Talk 7: Gray Tensor Product for $(\infty,2)$-Categories.

In talk 6, the Gray tensor product was defined for 2-Categories. In particular, the product gives the underlying 1-Category a symmetric monoidal structure, and its internal hom is given by pseudofunctors between 2-Categories.

This talk defines the $(\infty,2)$ generalisation of the gray tensor product. We use the implementation $sSet^{sc}$, the scaled simplicial sets, to model $(\infty,2)$-Cats. In section 0 we recall some necessary background definitions and facts about model categories and particularly the model on $sSet^{sc}$. In section 1 we define the gray tensor product on $sSet^{sc}$, and illustrates by a toy example how the internal hom object can generalise 2-functors and oplax natural transformations. In section 2 we show that the gray tensor product endows $sSet^{sc}$ with a symmetric monoidal structure, which is the main result of the paper [GHL21]. This is done by showing the gray tensor product is a (left) Quillen bifunctor.

Section 0. Background

Def A scaled simplicial set $(X, T) \in sSet^{sc}$ is a simplicial set X together with a subset $T \subseteq X_2$, called thin triangles, containing the degenerate ones. A map of scaled simplicial sets $f: (X, T_X) \to (Y, T_Y)$ is a map of simplicial set such that $f(T_X) \subseteq T_Y$. Denote by $X_\flat = (X, \deg_2(X))$ and by $X_\sharp = (X, X_2)$.

Def The following are the generating anodyne maps in $sSet^{sc}$:

(i) The inner horn inclusions
$$(\Lambda^m_i, \Delta^{\{i-1,i,i+1\}}) \to (\Delta^m, \Delta^{\{i-1,i,i+1\}}), \quad m \geq 2, \ 0 < i < m.$$

(ii) The map
$$(\Delta^4, T) \to (\Delta^4, T \cup \{\Delta^{\{0,3,4\}} \cup \Delta^{\{0,1,4\}}\}), \text{ where}$$
$$T := \{\Delta^{024}, \Delta^{123}, \Delta^{013}, \Delta^{134}, \Delta^{012}\}.$$

(iii) The maps
$$(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \Delta^{\{0,1,n\}}) \to (\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \Delta^{\{0,1,n\}})$$

Fact There is a left proper combinatorial model structure on $sSet^{sc}$ where

Cof : monomorphisms.

In particular $Cof = \ell(r(I))$ where I is the set containing $\partial\Delta^n \to \Delta^n$ and $\Delta^2_\flat \hookrightarrow \Delta^2_\sharp$.

Also every object is cofibrant because $\partial\Delta^0 = \phi$ is the initial object.

W and $W \cap Cof$ are given by the underlying Cisinski model structure.

Fibrant objects : The set which has lifting against all generating anodyne maps. They model the $(\infty,2)$-Cats and are called ∞-bicategories.

Naive fibrations are those with r.l.p against anodyne extensions. When domain and codomain are fibrant, they are precisely the fibrations. Also note an obj is fibrant if its unique map to the terminal object is naive fibrations.

Def A monoidal model category is a model category $\mathcal{C}$ with an associative and unital tensor product $(- \otimes -)$ that makes $\mathcal{C}$ a monodal category with the following additional properties:

1) $- \otimes -$ preserves colimits in each variable

2) (Pushout-product axiom).

For every pair of Cof $x \xrightarrow{f} y$, $z \xrightarrow{g} w$, the following pushout-product $f \hat{\otimes} g$ is a Cof:

$$\begin{tikzcd}
x \otimes z \arrow[r, "f \otimes id"] \arrow[d, "id \otimes g"'] & y \otimes z \arrow[d] \arrow[ddr, "id \otimes g"] & \\
x \otimes w \arrow[r] \arrow[drr, "f \otimes id"'] & x \otimes w \sqcup_{x \otimes z} y \otimes z \arrow[dr, "f \hat{\otimes} g"] & \\
& & y \otimes w
\end{tikzcd}$$

additionally, $f \hat{\otimes} g$ is w.e. when either f or g is w.e.

In particular $\_ \otimes \_$ is a Quillen bifunctor that has right adjoint with respect to both variables.

Justify? <u>Fact</u> If we take $\mathcal{C} = sSet^{sc}$, there is a simpler criterion to check that some functor $\_ \otimes \_$ satisfies pushout-product axiom: We need

1) $f \hat{\otimes} g$ is Cof when f and g are generating cofibrations.
2) $f \hat{\otimes} g$ is scaled anodyne when either f or g is generating scaled anodyne (note that anodyne extensions are $W \cap Cof$).

The verification of this fact relies on $\_ \otimes \_$ preserving colimits (so we can reduce to a generating set.) and a generalisation of the fact that for one-variable Quillen pair between model categories, the left adjoint sends anodyne extensions to w.e. iff the right adjoint sends fibration between fibrant objects to fibrations.

<u>Fact</u> If we loosely restrict from $sSet^{sc}$ to the cofibrant-fibrant objs, (i.e. the ∞-bicats), fibrations become naive fibrations and we have a Kan-complex-enriched category, i.e. an $(\infty,1)$-Cat of ∞-bicats. A Quillen bifunctor on $sSet^{sc}$ descends to a well-defined 2-var adjunction on this underlying $(\infty,1)$-Cat, such that homotopy colims are preserved in each variable. This enables us to define mapping objects between ∞-bicats, which serves as a generalisation of the mapping 2-Cat formed by (2-functors, oplax natural transformations, modifications).

Section 1. $\_\otimes\_$ on $sSet^{sc}$.

Def (Gray tensor product)

For a 2-simplex $\Delta^2 \xrightarrow{\sigma} X$ we say it degenerates along $\Delta^{\{i,i+1\}}$ if σ is degenerate and $\Delta^1 \underset{\partial_i^2}{\to} \Delta^2 \xrightarrow{\sigma} X$, (or the i-th face of σ) (or $\sigma|_{\Delta^{\{i,i+1\}}}$) is degenerate. This could mean either σ degenerates to a point, or σ factors through the surjective map $\Delta^2 \to \Delta^1$ that collapses the edge $\Delta^{\{i,i+1\}}$.

Let (X, T_X) and (Y, T_Y) be scaled simpl. sets. Define $(X \otimes Y, T)$ to have the underlying simplicial set $X \times Y$ and the following T:

$\sigma \in T$ iff 1) $\sigma \in T_X \times T_Y$ and

2) The image of σ in X degen. along $\Delta^{\{1,2\}}$ or the image of σ in Y degenerates along $\Delta^{\{0,1\}}$.

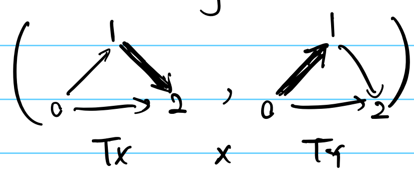


Fig 1.1

prop. $\_\otimes\_$ is associative, $\exists \varphi$. $((X, T_X) \otimes (Y, T_Y)) \otimes (Z, T_Z)$

$\varphi \; \wr\!\| \;$ natural

$(X, T_X) \otimes ((Y, T_Y) \otimes (Z, T_Z))$.

Pf. Take the natural isomorphism between Cartesian products (of simpl. sets). Then we have level-wise set theoretic iso. It remains to check φ preserves thin triangles. One can verify by hand using the definition of $\_\otimes\_$ that both $T_{((X,T_X)\otimes(Y,T_Y))\otimes(Z,T_Z)}$ and $T_{(X,T_X)\otimes((Y,T_Y)\otimes(Z,T_Z))}$ are tuples of the form $(\alpha, \beta, \gamma) \in T_X \times T_Y \times T_Z$ satisfying the following:

1) both α and β degenerates along $\Delta^{\{1,2\}}$;
2) both β and γ degenerate along $\Delta^{\{0,1\}}$;
3) α degenerates along $\Delta^{\{1,2\}}$ and γ degenerate along $\Delta^{\{0,1\}}$

$\square$

prop. $\_\otimes\_$ is unital with Δ^0 the unit.

Pf. This follows from the fact that $\{*\} \times A \simeq A$ for any set A. (Δ^0 has one point at each level, and a unique thin triangle that degen. to a point $\Rightarrow$ $T_{\Delta^0 \otimes (X, T_X)} \simeq T_X$, basically condition 2) for $T_{X \otimes Y}$ is void).

$\square$

__Remark__. From the definition one can see that $\_\otimes\_$ is in general not symm. But there is a natural iso $X \otimes Y \cong (Y^{op} \otimes X^{op})^{op}$.

__Remark__. Since the Cartesian product on sSet preserves colimits, it is
(really?) straightforward to verify by hand that $\_\otimes\_$ preserves colimits (one only needs to verify the thin triangles are consistent). By the Adjoint Functor theorem we have mapping objects:

$$\mathrm{Hom}(Y, \mathrm{Fun}^{gr}(X,Z)) \xleftarrow[\simeq]{\varphi_r} \mathrm{Hom}(X \otimes Y, Z) \xrightarrow[\sim]{\varphi_l} \mathrm{Hom}(X, \mathrm{Fun}^{opgr}(Y,Z)).$$

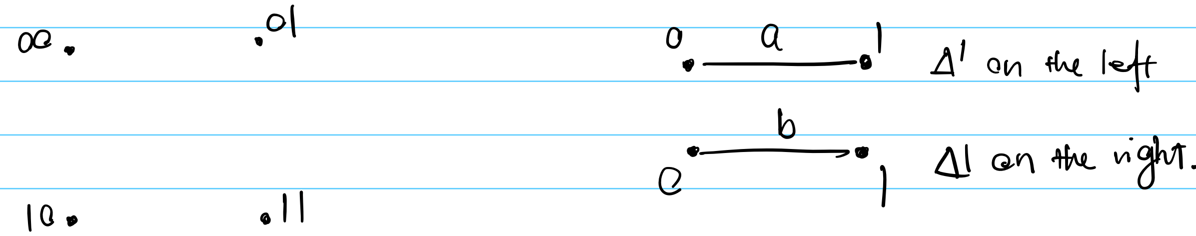
Fix $X = \Delta^n_b$. Then $\mathrm{Hom}(\Delta^n_b, \mathrm{Fun}^{opgr}(Y,Z)) \simeq (\mathrm{Fun}^{opgr}(Y,Z))_n$ by Yoneda, and since Δ^n_b only has degenerate 2-simplices thin, $\mathrm{Hom}_{sSet^{sc}}(\Delta^n_b, \_)$ is just $\mathrm{Hom}_{sSet}(\Delta^n, \_)$, without scaling. By φ_l we know an n-simplex of $\mathrm{Fun}^{opgr}(Y,Z)$ corresponds to a map of scaled simpl. sets

$$\Delta^n_b \otimes (Y, T_Y) \longrightarrow (Z, T_Z).$$

When Y, Z are ∞-bicats we can show $\mathrm{Fun}^{opgr}(Y,Z)$ is also ∞-bicat. Then we have the interpretation that:

- $n=0$: $\Delta^0 \otimes Y \to Z$ are functors between ∞-bicats
- $n=1$: oplax natural transformations.
- $n=2$: Modifications.

(Toy) __Example__. Consider $Y = \Delta^1$, Z any ∞-bicat. Then $\mathrm{Fun}^{opgr}(\Delta^1, Z)$ has oplax natural transformations of the form $\Delta^1 \otimes \Delta^1 \to Z$, a map of scaled simpl. sets. Then a 2-simplex of $\Delta^1 \otimes \Delta^1$ is induced by a map $[2] \to [1] \times [1]$ which preserves order in each component.



Vertices of $\Delta^1 \otimes \Delta^1$

Edges:	Source	Target
0b	$s(0b) = (s(0), s(b)) = 00$	01
1b	10	11
a0	00	10
a1	01	11
ab	00	11

$$\begin{tikzcd} 00 \arrow[r, "0b"] \arrow[d, "a0"'] \arrow[dr, "ab"] & 01 \arrow[d, "a1"] \\ 10 \arrow[r, "1b"'] & 11 \end{tikzcd}$$

(with $\sigma_1 \Nearrow$ above the diagonal ab and $\sigma_0 \Swarrow \cong$ below it)

Fig 1.2

Non-degen. Triangles:

$[2] \xrightarrow{\sigma_0} [1]\times[1]$
$0 \mapsto (0, 0)$
$1 \mapsto (1, 0)$
$2 \mapsto (1, 1)$

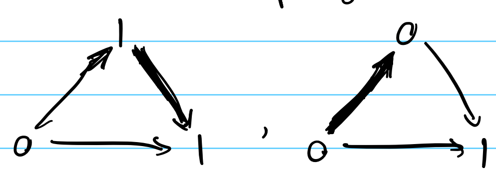
$[2] \xrightarrow{\sigma_1} [1]\times[1]$.
$0 \mapsto (0,0)$
$1 \mapsto (0,1)$
$2 \mapsto (1,1)$

$0 \mapsto$	$(1,0)$		$(0,1)$
$1 \mapsto$	$(1,1)$		$\vdots$
$2 \mapsto$	$\vdots$	X	X.

(Cannot start at $(1,0)$ or $(0,1)$).

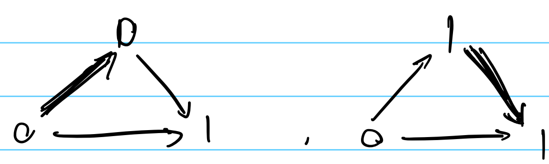
Thinness (w/ help of Fig 1.1):

σ_0:



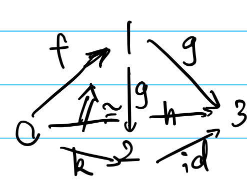
is thin!

σ_1:



NOT thin!

Fig 1.2 is an oplax commutative square in Z.
Then σ_0 specifies a well-defined composition that is "unique up to homotopy" by the lifting property of Z, since Z is $(\infty,2)$ and lifts against (inner) horn inclusion at dimension 3.



Λ^3_1 extended to Δ^3, $\Delta^{\{0,1,2\}}$ thin.
$\Rightarrow h \sim k$.

Data of σ_0 not so important. Fig 1.2 says "$1b \cdot a0$ commutes w $a1 \cdot 0b$ up to the 2-cell σ_1".

Now loosely consider Δ^1 and $\mathcal{Z}$ as 2-Cat (discard info of dim > 2). Then Δ^1 looks like $0 \xrightarrow[\circlearrowleft]{a} 1$, the 2-Cat with 2 vertices and the hom category $\underline{1}$ between them, while $\mathcal{Z}$ is arbitrary. Then a 2-functor from Δ^1 to $\mathcal{Z}$ is just a 1-mor in some hom cat of $\mathcal{Z}$:

$$F_0 \xrightarrow[\underset{1_{Fa}}{\circlearrowleft}]{Fa} F_1 .$$

Identify Δ^1 with the R.H.S. copy of $\Delta^1 \otimes \Delta^1$.

Then $\{0\} \times \Delta^1 \xrightarrow{0b} \mathcal{Z}$ and $\{1\} \times \Delta^1 \xrightarrow{1b} \mathcal{Z}$ give us $0b$, $1b$, the two 2-functors in this complex (ref. Fig 1.2). Then σ_1 is exactly a component cell of an oplax natural transformation from $0b$ to $1b$.

If we instead consider $\mathrm{Fun}^{gr}(Y, \mathcal{Z})$ we get lax natural transformations.

Since $\mathrm{Fun}^{opgr}(Y, \mathcal{Z})$ has higher dimensional data, we say it is a generalisation of the 2-Cat theoretic hom 2-Cat consisting of (2-functors, oplax-nat. trans., modifications).

Section 2. $\_\otimes\_$ is Quillen bifunctor.

This is the main result of [GHL21].

<u>prop.</u> Let $f : X \to Y$ be a monomorphism of scaled simplicial sets and $g : Z \to W$ be a scaled anodyne map. Then the pushout-products

$$f \hat{\otimes} g : X \otimes W \coprod_{X \otimes Z} Y \otimes Z \longrightarrow Y \otimes W \quad \text{and}$$

$$g \hat{\otimes} f : W \otimes X \coprod_{Z \otimes X} Z \otimes Y \longrightarrow W \otimes Y$$

are scaled anodyne. (or trivial cofibration, whenever the occasion arises).

To Lyne: (technically need $J\#$ ... but can I omit ?)

<u>Remark</u>. Once the prop is proven we have a Quillen bifunctor $\_\otimes\_$, since the general machinery on the model structure of $sSet^{sc}$ permits to consider generating monos and anodynes. (Also $\_\otimes\_$ preserves colim).

<u>Proof sketch</u>. As argued before we consider f to be generating mono and g to be generating anodyne. So there are cases to consider:

f : 1) $\partial\Delta^n_b \to \Delta^n_b \quad n \geq 0$

2) $\Delta^2_b \hookrightarrow \Delta^2_\#$.

g : 1) $(\Lambda^m_i, \Delta^{\{i-1,i,i+1\}}) \hookrightarrow (\Delta^m, \Delta^{\{i-1,i,i+1\}})$, $m \geq 2$, $0 < i < m$

2) $(\Delta^4, T) \hookrightarrow (\Delta^4, T \cup \ldots)$

3) $(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \Delta^{\{0,1,n\}}) \to (\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \Delta^{\{0,1,n\}})$ $n \geq 3$.

If f is $\partial\Delta^0 \to \Delta^0$, we have: ($\partial\Delta^0 = \phi$)

$$\begin{tikzcd}
\phi = (\phi \otimes Z) \arrow[r, "f \times id"] \arrow[d, "id \times g"] & \Delta^0 \otimes Z \arrow[d, "\cong Z"] \arrow[dr, bend left, "id \times g"] & \\
\phi = (\phi \otimes W) \arrow[r, "\phi \sqcup"] \arrow[rr, bend right, "f \times id"'] & \Delta^0 \otimes Z \arrow[r, dashed, "f \hat{\otimes} g = g"] & \Delta^0 \otimes W \cong W
\end{tikzcd}$$

for any g.

If f is $\Delta^2_b \hookrightarrow \Delta^2_\#$ we need some facts.

<u>Fact</u> Let $(X,T_X)\otimes(Y,T_Y) \in sSet^{sc}$. Let $T_{gr} := T_{X\otimes Y}$. (,).
(Reminder).
Let $T_- \subseteq T_{gr}$ be the subset having (σ_X, σ_Y) s.t. either both σ_X and σ_Y are degenerate or at least one of σ_X, σ_Y degenerates to a point. (Considering $T_{X\otimes Y}$ is either σ_X is or σ_Y is , this is a genuine restriction).

Not important.

Let $T_+ \supseteq T_{gr}$ be $(\sigma_X, \sigma_Y) \subseteq T_X \times T_Y$ such that either $\sigma_X|_{\Delta^{\{1,2\}}}$ or $\sigma_Y|_{\Delta^{\{0,1\}}}$ is degenerate. Note that if σ_X degenerates along $\Delta^{\{1,2\}}$ then $\sigma_X|_{\Delta^{\{1,2\}}}$ is degen. so indeed $T_{gr} \subseteq T_+$. The difference is σ_X itself is not required to be degenerate in T_+, say
$$\begin{tikzcd} & \bullet \arrow[dr, Rightarrow, "id"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "g"'] & & \bullet \end{tikzcd}$$
where $f \neq g$. This is not possible for T_{gr}.

The fact is if T_{gr} is replaced with T_- or T_+ the model structure on $sSet^{sc}$ will be equivalent. Also, we have:
$$(X\times Y, T_-) \hookrightarrow (X\otimes Y, T_{gr}) \hookrightarrow (X\times Y, T_+)$$
are both scaled anodyne.

$$\begin{tikzcd} \Delta^2_b \otimes Z \arrow[r, "f\times id"] \arrow[d, "id\times g"'] & \Delta^2_\# \otimes Z \arrow[d] \arrow[ddrr, bend left, "id\times g"] & & \\ \Delta^2_b \otimes W \arrow[r] \arrow[drr, bend right, "f\times id"'] & \Delta^2_b \otimes W \coprod_{\Delta^2_b\otimes Z} \Delta^2_\# \otimes Z \arrow[dr, dashed, "f\hat{\otimes} g"] & & \\ & & (\Delta^2_\# \otimes W, T_-) \arrow[r, hook] & (\Delta^2_\# \otimes W, T_{gr}) \end{tikzcd}$$

The simplicial set underlying $\Delta^2_b \otimes W \coprod_{\Delta^2_b\otimes Z} \Delta^2_\# \otimes Z$ and $\Delta^2_\# \otimes W$ are the same, since $Z \overset{g}{\hookrightarrow} W$ at the level of underlying simpl. set (recall g is gen. scaled anodyne). claim $f\hat{\otimes} g$ is iso.

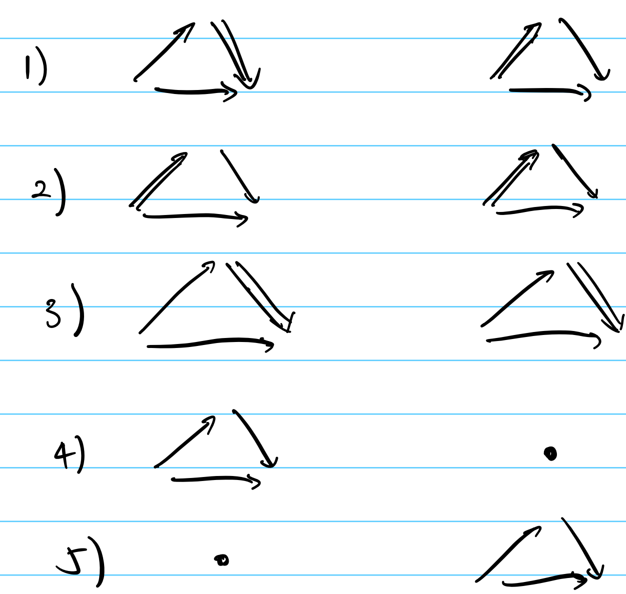
Observe $f\times id$ is already a bijection of simpl. sets, while $id\times g$ is possibly inclusion. but not bijective. The pushout-product then makes

$f\hat{\otimes}g$ a bijection. It remains to check thin simplices correspondence.

First note if σ in the domain is thin then $f\hat{\otimes}g(\sigma)$ is thin, because $T_{\Delta^2_\#} \supseteq T_{\Delta^2_b}$ clearly and $T_W \supseteq T_Z$ by different defs of g.

Now consider a thin triangle σ in $(\Delta^2_\# \otimes W, T_-)$. It has the form (σ_1, σ_2) and satisfy one of the following:

$P_1(\sigma_1)$ and $P_2(\sigma_2)$, where P_1, P_2 are logical predicates



Rmk. Both degen. But by pigeon-hole one of the degenerate edge must be $\Delta^{\{0,1\}}$ of σ_2 or $\Delta^{\{1,2\}}$ of σ_1.

Rmk. One degenerates to a point. The other is not necessarily degen.

In 1) 2) 3) (σ_1, σ_2) would be thin in $\Delta^2_b \otimes W$ of the domain, so it has thin preimage via $f \times id$.

In 4) (σ_1, σ_2) would also be thin in the domain, because $Z_0 \overset{g}{\simeq} W_0$ by defs of g (horn inclusion also has bijection on vertices). So it has thin preimage via $id \times g$.

In 5) the thin preimage is via $f \times id$.

Hence $f\hat{\otimes}g$ is isomorphism $\Rightarrow$ trivial cofibration. so the original map of T_{gr} is trivial cofib via the following 2-out-of-3:

$$\begin{tikzcd}
(\mathrm{dom}, T_-) \arrow[r, hook] \arrow[d, hook, "f\hat{\otimes}g_{T_-}"'] & (\mathrm{dom}, T_{gr}) \arrow[d, "f\hat{\otimes}g_{\mathrm{orig}}"] \\
(\mathrm{codom}, T_-) \arrow[r, hook] & (\mathrm{codom}, T_{gr})
\end{tikzcd}$$

So we are left with $f : \partial\Delta^n_b \to \Delta^n_b$, $n \geq 1$. We have 3 cases of g:

(A) g is horn inclusion $(\Lambda^m_i, T') \hookrightarrow (\Delta^m, T)$ where T is $\Delta^{\{i-1,i,i+1\}}$ singleton and $T' = T \cap \Lambda^m_i$. Picture:

$$
\begin{tikzcd}
\partial\Delta^n_b \otimes (\Lambda^m_i, T') \arrow[r, "f\times id"] \arrow[d, "id\times g"] & \Delta^n_b \otimes (\Lambda^m_i, T') \arrow[d] \arrow[ddr, bend left, "id\times g"] & \\
\partial\Delta^n_b \otimes (\Delta^m, T) \arrow[r] \arrow[ddr, bend right, "f\times id"'] & \partial\Delta^n_b \otimes (\Delta^m, T) \coprod_{\partial\Delta^n_b \otimes (\Lambda^m_i, T')} \Delta^n_b \otimes (\Lambda^m_i, T) =: (Z_0, M_0) \arrow[dd, dashed, "incl.\, \hat{\otimes}\, incl."] & \\
& & \\
& \Delta^n_b \otimes (\Delta^m, T) &
\end{tikzcd}
$$

Note that incl $\hat{\otimes}$ incl, i.e. $f \hat{\otimes} g$, is an inclusion of subsets at the level of simplicial set. To show $f \hat{\otimes} g$ is anodyne, the strategy is to build a filtration of incl $\hat{\otimes}$ incl and unite all simplices of $\Delta^n_b \otimes (\Delta^m, T)$, while keeping every step a scaled anodyne map, via a pushout of the anodyne $(\Lambda^k_p, T') \hookrightarrow (\Delta^k, T)$. Denote the domain by (Z_0, M_0) as above. Note that a pushout of scaled simplicial sets is levelwise pushout of sets with suitable scaling. A picture is as follows:

$$
\begin{tikzcd}
(\Lambda^{k_a}_{p_a}, T'_a) \arrow[r, "\sigma_a|_{\Lambda^k_p}"] \arrow[d, "anod."'] & (Z_{a-1}, M_{a-1}) \arrow[d] \\
(\Delta^{k_a}, T_a) \arrow[r, "\sigma_a"] & (Z_a, M_a) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

Here σ_a is the simplex we wish to add. We have already added $(a-1)$-simplices. Then we show $\sigma_a|_{\Lambda^k_p}$ is indeed scaled simplicial map, where $T_a \subseteq \Delta^k_a([2])$ is $\{\Delta^{\{p_a-1, p_a, p_a+1\}}\}$ and $T_a' = T_a \cap \Lambda^{k_a}_{p_a}$.

If this process stops at $\Delta^n_b \otimes (\Delta^m, T)$, i.e. there is a sequence S

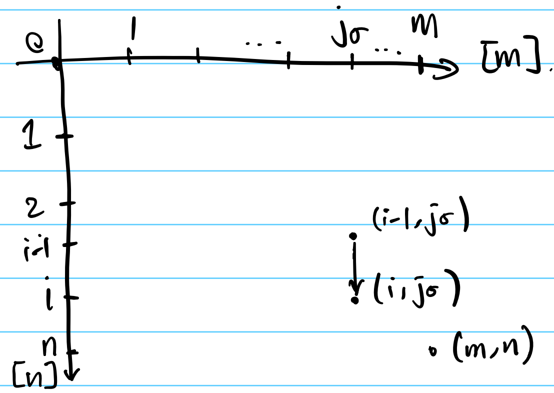
$$\sigma_1 < \sigma_2 < \dots < \sigma_l$$

such that $(Z_l, M_l) = \Delta^n_b \otimes (\Delta^m, T)$, then we are done.

We proceed to construct S.

Define the set S (no ordering S) to be the non-degenerate simplices $\sigma : \Delta^{k_\sigma} \to \Delta^n \times \Delta^m$ such that

1) σ is surjective in both components. This in particular means the order-preserving maps $[k_\sigma] \to [n]$ and $[k_\sigma] \to [m]$ induced by σ are surjective.
2) There is a $p_\sigma \geq 1$ such that $\sigma(p_\sigma - 1) = (i-1, j_\sigma)$ and $\sigma(p_\sigma) = (i, j_\sigma)$. for some $j_\sigma \in [m]$. Here it is helpful to visualise S as paths in a grid of size $n \times m$:



- A simplex σ corresponds to a path starting at origin
- Each step is to the right, or down, or diagonal ↘.
- No stopping allowed
- Must finish at (m,n).

For a σ to be in S is equivalent to saying it makes a vertical step downwards from $i-1$ to i in the $[n]$ component, as opposed to a diagonal step.

<u>claim 1</u>. S is finite.

This is because the set S is non-degenerate (the path does not stop). If k_σ is too big the simplex is bound to be degen. so k_σ is bounded upwards. $\square$

<u>claim 2</u>. Any simplex in our goal $\Delta^n \times \Delta^m$ is either in Z_0 or a face of a simplex in S.

Proof. Recall the def of Z_0: $(\partial \Delta^n \times \Delta^m) \sqcup (\Delta^n \times \Lambda^m_i)$ $(\Lambda^m_i \times \partial\Delta^n)$, any $\sigma : \Delta^k \to \Delta^n \times \Delta^m$ is NOT in Z_0 iff:

1) its image in Δ^n is surjective and
2) its image in Δ^m contains the face opposite to i.

Then we consider the following $[k] \to [m] \times [n]$: Let $p-1$ be the maximal element in $[k]$ sent to $i-1 \in [m]$. Then we have:

$$p-1 \mapsto (i-1, j)$$
$$p \mapsto (i, j') \qquad j' = j \text{ or } j+1$$

if $j'=j$, $\sigma \in S$. If $j'=j+1$ we construct a map $[k+1] \to [m] \times [n]$ in S, which σ factors through:

$$[k] \longrightarrow [k+1] \overset{\sigma'}{\longmapsto} [m]\times[n]$$
$$p-1 \mapsto p-1 \mapsto (i-1, j)$$
$$p \mapsto (i, j) \text{ or } (i-1, j+1).$$
$$p \mapsto p+1 \mapsto (i, j+1)$$

Then $\sigma' \in S$ and contains σ as a face. Depending on the properties of σ in the image on Δ^m there may be only 1 choice of σ' (precisely this happens when $\sigma|_{\Delta^m}$ is exactly the face opposite to i.). $\square$

Up to this point we have already a candidate set S that captures all non-degen. simplices of $\Delta^n \times \Delta^m$. All higher-dimensional degenerations follow automatically (because they factor through lower dimensional non-degenerate ones). It remains to establish an order on S and check the validity of pushouts (it is going to be an inductive check).

claim 3. There is an order on S.

Pf. We order S by increasing dimension k_σ and then on increasing order on j_σ. If two j_σ are the same, say $j_\sigma = j_{\sigma'}$ then it does not matter (indeed a partial order is enough for our inductive check; on Keradon there is a more sophisticated total order.). $\square$

claim 4. There is a filtration $(Z_0, M_0) \hookrightarrow (Z_1, M_1) \hookrightarrow \cdots \hookrightarrow (Z_\ell, M_\ell)$ over S and $(Z_\ell, M_\ell) = \Delta^n_b \otimes (\Delta^m, T)$.

(Reminder $T = \Delta^{\{i-1, i, i+1\}}$)

Pf. Copy picture:

$$\begin{tikzcd}
(\Lambda^{k_a}_{p_a}, T_a') \arrow[r, "\sigma_a|_{\Lambda^k_p}"] \arrow[d, "\text{anod.}"'] & (Z_{a-1}, M_{a-1}) \arrow[d] \\
(\Delta^{k_a}, T_a) \arrow[r, "\sigma_a"] & (Z_a, M_a) \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

Proceed to inductive check.

First WTS $\sigma_a(\Lambda^{k_a}_{p_a}) \subseteq Z_{a-1}$. Visualise σ_a as usual:

$$[k_a] \xrightarrow{\sigma_a} [m]\times[n]$$

$$\cancel{p_a - 2 \mapsto (\ldots, j \ldots)}$$

$$p_a - 1 \mapsto (i-1, j_a)$$

$$p_a \mapsto (i, j_a)$$

$$p_a + 1 \mapsto \ldots$$

For a face containing p_a, it is $\sigma_a \circ \partial_j^{k_a}$ for the j-th face. If $j \neq p_a - 1$ or p_a, then this face is in S and is smaller than σ_a (1-dim lower).

If $j = p_a - 1$, then the face is either in Z_0 or not. If not, then

$$p_a - 2 \mapsto (i-1, j_a - 1)$$
$$\cancel{p_a - 1 \mapsto (i-1, j_a)}$$
$$p_a \mapsto (i, j_a)$$

a) surj. in $[n]$

b) in $[m]$ contains face opp i.

because of a) $p_a - 2 \mapsto (\ldots, j_a - 1)$ or $(\ldots, j_a)$.

b) $p_a - 2 \mapsto (i-1, \ldots)$.

If $p_a - 2 \mapsto (i-1, j_a)$ this is invalid since σ_a is non-degen.

$\therefore$ $p_a - 2 \mapsto (i-1, j_a - 1)$.

Observe $p_a - 2 \mapsto (i-1, j_a - 1)$, $p_a \mapsto (i, j_a)$ is a diagonal step ↘. Then by the previous combinatorial observation it is the face of 2 simplices in S.

$$p_a - 2 \mapsto p - 2 \mapsto (i-1, j_a - 1)$$
$$p - 1 \mapsto (i-1, j_a) \text{ or } (i, j_a - 1).$$
$$p_a \mapsto p \mapsto (i, j_a)$$

we take $p - 1 \mapsto (i, j_a - 1)$ and the resulting simplex is before σ_a in S (lower in j-value).

One can verify the face opp. p_a is not in Z_{a-1}.

It is important to verify $\sigma_a|_{\Lambda^{k_a}_{p_a}}$ preserves scaling. For the only thin triangle $\Delta^{\{p_a - 1, p_a, p_a + 1\}}$, σ_a sends

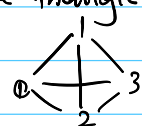
$$p_a - 1 \mapsto (\overset{[m]}{i-1}, \overset{[n]}{j_a})$$
$$p_a \mapsto (i, j_a)$$
$$p_a + 1 \mapsto (i+1, \_)$$

The $[n]$ component degenerates along $\Delta^{\{0,1\}}$. The $[m]$ component is thin in Δ^m. so the whole image is thin in $(\Delta^m, T) \otimes (\Delta^n_b)$, the ambient space. This establishes the validity of pushout in each step. so $Z_\ell = \Delta^m \times \Delta^n$.

It remains to check M_e = thin triangles in $(\Delta^m, T) \otimes \Delta^n_b$.
($\subseteq$) is clear because every step of filtration is sealed.
($\supseteq$) Given (σ_m, σ_n) thin, σ_m is either degen. or $\Delta^{\{i-1,i,i+1\}}$, and σ_n is degen. Copy def of (Z_0, M_0):

(order of Δ^m and Δ^n is wrong here)

$$(\Lambda^m_i, T') \otimes \Delta^n_b \sqcup (\Delta^m, T) \otimes \partial\Delta^n_b$$
$$\because\, ||$$
$$(Z_0, M_e)$$

σ_n is degen $\Rightarrow$ σ_n is thin in Δ^n_b. If $n \geq 2$ it is in $\partial\Delta^n_b$.
If $m \geq 3$, σ_m is degen $\Rightarrow$ $\sigma_m \subseteq \Lambda^m_i$ (Λ^m_i contains all edges)
σ_m is $\Delta^{\{i-1,i,i+1\}}$ $\Rightarrow$ $\sigma_m \subseteq \Lambda^m_i$ (if $m=2$ then $\Delta^{\{0,1,2\}}$ is the entire triangle which is beyond Λ^2_i)
(If $m=3$, , face is contained in inner horn)

In either cases $(\sigma_m, \sigma_n) \in M_0$, whence we are done.
In low dimensions where $m<2$ & $m<3$, one can compute by hand and use an additional step of pushout that includes the extra thin triangle. The detail is in [GHL21, prop 2.16].

(C) We first deal with $g : \left(\Lambda^m_0 \underset{\Delta^{\{0,1\}}}{\sqcup} \Delta^e, \Delta^{\{0,1,n\}}\right) \longrightarrow \left(\Delta^m \underset{\Delta^{\{0,1\}}}{\sqcup} \Delta^e, \Delta^{\{0,1,n\}}\right)$, $n \geq 3$.

This case is entirely similar to (A) where we similarly construct a filtration for $\Delta^m \times \Delta^n$, and then observe that $\left(\Delta^m \underset{\Delta^{\{0,1\}}}{\sqcup} \Delta^e\right)[k]$ is in bijection with $\Delta^m([k])$ for $k \geq 2$ for non-degenerates. Then one can make a filtration for $(\Delta^m \sqcup \Delta^e, \Delta^{\{0,1,n\}}) \otimes \Delta^n_b$ out of the filtration in (A).

(B) $g : (\Delta^4, T) \rightarrow (\Delta^4, T \cup \{\Delta^{034}, \Delta^{014}\})$
$T := \{\Delta^{024}, \Delta^{123}, \Delta^{013}, \Delta^{134}, \Delta^{012}\}$.

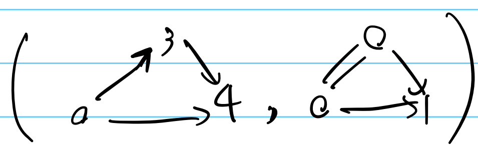
$$\begin{tikzcd}
(\Delta^4, T) \otimes \partial\Delta^n_b \arrow[r, "id \times f"] \arrow[d, "g \times id"'] & (\Delta^4, T) \otimes \Delta^n_b \arrow[d] \arrow[dd, bend left, "g \times id"] \\
(\Delta^4, T \cup \{034, 014\}) \otimes \partial\Delta^n_b \arrow[r] \arrow[dr, bend right, "id \times f"'] & (\Delta^4, T \cup \{034, 014\}) \otimes \partial\Delta^n_b \underset{(\Delta^4, T) \otimes \partial\Delta^n_b}{\sqcup} (\Delta^4, T) \otimes \Delta^n_b \arrow[d, "g \hat{\otimes} f"] \\
& (\Delta^4, T \cup \{034, 014\}) \otimes \Delta^n_b
\end{tikzcd}$$

Observe there is a bijection of simpl. sets for $g \times id$, so it is enough to check correspondence of thin triangles. (The part $\Delta^4 \times \partial\Delta^n$ is subsumed into g via the pushout).

For (σ_4, σ_n) thin in $(\Delta^4, T \cup \{034, 014\}) \otimes \Delta^n_b$, $\sigma_4 \in T \cup \{034, 014\} \cup \deg_2(\Delta^4)$ and $\sigma_n \in \deg_2(\Delta^n)$, and (σ_4, σ_n) satisfies .
If $n \geq 2$, then $\sigma_n \subseteq \partial\Delta^n_b$. (same as (A)), so we are done (The LHS of Z_0 has $\Delta^4, T \cup \{034, 014\}$). So we reduce to $n = 1$. Denote dom$(g \hat{\otimes} f)$ by $(\Delta^4 \times \Delta^1, T')$. Construct the following pushout:

$$\begin{tikzcd}
(\Delta^4, T) \arrow[r, "Id \times q"] \arrow[d, "\text{scaled anod.}"'] & (\Delta^4 \times \Delta^1, T') \arrow[d, "g \hat{\otimes} f"] \\
(\Delta^4, T \cup \{034, 014\}) \arrow[r, "Id \times q"] & (\Delta^4, T \cup \{034, 014\}) \otimes \Delta^1
\end{tikzcd}$$

Recall Z_0 is $(\Delta^4, T) \otimes \Delta^1 \amalg (\Delta^4, T \cup \{\ldots\}) \otimes \partial\Delta^1$. The thin triangles missing in (Z_0, M_0) is precisely the set $\{034, 014\} \times \{001\}$.
We have the following picture:

 $\notin M_0$.

To have $(034, 001)$ thin we require q to be image under q of $\Delta^{\{0,3,4\}}$ (and indeed also is $\Delta^{\{0,0,1\}}$, which is the missing thin triangle.

$0 \mapsto 0$
$1 \mapsto 0$
$2 \mapsto 0$. so the $\Delta^{\{0,1,4\}}$)
$3 \mapsto 0$
$4 \mapsto 1$

$g \hat{\otimes} f$ is scaled anodyne. Hence we conclude the entire prop. $\square$

<u>Corollary</u> If $\mathcal{C}$ is an ∞-bicat, K is a scaled simplicial set, then $\mathrm{Fun}^{gr}(K, \mathcal{C})$ and $\mathrm{Fun}^{opgr}(K, \mathcal{C})$ are ∞-bicats.

Proof. One verifies directly from definition:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\text{anod.}"'] & \mathrm{Fun}^{opgr}(K, \mathcal{C}) \\
B \arrow[ur, dashed, "f"']
\end{tikzcd}
\quad \overset{\text{adjunction}}{\longleftrightarrow} \quad
\begin{tikzcd}
A \otimes K \arrow[r] \arrow[d, "\sim"] & \mathcal{C} \\
B \otimes K \arrow[ur, dashed, "\exists"']
\end{tikzcd}$$

(wa Cof by prop.)

$\square$

Hence we now have a well-defined $-\otimes-$ on ∞-bicats, which generalises $-\otimes-$ on 2-Cats and gives rise to functors and oplax natural transforms in a meaningful way. We conclude this talk here.

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