Your Name: ________William Ard____________
(This is an INDIVIDUAL assignment)

CSC 4512, Optimization Approaches in CS: Algorithms and Applications

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Spring 2022 Semester

SUBMIT ALL PROJECTS Electronically via Moodle

A typical file name should have the following format:
"CSC4512_Spring2022_PROJ_1_YourFirstName_YourLastName.zip"
(Each submission involves more than one file; thus you will have to zip them into a single file.)

MAIN GOAL for PROJECT #1: Get familiarized with some important algorithmic issues regarding the solution of linear programming (LP) problems.

Today's date: Monday, January 31, 2022
Due date: Monday, February 21, 2022. By 10:00 PM of that day via MOODLE
Maximum grade points = 100
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ANNOUNCEMENT: Our TA is Augustine Orgah. His E-mail is: aorgah1@lsu.edu
CLEARLY EXPLAIN AND ORGANIZE YOUR ASNWERS! The TA may take points off otherwise.
Your answers must be presented sequentially.

NOTE: Always observe the Policy Statement for this course regarding Cheating / Academic Misconduct as it is stated on page 4 of the syllabus and described in the first day of classes.

1. Introduction

As it was discussed during the lectures, optimal solutions occur at corner points of the feasible region (FR). These corner points can be determined as the intersections of constraint lines (called hyperplanes, in general). If a problem has m constraints (including the non-negativity ones) and is defined on n variables, then we may have (m choose n) = $m!/(n!(m-n)!)$ intersections.

For instance, for the illustrative problem we studied in the lectures, we have $m = 6$ and $n = 2$. Thus, there are (6 choose 2) = $6!/(2!(6-2)!) = 15$ such intersection points. Out of these 15 points, only 7 are feasible (i.e., do not contradict any of the constraints) and thus correspond to corner points of the FR. Next, only one of them maximizes (optimizes) the objective function of that illustrative example. For the purpose of this project, assume that the LP problems will have up to two variables. Thus, your computer program will need to be able to solve systems of linear equations of size up to 2x2.

Therefore, one approach for solving any (simple) LP might be to first determine all possible intersections of the constraints. Next, determine which such intersection points are part of the FR and thus constitute corner points of the FR. The next step is to evaluate the objective function at each such corner point of the FR and identify the corner point(s) which makes the value of the objective function optimal. Finally, report the coordinates of the corner point(s) which correspond to the optimal value of the objective function as the optimal solution along with the corresponding value of the objective function.

2. Main Part

Develop a computer program that implements the above approach. Next, apply it to solve the following LP problems:

Problem #1

$$
\begin{array}{ll}
\text{Maximize } z = & 5x_1 + 6x_2 \\
\text{Subject to:} & 6x_1 + 4x_2 \leq 24 \\
 & x_1 + 2x_2 \leq 6 \\
 & -x_1 + x_2 \leq 1 \\
 & x_2 \leq 2 \\
\multicolumn{2}{l}{\text{and } x_1, x_2 \text{ are real numbers} \geq 0.}
\end{array}
$$

Problem #2

$$
\begin{array}{ll}
\text{Maximize } z = & 8x_1 + 10x_2 \\
\text{Subject to:} & 6x_1 + 4x_2 \leq 24 \\
 & x_1 + 2x_2 \leq 6 \\
 & -x_1 + x_2 \leq 1 \\
 & x_2 \leq 2 \\
\multicolumn{2}{l}{\text{and } x_1, x_2 \text{ are real numbers} \geq 0.}
\end{array}
$$

Problem #3

$$
\begin{array}{ll}
\text{Maximize } z = & 5x_1 + 6x_2 \\
\text{Subject to:} & 6x_1 + 4x_2 \leq 24 \\
 & x_1 + 2x_2 \leq 6 \\
 & -x_1 + x_2 \leq 1 \\
 & x_1 \geq 5 \\
 & x_2 \leq 2 \\
\multicolumn{2}{l}{\text{and } x_1, x_2 \text{ are real numbers} \geq 0.}
\end{array}
$$

Problem #4

$$
\begin{array}{ll}
\text{Maximize } z = & 10x_1 + 18x_2 \\
\text{Subject to:} & 3x_1 + 2x_2 \leq 12 \\
 & 2x_1 + 4x_2 \leq 12 \\
 & -x_1 + x_2 \leq 1 \\
 & 2x_2 \leq 4 \\
\multicolumn{2}{l}{\text{and } x_1, x_2 \text{ are real numbers} \geq 0.}
\end{array}
$$

Problem #5

Minimize $z = 6x_1$
Subject to:

$$5x_1 \geq 20$$
$$2x_1 \geq 30$$
$$x_1 \geq 6$$
$$3x_1 \geq 24$$

and x_1 is a real number ≥ 0.

Problem #6

Maximize $z = 3x_1 + 4x_2$
Subject to:

$$x_1 + 5x_2 \leq 20$$
$$x_1 \geq 2$$

and x_1, x_2 are real numbers ≥ 0.

Problem #7

Maximize $z = 6x_1$
Subject to:

$$5x_1 \leq 20$$

and x_1 is a real number ≥ 0.

3. Deliverables

Make sure your deliverables include the following items (use separate files if needed):

1. All source code (in a different file) with adequate info on how one can run it and reproduce your results. Write plenty of comments to explain your code.

2. To help with your presentation of the results, for each test problem you will need your program to first print out all the intersection points in an intuitive tabular manner. Show which ones are corner points of the FR and the value of the objective function at each intersection point (feasible or otherwise). Identify the optimal solution and report it properly. Present these results in your final report.

3. You may write comments regarding comparisons among the solutions of these problems.

4. What do you conclude of the previous solution approach for solving any LP problem?

Note: You will need to spend time to develop and debug your program and to run the test problems given in Section 2. You will also need time to prepare the solution reports requested above. Thus, start ASAP as this project may take more time than what you may anticipate now.

CSC 4512 – Project 1 **William Ard**

The software package developed for this project can solve LP problems with bounded feasible regions in one or two variables. The `lp_solver_function.py` contains the function used to solve the LP. Directions for how to use the solver is included in the docstring at the top of the file, and the code is explained step-by-step. The `get_solutions.py` file is used to solve all the problems detailed in the assignment. The optimal solution, and table of all possible solutions and intersections of constraints is printed to terminal, and a csv file is generated for each problem, containing a table of all constraint intersections, the value of the objective function at the intersection, and whether or not it is a solution in the feasible region. The code was written in Python 3, and the only dependency for the solver is NumPy.

Though not a requirement for the project, the file `plot_tool.py` can be used to generate plots of the feasible region and display the optimal point. SymPy and Matplotlib were used to generate the plots. The presentation of results is shown on the following pages.

Some notes regarding the project:

Problem 1 was benchmarked using Python 3's integrated profiler, cProfile, and the results were compared against SciPy's linear programing toolkit. The results are shown below:

```
Project Method
2983 function calls (2823 primitive calls) in 0.005 seconds

SciPy Interior Point
2802 function calls (2735 primitive calls) in 0.004 seconds

SciPy Simplex
3522 function calls (3468 primitive calls) in 0.004 seconds
```

The results were similar for all three methods, but SciPy's interior point method used the least number of function calls. The method proposed in this project is suitable for systems in only two variables. If the method were to be adapted to larger systems with multiple variables and constraints, the time complexity would be far higher than the Simplex method, as each constraint must be looped over, giving $O(n^2)$, additionally, for each loop, the intersection must be calculated--NumPy's linear system solver works on $O(n^3)$[1]; it's also necessary to check the intersections to verify that the solution is feasible—all of these things greatly increase complexity. Simplex method would be preferred as most Simplex algorithms have polynomial complexity[2].

[1] LAPACK Benchmark: www.netlib.org/lapack/lug/node71.hml

[2] Smoothed Analysis of Algorithms: Why the Simplex Algorithm Usually Takes Polynomial Time: https://arxiv.org/abs/cs/0111050

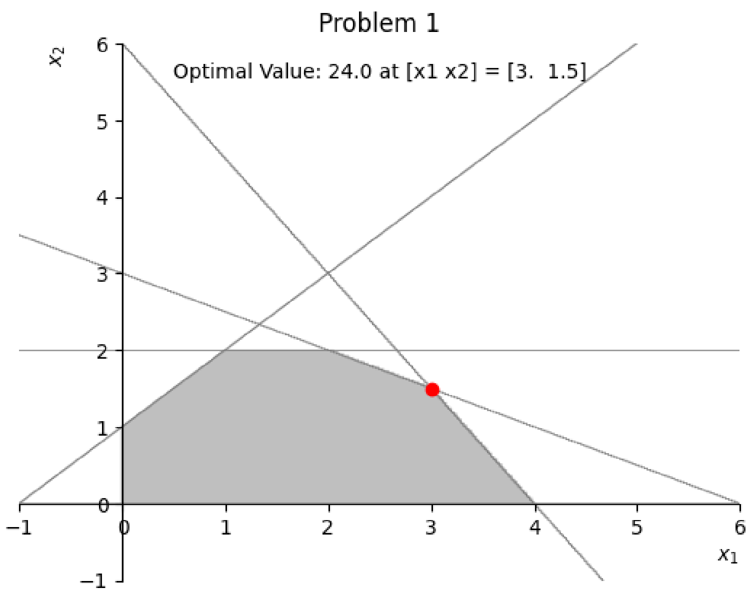
Problem #1

Maximize $z = 5x_1 + 6x_2$

Subject to:

$$6x_1 + 4x_2 \leq 24$$
$$x_1 + 2x_2 \leq 6$$
$$-x_1 + x_2 \leq 1$$
$$x_2 \leq 2$$

and x_1, x_2 are real numbers ≥ 0.

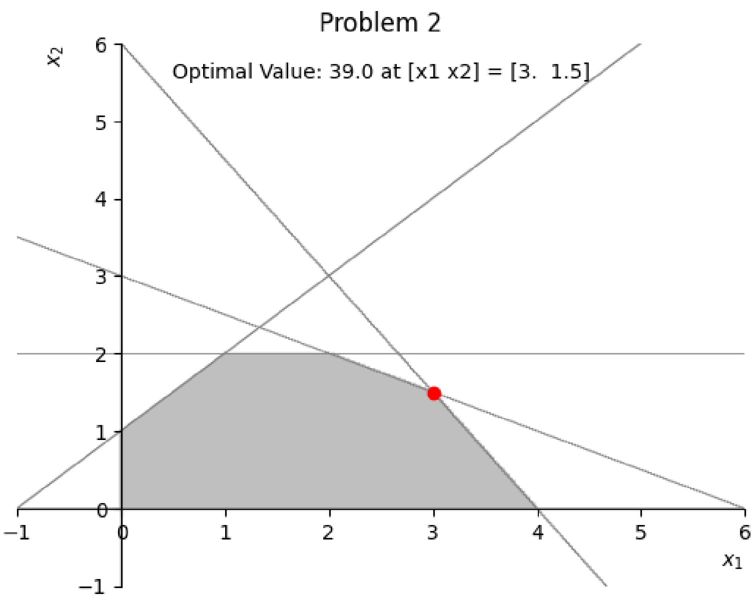


x_1	x_2	$z(x_1, x_2)$	**Feasible**
3	1.5	24	1
2	2	22	1
4	0	20	1
1	2	17	1
0	1	6	1
0	0	0	1
0	6	36	0
6	0	30	0
2	3	28	0
2.67	2	25.33	0
1.33	2.33	20.67	0
0	3	18	0
0	2	12	0
-1	0	-5	0

Comments: Optimal point is in the same location as the Reddy-Mikks problem from lectures.

Problem #2

Maximize $z = 8x_1 + 10x_2$
Subject to: $6x_1 + 4x_2 \leq 24$
$x_1 + 2x_2 \leq 6$
$-x_1 + x_2 \leq 1$
$x_2 \leq 2$
and x_1, x_2 are real numbers ≥ 0.



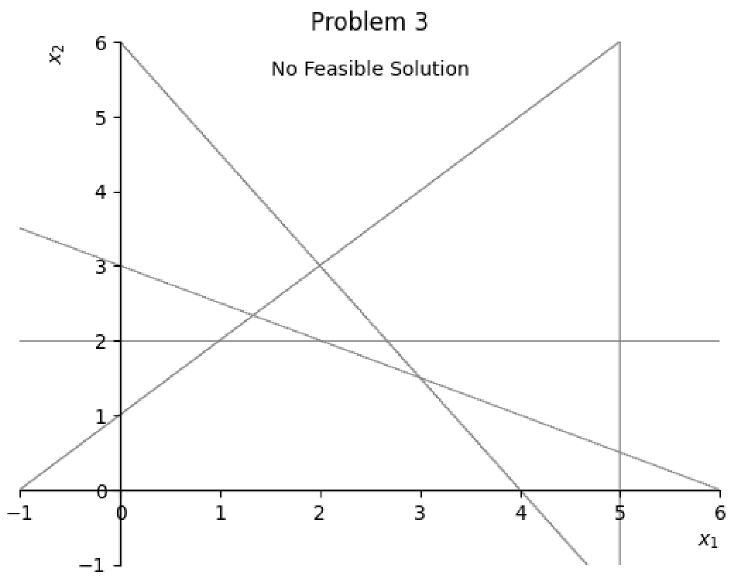
x_1	x_2	$z(x_1, x_2)$	Feasible
3	1.5	39	1
2	2	36	1
4	0	32	1
1	2	28	1
0	1	10	1
0	0	0	1
0	6	60	0
6	0	48	0
2	3	46	0
2.67	2	41.33	0
1.33	2.33	34	0
0	3	30	0
0	2	20	0
-1	0	-8	0

Comments: This has the same FR as problem 1, but with a different objective function. Optimal point is in the same place as Problem 1.

Problem #3

$$\begin{aligned} \text{Maximize } z = \; & 5x_1 + 6x_2 \\ \text{Subject to: } \; & 6x_1 + 4x_2 \le 24 \\ & x_1 + 2x_2 \le 6 \\ & -x_1 + x_2 \le 1 \\ & x_1 \ge 5 \\ & x_2 \le 2 \end{aligned}$$

and x_1, x_2 are real numbers ≥ 0.

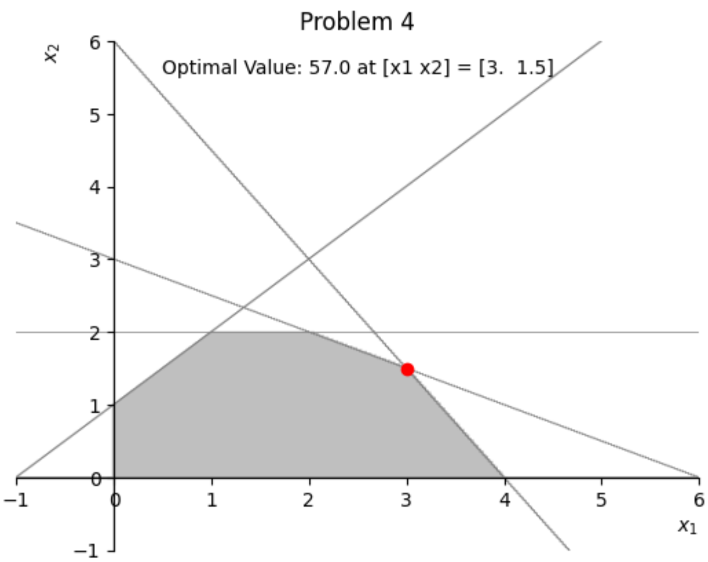


x_1	x_2	$z(x_1, x_2)$	Feasible
5	6	61	0
5	2	37	0
0	6	36	0
6	0	30	0
5	0.5	28	0
2	3	28	0
2.67	2	25.33	0
5	0	25	0
3	1.5	24	0
2	2	22	0
1.33	2.33	20.67	0
4	0	20	0
0	3	18	0
1	2	17	0
5	-1.5	16	0
0	2	12	0
0	1	6	0
0	0	0	0
-1	0	-5	0

Comments: The addition of $x_1 \ge 5$ removes the feasible region.

Problem #4

Maximize $z = 10x_1 + 18x_2$
Subject to: $3x_1 + 2x_2 \leq 12$
$2x_1 + 4x_2 \leq 12$
$-x_1 + x_2 \leq 1$
$2x_2 \leq 4$
and x_1, x_2 are real numbers ≥ 0.



x_1	x_2	$z(x_1, x_2)$	**Feasible**
3	1.5	57	1
2	2	56	1
1	2	46	1
4	0	40	1
0	1	18	1
0	0	0	1
0	6	108	0
2	3	74	0
2.67	2	62.67	0
6	0	60	0
1.33	2.33	55.33	0
0	3	54	0
0	2	36	0
-1	0	-10	0

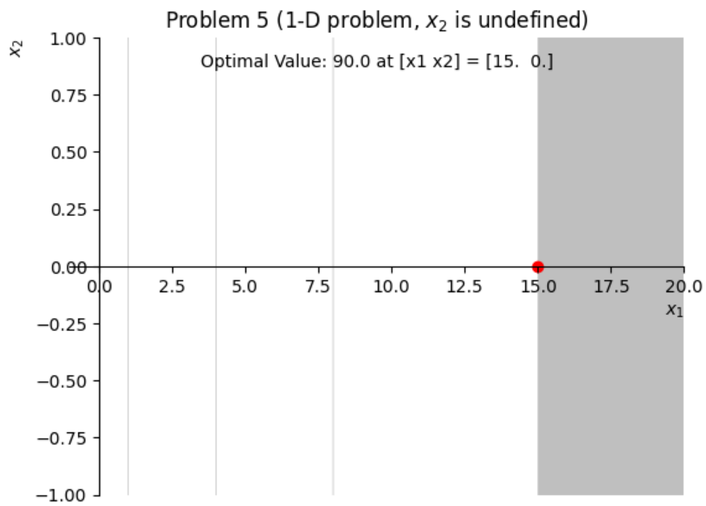
Comments: The constraints are the same as problem 3. Optimal point is in the same place as Problem 1 & 3.

Problem #5

Minimize $z = 6x_1$
Subject to:

$$5x_1 \geq 20$$
$$2x_1 \geq 30$$
$$x_1 \geq 6$$
$$3x_1 \geq 24$$

and x_1 is a real number ≥ 0.



x_1	x_2	$z(x_1, x_2)$	Feasible
15	0	90	1
8	0	48	0
4	0	24	0
1	0	6	0
0	0	0	0

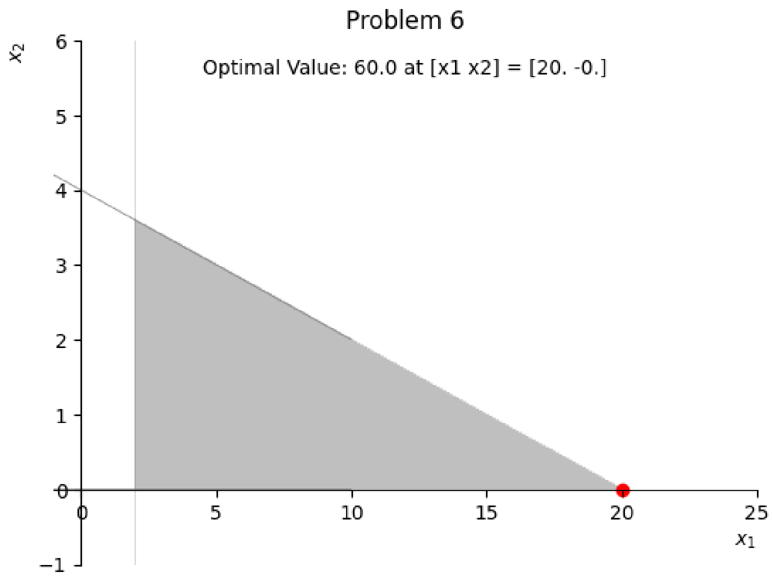
Comments: The plot is shown on two axes so that a feasible region can be illustrated. Note that x_2 is actually undefined. The problem has no upper bound.

Problem #6

Maximize $z = 3x_1 + 4x_2$
Subject to:

$$x_1 + 5x_2 \leq 20$$
$$x_1 \geq 2$$

and x_1, x_2 are real numbers ≥ 0.

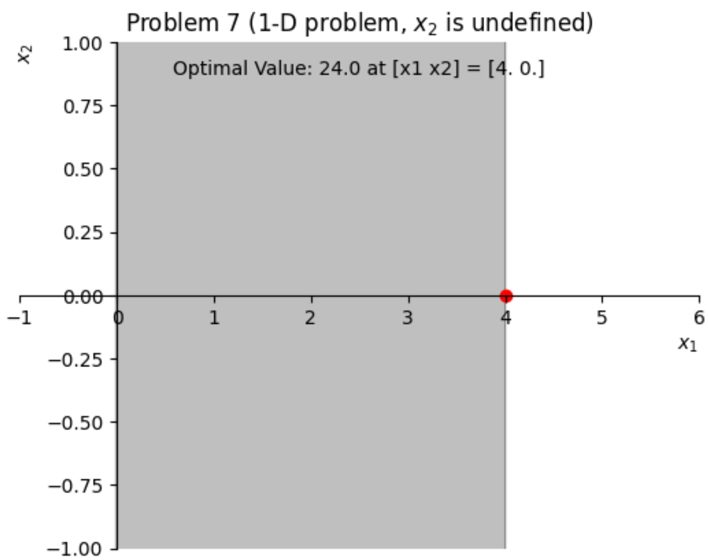


x_1	x_2	$z(x_1, x_2)$	**Feasible**
20	0	60	1
2	3.6	20.4	1
2	0	6	1
0	4	16	0
0	0	0	0

Comments: Optimal solution lies on the x_1 axis.

Problem #7

Maximize $z = 6x_1$
Subject to:
$$5x_1 \leq 20$$
and x_1 is a real number ≥ 0.



x_1	x_2	$z(x_1, x_2)$	**Feasible**
4	0	24	1
0	0	0	0

Comments: Problem is bounded between $x_1 = [0, 4]$.