# The Bayes Filter

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# *Readings for this class*

- Basic concepts in probabilities (pages 10 -15) from *Probabilistic Robotics*, Sebastian Thrun
- Bayes Filters (pages 23-31) from *Probabilistic Robotics*, Sebastian Thrun

# *Probabilistic Robotics*

- Robotics is the science of perceiving and manipulating the physical world through computer controlled mechanical devices.
- Probabilistic robotics is a relatively new approach to robotics that pays tribute to the uncertainty in robot perception and action.

# *Yes, it's all about probabilities...*

# *Problem Statement*

”Using sensory information to locate the robot in its environment is the most fundamental problem to providing a mobile robot with autonomous capabilities.” [Cox ’91]

# Problem Statement

- **Given**:
  - Map of the environment.
  - Sequence of sensors measurements.
- **Wanted**:
  - Estimate of the robot's position.
- **Problem classes:**
  - Position tracking (initial pose known)
  - Global localization (initial pose unknown)
  - Kidnapped robot problem (recovery)

Essentials of Probability Theory (whiteboard)

# Refresher

- Continuous spaces are characterized by random variables that can take on a continuum of values.
- In this course, we make the assumption that all continuous random variables possess probability density functions (PDFs)
- Common density function is *one-dimensional* ***normal distribution*** *with mean* $\mu$ *and variance* $\sigma^2$, given by the following Gaussian function:

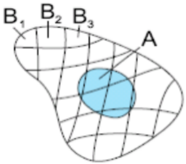
$$p(x) = (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\}$$

- The abbreviation of this is $\mathcal{N}(x; \mu, \sigma^2)$ which specifies the random variable, its mean and its variance.

**Total Probability Theorem**

- Let $B$ be a partition of the sample space $S$, $B = S$ and $B_i \cap B_j = \oslash$, $\forall\, i \neq j$, $\quad A = \bigcup_i^n (A \cap B_i)$

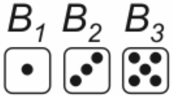
$$P(A) = \sum_{k=1}^{n} P(A \mid B_k) P(B_k)$$




(It expresses the total probability of an outcome which can be realized via several distinct events - hence the name)

## Total Probability Theorem - Example

- Imagine we toss a die. What is the probability of getting odd number?




$$P(A_{odd}) = \sum_{k=1}^{6} P(A \mid B_k)P(B_k)$$

$$P(A_{odd}) = 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} = \frac{3}{6}$$

# Refresher

**Bayes Rule**

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(A)}$$

*Example*: Imagine we toss a die. What is the probability of getting a 5 knowing we already got an odd number?

$$P(B_5 \mid A_{odd}) = \frac{P(A_{odd} \mid B_5)P(B_5)}{P(A_{odd})} = \frac{1 \cdot \frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

- Aposteriori knowledge: $P(B_5 \mid A_{odd})$
- Prior knowledge: $P(B_5)$
- Measurement: $P(A_{odd})$

# Refresher

**Bayes Rule** can be also written as:

$$P(B_i \mid A) = \eta P(A \mid B_i)P(B_i), \quad \text{where} \quad \eta = \frac{1}{P(A)}$$

Then, $\eta$ can be seen as a normalizer easily computed if $P(A \mid B_i)$ and $P(B_i)$ are known:

$$\eta = \frac{1}{P(A)} \quad \text{and} \quad P(A) = \sum_i P(A \mid B_i)P(B_i)$$

$$\Rightarrow \eta = \frac{1}{\sum_i P(A \mid B_i)P(B_i)}$$

In the previous example:

$$\eta = \frac{1}{\sum_{i=1}^{6} P(A_{odd}|B_i)P(B_i)} = \frac{1}{1\cdot\frac{1}{6}+0\cdot\frac{1}{6}+1\cdot\frac{1}{6}+0\cdot\frac{1}{6}+1\cdot\frac{1}{6}+0\cdot\frac{1}{6}} = 2$$

$$P(B_5 \mid A_{odd}) = \eta P(A_{odd} \mid B_5)P(B_5) = 2 \cdot (1 \cdot \frac{1}{6}) = \frac{1}{3s}$$

# The Bayes Filter for Probabilistic Robotics

The Bayes Filter plays a principle role in probabilistic Robotics.

- It probabilistically estimate a dynamic's system state from noisy observations.
- The state is the robot's location.
- State can be a simple 2D position or a 3D position, pitch, roll, yaw and linear and rotational velocities.
- Sensors provide observations about the state.

# The Bayes Filter for Probabilistic Robotics

Notations:

- $x_t$ - state of the robot at time $t$ (2D position, or 3D position + orientation, or 3D position+ orientation + velocities, ....)
- $z_t$ - observations (measurements) coming from the sensor
- $u_t$ - control/action/command executed by the robot (i.e: moved 2 m forward)
- $bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$ - robot belief at being at state $x_t$
- $\overline{bel}(x_t) = p(x_t \mid z_{1:t-1}, u_{1:t})$ - prior belief, prediction of state $x_t$
- $p(x_t \mid u_t, x_{t-1})$ - state transition probability
- $p(z_t \mid x_t)$ - measurement probability

# The Bayes Filter

**Algorithm Bayes_filter**($bel(x_{t-1}), u_t, z_t$)**:**
    for all $x_t$ do
        $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx$
        $bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$
    endfor
    return $bel(x_t)$

- $bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$ - robot belief at being at state $x_t$
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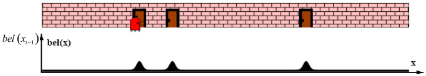
$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\ bel(x_{t-1})\ dx$ — Total Probability Theorem
$bel(x_t) = \eta\ p(z_t \mid x_t)\ \overline{bel}(x_t)$ — Bayes Rule
endfor
return $bel(x_t)$

$$bel(x_{t-1}) = p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})$$



# The Bayes Filter

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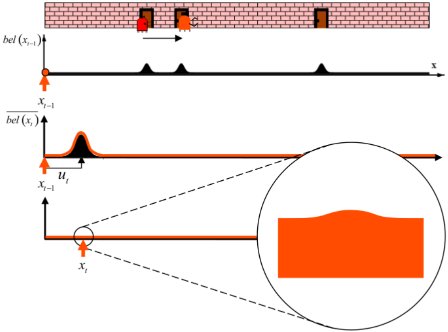

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*Total Probability Theorem*

*Bayes Rule*

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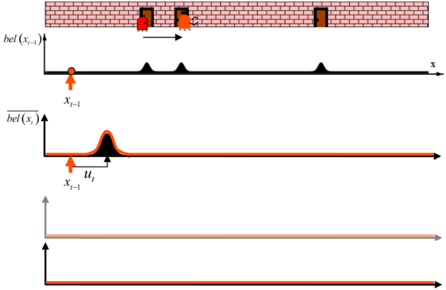

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$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$ *Bayes Rule*
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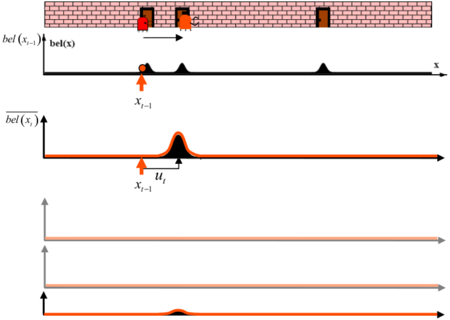

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*Total Probability Theorem*

*Bayes Rule*

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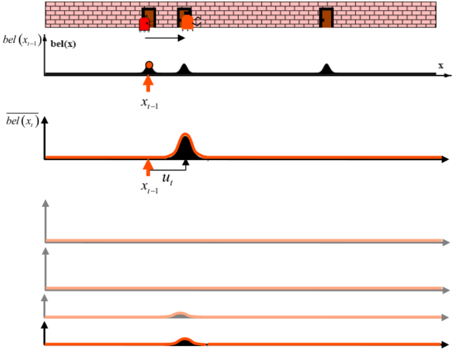

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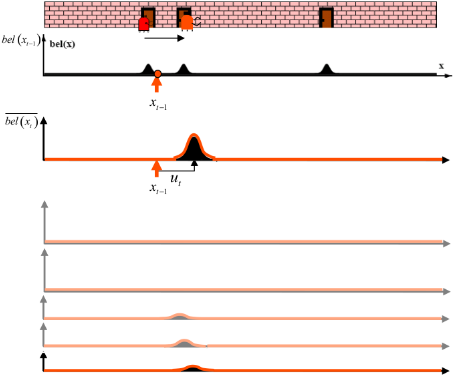

# *The Bayes Filter*

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  for all $x_t$ do
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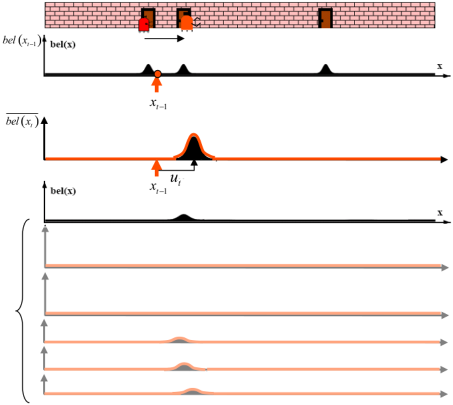
    $bel(x_t) = \eta\ p(z_t \mid x_t)\ \overline{bel}(x_t)$ — *Bayes Rule*
  endfor
  return $bel(x_t)$

$$bel(x_{t-1}) = p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})$$

$$p(x_t \mid u_t, x_{t-1})$$

$$=$$

$$\Sigma$$



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  endfor
  return $\boxed{bel(x_t)}$

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$$p(x_t \mid u_t, x_{t-1})$$

$$\overline{bel}(x_t) = p(x_t \mid z_{1:t-1}, u_{1:t})$$

$$p(z_t \mid x_t)$$

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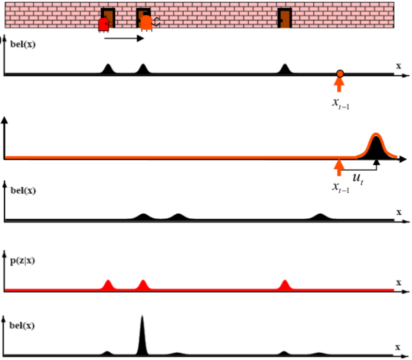
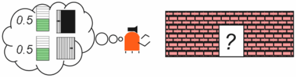


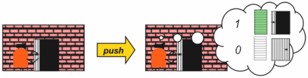
# Example: Is the door open or closed?



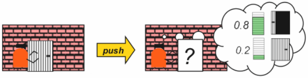

$$bel(X_0 = \text{open}) = 0.5$$
$$bel(X_0 = \text{closed}) = 0.5$$


$$p(Z_t = \text{sense\_open} \mid X_t = \text{is\_open}) = 0.6$$
$$p(Z_t = \text{sense\_closed} \mid X_t = \text{is\_open}) = 0.4$$

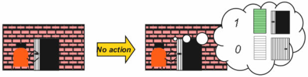

$$p(Z_t = \text{sense\_open} \mid X_t = \text{is\_closed}) = 0.2$$
$$p(Z_t = \text{sense\_closed} \mid X_t = \text{is\_closed}) = 0.8$$



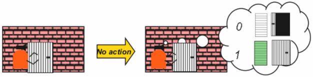

$$p(X_t = \text{is\_open} \mid U_t = \text{push}, X_{t\_1} = \text{is\_open}) = 1$$
$$p(X_t = \text{is\_closed} \mid U_t = \text{push}, X_{t\_1} = \text{is\_open}) = 0$$




$$p(X_t = \text{is\_open} \mid U_t = \text{push}, X_{t\_1} = \text{is\_closed}) = 0.8$$
$$p(X_t = \text{is\_closed} \mid U_t = \text{push}, X_{t\_1} = \text{is\_closed}) = 0.2$$



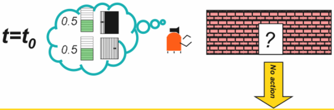

$$p(X_t = \text{is\_open} \mid U_t = \text{do\_nothing}, X_{t\_1} = \text{is\_open}) = 1$$
$$p(X_t = \text{is\_closed} \mid U_t = \text{do\_nothing}, X_{t\_1} = \text{is\_open}) = 0$$




$$p(X_t = \text{is\_open} \mid U_t = \text{do\_nothing}, X_{t\_1} = \text{is\_closed}) = 0$$
$$p(X_t = \text{is\_closed} \mid U_t = \text{do\_nothing}, X_{t\_1} = \text{is\_closed}) = 1$$

# *Example: Is the door open or closed?*

- Example




**Algorithm Bayes_filter**($bel(x_{t-1}), u_t, z_t$):
  for all $x_t$ do
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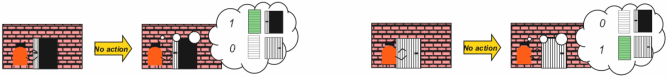
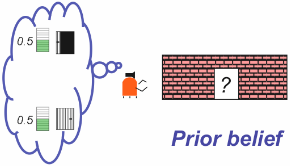
    $bel(x_t) = \eta\; p(z_t \mid x_t)\; \overline{bel}(x_t)$
  endfor
  return $bel(x_t)$

$u_1$=do_nothing




$$\begin{aligned}
&\overline{bel}(X_1 = \text{is_open}) \\
&= p(X_1 = \text{is_open} \mid U_1 = \text{do_nothing}, X_0 = \text{is_open})\; bel(X_0 = \text{is_open}) \\
&\quad + p(X_1 = \text{is_open} \mid U_1 = \text{do_nothing}, X_0 = \text{is_closed})\; bel(X_0 = \text{is_closed}) \\
&= 1 \cdot 0.5 + 0 \cdot 0.5 \;=\; 0.5 \qquad (2.46)
\end{aligned}$$

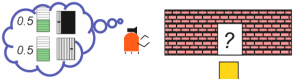
$$\begin{aligned}
&\overline{bel}(X_1 = \text{is_closed}) \\
&= p(X_1 = \text{is_closed} \mid U_1 = \text{do_nothing}, X_0 = \text{is_open})\; bel(X_0 = \text{is_open}) \\
&\quad + p(X_1 = \text{is_closed} \mid U_1 = \text{do_nothing}, X_0 = \text{is_closed})\; bel(X_0 = \text{is_closed}) \\
&= 0 \cdot 0.5 + 1 \cdot 0.5 \;=\; 0.5 \qquad (2.47)
\end{aligned}$$




***Prior belief***

# *Example: Is the door open or closed?*

### *Prior belief*

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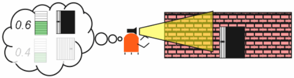
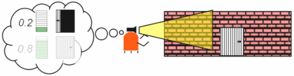
endfor
return $bel(x_t)$

*$z_1$=sense_open*



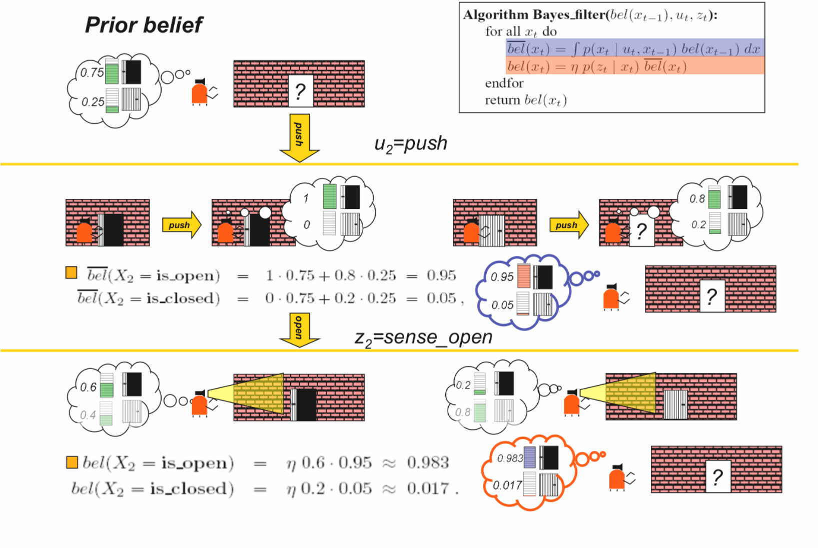




- $bel(X_1 = \text{is\_open})$
  $= \eta\, p(Z_1 = \text{sense\_open} \mid X_1 = \text{is\_open})\, \overline{bel}(X_1 = \text{is\_open})$
  $= \eta\, 0.6 \cdot 0.5 = \eta\, 0.3$ = 2.5 · 0.3 = 0.75
- $bel(X_1 = \text{is\_closed})$
  $= \eta\, p(Z_1 = \text{sense\_open} \mid X_1 = \text{is\_closed})\, \overline{bel}(X_1 = \text{is\_closed})$
  $= \eta\, 0.2 \cdot 0.5 = \eta\, 0.1$ = 2.5 · 0.1 = 0.25

Now, the normalizer can be computed $\eta = (0.3 + 0.1)^{-1} = 2.5$


***Belief at $t=t_1$***

# Example: Is the door open or closed?



# *Overview Bayes Filter*

- Belief of a robot is the posterior distribution over the state given all past measurements (z) and all past controls (u).
- Bayes Filter is the principal algorithm for calculating the belief in robotics.
- The Bayes Filter makes the markov assumption according to which the state is a complete summary of the past $\Rightarrow$ **belief is sufficient to represent the past history of the robot**

The Markov assumption postulates that past and future data are independent if one knows the current state.

# Overview Bayes Filter

- The Bayes Filter as shown is not efficient. We will study probabilistic algorithms that use tractable approximations to the Bayes Filter.
- The Non-Parametric Filters:
  - Histogram Filter
  - Particle Filter
- The Gaussian Filter:
  - Kalman Filter
  - Extended Kalman Filter
  - Unescended Kalman Filter
  - Information Filter

# *Questions?*