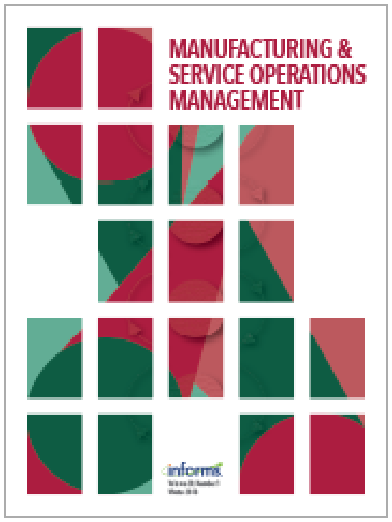

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Approximate Solutions of a Dynamic Forecast-Inventory Model

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In this paper we consider a dynamic forecast-inventory model with forecast updates, based on the martingale model of forecast evolution. Two types of updates are considered, additive and multiplicative. The formulation of the model results in a dynamic program with multidimensional state space. We derive some characteristics of optimal policies and also develop a computational approach to obtain approximate solutions. The approach is based on simulation and function approximation.

Key words: inventory; demand forecasts; approximate solutions
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1. Introduction

This paper analyzes an inventory model that explicitly includes demand forecasting. For the latter we use a variant of the martingale model of forecast evolution (MMFE) of Graves et al. (1986) and Heath and Jackson (1994).

We derive structural results describing how the optimal procurement policy depends on current forecasts of future demands. In particular, we define a new type of myopic policy and derive conditions for its optimality. These results enable us to address some theoretically and managerially important questions: How can forecast information be used sensibly within inventory-control methods? Which and how much information is really important? The results indicate, among other things, that near-term forecasts contain more significant information than long-term ones. Such results are useful in focusing forecasting and procurement efforts.

We also develop a computational technique to approximately solve the rather challenging dynamic programs to which such models give rise. The approach is based on simulation and functional approximation. Numerical tests indicate that the method is quite effective. Also, we discuss managerial insights obtained through the experiments.

The MMFE approach has several advantages. It is not a specific demand model, but rather a framework for representing the dynamics of forecasts. It is compatible with a wide range of forecasting techniques, including statistical and judgment-based methods. Also, it can represent nonstationary and correlated demands.

Several works have explored the use of MMFE-like models within larger models for operational decisions. Building on earlier work by Hausman (1969), Heath and Jackson (1994) introduced the MMFE. They also incorporated it into a simulation model, which contains a linear programming model for production and distribution planning. Graves et al. (1986) independently developed a model similar to the MMFE and used it within a production smoothing model. Graves et al. (1998) incorporated it into a system that mimics the logic of material requirements planning (MRP). Güllü (1996) analyzed the value of forecast information in procurement decisions, using a special case of the MMFE. Toktay and Wein (1999) analyzed a stationary capacitated inventory model with the MMFE, considering a particular class of policies called forecast-corrected base-stock policies. Gallego and Özer (2001) developed an inventory model with advance demand information. The MMFE can be interpreted as a case of their model. They obtained some structural results similar to ours.

An alternative approach is to employ a specific demand model based on a specific forecasting technique, such as the autoregressive moving average (ARMA) process. Works in this vein include Johnson and Thompson (1975), Miller (1986), Reyman (1989), and Graves (1999). Multistage models with ARMA demand were studied by Erkip et al. (1990) and Dong and Lee (2003).

Both the MMFE and the ARMA approaches are special cases of broader models that incorporate Markovian state variables, including those of Lovejoy (1992) and Song and Zipkin (1993). Indeed, some forecasting methods (e.g., ARIMA) cannot be represented in the MMFE framework; they require additional state variables. Still, the MMFE captures an important and interesting class of forecasting processes and is thus worth studying.

Also, there is a long line of research on models with nonstationary but independent demands. See Karlin (1960a, b), Veinott (1963, 1965), Schäl (1976), Morton (1978), Zipkin (1989), Morton and Pentico (1995), and Iida (1999). Some of our structural results generalize those found in this literature.

Section 2 briefly reviews the MMFE and formulates the model. Section 3 presents some structural properties of optimal policies. Section 4 develops a computational approach. Section 5 evaluates the computational approach with numerical experiments and presents managerial insights obtained through them. Finally, §6 concludes the paper and summarizes the results.

2. Formulation

Consider a finite-horizon, periodic-review model. The periods are numbered $1, 2, \ldots, T$. Demands are generated by a forecast-update process, described below. The lead time is zero for now. (Appendix A shows that the results extend to a positive lead time.) Unsatisfied demands are backlogged. The objective is total discounted expected costs, which include purchase costs, inventory-holding costs, and backlog-penalty costs. Those costs are linear.

At the beginning of a period an order decision is made, based on the current demand forecasts, and a corresponding purchase cost is paid. During the period the forecasts are updated, the order arrives, and customer demand occurs. At the end of the period the holding or penalty costs are charged.

2.1. The Martingale Model of Forecast Evolution

We maintain a forecast of demand in all future periods. The *initial forecasts* are given at the beginning of the first period. In each period we revise these forecasts by observing the *forecast-update vector*, which represents all new information about future demands.

Let $D_{t,s}$ denote the forecast at the *end* of period t of demand in period s, $s > t$, and $D_{t,t}$ the actual demand in period t. Let D_t^T denote the forecast vector at the *beginning* of period t; i.e., $D_t^T = (D_{t-1,t}, D_{t-1,t+1}, \ldots, D_{t-1,T})$. We consider two types of forecast updates: additive and multiplicative.

For additive updates, define $e_{t,s} \equiv D_{t,s} - D_{t-1,s}$, the *forecast update* in period t for the demand in period s. Fix $e_{t,s} = 0$, $s < t$. Assume $E[e_{t,s}] = 0$. This means that the updates embody new information available in period t but not before (see Heath and Jackson 1994, pp. 21–22). Let e_t^T denote the forecast-update vector at period t; i.e., $e_t^T = (e_{t,t}, e_{t,t+1}, \ldots, e_{t,T})$. The forecast-update vectors are independent over t. The updates within period t, however, are not necessarily independent over s.

For multiplicative updates, similarly define $e_{t,s} \equiv D_{t,s}/D_{t-1,s}$, and $e_{t,s} = 1$ for $s < t$. Here, $E[e_{t,s}] = 1$. Again, let e_t^T denote the forecast-update vector at period t. We assume that $D_{t,s}$ and $e_{t,s}$ are positive. Again, the updates are independent over t but not necessarily over s.

The initial forecasts can represent nonstationary demands, i.e., seasonal variations (periodic forecasts), trends (increasing or decreasing forecasts), and so on.

2.2. Dynamic Programming Formulation

Let

x_t: inventory position at the beginning of period t before ordering

y_t: inventory position at the beginning of period t after ordering

c_t, h_t, p_t: unit purchase cost, holding cost, and penalty cost in period t

c_{T+1}: salvage value at the end of period T

α: discount rate, $0 \le \alpha \le 1$.

Assume the cost factors are bounded: $c_- \le c_t \le c_+$, $0 \le h_t \le h_+$ for some constants $0 \le c_- \le c_+$ and $h_+ > 0$, and $p_t \ge 0$, for all t. The state at the beginning of period t is (x_t, D_t^T). The dynamics of the model are

$$x_{t+1} = y_t - (D_{t-1,t} \oplus e_{t,t}),$$
$$D_{t,s} = D_{t-1,s} \oplus e_{t,s} \quad \text{for } s = t, t+1, \ldots, T,$$

where $\oplus$ indicates the ordinary addition operator for additive updates and the ordinary multiplication operator for multiplicative updates.

Let $\tilde{J}_{t,T}(x, D_t^T)$ denote the optimal discounted expected cost from period t through T, given preorder inventory position x and demand-forecast vector D_t^T. Note that $\tilde{J}_{T+1,T}(x) = -c_{T+1}x$. We apply the standard transformation $J_{t,T}(x, D_t^T) \equiv c_t x + \tilde{J}_{t,T}(x, D_t^T)$ for $t = 1, 2, \ldots, T+1$. These functions obey the recursion

$$J_{t,T}(x, D_t^T) = \min_{y \ge x} \{ C_t(y, D_{t-1,t}) + \alpha E[J_{t+1,T}(y - D_{t,t}, D_{t+1}^T)] \},$$
$$J_{T+1,T}(x) \equiv 0, \tag{1}$$

where

$$C_t(y, D_{t-1,t}) \equiv (c_t - \alpha c_{t+1})y + h_t E(y - D_{t,t})^+ + p_t E(y - D_{t,t})^- + \alpha c_{t+1} D_{t-1,t}.$$

Let $\bar{h} \equiv h_+ + c_+ - \alpha c_-$. We hereafter call $C_t(y, D_{t-1,t})$ the one-period cost and $J_{t,T}(x, D_t^T)$ the optimal expected cost. Also define the auxiliary cost function $G_{t,T}(y, D_t^T)$, $t = 1, \ldots, T$, as follows:

$$G_{t,T}(y, D_t^T) = C_t(y, D_{t-1,t}) + \alpha E[J_{t+1,T}(y - D_{t,t}, D_{t+1}^T)]. \tag{2}$$

The expectations are taken with respect to the forecast-update vector e_t^T.

Remark. It is easily shown by induction on t that $G_{t,T}$ is convex in y, and a base-stock policy is optimal. The optimal base-stock level depends on t and the demand-forecast vector. Denote it by $y_{t,T}^*(D_t^T)$.

Remark. The optimal base-stock level is nondecreasing in the demand-forecast vector, that is, for two demand forecast vectors $D_t^{T,1}$ and $D_t^{T,2}$, if $D_t^{T,1} \ge D_t^{T,2}$, then $y_{t,T}^*(D_t^{T,1}) \ge y_{t,T}^*(D_t^{T,2})$. This too can be shown by induction on t.

Song and Zipkin (1993), Lovejoy (1992), and Gallego and Özer (2001) show analogous results for similar models.

Remark. For the case of one-period-ahead, additive updates, Güllü (1996) shows how to modify the inventory position to reduce the dimension of the state space by one. That technique can be used here as well, for the general case of additive updates.

We hereafter assume that all relevant functions are differentiable.

2.3. Examples

Consider the special case where only the forecasts for a fixed number of future periods are updated. This is a common practice, because it is often difficult, expensive, or impossible to update the forecasts for all future periods. We call this number the *forecast-update horizon* and denote it M. Thus,

$$D_{t-1,t+M-1} = D_{0,t+M-1},$$
$$D_{t-1,t+M} = D_{0,t+M}, \ldots, D_{t-1,T} = D_{0,T}.$$

Because the unchanged demand forecasts contain no new information, we can omit them from the state. The state at period t can be reduced to $(x_t, D_{t-1,t}, D_{t-1,t+1}, \ldots, D_{t-1,t+M-2})$.

Next, consider a variant of this scenario: The updates beyond M periods ahead are the same as the update for $M-1$ periods ahead; that is,

$$e_{t,s} = e_{t,t+M-2}, \quad \text{for } s > t+M-2.$$

We illustrate the MMFE with two simple examples.

Example. Nonstationary Independent Demands. Here the demands are independent over t but not necessarily identical. Let μ_t, $t = 1, \ldots, T$ be the mean demand in period t, and $D_{t,t} = \mu_t + \eta_t$, where the η_t are iid. This is a special case of the MMFE with

$$e_{t,t} = \eta_t,$$
$$e_{t,s} = 0 \quad \text{for } s \ge t+1, \quad \text{and}$$
$$D_{0,t} = \mu_t \quad \text{for all } t.$$

The forecast-update horizon is $M = 1$. When μ_t is constant over t, the model describes stationary independent demands.

Example. Stationary Demands with Two-Period Updates. Let μ be the mean demand and $D_{t,t} = \mu + e_{t-1,t} + e_{t,t}$ the demand in period t. This model was studied by Güllü (1996). In our terms

$$e_{t,s} = 0 \quad s \geq t+2,$$
$$D_{0,t} = \mu.$$

Here, $M = 2$. For example, this model can represent the moving-average process of lag 1, referred as to $MA(1)$. For that case $D_t = \mu + \eta_t + \theta_1 \eta_{t-1}$, where the η_t are iid $N(0, \sigma^2)$. In the terms of the MMFE,

$$e_{t,s} = 0 \quad s \geq t+2,$$
$$e_{t,t+1} = \theta_1 \eta_t,$$
$$e_{t,t} = \eta_t,$$
$$D_{0,t} = \mu.$$

3. Some Properties of Optimal Base-Stock Levels

This section develops some interesting and useful qualitative results about the behavior of the optimal policy. We show that the optimal base-stock levels monotonically decrease as the planning horizon gets longer. Consequently, the optimal base-stock levels for planning horizon T are upper bounds on those for longer planning horizons. Also, we develop lower bounds on the optimal base-stock levels. These lower bounds monotonically increase as the planning horizon gets longer. Thus, once we obtain the optimal base-stock levels and their lower bounds for planning horizon T, they provide upper and lower bounds on the optimal base-stock levels for any longer planning horizon. The difference between the upper and lower bounds decreases as the planning horizon gets longer. Under some conditions this difference converges to zero. We also develop an approximate, myopic-like policy.

3.1. Planning Horizon Effects and Lower Bounds

We first show that optimal base-stock levels for the T-horizon problem are larger than those for the $(T+1)$-horizon problem, obtained by extending the original problem by one period. (Proofs of the propositions below are given in Appendix B.)

Proposition 1. *For any $t \leq T$, D_t^T, and $D_{t-1,T+1}$,*

$$y^*_{t,T+1}(D_t^{T+1}) \leq y^*_{t,T}(D_t^T),$$

where $D_t^{T+1} = (D_t^T, D_{t-1,T+1})$.

At first glance, this result may seem to contradict Corollary 1 of Karlin (1960a). Karlin considered two sequences of demand distributions differing only in period $T+1$. One sequence has zero demand in period $T+1$, and the other has any nonnegative demand distribution. He showed that the optimal base-stock levels for the first sequence are smaller than those for the second. This situation is different from the one we consider, however. His model includes the same costs in period $T+1$ for both sequences, so his result is not quite a comparison of different planning horizons.

Next we develop lower bounds on the optimal base-stock levels. Assume $\alpha < 1$. For $k \leq T$ let

$$J^L_{k+1,k,T}(x) \equiv \frac{\bar{h}(1-\alpha^{T-k})}{1-\alpha} \cdot x. \tag{3}$$

For the k-horizon problem, using (3) as a terminal condition of the functional Equation (1), compute optimal base-stock levels and cost functions corresponding to $y^*_{t,k}(D_t^k)$, $J_{t,k}(x, D_t^k)$, and $G_{t,k}(y, D_t^k)$ for $t \leq k$, respectively. Denote them by $y^L_{t,k,T}(D_t^k)$, $J^L_{t,k,T}(x, D_t^k)$, and $G^L_{t,k,T}(y, D_t^k)$, respectively. We show that for any $k \leq T$, $y^L_{t,k,T}(D_t^k)$ is a lower bound on $y^*_{t,T}(D_t^T)$, and $y^L_{t,k,T}(D_t^k)$ increases as k gets larger.

Proposition 2. *For any $k \leq T$, all $t \leq k$, D_t^k and $D_{t-1,k+1}, \ldots, D_{t-1,T}$,*

$$y^L_{t,k,T}(D_t^k) \leq y^*_{t,T}(D_t^T),$$

where $D_t^T = (D_t^k, D_{t-1,k+1}, \ldots, D_{t-1,T})$.

Proposition 3. *For any $k \leq T$, all $t \leq k$, D_t^k and $D_{t-1,k+1}$,*

$$y^L_{t,k,T}(D_t^k) \leq y^L_{t,k+1,T}(D_t^{k+1}),$$

where $D_t^{k+1} = (D_t^k, D_{t-1,k+1})$.

Remark. Let $y^L_{t,k,\infty}(D_t^k)$ denote the optimal base-stock level in period t for the k-horizon problem with boundary condition (3), whose T is set to ∞. That is, set the constant in the right-hand side of (3) to

$\bar{h}/(1-\alpha)$. Similarly define $G^L_{t,k,\infty}$. One can show, as in Propositions 2 and 3, that $y^L_{t,k,\infty}(D^k_t)$ is a lower bound on the optimal base-stock level in period t for any finite planning horizon longer than k, and it increases in k.

3.2. Differences Between the Optimal Base-Stock Levels and the Lower Bounds

We showed that the optimal base-stock levels, $y^*_{t,k}(D^k_t)$, decrease as the planning horizon k gets longer and that the lower bounds, $y^L_{t,k,\infty}(D^k_t)$, increase. Next, we show that under mild conditions the differences between them decrease and converge to zero as k gets longer. Again, assume $\alpha < 1$.

Let $Y_{t,k}(D^k_t)$ denote the interval $[y^L_{t,k,\infty}(D^k_t), y^*_{t,k}(D^k_t)]$, and let

$$m_{t,k}(D^k_t) \equiv \min_{y \in Y_{t,k}(D^k_t)} \frac{\partial^2}{\partial y^2} C_t(y, D_{t-1,t}).$$

For any $t \le k$, y, and D^k_t,

$$\frac{\partial^2}{\partial y^2} G_{t,k}(y, D^k_t) \ge \frac{\partial^2}{\partial y^2} C_t(y, D_{t-1,t}),$$

as $J_{t+1,k}(x, D^k_{t+1})$ is convex in x, and thus for $y \in Y_{t,k}(D^k_t)$,

$$\frac{\partial^2}{\partial y^2} G_{t,k}(y, D^k_t) \ge m_{t,k}(D^k_t). \tag{4}$$

Similarly, for any $t \le k$, D^k_t and $y \in Y_{t,k}(D^k_t)$

$$\frac{\partial^2}{\partial y^2} G^L_{t,k,\infty}(y, D^k_t) \ge m_{t,k}(D^k_t). \tag{5}$$

Also, for any $t \le k$, y and D^k_t, let

$$\Delta\left(\frac{\partial}{\partial y} G_{t,k}(y, D^k_t)\right) \equiv \frac{\partial}{\partial y} G^L_{t,k,\infty}(y, D^k_t) - \frac{\partial}{\partial y} G_{t,k}(y, D^k_t)$$

$$\Delta\left(\frac{\partial}{\partial x} J_{t,k}(x, D^k_t)\right) \equiv \frac{\partial}{\partial x} J^L_{t,k,\infty}(x, D^k_t) - \frac{\partial}{\partial x} J_{t,k}(x, D^k_t),$$

$$\Delta y_{t,k}(D^k_t) \equiv y^*_{t,k}(D^k_t) - y^L_{t,k,\infty}(D^k_t).$$

Lemma 4. *For any $t \le k$ and D^k_t*
1. $\Delta(\partial J_{t,k}(x, D^k_t)/\partial x) \le \Delta(\partial G_{t,k}(x, D^k_t)/\partial x)$,
2. for $y \in Y_{t,k}(D^k_t)$,

$$m_{t,k}(D^k_t) \cdot \Delta y_{t,k}(D^k_t) \le \Delta(\partial G_{t,k}(y, D^k_t)/\partial y).$$

Lemma 5. *For any $t \le k$, y, and D^k_t*

$$\Delta\left(\frac{\partial}{\partial y} G_{t,k}(y, D^k_t)\right) \le \frac{\alpha^{k-t+1} h_+}{1-\alpha}.$$

Lemma 6. *For any $t \le l \le k$ and D^k_t, $m_{t,k}(D^k_t) \ge m_{t,l}(D^l_t)$.*

Proposition 7. *For any t and D^∞_t, if for some $l \ge t$ $m_{t,l}(D^l_t) > 0$, then*

$$\Delta y_{t,k}(D^k_t) \to 0 \quad \textit{as } k \to \infty.$$

Remark. The convergence rate of $\Delta y_{t,k}(D^k_t)$ to 0 is $O(\alpha^{k-t+1})$.

Remark. The condition of Proposition 7 holds for a wide range of update distributions. Note that, if the second derivative of the one-period cost C is larger than some positive constant δ over the interval $Y_{t,l}(D^l_t)$ for some l, then $m_{t,l}(D^l_t) > \delta$. For the additive-update case this second derivative is

$$(h_t + p_t) f_t(y - D_{t-1,t}),$$

where $f_t(\cdot)$ is the density function of $e_{t,t}$. If this function is larger than some constant δ over the interval $Y_{t,l}(D^l_t) - D_{t-1,t}$, the condition holds. This is true for most of the standard unimodal density functions. For the multiplicative update case the second derivative of C is

$$\frac{h_t + p_t}{D_{t-1,t}} \cdot f_t\left(\frac{y}{D_{t-1,t}}\right).$$

Again, the condition holds for most standard unimodal distributions with positive support.

3.3. A Myopic Policy

Next, we discuss conditions under which a myopic policy is optimal. We consider only the case of stationary, additive updates and stationary cost parameters.

Johnson and Thompson (1975) showed that such a policy is optimal for ARMA demand, assuming finite lower and upper limits on the demand in any period. Gallego and Özer (2001) showed the optimality of myopic policies for a model with advance demand information. Their result is the same as ours, but our proof is different.

From the definition of the one-period cost $C_t(y, D_{t-1,t})$,

$$C_t(y, D_{t-1,t}) = \dot{C}_t(y - D_{t-1,t}) + cD_{t-1,t}, \tag{6}$$

where

$$\dot{C}_t(z) = (h + (1-\alpha)c)E(z - e_{t,t})^+ + (p - (1-\alpha)c)E(z - e_{t,t})^-.$$

Let $\bar{y}_t$ minimize $\dot{C}_t(y)$. By stationarity $\bar{y}_t$ is constant over t, thus let $\bar{y} = \bar{y}_t$. Define a policy as follows:

$$y_t = \begin{cases} x_t & \text{for } x_t - D_{t-1,\,t} > \bar{y} \\ \bar{y}_t + D_{t-1,\,t} & \text{otherwise.} \end{cases}$$

We call the policy the *myopic forecast-centered base-stock policy* and $\bar{y}$ the *myopic forecast-centered base-stock level*.

Condition A. For any sample path of forecast updates, demands are nonnegative.

Proposition 8. *Under this condition, for the model with additive forecast updates and stationary cost parameters, the myopic forecast-centered base-stock policy is optimal.*

It is interesting to compare this result with the well-known myopia result of Veinott (1965). Let y_t^+ minimize $C_t(y, D_{t-1,\,t})$. This is the standard myopic base-stock level. Veinott (1965) showed that, when the $y_1^+, y_2^+, \ldots$ are nondecreasing, they are optimal. Now, $\bar{y}$ is different from y_t^+. Essentially, we decompose y_t^+ into two parts, the demand forecast $D_{t-1,\,t}$ and $\bar{y}$, that is, $y_t^+ = \bar{y} + D_{t-1,\,t}$. Thus, the result is different from Veinott's, although it is obviously in the same spirit.

4. A Computational Method

It is challenging to solve multidimensional dynamic programs. The usual approach for solving them approximately is to discretize the state space and then interpolate the solution. Johnson et al. (1993) used a tensor-product cubic spline interpolation. Chen et al. (1999) used experimental design methods to reduce the number of discretization points and the corresponding computational effort. Bertsekas and Tsitsiklis (1996) developed another approximation technique, in which the optimal expected-cost functions are evaluated with simulation and approximated with a neural network architecture. de Farias and Van Roy (2003) combined such methods with a linear-programming approach.

We consider here a method based on that of Bertsekas and Tsitsiklis (1996). Our method is different from theirs, however. Instead of a neural network, we use piecewise linear functions to approximate both $J_{t,\,T}(x, D_t^T)$ of (1) and $G_{t,\,T}(y, D_t^T)$ of (2). Because $G_{t,\,T}(y, D_t^T)$ is convex, its supporting hyperplanes are available to approximate it.

Let $\hat{J}_{t,\,T}(x, D_t^T)$ denote an approximation of $J_{t,\,T}(x, D_t^T)$ and $\widehat{G}_{t,\,T}(y, D_t^T)$ and $\widehat{\widehat{G}}_{t,\,T}(y, D_t^T)$ approximations of $G_{t,\,T}(y, D_t^T)$. In brief, we sequentially obtain the approximate functions $\hat{J}_{t,\,T}(x, D_t^T)$ and $\widehat{G}_{t,\,T}(y, D_t^T)$ and $\widehat{\widehat{G}}_{t,\,T}(y, D_t^T)$, working from the planning horizon backward in t. We use two types of approximation: One is functional approximation and the other is approximate expectation. For the former we need sample data. The sample data are obtained from the following equation, which defines the approximate auxiliary cost function $\widehat{G}_{t,\,T}(y, D_t^T)$:

$$\begin{aligned} &\widehat{G}_{t,\,T}(y, D_t^T) \\ &\qquad \equiv C_t(y, D_{t-1,\,t}) + E\big[\hat{J}_{t+1,\,T}\big(y - D_{t,\,t}, D_{t+1}^T\big)\big]. \end{aligned} \tag{7}$$

The expectation is taken with respect to the forecast update vector e_t^T. Approximate expectation is used to evaluate the expectation in (7). The details of the method are developed in the following subsections.

4.1. Piecewise Linear Approximation of the Auxiliary Cost Functions

Here we develop a method to approximate the auxiliary cost function $G_{t,\,T}(y, D_t^T)$ from below by a piecewise linear function, specifically by supporting hyperplanes.

Let I denote the number of sample points for the functional approximation and $(y^i, D_t^{T,\,i})$, $i = 1, \ldots, I$ the sample points themselves. We define the approximate functions $\widehat{G}_{t,\,T}(y, D_t^T)$, $\widehat{\widehat{G}}_{t,\,T}(y, D_t^T)$, and $\hat{J}_{t,\,T}(x, D_t^T)$ recursively. Suppose $\hat{J}_{t+1,\,T}(x, D_t^T)$ is already defined. The function $\widehat{G}_{t,\,T}(y, D_t^T)$ is defined as in (7). Let $g_{t,\,T}^i(y, D_t^T)$ denote the supporting hyperplane of $\widehat{G}_{t,\,T}(y, D_t^T)$ at sample point $(y^i, D_t^{T,\,i})$; that is,

$$g_{t,\,T}^i(y, D_t^T) \equiv a_t^i y + f_t^i D_t^{T'} + b_t^i,$$

where

$$\begin{aligned} a_t^i &\equiv \partial \widehat{G}_{t,\,T}(y^i, D_t^{T,\,i})/\partial y, \\ f_t^i &\equiv (f_{t,\,t}^i, f_{t,\,t+1}^i, \ldots, f_{t,\,T}^i), \\ f_{t,\,s}^i &\equiv \partial \widehat{G}_{t,\,T}(y^i, D_t^{T,\,i})/\partial D_{t-1,\,s} \quad \text{for } s = t, t+1, \ldots, T, \\ b_t^i &\equiv \widehat{G}_{t,\,T}(y^i, D_t^{T,\,i}) - a_t^i y - f_t^i D_t^{T,\,i'}. \end{aligned}$$

Then, define

$$\hat{\hat{G}}_{t,T}(y, D_t^T) \equiv \max_i g_{t,T}^i(y, D_t^T).$$

Finally,

$$\hat{J}_{t,T}(x, D_t^T) \equiv \min_{y \geq x} \hat{\hat{G}}_{t,T}(y, D_t^T).$$

Proposition 9.

1. *If* $\hat{J}_{T,T}(x, D_{T-1,T}) \leq J_{T,T}(x, D_{T-1,T})$, *then for any* $t \leq T-1$, x *and* D_t^T

$$\hat{J}_{t,T}(x, D_t^T) \leq J_{t,T}(x, D_t^T).$$

2. $\hat{J}_{t,T}(x, D_t^T)$ *is convex in* (x, D_t^T).

4.2. Simulation-Based Expectation Evaluation

It is time consuming to perform multidimensional integrations. We use Monte Carlo simulation to estimate the expectation in (7). Let N denote the number of samples and let $e_t^{T,n}$ denote the nth sample of e_t^T. Also, for given D_t^T and $n = 1, 2, \ldots, N$, let

$$D_{t,s}^n = D_{t-1,s} \oplus e_{t,s}^n \quad \text{for } s = t, t+1, \ldots, T,$$
$$D_{t+1}^{T,n} = (D_{t,t+1}^n, D_{t,t+2}^n, \ldots, D_{t,T}^n).$$

Then

$$E[\hat{J}_{t+1,T}(y - D_{t,t}, D_{t+1}^T)] \approx \frac{1}{N}\sum_{n=1}^N \hat{J}_{t+1,T}(y - D_{t,t}^n, D_{t+1}^{T,n}).$$

We still use the same notation for the approximate functions.

4.3. Algorithm

The above two approximation techniques are incorporated into a backward-induction algorithm to solve the multidimensional dynamic programming problem approximately. The procedure is summarized below.

Step 0. Set $t = T$; get random samples $(y^i, D_{T-1,T}^i)$, $i = 1, \ldots, I$; set

$$a_T^i = \partial C_T(y^i, D_{T-1,T}^i)/\partial y,$$
$$f_{T,T}^i = \partial C_T(y^i, D_{T-1,T}^i)/\partial D_{T-1,T}$$
$$b_T^i = C_T(y^i, D_{T-1,T}^i) - a_T^i y^i - f_{T,T}^i D_{T-1,T}^i.$$

Step 1. Set $t = t-1$; if $t == 0$, stop, or else get random samples $(y^i, D_t^{T,i})$, $i = 1, \ldots, I$; get random samples $e_t^{T,n}$, $n = 1, \ldots, N$.

Step 2. For each i estimate the partial derivatives of $\hat{G}_{t,T}(y, D_t^T)$ at $(y^i, D_t^{T,i})$ using samples $e_t^{T,n}$, $n = 1, \ldots, N$:

1. The derivative of $\hat{G}_{t,T}(y, D_t^T)$ with respect to y is

$$\frac{\partial C_t(y^i, D_{t-1,t}^i)}{\partial y} + \frac{1}{N}\sum_{n=1}^N \frac{\partial}{\partial y}\hat{J}_{t+1,T}(y^i - D_{t,t}^{i,n}, D_{t+1}^{T,i,n}).$$

Substitute this for a_t^i.

2. The derivative of $\hat{G}_{t,T}(y, D_t^T)$ with respect to $D_{t-1,t}$ is

$$\frac{\partial C_t(y^i, D_{t-1,t}^i)}{\partial D_{t-1,t}} - \frac{1}{N}\sum_{n=1}^N \frac{\partial}{\partial D_{t-1,t}}\hat{J}_{t+1,T}(y^i - D_{t,t}^{i,n}, D_{t+1}^{T,i,n}).$$

Substitute this for $f_{t,t}^i$.

3. The derivatives of $\hat{G}_{t,T}(y, D_t^T)$ with respect to $D_{t-1,s}$, $s \geq t+1$ are

$$\frac{1}{N}\sum_{n=1}^N \frac{\partial}{\partial D_{t-1,s}}\hat{J}_{t+1,T}(y^i - D_{t,t}^{i,n}, D_{t+1}^{T,i,n}).$$

Substitute this for $f_{t,s}^i$.

Also, set $b_t^i = \hat{G}_{t,T}(y^i, D_t^{T,i}) - a_t^i y^i - f_t^i D_t^{T,i'}$. Go to Step 1.

4.4. Performance Guarantees

We next discuss error bounds on the approximations. Suppose that the samples are rich enough so that, for each t,

$$\|\hat{\hat{G}}_{t,T}(y, D_t^T) - \hat{G}_{t,T}(y, D_t^T)\| \leq \epsilon,$$

where $\|\cdot\|$ is the supremum norm. We prove by induction on t that

$$\|\hat{J}_{t,T}(y, D_t^T) - J_{t,T}(y, D_t^T)\| \leq \frac{\epsilon(1-\alpha^{T-t+1})}{1-\alpha}.$$

Clearly, this holds for $t = T+1$. Assuming it holds for $t+1$, we obtain

$$\begin{aligned}
&\|\hat{G}_{t,T}(y, D_t^T) - G_{t,T}(y, D_t^T)\| \\
&\quad = \alpha\|E[\hat{J}_{t+1,T}(y - D_{t,t}, D_{t+1}^T) - J_{t+1,T}(y - D_{t,t}, D_{t+1}^T)]\| \\
&\quad \leq \alpha \cdot \frac{\epsilon(1-\alpha^{T-t})}{1-\alpha}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\|\hat{J}_{t,T}(x, D_t^T) - J_{t,T}(x, D_t^T)\| \\
&\quad = \left\|\min_{y \geq x} \hat{\hat{G}}_{t,T}(y, D_t^T) - \min_{y \geq x} G_{t,T}(y, D_t^T)\right\|
\end{aligned}$$

$$
\begin{aligned}
&\le \left\|\widehat{\widehat{G}}_{t,T}(y, D_t^T) - G_{t,T}(y, D_t^T)\right\| \\
&\le \left\|\widehat{\widehat{G}}_{t,T}(y, D_t^T) - \widehat{G}_{t,T}(y, D_t^T)\right\| \\
&\quad + \left\|\widehat{G}_{t,T}(y, D_t^T) - G_{t,T}(y, D_t^T)\right\| \\
&= \frac{\epsilon(1-\alpha^{T-t+1})}{1-\alpha}.
\end{aligned}
$$

4.5. Computational Complexity and a Modification

We discuss the computational complexity of the approximation algorithm. The problem size is measured by the number of sample points for functional approximation I, the number of samples for approximate expectation N, the forecast-update horizon M, and the planning horizon T.

The algorithm uses procedures to generate samples and to optimize the inventory position. These procedures are not specified above, and we consider their computational complexities as given. Let $\eta(M)$ and $\theta(I)$ denote the complexities of the generation of a sample point and the one-dimensional optimization, respectively.

Consider the effort required in the approximate backward induction of the algorithm along with its steps. In Step 1 the algorithm generates random samples for both functional approximation and approximate expectation, which requires effort $O(I\eta(M) + \eta(M)N)$. In Step 2, for each functional-approximation sample, the algorithm computes the partial derivatives of $\widehat{G}_{t,T}$, which requires effort $O(\theta(I)N + IMN)$. Thus, the total effort in Step 2 is $O(I\theta(I)N + I^2MN)$, so the effort in each period is $O(I\eta(M) + \eta(M)N + I\theta(I)N + I^2MN)$. So, the total effort over all periods is

$$O(I\eta(M)T + \eta(M)NT + I\theta(I)NT + I^2MNT). \tag{8}$$

Fixing I and N, (8) becomes $O(\eta(M)T + MT)$. Thus, the effort increases linearly with respect to T. Also, if $M = O(\eta(M))$ and $\eta(M)$ is polynomial of M, the effort increases polynomially with respect to M. However, note that I and N should probably increase as M increases in order to make the approximate solutions more accurate.

Next, we consider a modification of the approximation algorithm to reduce its computational time at the expense of the accuracy of the approximate solutions. Specifically, we approximate $\hat{J}_{t,T}$ by a piecewise linear function, as we did $\widehat{\widehat{G}}_{t,T}$. Let $\hat{\hat{J}}_{t,T}$ denote this approximation. Let I' denote the number of samples for the functional approximation, and assume $I' = O(I)$. Then, $\hat{\hat{J}}_{t,T}$ is piecewise linear convex and bounds $\hat{J}_{t,T}$ from below (thus, it bounds $J_{t,T}$ from below). The accuracy of $\hat{\hat{J}}_{t,T}$ is worse than that of $\hat{J}_{t,T}$. The computational complexity of computing $\hat{\hat{J}}_{t,T}$ is $O(I\theta(I) + I^2M)$. On the other hand, using $\hat{\hat{J}}_{t,T}$ instead of $\hat{J}_{t,T}$ in Step 2, the effort becomes $O(I^2MN)$. Thus, the total effort of the modified algorithm is

$$O(I\eta(M)T + \eta(M)NT + I\theta(I)T + I^2MNT). \tag{9}$$

The modification does not reduce the complexity with respect to M and T. However, the complexity with fixed M and T is $O(I\theta(I) + I^2N)$, as compared to $O(I\theta(I)N + I^2N)$ for the original method. For example, when $\theta(I) = O(I^2)$, the complexities of the original and modified algorithms are $O(I^3N)$ and $O(I^3 + I^2N)$, respectively. The difference between them can be significant for large I and N.

5. Numerical Experiments

We showed above that, under some conditions, the myopic forecast-centered policy is optimal for additive updates. Here we investigate by numerical experiments the quality of myopic policies for multiplicative updates, as well as the performance of the approximation algorithm.

We use the modified approximation algorithm to reduce the computation time at the expense of accuracy and then use the same sample points for functional approximation as those used for $\widehat{G}_{t,T}$. As we shall see later, the approximation errors of the algorithm are small.

The experiments include two patterns of initial demand forecasts, trends and cycles. The patterns are constructed as follows: At first the initial forecast for period 1 is 250. We then adjust that value and set the initial forecasts for periods 2–16 as follows:

Trend (T). Linear trends with six slopes, +10, +5, 0, −5, −10, and −25, numbered 1 through 6.

Cycle (C). Four types of cycle: none, $50\sin(2\pi t/12)$, $50\cos(2\pi t/12)$, and $-50\cos(2\pi t/12)$. These are numbered 1 through 4.

These patterns can produce nonpositive initial forecasts. Such values are reset to 1.

Other parameters of the model are

$$h_t = 2, \qquad p_t = 10, \qquad c_t = 0, \qquad \alpha = 1.0, \qquad M = 4.$$

The (multiplicative) forecast updates have multidimensional log-normal distributions. Let $\xi_{t,s} \equiv \log e_{t,s}$. The $\xi_{t,s}$, then, are joint-normally distributed. The covariance matrix is

$$\begin{pmatrix} 0.08 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.02 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.005 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0008 \end{pmatrix}, \tag{10}$$

and the mean vector is

$$(-0.04, -0.01, -0.0025, -0.0004).$$

For the expectation in (7) 500 samples are randomly chosen; i.e., $N = 500$. Also, 500 samples are used for functional approximation; i.e., $I = 500$.

To evaluate the policy obtained by the algorithm, we estimate the cost of the policy by simulation and compare the simulated cost with the approximate optimal expected cost $\hat{\hat{J}}_{1,T}(x, D_1^T)$. To estimate the expected total cost $E[\sum_{t=1}^{T} C_t(y_t, D_{t-1,t})]$, we randomly generate 300 sample paths of the inventory position and the forecast vector and denote them by $(y_t^i, D_t^{T,i})$, $t = 1, 2, \ldots, T$, $i = 1, 2, \ldots, 300$. To assess the error of functional approximation separately from the sampling error, we use two types of sampling for generation of the sample paths: One method samples uniformly from the samples used in the algorithm for expectation. The other takes new samples randomly following the log-normal distribution. We denote the former by Cost1 and the latter by Cost2. To compute them, we use the approximate expected one-period cost $\widehat{C}_t$.

We also estimate the expected total cost without this approximation. For that purpose, for each $(y_t^i, D_t^{T,i})$, we generate 1,000 samples of $e_{t,t}$ and denote them by $e_{t,t}^{ij}$, $j = 1, 2, \ldots, 1{,}000$. With those samples, we estimate the expected total cost by Cost3 $= \sum_{i=1}^{300} Z_i/300$, where

$$Z_i \equiv \sum_{t=1}^{T} \left(\frac{1}{1{,}000} \sum_{j=1}^{1{,}000} (h(y_t^i - D_{t-1,t}^i e_{t,t}^{ij})^+ + p(y_t^i - D_{t-1,t}^i e_{t,t}^{ij})^-) \right), \quad i = 1, 2, \ldots, 300.$$

This estimate is unbiased, as

$$\begin{aligned} &E[h(y_t^i - D_{t-1,t}^i e_{t,t}^{ij})^+ + p(y_t^i - D_{t-1,t}^i e_{t,t}^{ij})^-] \\ &\quad = E[E[h(y_t^i - D_{t-1,t}^i e_{t,t}^{ij})^+ \\ &\qquad + p(y_t^i - D_{t-1,t}^i e_{t,t}^{ij})^- \mid y_t^i, D_{t-1,t}^i]] \\ &\quad = E[C_t(y_t^i, D_{t-1,t}^i)]. \end{aligned}$$

As the Z_i, $i = 1, 2, \ldots, 300$ are iid, the standard deviation of the estimator is $\sigma(Z)/\sqrt{300}$.

Table 1 shows the approximate optimal base-stock levels, the approximate costs, and the corresponding estimates of the expected total costs for several planning horizons. (The standard deviations of the estimates are shown in parentheses.) Notice first that all the approximate costs are quite close to each other in most instances. Thus, the approximate optimal base-stock levels are quite close to the true optimal ones. Also, the myopic base-stock levels are very close to the approximate optimal ones.

Next, we explore the difference between early and late resolution of demand uncertainty. The examples above represent late resolution of uncertainty. To represent early resolution, we construct a new set of problems by reversing the order of the diagonal elements in the covariance matrix (10). Table 2 shows the results. They are similar to those of Table 1. Thus, myopic policies and the approximation algorithm perform well for both early and late resolution of demand uncertainty.

To see the impact of the discount factor α, the algorithm was tested for $\alpha = 0.99$. It performed well and the results are quite similar to those of Table 1.

Also, we investigate the effects of the forecast-update horizon M on the optimal base-stock levels. The forecast updates' distributions change as M changes. For $M = 4$, we start in period $t-3$ to update the forecast of demand in period t. For $M = 3$, we start updating in period $t-2$; the first forecast update is $e_{t-3,t} \cdot e_{t-2,t}$. Similarly, for $M = 2$, the first update is $e_{t-3,t} \cdot e_{t-2,t} \cdot e_{t-1,t}$. For $M = 1$ it is $e_{t-3,t} \cdot e_{t-2,t} \cdot e_{t-1,t} \cdot e_{t,t}$. The covariance matrix is obtained by truncating the original one (10), and the discount factor is $\alpha = 1.0$.

Tables 3 and 4 show the approximate optimal base-stock levels and the approximate expected costs for four-horizon problems with several update horizons M. We find that the policies for $M = 2, 3, 4$ are almost

Table 1 Approximate Base-Stock Levels and Approximate Costs: Base Case

			Planning horizon				
T	*C*	Myopic	2	3	4	8	16
1	1	315.2	315.2	315.2	315.2	315.2	315.2
		Approx. cost	492.3	753.1	1,021.0	2,192.1	4,989.5
		Cost1	492.6 (2.1)	753.4 (3.2)	1,025.0 (4.3)	2,199.2 (6.8)	4,989.9 (11.6)
		Cost2	490.5 (2.0)	747.8 (3.0)	1,013.8 (3.9)	2,177.1 (6.8)	4,977.0 (11.8)
		Cost3	490.8 (2.2)	749.0 (3.2)	1,015.9 (4.0)	2,178.3 (6.9)	4,987.8 (12.0)
	2	346.8	347.2	347.2	347.2	347.2	347.2
		Approx. cost	558.4	867.4	1,176.8	2,306.7	5,144.2
		Cost1	556.0 (2.4)	872.1 (3.8)	1,173.5 (4.8)	2,297.9 (6.9)	5,148.0 (12.0)
		Cost2	556.3 (2.3)	861.1 (3.6)	1,168.3 (4.5)	2,290.3 (7.1)	5,131.3 (12.1)
		Cost3	557.5 (2.4)	860.3 (3.7)	1,171.6 (4.7)	2,293.7 (7.3)	5,139.1 (12.5)
	3	369.8	370.3	370.3	370.3	370.3	370.3
		Approx. cost	558.2	818.9	1,063.0	2,078.6	5,031.3
		Cost1	555.0 (2.3)	816.4 (3.4)	1,059.7 (3.8)	2,074.8 (5.9)	5,043.9 (11.5)
		Cost2	556.2 (2.2)	813.4 (3.1)	1,055.7 (3.8)	2,064.7 (6.2)	5,018.7 (11.9)
		Cost3	557.9 (2.3)	813.1 (3.3)	1,058.1 (4.0)	2,066.4 (6.3)	5,021.0 (12.4)
	4	260.6	260.3	260.3	260.3	260.3	260.3
		Approx. cost	426.5	687.3	979.1	2,305.7	4,947.8
		Cost1	422.6 (1.8)	686.9 (3.1)	981.5 (4.4)	2,304.0 (7.3)	4,948.0 (12.4)
		Cost2	424.9 (1.8)	682.1 (2.9)	971.9 (3.9)	2,289.5 (7.4)	4,935.2 (11.8)
		Cost3	426.0 (1.9)	681.4 (3.0)	972.7 (4.2)	2,295.9 (7.8)	4,945.9 (12.1)
2	1	315.2	315.2	315.2	315.2	315.2	315.2
		Approx. cost	487.5	738.6	992.1	2,057.7	4,414.8
		Cost1	488.7 (2.1)	744.5 (3.4)	998.1 (3.9)	2,069.3 (6.7)	4,418.8 (9.8)
		Cost2	485.7 (2.0)	733.4 (2.9)	985.2 (3.7)	2,043.6 (6.3)	4,402.3 (10.3)
		Cost3	487.0 (2.1)	734.2 (3.0)	986.2 (3.8)	2,043.3 (6.6)	4,409.4 (10.7)
	2	346.8	346.8	346.8	346.8	346.8	346.8
		Approx. cost	553.5	852.9	1,147.9	2,172.3	4,569.3
		Cost1	553.2 (2.4)	852.8 (3.6)	1,145.8 (4.5)	2,173.9 (7.0)	4,581.8 (9.6)
		Cost2	551.4 (2.3)	846.8 (3.5)	1,139.7 (4.4)	2,156.8 (6.6)	4,556.9 (10.6)
		Cost3	554.1 (2.4)	848.8 (3.7)	1,140.8 (4.4)	2,160.8 (6.8)	4,560.4 (10.8)
	3	369.8	369.8	369.8	369.8	369.8	369.8
		Approx. cost	553.3	804.4	1,034.1	1,944.3	4,456.6
		Cost1	550.6 (2.1)	801.4 (3.1)	1,035.4 (3.9)	1,943.0 (5.8)	4,450.9 (10.6)
		Cost2	551.4 (2.1)	799.1 (3.1)	1,027.2 (3.7)	1,931.2 (5.7)	4,444.1 (10.4)
		Cost3	550.9 (2.3)	798.4 (3.2)	1,028.8 (3.8)	1,932.2 (5.8)	4,451.3 (10.7)
	4	260.6	260.6	260.6	260.6	260.6	260.6
		Approx. cost	421.7	672.8	950.3	2,171.3	4,373.1
		Cost1	422.2 (1.8)	668.0 (3.0)	950.6 (3.8)	2,173.1 (7.4)	4,387.1 (9.9)
		Cost2	420.0 (1.8)	667.8 (2.8)	943.3 (3.8)	2,156.0 (6.9)	4,360.7 (10.4)
		Cost3	419.2 (1.8)	668.3 (2.9)	945.2 (3.9)	2,160.6 (7.1)	4,371.3 (10.7)
3	1	315.2	315.6	315.6	315.6	315.6	315.6
		Approx. cost	482.6	724.1	963.3	1,923.3	3,840.0
		Cost1	480.0 (1.9)	726.0 (2.9)	962.5 (3.7)	1,924.8 (5.6)	3,836.1 (8.9)
		Cost2	480.9 (1.9)	719.1 (2.8)	956.7 (3.6)	1,910.2 (5.8)	3,827.9 (8.9)
		Cost3	481.3 (2.1)	720.4 (2.9)	957.6 (3.6)	1,912.3 (5.9)	3,831.8 (8.9)
	2	346.8	347.2	347.2	347.2	347.2	347.2
		Approx. cost	548.7	838.4	1,119.0	2,037.9	3,994.5
		Cost1	551.0 (2.4)	840.0 (3.5)	1,120.5 (4.5)	2,030.7 (5.7)	3,994.9 (8.9)
		Cost2	546.6 (2.2)	832.4 (3.4)	1,111.1 (4.2)	2,023.4 (6.1)	3,982.2 (9.2)
		Cost3	548.7 (2.4)	834.5 (3.5)	1,113.0 (4.4)	2,026.7 (6.3)	3,985.4 (9.6)

Table 1 (cont'd.)

			Planning horizon				
T	*C*	Myopic	2	3	4	8	16
	3	369.8	369.8	369.8	369.8	369.8	369.8
		Approx. cost	548.5	789.9	1,005.2	1,809.9	3,881.9
		Cost1	547.7 (2.2)	794.3 (3.3)	1,005.9 (4.1)	1,810.1 (5.2)	3,865.5 (8.4)
		Cost2	546.6 (2.1)	784.7 (3.0)	998.5 (3.5)	1,797.7 (5.2)	3,869.5 (9.0)
		Cost3	547.5 (2.2)	785.2 (3.1)	999.0 (3.6)	1,801.5 (5.2)	3,875.3 (9.1)
	4	260.6	260.6	260.6	260.6	260.6	260.6
		Approx. cost	416.8	658.3	921.3	2,036.8	3,798.3
		Cost1	416.0 (1.9)	657.3 (2.7)	923.1 (3.8)	2,039.6 (6.8)	3,815.3 (8.7)
		Cost2	415.2 (1.7)	653.4 (2.7)	914.7 (3.6)	2,022.5 (6.4)	3,786.1 (9.0)
		Cost3	416.3 (1.8)	654.4 (2.8)	916.0 (3.7)	2,024.6 (6.6)	3,792.3 (9.3)
4	1	315.2	315.2	315.3	315.3	315.3	315.3
		Approx. cost	477.8	709.6	934.5	1,789.1	3,265.3
		Cost1	476.8 (2.0)	711.0 (3.1)	928.9 (3.5)	1,791.6 (5.2)	3,275.9 (7.1)
		Cost2	476.1 (1.9)	704.7 (2.8)	928.0 (3.4)	1,776.6 (5.3)	3,253.2 (7.5)
		Cost3	475.6 (2.0)	704.5 (2.9)	929.0 (3.4)	1,778.6 (5.5)	3,262.5 (7.7)
	2	346.8	346.8	346.8	346.8	346.8	346.8
		Approx. cost	543.8	823.9	1,090.2	1,903.5	3,419.8
		Cost1	543.2 (2.2)	825.1 (3.5)	1,095.5 (4.4)	1,905.0 (5.8)	3,422.9 (8.3)
		Cost2	541.8 (2.2)	818.1 (3.3)	1,082.6 (4.1)	1,890.0 (5.7)	3,407.7 (7.8)
		Cost3	542.3 (2.4)	819.3 (3.5)	1,082.6 (4.2)	1,892.8 (5.7)	3,411.7 (8.1)
	3	369.8	369.4	369.4	369.4	369.4	369.4
		Approx. cost	543.6	775.4	976.4	1,675.5	3,307.1
		Cost1	543.2 (2.2)	776.5 (2.9)	976.9 (3.5)	1,680.3 (5.1)	3,314.2 (7.5)
		Cost2	541.8 (2.1)	770.4 (2.9)	970.0 (3.4)	1,664.3 (4.8)	3,295.0 (7.5)
		Cost3	543.1 (2.3)	772.5 (3.1)	969.9 (3.6)	1,665.0 (5.0)	3,297.9 (7.8)
	4	260.6	260.3	260.3	260.3	260.3	260.3
		Approx. cost	411.9	643.7	892.5	1,902.5	3,223.5
		Cost1	411.2 (1.7)	643.6 (2.8)	892.6 (3.7)	1,895.4 (5.6)	3,238.2 (7.7)
		Cost2	410.4 (1.7)	639.1 (2.6)	886.1 (3.5)	1,889.1 (5.9)	3,211.6 (7.6)
		Cost3	409.6 (1.7)	639.0 (2.8)	888.4 (3.6)	1,893.3 (6.0)	3,218.1 (7.8)
5	1	315.2	315.2	315.3	315.3	315.3	315.3
		Approx. cost	472.9	695.1	905.6	1,654.6	2,690.5
		Cost1	472.5 (1.8)	698.5 (2.8)	902.6 (3.2)	1,652.7 (5.0)	2,686.9 (5.9)
		Cost2	471.3 (1.8)	690.4 (2.7)	899.4 (3.3)	1,643.2 (4.9)	2,678.7 (6.2)
		Cost3	471.2 (2.0)	691.0 (2.8)	901.7 (3.4)	1,645.7 (5.1)	2,682.9 (6.4)
	2	346.8	346.8	346.8	346.8	346.8	346.8
		Approx. cost	538.9	809.4	1,061.3	1,769.1	2,845.1
		Cost1	542.9 (2.3)	811.5 (3.4)	1,058.0 (4.2)	1,763.2 (5.5)	2,856.5 (7.0)
		Cost2	537.0 (2.2)	803.7 (3.2)	1,053.9 (3.9)	1,756.5 (5.3)	2,833.1 (6.5)
		Cost3	537.8 (2.3)	803.8 (3.2)	1,055.9 (4.0)	1,757.0 (5.3)	2,833.5 (6.7)
	3	369.8	370.3	370.3	370.3	370.3	370.3
		Approx. cost	538.8	760.9	947.5	1,541.1	2,732.4
		Cost1	537.6 (2.2)	759.8 (2.9)	949.6 (3.5)	1,534.7 (4.7)	2,735.0 (5.9)
		Cost2	536.9 (2.0)	756.1 (2.8)	941.4 (3.3)	1,530.8 (4.3)	2,720.6 (6.2)
		Cost3	537.8 (2.2)	758.1 (3.0)	942.5 (3.5)	1,531.7 (4.4)	2,720.5 (6.4)
	4	260.6	261.3	261.3	261.3	261.3	261.3
		Approx. cost	407.1	629.1	863.5	1,768.0	2,648.5
		Cost1	409.4 (1.9)	633.0 (2.8)	867.2 (3.6)	1,761.4 (5.2)	2,647.0 (5.9)
		Cost2	405.6 (1.6)	624.7 (2.5)	857.7 (3.3)	1,755.8 (5.5)	2,637.3 (6.4)
		Cost3	406.2 (1.8)	625.8 (2.6)	860.4 (3.4)	1,758.1 (5.6)	2,637.9 (6.3)

Table 1 (cont'd.)

			Planning horizon				
T	C	Myopic	2	3	4	8	16
6	1	315.2	315.2	315.2	315.2	315.2	315.2
		Approx. cost	458.4	651.5	819.0	1,251.4	1,361.1
		Cost1	461.0 (1.7)	650.9 (2.4)	818.2 (3.1)	1,249.9 (3.6)	1,367.9 (4.3)
		Cost2	456.8 (1.7)	647.4 (2.4)	813.7 (2.8)	1,242.9 (3.6)	1,349.1 (4.1)
		Cost3	457.0 (1.9)	647.8 (2.6)	814.4 (2.9)	1,243.7 (3.7)	1,349.8 (4.2)
	2	346.8	346.8	346.8	346.8	346.8	346.8
		Approx. cost	524.4	765.9	974.8	1,368.7	1,432.3
		Cost1	526.8 (2.0)	770.4 (3.2)	972.1 (3.7)	1,366.6 (4.2)	1,430.4 (5.4)
		Cost2	522.5 (2.1)	760.7 (2.9)	968.2 (3.5)	1,359.2 (4.2)	1,421.1 (6.4)
		Cost3	523.3 (2.2)	763.9 (3.1)	968.1 (3.6)	1,361.9 (4.3)	1,424.3 (6.6)
	3	369.8	369.8	369.8	369.8	369.8	369.8
		Approx. cost	524.2	717.4	861.0	1,138.1	1,326.9
		Cost1	524.5 (2.1)	714.4 (2.6)	861.9 (3.1)	1,129.8 (3.5)	1,330.2 (3.7)
		Cost2	522.5 (1.9)	713.0 (2.5)	855.7 (2.9)	1,130.6 (3.2)	1,319.9 (3.6)
		Cost3	524.7 (2.0)	714.5 (2.7)	857.2 (3.0)	1,131.6 (3.2)	1,322.4 (3.7)
	4	260.6	260.6	260.6	260.6	260.6	260.6
		Approx. cost	392.5	585.7	777.1	1,365.0	1,497.8
		Cost1	392.8 (1.6)	594.0 (2.5)	782.1 (3.1)	1,367.0 (4.2)	1,495.9 (6.5)
		Cost2	391.2 (1.5)	581.7 (2.3)	771.7 (2.9)	1,355.5 (4.1)	1,484.8 (6.5)
		Cost3	391.8 (1.7)	582.3 (2.3)	772.8 (2.9)	1,358.8 (4.3)	1,487.8 (6.6)

the same; only the policy for $M = 1$ is larger than the others. The same feature appears in the approximate expected costs. Thus, although all the demand forecasts affect costs, the one-period-ahead forecasts are the most significant. The myopic policies use only that information.

Tables 5 and 6 show analogous results for eight-horizon problems. We see the same features as in Tables 3 and 4.

6. Conclusions

We considered a dynamic forecast-inventory model based on the MMFE. Two types of forecast updates were considered: additive and multiplicative. We formulated the model as a dynamic program with multidimensional state space and showed that a base-stock policy, depending on the current forecast, is optimal. Also, the optimal base-stock levels decrease monotonically as the planning horizon gets longer. So the optimal base-stock levels for one planning horizon are upper bounds on those for any longer planning horizon. Then, we developed lower bounds on the optimal base-stock levels. Under mild conditions the differences between the upper and lower bounds decrease exponentially as the planning horizon gets longer. These results imply that policies based on near-term forecasts perform well.

We also considered conditions under which a type of myopic policy is optimal. Assuming nonnegative demands, this myopic policy is optimal for additive updates with stationary data.

A computational approach was developed to solve the problem approximately. The approach is based on two techniques: functional approximation and simulation. We used piecewise linear functions to approximate auxiliary cost functions. Because this approximation preserves convexity, optimization remains easy.

Numerical experiments showed that the approximation techniques are quite accurate. Also, even when the myopic policy is not guaranteed to be optimal, it performs quite well. That remains so even with heavily nonstationary demands. These results also support the notion that the information embodied in near-term forecasts is more significant than that in longer-term forecasts.

Table 2 Approximate Base-Stock Levels and Approximation Errors: Early Resolution of Demand Uncertainty

			Planning horizon				
T	*C*	Myopic	2	3	4	8	16
1	1	256.9	256.7	256.7	256.7	256.7	256.7
		Approx. cost	43.8	66.8	91.8	196.0	448.0
		Cost1	43.9 (0.1)	67.2 (0.2)	92.5 (0.5)	197.2 (1.1)	453.0 (2.3)
		Cost2	43.9 (0.1)	67.0 (0.2)	91.0 (0.5)	197.2 (1.1)	451.8 (2.1)
		Cost3	43.8 (0.1)	66.9 (0.2)	91.0 (0.5)	196.8 (1.1)	451.2 (2.1)
	2	282.5	282.7	282.7	282.7	282.7	282.7
		Approx. cost	49.7	77.1	105.3	205.9	461.7
		Cost1	49.8 (0.1)	77.2 (0.3)	107.0 (0.6)	206.0 (1.2)	465.5 (2.1)
		Cost2	49.7 (0.1)	77.0 (0.3)	104.8 (0.6)	206.9 (1.1)	465.0 (2.1)
		Cost3	49.6 (0.1)	76.9 (0.3)	104.7 (0.6)	206.7 (1.1)	464.1 (2.1)
	3	301.3	301.4	301.4	301.4	301.4	301.4
		Approx. cost	49.7	72.9	95.6	185.8	451.3
		Cost1	49.8 (0.1)	73.0 (0.2)	96.0 (0.5)	187.3 (1.0)	453.2 (1.9)
		Cost2	49.7 (0.1)	72.7 (0.2)	94.6 (0.5)	186.6 (1.0)	455.0 (2.1)
		Cost3	49.5 (0.1)	72.6 (0.2)	94.4 (0.5)	186.3 (1.0)	453.9 (2.1)
	4	212.4	212.4	212.4	212.4	212.4	212.4
		Approx. cost	38.0	61.1	88.2	206.4	444.5
		Cost1	37.9 (0.1)	61.3 (0.2)	87.9 (0.6)	203.9 (1.2)	446.1 (2.0)
		Cost2	38.0 (0.1)	61.0 (0.2)	87.1 (0.5)	207.2 (1.2)	448.1 (2.1)
		Cost3	38.1 (0.1)	60.9 (0.2)	86.9 (0.5)	207.0 (1.3)	447.2 (2.1)
2	1	256.9	257.0	257.0	257.0	257.0	257.0
		Approx. cost	43.4	65.5	89.0	184.0	396.0
		Cost1	43.6 (0.1)	65.6 (0.2)	88.5 (0.5)	185.2 (1.0)	395.7 (1.8)
		Cost2	43.4 (0.1)	65.6 (0.2)	88.3 (0.5)	184.7 (1.0)	399.0 (1.8)
		Cost3	43.3 (0.1)	65.4 (0.2)	88.3 (0.5)	184.2 (1.0)	397.7 (1.8)
	2	282.5	283.4	283.4	283.4	283.4	283.4
		Approx. cost	49.3	75.8	103.3	194.6	411.2
		Cost1	49.4 (0.1)	75.7 (0.3)	104.6 (0.6)	196.1 (1.0)	411.6 (1.7)
		Cost2	49.4 (0.1)	75.8 (0.3)	102.3 (0.6)	194.9 (1.0)	412.9 (1.8)
		Cost3	49.4 (0.1)	75.7 (0.3)	102.1 (0.6)	194.8 (1.0)	412.0 (1.8)
	3	301.3	301.4	301.4	301.4	301.4	301.4
		Approx. cost	49.2	71.5	92.8	173.9	400.6
		Cost1	49.2 (0.1)	71.3 (0.2)	93.8 (0.4)	174.0 (0.9)	401.9 (1.6)
		Cost2	49.3 (0.1)	71.4 (0.2)	92.1 (0.4)	174.5 (0.9)	402.5 (1.8)
		Cost3	49.1 (0.1)	71.2 (0.2)	91.8 (0.4)	174.3 (0.9)	401.6 (1.8)
	4	212.4	212.4	212.4	212.4	212.4	212.4
		Approx. cost	37.6	59.9	85.7	194.4	392.8
		Cost1	37.7 (0.1)	60.0 (0.2)	85.7 (0.5)	193.7 (1.1)	396.1 (1.7)
		Cost2	37.5 (0.1)	59.7 (0.2)	84.6 (0.5)	194.9 (1.1)	395.0 (1.8)
		Cost3	37.5 (0.1)	59.6 (0.2)	84.3 (0.5)	194.6 (1.1)	394.1 (1.8)
3	1	256.9	256.9	256.9	256.9	256.9	256.9
		Approx. cost	43.0	63.7	85.9	171.3	344.4
		Cost1	43.0 (0.1)	64.7 (0.2)	86.8 (0.5)	172.0 (0.9)	344.8 (1.3)
		Cost2	43.0 (0.1)	64.3 (0.2)	85.8 (0.4)	172.6 (0.9)	346.6 (1.5)
		Cost3	43.0 (0.1)	64.3 (0.2)	85.6 (0.4)	172.3 (0.9)	345.8 (1.5)
	2	282.5	283.6	283.6	283.6	283.6	283.6
		Approx. cost	48.7	74.5	100.6	182.5	358.9
		Cost1	49.1 (0.1)	75.0 (0.3)	100.9 (0.5)	182.3 (0.9)	363.7 (1.5)
		Cost2	49.0 (0.1)	74.6 (0.3)	99.8 (0.5)	183.0 (0.9)	361.1 (1.6)
		Cost3	49.0 (0.1)	74.4 (0.3)	99.5 (0.5)	182.6 (0.9)	360.3 (1.6)

Table 2 (cont'd.)

			Planning horizon				
T	*C*	Myopic	2	3	4	8	16
	3	301.3	300.7	300.7	300.7	300.7	300.7
		Approx. cost	48.8	70.2	90.2	162.0	348.9
		Cost1	49.0 (0.1)	70.4 (0.2)	90.4 (0.4)	163.3 (0.8)	350.4 (1.5)
		Cost2	49.0 (0.1)	70.4 (0.2)	89.7 (0.4)	162.7 (0.8)	350.6 (1.5)
		Cost3	48.8 (0.1)	70.2 (0.2)	89.4 (0.4)	162.2 (0.8)	350.1 (1.5)
	4	212.4	212.9	212.9	212.9	212.9	212.9
		Approx. cost	37.1	58.3	82.8	182.1	340.8
		Cost1	37.1 (0.1)	59.3 (0.2)	82.8 (0.5)	183.4 (1.0)	339.2 (1.4)
		Cost2	37.2 (0.1)	58.5 (0.2)	82.0 (0.5)	182.8 (1.0)	342.9 (1.6)
		Cost3	37.1 (0.1)	58.4 (0.2)	81.9 (0.5)	182.5 (1.1)	342.0 (1.6)
4	1	256.9	257.1	257.1	257.1	257.1	257.1
		Approx. cost	42.5	63.1	84.0	160.2	293.2
		Cost1	42.6 (0.1)	63.5 (0.2)	83.6 (0.4)	160.6 (0.8)	294.6 (1.2)
		Cost2	42.6 (0.1)	63.0 (0.2)	83.2 (0.4)	160.5 (0.8)	294.7 (1.2)
		Cost3	42.6 (0.1)	62.8 (0.2)	82.8 (0.4)	160.3 (0.8)	294.2 (1.3)
	2	282.5	282.3	282.3	282.3	282.3	282.3
		Approx. cost	48.4	73.1	97.7	170.1	307.0
		Cost1	48.6 (0.1)	74.0 (0.3)	98.6 (0.5)	172.0 (0.9)	310.4 (1.3)
		Cost2	48.5 (0.1)	73.4 (0.3)	97.3 (0.5)	170.9 (0.8)	309.2 (1.3)
		Cost3	48.4 (0.1)	73.3 (0.3)	97.1 (0.5)	170.5 (0.8)	308.6 (1.3)
	3	301.3	301.1	301.1	301.1	301.1	301.1
		Approx. cost	48.5	69.0	87.7	150.1	296.5
		Cost1	48.5 (0.1)	68.9 (0.2)	88.1 (0.4)	150.5 (0.7)	299.0 (1.3)
		Cost2	48.5 (0.1)	69.0 (0.2)	87.0 (0.4)	150.3 (0.7)	298.7 (1.2)
		Cost3	48.4 (0.1)	68.9 (0.2)	86.8 (0.4)	149.9 (0.7)	298.0 (1.2)
	4	212.4	212.4	212.4	212.4	212.4	212.4
		Approx. cost	36.7	57.2	80.3	170.3	289.6
		Cost1	36.8 (0.1)	57.5 (0.2)	81.1 (0.5)	170.2 (1.0)	289.5 (1.3)
		Cost2	36.7 (0.1)	57.1 (0.2)	79.5 (0.5)	170.7 (1.0)	290.7 (1.3)
		Cost3	36.7 (0.1)	57.0 (0.2)	79.3 (0.5)	170.3 (1.0)	290.1 (1.3)
5	1	256.9	257.0	257.0	257.0	257.0	257.0
		Approx. cost	42.1	61.9	81.5	148.1	241.6
		Cost1	42.1 (0.1)	61.6 (0.2)	81.9 (0.4)	147.9 (0.7)	241.2 (1.0)
		Cost2	42.1 (0.1)	61.7 (0.2)	80.6 (0.4)	148.4 (0.7)	242.3 (1.0)
		Cost3	42.2 (0.1)	61.6 (0.2)	80.4 (0.4)	148.1 (0.7)	241.9 (1.0)
	2	282.5	282.4	282.4	282.4	282.4	282.4
		Approx. cost	48.0	72.0	95.4	158.5	255.7
		Cost1	48.0 (0.1)	71.8 (0.2)	96.1 (0.5)	157.7 (0.8)	257.2 (1.0)
		Cost2	48.0 (0.1)	71.8 (0.2)	94.5 (0.5)	158.3 (0.7)	256.1 (1.0)
		Cost3	48.0 (0.1)	71.7 (0.3)	94.3 (0.5)	157.9 (0.7)	255.6 (1.0)
	3	301.3	301.6	301.6	301.6	301.6	301.6
		Approx. cost	47.9	67.6	84.6	137.2	244.5
		Cost1	48.1 (0.1)	67.7 (0.2)	85.3 (0.4)	139.3 (0.6)	246.2 (1.1)
		Cost2	48.0 (0.1)	67.6 (0.2)	84.4 (0.4)	138.1 (0.6)	246.0 (1.0)
		Cost3	48.0 (0.1)	67.4 (0.2)	84.2 (0.4)	137.8 (0.6)	245.6 (1.0)
	4	212.4	212.0	212.0	212.0	212.0	212.0
		Approx. cost	36.3	56.0	77.7	158.0	237.4
		Cost1	36.5 (0.1)	56.3 (0.2)	77.4 (0.4)	158.4 (0.9)	236.8 (1.0)
		Cost2	36.3 (0.1)	55.9 (0.2)	76.9 (0.4)	158.7 (0.9)	238.6 (1.0)
		Cost3	36.2 (0.1)	55.8 (0.2)	76.7 (0.4)	158.4 (0.9)	238.0 (1.0)

Table 2 (cont'd.)

			Planning horizon				
T	*C*	Myopic	2	3	4	8	16
6	1	256.9	256.3	256.3	256.3	256.3	256.3
		Approx. cost	40.8	58.0	73.6	111.9	118.9
		Cost1	41.0 (0.1)	58.2 (0.2)	73.9 (0.3)	111.6 (0.5)	120.2 (0.5)
		Cost2	40.9 (0.1)	58.0 (0.2)	73.0 (0.3)	112.0 (0.5)	119.6 (0.5)
		Cost3	40.9 (0.1)	57.8 (0.2)	72.9 (0.3)	111.9 (0.5)	119.3 (0.5)
	2	282.5	282.4	282.4	282.4	282.4	282.4
		Approx. cost	46.7	68.0	87.4	122.4	123.3
		Cost1	46.7 (0.1)	68.2 (0.2)	88.0 (0.4)	122.5 (0.6)	123.8 (0.5)
		Cost2	46.7 (0.1)	68.0 (0.2)	86.8 (0.4)	121.9 (0.5)	123.4 (0.5)
		Cost3	46.6 (0.1)	67.9 (0.2)	86.6 (0.4)	121.6 (0.5)	123.2 (0.5)
	3	301.3	302.1	302.1	302.1	302.1	302.1
		Approx. cost	46.6	63.7	77.1	101.8	117.0
		Cost1	46.8 (0.1)	63.9 (0.2)	77.5 (0.3)	102.2 (0.4)	118.1 (0.5)
		Cost2	46.8 (0.1)	63.8 (0.2)	76.7 (0.3)	101.7 (0.4)	117.1 (0.4)
		Cost3	46.7 (0.1)	63.7 (0.2)	76.6 (0.3)	101.5 (0.4)	116.9 (0.4)
	4	212.4	212.7	212.7	212.7	212.7	212.7
		Approx. cost	35.0	52.1	69.9	121.7	128.1
		Cost1	35.1 (0.1)	52.4 (0.2)	70.0 (0.4)	123.1 (0.6)	129.1 (0.7)
		Cost2	35.0 (0.1)	52.1 (0.2)	69.3 (0.4)	122.3 (0.6)	129.2 (0.6)
		Cost3	34.9 (0.1)	52.1 (0.2)	69.1 (0.4)	122.1 (0.6)	128.9 (0.6)

Table 3 Approximate Base-Stock Levels (Four-Horizon)

		M			
T	*C*	1	2	3	4
1	1	325.8	315.2	315.2	315.2
	2	357.4	347.1	347.2	347.2
	3	382.3	370.3	370.3	370.3
	4	269.4	260.3	260.3	260.3
2	1	325.8	315.2	315.2	315.2
	2	358.4	346.8	346.8	346.8
	3	382.3	369.8	369.8	369.8
	4	269.4	260.4	260.6	260.6
3	1	325.8	315.6	315.6	315.6
	2	359.5	347.2	347.2	347.2
	3	382.2	369.8	369.8	369.8
	4	269.4	260.5	260.6	260.6
4	1	325.3	315.4	315.3	315.3
	2	357.9	346.8	346.8	346.8
	3	380.6	369.4	369.4	369.4
	4	269.0	260.3	260.3	260.3
5	1	325.8	315.2	315.3	315.3
	2	357.4	346.8	346.8	346.8
	3	382.3	370.8	370.3	370.3
	4	269.0	261.6	261.3	261.3
6	1	326.8	315.2	315.2	315.2
	2	358.4	346.6	346.8	346.8
	3	382.2	369.8	369.8	369.8
	4	269.0	260.5	260.6	260.6

Table 4 Approximate Expected Costs (Four-Horizon)

		M			
T	*C*	1	2	3	4
1	1	1,191.4	1,007.6	1,021.3	1,021.0
	2	1,373.1	1,173.1	1,177.1	1,176.8
	3	1,240.2	1,059.5	1,063.2	1,063.0
	4	1,142.8	975.5	979.4	979.1
2	1	1,157.8	986.8	992.4	992.1
	2	1,339.5	1,144.5	1,148.2	1,147.9
	3	1,206.5	1,030.1	1,034.3	1,034.1
	4	1,109.1	947.2	950.5	950.3
3	1	1,124.1	958.5	963.5	963.3
	2	1,305.7	1,111.1	1,119.3	1,119.0
	3	1,172.7	1,001.8	1,005.5	1,005.2
	4	1,075.4	911.5	921.6	921.3
4	1	1,090.3	931.1	934.7	934.5
	2	1,272.0	1,084.0	1,090.5	1,090.2
	3	1,138.9	971.1	976.6	976.4
	4	1,041.7	887.8	892.8	892.5
5	1	1,056.6	900.2	905.8	905.6
	2	1,238.2	1,058.1	1,061.6	1,061.3
	3	1,105.3	943.6	947.7	947.5
	4	1,007.9	852.6	863.7	863.5
6	1	955.5	813.5	819.1	819.0
	2	1,137.1	970.8	975.0	974.8
	3	1,004.1	858.1	861.1	861.0
	4	906.8	773.1	777.3	777.1

Table 5 Approximate Base-Stock Levels (Eight-Horizon)

		M			
T	*C*	1	2	3	4
1	1	325.8	315.3	315.2	315.2
	2	357.4	346.9	347.2	347.2
	3	382.3	370.4	370.3	370.3
	4	269.4	260.3	260.3	260.3
2	1	325.8	315.2	315.2	315.2
	2	358.4	346.8	346.8	346.8
	3	382.3	369.8	369.8	369.8
	4	269.4	260.6	260.6	260.6
3	1	325.8	315.6	315.6	315.6
	2	359.5	347.2	347.2	347.2
	3	382.2	369.9	369.8	369.8
	4	269.4	260.5	260.6	260.6
4	1	325.3	315.3	315.3	315.3
	2	357.9	346.8	346.8	346.8
	3	380.6	369.3	369.4	369.4
	4	269.0	260.3	260.3	260.3
5	1	325.8	315.2	315.3	315.3
	2	357.4	346.8	346.8	346.8
	3	382.3	370.8	370.3	370.3
	4	269.0	261.6	261.3	261.3
6	1	326.8	315.2	315.2	315.2
	2	358.4	346.6	346.8	346.8
	3	382.2	369.8	369.8	369.8
	4	269.0	260.4	260.6	260.6

Table 6 Approximate Expected Costs (Eight-Horizon)

		M			
T	*C*	1	2	3	4
1	1	2,562.7	2,158.3	2,191.6	2,192.1
	2	2,695.7	2,292.1	2,306.4	2,306.7
	3	2,429.9	2,059.6	2,078.1	2,078.6
	4	2,695.9	2,288.3	2,305.2	2,305.7
2	1	2,405.5	2,042.7	2,057.3	2,057.7
	2	2,538.5	2,153.4	2,172.0	2,172.3
	3	2,272.5	1,930.8	1,943.9	1,944.3
	4	2,538.3	2,157.5	2,170.9	2,171.3
3	1	2,248.2	1,909.9	1,923.0	1,923.3
	2	2,381.1	2,016.6	2,037.7	2,037.9
	3	2,115.0	1,797.8	1,809.6	1,809.9
	4	2,381.1	2,014.9	2,036.5	2,036.8
4	1	2,090.7	1,777.4	1,788.8	1,789.1
	2	2,223.7	1,886.8	1,903.4	1,903.5
	3	1,957.6	1,663.6	1,675.3	1,675.5
	4	2,223.7	1,884.1	1,902.2	1,902.5
5	1	1,933.4	1,642.7	1,654.4	1,654.6
	2	2,066.3	1,757.6	1,769.1	1,769.1
	3	1,800.4	1,531.9	1,541.0	1,541.1
	4	2,066.3	1,747.2	1,767.7	1,768.0
6	1	1,461.3	1,241.6	1,251.5	1,251.4
	2	1,587.2	1,360.1	1,368.9	1,368.7
	3	1,328.2	1,128.6	1,138.1	1,138.1
	4	1,594.2	1,355.9	1,365.0	1,365.0

Appendix

A. Constant Lead Time—Rigorous Argument

We provide a rigorous justification for using the inventory position as a state variable. The approach follows Zipkin (2000, pp. 407–408).

Suppose the lead time is $L-1$. Also, assume that we pay the order cost, not when we place an order but when we receive it, i.e., L periods later. At the beginning of period t, the available information is the current net inventory, outstanding orders, and demand forecasts. Let

$$z_t^l = \text{order placed } l \text{ periods ago, i.e., at time } t-l, \quad l = 1, \ldots, L-1$$

$$\mathbf{z}_t = [z_t^l]_{l=1}^{L-1},$$

and let $\tilde{x}_t$ denote the net inventory. The state is the triple $[\tilde{x}_t, \mathbf{z}_t, D_t^{T+L-1}]$. The dynamics are

$$\begin{aligned} D_{t,s} &= D_{t-1,s} \oplus e_{t,s} \quad \text{for } s = t, t+1, \ldots, T, \\ \tilde{x}_{t+1} &= \tilde{x}_t + z_t^{L-1} - D_{t,t} \\ \mathbf{z}_{t+1} &= [z_t, z_t^1, \ldots, z_t^{L-2}]. \end{aligned}$$

Define the functions

$$C_t^0(y, D_t^{T+L-1}) \equiv \bar{C}_t(y),$$

$$C_t^l(y, D_t^{T+L-1}) \equiv \alpha^l E\left[\bar{C}_{t+l}\left(y - \sum_{i=t}^{t+l-1}\left(D_{t-1,i} \oplus \bigoplus_{j=t}^{i} e_{j,i}\right)\right)\right] \quad \text{for } l \geq 1,$$

where

$$\bar{C}_t(y) \equiv h_{t-1} \cdot (y)^+ + p_{t-1} \cdot (y)^-,$$

$h_0 = p_0 = 0$ and $\bigoplus_{i=1}^n a_i \equiv a_1 \oplus a_2 \oplus \cdots \oplus a_n$. The expectation is taken at the beginning of period t. Let $\vec{J}_{t,T}(\tilde{x}_t, \mathbf{z}_t, D_t^{T+L-1})$ denote optimal expected cost from period t onward, starting in state $(\tilde{x}_t, \mathbf{z}_t, D_t^{T+L-1})$ and define

$$\begin{aligned} &\vec{C}_t(\tilde{x}_t, \mathbf{z}_t, D_t^{T+L-1}) \\ &\quad \equiv C_t^0(\tilde{x}_t, D_t^{T+L-1}) + C_t^1(\tilde{x}_t + z_t^{L-1}, D_t^{T+L-1}) \\ &\qquad + C_t^2(\tilde{x}_t + z_t^{L-1} + z_t^{L-2}, D_t^{T+L-1}) + \cdots + C_t^{L-1}(x_t, D_t^{T+L-1}), \end{aligned}$$

where x_t is the inventory position at the beginning of period t. Because the remaining costs after the end of period

T are given by $\vec{C}_{T+1}(\tilde{x}_{T+1}, \mathbf{z}_{T+1}, D_{T+1}^{T+L-1})$, we set

$$\vec{J}_{T+1,T}(\tilde{x}_{T+1}, \mathbf{z}_{T+1}, D_{T+1}^{T+L-1}) = \vec{C}_{T+1}(\tilde{x}_{T+1}, \mathbf{z}_{T+1}, D_{T+1}^{T+L-1}) - \alpha^L c_{T+1}\left(x_{T+1} - \sum_{i=1}^{L-1} D_{T,T+i}\right).$$

For $t \le T$ we obtain the following functional equation:

$$\vec{J}_{t,T}(\tilde{x}_t, \mathbf{z}_t, D_t^{T+L-1}) = \min_{z\ge 0}\{C_t^0(\tilde{x}_t, D_t^{T+L-1}) + \alpha^L c_t z + \alpha E[\vec{J}_{t+1,T}(\tilde{x}_t + z_t^{L-1} - D_{t,t}, (z, z_t^1, \ldots, z_t^{L-2}), D_{t+1}^{T+L-1})]\}.$$

The expectation is taken with respect to the forecast update vector e_t^T.

Proposition 10. $\vec{J}_{t,T}(\tilde{x}_t, \mathbf{z}_t, D_t^{T+L-1}) = \vec{C}_t(\tilde{x}_t, \mathbf{z}_t, D_t^{T+L-1}) + \tilde{J}_{t,T}(x_t, D_t^{T+L-1})$.

Proof. We show the result by induction on t. The result holds for $t = T+1$, by

$$\tilde{J}_{T+1,T}(x_{T+1}, D_{T+1}^{T+L-1}) = -\alpha^L c_{T+1}\left(x_{T+1} - \sum_{i=1}^{L-1} D_{T,T+i}\right).$$

Suppose that the result holds for $t+1$. Using this induction hypothesis and the fact that

$$C_t^l(y, D_t^{T+L-1}) = \alpha E[C_{t+1}^{l-1}(y - D_{t,t}, D_{t+1}^{T+L-1})],$$

$$\begin{aligned}
&\vec{J}_{t,T}(\tilde{x}_t, \mathbf{z}_t, D_t^{T+L-1})\\
&= \min_{z\ge 0}\{C_t^0(\tilde{x}_t, D_t^{T+L-1}) + \alpha^L c_t z\\
&\quad + \alpha E[\vec{C}_{t+1}(\tilde{x}_t + z_t^{L-1} - D_{t,t}, (z, z_t^1, z_t^2, \ldots, z_t^{L-2}), D_{t+1}^{T+L-1})]\\
&\quad + \alpha E[\tilde{J}_{t+1,T}(x_t + z - D_{t,t}, D_{t+1}^{T+L-1})]\}\\
&= \vec{C}_t(\tilde{x}_t, \mathbf{z}_t, D_t^{T+L-1}) + \min_{z\ge 0}\{\alpha^L c_t z + C_t^L(x_t + z, D_t^{T+L-1})\\
&\quad + \alpha E[\tilde{J}_{t+1,T}(x_t + z - D_{t,t}, D_{t+1}^{T+L-1})]\}\\
&= \vec{C}_t(\tilde{x}_t, \mathbf{z}_t, D_t^{T+L-1}) + \tilde{J}_{t,T}(x_t, D_t^{T+L-1}). \quad \square
\end{aligned}$$

B. Proofs of Propositions and Lemmas in §3

Proof of Proposition 1. It is sufficient to show that

$$\frac{\partial}{\partial y} G_{t,T+1}(y, D_t^{T+1}) \ge \frac{\partial}{\partial y} G_{t,T}(y, D_t^T).$$

This is shown by induction on t. For $t = T$, since $C_{T+1}(y, D_{T,T+1})$ is convex in y, $\partial J_{T+1,T+1}(x, D_{T,T+1})/\partial x \ge 0$. Thus the result holds for $t = T$.

Suppose the result holds for $t+1$. Then,

$$\frac{\partial}{\partial x} J_{t+1,T+1}(x, D_{t+1}^{T+1}) \ge \frac{\partial}{\partial x} J_{t+1,T}(x, D_{t+1}^T),$$

where $D_{t+1}^{T+1} = (D_{t+1}^T, D_{t,T+1})$. Also, one can show that $\partial J_{t+1,T+1}/\partial x$ and $\partial J_{t+1,T}/\partial x$ are bounded above by $\bar{h}(1 - \alpha^{T-t+1})/(1-\alpha)$ and $\bar{h}(1-\alpha^{T-t})/(1-\alpha)$, respectively, and bounded below by 0. Thus, we can exchange the expectation and the differential by the dominated convergence theorem. It follows that

$$\frac{\partial}{\partial y} E[J_{t+1,T+1}(y - D_{t,t}, D_{t+1}^{T+1})] \ge \frac{\partial}{\partial y} E[J_{t+1,T}(y - D_{t,t}, D_{t+1}^T)],$$

and so the result holds for t. $\square$

Proof of Proposition 2. For any $k \le T$ we show both the result and the following inequality simultaneously, by induction on t. For all $t \le k$, y, D_t^k and $D_{t-1,k+1}, \ldots, D_{t-1,T}$,

$$\frac{\partial}{\partial y} G_{t,k,T}^L(y, D_t^k) \ge \frac{\partial}{\partial y} G_{t,T}(y, D_t^T), \tag{11}$$

where $D_t^T = (D_t^k, D_{t-1,k+1}, \ldots, D_{t-1,T})$.

First, for any $k \le T$, y and D_k^T,

$$\begin{aligned}
&\frac{\partial}{\partial y} G_{k,T}(y, D_k^T)\\
&= \frac{\partial}{\partial y} C_k(y, D_{k-1,k}) + \alpha E\left[\frac{\partial}{\partial y} J_{k+1,T}(y - D_{k,k}, D_{k+1}^T)\right]\\
&\le \frac{\partial}{\partial y} C_k(y, D_{k-1,k}) + \alpha \cdot \frac{\bar{h}(1-\alpha^{T-k})}{1-\alpha}\\
&= \frac{d}{dy} G_{k,k,T}^L(y, D_k^k),
\end{aligned}$$

as $\partial J_{k,T}(x, D_k^T)/\partial x$ is bounded above by $\bar{h}(1-\alpha^{T-k})/(1-\alpha)$. Thus, $y_{k,k,T}^L(D_k^k) \le y_{k,T}^*(D_k^T)$.

Second, suppose that both the result and Equation (11) hold for $t+1$. Because

$$\frac{\partial}{\partial x} J_{t+1,k,T}^L(x, D_{t+1}^k) = \begin{cases} 0 & \text{for } x \le y_{t+1,k,T}^L(D_{t+1}^k) \\ \partial G_{t+1,k,T}^L(x, D_{t+1}^k)/\partial x & \text{for } x > y_{t+1,k,T}^L(D_{t+1}^k), \end{cases}$$

and

$$\frac{\partial}{\partial x} J_{t+1,T}(x, D_{t+1}^T) = \begin{cases} 0 & \text{for } x \le y_{t+1,T}^*(D_{t+1}^T) \\ \partial G_{t+1,T}(x, D_{t+1}^T)/\partial x & \text{for } x > y_{t+1,T}^*(D_{t+1}^T), \end{cases}$$

we obtain

$$\begin{aligned}
&\frac{\partial}{\partial y} G_{t,k,T}^L(y, D_t^k)\\
&= \frac{\partial}{\partial y} C_t(y, D_{t-1,t}) + E\left[\frac{\partial}{\partial y} J_{t+1,k,T}^L(y - D_{t,t}, D_{t+1}^k)\right]\\
&\ge \frac{\partial}{\partial y} C_t(y, D_{t-1,t}) + E\left[\frac{\partial}{\partial y} J_{t+1,T}(y - D_{t,t}, D_{t+1}^T)\right]\\
&= \frac{\partial}{\partial y} G_{t,T}(y, D_t^T).
\end{aligned}$$

Thus, the result holds for t. $\square$

Proof of Proposition 3. The argument is similar to the proof of Proposition 1. □

Proof of Lemma 4.

1. Since for any $t \le k$, x and D_t^k

$$J_{t,k}(x, D_t^k) = \begin{cases} G_{t,k}(y^*_{t,k}(D_t^k), D_t^k) & \text{for } x \le y^*_{t,k}(D_t^k) \\ G_{t,k}(x, D_t^k) & \text{for } x > y^*_{t,k}(D_t^k) \end{cases}$$

and

$$J^L_{t,k,\infty}(x, D_t^k) = \begin{cases} G^L_{t,k,\infty}(y^L_{t,k,\infty}(D_t^k), D_t^k) & \text{for } x \le y^L_{t,k,\infty}(D_t^k) \\ G^L_{t,k,\infty}(x, D_t^k) & \text{for } x > y^L_{t,k,\infty}(D_t^k), \end{cases}$$

we obtain

$$\begin{aligned} &\frac{\partial}{\partial x} J^L_{t,k,\infty}(x, D_t^k) - \frac{\partial}{\partial x} J_{t,k}(x, D_t^k) \\ &\quad = \begin{cases} 0 & \text{for } x \le y^L_{t,k,\infty}(D_t^k) \\ \partial G^L_{t,k,\infty}(x, D_t^k)/\partial x & \\ \qquad \text{for } y^L_{t,k,\infty}(D_t^k) < x \le y^*_{t,k}(D_t^k) & \\ \partial G^L_{t,k,\infty}(x, D_t^k)/\partial x - \partial G_{t,k}(x, D_t^k)/\partial x & \\ \qquad \text{for } x > y^*_{t,k}(D_t^k) & \end{cases} \\ &\quad \le \frac{\partial}{\partial x} G^L_{t,k,\infty}(x, D_t^k) - \frac{\partial}{\partial x} G_{t,k}(x, D_t^k) \\ &\quad = \Delta\left(\frac{\partial}{\partial x} G_{t,k}(x, D_t^k)\right). \end{aligned}$$

2. From (4) and (5) we obtain for $y \in Y_{t,k}(D_t^k)$

$$\begin{aligned} &(y - y^L_{t,k,\infty}(D_t^k)) m_{t,k}(D_t^k) \\ &\quad \le \frac{\partial}{\partial y} G^L_{t,k,\infty}(y, D_t^k) - \frac{\partial}{\partial y} G^L_{t,k,\infty}(y^L_{t,k,\infty}(D_t^k), D_t^k) \\ &\quad = \frac{\partial}{\partial y} G^L_{t,k,\infty}(y, D_t^k) \end{aligned}$$

and

$$\begin{aligned} &(y^*_{t,k}(D_t^k) - y) m_{t,k}(D_t^k) \\ &\quad \le \frac{\partial}{\partial y} G_{t,k}(y^*_{t,k}(D_t^k), D_t^k) - \frac{\partial}{\partial y} G_{t,k}(y, D_t^k) \\ &\quad = -\frac{\partial}{\partial y} G_{t,k}(y, D_t^k). \end{aligned}$$

Thus,

$$(y^*_{t,k}(D_t^k) - y^L_{t,k,\infty}(D_t^k)) m_{t,k}(D_t^k) \le \Delta\left(\frac{\partial}{\partial y} G_{t,k}(y, D_t^k)\right). \quad \square$$

Proof of Lemma 5. By induction on t. For $t = k$,

$$\begin{aligned} \Delta\left(\frac{\partial}{\partial y} G_{k,k}(y, D_{k-1,k})\right) &= \alpha E\left[\Delta\left(\frac{\partial}{\partial y} J_{k+1,k}(y - D_{k,k})\right)\right] \\ &= \frac{\alpha \bar{h}}{1-\alpha}. \end{aligned}$$

Suppose that the result holds for $t+1$. Then,

$$\begin{aligned} \Delta\left(\frac{\partial}{\partial y} G_{t,k}(y, D_t^k)\right) &= \alpha E\left[\Delta\left(\frac{\partial}{\partial y} J_{t+1,k}(y - D_{t,t}, D_{t+1}^k)\right)\right] \\ &\le \alpha E\left[\Delta\left(\frac{\partial}{\partial y} G_{t+1,k}(y - D_{t,t}, D_{t+1}^k)\right)\right] \\ &\le \alpha \cdot \frac{\alpha^{k-t}\bar{h}}{1-\alpha} = \frac{\alpha^{k-t+1}\bar{h}}{1-\alpha}. \end{aligned}$$

The first inequality comes from Lemma 4 and the second from the induction hypothesis. □

Proof of Lemma 6. Since $y^L_{t,k,\infty}(D_t^k)$ increases in k and $y^*_{t,k}(D_t^k)$ decreases, the interval $Y_{t,k}(D_t^k)$ shrinks. Thus, $m_{t,k}(D_t^k) \ge m_{t,l}(D_t^l)$. □

Proof of Proposition 7. From Lemma 6, $m_{t,k}(D_t^k) \ge m_{t,l}(D_t^l) > 0$ for all $k \ge l$, which guarantees that $\{m_{t,k}(D_t^k), k \ge l\}$ is bounded below. Thus, the result follows from Lemmas 4 and 5. □

Proof of Proposition 8. To show the optimality of this policy, it is sufficient from Veinott (1965) to show that it is always possible to raise the inventory position to the myopic forecast-centered base-stock level at the beginning of each period. We show this by induction on t. At the beginning of period 1 $y_1 = \bar{y}_1 + D_{0,1}$. At the end of period 1 the inventory position x_2 is

$$y_1 - D_{1,1} = \bar{y} + D_{0,1} - (D_{0,1} + e_{1,1}) = \bar{y} - e_{1,1}.$$

Then

$$\bar{y} + D_{1,2} - x_2 = \bar{y} + D_{0,2} + e_{1,2} - (\bar{y} - e_{1,1}) = D_{0,2} + e_{1,1} + e_{1,2} \ge 0.$$

The inequality holds by stationarity and Condition A. Thus the result holds for $t = 2$.

Now, suppose that the result holds for t; that is, $y_t = \bar{y}_t + D_{t-1,t}$. Then at the end of period t the inventory position x_t is

$$y_t - D_{t,t} = \bar{y} + D_{t-1,t} - (D_{t-1,t} + e_{t,t}) = \bar{y} - e_{t,t}.$$

Then

$$\bar{y} + D_{t,t+1} - x_t = D_{t,t+1} + e_{t,t} \ge 0.$$

Again, the inequality holds by stationarity and Condition A. Thus, the result holds for $t+1$. □

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