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An Information Based Multiechelon Inventory System with Emergency Orders

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AN INFORMATION BASED MULTIECHELON INVENTORY SYSTEM WITH EMERGENCY ORDERS

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In this paper we study a (S-1, S) type multiechelon inventory system where all the stocking locations have the option to replenish their inventory through either a normal or a more expensive emergency resupply channel. When ordering a unit, each stocking location decides which channel to use based on its inventory level and the remaining leadtimes of the its outstanding orders. We consider the implications of this policy by developing expressions for the operating characteristics of the system and describing a procedure for finding the optimal policy parameters. A numerical study presented in the paper indicates that when alternative resupply modes are available our policy, which incorporates the information on the remaining leadtimes of the outstanding orders in selecting the resupply mode, can result in considerable cost savings when compared to policies which allow a single resupply mode.

Multiechelon inventory systems are widely employed to distribute products to customers over extensive geographical areas. Given the importance of these systems, many researchers have studied their operating characteristics under a variety of conditions and assumptions; for example, see Sherbrooke (1968), Duermeyer and Schwarz (1981), Moinzadeh and Lee (1986), Svoronos and Zipkin (1988), Graves (1985), Zipkin and Svoronos (1990), Axsater (1990), and Nahmias (1981) for a survey. Few researchers, however, have studied multiechelon inventory systems with more than a single resupply mode, even though multiple resupply modes commonly exist in practice (e.g., one may choose to ship the product by overnight express instead of a "normal" truck-based service).

In this paper, we study a two-echelon inventory system consisting of a single warehouse and M retailers. Demand at each retailer is random and is assumed to follow a Poisson distribution. Each stocking location (i.e., the warehouse and the retailers) uses a (S-1, S) ordering policy. When a demand occurs, each stocking location places an order through either a normal or an emergency channel. If the latter channel is used, the shipping time is reduced but the stocking center pays a higher cost for the unit.

The importance of incorporating ordering policies which consider multiple resupply modes in multiechelon inventory/distribution systems has been discussed in previous works (e.g., see Graves 1985). The system studied here extends previous research which examined the issue of expediting. In a single location setting, these include Barankin (1961), Daniel (1962), Fukuda (1964), Wittmore and Saunders (1977), and Moinzadeh and Nahmias (1988). Other studies which consider emergency replenishments in multiechelon systems include Muckstadt and Thomas (1981), Aggarwal and Moinzadeh (1990), and Pyke and Cohen (1990). Recently, Moinzadeh and Schmidt (1991) considered a single location (S-1, S) inventory system with two options for resupply, normal, and emergency. Emergency orders arrive in a shorter time but at a premium cost. Assuming that leadtimes are known and constant, they proposed an order/expediting policy which incorporates the age of the outstanding orders. The steady state behavior of the system was studied, and some optimization results were presented. The model in this paper adopts the same class of policies proposed by Moinzadeh and Schmidt (1991), but extends their results to multiple stocking sites.

The rest of this paper is organized as follows. In the first section the basic model is described. Section 2 is devoted to the derivation of expressions for the operating characteristics of the system. In Section 3, the properties of the expected cost rate are investigated and an algorithm for finding the optimal policy parameters is developed. Finally, computational results and concluding remarks are presented and discussed in the fourth and fifth sections.

1. THE MODEL

Consider a two-echelon inventory system consisting of a warehouse and M retail centers each stocking a common product. Demand occurs at the retail centers according to a Poisson process with a mean rate λ_i for retailer i ($i = 1, \ldots, M$). If the retail center has the product, the demand is filled; otherwise, it is backordered. Whenever a demand occurs at the retail center, an order is immediately placed with the warehouse.[1] When placing an order, the retail center can expedite the shipping/transportation time from the

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warehouse by placing an emergency (as opposed to normal) order at a higher cost. In a similar fashion, if the warehouse has the stock on hand, it fills the retailer's order immediately; otherwise, it backorders the request, and the order faces a delay. The normal and emergency shipping times to retailer i are NT_i and ET_i, respectively (where NT_i and ET_i are known constants and $ET_i < NT_i$). When the warehouse receives an order from a retail center, it places an order with an outside supplier which carries an ample supply of the product. Similar to the retail centers, the warehouse also has the option of placing a normal or an emergency order at an additional cost with the outside supplier. The normal and emergency shipping times in this case are NT_0 and ET_0, respectively, where $ET_0 < NT_0$ and NT_0 and ET_0 are known constants.

We define the time between the placement of an order by a retail center until its actual delivery time as the order leadtime. The order leadtime is made up of two parts: the random delay, τ, defined as the time between the placement of an order by the retail center i and the release of the unit from the warehouse to fill the order, and the deterministic shipping time from the warehouse to retail center i. The shipping times from the outside source to the warehouse are deterministic; therefore the delay, τ, is always known at the warehouse. This information is communicated by the warehouse to the retailer at the time of ordering and is used by the retailer in determining the type of the order (normal or emergency). It should be emphasized that the type of the order will affect only the shipping/transportation times from the warehouse to the retailer and will have no effect on the delay experienced by the order. That is, orders are processed and dispatched from the warehouse on a first-come-first-served basis, regardless of type.[2] The order/expediting policy at each retailer is a modified one-for-one (S-1, S) type policy similar to the one discussed by Moinzadeh and Schmidt (1991), which may be described as follows.

Let S_i be the maximum stocking level at the retail center i and y_i be the minimum number of orders outstanding at retailer i that triggers the consideration of an emergency order. When a demand occurs and the inventory level is above the trigger level $S_i - y_i$, the retailer places a normal replenishment order with the warehouse. Otherwise, the retailer considers the remaining leadtime of its outstanding orders and places an emergency order if it would raise the inventory above the trigger level in the most expedient fashion. The above policy can be formally stated as follows:

Let n_i be the number of orders outstanding just before a demand occurs and x_j be the remaining leadtime of the jth oldest order outstanding at the retailer i. Then, when a demand occurs at retailer i:

$$\begin{array}{ll} \text{(i)} & \text{If } n_i < y_i, \text{ place a normal order.} \\ \text{(ii)} & \text{If } n_i \geq y_i \text{ and } x_{n_i - y_i + 1} \leq ET_i + \tau, \text{ place a normal order.} \\ \text{(iii)} & \text{If } n_i \geq y_i \text{ and } x_{n_i - y_i + 1} > ET_i + \tau, \text{ place an emergency order.} \end{array} \tag{1}$$

The form of the order/expediting policy at the warehouse is the same as the retail centers, except that there is no delay at the outside supplier. We assume that the inventory level which triggers the placement of an emergency order, $S_i - y_i$, is always greater than or equal to zero at all locations (or $S_i \geq y_i$).[3] This assumption is reasonable when backorder costs are significant when compared to the additional cost of placing emergency orders.

Finally, we would like to point out that the model described above may also represent a repairable inventory system similar to Graves (1985) and Sherbrooke (1968) with ample number of repair channels at the repair depot and two modes of repair and shipping times available to the system.

2. DERIVATION OF THE OPERATING CHARACTERISTICS

We now proceed by deriving the operating characteristics of the system. First, we define the following functions:

$$f(j, x) = \frac{x^j}{j!},$$

$$F(n, x) = \sum_{j=0}^{n} f(j, x).$$

Note that the warehouse behaves similar to the single-location model studied by Moinzadeh and Schmidt (1991). Let $P_{n_0 0}$ be the steady state probability of having n_0 orders outstanding at the warehouse. Then, from Moinzadeh and Schmidt (1991),[4] we have

$$P_{n_0 0} = \begin{cases} f(n_0, \lambda_0 NT_0) P_{00}, & n_0 < y_0, \\ \left[\sum_{j=0}^{y_0} \{ f(j, \lambda_0 (NT_0 - ET_0)) f(n_0 - j, \lambda_0 ET_0) \} \right] P_{00}, & n_0 \geq y_0, \end{cases} \tag{2}$$

where $\lambda_0 = \sum_{i=1}^{M} \lambda_i$ and P_{00} is the probability of having no orders outstanding at the warehouse, determined by normalizing the probabilities and can be expressed as:

$$P_{00} = [e^{\lambda_0 ET_0} F(y_0, \lambda_0 (NT_0 - ET_0))]^{-1}. \tag{3}$$

The above results can be explained through the following intuitive argument. At any point in time, t, all the outstanding orders which were placed by the warehouse to the outside supplier prior to $t - NT_0$ are delivered by t. Therefore, to find the probability of having n_0 orders outstanding at t, $P_{n_0 0}$, we only need to consider the orders which were placed in $[t - NT_0, t)$ in the following cases:

(i) For $n_0 < y_0$, $P_{n_0 0}$ is simply the product of the probability of having n_0 units demanded in $[t - NT_0, t)$ and a normalizing constant.

(ii) For $n_0 \geq y_0$, $P_{n_0 0}$ is equal to the product of a normalizing constant and the probability of having a total of n_0 units demanded in $[t - NT_0, t)$ with at most y_0 units (resulting in normal orders) of them occuring in $[t - NT_0, t - ET_0)$.

The interested reader is referred to Moinzadeh and Schmidt (1991, p. 309–310) for the details of the derivation.

To derive the probability distribution of the number of orders outstanding at the retailers, we first find the distribution of the delay at the warehouse. Since $S_0 \geqslant y_0$, then when the retail center places an order at the warehouse, the delay (if any) to be experienced by the order will be known at the retailer and, therefore, the promised delivery time of the order will not be changed subsequently. Furthermore, the delay takes values on the interval $[0, ET_0]$. At the end points it may have a positive probability mass, denoted by $g_1(\cdot)$, while on the interior it takes on a probability density, denoted by $g(\cdot)$. We derive the distribution of the delay at the warehouse in the following proposition:

Proposition 1. *Let*

$$g_1(t) = \begin{cases} \sum_{i=0}^{S_0-1} P_{i0}, & t = 0, \\ P_{00} e^{\lambda_0 ET_0} f(S_0, \lambda_0(NT_0 - ET_0)), & t = ET_0. \end{cases}$$

(*a*) *For* $S_0 > y_0$, *the delay at the warehouse takes values on the interval* $[0, ET_0)$. *Furthermore, its distribution has a probability mass of* $g_1(0)$ *at* $t = 0$ *and follows the probability density function* $g(t)$ *elsewhere, where*:

$$g(t) = \begin{cases} \lambda_0 P_{00} e^{\lambda_0 t} \sum_{j=0}^{y_0} f(j, \lambda_0(NT_0 - ET_0)) \\ \quad \cdot f(S_0 - 1 - j, \lambda_0(ET_0 - t)), & 0 < t < ET_0, \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

(*b*) *For* $S_0 = y_0$, *the delay at the warehouse takes values on the interval* $[0, ET_0)$. *Furthermore, its distribution has probability masses of* $g_1(0)$ *and* $g_1(ET_0)$ *at* $t = 0$ *and* ET_0, *respectively, and follows the probability density* $g(t)$ *elsewhere, where*:

$$g(t) = \begin{cases} \lambda_0 P_{00} e^{\lambda_0 t} f(S_0 - 1, \lambda_0(NT_0 - t)), & 0 < t < ET_0, \\ 0, \quad \text{otherwise}. \end{cases} \tag{5}$$

Proof. (a) Let N_0 be the number of orders outstanding at the warehouse just before the placement of an order by a retail center. We consider the following cases:

Case (*i*): If $N_0 < S_0$, then the delay, τ, will always be zero and, therefore, it has a probability mass equal to $g_1(0)$.

Case (*ii*): If $N_0 \geqslant S_0$, then the delay, τ, takes values on the interval $(0, ET_0)$. To find the distribution of the delay, we consider the remaining leadtimes of the outstanding orders at the warehouse before the arrival of the order, as shown in Figure 1(a). Once a demand occurs at a retail center, an order is immediately placed at the warehouse. The warehouse, in turn, will place either a normal or an

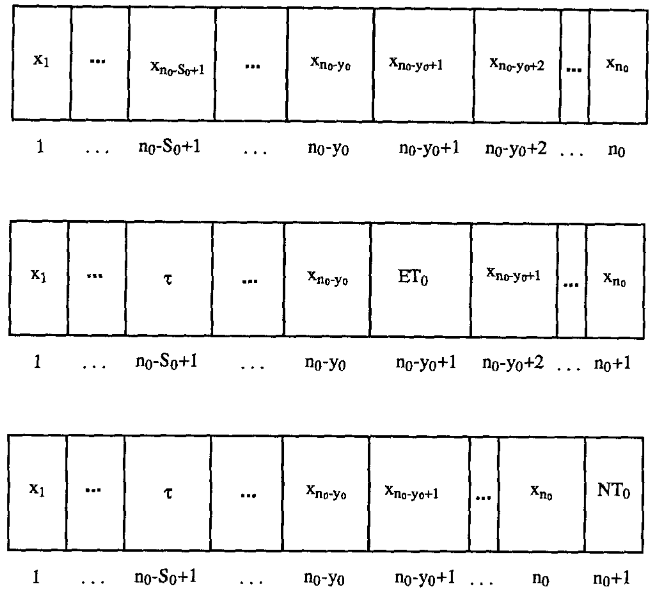

Figure 1. Remaining leadtimes of orders outstanding at the warehouse.
(a) Leadtimes at the warehouse before an order placement.
(b) Leadtimes at the warehouse after an emergency order is placed.
(c) Leadtimes at the warehouse after a normal order is placed.

emergency order with the outside supplier and the new remaining order leadtimes at the warehouse will be as shown in Figure 1(b and c).

As can be seen in Figure 1, τ, will be the remaining leadtime of the $(n_0 - S_0 + 1)$st order outstanding. It must also be noted that τ will always be less than ET_0 as,

$$x_{n_0 - S_0 + 1} \leqslant x_{n_0 - y_0} < ET_0.$$

The above relationship holds since $S_0 > y_0$. If $x_{n_0 y_0+1} > ET_0$, then the warehouse places an emergency order. Thus,

$$g(t) = \int_0^t \cdots \int_{x_{n_0-S_0-1}}^{t} \int_{x_{n_0-S_0+2}=t}^{ET_0} \cdots \int_{x_{n_0-y_0-1}}^{ET_0} \int_{x_{n_0-y_0+1}=ET_0}^{NT_0} \cdots \int_{x_{n_0-1}}^{NT_0} P_{n_0}(x_1, \ldots, x_{n_0})\, dx_{n_0} \ldots dx_{n_0-y_0}\, dx_{n_0-y_0+1} \ldots dx_{n_0-S_0+2}\, dx_{n_0-S_0} \ldots dx_1 \quad \text{for } x_{n_0-y_0+1} > ET_0. \tag{7}$$

Moinzadeh and Schmidt (1991, p. 310) have shown that $P_{n_0}(x_1, \ldots, x_{n_0}) = \lambda_0^{n_0} P_{00}$. Substituting in (7) and simplifying, we get

$$g(t) = \lambda_0 P_{00} f(n_0 - S_0, \lambda_0 t) \cdot f(S_0 - y_0 - 1, \lambda_0(ET_0 - t)) \cdot f(y_0, \lambda_0(NT_0 - ET_0)) \quad \text{for } x_{n_0-y_0+1} > ET_0. \tag{8}$$

If $x_{n_0-y_0+1} \leqslant ET_0$, then the warehouse places a normal order. Therefore,

$$g(t) = \int_0^t \cdots \int_{x_{n_0-S_0-1}}^t \int_{x_{n_0-S_0+2}=t}^{ET_0} \cdots \int_{x_{n_0-y_0}}^{ET_0} \int_{x_{n_0-y_0+1}}^{NT_0} \cdots \int_{x_{n_0-1}}^{NT_0} P_{n_0}(x_1, \ldots, x_{n_0})\, dx_{n_0} \ldots dx_{n_0-y_0+1}\, dx_{n_0-y_0+2} \ldots dx_{n_0-S_0+2}\, dx_{n_0-S_0} \ldots dx_1$$

$$\text{for } x_{n_0-y_0+1} \leqslant ET_0. \tag{9}$$

Define:

$$\begin{aligned}
\gamma(S_0, y_0, t) &= \int_{x_{n_0-S_0+2}=t}^{ET_0} \cdots \int_{x_{n_0-y_0}}^{ET_0} \int_{x_{n_0-y_0+1}}^{NT_0} \cdots \int_{x_{n_0-1}}^{NT_0} dx_{n_0} \ldots dx_{n_0-S_0+2} \\
&= \int_{x_{n_0-S_0+2}=t}^{ET_0} \cdots \int_{x_{n_0-y_0}}^{ET_0} f(y_0, NT_0 - x_{n_0-y_0+1}) \cdot dx_{n_0-y_0+1} \ldots dx_{n_0-S_0+2} \\
&= f(S_0 - 1, NT_0 - t) - \sum_{j=0}^{S_0-y_0} f(y_0 + j + 1, NT_0 - ET_0) \cdot f(S_0 - y_0 - j, ET_0 - t).
\end{aligned}$$

We can now rewrite (9) as:

$$g(t) = \lambda_0^{n_0} P_{00} f(n_0 - S_0, t)\gamma(S_0, y_0, t) \quad \text{for } x_{n_0-y_0+1} \leqslant ET_0. \tag{10}$$

Adding (8) and (10) and summing over the range of $n_0 \geqslant S_0$, after some algebraic manipulation, (4) can be obtained.

(b) The proof can be established using a similar line of argument as in (a). □

We now derive an expression for the probability distribution of the number of orders outstanding at the retailers in the following proposition.

Proposition 2. *The probability distribution of the number of orders outstanding at retailer i, P_{ni}, can be written as:*

$$P_{ni} = \begin{cases} E_\tau[f(n, \lambda_i(NT_i + \tau))P_{0i}(\tau)], & n < y_i, \\ E_\tau\left[\sum_{j=0}^{y_i}\{f(j, \lambda_i(NT_i - ET_i)) \cdot f(n - j, \lambda_i(ET_i + \tau))\}P_{0i}(\tau)\right], & n \geqslant y_i, \end{cases} \tag{11}$$

where

$$P_{0i}(\tau) = [e^{\lambda_i(ET_i + \tau)} F(y_i, \lambda_i(NT_i - ET_i))]^{-1}. \tag{12}$$

$P_{0i}(\tau)$ is defined as the conditional probability of having no orders outstanding at retailer i for a given τ and $E_\tau(\cdot)$ is the expectation over τ.

Proof. See the appendix.

3. DETERMINATION OF OPTIMAL STOCKING AND TRIGGER LEVELS

In this section we derive an expression for the expected total cost rate and present a procedure for finding the optimal policy parameters of the system at all locations. Let

h = unit holding cost/time,
π = unit backorder cost/time,
C_{Ei} = unit cost of an emergency replenishment at location i,
C_{Ni} = unit cost of a normal replenishment at location i,
$TC_i(S_i, y_i|\tau)$ = expected total cost/time at retailer i for a given value of τ and fixed S_0 and y_0, $i = 1, \ldots, M$,
$TC_i(S_i, y_i)$ = expected total cost/time at retailer i for fixed S_0 and y_0, $i = 1, \ldots, M$, and
$TC(\mathbf{S}, \mathbf{y}, S_0, y_0)$ = expected total cost/time of the system, where $\mathbf{S} = (S_1, S_2, \ldots, S_M)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_M)$.

We also define the following functions similar to Moinzadeh and Schmidt (1991, p. 312). Let

$$\begin{aligned}
I(n, x) &= \sum_{k=0}^{n} (n - k) f(k, x), \\
\cdot\ J(n, y_i|\tau) &= I[n, \lambda_i(NT_i + \tau)] - \sum_{j=y_i+1}^{n} f(j, \lambda_i(NT_i - ET_i)) \cdot I[n - j, \lambda_i(ET_i + \tau)] \\
&= \sum_{j=0}^{y_i} f(j, \lambda_i(NT_i - ET_i)) \cdot I[n - j, \lambda_i(ET_i + \tau)].
\end{aligned}$$

Clearly,

$$TC_i(S_i, y_i) = E_\tau[TC_i(S_i, y_i|\tau)],$$

where $TC_i(S_i, y_i|\tau)$ is given in Moinzadeh and Schmidt (1991, p. 312) and can be expressed as:

$$\begin{aligned}
TC_i(S_i, y_i|\tau) &= C_{Ni}\lambda_i + \lambda_i[C_{Ei} - C_{Ni} - \pi(NT_i - ET_i)] \cdot \frac{f(y_i, \lambda_i(NT_i - ET_i))}{F[y_i, \lambda_i(NT_i - ET_i)]} \\
&\quad - \pi[S_i - \lambda_i(NT_i + \tau)] + (h + \pi)P_{0i}(\tau)J(S_i, y_i|\tau).
\end{aligned}$$

The expected total cost/time of the system can now be written as:

$$\begin{aligned}
TC(\mathbf{S}, \mathbf{y}, S_0, y_0) &= C_{N0}\lambda_0 + (C_{N0} - C_{E0})\lambda_0 \cdot \frac{f(y_0, \lambda_0(NT_0 - ET_0))}{F(y_0 - 1, \lambda_0(NT_0 - ET_0))} \\
&\quad + hP_{00}J(S_0, y_0|0) + \sum_{i=1}^{M} E_\tau[TC_i(S_i, y_i|\tau)].
\end{aligned} \tag{13}$$

Note that (13) is separable in i, since the distribution of the delay depends only on S_0 and y_0. Therefore, in order to obtain the optimal system parameters, we first fix S_0 and y_0 and optimize over **S** and **y**, and then search for the optimal values of S_0 and y_0.

Proposition 3. *For fixed values of S_0, y_0, and y_i, $\mathrm{TC}_i(S_i, y_i)$ is convex in S_i and the value of S_i which minimizes $\mathrm{TC}_i(S_i, y_i)$, $S_i^*(y_i)$, satisfies:*

$$E_\tau[\Delta_{S_i} J(S_i - 1, y_i | \tau) P_{0i}(\tau)] < \frac{\pi}{\pi + h}, \tag{14}$$

$$E_\tau[\Delta_{S_i} J(S_i, y_i | \tau)] P_{0i}(\tau)] \geq \frac{\pi}{\pi + h}. \tag{15}$$

Furthermore, if $(C_{Ei} - C_{Ni})/\pi(NT_i - ET_i) \geq 1$, then it is never optimal for retail center i to place emergency orders with the warehouse.

Proof. Follows arguments similar to those presented in Proposition 4 in Moinzadeh and Schmidt (1991).

We note that left-hand sides of (14) and (15) are evaluated through numerical integration. Thus far, we can summarize our results as follows. For given values S_0 and y_0, we first determine $S_i^*(y_i)$ using Proposition 3. Then, we search for the optimal y_i starting at zero by evaluating $\mathrm{TC}_i(S_i^*(y_i), y_i)$ until a local minima is reached.[5] Since $\mathrm{TC}_i(S_i^*(y_i), y_i)$ may not necessarily be convex in y_i, we continue the search by examining a few values beyond the first observed local minima. Algorithm 1 states the exact steps in obtaining the optimal values for S_i and y_i for fixed values S_0 and y_0.

Algorithm 1

Step 1. Initialization

Set $y_i = 0$
$S_i(0) = 0$
$\mathrm{TC}_0 = \mathrm{TC}_i(S_i(0), 0)$
COUNT $= 0$

Step 2. Optimization of S_i

Set $S_i = S_i(y_i)$
$y_i = y_i + 1$
Increment S_i until (14) and (15) are satisfied
Set $S_i(y_i) = S_i$
$\mathrm{TC}_{y_i} = \mathrm{TC}_i(S_i, y_i)$

Step 3. Termination Test

If $\mathrm{TC}_{y_i} > \mathrm{TC}_{y_i - 1}$ then COUNT = COUNT + 1
If COUNT = COUNT_MAX then Stop
else Go To step 2

For the 360 test cases evaluated (see Section 4), we used a value of COUNT_MAX equal to 5.

Next, we focus on the optimization over S_0 for fixed values of y_0. For a given value of y_0, $\mathrm{TC}(\mathbf{S}^*, \mathbf{y}^*, S_0, y_0)$ may not necessarily be convex in S_0. However, for a fixed value of y_0, an upper bound for S_0^* can be found using the following two propositions.

Proposition 4. *For fixed values of y_0, $\mathbf{y} = \mathbf{S} = \mathbf{0}$, the optimal value of S_0, say $\overline{S_0}(y_0)$, must satisfy the following conditions:*

$$\Delta J_{S_0}(\overline{S_0}(y_0) - 1, y_0 | 0) P_{00} < \frac{\pi}{\pi + h}, \tag{16}$$

$$\Delta J_{S_0}(\overline{S_0}(y_0), y_0 | 0) P_{00} \geq \frac{\pi}{\pi + h}. \tag{17}$$

Proof. See the appendix.

Proposition 5. *Let $S_i^*(S_0)$ be the optimal value of the S_i for given values of y_i, S_0, and y_0. Then $S_i^*(S_0)$ is nonincreasing in S_0.*

Proof. See the appendix.

Corollary 1. $\overline{S_0}(y_0)$, which is obtained from (16) and (17), is an upper bound for $S_0^*(y_0)$.

Now for a fixed y_0, we vary S_0 over $y_0 \leq S_0 \leq \overline{S_0}(y_0)$, and obtain $S_0^*(y_0)$ yielding the lowest $\mathrm{TC}(\mathbf{S}^*, \mathbf{y}^*, S_0, y_0)$, which is evaluated by Algorithm 1. The search over y_0 is conducted using the same concepts discussed in Algorithm 1 for finding the optimal y_i.

4. COMPUTATIONAL RESULTS

In order to study the effectiveness of our model, we applied the optimization scheme developed in Section 3 to a set of test problems. In all, we evaluate 360 test cases composed of all possible combinations of the following parameters:

M:	10
λ_i:	0.1 (identical retail centers)
h:	1.0
C_{N0}:	1.0
ET_0:	1.0
NT/ET:	2, 6, 10 (same for all sites)
NT_i/NT_0:	1.2, 1.5
$\pi/\pi + h$:	0.75, 0.80, 0.85, 0.90, 0.95
C_{Ni}/C_{N0}:	1.5, 2.0
$C_E - C_N/\pi(NT - ET)$:	0.05, 0.10, 0.15, 0.20, 0.25, 0.30 (same for all sites)

In addition, we also computed the optimal expected total cost rate for three other policies, P1, P2, and P3. Each of these three policies is restricted to a single resupply mode, as in METRIC type models (for example, see Sherbrooke 1968 and Graves 1985). Under policy P1, only normal orders can be placed. Policy P2 is similar, but only emergency orders are allowed. Policy P3 selects the best of these two policies for each case. For each policy, we computed the average and the maximum percent deviations in the expected total cost rate from our policy. The results of these comparisons are presented in Table I. We note that the optimal average cost obtained by solving the METRIC type model may actually be lower than the one obtained from our policy, since it assumes that $y \to \infty$ (i.e., no

Table I
Comparison of Our Policy Against Policies P1, P2, and P3 as the Cost Parameters Are Varied

		% Deviation* in our policy vs.					
		Policy P1		Policy P2		Policy P3	
$\pi/(\pi + h)$	$C_E - C_N/$ $\pi(NT - ET)$	AVG	MAX	AVG	MAX	AVG	MAX
	0.05	76.72	115.47	0.22	0.69	0.22	0.69
	0.10	58.94	82.32	1.51	3.40	1.51	3.40
0.75	0.15	45.18	58.26	2.62	5.57	2.62	5.57
	0.20	34.03	44.79	3.51	7.24	3.51	7.24
	0.25	25.15	33.87	4.56	9.24	4.56	9.24
	0.30	17.91	25.52	5.75	11.19	5.75	11.19
	0.05	70.18	103.27	1.00	2.17	1.00	2.17
	0.10	52.48	70.54	3.20	6.39	3.20	6.39
0.80	0.15	39.68	47.72	5.57	9.81	5.57	9.81
	0.20	29.63	35.27	7.82	13.55	7.82	13.55
	0.25	21.16	25.93	9.69	16.55	9.69	16.55
	0.30	13.90	19.88	11.27	19.02	7.39	16.01
	0.05	61.75	91.73	4.05	5.17	4.05	5.17
	0.10	44.49	59.64	7.64	10.72	7.64	10.72
0.85	0.15	31.95	38.38	10.92	16.61	10.92	16.61
	0.20	21.89	31.30	13.59	21.18	13.58	21.10
	0.25	13.61	28.43	15.82	24.81	8.30	11.06
	0.30	6.90	25.68	18.02	27.75	2.88	12.44
	0.05	49.85	82.25	2.55	5.67	2.55	5.67
	0.10	32.72	50.78	7.86	15.03	7.86	15.03
0.90	0.15	20.78	29.05	12.83	22.42	11.81	22.42
	0.20	12.78	23.24	18.62	27.98	7.36	12.80
	0.25	7.31	19.65	25.28	36.90	3.44	5.08
	0.30	3.50	16.27	32.84	50.11	1.08	6.25
	0.05	44.42	94.20	21.46	28.31	17.59	28.31
	0.10	31.84	67.42	37.06	48.32	25.66	44.74
0.95	0.15	25.85	56.73	54.33	71.98	25.85	56.73
	0.20	20.70	47.62	70.11	96.35	20.70	47.62
	0.25	16.30	40.11	84.79	119.49	16.30	40.11
	0.30	12.32	33.33	98.29	142.22	12.32	33.33
Overall		31.36	115.47	19.76	142.22	8.42	56.73

*% Deviation = TC(P1 or P2 or P3) − TC(our policy)/TC(our policy) * 100

expediting), whereas our model restricts the value of y to be less than or equal to S. However, in all the test cases considered in this study, our policy outperformed the ones obtained by solving the METRIC type model.

As evident in Table I, each of the three policies examined may result in a significant cost deviation when compared to our policy. Specifically, the average (maximum) percent deviation for P1, P2, and P3 from our policy was found to be 31.36% (115.47%), 19.76% (142.22%), and 8.42% (56.73%), respectively. In summary, among the three policies considered, P3 performs better, as expected. Furthermore, one may find policy P1 (P2) reasonable, when the cost of emergency orders represented by the ratio $(C_E - C_N)/\pi(NT - ET)$, increases (decreases). Policy P3 performs well when $(C_E - C_N)/\pi(NT - ET)$ takes small or large values. The policy becomes ineffective for intermediate values of $(C_E - C_N)/\pi(NT - ET)$. Finally, we close this section by noting that the average computational requirement for solving a problem using our policy which involves searching for the optimal values of y_0, $\mathbf{y}$, S_0 and $\mathbf{S}$ was found to be 21.39 CPU seconds, compared to 5.96 CPU seconds for P1 in which the search was only performed for the optimal values of S_0 and $\mathbf{S}$.[6]

5. CONCLUDING REMARKS

In this paper, we considered a multiechelon inventory system with two modes of resupply, normal, and emergency. We proposed and tested an order/expediting policy which uses the information about the remaining leadtimes of the orders in the pipeline. The result of our numerical experiment indicates that our policy outperforms other policies which are based on a single mode of resupply. We remark that, in our model, expediting was only modelled in the shipping/transportation times from the warehouse to each location and no priority dispatching policies were considered as a part of expediting at the warehouse. A valuable extension to our work will be the development and the analysis of information based policies similar to ours in studying models with some type of priority dispatching at the warehouse.

APPENDIX

Proof of Proposition 2. Let
$P_t(n, x_1, \ldots, x_n|\tau)$ = probability density of having n orders outstanding with the remaining leadtimes $(x_1, \ldots, x_n)$ at

retailer i, given that the latest information on the delay at the warehouse is τ.

Since orders of the same type are received in the same sequence they were placed at each retailer, at epochs where no demand occurs at i, we have:

$$x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n < NT_i + \tau. \tag{A1.a}$$

In addition, $n > y_i$ we have:

$$x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{n-y_i} < ET_i + \tau \quad \text{if } n > y_i. \tag{A1.b}$$

We first consider the evolution of the system at the times when no demand occurs. Using an approach similar to Moinzadeh and Schmidt (1991, refer to the Section 1 of this paper for more details), the system of partial differential equations for $P_i(n, x_1, \ldots, x_n|\tau)$ can be written as:

$$\lambda_i P_i(0|\tau) = P_i(1, 0|\tau), \tag{A2.a}$$

$$\lambda_i P_i(n, x_1, \ldots, x_n|\tau) - \sum_{j=1}^{n} \partial P_i(n, x_1, \ldots, x_n|\tau) \cdot P_i(n+1, 0, x_i \ldots, x_n|\tau) \quad n > 0. \tag{A2.b}$$

The boundary conditions for the above system of partial differential equations are obtained by considering the demand epochs at retailer i. Suppose τ is the delay which is to be experienced by the order placed by i as a result of a demand. Using the fact that $S_0 \geqslant y_0$ and noting that (A1) holds, in a similar fashion to Moinzadeh and Schmidt (1991) we get:

$$P_i(n, x_1, \ldots, x_{n-1}, NT_i + \tau|\tau) = \begin{cases} \lambda_i P_i(n-1, x_1, \ldots, x_{n-1}|\tau), \\ \qquad (n<y_i) \text{ or } (n \geqslant y_i \ \& \ x_{n-y_i} \leqslant ET_i + \tau), \\ 0, \qquad \text{otherwise,} \end{cases}$$

and

$$P_i(n, x_1, \ldots, x_{n-y_i}, ET_i + \tau, x_{n-y_i+1}, \ldots, x_{n-1}|\tau) = \begin{cases} \lambda_i P_i(n-1, x_1, \ldots, x_{n-y_i}, x_{n-y_i+1}, \ldots, x_{n-1}|\tau), \\ \qquad n > y_i \ \& \ x_{n-y_i+1} > ET_i + \tau, \\ 0, \qquad \text{otherwise.} \end{cases}$$

The above boundary equations simply represent the transition to the boundary from the neighboring state when a demand occurs. A solution to the above system of partial differential equations and their boundary conditions is:

$$P_i(n, x_1, \ldots, x_n|\tau) = \lambda_i^n P_{0i}(\tau), \quad \text{where } P_{0i}(\tau) = P_i(0|\tau)$$

is the normalizing constant and denotes the probability of having no orders outstanding at retailer i for a given value of τ. The conditional probability of having n orders outstanding at retailer i for a given value of τ, $P_{ni}(\tau)$ can now be obtained by integrating over the possible ranges of x_j ($j = 1, \ldots, n$) which upon simplification can be written as:

$$P_{ni}(\tau) = \begin{cases} f(n, \lambda_i(NT_i + \tau)) P_{0i}(\tau), \\ \qquad n < y_i, \\ \sum_{j=0}^{y_i} \{ f(j, \lambda_i(NT_i - ET_i)) f(n-j, \lambda_i(ET_i + \tau)) \} P_{0i}(\tau), \\ \qquad n \geqslant y_i. \end{cases} \tag{A3}$$

The proof can now be established using (A3) and the laws of probability. □

Proof of Proposition 4. We define

$$\Phi(S_0) = \mathrm{TC}(0, 0, S_0, y_0).$$

Then, from (13) we have,

$$\Phi(S_0) = hP_{00}J(S_0, y_0|0) + C_{N0}\lambda_0 + (C_{N0} - C_{E0})\lambda_0 \cdot \frac{f(y_0, \lambda_0(NT_0 - ET_0))}{F[y_0 - 1, \lambda_0(NT_0 - ET_0)]} + \sum_{i=1}^{M} [C_{Ei}\lambda_i + \pi\lambda_i ET_i + \pi\lambda_i E(\tau)]. \tag{A4}$$

Now

$$\Delta\Phi(S_0) = hP_{00}\Delta_{S_0}J(S_0, y_0|0) + \pi\lambda_0\Delta_{S_0}E(\tau) \tag{A5}$$

where

$$E(\tau) = \frac{1}{\lambda_0} \sum_{n_0=S_0}^{\infty} n_0 P_{n_0 0}.$$

Thus, using (2) we can show that:

$$\Delta_{S_0}E(\tau) = \frac{-P_{00}}{\lambda_0} \sum_{j=0}^{y_0} \left[F(j, \lambda_0(NT_0 - ET_0)) \cdot \sum_{k=S_0-j+1}^{\infty} f(k, \lambda_0 ET_0) \right]$$
$$= \frac{-1}{\lambda_0} + \frac{P_{00}}{\lambda_0} \Delta_{S_0}J(S_0, y_0|0). \tag{A6}$$

Employing (A6) in (A5), we get

$$\Delta\Phi(S_0) = (h + \pi)P_{00}\Delta_{S_0}J(S_0, y_0|0) - \pi$$

and

$$\Delta^2\Phi(S_0) = (h + \pi)P_{00}\Delta^2_{S_0}J(S_0, y_0|0) \geqslant 0.$$

Thus, $\Phi(S_0)$ is convex in S_0 and, therefore, $\overline{S_0}(y_0)$ must satisfy (16) and (17). The proof is now complete. □

Proof of Proposition 5. We prove the proposition only for the case when $S_0 > y_0$ here, since the proof for the case when $S_0 = y_0$ follows a similar line of argument. Let $S_1 = S_i^*(S_0)$ and $S_2 = S_i^*(S_0 + 1)$. We would like to show that $S_2 \leqslant S_1$. Using the results obtained in Proposition 3, with some effort, we get

$$[\Delta_{S_t} J(S_2, y_t|0) P_{0t}(0)] \sum_{t=0}^{S_0} P_{0t} - [\Delta_{S_t} J(S_1 - 1, y_t|0) P_{0t}(0)]$$
$$\cdot \sum_{t=0}^{S_0-1} P_{t0} + \int_{t=0}^{ET_0} P_{0t}(t)[\Delta_{S_t} J(S_2, y_t|t)$$
$$\cdot g_{S_0+1}(t) - \Delta_{S_t} J(S_1 - 1, y_t|t)$$
$$\cdot g_{S_0}(t)]\, dt > 0, \qquad \text{(A7)}$$

where $g_{S_0}(t)$ is the density function of the delay, τ, for a given value of S_0.

Suppose that $S_1 < S_2$. Then, we will show that (A7) cannot hold (proof by contradiction). Let G represent the left hand side of the inequality (A7). If $S_1 < S_2$, we have:

$$\Delta_{S_t} J(S_2, y_t|t) - \Delta_{S_t} J(S_1 - 1, y_t|t) > 0. \qquad \text{(A8)}$$

Furthermore, we can easily show that:

$$P_{0t}(t) \Delta_S J(S, y_t|t) \leq 1. \qquad \text{(A9)}$$

Using (A8) and then (A9), we have

$$G < P_{0S_0}[P_{0t}(0) \Delta_{S_t} J(S_2, y_t|0)]$$
$$+ \int_{t=0}^{ET_0} P_{0t}(t) \Delta_{S_t} J(S_2, y_t|t)[g_{S_0+1}(t) - g_{S_0}(t)]\, dt$$
$$< P_{0S_0} + \int_{t=0}^{ET_0} [g_{S_0+1}(t) - g_{S_0}(t)]\, dt$$
$$< P_{0S_0} + \left[1 - \sum_{i=0}^{S_0} P_{t0}\right] - \left[1 - \sum_{t=0}^{S_0-1} P_{t0}\right] = 0,$$

which is a contradiction. Hence, (A7) can only be satisfied when $S_2 \leq S_1$. □

ENDNOTES

1. It is assumed that the retailers can replenish their stock only from the warehouse.
2. Though we do not address the issue of priority dispatching of orders at the warehouse in this paper, it is quite interesting to consider policies where the warehouse gives a higher priority to emergency orders as opposed to normal orders as an extension to this work.
3. In fact, our analysis requires only $S_0 \geq y_0$, and the results presented in this paper still hold when the assumption is relaxed at the retailers.
4. In Moinzadeh and Schmidt (1991), the results were presented (see p. 310) for the net inventory levels.
5. Numerical integration is used to evaluate $TC_i(S_i^*(y_i), y_i)$.
6. All the computations were performed on a VAX 6530 DECnet system. Also, in our computations, a 20-step increment was used when evaluating the integrations over the interval $[0, ET_0]$.

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