W') \arrow[r, "\tilde{\mu'}"] & P \times_{\rho_{W'}} W'\end{tikzcd}$$

Remark. Mentally, one thinks of $P \times_{\rho_A} A$ as a principal A-bundle. Then the above lemma is analogous to lemma 2.4.5.

The next goal is to characterise the equivalence classes of indecomposable representations of $\mathbb{C}l_n$ for all n. This helps determine the number of different complex spinor bundles. To start we need a globalise decomposable representations to bundles.

Lemma 2.4.9. Suppose $\rho : G \to W$ is decomposable and $\rho = \rho_1 \oplus \rho_2$ is a sum of representations in W_1 and W_2 respectively, where $W = W_1 \oplus W_2$. Let P be a principal G-bundle. Then there is an isomorphism of vector bundles

$$P \times_\rho W \cong P \times_{\rho_1} W_1 \oplus P \times_{\rho_2} W_2$$

Proof. Write $S = P \times_\rho W$, $S_1 = P \times_{\rho_1} W_1$, $S_2 = P \times_{\rho_2} W_2$. We attempt to prove a generalised version of lemma 2.4.7. The (non-canonical) isomorphism is going to be

$$\begin{aligned}f : P \times_{\rho_1} W_1 \oplus P \times_{\rho_2} W_2 &\to P \times_\rho W\\ f : (\llbracket a, v\rrbracket, \llbracket a, w\rrbracket) &\mapsto \llbracket a, v + w\rrbracket\end{aligned} \tag{2.7}$$

for any choice of $a \in P$, but in particular the same for both components on the LHS. Denote trivialising neighbourhoods U, V on M, and trivialisation φ_U, φ_V for the principal bundle P. Then for the choice of $a \in P$, we can write the transition map for S as

$$(p, \rho(\varphi_U(a))v) \mapsto \llbracket a|_p, v\rrbracket \mapsto (p, \rho(\varphi_V(a))v)$$

Setting $v' = \rho((\varphi_U(a))v)$, we see that the transition map with $a \in P$ is

$$v' \mapsto \rho(\varphi_V(a)\varphi_U(a^{-1}))v'$$

Next we see a local section of $S_1 \oplus S_2$ has the form

$$s(p) = (p, \rho_1(\varphi_U(a))v, \rho_2(\varphi_U(a))w) \in U \times (W_1 \oplus W_2)$$

If we trivialise f, it has the form

$$f(p, \rho_1(\varphi_U(a))v, \rho_2(\varphi_U(a))w) = (p, \rho_1(\varphi_U(a))v + \rho_2(\varphi_U(a))w)$$

Choose $a \in P$ for both S_1 and S_2. Apply a transition map to the local section s, we get

$$(p, \rho_1(\varphi_V(a))v, \rho_2(\varphi_V(a))w)$$

so the transition map is

$$(v', w') \mapsto (\rho_1(\varphi_V(a)\varphi_U(a^{-1}))v', \rho_2(\varphi_V(a)\varphi_U(a^{-1}))w')$$

Since

$$\rho_1(\varphi_V(a)\varphi_U(a^{-1}))v' + \rho_2(\varphi_V(a)\varphi_U(a^{-1}))w' = \rho(\varphi_V(a)\varphi_U(a^{-1}))(v' + w')$$

and $f(p, (v', w')) = (p, v'+w')$ (with choice of $e_G \in P$), the local isomorphism f in fact commutes with the transition maps. Therefore we have a vector bundle isomorphism. □

2.5 Spin structures and spinor bundles

We have established a theory of principal bundles and associated bundles in the previous section. A Spin structure is a instance of a principal bundle, while a spinor bundle is an associated vector bundle with respect to the spinor representation.

There are many equivalent definition for a manifold to be spin. A default one is as follows.

Definition 2.5.1. Let M be an oriented manifold. Let $P^{SO}(M)$ be its oriented frame bundle. Let $\rho : \mathrm{Spin}(n) \to SO(n)$ be the canonical covering. A **spin structure** on M is a double cover $\bar{\rho} : P^{\mathrm{Spin}}(M) \to P^{SO}(M)$ where $P^{\mathrm{Spin}}(M)$ is a principal $\mathrm{Spin}(n)$-bundle, which satisfies the commutativity:

$$\begin{tikzcd}
P^{\mathrm{Spin}}(M) \times \mathrm{Spin}(n) \arrow[r, "(\cdot)"] \arrow[d, "\bar{\rho}\times\rho"] & P^{\mathrm{Spin}}(M) \arrow[d, "\rho"] \arrow[dr, "\pi"] & \\
P^{SO}(M) \times SO(n) \arrow[r, "(\cdot)"] & P^{SO}(M) \arrow[r, "\pi"] & M
\end{tikzcd}$$

where horizontal arrows are group actions on the principal bundles.

M is **spinnable** if there is a spin structure on M. M is **spin** if M is spinnable and a spin structure has been chosen on M.

Remark. We can deduce from the equivalence between frame and tangent bundles that the transition map on TM for a spin manifold can be lifted to $\mathrm{Spin}(n)$. This is one of the other definitions of spin.

A question is how spin structures behave under product. To answer this question we first need to show what a frame bundle on a product manifold looks like.

Lemma 2.5.2. Let M, N be oriented manifolds. Let $SO(m) \times SO(n) \to SO(m+n)$ be the canonical inclusion.

$$P^{SO}(M \times N) \cong (P^{SO}(M) \times P^{SO}(N)) \times_{SO(m)\times SO(n)} SO(m+n)$$

Proof. $P^{SO}(M \times N)$ is trivialised on coordinate open sets $U \times V \subset M \times N$. The RHS is trivialised following the product bundle $P^{SO}(M) \times P^{SO}(N)$ which is trivialised on the same open cover.

Fix $(p, q) \in M \times N$. Let $f : \mathbb{R}^{m+n} \to T_{(p,q)}(M \times N)$ be an orientation preserving isometry. Then $f \in P^{SO}(M \times N)$. Let $e_1, \ldots, e_{m+n}$ be the canonical basis of $\mathbb{R}^{m+n}$. By assumption $\{f(e_i)\}$ is a positively oriented orthonormal set in $T_{(p,q)}(M \times N)$.

Claim 2.5.3. $f(e_i) \in T_pM \cup T_qN$ for all i.

First notice that at least one e_i has this property. Observe that

$$\langle f(e_i)\rangle^\perp = f(\langle e_i\rangle^\perp) = f(\langle e_1, \ldots, \hat{e_i}, \ldots, e_{m+n}\rangle)$$

f restricts to an orientation-preserving isometry on this subspace. Induction gives the claim.

Then we have $\mathbb{R}^m \cong f^{-1}(T_pM) \xrightarrow{f|_{f^{-1}(T_pM)}} T_pM$, where we identify $T_{(p,q)}(M \times N)$ with $T_pM \oplus T_qN$. This way we have an induced orientation-preserving isometry from $\mathbb{R}^m$ to T_pM. Similarly we have $f|_{f^{-1}(T_qN)}$. We write $f_M = f|_{f^{-1}(T_pM)}$ and $f_N = f|_{f^{-1}(T_qN)}$. Define the bundle map

$$\begin{aligned} \phi : P^{SO}(M \times N) &\to (P^{SO}(M) \times P^{SO}(N)) \times_{SO(m)\times SO(n)} SO(m+n) \\ \phi(f) &= [\![(f_M, f_N), I_{m+n}]\!] \end{aligned}$$

One can check that this map is smooth, and pointwise a linear isomorphism. Hence it is a bundle isomorphism. $\square$

Lemma 2.5.4. Let M be a spin manifold of dimension m, and N a spin manifold of dimension n. Then there is an action of $\mathrm{Spin}(m)\times\mathrm{Spin}(n)$ on the space $P^{\mathrm{Spin}}(M)\times P^{\mathrm{Spin}}(N)$. There is an inclusion of groups

$$\mathrm{Spin}(m) \times \mathrm{Spin}(n) \to \mathrm{Spin}(m+n)$$

$M \times N$ is spin, and a spin structure is given by

$$(P^{\mathrm{Spin}}(M) \times P^{\mathrm{Spin}}(N)) \times_{\mathrm{Spin}(m)\times\mathrm{Spin}(n)} \mathrm{Spin}(m+n)$$

Proof sketch. The fact that there is an inclusion of groups $\mathrm{Spin}(m) \times \mathrm{Spin}(n) \hookrightarrow \mathrm{Spin}(m+n)$ follows from Clifford algebra inclusions, and the fact that $\mathrm{Spin}(m)$ commutes with $\mathrm{Spin}(n)$ in the bigger Clifford algebra $\mathbb{C}l_{m+n}$.

We need only show that the RHS is a double-cover of the product frame bundle characterised in lemma 2.5.2. First note that both the RHS and the product frame bundle are trivialised on the same ground, i.e. coordinate open sets of the form $U \times V$. To find a covering map we write the respective spin structure coverings as $\bar{\rho}_M : P^{\mathrm{Spin}}(M) \to P^{SO}(M)$ and $\bar{\rho}_N : P^{\mathrm{Spin}}(N) \to P^{SO}(N)$, and the spin group covering as $\rho : \mathrm{Spin}(m+n) \to SO(m+n)$. Then we can easily check that the map $\bar{\rho}_M \times \bar{\rho}_N \times \rho$ descends to a map on the associated bundles. We argue that this map is a double cover. For a point in the product frame bundle, of the form $[\![(f_M, f_N), a]\!]$, where f_M sits above $p \in M$ and f_N is above $q \in N$, we can find neighbourhoods $U \times SO(m)$ and $V \times SO(n)$ that trivialises $P^{SO}(M)$ and $P^{SO}(N)$ respectively.

With some effort, one can show $\bar{\rho}_M$ can be locally trivialised to a double cover $\bar{\rho}_M|_U = id_U \times \rho : U \times \mathrm{Spin}(m) \to U \times SO(m)$. Doing the same for V, the bundle can be trivialised to

$$(U \times V \times \mathrm{Spin}(m) \times \mathrm{Spin}(n)) \times_{\mathrm{Spin}(m)\times\mathrm{Spin}(n)} \mathrm{Spin}(m+n)$$

where the action of $\mathrm{Spin}(m) \times \mathrm{Spin}(n)$ on $U \times V \times \mathrm{Spin}(m) \times \mathrm{Spin}(n)$ is right multiplication. Then the space is in fact diffeomorphic to

$$U \times V \times \mathrm{Spin}(m+n)$$

and the map $\bar{\rho_M} \times \bar{\rho_N} \times \rho$ descends to the covering map $id \times \rho$. □

Next we form the spinor bundle using the associated bundle construction. In the following, let $SO(F)$ denote the orientation-preserving homeomorphisms on F.

Definition 2.5.5. Let M be a spin manifold with a fixed spin structure $P^{\mathrm{Spin}}(M)$. A **real spinor bundle** on M is a fiber bundle of the form

$$P^{\mathrm{Spin}}(M) \times_f F$$

where F is a left Cl_n-module and $f : \mathrm{Spin}(n) \to SO(F)$ is given by the left multiplication of $\mathrm{Spin}(n) \subset Cl_n^0$ on F. A **complex spinor bundle** is of the same form, except F is a left $\mathbb{C}l_n$-module and $f : \mathrm{Spin}(n) \to SO(F)$ is left multiplication of $\mathrm{Spin}(n) \subset Cl_n \subset \mathbb{C}l_n$.

Remark. Recall the definition of a Clifford algebra bundle $\mathbb{C}l(M)$, introduced in Example 5. It is important that the Spin(n)-representation in the above definition comes from a Clifford algebra representation so that one can realise the spinor bundle ΣM, as a bundle of modules over the Clifford algebra bundle. Let $P = P^{\mathrm{Spin}}(M)$, $A = \mathbb{C}l(M) = P^{\mathrm{Spin}}(M) \times_{Ad} \mathbb{C}l_n$ and $W = \Sigma M = P^{\mathrm{Spin}}(M) \times_\sigma \Sigma_n$. We check that for $(p, a, w) \in P \times A \times W$,

$$\begin{aligned}
&g\mu(p, a, w) = g(p, aw) = (pg^{-1}, gaw) \\
&\mu g(p, a, w) = \mu(pg^{-1}, Ad_g(a), \sigma(g)w) = (pg^{-1}, gag^{-1}, gw) = (pg^{-1}, gaw)
\end{aligned}$$

Therefore ΣM is a bundle of modules over the bundle of algebras $\mathbb{C}l(M)$ by lemma 2.4.7.

Example 6. Suppose M is spin of dimension n. Fix a spin structure $P^{\mathrm{Spin}}(M)$. Consider the spinor representation $\sigma_n : \mathrm{Spin}(n) \to \Sigma_n$, as introduced in definition 2.3.11. The associated vector bundle $\Sigma M := P^{\mathrm{Spin}}(M) \times_{\sigma_n} \Sigma_n$ is called the **spinor bundle** on M. This should be taken as the canonical formulation of spinor bundle. Note that when n is odd the definition involves Σ_{n+1}.

Definition 2.5.6. We say that a spinor bundle is **indecomposable** if its pointwise action is an irreducible representation of the Clifford algebra.

Remark. It is a bit odd that irreducibility of spinor bundle is defined for the ambient Clifford algebra action but not the spinor representation. If a spinor representation is irreducible, then the underlying Clifford multiplication must also be irreducible. One verifies that the spinor representation splits in even dimensions as in definition 2.3.8.

By the above remark and lemma 2.4.9, we can inductively decompose an algebra representation until the underlying pointwise action, which is a Clifford algebra representation, is irreducible. Any spinor bundle S has a decomposition $S = S_1 \oplus \cdots \oplus S_n$ where each S_i is an indecomposable spinor bundle. Hence reducible Clifford multiplication implies reducible spinor representation. This is in keeping with our canonical spinor bundle in Example 6, where in even n the spinor representation is reducible but Clifford multiplication is irreducible.

Next we consider equivalence classes of irreducible representations of $\mathbb{C}l_n$.

Definition 2.5.7. A complex spinor bundle is $\mathbb{Z}_2$**-graded** if its fiber is a $\mathbb{Z}_2$-graded $\mathbb{C}l_n$-module.

[LM89] provides a characterisation of the number of distinct equivalence classes of representations for Clifford algebras (real or complex, graded or ungraded). We quote the results below.

Lemma 2.5.8. The number of inequivalent spinor bundles (real/complex, ungraded/graded) depends on the dimension of the manifold M as follows:

dim (mod 8)	$\mathbb{R}$ ungraded	$\mathbb{C}$ ungraded	$\mathbb{R}$ graded	$\mathbb{C}$ graded
1	1	2	1	1
2	1	1	1	2
3	2	2	1	1
4	1	1	2	2
5	1	2	1	1
6	1	1	1	2
7	2	2	1	1
8	1	1	2	2

Proof. [LM89, Proposition 3.9]. □

Lemma 2.5.9. Let n be an odd number. Then spinor representation σ_n is independent of the choice of the irreducible representation of $\mathbb{C}l_n$ (of which there are 2).

Proof. [LM89, Proposition 5.15]. □

The reason we have gone such lengths to introduce the theory of Clifford algebras and spinor bundles is the following special case.

Lemma 2.5.10. Let M be a spin manifold of even dimension n, with a fixed spin structure. Let $M \times S^1$ be the product manifold and $p : M \times S^1 \to M$ be the canonical projection map. Let $S^1 \sqcup S^1 \to S^1 = P^{SO}(S^1)$ be the chosen spin structure on S^1. Let ΣM be the canonical spinor bundle on M. Then the pullback bundle $p^*\Sigma M$ is isomorphic to the canonical spinor bundle $\Sigma(M \times S^1)$.

Proof. We need to establish a global action from $\mathbb{C}l(M \times S^1)$ on $p^*\Sigma M$ using a pointwise action from $\mathbb{C}l_{n+1}$. If we can show this pointwise action is irreducible, then by lemma 2.5.9 we can restrict to $\mathrm{Spin}(n+1)$ to get the equivalent spinor representation. Combined with the representation $Ad : \mathrm{Spin}(n+1) \to GL(\mathbb{C}l_{n+1})$, we have all ingredients to apply lemma 2.4.8.

Let $e_1, \dots, e_{n+1}$ be the standard basis of $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$, with the standard inner product. Let $\omega = i^{\frac{n}{2}} e_1 \dots e_n \in \mathbb{C}l_n$ be the complex volume element of dimension n. Since n is even we write $n = 2m$. Note that

$$\omega^2 = i^{2m}(-1)^{m(2m+1)} = (-1)^{2m^2+2m} = 1$$

Consider $\mathbb{C}l_{n+1} = (Cl_n \otimes Cl_1) \otimes \mathbb{C}$. Define the following pointwise action $(\bullet)$, which is based on the action of Cl_n on Σ_n, denoted $(\cdot)$:

$$\begin{aligned} (\bullet) : \mathbb{C}l_{n+1} \times \Sigma_n &\to \Sigma_n \\ e_i \bullet \phi &= e_i \cdot \phi \quad \forall i < n+1 \\ e_{n+1} \bullet \phi &= i\omega \cdot \phi \end{aligned} \tag{2.8}$$

This is defined on $\mathbb{R}^{n+1} \subset \mathbb{C}l_{n+1}$ so we need to verify the Clifford relation:

$$\begin{aligned} e_i \bullet (e_i \bullet \phi) &= e_i \cdot e_i \cdot \phi = -\phi \\ e_{n+1} \bullet (e_{n+1} \bullet \phi) &= -\omega^2 \phi = -\phi \end{aligned}$$

Hence $(\bullet)$ extends to an algebra homomorphism $(\bullet) : \mathbb{C}l_{n+1} \to \mathrm{End}_{\mathbb{C}}(\Sigma_n)$.

We claim the representation $(\bullet)$ is irreducible. To do so we show it restricts to an irreducible representation of the subalgebra $\mathbb{C}l^0_{n+1} \cong \mathbb{C}l_n$. So consider the subalgebra $\mathbb{C}l_n$ which embeds by $\alpha : e_i \mapsto e_i \cdot e_{n+1}$. The representation on the subalgebra is

$$\sigma(e_i)(\phi) = (e_i \cdot e_{n+1}) \bullet \phi = e_i \cdot i\omega \cdot \phi$$

Note that $e_i \cdot \omega = -\omega \cdot e_i$ since n even. Hence the above becomes $-i\omega \cdot e_i \cdot \phi$. Let $m : \mathbb{C}l_n \to \mathrm{End}(\Sigma_n)$ be the constant representation $m(v)(\phi) = i\omega \cdot \phi$. Then we have that the multiplication of homomorphisms, considered as a composition in $\mathrm{End}(\Sigma_n)$, given by

$$v \in \mathbb{C}l_n \mapsto m(v) \circ \sigma(v) \in \mathrm{End}(\Sigma_n)$$

equals the standard Clifford multiplication on dimension n, which is irreducible. Hence if we have a splitting $\Sigma_n = \Sigma_{n,1} \oplus \Sigma_{n,2}$ where $\sigma = \sigma_1 \oplus \sigma_2$ acts respectively, we deduce

$$\begin{aligned} m(v)(\sigma(v)(\phi_1 + \phi_2)) &= m(v)\sigma_1(v)(\phi_1) + m(v)\sigma_2(v)(\phi_2) \\ &= m(v)\sigma(v)(\phi_1) + m(v)\sigma(v)(\phi_2) \\ &= \sigma_n(v)(\phi_1) + \sigma_n(v)(\phi_2) \end{aligned}$$

In particular σ_n would be reducible. Contradiction.

The above claim, combined with lemma 2.5.9, tells us this representation restricted to $\mathrm{Spin}(n+1)$ is equivalent to the spinor representation. So we need to check this representation satisfies the commutativity relation (2.6). Recall:

- $\mathrm{Spin}(n+1)$ acts on $\mathbb{C}l_{n+1}$ by Ad
- $\mathbb{C}l_{n+1}$ acts on Σ_n by $(\bullet)$

- Spin_{n+1} implicitly acts on Σ_n by inclusion into $\mathbb{C}l_{n+1}$

So our claim is $\forall g \in \mathrm{Spin}(n+1)$, $\forall \phi \in \Sigma_n$, $\forall v \in \mathbb{C}l_{n+1}$

$$Ad_g(v) \bullet (g \bullet \phi) = g \bullet (v \bullet \phi)$$

It is enough to check on $v = e_i$ for $i \in \{1, \dots, n+1\}$ and $g = e_j e_k \in \mathrm{Spin}(n+1)$. We assume $j \neq k$, since if $j = k$ then $g = -1$ and the statement is clear. There are 9 cases; we verify the first 3 and leave the rest.

<u>Case 1: i, j, k are distinct and all not equal to $n+1$</u>

$LHS = e_j e_k e_i (e_j e_k)^{-1} e_j e_k \phi = e_j e_k e_i \phi = RHS.$

<u>Case 2: $i = j \neq k$ and all not equal to $n+1$</u>

$LHS = e_j e_k e_j (e_j e_k)^{-1} e_j e_k \phi = e_j e_k e_j \phi = RHS.$

<u>Case 3: $i = k \neq j$ and all not equal to $n+1$</u>

$LHS = e_j e_k e_k (e_j e_k)^{-1} e_j e_k \phi = -e_j \phi$, $RHS = e_j e_k e_k \phi = -e_j \phi = LHS.$

<u>Case 4:i, j, k are distinct, and $i = n+1$</u>

<u>Case 5:i, j, k are distinct, and $j = n+1$ or $k = n+1$</u>

<u>Case 6:$i = j \neq k$, $i = j = n+1$</u>

<u>Case 7:$i = j \neq k$, $k = n+1$</u>

<u>Case 8:$i = k \neq j$, $i = k = n+1$</u>

<u>Case 9:$i = k \neq j$, $j = n+1$</u>

We need to show that $p^*\Sigma M$ is isomorphic to the associated bundle $P^{\mathrm{Spin}}(M \times S^1) \times_{(\bullet)} \Sigma_n$. Note $S^1 \sqcup S^1$ is the spin structure chosen on S^1, so $P^{\mathrm{Spin}}(S^1)$ has trivialising cover $\{S^1\}$. Let $\mathcal{U}$ be the trivialising cover for $P^{\mathrm{Spin}}(M)$. The product spin structure $P^{\mathrm{Spin}}(M \times S^1)$ has trivialising cover $\{U \times S^1 : U \in \mathcal{U}\}$, which is the same as that for $p^*\Sigma M$. Suppose $U \times S^1, V \times S^1 \subset M \times S^1$ are trivialising neighbourhoods for both bundles. We adopt lemma 2.5.4 to see that $P^{\mathrm{Spin}}(M \times S^1)$ is diffeomorphic to

$$(P^{\mathrm{Spin}}(M) \times (S^1 \sqcup S^1)) \times_{\mathrm{Spin}(n) \times \mathrm{Spin}(1)} \mathrm{Spin}(n+1)$$

<u>Calculation for the transition map of $P^{\mathrm{Spin}}(M \times S^1) \times_{(\bullet)} \Sigma_n$</u>

Suppose we have trivialisations $\{f_U\}$ for $P^{\mathrm{Spin}}(M \times S^1)$ over the cover $\{U \times S^1\}$. Suppose additionally $\{\varphi_U\}$ trivialises $p_1^* P^{\mathrm{Spin}}(M)$ over $\{U \times S^1\}$ and ψ trivialises $p_2^* P^{\mathrm{Spin}}(S^1) = M \times S^1 \times \mathbb{Z}_2$ globally. Take $h : \mathrm{Spin}(n) \times \mathrm{Spin}(1) \to \mathrm{Spin}(n+1)$ to be the canonical inclusion given by $h(a, b) = a \cdot b$.

$$\begin{aligned} P^{\mathrm{Spin}}(M \times S^1) &\xrightarrow{f_U} U \times S^1 \times \mathrm{Spin}(n+1) \\ [\![(a|_x, b|_t), v]\!] &\mapsto ((x, t), \varphi_U(a) \cdot \psi(b) \cdot v) \end{aligned}$$

Then let $\{g_U\}$ trivialise $P^{\mathrm{Spin}}(M \times S^1) \times_{(\bullet)} \Sigma_n$. We have

$$\begin{aligned} V \times S^1 \times \Sigma_n &\xrightarrow{g_V^{-1}} P^{\mathrm{Spin}}(M \times S^1) \times_{(\bullet)} \Sigma_n \xrightarrow{g_U} U \times S^1 \times \Sigma_n \\ ((x, t), y) &\mapsto [\![[\![(a|_x, b|_t, v)]\!], (\phi_V(a) \cdot \psi(b) v)^{-1} y]\!] \mapsto ((x, t), (\varphi_U(a) \psi(b) v)(\varphi_V(a) \psi(b) v)^{-1} y) \\ &= ((x, t), \varphi_U(a) \varphi_V(a)^{-1} y) \end{aligned}$$

Calculation for the transition map of $p^*\Sigma M$

Similarly let $\{\varphi_U\}$ trivialise $p_1^* P^{\mathrm{Spin}}(M)$ over $\{U \times S^1\}$. Let $\{\Psi_U\}$ trivialise $p^*\Sigma_M$ and $\{\psi_U\}$ trivialise ΣM over $\{U\}$. We have the transition maps

$$V \times S^1 \times \Sigma_n \xrightarrow{\Psi_V^{-1}} p^*\Sigma M \xrightarrow{\Psi_U} U \times S^1 \times \Sigma_n$$
$$((x,t),y) \mapsto ((x,t), \psi_V^{-1} y) \to ((x,t), \psi_U \psi_V^{-1} y)$$

Suppose we choose $a|_x \in P^{\mathrm{Spin}}(M)|_x = p_1^* P^{\mathrm{Spin}}(M)|_{(x,t)}$, we have $\psi_U([\![a|_x, w]\!]) = (x, \varphi_U(a) \cdot w)$.

Therefore, given the same choice of principal bundle elements, the transition maps of the two bundles are the same. Hence $p^*\Sigma M$ is isomorphic to the spinor bundle $\Sigma(M \times S^1)$ with the Clifford multiplication $(\bullet)$. □

Remark. There is a similar construction when we attempt to restrict a spinor bundle to the boundary of a manifold and compare it to the spinor bundle OF the boundary. This will be discussed in the next section after Dirac operator is introduced.

2.6 Dirac operators

We define the classical Dirac operators, which are a specific type of Dirac-type operators, on manifolds with boundary. We relate the classical Dirac operator *of* the boundary (which is a codimension 1 submanifold) to that of the whole manifold.

Let E be a vector bundle over M. A connection on E has the type $\nabla : \Gamma(E) \times \Gamma(TM) \to \Gamma(E)$. It is a derivation over $C^\infty(M)$ in the first component and tensorial in the second. Hence it induces a differential operator of 1st order of the type $\Gamma(E) \to \Gamma(\mathrm{Hom}(TM, E))$. Observe that $\mathrm{Hom}(TM, E) \cong T^*M \otimes E$. Then the connection can be shown to satisfy the Leibniz rule:

$$\nabla(fu) = df \otimes u + f \nabla u$$

which is equivalent to the Leibniz rule introduced in the beginning of this chapter.

We are going to need the following fact.

Lemma 2.6.1. Let S be the spinor bundle on M. There exists a connection ∇ on S that is compatible with the Clifford multiplication $(\cdot)$ of $\mathbb{C}l(M)$, restricted to elements of TM on S:

$$\nabla_Y(X \cdot v) = (\nabla_Y^{LC} X) \cdot v + X \cdot \nabla_Y v$$

for any $X, Y \in TM$ and $v \in S$.

Proof. See [Bä11, Chapter 2]. □

Remark. We call ∇ the **spinor connection** on M.

Suppose M is closed of even dimension n. Let E be a vector bundle over M, and let S be the spinor bundle over M. Recall that in even dimension we have the decomposition of the spinor space $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$, by the discussion around definition 2.3.8. Let $S = P^{\mathrm{Spin}}(M) \times_{\sigma_n} \Sigma_n$ be the canonical spinor bundle. Since the spinor representation has decomposition $\sigma_n = \sigma_n^+ \oplus \sigma_n^-$, by lemma 2.4.9 we have a decomposition $S = S^+ \oplus S^-$, where $S^\pm = P^{\mathrm{Spin}}(M) \times_{\sigma_n^\pm} \Sigma_n^\pm$. Let $S \otimes E = S^+ \otimes E \oplus S^- \otimes E$ be the twisted spinor bundle with E. Then there is another fact we will need:

Lemma 2.6.2. The spinor connection ∇ on M preserves chirality, i.e. it maps sections of $S^{\pm}$ to $S^{\pm}$.

Proof. Follows from a local formula of the spinor connection. See for example [Bä11, Formula (2.21)]. □

If A is a field of homomorphisms of type $\Gamma(T^*M \otimes E) \to \Gamma(E)$, we can form the operator $D = A \circ \nabla$. Given a section e and a local tangent frame $b_1, \ldots, b_n$, the isomorphism between $\mathrm{Hom}(V, W) \cong V^* \otimes W$ given by $f \mapsto \sum_i b_i^* \otimes f(b_i)$ gives the local expression $\nabla e = \sum_i b_i^* \otimes \nabla_{b_i} e$. Then $De = \sum_i A(b_i^* \otimes \nabla_{b_i} e)$. If we fix $E = \Sigma M$ the spinor bundle, and A to be the Clifford multiplication on the spinor bundle, we get the following definition of **classical Dirac operator**.

Definition 2.6.3. The **classical Dirac operator** D on the spinor bundle ΣM over a spin manifold M is given by the composition $D = A \circ \nabla$, where A is Clifford multiplication on ΣM and ∇ is the spinor connection. Locally with respect to a frame $b_1, \ldots, b_n$ it is given by the expression

$$De = \sum_{i=1}^{n} b_i \cdot \nabla_{b_i} e$$

for $e \in \Gamma(\Sigma M)$.

In the twisted case $\Sigma M \otimes E$, the **twisted Dirac operator** with E, often denoted D_E, is given by the same definition with ∇ replaced with the twisted spinor connection.

Remark. The Dirac operators are independent of the choice of orthonormal basis.

Remark. Again suppose M is closed of even dimension. With the knowledge that ∇ preserves chirality of spinors, we see that D_E reverses chirality. Therefore with a twisted spinor bundle $S \otimes E$, D_E is of the form

$$D_E = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

where $D^{\pm} : \Gamma(S^{\pm} \otimes E) \to \Gamma(S^{\mp} \otimes E)$. Since D_E is formally self-adjoint, we have $(D^+ + D^-)^* = D^+ + D^-$ so $(D^+)^* = D^-$ and $(D^-)^* = D^+$. Therefore the index of D^+ can be expressed as

$$\dim \ker D^+ - \dim \ker D^-$$

We say that D^+ is the **positive part** of D_E and D^- is the **negative part**.

Remark. The index of a Dirac-type operator (or more generally a Fredholm operator) is a rather rigid object. Consider the Banach space of bounded linear operators between two Hilbert spaces $\mathcal{B}(V, W)$ equipped with operator norm. Let $\mathcal{F}$ be the subspace of Fredholm operators. One can show that for $T \in \mathcal{F}$, $T + fId$ for any smooth function f is still in $\mathcal{F}$. Since $T + fId$ is in the same path component as T, by [LM89, Proposition III.7.1] we have $\operatorname{ind} T = \operatorname{ind}(T + fId)$.

Fix a codimension 1 submanifold $N \subset M$. In this case we say N is a **hypersurface** of M. We would like to understand whether N is spinnable; then, if true, how the spinor bundle and classical Dirac operator on N look like.

Lemma 2.6.4. Any hypersurface N of a spin manifold M is spin.

Proof. M is spin so is orientable, so N is orientable. Choose a unit normal vector field $\nu \in \Gamma(TN)$. Denote $\dim M = n+1$ and $\dim N = n$. Using the injection $i : SO(n) \to SO(n+1)$ given by

$$i : A \mapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix} \tag{2.9}$$

We have a smooth embedding of oriented frame bundle $\iota : P^{SO}(N) \to P^{SO}(M)$, where at each point $x \in N$ we have $T_xM = T_xN \oplus \langle \nu_x \rangle$

$$\iota : (h : T_xN \to T_xN) \mapsto (h' : T_xM \to T_xM) \tag{2.10}$$

where $h'(\nu_x) = \nu_x$ and $h'|_{T_xN} = h$. Consider the double covering map $P^{\text{Spin}}(M) \xrightarrow{\bar{\rho}} P^{SO}(M)$ and define $P^{\text{Spin}}(N) := \bar{\rho}^{-1}(\iota(P^{SO}(N))) \subset P^{\text{Spin}}(M)$.

Claim 2.6.5. $\bar{\rho} : P^{\text{Spin}}(N) \to P^{SO}(N) \subset P^{SO}(M)$ is a spin structure on N.

proof of claim. From covering space theory, $\bar{\rho} : P^{\text{Spin}}(N) \to P^{SO}(N)$ is clearly a covering map of degree 2. The action on $P^{\text{Spin}}(N)$ by $\text{Spin}(n)$ is induced from the action on $P^{\text{Spin}}(M)$ by $\text{Spin}(n+1)$: First we embed $\text{Spin}(n) \to \text{Spin}(n+1)$ by restricting the Clifford algebra embedding $j : \mathbb{C}l_n \to \mathbb{C}l^0_{n+1} \subset \mathbb{C}l_{n+1}$ given by $v \mapsto v \cdot e_{n+1}$, where the embedding on the generating vector space is $\mathbb{R}^n \subset \mathbb{C}l_n \hookrightarrow \mathbb{R}^{n+1} \subset \mathbb{C}l_{n+1}$ is given by $e_i \mapsto e_i \in \mathbb{R}^{n+1}$. Notice that $j(v_1 \dots v_{2m}) = v_1 \cdot e_{n+1} \dots v_{2m} \cdot e_{n+1} = v_1 \dots v_{2m} \in \text{Spin}(n+1)$. Denote for all k the spin representation $\rho : \text{Spin}(k) \to SO(k)$. Then we have the following commutativity which is easy to check:

$$\begin{array}{ccc} \text{Spin}(n) & \xrightarrow{j|_{\text{Spin}(n)}} & \text{Spin}(n+1) \\ \big\downarrow \rho & & \big\downarrow \rho \\ SO(n) & \xrightarrow{i} & SO(n+1) \end{array}$$

We also have that the action of $SO(n)$ on $P^{SO}(N)$ is compatible with the action of $SO(n+1)$ on $P^{SO}(M)$ when embedded using i and ι. It remains to check the commutativity of spin structure:

$$\begin{array}{ccc} P^{\text{Spin}}(N) \times \text{Spin}(n) & \longrightarrow & P^{\text{Spin}}(N) \\ \big\downarrow \bar{\rho}\times\rho & & \big\downarrow \bar{\rho} \\ P^{SO}(N) \times SO(n) & \longrightarrow & P^{SO}(N) \end{array}$$

The action on the upper horizontal is $(H, v) \mapsto H \cdot j(v)$, where $(\cdot)$ is the action on $P^{\text{Spin}}(M)$. We have the additional task of showing this image is indeed in $P^{\text{Spin}}(N)$. Note $H \in P^{\text{Spin}}(N)$ so $\bar{\rho}(H) = \iota(h)$ for some $h \in P^{SO}(N)$. Now view everything in M and get

$$\begin{aligned} \bar{\rho}(H \cdot j(v)) &= \bar{\rho}(H) \cdot \rho(j(v)) \\ &= \bar{\rho}(H) \cdot i(\rho(v)) \\ &= \iota(h) \cdot i(\rho(v)) \\ &= h \cdot \rho(v) \in P^{SO}(N) \end{aligned}$$

where in the last step the action is on $P^{SO}(N)$. Hence we have verified the commutativity and also showed that $H \cdot j(v) \in P^{\text{Spin}}(N)$. $\square$

It follows immediately from the claim that N is spin. $\square$

For the following arguments we fix the spin structure $P^{\mathrm{Spin}}(N)$ as defined above. On one hand we have the canonical spinor bundle on N, $\Sigma N = P^{\mathrm{Spin}}(N) \times_{\sigma_n} \Sigma_n$; on the other hand we have the restriction $\Sigma M|_N$.

Lemma 2.6.6. Let $\dim N = n$. When n is odd, $\Sigma N \cong \Sigma^+ M|_N$ as vector bundles. When n is even, $\Sigma N \cong \Sigma M|_N$.

Proof. Consider n even. The odd case is similar. Using the uniqueness of representations of Clifford algebras, see [LM89, Chapter I.5], one can show that the Clifford multiplication given by $\mathbb{C}l(N)$ on $\Sigma_{n-1} \cong \Sigma^+_{n+1}$ is compatible with the Clifford multiplication by $\mathbb{C}l(M)$ via the isomorphism. Then note that $\Sigma_n := \Sigma^+_{n+1}$. A more explicit argument can be found in [Bä11, Section 2.6]. $\square$

2.7 Several connection and curvature identities

We record a few well-known facts in Riemannian geometry.

Lemma 2.7.1 (Scalar curvature of warped products)**.** Let $S^{n-1} \times [\theta_-, \theta_+]$ be equipped with the warped product metric $g_0 = g_{S^{n-1}} + \rho(\theta)^2 d\theta^2$. Then

$$\mathrm{scal}^{g_0} = (n-1)\left(-2\frac{\rho''(\theta)}{\rho(\theta)} + (n-2)\frac{1-\rho'(\theta)}{\rho(\theta)^2}\right)$$

Proof. A proof can be found in [Pet16, Section 4.2.3]. $\square$

Lemma 2.7.2 (Spinor curvature tensor local formula)**.**

$$R^S(X,Y)\phi = -\frac{1}{4}\sum_{i=1}^{n}(R(X,Y)b_i)\cdot b_i \cdot \phi \tag{2.11}$$

Proof. See [Bä11, Lemma 2.4.13]. $\square$

Lemma 2.7.3 (Riemannian curvature tensor formula for sphere)**.** Let $\langle .,. \rangle$ be the canonical metric on $S^n \subset \mathbb{R}^{n+1}$. Let R be the Riemannian curvature tensor with respect to $\langle .,. \rangle$. Then R satisfies

$$R(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y \tag{2.12}$$

Proof. This is common knowledge in Riemannian geometry. $\square$

Lemma 2.7.4. Let E, F be vector bundles on M, with connections ∇^E and ∇^F respectively. Then there is a connection $\nabla^{E\otimes F}$ on $E \otimes F$ given on local sections $e \otimes f$ by

$$\nabla^{E\otimes F}(e\otimes f) = (\nabla^E e)\otimes f + e \otimes (\nabla^F f)$$

Proof. Simply verify the Leibniz rule. Tensoriality in the first component is clear by linearity. $\square$

Lemma 2.7.5. Let E, F be vector bundles on M, with connections ∇^E and ∇^F respectively. Let R^E and R^F, $R^{E\otimes F}$ be the associated curvature tensors. Then for local section $v \otimes w$ and vector fields X, Y we have

$$R^{E\otimes F}_{XY}(v \otimes w) = R^E_{XY} v \otimes w + v \otimes R^F_{XY} w$$

Proof. This is an easy consequence of lemma 2.7.4. □

Definition 2.7.6. Let $S \otimes E$ be a twisted spinor bundle. The canonical section of $\mathrm{End}(S \otimes E)$ given by

$$\mathcal{R}^{S\otimes E}(\sigma \otimes v) = \frac{1}{2} \sum_{j.k=1}^{n} e_j \cdot e_k \cdot (\sigma \otimes R^E_{e_j e_k}(v))$$

is called the **Bochner curvature tensor** of $S \otimes E$.

Remark. One might denote $\mathcal{R}^{S\otimes E}$ by $\mathcal{R}^E$ if the context is clear.

Remark. The Bochner curvature tensor arises as a result of a non-flat connection on E. If E has a flat connection, $R^E = 0$ and then $\mathcal{R}^E = 0$.

Lemma 2.7.7. Let $S \otimes E$ be a twisted spinor bundle, $R^{S\otimes E}$ be its curvature tensor. Let $\mathcal{R}^{S\otimes E}$ be its Bochner curvature tensor. Then

$$\frac{1}{4} \operatorname{scal} + \mathcal{R}^{S\otimes E} = \frac{1}{2} \sum_{i,j=1}^{n} e_i \cdot e_j \cdot R^{S\otimes E}_{e_i e_j}$$

Proof. Check locally for a section $\sigma \otimes v$. Expand the RHS, get

$$\begin{aligned}
\frac{1}{2} \sum_{j,k=1}^{n} e_j e_k R^{S\otimes E}_{e_j e_k}(\sigma \otimes v) &= \frac{1}{2} \sum_{j,k=1}^{n} e_j e_k (R^S_{e_j e_k} \sigma \otimes v + \sigma \otimes R^E_{e_j e_k} v) \\
&= \mathcal{R}^E(\sigma \otimes v) + \frac{1}{2} \sum_{j,k=1}^{n} e_j e_k R^S_{e_j e_k} \sigma \otimes v
\end{aligned}$$

Then we use lemma 2.11 to calculate

$$\begin{aligned}
\frac{1}{2} \sum_{j,k=1}^{n} e_j e_k R^S_{e_j e_k} &= -\frac{1}{8} \sum_{j,k=1}^{n} e_j e_k \sum_{l=1}^{n} R_{e_j e_k} e_l \cdot e_l \\
&= -\frac{1}{8} \sum_{j,k,l,m=1}^{n} e_j e_k \langle R_{e_j e_k} e_l, e_m \rangle e_m \cdot e_l \\
&= -\frac{1}{8} \sum_{l=1}^{n} \left(\sum_{j,k,m=1}^{n} \langle R_{e_j e_k} e_l, e_m \rangle e_j e_k e_m \right) e_l
\end{aligned}$$

For the middle sum over j, k, m we can assume $j \neq k$. When j, k, m are all distinct we consider groups of triples $(j, k, m), (k, m, j), (m, j, k)$. We know that each triple belongs to one and only one combination. Then $e_j e_k e_m = e_k e_m e_j = e_m e_j e_k$. So these terms sum to

$$-\langle R_{e_j e_k} e_m + R_{e_k e_m} e_j + R_{e_m e_j} e_k, e_l \rangle e_j e_k e_m$$

which is 0 by Bianchi identity.

So the rest of the terms simplifies to

$$\begin{aligned}
&\frac{1}{8}\sum_{l=1}^{n}\left(\sum_{j,k}\langle R_{e_je_k}e_j, e_l\rangle e_je_ke_j + \sum_{j,k}\langle R_{e_je_k}e_k, e_l\rangle e_je_ke_k\right)e_l \\
=&\frac{1}{8}\sum_{l=1}^{n}\left(\sum_{j,k}\langle R_{e_je_k}e_j, e_l\rangle e_je_ke_j + \sum_{j,k}\langle R_{e_je_k}e_k, e_l\rangle e_je_ke_k\right)e_l \\
=&\frac{1}{4}\sum_{l,j,k=1}^{n}\langle R_{e_je_k}e_j, e_l\rangle e_ke_l \\
=&-\frac{1}{4}\sum_{l,j,k=1}^{n}\langle R_{e_je_k}e_l, e_j\rangle e_ke_l \\
=&-\frac{1}{4}\sum_{l,k=1}^{n}\mathrm{ric}(e_k, e_l)e_ke_l \\
=&-\frac{1}{4}\sum_{k=1}^{n}\mathrm{Ric}(e_k)\cdot e_k \\
=&\frac{1}{4}\mathrm{scal}
\end{aligned}$$

This proves the lemma. □

The Weitzenböck formula is the main technical tool for proving results about scalar curvature in spin geometry.

Lemma 2.7.8 (Weitzenböck formula)**.** Let M be a spin manifold, S the spinor bundle on M and E another vector bundle on M with a connection. Let $\nabla^{S\otimes E}$ denote the connection on $S\otimes E$. Then the twisted Dirac operator D_E satisfies

$$D_E^2 = (\nabla^{S\otimes E})^*\nabla^{S\otimes E} + \frac{1}{4}\,\mathrm{scal} + \mathcal{R}^E \tag{2.13}$$

where $\mathcal{R}^E = \mathcal{R}^{S\otimes E}$ is the Bochner curvature.

Proof. Choose a normal frame $\{e_i\}$. Then

$$\begin{aligned}
D^2 v &= \sum_{j=1}^n e_j \cdot \nabla_{e_j} \left(\sum_{i=1}^n e_i \cdot \nabla_{e_i} v \right) \\
&= \sum_{i,j=1}^n e_j \cdot \nabla_{e_j} (e_i \cdot \nabla_{e_i} v) \\
&= \sum_{i,j=1}^n e_j \cdot e_i \cdot \nabla_{e_j} \nabla_{e_i} v \\
&= \sum_{i,j=1}^n e_j \cdot e_i \cdot \nabla^2_{e_j e_i} v \\
&= -\sum_{i=1}^n \nabla^2_{e_i e_i} v + \sum_{i<j} e_j e_i (\nabla^2_{e_j e_i} - \nabla^2_{e_i e_j}) v \\
&= \nabla^* \nabla v + \sum_{i<j} e_j e_i R^{S \otimes E}_{e_j e_i} v \\
&= \nabla^* \nabla v + \mathcal{R}^E v + \frac{1}{4} \mathrm{scal} \cdot v
\end{aligned}$$

□

Remark. The Atiyah-Singer index theorem is a totally non-trivial theorem relating the topological index of a vector bundle and the analytical index of the Dirac operator on it. There are a few generalisations. We state a basic version below. We are not concerned with the proof of this theorem in this thesis; the biggest use of this theorem would be to deduce existence of non-trivial elements in the kernel of the Dirac operator, i.e. non-vanishing harmonic spinor fields. A typical reference is for example [LM89].

Theorem 2.7.9 (Atiyah-Singer)**.** Let M be a closed Riemannian spin manifold of even dimension and E be a complex super vector bundle with a Hermitian metric. Let D_E be the twisted Dirac operator associated with E. Let D_E^+ be the positive part of the operator. Then

$$\mathrm{ind}(D_E^+) = \langle \hat{A}(M) \cup ch(E), [M] \rangle \tag{2.14}$$

2.8 A little topology lemma

We will need a lemma to deduce that in specific settings local Riemannian isometries can be strengthened to global ones.

Lemma 2.8.1. Let $f : M \to N$ be a local Riemannian isometry. Suppose M is complete and N is simply connected. Then f is a global isometry.

Proof. It suffices to prove that f is injective. By a lemma of Ambrose [GHL04, Proposition 2.106], we have that f is a covering map. Since N is simply connected its universal cover is N. By universal property of universal covers there exists a map $\phi : N \to M$ smooth such that $f \circ \phi = id_N$. So ϕ is injective. So ϕ is bijective. So f is bijective. □

2.9 Boundary conditions for elliptic operators of first order

We will need some delicate machinery about elliptic differential operators on manifolds with boundary.

The setting for this section is an elliptic differential operator of order 1 acting on a manifold M with non-empty boundary. It is a fact that its index is always Fredholm, so its index is well-defined. We denote such an operator by P. Consider the following boundary value problem:

$$\begin{cases} Pu = 0 \\ u = i\nu \cdot u \quad \text{on } \partial M \end{cases} \tag{2.15}$$

where $\cdot$ is Clifford multiplication and ν is a chosen outward normal vector field on M. Note that solving for this system is equivalent to finding the kernel of P with a certain restricted domain $C_B = \{u \in dom(P) : u|_{\partial M} = i\nu \cdot u\}$, which is a closed subset of $dom(P)$. A question to ask at this point is what the adjoint of $P|_{C_B}$ looks like, or whether it can also be represented by some **adjoint boundary condition**. It turns out that such boundary conditions do exist, and to formulate them precisely takes quite some effort. In this section we will state the intuitive results and leave the technical details to [BB11].

2.9.1 Maximal and minimal extensions

We fix some notations. Suppose E, F are vector bundles over M. Let $C_c^\infty(M, E)$ denote the compactly supported sections of E over M. Let $C_{cc}^\infty(M, E)$ denote the sections of E over M which have compact support in the interior of M. Let $P : C^\infty(M, E) \to C^\infty(M, F)$ be a differential operator. Let $\sigma_P(\xi) \in C^\infty(\mathrm{Hom}(E, F))$ denote the field of homomorphisms given by the principal symbol of P with respect to a covector field ξ. Let $P^* : C^\infty(M, F) \to C^\infty(M, E)$ denote the formal adjoint of P.

Definition 2.9.1. The **graph norm** of $\phi \in L^2(M, E)$ is defined by

$$||\phi||_P^2 \coloneqq ||\phi||_{L^2(M)}^2 + ||P\phi||_{L^2(M)}^2$$

Remark. The reason for the name is $||\phi||_P^2 = ||(\phi, P\phi)||_{L^2(M)}^2$, where the RHS is the L^2-norm of the image under the map $\phi \mapsto (\phi, P\phi) \in Im(P) \subset L^2(M, E) \times L^2(M, E)$.

We denote by $P|_{cc}$ the operator P restricted to compactly supported sections in the interior.

Definition 2.9.2 (maximal extension)**.** The **maximal extension** of P, denoted by $P_{\max}$, is the functional analytic adjoint of $P^*|_{cc}$ under L^2-norm. Namely, for the following set

$$\begin{aligned} \mathrm{dom}(P_{\max}) \coloneqq \{\phi \in L^2(M, E) : \exists u \in L^2(M, F), \forall \psi \in C_{cc}^\infty(M, F) \\ (\phi, \psi)_{L^2} = (u, P^*\psi)_{L^2}\} \end{aligned}$$

Then u is uniquely determined (a.e.) for each $\phi \in \mathrm{dom}(P_{\max})$. We say $P_{\max}\phi = u$.

It is easy to check that if $C^\infty(M, E) \subset \mathrm{dom}(P_{\max})$. If ϕ is smooth, then $P_{\max}\phi = P\phi$. If $\phi_n \to \phi$ in L^2 and $P\phi_n \to u$ in L^2, then for all $\psi \in C^\infty(M, F)$ we have $(u, \psi) = \lim_{n\to\infty}(P\phi_n, \psi) = (\phi, P^*\psi)$. Hence $u = P_{\max}\phi$.

Definition 2.9.3 (minimal extension)**.** The domain of the **minimal extension** of P, denoted $\mathrm{dom}(P_{\min})$, is given by $\mathrm{dom}(P_{\min}) := \overline{C^\infty_{cc}(M,E)}^{||\cdot||_P}$, the completion with respect to the graph norm.

We claim that $\mathrm{dom}(P_{\min}) \subset \mathrm{dom}(P_{\max})$. First we notice that $C^\infty_{cc}(M,E) \subset L^2(M,E)$, and since $L^2(M,E)$ is a Hilbert space (in particular complete), any Cauchy sequence in $||\cdot||_P$ is Cauchy in $||\cdot||_{L^2}$ and converges in L^2. Hence we have an embedding $\mathrm{dom}(P_{\min}) \subset L^2(M,E)$. Then note that if $\phi_n \to \phi$ in $||\cdot||_P$, both ϕ_n and $P\phi_n$ converges in L^2, so the previous discussion tells us $\phi \in \mathrm{dom}(P_{\max})$. This shows the claim.

One might wonder if there is a dense subspace of $\mathrm{dom}(P_{\max})$, like how $C^\infty_{cc}(M,E)$ is dense in $\mathrm{dom}(P_{\min})$, by definition. The answer is yes, but we do not prove it here.

Lemma 2.9.4. $C^\infty_c(M,E)$ is dense in $\mathrm{dom}(P_{\max})$ with respect to the graph norm $||\cdot||_P$.

Proof. A delicate argument using Sobolev norms. See [Bär-Ballman]. □

2.9.2 The restriction map

Next we consider everything restricted to the boundary. The most basic restriction map is the following:

$$\begin{aligned} \mathscr{R} : C^\infty_c(M,E) &\to C^\infty(\partial M, E) \\ \phi &\mapsto \phi|_{\partial M} \end{aligned}$$

where on the boundary the bundle E is in fact $E|_{\partial M}$.

Next we are going to make hand-wavy claims about Dirac-type operators and an extension of the above restriction map to $\mathrm{dom}(P_{\max})$. We write D for a Dirac-type operator instead of P. Assume that M has compact boundary ∂M. Then there is a collar neighbourhood U of ∂M in M such that U is diffeomorphic to $\partial M \times [0,\varepsilon)$. We suppose M is a Riemannian spin manifold, so it comes with a canonical volume form vol. Write $Z_I = I \times \partial M$ for $I \subset \mathbb{R}$ any interval. We fix a diffeomorphism from $Z_{[0,\varepsilon)}$ to U:

$$F = (t,f) : U \to Z_{[0,\varepsilon)}$$

Let vol_1 be the volume form inherited by ∂M. We have that $(F^{-1})^* vol = dt \wedge vol_1$ on $Z_{[0,\varepsilon)}$. Via F we can construct a vector bundle isomorphism between $E|_U$ and $E|_{\partial M} \times [0,\varepsilon)$. Sections over $E|_U$ can be viewed as functions of $(t,x) \in [0,\varepsilon) \times \partial M$. If P is an elliptic differential operator of order 1 from E to F, we have a "orthogonal decomposition" of P over U into a derivative in the t direction, a derivative over ∂M, plus some remainder terms.

Lemma 2.9.5. Let $D : C^\infty(M,E) \to C^\infty(M,F)$ be an elliptic differential operator of order one. If D is boundary symmetric, then there are formally self-adjoint elliptic differential operators

$$A : C^\infty(\partial M, E) \to C^\infty(\partial M, E) \quad \textit{and} \quad \tilde{A} : C^\infty(\partial M, F) \to C^\infty(\partial M, F)$$

such that over $Z_{[0,\varepsilon)}$,

$$D = \sigma_t\left(\frac{\partial}{\partial t} + A + R_t\right)$$

$$D^* = -\sigma_t^*(\frac{\partial}{\partial t} + \tilde{A} + \tilde{R}_t)$$

where R_t and $\tilde{R}_t$ are families of differential operators of order at most one with coefficients depending smoothly on $t \in [0, \varepsilon)$.

Next fix a first-order elliptic operator D on M. By the above normal form lemma we have the corresponding A and $\tilde{A}$ on $E|_{\partial M}$ and $F|_{\partial M}$. Since A is elliptic, by the spectral theorem A has discrete spectrum with eigenvalues λ_j. Fix a L^2-orthonormal eigen-basis $\{\phi_j : j \in \mathbb{Z}\}$ of A. Then one can define the Sobolev norm for any $s \in \mathbb{R}$, on $C^\infty(\partial M, E)$, via functional calculus:

$$||\phi||^2_{H^s(\partial M)} \coloneqq ||(id + A^2)^{s/2}||^2_{L^2(\partial M)}$$

One can show that when s is a nonnegative integer, this definition is equivalent as norms to the **basic Sobolev norm**:

$$||\phi||^2_{H^s} = ||\phi||^2_{L^2} + ||\nabla\phi||^2_{L^2} + \cdots + ||\nabla^k\phi||^2_{L^2}$$

For an interval $J \subset \mathbb{R}$, let Q_J be the spectral projection of A, given by

$$Q_J : \sum_{j \in \mathbb{Z}} a_j\phi_j \mapsto \sum_{\lambda_j \in J} a_j\phi_j$$

Fix $\Lambda \in \mathbb{R}$ and define the $\check{H}$-norm

$$||\phi||^2_{\check{H}(A)} \coloneqq ||Q_{(-\infty,\Lambda]}\phi||^2_{H^{1/2}(\partial M)} + ||Q_{(\Lambda,\infty)}\phi||^2_{H^{-1/2}(\partial M)}$$

One can show that this definition is independent of the choice of Λ.

Let $H^s_J(A) \coloneqq Q_J(H^s(\partial M, E))$.

Let $\check{H}(A) \coloneqq H^{-1/2}_{(-\infty,\Lambda]}(A) \oplus H^{1/2}_{(\Lambda,\infty)}(A)$.

This $\check{H}(A)$ finally gives the correct setting to work with boundary conditions rigorously. We have the following lemma:

Lemma 2.9.6. The restriction map $\mathscr{R} : C^\infty(M, E) \to C^\infty(\partial M, E)$ extends uniquely to a surjective bounded linear map $\mathscr{R} : \mathrm{dom}(D_{\max}) \to \check{H}(A)$.

Remark. The smooth subspace $C^\infty(\partial M, E)$ is contained in $\check{H}(A)$ because $C^\infty(\partial M, E) \subset \bigcap_s H^s(M, E)$, and it makes sense to take their images under Q_J.

2.9.3 (Adjoint) boundary conditions

While the backdrop against which everything occurs is $L^2(M, E)$, the largest domain of D under consideration is $\mathrm{dom}(D_{\max})$, and the smallest is $\mathrm{dom}(D_{\min})$. Note that $\mathrm{dom}(D_{\max})$ is the completion of $C_c^\infty(M, E)$, while $\mathrm{dom}(D_{\min})$ is the completion of $C_{cc}^\infty(M, E)$.

Definition 2.9.7. A **boundary condition** B is a closed subspace of $\check{H}(A)$.

By the first isomorphism theorem, a boundary condition B corresponds with a closed subspace of $\mathrm{dom}(D_{\max})$ containing $\ker \mathscr{R}$. We first prove a statement saying that $D_{\min}$ is the closed extension of D_{cc} with Dirichlet boundary condition.

Let $H^{\mathrm{fin}}(A) := \{\phi = \sum_{j\in\mathbb{Z}} a_j\phi_j | a_j = 0 \text{ for all but finitely many } j\}$. The idea is the space of finite Fourier series.

We define the extension map $\mathscr{E} : H^{\mathrm{fin}}(A) \to C_c^\infty(Z_{[0,\infty)}, E)$ by

$$(\mathscr{E}\phi)(t) := \chi(t) \cdot \exp(-t|A|)\phi$$

where the cut function $\chi : \mathbb{R} \to \mathbb{R}$ is smooth and is given by

$$\chi(t) = \begin{cases} 1 & \text{for all } t \leq \frac{r}{3} \\ 0 & \text{for all } t \geq \frac{2r}{3} \end{cases} \tag{2.16}$$

for some fixed positive real r. Unraveling the functional calculus, we see that for $\phi = \sum_{j=-n}^{n} a_j\phi_j \in H^{\mathrm{fin}}(A)$,

$$\mathscr{E}\phi(t,x) = \chi(t)\exp(-t|A|)\phi(x) = \chi(t)\sum_{j=-n}^{n} a_j(\exp(-t|\lambda_j|))\phi_j(x)$$

which is smooth and has compact support over $Z_{[0,\infty)}$ and exponentially decays as $t \to \infty$.

Lemma 2.9.8. $\mathrm{dom}(D_{\min}) = \{\phi \in \mathrm{dom}(D_{\max}) : \phi|_{\partial M} = 0\}$.

For any boundary condition $B \subset \check{H}(A)$, define the operator $D_{B\max}$ to be the restriction of $D_{\max}$ to the following domain:

$$\mathrm{dom}(D_{B\max}) := \{\phi \in \mathrm{dom}(D_{\max}) : \mathscr{R}\phi \in B\}$$

Lemma 2.9.9. For any closed extension of D betweeen D_{cc} and $D_{\max}$ is of the form $D_{B\max}$, where $B \subset \check{H}(A)$ is a closed subspace.

Proof. Suppose $\bar{D}$ is a closed extension of D_{cc}. Then

$$\mathrm{dom}(D_{\min}) = \{\phi \in \mathrm{dom}(D_{\max}) : \phi|_{\partial M} = 0\} \subset \mathrm{dom}(\bar{D})$$

where $\mathrm{dom}(\bar{D}) \subset \mathrm{dom}(D_{\max})$ is a closed subspace. Take the restriction of the restriction map $\mathscr{R}|_{\mathrm{dom}(\bar{D})} : \mathrm{dom}(\bar{D}) \to \check{H}(A)$. We define

$$B := \mathrm{Im}\,\mathscr{R}|_{\mathrm{dom}(\bar{D})} \subset \check{H}(A)$$

Claim that

$$\mathrm{dom}(\bar{D}) = \{\phi \in \mathrm{dom}(D_{\max}) : \mathscr{R}(\phi) \in B\}$$

Take $\Phi \in \mathrm{dom}(D_{\max})$ such that $\mathscr{R}\phi \in B$. By definition of B there exists $\phi' \in \mathrm{dom}(\bar{D})$ such that $\mathscr{R}\phi = \mathscr{R}\phi'$. Then $\phi - \phi' \in \mathrm{dom}(D_{\min})$, then $\phi \in \mathrm{dom}(\bar{D})$.

In particular, for any $\varphi \in B$, the extension $\mathscr{E}\varphi \in \mathrm{dom}(\bar{D})$.

Claim that B is a closed subspace. Suppose $\varphi_n \to \varphi$ in $\check{H}(A)$, where $\varphi_n \in B$. Then $\mathscr{E}\varphi_n$ is contained in $\mathrm{dom}(\bar{D})$ and converges to $\mathscr{E}\varphi$ in $\mathrm{dom}(D_{\max})$. Since $\mathrm{dom}(\bar{D})$ is closed, $\mathscr{E}\varphi \in \mathrm{dom}(\bar{D})$. Then $\mathscr{R}(\mathscr{E}\varphi) = \varphi \in B$. □

Remark. From this lemma we see that the generic way of formulating a boundary condition is to restrict the domain of the operator to a closed subset $\mathrm{dom}(\bar{D})$, with the boundary condition $\mathscr{R}\phi \in B$. This explains why B is named this way.

Next we write the functional analytic adjoint of any operator P (from vector bundles E to F) as $(P)^{\mathrm{ad}}$. It has domain

$$\{\psi \in L^2(M,F) : \exists u \in L^2(M,E)\ \forall \phi \in \mathrm{dom}(P)\ (P\phi,\psi) = (\phi,u)\}$$

u is uniquely determined a.e. and we write $u = (P)^{\mathrm{ad}}\psi$.

Now consider the extension $D_{cc} \subset D_{\max}$. One can check that $(D_{\max})^{\mathrm{ad}} \subset (D_{cc})^{\mathrm{ad}}$. Claim that $(D_{cc})^{\mathrm{ad}} = D^*{}_{\max}$. By the definition of maximal extension $D^*{}_{\max} = (D^{**}|_{cc})^{\mathrm{ad}} = (D_{cc})^{\mathrm{ad}}$.

We have the formula

Lemma 2.9.10. Let $\phi \in \mathrm{dom}(D_{\max})$ and $\psi \in \mathrm{dom}(D^*{}_{\max})$. Then

$$(D_{\max}\phi,\psi) - (\phi, D^*{}_{\max}\psi) = -(\sigma_0\mathscr{R}\phi, \mathscr{R}\psi)$$

Consider the adjoint of an operator with boundary condition, $(D_{B\max})^{\mathrm{ad}}$. The domain would be

$$\begin{aligned}\{\psi \in L^2(M,R) : \exists u \in L^2(M,E)\ \forall \phi \in \mathrm{dom}(D_{B\max})\\ (D_{B\max}\phi,\psi) = (\phi,u)\}\end{aligned}$$

Note that $\mathrm{dom}((D_{B\max})^{\mathrm{ad}}) \subset \mathrm{dom}(D^*{}_{\max})$. Then if $\psi \in \mathrm{dom}((D_{B\max})^{\mathrm{ad}})$, u is determined a.e. and is in fact $D^*{}_{\max}\psi$. Also since $D_{B\max} \subset D_{\max}$, we have by the formula of lemma 2.9.10 the equivalent condition

$$\forall \phi \in \mathrm{dom}(D_{B\max}) \quad (\sigma_0\mathscr{R}\phi, \mathscr{R}\psi) = 0$$

Hence we have

Lemma 2.9.11.

$$\mathrm{dom}((D_{B\max})^{\mathrm{ad}}) = \{\psi \in \mathrm{dom}(D^*{}_{\max}) : (\sigma_0\mathscr{R}\phi, \mathscr{R}\psi) = 0 \quad \forall \phi \in \mathrm{dom}(D_{B\max})\}$$

Given a boundary condition B, consider the boundary condition

$$B^{\mathrm{ad}} := \{\psi \in \check{H}(\tilde{A}) : (\sigma_0\phi,\psi) = 0 \quad \forall \phi \in B\}$$

The operator $D^*_{B^{\mathrm{ad}}\max}$ has domain

$$\{\phi \in \mathrm{dom}(D^*{}_{\max}) : \mathscr{R}\psi \in B^{\mathrm{ad}}\}$$

and

$$\begin{aligned}&\mathscr{R}\psi \in B^{\mathrm{ad}}\\ \iff &(\sigma_0\phi, \mathscr{R}\psi) = 0 \quad \forall \phi \in B\\ \iff &(\sigma_0\mathscr{R}v, \mathscr{R}\psi) = 0 \quad \forall v \in \mathrm{dom}(D_{B\max})\end{aligned}$$

which is the same condition as $(D_{B\max})^{\mathrm{ad}}$. Hence

$$(D_{B\max})^{\mathrm{ad}} = D^*_{B^{\mathrm{ad}}\max}$$

In words, we say the adjoint of the operator extended with boundary condition B is the formal adjoint extended with the **adjoint boundary condition** B^{ad}.

When $E = F$, i.e. D maps sections of E to itself, then $\sigma_0 = id$, $\check{H}(A) = \check{H}(\tilde{A})$ and we have a nice formula to calculate the adjoint boundary condition.

Define the operator D_B with domain

$$H^1_D(M, E; B) \coloneqq \{\phi \in H^1_D(M, E) : \mathscr{R}\phi \in B\}$$

which is effectively a restriction of $D_{B\max}$. The assumption $H^1_D(M, E)$ adds to the regularity of ϕ near the boundary and overall C^1. Recall that $H^{\frac{1}{2}}(\partial M, E) \subset \check{H}(A)$. We have a special property for boundary conditions that fall in $H^{\frac{1}{2}}(\partial M, E)$:

Lemma 2.9.12. Let $B \subset \check{H}(A)$ be a boundary condition. Then $B \subset H^{\frac{1}{2}}(\partial M, E)$ if and only if $D_B = D_{B\max}$.

We say that B is **elliptic** if $B \subset H^{\frac{1}{2}}(\partial M, E)$.

Lemma 2.9.13. If $B \subset H^{\frac{1}{2}}(M, E)$ is an elliptic boundary condition, then $B^{\mathrm{ad}} \subset H^{\frac{1}{2}}(M, F)$ is also elliptic.

Theorem 2.9.14. Let D be a Dirac-type operator. Let B be an elliptic boundary condition for D. Then

$$D_B : H^1(M, E; B) \to L^2(M, F)$$

is a Fredholm operator and $\operatorname{ind} D_B = \dim\ker D_B - \dim\ker (D^*)_{B^{\mathrm{ad}}} \in \mathbb{Z}$.

Note that $(D^*)_{B^{\mathrm{ad}}} = (D^*)_{B^{\mathrm{ad}}\max} = (D_{B\max})^{\mathrm{ad}} = (D_B)^{\mathrm{ad}}$.

We compute that

$$\operatorname{ind} D^*_{B^{\mathrm{ad}}} = \dim\ker (D^*)_{B^{\mathrm{ad}}} - \dim\ker ((D^*)_{B^{\mathrm{ad}}})^*_B = \dim\ker (D^*)_{B^{\mathrm{ad}}} - \dim\ker D_B$$

. Then we have

Lemma 2.9.15. Let D be a Dirac-type operator and let B be an elliptic boundary condition for D. Let B^{ad} be its adjoint boundary condition. Then

$$\operatorname{ind} D^*_{B^{\mathrm{ad}}} = -\operatorname{ind} D_B$$

Example 7. Consider the spinor bundle S, possibly twisted, on a Riemannian spin manifold M. The twisted Dirac operator D can be equipped with a closed extension D_1 and D_2 with respective boundary conditions that are ± 1 eigensections of the *chirality operator* $i\nu \cdot (-)$. In other words,

$$\operatorname{dom}(D_1) = \{s \in D_{\max} : s = i\nu \cdot s \text{ on } \partial M\}$$
$$\operatorname{dom}(D_2) = \{s \in D_{\max} : s = -i\nu \cdot s \text{ on } \partial M\}$$

It is a fact that these are elliptic boundary conditions. By the adjointness formula (3.36), we have for $\phi \in \mathrm{dom}(D_1)$, $\psi \in \mathrm{dom}(D_2)$, $D_1\phi = D_{\max}\phi$ and $D_2\psi = D_{\max}\psi$ and

$$\int_M \langle D_1\phi, \psi\rangle - \langle \phi, D_2\psi\rangle = \int_{\partial M} \langle \nu \cdot \phi, \psi\rangle$$

Claim that $\langle \nu\phi, \psi\rangle = 0$ on ∂M. By the boundary conditions we have

$$\begin{aligned}\langle \nu \cdot \phi, \psi\rangle =& \langle \nu \cdot (i\nu \cdot \phi), -i\nu \cdot \psi\rangle \\ =& -\langle -\phi, \nu \cdot \psi\rangle \\ =& -\langle \nu \cdot \phi, \psi\rangle\end{aligned}$$

which proves the claim.

Hence D_1 and D_2 are adjoint to each other and $\mathrm{ind}\, D_1 + \mathrm{ind}\, D_2 = 0$.

Chapter 3

The main result: a Llarull-type estimate

3.1 Motivating example

We describe the general setting. Let S^n be the n-sphere. Consider the complement $S^n \backslash A$ where $A \subset S^n$ is a closed subset. There is a canonical metric g_0 on $S^n \backslash A$, namely the restriction of the spherical metric on S^n. Consider another metric g on $S^n \backslash A$. We require that g satisfies the following:

(1) $g \geq g_0$
(2) $\mathrm{scal}^g \geq \mathrm{scal}^{g_0}$

To see that these are opposing conditions, take $\lambda^2 g$ and we have an inverse relationship between metric and scalar curvature $\mathrm{scal}^{\lambda^2 g} = \lambda^{-2}\,\mathrm{scal}^g$. If λ^2 is too large, condition (2) above would be violated. In this sense, (1) and (2) work against each other. The following is the main question regarding the rigidity of g which can be formulated for any closed $A \subset S^n$.

Question 1. Is $g = g_0$?

Elementary considerations allow us to exclude certain A's that go beyond the northern hemisphere, as seen in the following example.

Example 8. Let $S^n \subset \mathbb{R}^n$ be given by hyperspherical local coordinates $(\theta, \varphi_1, \ldots, \varphi_{n-1})$. A coordinate patch is given by

$$\begin{aligned}(\theta, \varphi_1, \cdots, \varphi_{n-1}) \overset{x}{\mapsto} (&\cos\varphi_1 \cos\varphi_2 \cos\varphi_3 \ldots \cos\varphi_{n-1} \cos\theta, \\ &\sin\varphi_1 \cos\varphi_2 \cos\varphi_3 \ldots \cos\varphi_{n-1} \cos\theta, \\ &\sin\varphi_2 \cos\varphi_3 \ldots \cos\theta, \\ &\ldots \\ &\sin\theta)\end{aligned}$$

where $\varphi_i \in (0, 2\pi), \theta \in (-\pi/2, \pi/2)$. θ represents the vertical angle, which can be visualised by reducing to the usual case $n = 2$. Fix a small $\varepsilon > 0$. Let $A = x(\{(\theta, \varphi_1, \cdots, \varphi_{n-1}) : -\frac{\pi}{2} < \theta \leq \varepsilon\}) \subset S^n \backslash \{\pm(0, \ldots, 0, 1)\}$, so that $S^n \backslash A \subset S^n$ is an open submanifold having the shape of "a cap that goes down from the north-pole until slightly short of the equator". Define $F : (\varepsilon, \frac{\pi}{2}) \to (\frac{\varepsilon}{2}, \frac{\pi}{2})$ to be smooth with $F' = 1$ at the beginning and $F' > 1$ slightly at the end.

Let $f : S^n \backslash A \to S^n$ be given by

$$f(x(\theta, \varphi_1, \cdots, \varphi_{n-1})) = \tilde{x}(F(\theta), \varphi_1, \cdots, \varphi_{n-1})$$
$$f((0, \ldots, 0, 1)) = (0, \ldots, 0, 1)$$

locally for $\varphi_i \in (0, 2\pi)$. Here $\tilde{x}$ is the coordinate patch we give to S^n of codomain of f, which acts on coordinates $(\Theta, \phi_1, \ldots, \phi_{n-1})$, with the exact same definition as x.

The derivative of f is given by $dx^{-1} \circ df \circ dx = d(x^{-1} \circ f \circ x)$ which has $n \times n$ Jacobian

$$\begin{bmatrix} F'(\theta) & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & & \ddots & \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

Let $g = f^* g_0$. Note that $\mathrm{scal}^g = \mathrm{scal}^{g_0} \circ f = n(n-1)$. We check $g \geq g_0$. It is enough to check on coordinate vector fields $df(\frac{\partial}{\partial \theta})$ and $df(\frac{\partial}{\partial \varphi_i})$. The first fundamental forms on both sides are respectively

$$g_0(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}) = g_0(\frac{\partial}{\partial \Theta}, \frac{\partial}{\partial \Theta}) = 1,$$
$$g_0(\frac{\partial}{\partial \varphi_i}, \frac{\partial}{\partial \varphi_i}) = \cos^2 \theta \prod_{j \neq i} \cos^2 \varphi_j,$$
$$g_0(\frac{\partial}{\partial \phi_i}, \frac{\partial}{\partial \phi_i}) = \cos^2 \Theta \prod_{j \neq i} \cos^2 \phi_j$$

hence $f^* g_0$ is

$$g_0(df(\frac{\partial}{\partial \theta}), df(\frac{\partial}{\partial \theta})) = (F'(\theta))^2 \geq g_0(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}) \tag{3.1}$$
$$g_0(df(\frac{\partial}{\partial \varphi_i}), df(\frac{\partial}{\partial \varphi_i})) = \cos^2 F(\theta) \prod_{j \neq i} \cos^2 \varphi_j \geq g_0(\frac{\partial}{\partial \varphi_i}, \frac{\partial}{\partial \varphi_i}) \tag{3.2}$$

Therefore $g > g_0$ and we know A cannot be too large in the sense of this example.

Notice that the inequality (3.2) holds only because F has the property that $F(\theta) < 0$ and $\cos^2 F(\theta) \geq \cos^2(\theta)$ for $\theta \in (-\frac{\pi}{2}, -\varepsilon)$. It breaks down if $S^n \backslash A$ is exactly the northern hemisphere or goes south of the equator, because $\cos^2 \theta$ decreases when θ decreases below 0.

Therefore we restrict our attention to finite A; in fact we consider the most elementary cases where $A = \emptyset$, $A = \{p\}$ (single point) and $A = \{\pm p\}$ (antipodal points). The solution of the empty-set case is a well-known result by Llarull, see [Lla88]. Gromov conjectured the antipodal-points case in his popular Four Lectures, see [Gro23]. The answer to **Question 1** is again a yes, see [BBHW24], which is the main point of reference of this thesis.

3.2 The classical Llarull estimate

We present Llarull's solution to the $A = \emptyset$ case of **Question 1** following [Lla88]. This is the most basic case and helps motivate subsequent constructions. Llarull's theorem is stated as follows.

Theorem 3.2.1 (Llarull, 1998)**.** Let M be a connected, compact, spin Riemannian manifold of dimension n. Suppose M has a Riemannian metric g. Let g_0 be the spherical metric on S^n. Suppose $f : (M, g) \to (S^n, g_0)$ is a smooth map that satisfies the following:

(1) *deg* $f \neq 0$
(2) f is 1-Lipschitz
(3) $\mathrm{scal}^g \geq \mathrm{scal}^{g_0} \circ f = n(n-1)$

then f is a Riemannian isometry.

Remark. This solves the $A = \emptyset$ case because if there were a metric g on S^n that has $g \geq g_0$ and $\mathrm{scal}^g \geq \mathrm{scal}^{g_0}$ then the identity map from (S^n, g) to (S^n, g_0) will satisfy the conditions of Theorem 3.2.1.

The strategy is as follows. Since S^n is simply connected and compact, we can deduce that f is global isometry from f being local isometry by Lemma 2.8.1. By inverse function theorem we can reduce to pointwise isometry. The proof of pointwise isometry relies on a genius use of spinors and the index theorem. For a first step we use the following lemma,

Take a regular point $x \in M$. The differential $df_x : (T_xM, g) \to (T_{f(x)}S^n, g_0)$ admits a singular value decomposition with respect to a g-orthonormal basis $\{e_i\}$ on T_xM and a g_0-orthonormal basis $\{\varepsilon_i\}$ on $T_{f(x)}S^n$ with positive singular values arranged in decreasing order, say $\mu_1 \geq \mu_2 \geq \dots \mu_n > 0$. That is to say we have

$$df_x(e_i) = \mu_i \varepsilon_i \quad \forall i \in [1, n] \tag{3.3}$$

Since f is 1-Lipschitz, we have $g_0(df_x(e_i), df_x(e_i)) = \mu_i^2 \leq g(e_i, e_i) = 1$, Hence $\mu_i \leq 1$ for each i. If we can show $\mu_i \geq 1$, then df_x is an isometry and by our strategy we are done. We prove this in subsection 3.2.3, after some preparations.

3.2.1 An estimate of curvature term

Recall the Weitzenböck formula (2.13)

$$D_E^2 = (\nabla^{S\otimes E})^* \nabla^{S\otimes E} + \frac{1}{4}\,\mathrm{scal} + \mathcal{R}^E$$

where locally

$$\mathcal{R}^E(\sigma \otimes v) = \frac{1}{2}\sum_{i,j=1}^{n} b_i \cdot b_j \cdot \sigma \otimes R^E(b_i, b_j)v$$

which holds for any vector bundle E on M. Let E_0 be the spinor bundle on S^n and $E = f^*E_0$. In this case we can compute R^E explicitly and derive an estimate about $\mathcal{R}^E$, see lemma 3.2.3.

Lemma 3.2.2. When $i \neq j$, the curvature tensor R^E with respect to $E = f^*E_0$ is given by

$$R^E(e_i, e_j)v = \frac{1}{2}\mu_i\mu_j\varepsilon_j \cdot \varepsilon_i \cdot df_x(v) \tag{3.4}$$

where $\cdot$ denotes Clifford multiplication.

When $i = j$, then

$$R^E(e_i, e_i)v = 0$$

Proof. Since $E = f^*E_0$, $R^E = R_0^E \circ df_x$ in the sense that $R^E(e_i, e_j)v = R^{E_0}(f_*e_i, f_*e_j)f_*v$. By the local formula (2.11) we have

$$\begin{aligned}
R^{E_0}(f_*e_i, f_*e_j)f_*v &= -\frac{1}{4}\sum_{k=1}^{n}(R(f_*e_i, f_*e_j)\varepsilon_k) \cdot \varepsilon_k \cdot f_*v \\
&= -\frac{1}{4}\sum_{k=1}^{n}\sum_{l=1}^{n} g_0(R(f_*e_i, f_*e_j)\varepsilon_k, \varepsilon_l)\varepsilon_l \cdot \varepsilon_k \cdot f_*v \\
&= \frac{1}{4}\sum_{k=1}^{n}\sum_{l=1}^{n} g_0(R(f_*e_i, f_*e_j)\varepsilon_k, \varepsilon_l)\varepsilon_k \cdot \varepsilon_l \cdot f_*v
\end{aligned}$$

where R is the Riemannian curvature tensor on S^n. Then by the spherical curvature tensor formula (2.12)

$$\frac{1}{4}\sum_{k=1}^{n}\sum_{l=1}^{n} g_0(R(f_*e_i, f_*e_j)\varepsilon_k, \varepsilon_l)\varepsilon_k \cdot \varepsilon_l \cdot f_*v \tag{3.5}$$

$$=\frac{1}{4}\sum_{k=1}^{n}\sum_{l=1}^{n} \mu_i\mu_j(g_0(\varepsilon_j, \varepsilon_k)g_0(\varepsilon_i, \varepsilon_l) - g_0(\varepsilon_i, \varepsilon_k)g_0(\varepsilon_j, \varepsilon_l))\varepsilon_k \cdot \varepsilon_l \cdot f_*v \tag{3.6}$$

$$=\frac{1}{2}\mu_i\mu_j\varepsilon_j \cdot \varepsilon_i \cdot f_*v \tag{3.7}$$

Notice that in (3.6) we have $g_0(\varepsilon_i, \varepsilon_k) = \delta_{ik}$ because of orthonormality. This takes care of the $i \neq j$ case. When $i = j$, the difference from the above is (3.6) vanishes uniformly, unlike in the $i \neq j$ case where the summand of index $(k, l) = (i, j)$ survives. $\square$

Using formula for the Bochner curvature tensor, together with lemma 2.11, we have the following estimate of $\langle \mathcal{R}^E\phi, \phi\rangle$.

Lemma 3.2.3. Let $E = f^*E_0$ and $\mu_1, \ldots, \mu_n$ be the positive singular values of df_x. We have

$$\langle \mathcal{R}^E\phi, \phi\rangle \geq -\frac{1}{4}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \mu_i\mu_j\langle\phi, \phi\rangle \tag{3.8}$$

for any section $\phi \in C^\infty(S \otimes E)$.

Proof. This is a pointwise condition so we consider an orthonormal local frame of the bundle $S \otimes E$, which is given by $\{\sigma_\alpha \otimes v_\beta\}_{\alpha,\beta}$. We write $\phi = \sum_{\alpha,\beta} a_{\alpha\beta}\sigma_\alpha \otimes v_\beta$. Then by formula (2.11) and (3.4) we have

$$\langle \mathcal{R}^E\phi, \phi\rangle \tag{3.9}$$

$$=\langle \frac{1}{2}\sum_{i\neq j}\sum_{\alpha,\beta} a_{\alpha\beta} e_i\cdot e_j\cdot\sigma_\alpha \otimes \frac{1}{2}\mu_i\mu_j\varepsilon_j\cdot\varepsilon_i\cdot v_\beta, \sum_{k,l} a_{kl}\sigma_k\otimes v_l\rangle \tag{3.10}$$

$$=\frac{1}{4}\sum_{i\neq j}\mu_i\mu_j\langle\sum_{\alpha,\beta} a_{\alpha\beta} e_i\cdot e_j\cdot\sigma_\alpha\otimes\varepsilon_j\cdot\varepsilon_i\cdot v_\beta, \sum_{k,l} a_{kl}\sigma_k\otimes v_l\rangle \tag{3.11}$$

$$=\frac{1}{4}\sum_{i\neq j}\mu_i\mu_j\sum_{\alpha,\beta,k,l} a_{\alpha\beta}a_{kl}\langle e_i\cdot e_j\cdot\sigma_\alpha, \sigma_k\rangle\langle\varepsilon_j\cdot\varepsilon_i\cdot v_\beta, v_l\rangle \tag{3.12}$$

To bound this term we consider the invariance of the basis of spinor space under Clifford multiplication by the spin group. Fix a canonical basis $\{e_i\}$ and $\{\varepsilon_j\}$ for the spinor bundles. Consider the Clifford multiplication $(e_i\cdot e_j)\,\cdot : S\to S$ which permutes this basis. Similarly $(\varepsilon_j\cdot\varepsilon_i)\,\cdot : E\to E$ permutes the basis. Then we can write ϕ in four equivalent ways:

- $\phi = \sum_{\alpha,\beta} a_{\alpha\beta}\sigma_\alpha\otimes v_\beta$
- $\phi = \sum_{\alpha,\beta} a_{\alpha'\beta} e_i\cdot e_j\cdot\sigma_\alpha\otimes v_\beta$
- $\phi = \sum_{\alpha,\beta} a_{\alpha\beta'}\sigma_\alpha\otimes\varepsilon_j\cdot\varepsilon_i\cdot v_\beta$
- $\phi = \sum_{\alpha,\beta} a_{\alpha'\beta'} e_i\cdot e_j\cdot\sigma_\alpha\otimes\varepsilon_j\cdot\varepsilon_i\cdot v_\beta$

Let us abbreviate the Clifford multiplication as $ab = a\cdot b$. With $(e_ie_j)^2 = (\varepsilon_j\varepsilon_i)^2 = -1$, we then have

$$\langle 4\sum_{\alpha,\beta} a_{\alpha\beta}e_ie_j\sigma_\alpha\otimes\varepsilon_j\varepsilon_iv_\beta, \sum_{k,l} a_{kl}\sigma_k\otimes v_l\rangle \tag{3.13}$$

$$=\langle\sum_{\alpha,\beta} a_{\alpha\beta}e_ie_j\sigma_\alpha\otimes\varepsilon_j\varepsilon_iv_\beta - \sum_{\alpha,\beta} a_{\alpha'\beta}\sigma_\alpha\otimes\varepsilon_j\varepsilon_iv_\beta \tag{3.14}$$

$$-\sum_{\alpha,\beta} a_{\alpha\beta'}e_ie_j\sigma_\alpha\otimes v_\beta + \sum_{\alpha,\beta} a_{\alpha'\beta'}\sigma_\alpha\otimes v_\beta, \sum_{k,l} a_{kl}\sigma_k\otimes v_l\rangle \tag{3.15}$$

Fix a pair (α,β). Note that tensoring orthogonal bases gives us an orthogonal basis of the tensor product (with respect to the induced inner product). Then we have

$$\langle a_{\alpha\beta}e_ie_j\sigma_\alpha\otimes\varepsilon_j\varepsilon_iv_\beta, \sum_{kl} a_{kl}\sigma_k\otimes v_l\rangle = \langle a_{\alpha\beta}e_ie_j\sigma_\alpha\otimes\varepsilon_j\varepsilon_iv_\beta, \sum_{kl} a_{k'l'}e_ie_j\sigma_k\otimes\varepsilon_j\varepsilon_iv_l\rangle = a_{\alpha\beta}a_{\alpha'\beta'}$$

while the other three summands can be done similarly. Then (3.13) reduces to

$$\begin{aligned}&a_{\alpha\beta}a_{\alpha'\beta'} + a_{\alpha'\beta'}a_{\alpha\beta} - a_{\alpha'\beta}a_{\alpha\beta'} - a_{\alpha\beta'}a_{\alpha'\beta}\\ =&2a_{\alpha\beta}a_{\alpha'\beta'} - 2a_{\alpha'\beta}a_{\alpha\beta'}\\ \geq&-a_{\alpha\beta}^2 - a_{\alpha'\beta'}^2 - a_{\alpha'\beta}^2 - a_{\alpha\beta'}^2\end{aligned}$$

Since $|\phi|^2 = \sum_{\alpha,\beta} a_{\alpha\beta}^2 = \sum_{\alpha,\beta} a_{\alpha'\beta'}^2 = \sum_{\alpha,\beta} a_{\alpha'\beta}^2 = \sum_{\alpha,\beta} a_{\alpha\beta'}^2$, summing over all (α,β) we get

$$4\langle\mathcal{R}^E\phi,\phi\rangle \geq -\sum_{i\neq j}\mu_i\mu_j|\phi|^2$$

and the desired inequality follows. □

3.2.2 An index calculation

Suppose n is even. We continue with the notations of the previous section. So $E = f^*E_0$ is the pullback bundle of the spinor bundle E_0 on S^n. Using the even dimensional volume element ω there is a splitting $E_0 = E_0^+ \oplus E_0^-$. So $E = f^*E_0^+ \oplus f^*E_0^-$. Denote $f^*E_0^+$ by E^+ and $f^*E_0^-$ by E^-. The twisted Dirac operator $D_E : \Gamma(S \otimes E^+ \oplus S \otimes E^-) \to \Gamma(S \otimes E^+ \oplus S \otimes E^-)$ is known to preserve the direct sum.

Hence we can restrict to $D_E|_{S\otimes E^+} = D_{E^+}$. Note also the splitting of spinor bundle on M, which is $S \otimes E^+ = S^+ \otimes E^+ \oplus S^- \otimes E^+$. The twisted Dirac $D_{E^+} : \Gamma(S^\pm \otimes E^+) \to \Gamma(S^\mp \otimes E^+)$ reverses the chirality and also splits into $D_{E^+} = D^+_{E^+} \oplus D^-_{E^+}$, and the index is given by $ind(D_{E^+}) = \dim\ker D^+_{E^+} - \dim\ker D^-_{E^+}$.

Since the manifold M is assumed to have positive scalar curvature, by the untwisted Lichnerowicz formula

$$D^2 = \frac{1}{4}\mathrm{scal} + \nabla^*\nabla$$

One easily sees that $\ker D = 0$ and hence $\mathrm{ind}(D_+) = \hat{\mathcal{A}}(M) = 0$.

Lemma 3.2.4. $\ker D_E \neq 0$.

Before we prove this lemma, let us establish a fact about characteristic number on even dimensional spheres.

Lemma 3.2.5. Let S^n be an even dimensional sphere and E_0^+ be the positive part of its spinor bundle. With $n = 2m$, we have $\langle c_m(E_0^+), [S^n]\rangle \neq 0$.

Proof. This is a relatively well-known fact. An example reference is [Lla88, Appendix 1]. □

We are now ready to prove lemma 3.2.4.

Proof of lemma 3.2.4. Assume $\ker D_E = 0$. Then we have the relation $\ker D^+_{E^+} \oplus \ker D^-_{E^+} = \ker D_{E^+} \subset \ker D_E = 0$. Hence $ind(D^+_{E^+}) = \dim\ker D^+_{E^+} - \dim\ker D^-_{E^+} = 0$. By the Atiyah-Singer index theorem 2.7.9 we know

$$0 = ind(D^+_{E^+}) = \langle ch(E^+) \cup \hat{A}(M), [M]\rangle = \langle [ch(E^+) \cup \hat{A}(M)]_n, [M]\rangle \tag{3.16}$$

We note that $E^+ = f^*E_0^+$ is the pullback of the positive part of the spinor bundle on S^n. Writing $n = 2m$, The formula on Chern character (A.3) yields

$$ch(E^+) = 2^{m-1} + c_m(E^+) = 2^{m-1} + f^*c_m(E_0^+) \tag{3.17}$$

Plug (3.17) into (3.16) and use that fact $\hat{\mathcal{A}}(M) = 0$ we have

$$0 = (\deg f)\langle c_m(E_0^+), [S^n]\rangle + 2^{m-1}\hat{\mathcal{A}}(M) = (\deg f)\langle c_m(E_0^+), [S^n]\rangle \tag{3.18}$$

Since $\deg f \neq 0$ by assumption, we have $\langle c_m(E_0^+), [S^n]\rangle = 0$. But this contradicts lemma 3.2.5. □

Lemma 3.2.6. There is a section $\phi \in C^\infty(M, S \otimes E)$ such that $\phi \neq 0$ at each point of M and $D_E\phi = 0$.

Proof. Recall that a parallel section ϕ, i.e. $\nabla\phi = 0$, satisfies that if $\phi = 0$ at some point then $\phi \equiv 0$. Therefore a nonzero section in $\ker D_E$ gotten from lemma 3.2.4 works as needed. □

3.2.3 Proof of Llarull theorem

In this section we proceed to prove theorem 3.2.1.

Proof of theorem 3.2.1. First consider the case of even n.

By lemma 3.2.6 take a smooth section ϕ that is everywhere nonzero and satisfies $D_E\phi = 0$. Apply pointwise inner product with ϕ to the Weitzenböck formula (2.13) we get

$$\langle D_E^2\phi, \phi\rangle = |\nabla\phi|^2 + \frac{1}{4}\operatorname{scal}^g\langle\phi, \phi\rangle + \langle\mathcal{R}^E\phi, \phi\rangle$$

Note LHS=0 and by lemma 3.2.3 we have the inequality

$$0 \geq |\nabla\phi|^2 + \frac{1}{4}(\operatorname{scal}^g - \sum_{i\neq j}\mu_i\mu_j)\langle\phi, \phi\rangle$$

Since $0 < \mu_i \leq 1$ we have $\sum_{i\neq j}\mu_i\mu_j \leq n(n-1)$. But $\operatorname{scal}^g \geq n(n-1)$. So the RHS of the above inequality is a nonnegative quantity, hence must be 0. So ϕ is a parallel section and $\operatorname{scal}^g = n(n-1)$. Substitute this back to the inequality we have

$$0 \geq \frac{1}{4}(n(n-1) - \sum_{i\neq j}\mu_i\mu_j)\langle\phi, \phi\rangle$$
$$\sum_{i\neq j}\mu_i\mu_j \geq n(n-1)$$

hence $\mu_j = 1$ and we have shown f is a local isometry. By lemma 2.8.1 f is a global isometry.

Next we consider the case of odd $n = 2m-1$. Ideally, we would like to to reuse the even dimensional argument wholesale; we work towards this direction and will discover some subtleties. Consider the product $M \times S^1$ which is $2m$ dimensional. We will use the following lemma:

Lemma 3.2.7. For each $n \geq 1$, there exsists a 1-contracting map $h : S^{n-1} \times S^1 \to S^n$, where $S^{n-1} \times S^1$ is equipped with the usual product metric.

Proof. Let us consider the simpler case $n = 2$. Parametrise $S^2 = \{x^2 + y^2 + (z-1)^2 = 0\} \subset \mathbb{R}^3$ in the following way:

$$\begin{aligned} P : [0,\pi] \times [0, 2\pi] \to & S^2 \\ P : (\alpha, \theta) \mapsto & (x = \cos\theta(\sin\theta - \sin\theta\cos\alpha), \\ & y = \sin\theta\sin\alpha, \\ & z = \sin\theta(\sin\theta - \sin\theta\cos\alpha)) \end{aligned}$$

Let us show h is 1-contracting on the points $[(x,t)]$ where $t \neq 0, 2\pi$. We equip $S^1 \times S^1$ with the product metric, which can be identified with the metric inherited from the embedding

$$Q : [0,2\pi] \times [0,2\pi] \to S^1 \times S^1 \subset \mathbb{C}^2 = \mathbb{R}^4$$
$$Q : (\alpha, t) \mapsto (\cos\alpha, \sin\alpha, \cos t, \sin t)$$

The idea is to "wrap" the torus around S^2 once, while fixing a point. Hence we define

$$\begin{aligned} R : [0,2\pi] \times [0,2\pi] \to & S^2 \subset \mathbb{R}^3 \\ R(\alpha,t) \mapsto & (x = \cos\frac{t}{2}(\sin\frac{t}{2} - \sin\frac{t}{2}\cos\alpha), \\ & y = \sin\frac{t}{2}\sin\alpha, \\ & z = \sin\frac{t}{2}(\sin\frac{t}{2} - \sin\frac{t}{2}\cos\alpha)) \end{aligned}$$

Remove the boundaries from the domains of R, Q and we have local charts of $S^1 \times S^1$ and S^2. Define the local map

$$h : S^1 \times S^1 \to S^2$$
$$h = R \circ Q^{-1}$$

h is obviously smooth when restricted to the open subset $S^1 \times S^1 - \{*\}$, where $*$ is the point $Q((0,0))$. We omit the proof that h is smooth on $Q((0,0))$.

Let us demonstrate h is 1-contracting on $S^1 \times S^1 - \{*\}$. We have the following relations:

$$h_*(Q_*(e_1)) = R_*(e_1)$$
$$h_*(Q_*(e_2)) = R_*(e_2)$$

It is an amusing little exercise to check that $\{Q_*(e_1), Q_*(e_2)\}$ forms an orthonormal basis of $T_x(S^1 \times S^1 - \{*\})$, whatever the point x is. Now we differentiate to get

$$R_\alpha = \left(\cos\frac{t}{2}\sin\frac{t}{2}\sin\alpha, \sin\frac{t}{2}\cos\alpha, \sin^2\frac{t}{2}\sin\alpha\right)$$
$$R_t = \left((1-\cos\alpha)(-\frac{1}{2}\sin^2\frac{t}{2} + \frac{1}{2}\cos^2\frac{t}{2}), \frac{1}{2}\cos\frac{t}{2}\sin\alpha, (1-\cos\alpha)(\sin\frac{t}{2}\cos\frac{t}{2})\right)$$

Here are the lengths of R_α and R_t in standard metric:

$$\langle R_\alpha, R_\alpha \rangle = \sin^2\frac{t}{2}\left(1 + \cos^2\frac{t}{2}\sin^2\alpha\right)$$
$$\langle R_t, R_t \rangle = \frac{1}{2} - \frac{1}{2}\cos\alpha$$

both are ≤ 1. So h is 1-contracting. It should be noted that h is NOT 1-Lipschitz, because $\langle R_\alpha, R_t \rangle \not\equiv 0$.

This could be generalised to higher dimensions with hyperspherical coordinates. $\square$

By lemma 3.2.7 we can take a 1-Lipschitz map $h : S^{n-1} \times S^1$. We consider the product $(M \times S^1, g + r^2 g_{S^1})$, where the usual metric g_{S^1} on the S^1 component is scaled by a radius factor r^2, for some $r > 1$. Let $\tilde{f}$ be given by the composition:

$$\tilde{f} : M \times S^1 \xrightarrow{f \times id} S^{2m-1} \times S^1 \xrightarrow{h} S^{2m} \tag{3.19}$$

We compute:

$$||\tilde{f}_*(v, w)|| = ||h_*(f_*(v), w)|| \leq ||f_*(v)|| + ||w||_{g_{S^1}} < ||v|| + r||w||_{g_{S^1}} = ||(v, w)||_{g+r^2 g_{S^1}} \tag{3.20}$$

hence $\tilde{f}$ is 1-Lipschitz.

One may wonder why we add the factor r. The reason is scalar curvature bound. Note that if $r = 1$, we have $\mathrm{scal}^{g_{S^1}} = 0$ and $\mathrm{scal}^{g+g_{S^1}} = \mathrm{scal}^g \geq (2m-1)(2m-2)$ by assumption. But the codomain of $\tilde{f}$ is S^{2m} which has scalar curvature $2m(2m-1)$. Thus we cannot directly use the argument for even-n case to conclude $\tilde{f}$ is isometry. It will turn out that r remedies this problem by giving us a finer estimate of curvature term.

Consider the spinor bundle S on $M \times S^1$, where we fix the spin structure $S^1 \sqcup S^1 \to S^1$. We similarly construct the twisted spinor bundle $E = S \otimes \tilde{f}^* E_0$, where E_0 is the spinor bundle on S^{2m}. We denote $E^+ = \tilde{f}^* E_0^+$, as before.

We follow the same argument as the even case and start from an evaluation of the curvature term in the Weitzenböck formula such as (3.12). Precisely, we start with a Singular Value Decomposition of $d\tilde{f}_{(x,t)}$:

- There exists orthonormal basis $\{e_1, \ldots, e_{2m-1}\} \cup \{e_{2m}\}$ of $T_{(x,t)} M \times S^1 \cong T_x M \oplus T_t S^1$ such that $\{e_1, \ldots, e_{2m-1}\}$ is an orthonormal basis of $T_x M$ and e_{2m} is a unit tangent vector in $T_t S^1$;
- there exists orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_{2m}\}$, of $T_{\tilde{f}(x,t)} S^{2m}$;
- there exists positive constants $\lambda_1, \ldots, \lambda_{2m}$ such that $d\tilde{f}_x(e_i) = \lambda_i \varepsilon_i$ for each $i \in \{1, \ldots, 2m\}$

We easily obtain from 1-Lipschitz property of $\tilde{f}$ that $\lambda_i \leq 1$ for $i \in \{1, \ldots, 2m-1\}$. For $2m$, note that $||e_{2m}||_{r^2 g_{S^1}} = 1$. So $||e_{2m}||_{g_{S^1}} = \frac{1}{r}$. Hence $\lambda_{2m} = ||\tilde{f}_*(e_{2m})|| = ||h_*(e_{2m})|| \leq ||e_{2m}||_{g_{S^1}} = \frac{1}{r}$. Intuitively we see that $\tilde{f}$ contracts e_{2m} by r.

The spinor bundle on $M \times S^1$ is 2^m dimensional. The pullback bundle E is also 2^m dimensional. We apply the calculation of (3.12) and obtain the following:

Lemma 3.2.8. Let $\mathcal{R}^E$ be the curvature term in the Weitzenböck formula associated to the Dirac operator D_E. Given a local frame of $S \otimes E$, $\{\sigma_\alpha \otimes v_\beta : 1 \leq \alpha \leq 2^m,\ 1 \leq \beta \leq 2^{m-1}\}$, we have

$$\langle \mathcal{R}^E \phi, \phi \rangle = \frac{1}{4} \sum_{i \neq j = 1}^{2m} \lambda_i \lambda_j \sum_{\alpha, \beta, k, l = 1}^{2^m} a_{\alpha\beta} a_{kl} \langle e_i \cdot e_j \cdot \sigma_\alpha, \sigma_k \rangle \langle \varepsilon_j \cdot \varepsilon_i \cdot v_\beta, v_l \rangle \tag{3.21}$$

Then (3.21) becomes

$$\begin{aligned}\langle \mathcal{R}^E\phi,\phi\rangle &= \frac{1}{4}\sum_{i\neq j=1}^{2m-1}\lambda_i\lambda_j\sum_{\alpha,\beta,k,l=1}^{2^m} a_{\alpha\beta}a_{kl}\langle e_i\cdot e_j\cdot\sigma_\alpha,\sigma_k\rangle\langle\varepsilon_j\cdot\varepsilon_i\cdot v_\beta,v_l\rangle\\ &+\frac{1}{4}\sum_{i=1}^{2m-1}\lambda_i\lambda_{2m}\sum_{\alpha,\beta,k,l=1}^{2^m} a_{\alpha\beta}a_{kl}\langle e_i\cdot e_{2m}\cdot\sigma_\alpha,\sigma_k\rangle\langle\varepsilon_{2m}\cdot\varepsilon_i\cdot v_\beta,v_l\rangle\\ &+\frac{1}{4}\sum_{j=1}^{2m-1}\lambda_{2m}\lambda_j\sum_{\alpha,\beta,k,l=1}^{2^m} a_{\alpha\beta}a_{kl}\langle e_{2m}\cdot e_j\cdot\sigma_\alpha,\sigma_k\rangle\langle\varepsilon_j\cdot\varepsilon_{2m}\cdot v_\beta,v_l\rangle\end{aligned}$$

where we separate the index $2m$ from the rest. Then from the proof of lemma 3.2.3 we consider a single pair of unequal $(i.j)$, neither equal to $2m$. Using $\lambda_i\leq 1$ in this range, we obtain

$$\langle 4\sum_{\alpha,\beta}a_{\alpha\beta}e_ie_j\sigma_\alpha\otimes\varepsilon_j\varepsilon_i v_\beta,\sum_{k,l}a_{k.l}a_{k,l}\sigma_k\otimes v_l\rangle\geq -|\phi|^2 \tag{3.22}$$

If either i or j is $2m$, we use $\lambda_{2m}\leq\frac{1}{4}$ and get

$$\langle 4\sum_{\alpha,\beta}a_{\alpha\beta}e_ie_j\sigma_\alpha\otimes\varepsilon_j\varepsilon_i v_\beta,\sum_{k,l}a_{k.l}a_{k,l}\sigma_k\otimes v_l\rangle\geq -\frac{1}{r}|\phi|^2 \tag{3.23}$$

combining the above we get

$$\langle\mathcal{R}^E\phi,\phi\rangle\geq -\frac{1}{4}(2m-1)(2m-2)|\phi|^2-\frac{1}{2r}(2m-1)|\phi|^2 \tag{3.24}$$

Substitute this into Weitzenböck we get

$$\langle D_E^2\phi,\phi\rangle\geq|\nabla\phi|^2+(\frac{1}{4}\operatorname{scal}^g-\frac{1}{4}(2m-1)(2m-2)-\frac{1}{2r}(2m-1))|\phi|^2 \tag{3.25}$$

Next note that we can apply the index calculation of section 3.2.2 directly and see from there that $ind(D^+_{E^+})\neq 0$ and hence $\ker D_E\neq 0$ by lemma 3.2.4. Plug in a nonzero section $\phi\in\ker D_E$ we have

$$0\geq|\nabla\phi|^2+(\frac{1}{4}\operatorname{scal}^g-\frac{1}{4}(2m-1)(2m-2)-\frac{1}{2r}(2m-1))|\phi|^2 \tag{3.26}$$

We look at a point $x_0\in M\times S^1$ such that $\phi(x_0)\neq 0$. Note $\frac{1}{4}\operatorname{scal}^g-\frac{1}{4}(2m-1)(2m-2)\geq 0$. Since (3.26) holds for any $r>1$, we must have $\operatorname{scal}^g=(2m-1)(2m-2)$ and $\nabla\phi(x_0)=0$. If $x_0\in M\times S^1$ such that $\phi(x_0)=0$, then $\nabla\phi(x_0)=0$ again. Hence ϕ is parallel and by assumption is everywhere nonzero. Substitute $\nabla\phi\equiv 0,\operatorname{scal}^g=(2m-1)(2m-2)$ and note that we in fact have $0\geq(\frac{1}{4}(2m-1)(2m-2)-\frac{1}{4}\sum_{i\neq j}^{2m-1}\lambda_i\lambda_j-\frac{1}{2r}(2m-1))$ at (x,t), where λ_i is defined. Then obviously $\lambda_i=1$ for each $i\in\{1,\dots,2m-1\}$.

We combine the facts that h and f are both 1-Lipschitz with $\lambda_i=1$ to obtain the following two pinched inequalities:

- $1=||\tilde{f}_*(e_i)||=||h_*(f_*(e_i))||\leq||f_*(e_i)||\leq||e_i||=1$; and
- $0=g_{S^{2m}}(\tilde{f}_*e_i,\tilde{f}_*e_j)\leq g_{S^{2m-1}}(f_*e_i,f_*e_j)\leq g(e_i,e_j)=0$

Hence f sends $\{e_i\}$ to an orthonormal basis of T_xS^{2m-1} and is thus a local and global Riemannian isometry. $\square$

3.3 The Bär-Brendle-Hanke-Wang estimate

In [BBHW24], Bär-Brendle-Hanke-Wang solves **Question 1** in the antipodal points case $A = \{\pm p\}$. Their strategy is to first remove two open discs in antipodal positions, and regard $S^n \backslash (D^n \sqcup D^n)$ as a cylinder with a warped product metric. Then the antipodal points case is solved as a limit of the disc case by letting the radius of the discs go to 0. This section presents the disc case.

3.3.1 Statement of the theorem

In this subsection we state the main theorem 1.1 of [BBHW] and derive some preliminary lemmas we need later.

Definition 3.3.1. Let $D \subset \mathbb{R}$ be a convex subset. A smooth function $f : D \to \mathbb{R}$ is **concave** if for any $x, y \in D$ and any $t \in [0, 1]$ we have $f((1-t)x + ty \geq (1-t)f(x) + tf(y)$. If further the inequality is strict, f is said to be **strictly concave**.

Remark. One can show that f is strictly concave if and only if $f'' < 0$.

The main statement for the disc case is the following.

Theorem 3.3.2. Let $n > 2$ and let $\rho : [\theta_-, \theta_+] \to \mathbb{R}$ be a positive smooth function such that $log(\rho)$ is strictly concave. Let g_0 denote the warped product metric $g_0 = d\theta \otimes d\theta + \rho(\theta)^2 g^{S^{n-1}}$ on $[\theta_-, \theta_+] \times S^{n-1}$. Let M be a compact connected spin manifold of dimension n with boundary ∂M. Let g be a Riemmannian metric on M. Suppose that $\Phi : (M, g) \to ([\theta_-, \theta_+] \times S^{n-1}, g_0)$ is a smooth map with the following properties:

(1) $\Phi(\partial M) \subset S^{n-1} \times \{\theta_-, \theta_+\}$
(2) Φ has nonzero degree
(3) Φ is 1-Lipschitz
(4) $\mathrm{scal}^g \geq \mathrm{scal}^{g_0} \circ \Phi$ at each point in M
(5) Mean curvature $H_g \geq H_{g_0} \circ \Phi$ at each point in ∂M

Then Φ is a Riemannian isometry.

Here is a summary of the idea of proof. A strategy similar to that of Llarull can be employed here, but one needs to consider the boundary of the manifold. The idea is to consider the Dirac operator "adapted to" the boundary, which is then a closed manifold of dimension $\dim M - 1$. The index theorem tells us there is a parallel section with respect to a modified operator, which puts constraint on the singular values of the derivative of Φ. In case of even dimensional M, we have to deal with ∂M being odd-dimensional and its Dirac operator having index 0. We can take product with S^1 similar to the proof of section 3.2.3.

We set up some notations first.

Let $\psi : [\theta_-, \theta_+] \to \mathbb{R}$ be the log-derivative of the warping function ρ, i.e. $\psi(\theta) = \frac{\rho'(\theta)}{\rho(\theta)}$. $\log \rho$ is strictly concave so $-\psi' > 0$ on $[\theta_-, \theta_+]$. Define $\Psi : M \to \mathbb{R}$ by $\Psi = \psi(\Theta)$.

Write $\Phi = (\varphi, \Theta)$, where

$$\varphi = p_{S^{n-1}} \circ \Phi \tag{3.27}$$
$$\Theta = p_{[\theta_-, \theta_+]} \circ \Phi \tag{3.28}$$

are the respective projections. Let $\partial_- M = \Phi^{-1}(\{\theta_-\} \times S^{n-1})$, $\partial_+ M = \Phi^{-1}(\{\theta_+\} \times S^{n-1})$. Since Φ maps ∂M to $S^{n-1} \times \{\theta_-, \theta_+\}$, it is easy to check that $\partial M = \partial_+ M \sqcup \partial_- M$.

Lemma 3.3.3. For all $n \in \mathbb{N} - \{0\}$, there exists a smooth map $h : S^{n-1} \times S^1 \to S^n$ that satisfies

$$h^* g_{S^n} \leq g_{S^{n-1}} + 4 g_{S^1}$$

Proof. This is an analogous statement to lemma 3.2.7. Note that we forgo 1-contracting to get 4-Lipschitz. As this does not lie in the technical heart of this thesis, we refer the conscientious reader to [BBHW24, Lemma 2.3] for a proof. □

Remark. The constant 4 here does not make a difference. As the argument develops, we will restrict h to a fiber $S^{n-1} \times \{t\}$.

Take a map $h : S^{n-1} \times S^1 \to S^n$ from lemma 3.3.3. Let $\tilde{f}$ be given by:

$$\tilde{f} : M \times S^1 \xrightarrow{\varphi \times id} S^{n-1} \times S^1 \xrightarrow{h} S^n \tag{3.29}$$

where $\varphi : M \to S^{n-1}$ is the projection defined by (3.27).

In the proof of classical Llarull's estimate, one would like to invoke Weitzenböck formula, which would involve integration on $\partial \tilde{M}$. Note that we assume ∂M is closed, so we can try to relate the argument to the index of the Dirac operator on $\partial \tilde{M}$.

We fix a vector field $T = \frac{1}{r} \frac{1}{\partial t}$ on $\tilde{M}$, where $t \mapsto (\cos t, \sin t)$ is the canonical coordinate on S^1-component of $\tilde{M}$.

We can write the warped product on $[\theta_-, \theta_+] \times S^{n-1}$ as $d\theta \otimes d\theta + \rho(\theta)^2 g_{S^{n-1}}$. Since Φ is 1-Lipschitz, we have

$$g \geq \Phi^*(d\theta \otimes d\theta + \rho(\theta)^2 g_{S^{n-1}})$$

For a tangent vector $X \in T_p M$, $d\Phi(X) = (d\varphi(X), d\Theta(X))$. Then $d\theta((d\varphi(X), d\Theta(X))) = d\Theta(X)$. Hence $\Phi^*(d\theta \otimes d\theta + \rho(\theta)^2 g_{S^{n-1}})(X) = d\Theta(X)^2 + \rho(\Theta)^2 g_{S^{n-1}}(d\varphi(X), d\varphi(X))$. Therefore we can write

$$g \geq d\Theta \otimes d\Theta + \rho(\Theta)^2 \varphi^* g_{S^{n-1}} \tag{3.30}$$

Take the gradient vector of Θ, denoted by $\nabla\Theta$. We have that $d\Theta(\nabla\Theta) = g(\nabla\Theta, \nabla\Theta) = |\nabla\Theta|^2$. Evaluate the inequality (3.30) at $\nabla\Theta$ we get

$$|\nabla\Theta|^2 \geq |\nabla\Theta|^4 + \rho(\Theta)^2 |d\varphi(\nabla\Theta)|^2_{g_{S^{n-1}}}$$

Note that ρ is assumed to be positive. Hence we have proven

Lemma 3.3.4. $|\nabla\Theta| \leq 1$, and the inequality is strict unless $d\varphi(\nabla\Theta) = 0$.

By lemma 2.7.1, the scalar curvature of the warped product g_0 on $[\theta_-, \theta_+] \times S^{n-1}$ can be calculated as

$$\mathrm{scal}^{g_0} = (n-1)\left(-2\frac{\rho''(\theta)}{\rho(\theta)} + (n-2)\frac{1-\rho'(\theta)^2}{\rho(\theta)^2}\right) \tag{3.31}$$

Since $g \geq g_0$ we can use the definition of ψ to calculate the following:

Lemma 3.3.5.

$$\mathrm{scal}^g \geq (n-1)\left(-2\psi'(\Theta) - n\psi(\theta)^2 + (n-2)\frac{1}{\rho(\theta)^2}\right)$$

Proof. Follows from the definition $\psi(\theta) = \frac{\rho'(\theta)}{\rho(\theta)}$. □

3.4 The holographic index theorem

On a manifold Q with non-empty boundary, we have seen in lemma 2.6.6 that the spinor bundle of ∂Q is related to the restriction $\Sigma Q|_{\partial Q}$ depending on the dimension of Q. Let D be the Dirac operator on ΣQ. Let D^∂ be the Dirac operator on $\Sigma(\partial Q)$. Let ν be a fixed outward normal vector field about ∂Q on Q.

Theorem 3.4.1 (Holographic index theorem)**.** Let $N_1, \ldots, N_k$ be the connected components of ∂Q. Suppose that $\varepsilon_1, \ldots, \varepsilon_k$ are integers in $\{-1, 1\}$. Suppose S^+ is the subbundle of $\Sigma Q|_{\partial Q}$ such that at $x \in N_j$ the fiber is the eigenspace of $(i\nu\cdot)$ with eigenvalue ε_j. Similarly, suppose S^- is defined with eigenvalue $-\varepsilon_j$. Then $\Sigma Q|_{\partial Q} = S^+ \oplus S^-$ and

$$\mathrm{ind}(D : C^\infty_+(Q, \Sigma Q) \to C^\infty(Q, \Sigma Q)) = \sum_{\varepsilon_j = 1} \mathrm{ind}(D^\partial : C^\infty(N_j, S^+) \to C^\infty(N_j, S^-))$$

Proof. See [BBHW24, Appendix B, Corollary B.3] for a more general edition. □

If Q has odd dimension, then ∂Q has even dimension, and $\Sigma(\partial Q) = \Sigma Q|_{\partial Q}$. The complex volume form ω in $\mathbb{C}l(\partial Q)$ splits $\Sigma(\partial Q)$ into ± 1 eigen-subbundles. Somehow this is equivalent to splitting by $i\nu$ in $\Sigma Q|_{\partial Q}$.

3.5 The perturbed connection

Let $\tilde{P}$ be given by

$$\tilde{P}_X u = \nabla^{\tilde{S} \otimes E^+}_{X - \langle X, T\rangle T} u + \frac{i}{2} \Psi \cdot (X - \langle X, T\rangle T) \cdot u \tag{3.32}$$

This looks like a connection but actually is not. We will need it later on.

3.6 Proof of even-dimensional case

Consider the case where n is even. Fix a real number $r > 1$. Write $\tilde{M} = M \times S^1$ with the metric $g + r^2 g_{S^1}$. This is similar to the odd-dimensional case of 3.2.3.

Let $\partial\tilde{M} = \partial_-\tilde{M} \sqcup \partial_+\tilde{M}$.

Let $p : \tilde{M} \to M$ be the projection to M. Fix the spinor bundle $S \to M$. Consider the pullback $\tilde{S} := p^* S$ on $\tilde{M}$. By lemma 2.5.10, $\tilde{S}$ is isomorphic as associated vector bundles to the spinor bundle on $\tilde{M}$, where the factor S^1 is equipped with the spin structure $S^1 \sqcup S^1 \to S^1$.

Let $E_0 \to S^n$ be the spinor bundle on S^n. Since n is even, the spinor representation is decomposable and there is a decomposition $E_0 = E_0^+ \oplus E_0^-$ by lemma 2.4.9. Let $E = \tilde{f}^* E_0$ be the pullback of E_0 under $\tilde{f}$, which satisfies $E = \tilde{f}^*(E_0^+ \oplus E_0^-) = \tilde{f}^* E_0^+ \oplus \tilde{f}^* E_0^-$. Write $E^+ = \tilde{f}^* E_0^+$ and $E^- = \tilde{f}^* E_0^-$. The bundle $\tilde{S} \otimes E^+$ is a twisted spinor bundle on $\tilde{M}$. Let $\nabla^{\tilde{S}\otimes E^+}$ be the twisted spinor connection on $\tilde{S} \otimes E^+$, where the connection on E^+ is the pullback connection induced from that of E_0^+. As in the classical case, we have the twisted Dirac operator D_{E^+}.

Choose an outward unit normal vector field ν on $\tilde{M}$. Given a local orthonormal basis $e_1, \ldots, e_n$ of $T\partial\tilde{M}$ (note that M is n-dimensional, so $\tilde{M}$ is $(n+1)$-dimensional, so $\partial\tilde{M}$ is n-dimensional), define the **boundary Dirac operator** acting on sections of $\tilde{S} \otimes E^+$ by

$$D^{\partial\tilde{M}} u = \sum_{i=1}^{n} \nu \cdot e_i \cdot \nabla_{e_i}^{\tilde{S}\otimes E^+} u + \frac{1}{2} H u \tag{3.33}$$

By the discussion on hypersurfaces in [Bä11, Section 2.6], this operator is well-defined on the whole boundary $\partial\tilde{M}$, and corresponds to the classical Dirac operator on $\Sigma\partial\tilde{M}$, which is identified with $\Sigma\tilde{M}|_{\partial\tilde{M}}$.

3.6.1 An adjointness formula for Dirac operator

We perform a computation similar to the proof of self-adjointness of the Dirac operator on a closed manifold. The result is a long expression involving integration on the boundary. The reader is advised to jump straight to Lemma 3.6.2, and never refer to the workings in this subsection unless in absolute need.

For fixed twisted spinor fields V, W, fix a vector field X on $\tilde{M}$ such that for any vector field Y we have

$$\langle X, Y \rangle = \langle Y \cdot V, W \rangle \tag{3.34}$$

in other words we take the dual of the 1-form $\langle (-) \cdot V, W \rangle_{\tilde{S}\otimes E^+}$ with respect to the metric on TM. Denote the Levi-Civita connection as ∇^{LC} and the twisted spinor connection as ∇, we compute the divergence of X:

$$\begin{aligned}
\operatorname{div} X &= \sum_i \langle \nabla_{e_i}^{LC} X, e_i \rangle \\
&= \sum_i (\partial_i \langle X, e_i \rangle - \langle X, \nabla_{e_i}^{LC} e_i \rangle) \\
&= \sum_i (\partial_i \langle e_i \cdot V, W \rangle - \langle (\nabla_{e_i}^{LC} e_i) \cdot V, W \rangle) \\
&= \sum_i (\langle \nabla_{e_i}^{LC} e_i \cdot V, W \rangle + \langle e_i \cdot \nabla_{e_i} V, W \rangle + \langle e_i \cdot V, \nabla_{e_i} W \rangle - \langle \nabla_{e_i}^{LC} e_i \cdot V, W \rangle) \\
&= \sum_i (\langle e_i \cdot \nabla_{e_i} V, W \rangle - \langle V, e_i \cdot \nabla_{e_i} W \rangle) \\
&= \langle D_{E^+} V, W \rangle - \langle V, D_{E^+} W \rangle
\end{aligned}$$

Integrate both sides. By the divergence theorem we have

$$\int_{\partial\tilde{M}} \langle X, \nu \rangle = \int_{\tilde{M}} \langle D_{E^+} V, W \rangle - \langle V, D_{E^+} W \rangle \tag{3.35}$$

$$\int_{\partial\tilde{M}} \langle \nu \cdot V, W \rangle = \int_{\tilde{M}} \langle D_{E^+} V, W \rangle - \langle V, D_{E^+} W \rangle \tag{3.36}$$

for some outward normal ν. Let $W = D_{E^+}V$ we get

$$\int_{\partial\tilde{M}} \langle \nu \cdot V, D_{E^+}V\rangle = \int_{\tilde{M}} |D_{E^+}V|^2 - \langle V, D_{E^+}^2 V\rangle \tag{3.37}$$

We use a similar technique to derive a local formula for the connection Laplacian $(\nabla^{\tilde{S}\otimes E^+})^*\nabla^{\tilde{S}\otimes E^+}$ for $\tilde{S}\otimes E^+$. Let V,W be fixed twisted spinor fields and X be the vector field dual to the 1-form $\langle V, \nabla_{(-)}W\rangle_{\tilde{S}\otimes E^+}$. We assume that we are compactly supported by orthonormal tangent frames $e_1, \dots, e_n$. Then

$$\begin{aligned}
\operatorname{div} X &= \sum_i \langle \nabla^{LC}_{e_i} X, e_i\rangle && (3.38)\\
&= \sum_i (\partial_i \langle V, \nabla_{e_i} W\rangle - \langle V, \nabla_{\nabla^{LC}_{e_i} e_i} W\rangle) && (3.39)\\
&= \sum_i (\langle \nabla_{e_i} V, \nabla_{e_i} W\rangle + \langle V, \nabla_{e_i}\nabla_{e_i} W\rangle - \langle V, \nabla_{\nabla^{LC}_{e_i} e_i} W\rangle) && (3.40)\\
&= \langle \nabla V, \nabla W\rangle + \sum_i \langle V, (\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla^{LC}_{e_i} e_i}) W\rangle && (3.41)
\end{aligned}$$

where $\sum_i \langle \nabla_{e_i} V, \nabla_{e_i} W\rangle$ is equals the induced metric $\langle \nabla V, \nabla W\rangle$ on the tensor product bundle. Integrate on both sides and use divergence theorem we get

$$\begin{aligned}
\int_{\partial\tilde{M}} \langle X, \nu\rangle &= (\nabla V, \nabla W)_{L^2} + (V, \sum_i (\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla^{LC}_{e_i} e_i}) W)_{L^2} && (3.42)\\
\int_{\partial\tilde{M}} \langle V, \nabla_\nu W\rangle &= (\nabla V, \nabla W)_{L^2} + (V, \sum_i (\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla^{LC}_{e_i} e_i}) W)_{L^2} && (3.43)\\
\int_{\partial\tilde{M}} \langle V, \nabla_\nu W\rangle &= (\nabla V, \nabla W)_{L^2} - (V, \nabla^*\nabla W)_{L^2} && (3.44)
\end{aligned}$$

Next we substitute the Weitzenböck formula into the adjointness identity (3.37):

$$\int_{\partial\tilde{M}} \langle \nu \cdot V, D_{E^+}V\rangle = \int_{\tilde{M}} |D_{E^+}V|^2 - \langle V, \nabla^*\nabla V + \frac{1}{4}\operatorname{scal}^g V + \mathcal{R}^{E^+} V\rangle \tag{3.45}$$

Next we substitute the boundary connection Laplacian identity (3.44) to the above:

$$\begin{aligned}
&\int_{\tilde{M}} |D_{E^+}V|^2 - \int_{\tilde{M}} |\nabla V|^2 - \frac{1}{4}\int_{\tilde{M}} \operatorname{scal}^g |V|^2 - \int_{\tilde{M}} \langle V, \mathcal{R}^{E^+} V\rangle\\
&= -\int_{\partial\tilde{M}} \langle V, \nu \cdot D_{E^+}V\rangle - \int_{\partial\tilde{M}} \langle V, \nabla_\nu V\rangle
\end{aligned}$$

We use the definition for boundary Dirac operator (3.33) to get

$$\nu \cdot D_{E^+} u = D^{\partial\tilde{M}} u - \nabla_\nu^{\tilde{S}\otimes E^+} u - \frac{1}{2} H u$$

on $\partial\tilde{M}$ only. Nevertheless we can substitute $V = u$ into the integration on $\partial\tilde{M}$ and get

$$\begin{aligned}
&\int_{\tilde{M}} |D_{E^+}V|^2 - \int_{\tilde{M}} |\nabla V|^2 - \frac{1}{4}\int_{\tilde{M}} \operatorname{scal}^g |V|^2 - \int_{\tilde{M}} \langle V, \mathcal{R}^{E^+} V\rangle\\
&= -\int_{\partial\tilde{M}} \langle V, D^{\partial\tilde{M}} V - \nabla_\nu^{\tilde{S}\otimes E^+} V - \frac{1}{2} H V\rangle - \int_{\partial\tilde{M}} \langle V, \nabla_\nu V\rangle\\
&= -\int_{\partial\tilde{M}} \langle V, D^{\partial\tilde{M}} V\rangle + \frac{1}{2}\int_{\partial\tilde{M}} H|V|^2\\
&= -\frac{1}{2}\int_{\partial\tilde{M}} \langle V, D^{\partial\tilde{M}} V\rangle - \frac{1}{2}\int_{\partial\tilde{M}} \langle D^{\partial\tilde{M}} V, V\rangle + \frac{1}{2}\int_{\partial\tilde{M}} H|V|^2
\end{aligned} \tag{3.46}$$

where in the last step we used the fact that $D^{\partial\tilde{M}}$ is self-adjoint.

Lemma 3.6.1. $D^{\partial\tilde{M}}$ anti-commutes with Clifford multiplication by ν.

Proof. Let $e_1, \ldots, e_n$ be an orthonormal frame of $\partial\tilde{M}$. Write $D = D^{\partial\tilde{M}}$. Then

$$\begin{aligned}
\nu \cdot Du =& \nu \cdot \sum_{i=1}^{n} \nu \cdot e_i \cdot \nabla_{e_i} u + \frac{1}{2} H\nu \cdot u = -\sum_{i=1}^{n} e_i \cdot \nabla_{e_i} u + \frac{1}{2} H\nu \cdot u \\
D\nu \cdot u =& \sum_{i=1}^{n} \nu \cdot e_i \cdot \nabla_{e_i}(\nu \cdot u) + \frac{1}{2} H(\nu \cdot u) \\
=& \sum_{i=1}^{n} (\nu \cdot e_i \cdot \nabla^{LC}_{e_i} \nu \cdot u + e_i \cdot \nabla_{e_i} u) + \frac{1}{2} H\nu \cdot u \\
D\nu \cdot u + \nu \cdot Du =& H\nu \cdot u + \sum_{i=1}^{n} \nu \cdot e_i \cdot \nabla^{LC}_{e_i} \nu \cdot u
\end{aligned}$$

Then we make $\{e_i\}$ an orthogonal eigenbasis of the Weingarten map S. This is valid because the definition of D is independent of the choice of orthonormal basis. Then

$$\begin{aligned}
\sum_{i=1}^{n} \nu \cdot e_i \cdot \nabla^{LC}_{e_i} \nu \cdot u =& \sum_{i=1}^{n} \nu \cdot e_i \cdot S(e_i) \cdot u \\
=& \sum_{i=1}^{n} \nu \cdot e_i \cdot \lambda_i e_i \cdot u \\
=& -(\mathrm{tr} S)\nu \cdot u \\
=& -H\nu \cdot u
\end{aligned}$$

where λ_i are the eigenvalues of S. The lemma then follows immediately. □

We get from the above assertion, together with self-adjointness that

$$\langle \nu \cdot u, D^{\partial\tilde{M}} u \rangle = \langle D^{\partial\tilde{M}}(\nu \cdot u), u \rangle = -\langle \nu \cdot D^{\partial\tilde{M}} u, u \rangle = \langle D^{\partial\tilde{M}} u, \nu \cdot u \rangle$$

Hence

$$\langle D^{\partial\tilde{M}} u, \nu \cdot u \rangle - \langle \nu \cdot u, D^{\partial\tilde{M}} u \rangle = 0$$

$$\langle D^{\partial\tilde{M}} u, i\nu \cdot u \rangle + \langle i\nu \cdot u, D^{\partial\tilde{M}} u \rangle = 0$$

$$\int_{\partial_\pm \tilde{M}} \langle D^{\partial\tilde{M}} u, i\nu \cdot u \rangle + \int_{\partial_\pm \tilde{M}} \langle i\nu \cdot u, D^{\partial\tilde{M}} u \rangle = 0 \tag{3.47}$$

Next we want to relate $\tilde{P}$ with D_{E^+}. Given a local orthonormal basis $e_1, \ldots, e_n, T$ of $T\tilde{M}$, we calculate

$$
\begin{aligned}
|\tilde{P}u|^2 =& \sum_{i=1}^{n} |\tilde{P}_{e_i}u|^2 + |\tilde{P}_T u|^2 \\
=& \sum_{i=1}^{n} |\nabla_{e_i}u + \frac{i}{2}\Psi \cdot e_i \cdot u|^2 \\
=& |\nabla u|^2 - |\nabla_T u|^2 + \sum_{i=1}^{n}\langle \nabla_{e_i}u, \frac{i}{2}\Psi \cdot e_i \cdot u\rangle + \sum_{i=1}^{n}\langle \frac{i}{2}\Psi \cdot e_i \cdot u, \nabla_{e_i}u\rangle + \frac{n}{4}\Psi^2|u|^2 \\
=& |\nabla u|^2 - |\nabla_T u|^2 + \frac{n}{4}\Psi^2|u|^2 + \frac{i}{2}\Psi\sum_{i=1}^{n}\langle \nabla_{e_i}u, e_i \cdot u\rangle - \frac{i}{2}\Psi\sum_{i=1}^{n}\langle e_i \cdot u, \nabla_{e_i}u\rangle \\
=& |\nabla u|^2 - |\nabla_T u|^2 + \frac{n}{4}\Psi^2|u|^2 - \frac{i}{2}\Psi\langle D_{E^+}u - T\cdot\nabla_T u, u\rangle + \frac{i}{2}\Psi\langle u, D_{E^+}u - T\cdot\nabla_T u\rangle \\
=& |\nabla u|^2 - |\nabla_T u|^2 + \frac{n}{4}\Psi^2|u|^2 \\
& - \frac{i}{2}\Psi\langle D_{E^+}u, u\rangle + \frac{i}{2}\Psi\langle u, D_{E^+}u\rangle + \frac{i}{2}\Psi\partial_T\langle T\cdot u, u\rangle - \frac{i}{2}\Psi\langle (\nabla^{LC}_T T)\cdot u, u\rangle \\
=& |\nabla u|^2 - |\nabla_T u|^2 + \frac{n}{4}\Psi^2|u|^2 - \frac{i}{2}\Psi\langle D_{E^+}u, u\rangle + \frac{i}{2}\Psi\langle u, D_{E^+}u\rangle + \frac{i}{2}\Psi\partial_T\langle T\cdot u, u\rangle
\end{aligned}
$$

where in the last step we used the obvious fact that T is a parallel vector field along the S^1 factor; in fact T is parallel.

Integrating over $\tilde{M}$ we get

$$
\int_{\tilde{M}} |\tilde{P}u|^2 = \int_{\tilde{M}} |\nabla u|^2 - |\nabla_T u|^2 + \frac{n}{4}\Psi^2|u|^2 - \frac{i}{2}\Psi\langle D_{E^+}u, u\rangle + \frac{i}{2}\Psi\langle u, D_{E^+}u\rangle + \frac{i}{2}\Psi\partial_T\langle T\cdot u, u\rangle \tag{3.48}
$$

Using divergence theorem, we calculate that the last term is

$$
\int_{\tilde{M}} \Psi\partial_T\langle T\cdot u, u\rangle = \int_{\partial\tilde{M}} \Psi\langle T\cdot u, u\rangle\langle T, \nu\rangle - \int_{\tilde{M}} \partial_T(\Psi)\langle T\cdot u, u\rangle + \Psi\langle T\cdot u, u\rangle \operatorname{div} T = 0
$$

where we use the observations that $\langle T, \nu\rangle = 0, \operatorname{div} T = 0, \partial_T(\Psi) = 0$.

Substitute (3.47) into (3.46). But we separate the integration on $\partial\tilde{M}$ by $\partial_+\tilde{M}$ and $\partial_-\tilde{M}$. We get

$$
\begin{aligned}
& \int_{\tilde{M}} |D_{E^+}V|^2 - \int_{\tilde{M}} |\nabla V|^2 - \frac{1}{4}\int_{\tilde{M}} \operatorname{scal}^g |V|^2 - \int_{\tilde{M}} \langle V, \mathcal{R}^{E^+}V\rangle \\
=& -\frac{1}{2}\int_{\partial_+\tilde{M}} \langle V + i\nu\cdot V, D^{\partial\tilde{M}}V\rangle - \frac{1}{2}\int_{\partial_+\tilde{M}} \langle D^{\partial\tilde{M}}V, V + i\nu\cdot V\rangle \\
& -\frac{1}{2}\int_{\partial_-\tilde{M}} \langle V - i\nu\cdot V, D^{\partial\tilde{M}}V\rangle - \frac{1}{2}\int_{\partial_-\tilde{M}} \langle D^{\partial\tilde{M}}V, V - i\nu\cdot V\rangle + \frac{1}{2}\int_{\partial\tilde{M}} H|V|^2
\end{aligned} \tag{3.49}
$$

Then we substitute equation (3.48) into the above and get

$$\begin{aligned}
&\int_{\tilde{M}} |D_{E^+}V|^2 - \int_{\tilde{M}} |\tilde{P}V|^2 - \frac{1}{4}\int_{\tilde{M}} \mathrm{scal}^g\,|V|^2 - \int_{\tilde{M}} \langle V, \mathcal{R}^{E^+}V\rangle - \int_{\tilde{M}} |\nabla_T V|^2 \\
&+ \frac{n}{4}\int_{\tilde{M}} \Psi^2|V|^2 - \frac{i}{2}\int_{\tilde{M}} \Psi\langle D_{E^+}V, V\rangle + \frac{i}{2}\int_{\tilde{M}} \Psi\langle V, D_{E^+}V\rangle \\
=& -\frac{1}{2}\int_{\partial_+\tilde{M}} \langle V + i\nu\cdot V, D^{\partial\tilde{M}}V\rangle - \frac{1}{2}\int_{\partial_+\tilde{M}} \langle D^{\partial\tilde{M}}V, V + i\nu\cdot V\rangle \\
&- \frac{1}{2}\int_{\partial_-\tilde{M}} \langle V - i\nu\cdot V, D^{\partial\tilde{M}}V\rangle - \frac{1}{2}\int_{\partial_-\tilde{M}} \langle D^{\partial\tilde{M}}V, V - i\nu\cdot V\rangle + \frac{1}{2}\int_{\partial\tilde{M}} H|V|^2
\end{aligned} \tag{3.50}$$

We apply divergence theorem to the terms

$$-\frac{i}{2}\int_{\tilde{M}} \Psi\langle D_{E^+}V, V\rangle + \frac{i}{2}\int_{\tilde{M}} \Psi\langle V, D_{E^+}V\rangle$$

Recall the advanced version of divergence theorem (3.36)

$$\int_{\partial\tilde{M}} \langle \nu\cdot V, W\rangle = \int_{\tilde{M}} \langle D_{E^+}V, W\rangle - \langle V, D_{E^+}W\rangle$$

We calculate

$$D(\Psi u) = \Psi Du + \sum_i e_i\cdot(\partial_{e_i}\Psi)u = \Psi Du + (\nabla\Psi)\cdot u$$

Then

$$\int_{\tilde{M}} \langle \Psi Du, u\rangle + \int_{\tilde{M}} \langle (\nabla\Psi)\cdot u, u\rangle - \int_{\tilde{M}} \langle \Psi u, Du\rangle = \int_{\partial\tilde{M}} \Psi\langle \nu\cdot u, u\rangle$$

Multiply both sides by $-\frac{i(n-1)}{2}$ and add to equation (3.50) we get

$$\begin{aligned}
&\int_{\tilde{M}} |D_{E^+}u|^2 - \int_{\tilde{M}} |\tilde{P}u|^2 - \frac{1}{4}\int_{\tilde{M}} \mathrm{scal}^g\,|u|^2 - \int_{\tilde{M}} \langle u, \mathcal{R}^{E^+}u\rangle - \int_{\tilde{M}} |\nabla_T u|^2 \\
&+ \frac{n}{4}\int_{\tilde{M}} \Psi^2|u|^2 - \frac{in}{2}\int_{\tilde{M}} \Psi\langle D_{E^+}u, u\rangle + \frac{in}{2}\int_{\tilde{M}} \Psi\langle u, D_{E^+}u\rangle \\
&- \frac{i(n-1)}{2}\int_{\partial\tilde{M}} \Psi\langle (\nabla\Psi)\cdot u, u\rangle \\
=& -\frac{1}{2}\int_{\partial_+\tilde{M}} \langle u + i\nu\cdot u, D^{\partial\tilde{M}}u\rangle - \frac{1}{2}\int_{\partial_+\tilde{M}} \langle D^{\partial\tilde{M}}u, u + i\nu\cdot u\rangle \\
&- \frac{1}{2}\int_{\partial_-\tilde{M}} \langle u - i\nu\cdot u, D^{\partial\tilde{M}}u\rangle - \frac{1}{2}\int_{\partial_-\tilde{M}} \langle D^{\partial\tilde{M}}u, u - i\nu\cdot u\rangle + \frac{1}{2}\int_{\partial\tilde{M}} H|u|^2 \\
&- \frac{i(n-1)}{2}\int_{\partial\tilde{M}} \Psi\langle \nu\cdot u, u\rangle
\end{aligned} \tag{3.51}$$

We manipulate on the RHS

$$
\begin{aligned}
&\frac{1}{2}\int_{\partial\tilde{M}} H|u|^2 - \frac{i(n-1)}{2}\int_{\partial\tilde{M}} \Psi\langle\nu\cdot u, u\rangle \\
&= \frac{1}{2}\int_{\partial_+\tilde{M}}(H-(n-1)\Psi)|u|^2 + \frac{1}{2}\int_{\partial_-\tilde{M}}(H+(n-1)\Psi)|u|^2 + \frac{n-1}{2}\int_{\partial_+\tilde{M}}\Psi\langle u,u\rangle \\
&+ \frac{n-1}{2}\int_{\partial_+\tilde{M}}\Psi\langle i\nu\cdot u,u\rangle - \frac{n-1}{2}\int_{\partial_-\tilde{M}}\Psi\langle -i\nu\cdot u,u\rangle - \frac{n-1}{2}\int_{\partial_-\tilde{M}}\Psi\langle u,u\rangle \\
&= \frac{1}{2}\int_{\partial_+\tilde{M}}(H-(n-1)\Psi)|u|^2 + \frac{1}{2}\int_{\partial_-\tilde{M}}(H+(n-1)\Psi)|u|^2 \\
&+ \frac{n-1}{2}\int_{\partial_+\tilde{M}}\Psi\langle u+i\nu\cdot u,u\rangle - \frac{n-1}{2}\int_{\partial_-\tilde{M}}\Psi\langle u-i\nu\cdot u,u\rangle
\end{aligned}
\tag{3.52}
$$

Substitute back into (3.51), we get

$$
\begin{aligned}
&\int_{\tilde{M}}|D_{E^+}u|^2 - \int_{\tilde{M}}|\tilde{P}u|^2 - \frac{1}{4}\int_{\tilde{M}}\mathrm{scal}^g|u|^2 - \int_{\tilde{M}}\langle u,\mathcal{R}^{E^+}u\rangle - \int_{\tilde{M}}|\nabla_T u|^2 \\
&+ \frac{n}{4}\int_{\tilde{M}}\Psi^2|u|^2 - \frac{in}{2}\int_{\tilde{M}}\Psi\langle D_{E^+}u,u\rangle + \frac{in}{2}\int_{\tilde{M}}\Psi\langle u, D_{E^+}u\rangle \\
&- \frac{i(n-1)}{2}\int_{\partial\tilde{M}}\Psi\langle(\nabla\Psi)\cdot u,u\rangle \\
&= -\frac{1}{2}\int_{\partial_+\tilde{M}}\langle u+i\nu\cdot u, D^{\partial\tilde{M}}u\rangle - \frac{1}{2}\int_{\partial_+\tilde{M}}\langle D^{\partial\tilde{M}}u, u+i\nu\cdot u\rangle \\
&+ \frac{1}{2}\int_{\partial_+\tilde{M}}(H-(n-1)\Psi)|u|^2 + \frac{n-1}{2}\int_{\partial_+\tilde{M}}\Psi\langle u+i\nu\cdot u,u\rangle \\
&- \frac{1}{2}\int_{\partial_-\tilde{M}}\langle u-i\nu\cdot u, D^{\partial\tilde{M}}u\rangle - \frac{1}{2}\int_{\partial_-\tilde{M}}\langle D^{\partial\tilde{M}}u, u-i\nu\cdot u\rangle \\
&+ \frac{1}{2}\int_{\partial_-\tilde{M}}(H+(n-1)\Psi)|u|^2 - \frac{n-1}{2}\int_{\partial_-\tilde{M}}\Psi\langle u-i\nu\cdot u,u\rangle
\end{aligned}
\tag{3.53}
$$

Notice that

$$
\begin{aligned}
&|Du|^2 - \frac{in}{2}\Psi\langle Du,u\rangle + \frac{in}{2}\Psi\langle u,Du\rangle + \frac{n}{4}\Psi^2|u|^2 \\
=&|Du|^2 - \frac{in}{2}\Psi\langle Du,u\rangle + \frac{in}{2}\Psi\langle u,Du\rangle + \frac{n^2}{4}\Psi^2|u|^2 - \frac{n(n-1)}{4}\Psi^2|u|^2 \\
=&\langle Du,Du\rangle + \langle Du, -\frac{in}{2}\Psi u\rangle + \langle -\frac{in}{2}\Psi u, Du\rangle + \langle -\frac{in}{2}\Psi u, -\frac{in}{2}\Psi u\rangle - \frac{n(n-1)}{4}\Psi^2|u|^2 \\
=&|Du - \frac{in}{2}\Psi u|^2 - \frac{n(n-1)}{4}\Psi^2|u|^2
\end{aligned}
\tag{3.54}
$$

Finally, substituting (3.54) into (3.53), we arrive at the fruitful expression relating integration on $\tilde{M}$ and the boundary $\partial\tilde{M}$, involving the Dirac operator D_{E^+} and the boundary Dirac operator $D^{\partial\tilde{M}}$.

Lemma 3.6.2. Let $u \in C^\infty(\tilde{S} \otimes E^+)$ over $\tilde{M}$. Then

$$\begin{aligned}
&\int_{\tilde{M}} |D_{E^+}u - \frac{in}{2}\Psi u|^2 - \int_{\tilde{M}} |\tilde{P}u|^2 - \int_{\tilde{M}} |\nabla_T^{\tilde{S}\otimes E^+} u|^2 - \frac{1}{4}\int_{\tilde{M}} \mathrm{scal}^g\, |u|^2 \\
&- \int_{\tilde{M}} \langle \mathcal{R}^{E^+} u, u\rangle - \frac{n(n-1)}{4}\int_{\tilde{M}} \Psi^2 |u|^2 + \frac{i(n-1)}{2}\int_{\tilde{M}} \langle (\nabla\Psi)\cdot u, u\rangle \\
&= -\frac{1}{2}\int_{\partial_+\tilde{M}} \langle D^{\partial\tilde{M}} u, u + i\nu\cdot u\rangle - \frac{1}{2}\int_{\partial_+\tilde{M}} \langle u + i\nu\cdot u, D^{\partial\tilde{M}} u\rangle \\
&+ \frac{1}{2}\int_{\partial_+\tilde{M}} (H - (n-1)\Psi)|u|^2 + \frac{n-1}{2}\int_{\partial_+\tilde{M}} \Psi\langle u + i\nu\cdot u, u\rangle \\
&- \frac{1}{2}\int_{\partial_-\tilde{M}} \langle D^{\partial\tilde{M}} u, u - i\nu\cdot u\rangle - \frac{1}{2}\int_{\partial_-\tilde{M}} \langle u - i\nu\cdot u, D^{\partial\tilde{M}} u\rangle \\
&+ \frac{1}{2}\int_{\partial_-\tilde{M}} (H + (n-1)\Psi)|u|^2 - \frac{n-1}{2}\int_{\partial_-\tilde{M}} \Psi\langle u - i\nu\cdot u, u\rangle
\end{aligned} \tag{3.55}$$

3.6.2 Index calculation

We need an index calculation similar to that of Llarull's as described in section 3.2.2.

Lemma 3.6.3. Consider the index of the following operators:

(1) Let ind_1 denote the index of the Dirac operator on $\tilde{S} \otimes E^+$ with boundary conditions $u + i\nu\cdot u = 0$ on $\partial_+\tilde{M}$ and $u - i\nu\cdot u = 0$ on $\partial_-\tilde{M}$.
(2) Let ind_2 denote the index of the Dirac operator on $\tilde{S} \otimes E^+$ with boundary conditions $u - i\nu\cdot u = 0$ on $\partial_+\tilde{M}$ and $u + i\nu\cdot u = 0$ on $\partial_-\tilde{M}$.
(3) Let ind_3 denote the index of the Dirac operator on $\tilde{S} \otimes E^-$ with boundary conditions $u + i\nu\cdot u = 0$ on $\partial_+\tilde{M}$ and $u - i\nu\cdot u = 0$ on $\partial_-\tilde{M}$.
(4) Let ind_4 denote the index of the Dirac operator on $\tilde{S} \otimes E^-$ with boundary conditions $u - i\nu\cdot u = 0$ on $\partial_+\tilde{M}$ and $u + i\nu\cdot u = 0$ on $\partial_-\tilde{M}$.

Proof. By the calculation in **Example 7**, we have $\mathrm{ind}_1 + \mathrm{ind}_2 = 0$ because the pair of operators have boundary conditions adjoint to each other.

Suppose that $\mathrm{ind}_1 = \mathrm{ind}_2 = \mathrm{ind}_3 = \mathrm{ind}_4 = 0$. The spinor bundle on $\partial_-\tilde{M}$ is isomorphic to $\tilde{S}|_{\partial_-\tilde{M}}$. Since $\partial_-\tilde{M}$ is even-dimensional, we have a decomposition of the spinor bundle $\tilde{S} = S^+ \oplus S^-$. We identify restriction of a vector bundle with pullback via inclusion. Then we can calculate

$$\begin{aligned}
(\tilde{S} \otimes E^+)|_{\partial_-\tilde{M}} &= \tilde{S}|_{\partial_-\tilde{M}} \otimes \tilde{f}^* E_0^+|_{\partial_-\tilde{M}} \\
&= (S^+ \oplus S^-) \otimes (\tilde{f}|_{\partial_-\tilde{M}})^* E_0^+ \\
&= S^+ \otimes (\tilde{f}|_{\partial_-\tilde{M}})^* E_0^+ \oplus S^- \otimes (\tilde{f}|_{\partial_-\tilde{M}})^* E_0^+
\end{aligned}$$

The same calculation applies to $\partial_+\tilde{M}$.

Let $e_1, \ldots, e_n, \nu$ be a local orthonormal frame over a neighbourhood meeting $\partial_-\tilde{M}$. Recall that $\mathbb{C}l_{n+1}$ acts on the spinor space Σ_n via ($\bullet$), where ν acts by the complex volume element ω. The splitting $S^+ \oplus S^-$ can be regarded as a splitting by the eigensections of ω, hence by the eigensections of ν (or $i\nu$) when embedded into $\mathbb{C}l_{n+1}$. Recall that the eigenvalues are ± 1, with Σ^+ being the $+1$ part.

Now consider the Dirac operator associated to $\mathrm{ind}_1 = 0$. By theorem 3.4.1 this index equals the sum of indices of boundary Dirac operators on connected components of $\partial\tilde{M}$ where S^+ is the $+1$ eigenbundle of $i\nu$. Of the two components $\partial_\pm\tilde{M}$, $\partial_-\tilde{M}$ has this property, since we have boundary condition $u = i\nu \cdot u$ on it. Hence the boundary Dirac operator on $\partial_-\tilde{M}$ that goes from $S^+ \otimes (\tilde{f}|_{\partial_-\tilde{M}})^* E_0^+$ to $S^- \otimes (\tilde{f}|_{\partial_-\tilde{M}})^* E_0^+$ has index 0.

On the other hand, we can compute this index using the Atiyah-Singer index theorem. Denote the boundary Dirac operator as $D^{\partial_-\tilde{M}}$. We have

$$\begin{aligned} 0 = \mathrm{ind}(D_+^{\partial_-\tilde{M}}) &= \langle \hat{A}(\partial_- M) \cup ch((\tilde{f}|_{\partial_-\tilde{M}})^* E_0^+), [\partial_- M] \rangle \\ &= \mathrm{rk}\, E_0^+ \cdot \hat{\mathcal{A}}(\partial_-\tilde{M}) + \deg(\tilde{f}|_{\partial_-\tilde{M}}) \cdot \langle ch(E_0^+), [S^n] \rangle \end{aligned}$$

where we have written $ch(E_0^+)$ instead of the top Chern class when computing the pairing with $[S^n]$.

The same argument can be used on $\mathrm{ind}_3 = 0$ to obtain

$$0 = \mathrm{rk}\, E_0^- \cdot \hat{\mathcal{A}}(\partial_-\tilde{M}) + \deg(\tilde{f}|_{\partial_-\tilde{M}}) \cdot \langle ch(E_0^-), [S^n] \rangle$$

Subtract the two equations, we get

$$0 = \deg(\tilde{f}|_{\partial_-\tilde{M}}) \cdot \langle ch(E_0^+) - ch(E_0^-), [S^n] \rangle$$

By a familiar fact of [LM89, Proposition 11.24, Chapter III], $\langle ch(E_0^+) - ch(E_0^-), [S^n] \rangle \neq 0$ because it is a non-zero multiple of the Euler characteristic $\chi(S^n) = 2$. So we are forced to say $\deg(\tilde{f}|_{\partial_-\tilde{M}}) = 0$. But $\tilde{f} = h \circ (\varphi|_{\partial_- M} \times id)$. This is a contradiction as both h and $\varphi|_{\partial_- M} \times id$ have non-zero degrees, by lemma 3.6.4 below. □

Lemma 3.6.4. Let $\varphi : M \to S^{n-1}$ be defined as the projection $p_{S^{n-1}} \circ \Phi$, see definition (3.27). Then $\deg(\Phi) = \pm \deg(\varphi|_{\partial_- M})$. In particular $\varphi|_{\partial_- M}$ has non-zero degree.

Proof. Recall that we write $\Phi = (\varphi, \Theta)$.

By a connectedness argument, we have that $\partial_- M$ is a union of connected components of ∂M. In other words, if $p \in \partial M \cap \Phi^{-1}(S^{n-1} \times \{\theta_-\})$, then the connected component of p in ∂M all maps to θ_-.

Let us homotope Φ to a more desirable form. Recall that $\Phi(\partial_- M) \subseteq S^{n-1} \times \{\theta_-\}$. The containment may be strict, and $\Phi^{-1}(S^{n-1} \times \{\theta_-\}) = \Theta^{-1}(\{\theta_-\})$ is a closed neighbourhood of $\partial_- M$. Since M is orientable, let us assume $\Theta^{-1}(\{\theta_-\})$ is a compact subset of a collar neighbourhood of ∂M, i.e. there exists a smooth embedding $\iota : \partial_- M \times [0, \epsilon) \to U \subset M$ where U is an open neighbourhood of $\partial_- M$, and $\Theta^{-1}(\{\theta_-\}) \subset U$. Note that U deformation retracts to $\partial_- M$. Then $\Phi|_U$ can be expressed as

$$\Phi(x, t) = (\varphi(x, t), \Theta(x, t))$$

Consider the function Θ. For each $x \in \partial_- M$ there is a maximal t_x such that $\Theta(x, t_x) = 0$. Let $\Theta_c : \partial_- M \times [0, \epsilon) \to [\theta_-, \theta_+]$ be a homotopy of maps with respect to c, given by

$$\Theta_c(x, t) = \begin{cases} \Theta\left(x, \dfrac{t}{c}\right) & 0 \le t < ct_x \\ \Theta\left(x, t_x + \dfrac{t - ct_x}{\epsilon - ct_x}(\epsilon - t_x)\right) & ct_x \le t \le \epsilon \end{cases}$$

In particular, $\Theta_0(x,t) = \Theta(x, t_x + \frac{t}{\epsilon}(\epsilon - t_x))$ interpolates Θ from t_x to ϵ.

Define $\Phi_0 := (\varphi, \Theta_0)$. Note $\Phi \sim \Phi_0$, relative ∂M. Hence $\deg(\Phi) = \deg(\Phi_0)$. One can easily check that $\Phi_0^{-1}(S^{n-1} \times \{\theta_-\}) = \partial_- M$. By local degree formula, fix a regular point $p \in \partial_- M$, we have

$$\deg(\Phi_0) = \sum_{x \in \Phi_0^{-1}(\Phi_0(p))} \operatorname{sgn} \det(d_x \Phi_0)$$

By construction, $x \in \partial_- M$. Fix a basis $e_1, \ldots, e_{n-1}, \nu$ on $T_x M$, where $e_1, \ldots, e_{n-1}$ is ON basis of $T_x \partial_- M$ and ν is an outward normal vector field. We have that $d_x \Phi_0(e_i) \in TS^{n-1} \subset T_{(\varphi(x),\theta_-)}(S^{n-1} \times [\theta_-, \theta_+])$ (consider for example a curve in $\partial_- M$ with tangent e_i). Hence with respect to this basis, and a canonically chosen basis of $T_{(\varphi(x),\theta_-)}(S^{n-1} \times [\theta_-, \theta_+])$, $d_x \Phi_0$ has the form

$$\begin{pmatrix} d_x(\varphi|_{\partial_- M}) & * \\ 0 & d_x \Theta_0 \end{pmatrix}$$

which has determinant $\det(d_x \varphi|_{\partial_- M}) \det(d_x \Theta_0)$. We can easily make $\operatorname{sgn}(\det(d_x \Theta_0)) = \pm 1$. Plugging back into the local degree formula, we have

$$\deg(\Phi_0) = \pm \sum_{x \in \Phi_0^{-1}(\Phi_0(p))} \operatorname{sgn} \det(d_x(\varphi|_{\partial_- M})) = \pm \deg(\varphi|_{\partial_- M})$$

□

3.6.3 Existence of parallel spinor field

One might wonder why we manipulated (3.55) to include terms such as $u \pm i\nu \cdot u$ on $\partial \tilde{M}$. This is so that we have a pair of adjoint boundary conditions as discussed in section 2.9.3.

Lemma 3.6.5. Suppose r is sufficiently large. Then there is $t_0 \in S^1$ and a section $u \in C^\infty(\tilde{M}, \tilde{S} \otimes E^+)$ such that

$$\int_{M \times \{t_0\}} |u|^2 = 1$$

and

$$\int_{M \times \{t_0\}} |\tilde{P} u|^2 \leq \frac{n-1}{r}$$

Proof. By lemma 3.6.3, and the homotopy invariance of index discussed in Remark 2.6, we are able to find u such that

(1) u does not vanish identically
(2) $D_{E^+} u - \frac{in}{2} \Psi u = 0$ on $\tilde{M}$
(3) $u + i\nu \cdot u$ on $\partial_+ \tilde{M}$ and $u - i\nu \cdot u$ on $\partial_- \tilde{M}$

Let us plug in this u to equation (3.55). Note that the corresponding terms involving $u \pm i\nu \cdot u$ all vanish, the term $|D_{E^+} u - \frac{in}{2} \Psi u|$ vanishes, which yields

$$\begin{aligned} &- \int_{\tilde{M}} |\tilde{P} u|^2 - \int_{\tilde{M}} |\nabla_T^{\tilde{S} \otimes E^+} u|^2 - \frac{1}{4} \int_{\tilde{M}} \operatorname{scal}^g |u|^2 \\ &- \int_{\tilde{M}} \langle \mathcal{R}^{E^+} u, u \rangle - \frac{n(n-1)}{4} \int_{\tilde{M}} \Psi^2 |s|^2 + \frac{i(n-1)}{2} \int_{\tilde{M}} \langle (\nabla \Psi) \cdot u, u \rangle \\ &= \frac{1}{2} \int_{\partial_+ \tilde{M}} (H - (n-1)\Psi)|u|^2 + \frac{1}{2} \int_{\partial_- \tilde{M}} (H + (n-1)\Psi)|u|^2 \end{aligned} \tag{3.56}$$

By the assumption on mean curvature, which is made in theorem 3.3.2(5), we have

$$H - (n-1)\Psi = H - (n-1)\frac{\rho'(\theta_+)}{\rho(\theta_+)} \geq 0 \tag{3.57}$$

on $\partial_+ \tilde{M}$ and

$$H + (n-1)\Psi = H + (n-1)\frac{\rho'(\theta_+)}{\rho(\theta_+)} \geq 0 \tag{3.58}$$

on $\partial_- \tilde{M}$. Hence the RHS on (3.56) is nonnegative.

Using (3.30) we have the metric inequality

$$g + r^2 g_{S^1} \geq \rho(\Theta)^2 \varphi^* g_{S^{n-1}} + r^2 g_{S^1}$$

Recall our definition of $\tilde{f}$

$$\tilde{f} : M \times S^1 \xrightarrow{\varphi \times id} S^{n-1} \times S^1 \xrightarrow{h} S^n$$

where h satisfies $h^* g_{S^n} \leq g_{S^{n-1}} + 4g_{S^1}$, where S^1 has the canonical metric in $S^{n-1} \times S^1$ but $r^2 g_{S^1}$ on $M \times S^1$. So $g_{S^{n-1}} + g_{S^1} \geq h^* g_{S^n}$. Then

$$\tilde{f}^* g_{S^n} = (\varphi \times id)^* h^* g_{S^n} \leq (\varphi \times id)^* (g_{S^{n-1}} + 4g_{S^1}) \leq \varphi^* g_{S^{n-1}} + 4g_{S^1} \tag{3.59}$$

Now if $r \geq 2\rho(\Theta)$, we have $\rho(\Theta)^2 \varphi^* g_{S^{n-1}} + r^2 g_{S^1} \geq \rho(\Theta)^2 (\varphi^* g_{S^{n-1}} + 4g_{S^1})$, hence

$$\varphi^* g_{S^{n-1}} + 4g_{S^1} \leq \frac{1}{\rho(\Theta)^2}(g + r^2 g_{S^1}) \tag{3.60}$$

We proceed with a strategy similar to the singular value decomposition performed in section 3.2. Fix a regular point $(x,t) \in M \times S^1$. Let $\mu_1 \geq \cdots \geq \mu_{n+1}$ be singular values of the linear map $d\tilde{f}_{(x,t)} : T_{(x,t)}\tilde{M} \to T_{\tilde{f}(x,t)} S^n$. Since this map factors through $TS^{n-1} \times S^1$ which is rank n, and (x,t) is regular, we get that $\mu_{n+1} = 0$ and $\mu_i > 0$ when $i = 1, \ldots, n$.

Let us look at the eigenvectors of $d\tilde{f}_{(x,t)}$. If $T_t S^1$ is in the kernel, then $id_*(T_t S^1) = T_t S^1 \subset \ker h_*$. Then with respect to a regular point in the codomain S^n. h_* is of rank $n-1$ and hence not invertible. Since h is designed to be a bijection on all but one point in the target, $\deg h = 0$ by the local degree formula, contradicting the assumption that $\deg h = \pm 1$. Take $T_{(x,t)}\tilde{M}/\ker d\tilde{f}_{(x,t)}$ which is n-dimensional. Note that $T_t S^1 \hookrightarrow T_{(x,t)}\tilde{M}/\ker d\tilde{f}_{(x,t)}$.

We can arrange that:

- $e_1, \ldots, e_{n-1}$ is a g-orthonormal linearly independent set of $T_x M$ and e_n is unit tangent vector in $T_t S^1$ with respect to $r^2 g_{S^1}$. Additionally, $e_1, \ldots, e_{n-1}, e_n$ is an orthonormal basis of $T_{(x,t)}\tilde{M}/\ker d\tilde{f}_{(x,t)}$.
- $\varepsilon_1, \ldots, \varepsilon_n$ is a g_{S^n}-orthonormal basis of $T_{\tilde{f}(x,t)} S^n$.
- $d\tilde{f}_{(x,t)}(e_i) = \mu_i \varepsilon_i$ for each $i \in \{1, \ldots, n\}$.

From estimates (3.59) and (3.60) we deduce that $\forall i \in \{1, \ldots, n-1\}$

$$\mu_i \leq \frac{1}{\rho(\Theta)}$$

and

$$\mu_n \leq \frac{2}{r}$$

Lemma 3.2.3 easily generalises to this setting, which states that

Lemma 3.6.6. Let $\mu_1, \dots, \mu_n$ be the positive singular values of $d\tilde{f}_{(x,t)}$. We have

$$\langle \mathcal{R}^{E^+}\phi, \phi\rangle \geq -\frac{1}{4}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \mu_i\mu_j \langle \phi, \phi\rangle \tag{3.61}$$

for any section $\phi \in C^\infty(\tilde{S}\otimes E^+)$.

Then we have the following estimate:

$$-\langle \mathcal{R}^{E^+}\phi, \phi\rangle \leq \frac{(n-1)(n-2)}{4}\frac{1}{\rho(\Theta)^2}|\phi|^2 + \frac{(n-1)}{r}\frac{1}{\rho(\Theta)}|\phi|^2 \tag{3.62}$$

In the equation (3.56), we note that $\Psi = \psi\circ\Theta$. We can bound the term $\langle(\nabla\Psi)\cdot u, u\rangle \leq |\nabla\Psi||u|^2$ by the inequality (2.3).

We gather from all the information so far on equation (3.56). We get

$$\begin{aligned}
&\int_{\tilde{M}} |\tilde{P}u|^2 + \int_{\tilde{M}} |\nabla_T u|^2 + \frac{1}{4}\int_{\tilde{M}} \operatorname{scal}^g |u|^2 \\
&+ \int_{\tilde{M}} \langle \mathcal{R}^{E^+}u, u\rangle + \frac{n(n-1)}{4}\int_{\tilde{M}} \Psi^2|u|^2 - \frac{(n-1)}{2}\int_{\tilde{M}} \langle(\nabla\Psi)\cdot u, iu\rangle \leq 0
\end{aligned}$$

$$\begin{aligned}
&\int_{\tilde{M}} |\tilde{P}u|^2 + \int_{\tilde{M}} |\nabla_T u|^2 \\
&\leq \int_{\tilde{M}} -\frac{1}{4}\operatorname{scal}^g |u|^2 - \int_{\tilde{M}} \langle \mathcal{R}^{E^+}u, u\rangle - \frac{n(n-1)}{4}\int_{\tilde{M}} \Psi^2|u|^2 + \frac{(n-1)}{2}\int_{\tilde{M}} \langle(\nabla\Psi)\cdot u, iu\rangle
\end{aligned}$$

On RHS we have

$$\begin{aligned}
&\int_{\tilde{M}} -\frac{1}{4}\operatorname{scal}^g |u|^2 - \int_{\tilde{M}} \langle \mathcal{R}^{E^+}u, u\rangle - \frac{n(n-1)}{4}\int_{\tilde{M}} \Psi^2|u|^2 + \frac{(n-1)}{2}\int_{\tilde{M}} \langle(\nabla\Psi)\cdot u, iu\rangle \\
&\leq \int_{\tilde{M}} \left[-\frac{1}{4}\operatorname{scal}^g |u|^2 + \frac{(n-1)(n-2)}{4}\frac{1}{\rho(\Theta)^2}|u|^2 + \frac{(n-1)}{r}\frac{1}{\rho(\Theta)}|u|^2 \right. \\
&\left. + \frac{n(n-1)}{4}|\Psi|^2|u|^2 + \frac{(n-1)}{2}|\nabla\Psi||u|^2\right]
\end{aligned} \tag{3.63}$$

We estimate $|\nabla\Psi|$. Recall by lemma 3.3.4 we have $|\nabla\Theta| \leq 1$. Since $\Psi = \psi\circ\Theta$ we have $\nabla\Psi = \psi'(\Theta)\nabla\Theta$ and $|\nabla\Psi| = |\psi'(\Theta)||\nabla\Theta| \leq |\psi'(\Theta)| = -\psi'(\Theta)$ by log-concavity. Hence the RHS of the above inequality becomes

$$\begin{aligned}
&\leq \int_{\tilde{M}} \left[-\frac{1}{4}\operatorname{scal}^g |u|^2 + \frac{(n-1)(n-2)}{4}\frac{1}{\rho(\Theta)^2}|u|^2 + \frac{(n-1)}{r}\frac{1}{\rho(\Theta)}|u|^2 \right. \\
&\left. + \frac{n(n-1)}{4}\psi(\Theta)^2|u|^2 - \frac{(n-1)}{2}\psi'(\Theta)|u|^2\right] \\
&\leq \frac{n-1}{r}\int_{\tilde{M}} \frac{1}{\rho(\Theta)}|u|^2
\end{aligned} \tag{3.64}$$

where in the last step we used lemma 3.3.5. Then we can find a t_0 such that

$$\int_{M\times\{t_0\}} |\tilde{P}u|^2 \leq \frac{n-1}{r}\int_{M\times\{t_0\}} \frac{1}{\rho(\Theta)}|u|^2$$

such that u is not uniformly 0. Hence we can normalise u to $||u||^2_{L^2(M)} = \frac{1}{\int_{M\times\{t_0\}} \frac{1}{\rho(\Theta)}}$ and get

$$\int_{M\times\{t_0\}} |\tilde{P}u|^2 \leq \frac{n-1}{r} \int_{M\times\{t_0\}} \frac{1}{\rho(\Theta)} \int_{M\times\{t_0\}} |u|^2 = \frac{n-1}{r}$$

□

The flexibility of lemma3.6.5 is that r is arbitrarily large and the estimate is bound by inverse of r. Hence we can take larger and larger r to yield a spinor field on M that is "$\tilde{P}$-parallel".

Lemma 3.6.7. There exists a $t \in S^1$ with the following property. Let $f : M \to S^n$ be defined by $\tilde{f}|_{M\times\{t\}}$. Let E denote the pullback of E_0^+ under f. There is a section $s \in H^1(M, S \otimes E)$ such that

$$\int_M |s|^2 = 1$$

and

$$\nabla_X^{S\otimes E} s + \frac{i}{2}\psi(\Theta) X \cdot s = 0$$

for every vector field X. $\nabla^{S\otimes E}$ is the spinor connection on $S \otimes E$.

Proof. Consider a sequence $r_l \to \infty$. By lemma 3.6.5 we have $t_l \in S^1$ and $u_l \in C^\infty(\tilde{M}, \tilde{S} \otimes E^+)$ such that

$$\int_{M\times\{t_l\}} |u_l|^2 = 1$$

and

$$\int_{M\times\{t_l\}} |\tilde{P}u_l|^2 \leq \frac{n-1}{r_l}$$

We may assume that t_l converges to an element t. Define $f : M \to S^n$ and $f^l : M \to S^n$ by

$$f = \tilde{f}|_{M\times\{t\}}, \quad f^l = \tilde{f}|_{M\times\{t_l\}}$$

There is a pullback bundle $E^l = (f^l)^* E_0^+$ for each l. Note that $\tilde{S} \otimes E^+|_{M\times\{t_l\}} = S \otimes E^l$ so we can restrict u_l to $M \times \{t_l\}$, which is identified with M. Write $s_l = u|_{M\times\{t_l\}} \in C^\infty(M, S \otimes E^l)$, we have

$$\int_M |s_l|^2 = 1$$

Let $e_1, \dots, e_n, T$ be an orthonormal frame on $\tilde{M}$, where $e_1, \dots, e_n$ forms an orthonormal frame of M. Note that $|\tilde{P}u|^2 = \sum_{i=1}^n |\tilde{P}_{e_i} u|^2 + |\tilde{P}_T u|^2 = \sum_{i=1}^n |\tilde{P}_{e_i} u|^2 = \sum_{i=1}^n |\nabla_{e_i}^{\tilde{S}\otimes E^+} u + \frac{i}{2}\Psi e_i \cdot u|^2$. Then $\nabla_{e_i}^{\tilde{S}\otimes E^+} u = \nabla_{e_i}^{S\otimes E^l} u$ when $\{t_l\}$ is contained in the support of the local frame. We get

$$\int_M \sum_{i=1}^n |\nabla_{e_i}^{S\otimes E^l} s_l + \frac{i}{2}\Psi e_i \cdot s_l|^2 \leq \frac{n-1}{r_l} \tag{3.65}$$

Define a map $\sigma^{(l)} : E^l \to E^+$ to be pointwise given by the parallel transport along the shortest geodesic from $f^l(x)$ to $f(x)$. Then the bundle map $id \otimes \sigma^{(l)}$ is a bundle isometry from $S \otimes E^l$ to $S \otimes E$.

The sections $(id \otimes \sigma^{(l)})s_l$ satisfies

$$\nabla^{S\otimes E}((id \otimes \sigma^{(l)})s_l) = id \otimes \sigma^{(l)}(\nabla^{S\otimes E^{(l)}} s_l + A^{(l)} s_l) \tag{3.66}$$

for a field of endomorphisms $A^{(l)}$ in $\Gamma(\mathrm{End}(S \otimes E^l))$. This can be shown by calculating the linearity of $\nabla \circ (id \otimes \sigma^{(l)}) - (id \otimes \sigma^{(l)}) \circ \nabla$, in particular with respect to $C^\infty(M)$. Alternatively one can calculate principal symbols.

Since $t_l \to t$ we have $f_l \to f$ smoothly. Then $A^{(l)} \to 0$ as $l \to \infty$.

We write $\nabla^\Psi_X s = \nabla^{S\otimes E} s + \frac{i}{2}\Psi X \cdot s$.

By (3.66), $\nabla^{S\otimes E}((id\otimes\sigma^{(l)})s_l)$ is L^2. Hence $(id\otimes\sigma^{(l)})s_l$ converges weakly to some $s \in H^1(M, S\otimes E)$. We claim that

$$\int_M \langle (\nabla^\Psi)^* \phi, s\rangle = 0 \tag{3.67}$$

for every $\phi \in C^\infty_c(M, S \otimes E)$.

We expand

$$\begin{aligned} \langle \phi, \nabla^\Psi_X (id \otimes \sigma^{(l)}) s_l\rangle =& \langle \phi, \nabla^{S\otimes E}_X ((id \otimes \sigma^{(l)}) s_l) + \frac{i}{2}\Psi X \cdot (id \otimes \sigma^{(l)}) s_l\rangle \\ =& \langle \phi, id \otimes \sigma^{(l)}(\nabla^{S\otimes E^{(l)}}_X s_l + A^{(l)} s_l) + \frac{i}{2}\Psi X \cdot (id \otimes \sigma^{(l)}) s_l\rangle \end{aligned}$$

Note that $X\cdot$ is the Clifford multiplication which acts only on S. Hence $X \cdot (id \otimes \sigma^{(l)}) s_l = (id \otimes \sigma^{(l)}) X \cdot s_l$. Hence the above inner product reduces to

$$\langle \phi, id \otimes \sigma^{(l)} \left(\nabla^{S\otimes E^{(l)}}_X s_l + A^{(l)} s_l + \frac{i}{2}\Psi X \cdot s_l \right)\rangle$$

Substitute into (3.67) we get

$$\begin{aligned} & \lim_{l\to\infty} \int_M \langle \phi, id \otimes \sigma^{(l)} \left(\nabla^{S\otimes E^{(l)}}_X s_l + A^{(l)} s_l + \frac{i}{2}\Psi X \cdot s_l \right)\rangle \\ \le & \lim_{l\to\infty} ||\phi|| \int_M |id \otimes \sigma^{(l)} \left(\nabla^{S\otimes E^{(l)}}_X s_l + A^{(l)} s_l + \frac{i}{2}\Psi X \cdot s_l \right)| \\ = & \lim_{l\to\infty} ||\phi|| \int_M |\nabla^{S\otimes E^{(l)}}_X s_l + A^{(l)} s_l + \frac{i}{2}\Psi X \cdot s_l| \\ =& 0 \end{aligned}$$

where in the last step we have used $A^{(l)} \to 0$ and (3.65). This proves the claim.

With the claim, we know $\nabla^\Psi s = 0$ holds in the weak sense. By standard elliptic estimates, s is smooth and the equation holds classically. □

3.6.4 The perturbed connection ∇^Ψ

In the above argument we considered an operator similar to $\tilde{P}$ on $\tilde{M}$, namely the operator ∇^Ψ. It operates on $C^\infty(M, S \otimes E)$ and has the form

$$\nabla^\Psi_X s = \nabla^{S\otimes E}_X s + \frac{i}{2}\Psi X \cdot s$$

We establish a few lemmas about the curvature terms of ∇^Ψ, both in the ordinary sense R^{∇^Ψ} and in the sense of the Bochner identity $\mathcal{R}^{\nabla^\Psi}$. Let

$$R^{\nabla^\Psi}_{XY} \coloneqq \nabla^\Psi_X \nabla^\Psi_Y - \nabla^\Psi_Y \nabla^\Psi_X - \nabla^\Psi_{[X,Y]}$$

Lemma 3.6.8.

$$R^{\nabla^\Psi} s = R^{S\otimes E}_{XY} s + \frac{i}{2}((\partial_X \Psi)Y - (\partial_Y \Psi)X)s - \frac{1}{4}\Psi^2(XY - YX)s$$

Proof. Take a normal coordinate system of X, Y such that all Levi-Civita connection and Lie brackets vanish at a point. Compute

$$\begin{aligned}
R^{\nabla^\Psi} =& (\nabla^{S\otimes E}_X + \frac{i}{2}\Psi X)(\nabla^{S\otimes E}_Y + \frac{i}{2}\Psi Y) - (\nabla^{S\otimes E}_Y + \frac{i}{2}\Psi Y)(\nabla^{S\otimes E}_X + \frac{i}{2}\Psi X) \\
&- (\nabla^{S\otimes E}_{[X,Y]} + \frac{i}{2}\Psi XY) \\
=& R^{S\otimes E}_{XY} \\
&+ \frac{i}{2}\nabla^{S\otimes E}_X(\Psi Y) + \frac{i}{2}\Psi X(\nabla^{S\otimes E}_Y) \\
&- \frac{1}{4}\Psi^2 XY - \frac{i}{2}\nabla^{S\otimes E}_Y(\Psi X) - \frac{i}{2}\Psi Y(\nabla^{S\otimes E}_X) \\
&+ \frac{1}{4}\Psi^2 YX - \frac{i}{2}\Psi[X,Y] \\
=& R^{S\otimes E}_{XY} \\
&+ \frac{i}{2}((\partial_X\Psi)Y + \Psi(Y\nabla^{S\otimes E}_X) + \Psi X(\nabla^{S\otimes E}_Y) - (\partial_Y\Psi)X - \Psi X(\nabla^{S\otimes E}_Y) - \Psi Y(\nabla^{S\otimes E}_X))) \\
&- \frac{1}{4}\Psi^2(XY - YX) \\
=& R^{S\otimes E}_{XY} + \frac{i}{2}((\partial_X\Psi)Y - (\partial_Y\Psi)X) - \frac{1}{4}\Psi^2(XY - YX)
\end{aligned}$$

□

Next, much like the $\mathcal{R}$'s in other Bochner identities, for example lemma 2.7.7, let

$$\mathcal{R}^{\nabla^\Psi} \coloneqq \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} e_i e_j R^{\nabla^\Psi}_{e_i e_j} - \frac{1}{4} scal$$

To relate $\mathcal{R}^{\nabla^\Psi}$ to the un-perturbed version $\mathcal{R}^{S\otimes E}$, we have

Lemma 3.6.9.

$$\mathcal{R}^{\nabla^\Psi} = \mathcal{R}^{S\otimes E} - \frac{i(n-1)}{2}\nabla\Psi + \frac{n(n-1)}{4}\Psi^2$$

Proof. Evaluate the term R^{∇^Ψ} in the definition of $\mathcal{R}^{\nabla^\Psi}$:

$$\frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} e_i e_j R^{\nabla^\Psi}_{e_i e_j} = \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} e_i e_j (R^{S\otimes E}_{e_j e_j} + \frac{i}{2}((\partial_{e_i}\Psi)e_j - (\partial_{e_j}\Psi)e_i) - \frac{1}{4}\Psi^2(e_i e_j - e_j e_i))$$

Note that

$$\sum_{\substack{i,j=1\\i\neq j}}^{n} e_ie_j(\partial_{e_i}\Psi)e_j = \sum_{\substack{i,j=1\\i\neq j}}^{n} e_ie_j\langle\nabla\Psi, e_i\rangle e_j = -(n-1)\nabla\Psi$$

and that

$$\sum_{\substack{i,j=1\\i\neq j}}^{n} e_ie_j(\partial_{e_j}\Psi)e_i = -(n-1)\nabla\Psi$$

and that

$$\sum_{\substack{i,j=1\\i\neq j}}^{n} e_ie_j(e_ie_j - e_je_i) = -2n(n-1)$$

we deduce that

$$\frac{1}{2}\sum_{\substack{i,j=1\\i\neq j}}^{n} e_ie_jR^{\nabla^\Psi}_{e_ie_j} = \frac{1}{2}\sum_{\substack{i,j=1\\i\neq j}}^{n} e_ie_jR^{S\otimes E}_{e_ie_j} - \frac{i(n-1)}{2}\nabla\Psi + \frac{n(n-1)}{4}\Psi^2$$

so

$$\mathcal{R}^{\nabla^\Psi} = \mathcal{R}^{S\otimes E} - \frac{i(n-1)}{2}\nabla\Psi + \frac{n(n-1)}{4}\Psi^2$$

□

3.6.5 Completing the proof for n even

We are finally ready to prove Theorem 3.3.2 in the case where M is even-dimensional.

Proof of theorem 3.3.2, n even. Take $s \in C^\infty(M, S\otimes E)$ as defined in lemma 3.6.7. So $\nabla^\Psi s = 0$. Immediately we see $R^{\nabla^\Psi} s = 0$. By lemma 3.6.9 and inequality (2.3) we have

$$\begin{aligned}\langle\mathcal{R}^{S\otimes E}s, s\rangle &= -\frac{1}{4}scal|s|^2 + \frac{i(n-1)}{2}\langle\nabla\Psi\cdot s, s\rangle - \frac{n(n-1)}{4}\Psi^2|s|^2\\ &\leq -\frac{1}{4}scal|s|^2 + \frac{(n-1)}{2}|\nabla\Psi||s|^2 - \frac{n(n-1)}{4}\Psi^2|s|^2\\ &= -\frac{1}{4}scal|s|^2 - \frac{(n-1)}{2}\psi'(\Theta)|\nabla\Theta||s|^2 - \frac{n(n-1)}{4}\psi^2(\Theta)|s|^2\\ &\leq -\frac{1}{4}scal|s|^2 - \frac{(n-1)}{2}\psi'(\Theta)|s|^2 - \frac{n(n-1)}{4}\psi^2(\Theta)|s|^2\end{aligned}$$

From lemma 3.3.5 we have

$$-\frac{1}{4}scal \leq \frac{n-1}{2}\psi'(\Theta) + \frac{n(n-1)}{4}\psi(\Theta)^2 - \frac{(n-1)(n-2)}{4}\frac{1}{\rho(\Theta)^2}$$

Hence

$$\langle\mathcal{R}^{S\otimes E}s, s\rangle \leq -\frac{(n-1)(n-2)}{4}\frac{1}{\rho(\Theta)^2}|s|^2 \tag{3.68}$$

Take a regular point $x \in M = M\times\{t\}$. Recall the definition

$$f : M\times\{t\} \subset M\times S^1 \xrightarrow{\varphi} S^{n-1}\times\{t\} \xrightarrow{h|_{S^{n-1}\times\{t\}}} S^n$$

Fix a singular value decomposition of df_x with singular values $\mu_1 \geq \cdots \geq \mu_n$ and orthonormal basis e_i of T_xM and ε_i of $T_{f(x)}S^n$. $\mu_n = 0$ because df factors through TS^{n-1}. Now

$$f^*g_{S^n} = (h\varphi)^*g_{S^n} \leq \varphi^*g_{S^{n-1}} \leq \frac{1}{\rho(\Theta)^2}g$$

so we have $\mu_1, \ldots, \mu_{n-1} \leq \frac{1}{\rho(\Theta)}$. Lemma 3.2.3 tells us that

$$\langle \mathcal{R}^{S\otimes E}s, s\rangle \geq -\frac{1}{4}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \mu_i\mu_j|s|^2 \geq -\frac{(n-1)(n-2)}{4}\frac{1}{\rho(\Theta)^2}|s|^2$$

But we have established the reverse inequality (3.68). Hence by a pinching argument we see $\mu_1, \ldots, \mu_{n-1} = \frac{1}{\rho(\Theta)}$. We also see that

$$\frac{n-1}{2}\psi'(\Theta)|\nabla\Theta||s|^2 = \frac{n-1}{2}\psi'(\Theta)|s|^2$$

Therefore

$$|\nabla\Theta| = 1$$

Hence by lemma 3.3.4 we have that $d\varphi(\nabla\Theta) = 0$. Hence $\nabla\Theta \in \ker df_x$. Since df_x has a 1-dimensional kernel spanned by e_n, we conclude that e_n and $\nabla\Theta$ are linearly dependent.

Consider

$$\rho(\Theta)^2 f^*g_{S^n}(e_i, e_i)$$

for $i \neq n$. We see this is equal to $g(e_i, e_i)$. Consider the augmented inner product

$$\rho(\Theta)^2 f^*g_{S^n} + d\Theta \otimes d\Theta$$

Since $e_n = k\nabla\Theta$, we have $d\Theta(e_i) = \langle\nabla\Theta, e_i\rangle = 0$ for $i \neq n$. Then it is easy to check the above expression equals g when evaluated at e_i for each i. Hence

$$g = \rho(\Theta)^2 f^*g_{S^n} + d\Theta \otimes d\Theta$$

Note that when restricted to $S^{n-1} \times \{t\}$, h satisfies $h^*g_{S^n} \leq g_{S^{n-1}\times\{t\}} = g_{S^{n-1}}$.

Writing $f^* = \varphi^*h^*$, we have the inequality

$$g \leq \rho(\Theta)^2\varphi^*g_{S^{n-1}} + d\Theta \otimes d\Theta$$

but this shows

$$g = \rho(\Theta)^2\varphi^*g_{S^{n-1}} + d\Theta \otimes d\Theta = \Phi^*g_0$$

hence Φ is a local isometry. By lemma 2.8.1, Φ is a global isometry. $\square$

3.7 Proof of odd-dimensional case

When M is odd-dimensional, the proof proceeds in a similar manner as in the even-dimensional case, but since we do not have to take the product $M \times S^1$ and can directly consider adjointness formula for $D^{\partial M}$, we are in a more favourable position.

3.7.1 A simpler adjointness formula for Dirac operator

Note that the formulae from (3.46) to (3.53) hold verbatim except in the following places:

- Replace all $\tilde{M}$ with M, and $\partial_\pm \tilde{M}$ with $\partial_\pm M$.
- The definition of $\tilde{P}$ no longer has to minus T. In other words, we replace $\tilde{P}$ by the following P:
$$P_X u = \nabla_X^{S\otimes E} u + \frac{i}{2}\Psi X \cdot u$$
where $E = \varphi^* E_0^+$ and S is spinor bundle over M. This is in spirit similar to ∇^Ψ.
- As a consequence of the above, the term $|\nabla_T u|^2$ vanishes.
- The numerical constants (e.g. the $\frac{i(n-1)}{2}$, the $\frac{n(n-1)}{4}$) all remain the same, since they are not consequences of summing basis vectors of a different dimension, but rather follow from a "multiply both sides by $-\frac{i(n-1)}{2}$" operation before (3.51).

We have the following lemma analogous to lemma 3.55.

Lemma 3.7.1. Let $u \in C^\infty(M, S \otimes E)$ over M. Then

$$\begin{aligned}
&\int_M |D_E u - \frac{in}{2}\Psi u|^2 - \int_M |Pu|^2 - \frac{1}{4}\int_M \operatorname{scal}^g |u|^2 \\
&- \int_M \langle \mathcal{R}^E u, u\rangle - \frac{n(n-1)}{4}\int_M \Psi^2 |u|^2 + \frac{i(n-1)}{2}\int_M \langle (\nabla\Psi)\cdot u, u\rangle \\
&= -\frac{1}{2}\int_{\partial_+ M} \langle D^{\partial M} u, u + i\nu\cdot u\rangle - \frac{1}{2}\int_{\partial_+ M} \langle u + i\nu \cdot u, D^{\partial M} u\rangle \\
&+ \frac{1}{2}\int_{\partial_+ M} (H - (n-1)\Psi)|u|^2 + \frac{n-1}{2}\int_{\partial_+ M} \Psi\langle u + i\nu\cdot u, u\rangle \\
&- \frac{1}{2}\int_{\partial_- M} \langle D^{\partial M} u, u - i\nu\cdot u\rangle - \frac{1}{2}\int_{\partial_- M} \langle u - i\nu \cdot u, D^{\partial M} u\rangle \\
&+ \frac{1}{2}\int_{\partial_- M} (H + (n-1)\Psi)|u|^2 - \frac{n-1}{2}\int_{\partial_- M} \Psi\langle u - i\nu\cdot u, u\rangle
\end{aligned} \tag{3.69}$$

3.7.2 Index calculation

Similar to lemma 3.6.3, we have the following lemma. The proof is similarly the holographic index theorem.

Lemma 3.7.2. Consider the index of the following operators:

(1) Let ind_1 denote the index of the Dirac operator on $S\otimes\varphi^* E_0^+$ with boundary conditions $u + i\nu\cdot u = 0$ on $\partial_+ M$ and $u - i\nu \cdot u = 0$ on $\partial_- M$.
(2) Let ind_2 denote the index of the Dirac operator on $S\otimes\varphi^* E_0^+$ with boundary conditions $u - i\nu\cdot u = 0$ on $\partial_+ M$ and $u + i\nu \cdot u = 0$ on $\partial_- M$.
(3) Let ind_3 denote the index of the Dirac operator on $S\otimes\varphi^* E_0^-$ with boundary conditions $u + i\nu\cdot u = 0$ on $\partial_+ M$ and $u - i\nu \cdot u = 0$ on $\partial_- M$.
(4) Let ind_4 denote the index of the Dirac operator on $S\otimes\varphi^* E_0^-$ with boundary conditions $u - i\nu\cdot u = 0$ on $\partial_+ M$ and $u + i\nu \cdot u = 0$ on $\partial_- M$.

Then $\max\{\mathrm{ind}_1, \mathrm{ind}_2, \mathrm{ind}_3, \mathrm{ind}_4\} > 0$.

Proof. By holographic index theorem. □

3.7.3 Existence of parallel spinor field

Without loss of generality we can assume $\mathrm{ind}_1 > 0$ and write $E = \varphi^* E_0^+$. We prove a lemma analogous to lemma 3.6.7.

Lemma 3.7.3. There is a section $s \in C^\infty(M, S \otimes E)$ such that

$$P_X s = 0$$

for any vector field X.

Proof. We take a section $s \in C^\infty(M, S \otimes E)$ such that the following holds:

- s does not vanish on M
- $D_E s - \frac{in}{2}\Psi s = 0$
- $s = i\nu \cdot s$ on $\partial_- M$ and $s = -i\nu \cdot s$ on $\partial_+ M$

Then by lemma 3.69 we have:

$$\begin{aligned} &-\int_M |Ps|^2 - \frac{1}{4}\int_M \mathrm{scal}^g |s|^2 \\ &-\int_M \langle \mathcal{R}^{S\otimes E} s, s\rangle - \frac{n(n-1)}{4}\int_M \Psi^2 |s|^2 + \frac{i(n-1)}{2}\int_M \langle (\nabla\Psi)\cdot s, s\rangle \\ &= \frac{1}{2}\int_{\partial_+ M} (H - (n-1)\Psi)|s|^2 + \frac{1}{2}\int_{\partial_- M} (H + (n-1)\Psi)|s|^2 \end{aligned} \tag{3.70}$$

We have $H_{g_0} \circ \Phi = \pm(n-1)\Psi$. By assumption on mean curvature we see that RHS of (3.70) is nonnegative. Hence

$$-\int_M |Ps|^2 \geq \frac{1}{4}\int_M \mathrm{scal}^g |s|^2 + \int_M \langle \mathcal{R}^{S\otimes E} s, s\rangle + \frac{n(n-1)}{4}\int_M \Psi^2 |s|^2 - \frac{i(n-1)}{2}\int_M \langle (\nabla\Psi)\cdot s, s\rangle$$

By lemma 3.3.5 we have that

$$\frac{1}{4} scal \geq -\frac{(n-1)}{2}\psi'(\Theta) - \frac{n(n-1)}{4}\psi(\Theta)^2 + \frac{(n-2)}{4}\frac{1}{\rho(\Theta)^2}$$

Since

$$-\frac{i(n-1)}{2}\langle (\nabla\Psi)\cdot u, u\rangle$$

is a real number and $\psi' < 0$, using inequality (2.3) and lemma 3.3.4 we have

$$-\frac{i(n-1)}{2}\langle (\nabla\Psi)\cdot u, u\rangle \geq \frac{n-1}{2}\psi'(\Theta)|\nabla\Theta||u|^2 \geq \frac{n-1}{2}\psi'(\Theta)|u|^2$$

Hence we get

$$-\int_M |Pu|^2 \leq 0$$

which means

$$\int_M |Pu|^2 = 0$$

and

$$\nabla_X^{S\otimes E} s + \frac{i}{2}\Psi X \cdot s = 0$$

for any vector field X. $\square$

3.7.4 Completing the proof for n odd

Note that the results for the perturbed connection ∇^Ψ presented in section 3.6.3 are all valid on P, since they have exactly the same formula. We will quote the corresponding lemmas without proof.

Proof of theorem 3.3.2, n odd. Fix a regular point $x \in M$ and consider $\varphi : M \to S^{n-1}$. Take singular value decomposition of $d\varphi_x : T_xM \to T_{\varphi(x)}S^{n-1}$ with singular values $\mu_1 \geq \cdots \geq \mu_{n-1} \geq \mu_n$. We denote the g-orthonormal basis by $\{e_i : 1 \leq i \leq n\}$ and $g_{S^{n-1}}$-orthonormal basis by $\{\varepsilon_i : 1 \leq i \leq n-1\}$. Since $T_{\varphi(x)}S^{n-1}$ is dimension $n-1$ we have $\mu_n = 0$. By 1-Lipschitz we hvae $g \geq \rho(\Theta)^2\varphi^* g_{S^{n-1}}$, so for $i = 1, \ldots, n-1$ we have

$$\mu_i \leq \frac{1}{\rho(\Theta)}$$

Hence

$$\langle \mathcal{R}^E s, s\rangle \geq -\frac{(n-1)(n-2)}{4}\frac{1}{\rho(\Theta)^2}|s|^2$$

By exactly the same calculation for (3.68) we actually have equality

$$\langle \mathcal{R}^E s, s\rangle = -\frac{(n-1)(n-2)}{4}\frac{1}{\rho(\Theta)^2}|s|^2$$

whence $\mu_i = \frac{1}{\rho(\Theta)}$ for $i = 1, \ldots, n-1$. We also have $|\nabla\Theta| = 1$. Thus $d\varphi_x(\nabla\Theta) = 0$ by lemma 3.3.4. Hence $\nabla\Theta$ and e_n are linearly dependent, and for $i = 1, \ldots, n-1$, we have

$$0 = \langle \nabla\Theta, e_i\rangle = d\Theta(e_i)$$

Consider

$$g_0 \circ \Phi = \rho(\Theta)^2\varphi^* g_{S^{n-1}} + d\Theta \otimes d\Theta$$

On $e_1, \ldots, e_{n-1}$ we have $d\Theta(e_i)^2 = 0$ so $g_0 \circ \Phi(e_i, e_i) = 1$. On e_n we have $d\varphi(e_n) = kd\varphi(\nabla\Theta) = 0$, and $d\Theta(\nabla\Theta)^2 = |\nabla\Theta|^4 = 1$. Again, $g_0 \circ \Phi(e_n, e_n) = 1$. Hence $d\Phi$ is an isometry at x. Hence Φ is a global Riemannian isometry by lemma 2.8.1 since the domain is complete and the target is simply connected. $\square$

Appendices

Appendix A

Characteristic classes

Characteristic classes are cohomological, isomorphism invariants of vector bundles over manifolds. When M is compact and orientable with no boundary, we know that $H_{n-1}(M)$ is free, and via the universal coefficient theorem we have $H^n(M;\mathbb{Z})$ isomorphic to $Hom(H_n(M),\mathbb{Z})$. Characteristic classes take value in $H^*(M;R)$, and the evaluation of a class at the top degree n on the fundamental homology class $[M]$ often yields important **characteristic numbers** of the manifold. For example the Euler class yields the Euler characteristic.

Given a complex vector bundle E of dimension k over M, the **Chern class** $c_i(E)$ can be defined in various equivalent ways. The most prominent use case of this class is perhaps the Hirzebruch-Riemann-Roch theorem, which allows for a computation of the Euler characteristic of the sheaf cohomology of a complex vector bundle using the Chern and Todd classes. This computation is valuable in the sense that geometric questions can be formulated in terms of the existence of certain sections on a bundle, whose information can be gleaned from the bundle's sheaf cohomology.

We have the following axiomatic definition of Chern classes.

Definition A.0.1. Let E be a complex vector bundle of rank k equipped with a Hermitian metric, over a manifold M. The **Chern classes** $c_i(E) \in H^{2i}(M;\mathbb{Z})$ for $i = 0,\dots,k$ satisfy the following axioms:

(1) $c_0(E) = 1$ is the unit element and $c_i(E) = 0$ for $i > n$
(2) Given a continuous map $f : N \to M$ and f^*E the pullback vector bundle on N, we have $f^*c_i(E) = c_i(f^*E)$
(3) (multiplicativity) $c(E \oplus F) = c(E)c(F)$, where $c(E) = \sum_{i=0}^{k} c_i(E)$ is called the **total Chern class** of E.
(4) Let E_0 be the tautological line bundle over $\mathbb{C}P^n$. Let H be the Poincaré dual of the hypersurface $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$. Then the total Chern class of E_0 is $c(E_0) = 1 - H$.

We accept the following fact.

Lemma A.0.2. Chern classes exist and are unique.

Lemma A.0.3. $c_i(E \oplus F) = \sum_{l=0}^{i} c_l(E)c_{i-l}(F)$

Proof. Suppose E and F are of dimension m and n respectively. By multiplicativity of the Chern character we have $c(E\oplus F) = 1+\cdots+c_{m+n}(E\oplus F) = (1+\cdots+c_m(E))(1+\cdots+c_n(F))$. Expanding product $(1+\cdots+c_m(E))(1+\cdots+c_n(F))$ we immediately have the equality. $\square$

By this lemma, if $E = L_1\oplus\cdots\oplus L_k$ where L_i are line bundles, we have $c_i(E) = c_i(L_1\oplus\cdots\oplus L_k) = \sigma_i^{(k)}(x_1,\ldots,x_k)$ where $x_\alpha = c_1(L_\alpha)$. Here $\sigma_i^{(k)}$ is the **elementary symmetric polynomial** of degree i over k variables and has the form $\sigma_i^{(k)}(y_1,\ldots,y_k) = \sum_{1\leq\alpha_1<\cdots<\alpha_i\leq k} y_{\alpha_1}\cdots y_{\alpha_i}$.

Not every E is a sum of line bundles, but we have the following "splitting principle" which is almost as good.

Lemma A.0.4 (splitting principle)**.** Let E be a vector bundle of complex dimension k over M. There exists a manifold Y and a continuous map $f : Y \to M$ such that

(1) $f^* : H^*(M;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$ is injective
(2) the pullback f^*E splits, i.e.

$$f^*E \cong L_1 \oplus \cdots \oplus L_k$$

for complex line bundles L_i, $i = 1\ldots k$.

Thus, given any E, by the lemma above we have $f^*E = L_1\oplus\cdots\oplus L_k$ for some $f : Y \to M$. Then $c_i(f^*E) = \sigma_i^{(k)}(x_1,\ldots,x_k)$ where $x_\alpha = c_1(L_\alpha)$. By naturality we have $c_i(f^*E) = f^*(c_i(E))$ and since f^* is injective we can identity $c_i(E)$ with $\sigma_i^{(k)}(x_1,\ldots,x_k)$. Hence the total Chern class $c(E) = \sigma_0^{(k)}(x_1,\ldots,x_k)+\cdots+\sigma_k^{(k)}(x_1,\ldots,x_k) = \Pi_{i=0}^k(1+x_i)$. With this understood, we define the **Chern character** of E as follows.

Definition A.0.5. The **Chern character** of E, denoted $ch(E)$, is given by

$$e^{x_1}+\cdots+e^{x_k} = k+\sum_{i=0}^k x_i + \frac{1}{2}\sum_{i=0}^k x_i^2 + \ldots \tag{A.1}$$

which is a power series in the cohomology ring $H^*(M;\mathbb{Z})$.

Suppose $M = S^{2m}$ an even dimensional sphere. Then the cohomology ring $H^*(S^{2m};\mathbb{Z})$ is $\mathbb{Z}$ at degrees 0, $2m$ and is 0 elsewhere. Since $x_i \in H^{2i}(S^{2m};\mathbb{Z})$, expression (A.1) is greatly simplified to be

$$k+\frac{1}{m!}\sum_{i=0}^k x_i^m \tag{A.2}$$

We can use the **Newton's identities**:

Lemma A.0.6. Let $1\leq n\leq k$. $n\sigma_n^{(k)} = \sum_{i=0}^n(-1)^{i-1}\sigma_{n-1}^{(k)}s_i$.

By this identity, and the fact that $s_i(x_1,\ldots,x_k)$ on S^{2m} are mostly 0, expression (A.2) reduces to

$$ch(E) = k+m\cdot\sigma_m^{(k)}(x_1,\ldots,x_k)\frac{1}{m!} = k+\frac{1}{(m-1)!}c_k(E) \tag{A.3}$$

We will most often use this expression for Chern character in this article.

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