Operations Research

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Joint Inventory and Markdown Management for Perishable Goods with Strategic Consumer Behavior

Peng Hu, Stephen Shum, Man Yu

OPERATIONS RESEARCH
Vol. 64, No. 1, January–February 2016, pp. 118–134
ISSN 0030-364X (print) | ISSN 1526-5463 (online)

http://dx.doi.org/10.1287/opre.2015.1439

Joint Inventory and Markdown Management for Perishable Goods with Strategic Consumer Behavior

Peng Hu
School of Management, Huazhong University of Science and Technology, Wuhan, China 430074,
hu_peng@hust.edu.cn

Stephen Shum
College of Business, City University of Hong Kong, Kowloon Tong, Kowloon, Hong Kong,
swhshum@cityu.edu.hk

Man Yu
School of Business and Management, Hong Kong University of Science and Technology,
Clear Water Bay, Kowloon, Hong Kong, manyu@ust.hk

In this paper we formulate and analyze a novel model on a firm's dynamic inventory and markdown decisions for perishable goods. We consider a dynamic stochastic setting, where every period consists of two phases, clearance phase and regular-sales phase. In the clearance phase, the firm decides how much to order for regular sales, as well as whether to markdown some (or all) of the leftover inventory from the previous period that will be disposed otherwise. Since strategic consumers may buy the product during clearance sales for future consumption, markdown may cannibalize future sales at regular price. Hence, the firm needs to make a trade-off between product spoilage and intertemporal demand substitution. We show that the firm should either put all of the leftover inventory on discount or dispose all of it, and the choice depends on the amount of leftover inventory from the previous period. In particular, the firm should introduce markdown when the amount of leftover inventory is higher than a certain threshold, and dispose all otherwise. We also conduct numerical studies to further characterize the optimal policy, and to evaluate the loss of efficiency under static policies when compared to the optimal dynamic policy.

Keywords: revenue management; inventory; markdown; intertemporal substitution.
Subject classifications: inventory/production: perishable/aging items.
Area of review: Operations and Supply Chains.
History: Received November 2014; revisions received May 2015, August 2015; accepted September 2015. Published online in *Articles in Advance* December 30, 2015.

1. Introduction

This paper is motivated by a local bakery chain's decision problem in its day-to-day operations. Every day in the morning, fresh bakery products are delivered to each store based on the store manager's order, which is submitted in the previous evening. Because of the perishability nature of bakery products, each shop keeps on shelf only the food baked on that day. Anything unsold at the end of a day is returned to the factory for disposal. As consumption of bakery products occurs in the morning or early afternoon (for breakfast, lunch or afternoon tea), any item not sold by late afternoon will typically remain unsold and be returned for disposal at the end of the day. To reduce the amount of disposed products, the chain has started selling all products on display at a discount price every evening shortly before the store closes. While the regular price, the discount price, and the time to introduce discount are decided centrally on a long-term basis, how much to sell at the discount price and how much to order for the next day are determined every day at each store manager's discretion. Specifically, every day in the evening, each store manager needs to make ordering decision for the next day, as well as decide whether to sell some (or all) of the remaining products at the discount price or put them in the backroom for return and disposal. While there is a discount for everything consumers see on display in the store, the store manager may not have made all remaining items available during this promotion period.

The number of products to put on discount in each evening is not a trivial problem. On one hand, the markdown sales generate some revenue for the otherwise disposed food. Offering a price discount on those items can induce purchases from consumers who cannot afford the full price. On the other hand, markdown provides incentives for consumers to strategically forward buy. Products that are kept overnight are not fresh enough for selling but still suitable for consumption. Attracted by the price discount during markdown sales, some of the consumers who can afford the full price may stock up the product for consumption on the next day. Hence, any clearance sales on a given day has a negative impact on the revenue from selling at the full price on the next day.

The example of this local bakery chain does not stand alone. Intertemporal cannibalization is an important concern when a shop markdowns otherwise-disposed products at the end of a day. This practice of marking down products at the end of the day is very common for bakery products[1] and for poultry and meat items.[2] For example, grocery stores usually choose to either markdown or dispose rotisserie chickens left at the end of the day, because the process of refrigerating and reheating a rotisserie chicken is costly and the chicken will look significantly different after this overnight process, making it hard to attract sales on the next day. However, consumers who purchase it at a discount in the evening may store the chicken overnight and reheat it for consumption the next day. There are many other similar grocery products that are perishable in the sense that, if kept overnight, they are not fresh enough for selling but good enough for consumption. For example, it is no secret to buyers that in evenings many Japanese and Korean supermarkets slash the prices by as much as 50% for their unsold sushi, sashimi, salad, kimbap, and bentos made on the day. Consumers rush into the stores at the end of the day to grab the sushi boxes or bentos, which are great deals for lunch the next day.[3] Similar end-of-day deals can be found in food markets for live fish and fresh seafood.[4]

These examples exhibit two common characteristics. First, markdown and regular sales are separated as regular sales occur usually during the daytime whereas markdown sales happen in the evenings. Second, after the markdown sales, firms dispose all the unsold items of the day, even though they can still be consumed on the next day.[5] One reason for this practice is that some food items are strictly regulated with a mandatory "sell by" date, which dictates the end of the items' *shelf life*. After the sell by date, the items must be removed from shelf, even though they may still be edible past this date.[6] In addition, the items will deteriorate after the overnight storage, making it not sellable per the firms' high quality standards. This is especially true for highly competitive industries like bakeries or groceries, where freshness and quality are among the key competitive dimensions. It is thus not surprising that many firms adopt the policy to discard aging produce as "we only sell quality goods."[7] Nevertheless, since consumers may take advantage of end-of-day sales to purchase for future consumption, managers need to carefully take account of strategic customer behavior when coordinating markdown sales with inventory ordering decisions.

In this paper, we formulate a novel model in which the firm makes simultaneous decisions on markdown and inventory for perishable products, in the presence of intertemporal cannibalization and demand uncertainty. We consider a multiperiod stochastic setting, in which each period is divided into two phases, a clearance phase and a regular-sales phase. We label the time periods such that every period starts with the clearance phase. Any unit from the previous period has to be either sold or disposed by the end of the clearance phase, while leftover inventory at the end of the regular-sales phase can be carried over to the clearance phase of the next period. Demand in each period is random with some consumers preferring to buy in the clearance phase and others preferring purchases in the regular-sales phase. Consumption happens at the end of the regular-sales phase. Therefore, consumers who cannot get a unit in the clearance phase may stay in the market for the regular sales, while any unsatisfied demand during the regular-sales phase is lost. The firm decides how much leftover inventory from the previous period to sell in the clearance phase and places an order for delivery in the regular-sales phase at the beginning of each period.

One unique feature in our model is that the firm carries inventory from one period to another, but not from the clearance phase to the regular-sales phase of a period. On the other hand, consumers carry the product from one phase to another inside a period, but not from one period to another. This leads to correlation between the demand in the clearance phase and the demand in the regular-sales phase. In other words, markdown sales allow the firm to sell to consumers who cannot afford the product at the regular price, yet at the expense of a lower revenue from consumers who can afford the regular price but choose to buy early at the discount price. To the best of our knowledge, this is the first stochastic and dynamic inventory model facing strategic customer behavior with joint ordering and discounting decisions.

If demand uncertainty does not exist, the firm can always accurately plan its inventory. In this case, the problem becomes a trade-off between additional revenue from the consumers who cannot afford the full price and the revenue loss from the consumers who forward buy. The optimal strategy is naturally *bang-bang*, i.e., the firm either does not mark down or put all leftover inventory on discount, depending on the profit margin from selling to the two groups of consumers. Our results suggest that demand uncertainty reinforces the benefit of markdown and particularly, when demand becomes more volatile, it is more likely that markdown is optimal. This is because markdown sales reduce the overage risk that the firm will encounter during regular sales. In particular, when the markdown quantity increases, the chance of selling to an additional customer at the regular price decreases. Hence, the firm can order a smaller quantity of fresh products and the expected overage cost for the fresh products decreases. Furthermore, the marginal cost of cannibalization decreases and the marginal cost savings because of inventory overage increases when the markdown quantity increases. Thus, the optimal strategy in the presence of demand uncertainty is still bang-bang: the firm should either put all of the leftover inventory on sale or dispose all of it, and the choice depends on the amount of inventory on hand. In particular, there exists a threshold such that the firm should offer discount sales for all leftover inventory if the amount of leftover inventory is above this threshold, and dispose all inventory otherwise. This simple bang-bang optimal policy allows chain stores (such as the one in our motivating story) to develop clear guidelines for store managers to manage discount sales for leftover products.

One technical challenge in this problem is that traditional approaches in solving dynamic inventory models cannot be applied to our model. This is because the single-period profit function is neither convex nor concave but quasiconvex in the amount of leftover inventory to markdown. In addition, the intertemporal cannibalization leads to demand interdependence across the two phases in a period, which creates complicated dynamics in how inventory is carried over from one period to another. To the best of our knowledge, there does not exist any prior study on preserving *quasiconvexity* in a multiperiod *maximization* problem. However, by using a transformation technique and exploiting properties that are specific to our problem, we are able to prove that this bang-bang policy is optimal in the multiperiod case under some conditions.

To gain further insights on the optimal policy, we conduct extensive numerical explorations to examine how the optimal strategy depends on the level of demand uncertainty, as well as to evaluate loss of efficiency under some static policies in comparison with the optimal dynamic policy. Our results indicate that both the improvement of dynamic markdown policy over a static policy and the cost of ignoring intertemporal substitution effect can be quite significant. We extend our base model by considering micro models capturing individual consumers' purchasing choices and by incorporating endogenous markdown price and consumers' shopping costs. We show, in some cases through numerical studies, that our results are robust in these extensions.

The remainder of the paper is structured as follows. Section 2 reviews the literature. We introduce the model in §3, and present the analysis and results in §4. In §5 we examine several extensions. Finally, §6 provides concluding remarks. All proofs are relegated to the appendix, except Theorem 3 and Proposition 4, which are the main results of the paper.

2. Literature Review

The question of how much to sell at a discount is a classical question in capacity-based revenue management. Earliest work dates back to Littlewood (1972), who studies the problem of how to allocate a fixed number of capacity to demand for two fare classes, with presence of demand uncertainty. However, this and most of the other early works (for reviews of the early works, see Barnhart and Talluri 1997, McGill and van Ryzin 1999) do not consider cannibalization across demands for different fare classes, until the seminal work by Talluri and van Ryzin (2004). They consider the problem of how to allocate a fixed amount of capacity to sell at different prices when the availability of product at different prices may affect consumers' choice. Some papers extend the problem to the case with a network of resources (e.g., Gallego et al. 2004, Liu and van Ryzin 2008, Chaneton and Vulcano 2011). See Talluri (2012) for a recent review. There are two main differences between our paper and the existing papers in this stream of literature. First, most of these papers in this area assume fixed capacity and do not consider the inventory decisions of the firm, because the focus of these papers has been mostly on airline and hospitality industries. Second, these papers do not consider strategic customer behavior.

Our paper is also related to the stream of literature on stochastic inventory management. For reviews, we refer to Porteus (1990), Lee and Nahmias (1993), Zipkin (2000). Recent papers have incorporated pricing decisions (see Federgruen and Heching 1999, Chen and Simchi-Levi 2004, Adida and Perakis 2010, Chen et al. 2015), marketing and advertising decisions (Olsen and Parker (2008)), and transhipment and expediting decisions (Hu et al. 2008, Huggins and Olsen 2010). While some papers focus on nonperishable products, our paper is more related to the literature on perishable products (for recent reviews, see Karaesmen et al. 2008, Nahmias 2011). In the perishable-inventory literature, some studies consider the question of how much to sell at clearance price in addition to how much to order (see, e.g., Xue et al. 2012, Li et al. 2013, Li and Yu 2014). However, these papers assume that the demand for the regular-priced products and clearance-sales products do not overlap and hence there is not cannibalization between the two. As we will show, cannibalization results in *interdependent demand* for the regular-priced products and clearance-sales products, and has a huge impact on the technical complexity and results of the problem.

There is a rapid-growing stream of literature on strategic customer behavior, after the pioneering work of Besanko and Winston (1990) and Su (2007). For recent reviews, we refer to Shen and Su (2007), Aviv and Vulcano (2012). Different from papers that focus on the impact of other types of consumer behavior on operations management (see, e.g., recent papers on the effects of stochastic reference point (Baron et al. 2015) and of network effect (Hu and Wang 2014)), the research on strategic customer behavior evaluates the impact of intertemporal substitution. Particularly, most papers in this area focus on the case when consumers wait for price discount (i.e., demand is shifted forward from the current period to future periods), with very few papers studying consumer forward-buying behavior (i.e., demand is shifted backward from future periods to the current period). Shou et al. (2013) study a two-period model where consumers may stockpile because of the fear of a stock-out in the second period, and the firm makes inventory ordering decisions in both periods. As price is assumed to be the same across the two periods, Shou et al. (2013) do not consider markdown decision. Su (2010) studies the dynamic pricing problem of a firm in a multiperiod setting when consumers stockpile, and some papers consider the firm's joint rationing and pricing strategy (Xie and Shugan 2001, Liu and Shum 2013, Yu et al. 2014). Yet, none of these papers consider the stochastic dynamic inventory problem of the firm. Our work examines consumer forward buying behavior and its implications to a firm's markdown and inventory decisions. To the best of our knowledge, this is the first paper that considers a stochastic dynamic inventory model of a firm facing strategic customer behavior with joint ordering and discounting decisions.

3. Model

3.1. Formulation

We consider a firm selling a perishable product periodically, where each period (indexed by $n \in \{1, 2, \ldots\}$) is divided into two phases, a clearance phase followed by a regular-sales phase.[8] In the first phase of period n (i.e., the clearance sales on a given day), the firm has x units of inventory on hand and decides the amount $z_n \in [0, x]$ to sell at a clearance price p. This inventory cannot be carried over to the second phase, and all units unsold in the first phase have zero salvage value. In the meanwhile, the firm also decides the amount y_n^0 to produce for selling at a full price $r \geqslant p$ in the second phase (i.e., the regular sales of the next day). The per-unit production cost is c. What is not sold in the second phase of a period will remain until the first phase of the next period (i.e., the clearance sales of the next day). We assume that the firm's holding cost is zero. In Appendix C4 (available as supplemental material at http://www.dx.doi.org/10.1287/opre.2015.1439), we show that holding cost can be incorporated in the model by redefining the cost and revenue terms.

The aggregated demand (i.e., the total number of consumers who can afford at least the discount price p) in period n, denoted by M_n, is random, stationary, bounded, and independent of each other. Denote the upper and lower bounds of M_n by $M_{\max}$ and $M_{\min}$, respectively. Each consumer of a period demands to use one unit of the product in the second phase of that period and decides whether to purchase in the first phase or second phase. Note that duration of the first phase is much shorter than that of the second phase because phase 1 represents the short period of time (usually one or two hours) before store closure, and consumption only occurs in the second phase. That is, those who buy in the first phase will hold the product for use in the second phase. If consumers buy in the first phase, the value of the product will be discounted, since the product will be less fresh when they consume it. If they buy in the second phase, they can enjoy newly produced product but the price will be higher at r. Some consumers may prefer to purchase the product at the discount price, whereas other consumers may prefer to wait until the next morning for a fresh product. A portion of the consumers from the former group may purchase at the full price in phase 2 if they are not able to get the discounted product.

Without loss of generality, we abstract from the consumers' individual purchasing decisions and focus on their aggregated effects on the demand in the two phases (which is similar to the approach used in Honhon et al. 2010). Let α represent the portion of consumers who attempt to make purchases in the first phase. The remaining $1-\alpha$ portion of consumers disdain to buy in the first phase and only buy the fresh product in the second phase. Among the consumers who try to buy in the first phase, some can afford the regular price r but are attracted by the promotion in the first phase. Let ρ represent the portion of such consumers among the first-phase buyers. Note that ρ captures the magnitude of intertemporal substitution, i.e., the demand shifted from the regular-sales phase to the clearance phase because of markdown pricing. We focus on the case with $\rho > 0$. As we illustrate below, when $\rho = 0$, the two phases become independent of each other and the firm's markdown decision becomes straightforward. While we focus on aggregated demand in the base model, in §5 we formulate a micro model for individual consumer choice by constructing consumer utility function and comparing expected utilities from purchasing in either the first or the second phase. We show that such a micro model will also lead to segments of consumers differing in the timing of purchases and hence reduce to our base model with aggregated demand.

Since there is only limited supply z_n in the first phase, not all consumers who attempt to buy the product at price p, can obtain it. In such a case, products are randomly rationed, i.e., every customer has the same chance to get the product. Those consumers who get rationed out in the first phase and can afford the regular price will return for a second attempt in the second phase. That is, the demand in the second phase consists of those consumers who prefer purchases in the second phase (i.e., the $1-\alpha$ portion), as well as those who have been rationed in the first phase and can afford the regular price (i.e., a subset of the $\alpha\rho$ portion).

Because of the leadtime in inventory replenishment, the firm determines the inventory in each period y_n^0 at the same time when it decides the amount z_n to sell at the discount price. Owing to the market-size uncertainty, the ordering quantity y_n^0 may not perfectly match with the demand in the second phase. Leftover units, if any, are carried over to the first phase of the next period for markdown sales or disposal. However, since the consumption time of a consumer buying in one regular-sales phase cannot be delayed into the future, demand in the regular-sales phase cannot be backlogged into the future and hence all unmet demand in this phase is lost. The sequence of events is summarized in Figure 1.

Denote $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$, and $a^+ = a \vee 0$. For given $y_n^0 \in [0, \infty)$ and $z_n \in [0, x_n]$, the firm's expected profit in period n is as follows:

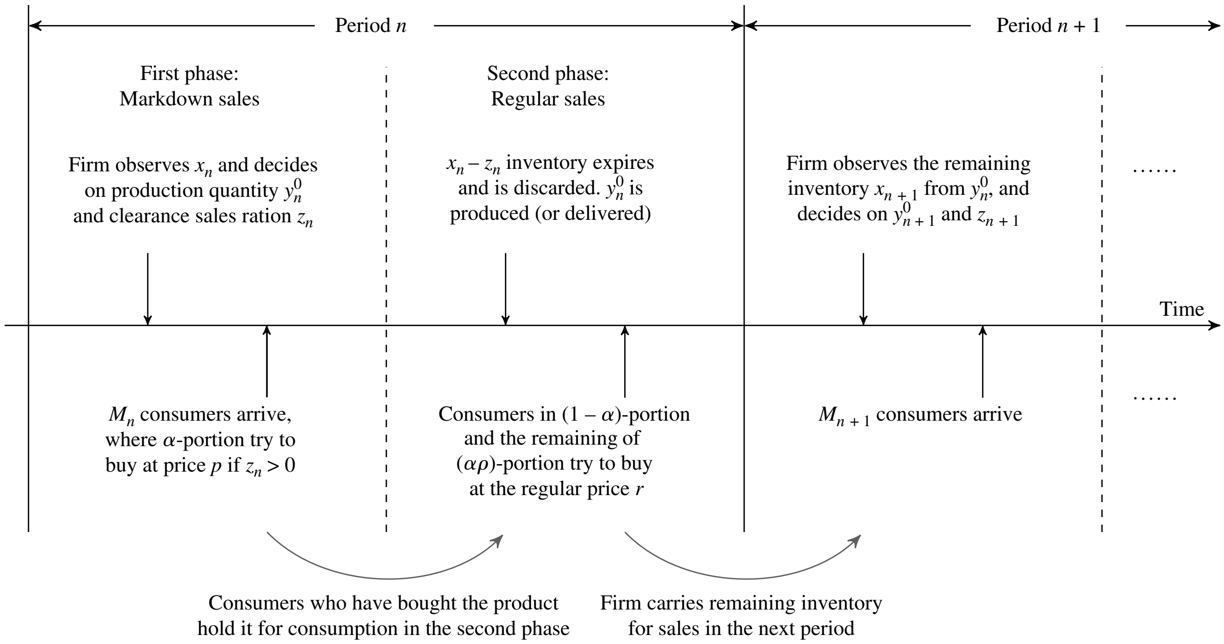
$$\pi_n^0(y_n^0, z_n) = p\mathbb{E}[z_n \wedge (\alpha M_n)] + r\mathbb{E}\{y_n^0 \wedge [(1-\alpha)M_n + \rho(\alpha M_n - z_n)^+]\} - c y_n^0.$$

For period $n+1$, the starting inventory is

$$x_{n+1} = [y_n^0 - (1-\alpha)M_n - \rho(\alpha M_n - z_n)^+]^+,$$

where the positive-part sign inside the square brackets corresponds to the fact that the product is perishable (i.e.,

Figure 1. Sequence of events.



excess supply in the first phase, if any, cannot be used to satisfy demand in the second phase), and the positive-part sign outside the square brackets reflects the lost-sales situation in the second phase.

The firm's objective is to maximize the total expected discounted profit over an infinite planning horizon by dynamically adjusting its inventory and markdown decisions periodically. Given an initial inventory x, the firm's long-term profit satisfies the Bellman equation

$$v(x) = \max_{y^0, z}\left\{\pi^0(y^0, z) + \gamma\,\mathbb{E}\,v(\tilde{x})\colon\ y^0 \geqslant 0, 0 \leqslant z \leqslant x\right\}, \qquad (1)$$

where $0 \leqslant \gamma < 1$ is the firm's discount factor, and by letting M_n be identically distributed as M,

$$\begin{aligned}\pi^0(y^0, z) &= p\,\mathbb{E}[z \wedge (\alpha M)] \\ &\quad + r\,\mathbb{E}\left\{y^0 \wedge [(1-\alpha)M + \rho(\alpha M - z)^+]\right\} - c y^0, \\ \tilde{x} &= [y^0 - (1-\alpha)M - \rho(\alpha M - z)^+]^+.\end{aligned}$$

Denote the optimal solution to problem (1) by $[y^{0*}(x), z^*(x)]$.

A distinguishing feature of the dynamic program (1) is that within each period, consumers hold products across two phases, whereas the firm does not; on the other hand, the firm carries inventory across two consecutive periods. Without intertemporal substitution, the firm's markdown decision is straightforward. To see why, note that if $\rho = 0$, the firm's single-period expected profit and system dynamics become $\pi^0(y^0, z) = p\,\mathbb{E}[z \wedge (\alpha M)] + r\,\mathbb{E}[y^0 \wedge (1-\alpha)M] - c y^0$ and $\tilde{x} = [y^0 - (1-\alpha)M]^+$, respectively. It can be shown that $z^*(x) = x$. Intuitively, if the clearance sales do not influence the demand during the regular sales (as the situation considered in the perishable inventory literature, e.g., Li and Yu 2014), then the two phases are decoupled and the firm should make all units available during the markdown sales. With intertemporal substitution, the leftover inventory x affects future periods through z and y^0. In essence, z affects the amount of consumers who successfully purchase in the markdown phase of the current period and hence the demand arriving in the regular-sales phase, which in turn determines the firm's inventory decision y^0 and the leftover inventory $\tilde{x}$. Because of this, a firm needs to take into consideration the dynamic effects on future profits when making markdown decisions.

Our primary interest is to characterize the firm's optimal joint policy and to evaluate the impact of intertemporal substitution in the dynamic setting. To facilitate our analysis, we first reformulate the problem by introducing some new notations and variables.

3.2. Reformulation

For notation simplicity, we introduce the following notations:

$$p_0 = p/\rho \text{ and } \beta = (1-\alpha)/\alpha,$$

where recall $\rho \leqslant 1$ and we can express $p = \rho p_0 \leqslant p_0$. It is worth noting that p_0 represents "adjusted" discount price and captures cannibalization of markdown sales on regular sales. To see why, suppose that market size is deterministic ($M_n = m$ with probability one). In such a case, the firm can always perfectly plan its inventory and the markdown decision becomes very simple, where the firm makes a trade-off solely between margin and probability of sales. When a consumer attempts to buy in the first phase, the firm has two choices: either serving the consumer and obtaining a profit of p right away, or shutting down the markdown sales and forcing the consumer to wait to buy in the second phase at the higher price r. Although the latter appears to be more profitable, the downside is that, with probability $1-\rho$, the consumer may not return to buy in the second phase. Thus, the firm loses the sales opportunity completely. Besides, the firm needs to pay a unit cost c for production in the second phase. Hence, the expected profit from the latter strategy is $(r-c)\rho$. Therefore, if $p \geqslant (r-c)\rho$, or equivalently $p_0 \geqslant r-c$, then the markdown sales is profitable and the firm should serve as many consumers as possible in the first phase, $z^*(x)=x$. Otherwise, the firm should not sell any product on discount and $z^*(x)=0$. In such a case, the optimal markdown strategy is of bang-bang type, since the profit function $\pi^0(y^0, z) = \rho p_0 z + (r-c)[(1-\alpha)m + \rho(\alpha m - z)]$ is linear in z when $z \in [0, x \wedge (\alpha m)]$. In particular, whether the firm should offer markdown sales is solely determined by the value of p_0. We formally state the result in the following proposition.

Proposition 1. *When demand is deterministic, $z^*(x)=x$ if $p_0 \geqslant r-c$ and $z^*(x)=0$ otherwise.*

When market uncertainty is present, as we shall see later, the firm's trade-off becomes more complex, but the role of p_0 is similar: as p_0 increases, markdown becomes more appealing to the firm. This is because a larger p_0 implies either a higher discount price or a weaker intertemporal cannibalization. The former means a higher revenue from clearance sales, whereas the latter implies that the clearance sales has less impact on the regular sales.

It would be convenient to introduce a transformed decision variable y:

$$y^0 = D(y, z) = \beta y + \rho(y-z)^+.$$

Obviously $D(y, z)$ is increasing in y and it is equal to 0 at $y=0$ for any fixed $z \geqslant 0$. It implies that for any production quantity $y^0 \geqslant 0$, there exists $y \geqslant 0$ such that $y^0 = D(y, z) \geqslant 0$; and for any $y \geqslant 0$, the corresponding production $y^0 = D(y, z) \geqslant 0$. Thus, to find the optimal inventory policy, it is equivalent to optimizing over $y \geqslant 0$. Also, note that $D(\alpha M, z) = \beta\alpha M + \rho(\alpha M - z)^+$ corresponds to the total demand to be fulfilled in the second phase. Hence, the firm's dynamic decision problem can be reformulated as

$$v(x) = \max_{y,z}\{g(y, z)\colon y \geqslant 0,\ z \in [0, x]\}, \tag{2}$$

where

$$g(y, z) = \pi(y, z) + \gamma\,\mathbb{E}\,v(\tilde{x}), \tag{3}$$

$$\pi(y, z) = \rho p_0\,\mathbb{E}[z \wedge (\alpha M)] - cD(y, z) + r\,\mathbb{E}\,D(y \wedge \alpha M, z), \tag{4}$$

$$\tilde{x} = [D(y, z) - D(\alpha M, z)]^+. \tag{5}$$

Although y is introduced mainly for analytical convenience, it has a neat economic interpretation: for a single-period problem, y is the amount of inventory prepared for αM consumers. To see why, note that in a single-period problem (or in the last period of a multiperiod problem), the firm has only one opportunity to sell the produced units y^0 and aims to satisfy the second-phase demand, with a total amount of $(1-\alpha)M + \rho(\alpha M - z)^+$. Since $y^0 = (1-\alpha)y/\alpha + \rho(y-z)^+$, y can be regarded as the total amount of inventory targeting for the αM consumers who prefer to purchase in the first phase. In particular, if the number of units on clearance sales exceeds the target stocking level in this phase, i.e., $z \geqslant y$, then there is no need to produce extra units for this group of consumers and $y^0 = (1-\alpha)y/\alpha$. On the other hand, if markdown sales is not offered, i.e., $z=0$, then the production is for both $(1-\alpha)M$ and a portion ρ of the αM groups, i.e., $y^0 = [(1-\alpha)/\alpha + \rho]y$.

Let $[y^*(x), z^*(x)]$ be the optimal solution to problem (2). It should be noticed that by a standard approach in dynamic programming, $v(x)$ is the limit of $v_n(x)$ as n goes to infinity, where $v_0(x)=0$ and $v_n(x)$ are inductively defined as below for all $n \geqslant 1$:

$$v_n(x) = \max_{y,z}\{\pi(y, z) + \gamma\mathbb{E}\,v_{n-1}(\tilde{x})\colon y \geqslant 0, 0 \leqslant z \leqslant x\}, \tag{6}$$

Furthermore, if we denote by $[y_n^*(x), z_n^*(x)]$ the optimal solution to the above problem, then $y^*(x)$ and $z^*(x)$ are the limits of $y_n^*(x)$ and $z_n^*(x)$ as n goes to infinity, respectively. Also notice that our main results remain valid in a finite-horizon setting.

To facilitate further analysis, define two auxiliary functions for $x \geqslant 0$ as below:

$$\theta(x) = r\,\mathbb{E}[x \wedge (\alpha M)] - cx,$$

$$\lambda(x) = (r-p_0)\,\mathbb{E}[x \wedge (\alpha M)] - cx.$$

where $\theta(x)$ is simply the newsvendor profit function for the first-phase demand αM and is clearly concave in x; and $\lambda(x)$ represents the difference between the newsvendor profit $\theta(x)$ and the profit from serving consumers αM with on-hand inventory x at the adjusted discounted price p_0. Define the (smallest) maximizers of $\theta(x)$ and $\lambda(x)$ as below:

$$\tau = \arg\max_{x \geqslant 0} \theta(x), \quad \sigma = \arg\max_{x \geqslant 0} \lambda(x).$$

4. Optimal Policy

4.1. Single-Period Problem

We start from the firm's problem in a single-period setting, i.e., $\gamma = 0$ in problem (2). We characterize $\pi(y, z)$ in the following proposition.

Proposition 2. (a) *$\pi(y, z)$ is increasing in y if $y \leqslant \tau$ and decreasing in y if $y \geqslant \tau$.*

(b) *$\pi(y, z)$ is increasing in z if $p_0 \geqslant r-c$. Otherwise it is first decreasing and then increasing in z.*

Proposition 2 implies that $\pi(y,z)$ is *quasiconcave* in y and *quasiconvex* in z. With this proposition, we characterize the firm's optimal policy in a single-period problem as below.

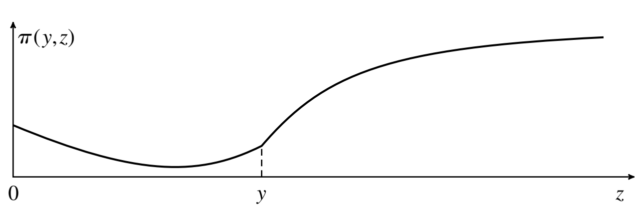
THEOREM 1. *For the single-period problem, $y^*(x)=\tau$ and either $z^*(x)=0$ or $z^*(x)=x$. In particular, $z^*(x)=x$ if $p_0 \geqslant r-c$ and $z^*(x)=0$ if $p_0 \leqslant [r\,\mathbb{E}(\tau\wedge\alpha M)-c\tau]/\mathbb{E}(\alpha M)$; otherwise, there exists a cutoff level s such that $z^*(x)=x$ if $x \geqslant s$ and $z^*(x)=0$ if $x<s$.*

Theorem 1 states that the optimal transformed production quantity in the single-period problem is simply the optimal stocking level τ for a newsvendor facing demand αM. This is in line with our earlier interpretation of the decision variable y. It is the amount of inventory prepared for the αM consumers who decide to come in the first phase. Meanwhile, similar to the solution in the deterministic-demand setting, the optimal markdown policy in the case with stochastic demand is also bang-bang for the single-period problem, i.e., the firm should either sell as much as possible in the markdown sales or does not offer discount at all, i.e., $z^*(x)=x$ or $z^*(x)=0$.

Nevertheless, there are two main differences when compared to the case with deterministic demand. First, when $p_0<r-c$, with uncertain demand it can sometimes be optimal to sell as much as possible in the markdown sales (i.e., $z^*(x)=x$), although such a strategy is never optimal under deterministic demand (ref. Proposition 1). This is because demand uncertainty brings in another benefit of markdown. In particular, because of the market uncertainty, any production for the second phase is exposed to overstocking risk: if the market size turns out to be small, the unit cannot be sold and the production cost c is wasted. By serving some consumers in the first phase, the firm can order a smaller quantity of fresh products and hence the expected overage cost for the fresh products decreases.

The second difference when compared to the case with deterministic demand is that, the profit function is no longer linear in z. It is in fact neither convex nor concave in z. Luckily, as illustrated in Figure 2, the profit function is quasiconvex in z, which preserves the bang-bang solution. To see the rationale behind the quasiconvex function, assume $p_0<r-c$ and consider two cases: if the markdown quantity z is higher than y, then $\pi(y,z)$ depends on z only via the term $\rho p_0\,\mathbb{E}[z\wedge(\alpha M)]$ by (4), which increases in z. Intuitively, as we noted earlier, when $z\geqslant y$, the markdown quantity is larger than the amount of inventory the firm needs to prepare for consumers in the α segment. In this case, the firm does not need to produce any additional unit for this segment of consumers and, consequently, the two phases are decoupled. Thus, the more the firm sells in the markdown phase (i.e., z increases), the higher the firm's profit is. On the other hand, if $z\leqslant y$, then $\pi(y,z)$ depends on z via $-\rho\lambda(z)$. The function $\lambda(z)$ captures the expected profit loss by selling up to z consumers in the first phase in comparison to having all consumers purchase with the full price in the second phase. Specifically, $\lambda(z)=(r-c-p_0)\,\mathbb{E}[z\wedge(\alpha M)]-c\,\mathbb{E}[z-(\alpha M)]^+$, where the first term is the loss in gross profit and equals to the difference in the margins, $r-c-p_0$, multiplied by the expected sales, and the second term represents the savings in overage cost by serving consumers with on-hand inventory in the first phase. When z increases, the probability of having a shortage in the first phase decreases, which implies a lower chance of having any of consumers from the αM group remaining in the second phase. Thus, even if z further increases, the expected loss in gross profit decreases because of a small chance of cannibalization actually occurring, but the expected savings in overage cost increases. In other words, the gross profit loss $(r-c-p_0)\,\mathbb{E}[z\wedge(\alpha M)]$ is concave in z and the overage cost saving $c\,\mathbb{E}[z-(\alpha M)]^+$ is convex in z. This means that the expected profit loss $\lambda(z)$ is concave, implying that the profit function is convex in z. Summarizing the two cases, the profit function is first convex (when $z\leqslant y$) and then increasing (when $z\geqslant y$), and thus is quasiconvex in z, resulting in the bang-bang solution.

Figure 2. The firm's single-period profit as a function of z, for a given y.



There are two special scenarios where the optimal markdown policy is independent of the initial inventory level. When the (adjusted) discount price is very high (i.e., $p_0\geqslant r-c$), markdown sales are even more profitable than the regular sales and thus the firm should never limit the markdown sales. In the other extreme, with a very low discount price (i.e., $p_0\leqslant[r\,\mathbb{E}(\tau\wedge\alpha M)-c\tau]/\mathbb{E}(\alpha M)$), the firm should sell only in the second phase. For intermediate values of the discount price, whether to allow the first-phase purchases is determined by the aforementioned trade-off between margin difference and cost saving, and thus depends on the initial on-hand inventory x. Interestingly, when market size is deterministic (which implies $\tau=\alpha M=\mathbb{E}(\alpha M)$), the two special cases span the whole domain of p, fully characterizing the optimal solution: $z^*(x)=x$ if $p_0\geqslant r-c$ and $z^*(x)=0$ otherwise. This coincides with our discussion in §3.2.

4.2. Infinite Horizon Problem

We now proceed to examine the firm's dynamic program in the infinite horizon setting. Compared to the single period setting, the problem is much more challenging because of the quasiconvexity of the single-period profit function and the complex system dynamics. We first characterize the profit-to-go function $v(x)$ and the objective function $g(y,z)$ in the following proposition.

PROPOSITION 3. (a) $v(x)$ *is increasing in* x.

(b) $g(y,z)$ *is increasing in* z *if* $z \geqslant y$, *and increasing in* y *if* $y \leqslant \tau$.

(c) *Both* $v(x)+\rho\lambda(x\vee\sigma)$ *and* $g(y,x)+\rho\lambda(x\vee\sigma)$ *are decreasing in* x *if* $y \geqslant \tau$.

Proposition 3(a) suggests that the firm's expected profit-to-go is increasing in the initial inventory level. This is intuitive because the firm can decide how much of these inventory to sell at discount and how much to return for disposal. Proposition 3(b) and 3(c) are useful in characterizing the firm's optimal policy. By Proposition 3(b), we immediately observe that $y^*(x) \geqslant \tau$ and either $z^*(x) < y^*(x)$ or $z^*(x)=x$. This narrows down the range of the optimal markdown quantity and ordering quantity. Furthermore, we prove the optimal markdown quantity in two special cases, as in the following theorem.

THEOREM 2. (a) $y^*(x) \geqslant \tau$ *and* $z^*(x) \notin [y^*(x), x)$; *and*

(b) $z^*(x)=x$, *if* $p_0 \geqslant r-c$ *and* $z^*(x)=0$, *if* $p_0 \leqslant [r\,\mathbb{E}(\tau\wedge\alpha M)-c\tau]/\mathbb{E}(\alpha M)$.

Theorems 2(a) and 1 imply that the target inventory level $y^*(x)$ is higher than or equal to that in the single-period setting, τ. It is because newly produced products not sold in a period can be sold in the next period. In addition, if the optimal amount of inventory to markdown $z^*(x)$ is larger than this target inventory level $y^*(x)$ such that the firm is not going to order anything for consumers coming in the clearance phase, the firm will simply put all leftover inventory in discount. Theorem 2(b) shows that the two special cases identified in Theorem 1 for the single-period problem can be extended to the dynamic problem. Particularly, independent of the leftover inventory level, if the (adjusted) margin from markdown sales dominates that from the regular sales ($p_0 \geqslant r-c$), then the firm should put all leftover inventory in discount. On the other hand, if the discounted margin is very low, the firm should not mark down any unit. The following theorem studies the case of intermediate markdown price.

THEOREM 3. *When* $[r\,\mathbb{E}(\tau\wedge\alpha M)-c\tau]/\mathbb{E}(\alpha M) \leqslant p_0 \leqslant r-c$, *we have the following statements.*

(a) *If* $v(x)+\mu(x)$ *is decreasing, and* M *follows a two-point distribution, then* $z^*(x)\in\{0,x\}$.

(b) *If* $z^*(x)\in\{0,x\}$, *then* $v(x)+\mu(x)$ *is decreasing in* x, *where* $\mu(x)=\rho\lambda(\sigma\vee x\wedge\alpha M_{\max})$.

PROOF. In preparation, we first show $\sigma\leqslant\tau$. Note that we can express $\sigma=x^*(-p_0)$ and $\tau=x^*(0)$ with $x^*(t)$ being the smallest maximizer of the function $f(x,t)=(r+t)\,\mathbb{E}(x\wedge\alpha M)-cx$ in terms of x. Since $\mathbb{E}(x\wedge\alpha M)$ is increasing in x, $f(x,t)$ is supermodular, implying that $x^*(-p_0)\leqslant x^*(0)$ by Theorem 2.8.2 in Topkis (1998).

Next we show that it leads no loss of optimality to assume $y\leqslant\alpha M_{\max}$ in problem (2), and that $v(x)=v(\alpha M_{\max})$ for any $x\geqslant\alpha M_{\max}$. In fact, given any $y>\alpha M_{\max}$, we can verify from (4) and (5) that $g(y,z)=\rho p_0\,\mathbb{E}[z\wedge(\alpha M)]+(r-c)\,\mathbb{E}\,D(\alpha M,z)+\mathbb{E}[\gamma v(\tilde{x})-c\tilde{x}]$, where $\tilde{x}=D(y,z)-D(\alpha M,z)$ is increasing in y. Notice that $\gamma v(x)-cx=[(\gamma-1)v(x)]+[v(x)+\rho\lambda(x\vee\sigma)]-[\rho\lambda(x\vee\sigma)+cx]$, where the first two bracketed terms on the right are decreasing in x by $\gamma\leqslant 1$ and Proposition 3(b), and

$$\rho\lambda(x\vee\sigma)+cx=\begin{cases}\rho\lambda(\sigma)+cx, & \text{if } x\leqslant\sigma;\\ \rho(r-p_0)\,\mathbb{E}[x\wedge(\alpha M)]+c(1-\rho)x, & \\ & \text{if } x\geqslant\sigma;\end{cases}$$

is increasing in x. Thus, $\gamma v(\tilde{x})-c\tilde{x}$ is decreasing in y, implying that without loss of optimality we can assume $y\leqslant\alpha M_{\max}$ in problem (2). For any $z\geqslant\alpha M_{\max}$, because $z\geqslant y$ by $y\leqslant\alpha M_{\max}$, we know from (4) and (5) that $g(y,z)=\beta\theta(y)+\rho p_0\,\mathbb{E}(\alpha M)+\gamma\,\mathbb{E}(\beta(y-\alpha M)^+)$ is independent of z, i.e., $g(y,z)=g(y,\alpha M_{\max})$. Thus, $v(x)=v(\alpha M_{\max})$ for any $x\geqslant\alpha M_{\max}$.

We are now ready to prove the theorem. By above discussions and Proposition 3(b), it suffices to restrict $\tau\leqslant y\leqslant\alpha M_{\max}$ and $0\leqslant z\leqslant x\leqslant\alpha M_{\max}$ in the following.

(a) Since M follows a two-point distribution, $\sigma,\tau\in\{\alpha M_{\min},\alpha M_{\max}\}$. If $\sigma=\alpha M_{\max}$, then $\tau=\alpha M_{\max}$ by $\sigma\leqslant\tau$, implying that $\mu(x)$ is a constant when $x\leqslant\alpha M_{\max}$. Together with Proposition 3(a) and the fact that $v(x)+\mu(x)$ is decreasing, it ensures that $v(x)$ is a constant, too. Thus, the myopic solution is optimal, i.e., $y^*(x)=\tau$ and $z^*(x)\in\{0,x\}$. When $\sigma=\alpha M_{\min}$, we shall prove the quasiconvexity of $g(y,z)$ in z as follows by distinguishing among the following three cases:

(i) If $z\leqslant\alpha M_{\min}$, then $z\leqslant y$ by $y\geqslant\tau\geqslant\alpha M_{\min}$. By (4) and (5), $\pi(y,z)$ depends on z via the term $\rho(p_0+c-r)z$ (which decreases in z) and $\tilde{x}$ is independent of z. Hence, $g(y,z)$ is decreasing in z.

(ii) If $z\in[\alpha M_{\min},y]$, then by (4) and the definition of $\mu(x)$,

$$g(y,z)-(\beta+\rho)\theta(y)=\gamma\,\mathbb{E}[v(\tilde{x})+\mu(\tilde{x})]-\mathbb{E}[\mu(z)+\gamma\mu(\tilde{x})],$$

where the first bracketed term on the right side is increasing in z by the fact that $v(x)+\mu(x)$ is decreasing in x and the monotonicity of $\tilde{x}$ in z. We next show $\mu(z)+\gamma\mu(\tilde{x})$ is decreasing in z, which immediately ensures that $g(y,z)$ is increasing z. In fact, since M follows a two-point distribution, one can be verified that $\mu(x)$ is linearly decreasing with, say, slope $-k\leqslant 0$ over $[\sigma,\alpha M_{\max}]$, and it is a constant for either $x\leqslant\sigma$ or $x\geqslant\alpha M_{\max}$. Thus, by $\sigma=\alpha M_{\min}\leqslant z\leqslant y\leqslant\alpha M_{\max}$ and (5),

$$\begin{aligned}&\mu(z)+\gamma\mu(\tilde{x})\\ &\quad=-kz+\gamma\mu\big\{(\beta+\rho)(y-\alpha M)^+-\rho(z-\alpha M)^+\big\},\end{aligned}$$

where the right side, regarded as a function of z, consists of linear pieces with slopes in $\{-k,-k+\gamma\rho k\}$. Since $0\leqslant\rho,\gamma\leqslant 1$, we have $\mu(z)+\gamma\mu(\tilde{x})$ decreasing in z, and thus $g(y,z)$ increases in z.

(iii) If $z\in[y,\alpha M_{\max}]$, then by (4) and (5), $\pi(y,z)$ depends on z via the term $\rho p_0\,\mathbb{E}[z\wedge(\alpha M)]$ (which increases in z) and $\tilde{x}$ is independent of z. Hence, $g(y,z)$ is increasing in z.

In summary, $g(y,z)$ is decreasing in z when $z \leqslant \alpha M_{\min}$ and then increasing in z when $z \geqslant \alpha M_{\min}$. Thus, $g(y,z)$ is quasiconvex in z, implying $z^*(x) \in \{0, x\}$.

(b) When $x \leqslant \sigma$, by Proposition 3(a, c) and $\lambda(x \vee \sigma) = \lambda(\sigma)$, we know that $v(x) + \mu(x)$ is a constant. When $\sigma \leqslant x \leqslant \alpha M_{\max}$, by $z^*(x) \in \{0, x\}$, $v(x) + \mu(x) = [u(0) + \rho\lambda(x)] \vee [u(x) + \rho\lambda(x)]$, where $u(z) = \max_y\{g(y,z)\colon \tau \leqslant y \leqslant \alpha M_{\max}\}$. Because $\lambda(x)$ is decreasing when $x \geqslant \sigma$, and $g(y,x) + \rho\lambda(x)$ is decreasing by Proposition 3(c), we conclude that $v(x) + \mu(x)$ is decreasing in x. $\square$

In Theorem 3 we proved that $z^*(x) \in \{0, x\}$ when the aggregate demand follows a two-point distribution. We now characterize how the optimal policy $z^*(x)$ depends on the state x.

PROPOSITION 4. *If $z^*(x) \in \{0, x\}$, then there exists a cutoff level s such that $z^*(x) = 0$ when $x < s$ and $z^*(x) = x$ when $x \geqslant s$, where $s \geqslant 0$ is decreasing in p.*

PROOF. From the definition of $\pi(y,z)$, $z = z^*(x)$ solves the problem

$$\max_z \big\{ p\,\mathbb{E}[(z \wedge (\alpha M)] + w(z)\colon z \in [0, x] \big\}, \tag{7}$$

where $w(z)$ given below is independent of p:

$$w(z) = \max_{y \geqslant 0} \big\{ r\,\mathbb{E}\,D(y \wedge \alpha M, z) - cD(y,z) + \gamma\,\mathbb{E}\,v(\tilde{x}) \big\}.$$

On the maximization problem (7), since its objective function is supermodular in (x, z, p), and the set $\{(x,z,p)\colon 0 \leqslant z \leqslant x\}$ forms a sublattice, its optimal solution $z^*(x)$ is increasing in x and p by in Topkis (1998, Theorem 2.8.2). Since $z^*(x) \in \{0, x\}$, by its monotonicity in x, the cutoff level exists and can be defined as $s = \sup\{x\colon z^*(x) = 0\}$. Furthermore, since $z^*(x)$ is increasing in p, we conclude from its definition that s is decreasing in p. $\square$

As shown in Proposition 4, there exists a cutoff level s for the on-hand inventory x, above which the firm offers discounted sales. Furthermore, as the discounted price p increases, the threshold s decreases, implying that the firm is more likely to sell in the first phase.

Theorem 3 and Proposition 4 prove the optimal markdown policy for the case when the aggregate demand follows a two-point distribution. There is significant challenge in extending the results to other aggregate demand distributions. As discussed before, the profit-to-go of the firm is not concave or convex but quasiconvex in the markdown quantity z. In addition, the demand interdependence across two phases in each period creates very complicated dynamics for inventory carried over between two consecutive periods. To the best of our knowledge, there does not exist any prior study that shows the preservation of quasiconvex functions in dynamic maximization problems. Similar to Harrison et al. (2012), our motivation for considering the simple two-point demand distribution is to gain insights into the basic issues involved by studying a setting that is analytically tractable. The analysis performed by Naddor (1978) suggests that optimal decisions of many inventory systems are sensitive to the mean and standard deviation but not the precise form of demand, and two-point distribution is usually the first to be considered in operations problems with demand uncertainty (see, for e.g., Chou et al. 2010, 2011). In §4.3, in addition to conducting numerical studies to get a better understanding of the optimal policy, we numerically test the robustness of our results when the aggregate demand follows multipoint distributions. Since all demand distributions can be approximated using a multipoint demand distribution, optimality of the bang-bang strategy in the multipoint demand distribution can provide confidence that the same policy is optimal for other demand distributions.

4.3. Numerical Illustration

To gain further insights on the optimal policy, we perform an extensive numerical study. We normalize the regular price r to unity and assume the discount factor $\gamma = 0.9$. In the majority of the study we focus on a two-point distribution of the aggregate demand M, given by $\Pr\{M = 1-\kappa\} = 1-q$ and $\Pr\{M = 1+\kappa\} = q$, where $q = 0.5$. In addition to the two-point distribution, we consider multipoint distributions in the last part of the study (elaborated below). We vary the following parameters: discounted price $p \in \{0.2, 0.4, 0.6, 0.8\}$, marginal cost $c \in \{0.2, 0.4\}$, demand uncertainty $\kappa \in \{0.1, 0.2, \ldots, 1\}$, first-phase demand portion $\alpha \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$, and proportion of first-phase buyers who can afford the regular price $\rho \in \{0.1, 0.2, 0.3, \ldots, 0.9, 1\}$, resulting in a total of 4,000 instances. For tractability, when solving the dynamic program, we approximate the continuous state space $x \in \mathbb{R}^+$ with a discrete one. Specifically, recall that the profit-to-go function $v(x)$ is a constant when $x \geqslant \alpha M_{\max}$ (see the proof of Theorem 3), it suffices to keep track of its value for $x \in [0, \alpha M_{\max}]$ only. Thus, we consider the discretized state space $\mathscr{X} = \{(i/N)\alpha M_{\max}\colon 0 \leqslant i \leqslant N\}$ for $N = 200$. In addition, $v(x)$ satisfying the Bellman equation (2) is approximated by $v_n(x)$ defined in (6), where the iteration stops when $\max_{x \in \mathscr{X}} |v_{n+1}(x) - v_n(x)| < 0.001$.

Effects of demand uncertainty κ. We calculate the cutoff level s for all 4,000 instances. These instances can be divided into 400 groups such that in each group κ varies from 0.1 to 1, while the other parameters are fixed. We observe that s decreases in κ for all groups. The left panel of Figure 3 shows a typical instance. In fact, we can prove this result for the single-period problem when market-size follows a two-point distribution (see Proposition 5 in Appendix B). This result suggests that markdown is more likely to be optimal as demand uncertainty increases. As discussed earlier, one benefit of markdown is that it reduces the overage risk encountered by the firm. This benefit is magnified as demand uncertainty increases, making markdown more preferable.

Effects of intertemporal substitution ρ. We divide the 4,000 instances into 400 groups such that in each group ρ varies from 0.1 to 1, while the other parameters are fixed. Interestingly, we find that in all 400 groups, the cutoff levels s are all increasing in ρ. The right panel of Figure 3 illustrates a typical pattern of the cutoff s in ρ. For the single-period problem, we can formally prove this result with any market-size distribution (see Proposition 5 in Appendix B). The intuition behind this is that, as ρ increases, the firm is less likely to offer the discounted sales. Recall that ρ captures magnitude of the intertemporal substitution, or the cannibalization effect of markdown pricing on future regular sales. As one may expect, the higher the ρ, the larger the impact of intertemporal substitution, and thus the more cautious the firm is toward markdown sales.

Coordination of inventory and markdown decisions. Figure 4 illustrates how the markdown quantity $z^*(x)$ and ordering quantity $y^{0*}(x)$ changes with the amount of inventory x. When x is small, the firm does not make any units available during markdown and hence the firm orders a fixed quantity independent of x. When x is large, the firm makes all inventory available during markdown. Hence, the demand in the regular-sales phase decreases in x and so does the ordering quantity. The most interesting observation is that there is a sudden drop in ordering quantity when inventory level increases. Such a sudden drop does not exist in a dynamic inventory model without a fixed ordering cost. This sudden drop exists here because the markdown quantity follows a bang-bang structure and hence there is a sudden drop in the demand in the regular-sales phase when inventory reaches the cutoff level s. As shown in the figure, the sudden drop occurs at a smaller x when κ increases, as a higher market uncertainty makes markdown sales more profitable.

Static vs. dynamic policy. In practice some firms may stick to a certain markdown policy and do not change it over time. Although some stores may never mark down leftover products to avoid complexity in management, some stores may mark down all leftover products for marketing purposes. To evaluate the benefit that the firm can enjoy by dynamically adjusting its markdown and inventory policy, we consider two static policies where $z^*(x)$ is fixed at either x or 0 over the entire planning horizon. Specifically, define the following two static policies:

(H_0): Fix $z^*(x) = 0$ for all x. Denote the corresponding profit-to-go as $v^0(x)$.

(H_x): Fix $z^*(x) = x$ for all x. Denote the corresponding profit-to-go as $v^x(x)$.

In addition, let $v(x)$ be the profit-to-go corresponding to the optimal dynamic policy. Define the firm's percentage loss in expected profit because of the static policy (in comparison to that under the optimal dynamic policy) by

$$\mathrm{LoE}_0 = \frac{1}{N}\sum_{y\in\mathscr{X}} \frac{v(y)-v^0(y)}{v(y)} \times 100\%,$$

$$\mathrm{LoE}_x = \frac{1}{N}\sum_{y\in\mathscr{X}} \frac{v(y)-v^x(y)}{v(y)} \times 100\%,$$

where LoE means *loss of efficiency*. We compute the LoE's for all possible values of the initial state.

Our numerical results indicate that the firm incurs substantial loss by adopting the static policies. Specifically, the average LoE across the 4,000 instances is 1.7% for the H_x policy and 12.4% for the H_0 policy. In the worst-case scenario, LoE of the H_x (H_0) policy can be as high as 27.0% (77.9%).

Our results also suggest that the cost of ignoring intertemporal substitution or strategic customer behavior can be very significant. As we noted earlier in §3.1, H_x would be the firm's optimal dynamic policy if there was no intertemporal substitution (i.e., $\rho = 0$). In other words, H_x is also the policy that the firm would apply when it completely ignored the substitution effect. As reported in the previous paragraph, the LoE from the H_x policy is quite significant.

Furthermore, as illustrated in the left panel of Figure 5, when the level of uncertainty κ increases, the performance of H_0 is worsened (i.e., LoE_0 increases). This is consistent with the aforementioned intuition that demand uncertainty

Figure 3. Cutoff level s as a function of demand uncertainty κ (left) for $(p, c, \alpha, \rho) = (0.4, 0.4, 0.9, 0.7)$, and intertemporal substitution level ρ (right) for $(p, c, \alpha, \kappa) = (0.2, 0.4, 0.1, 0.9)$.

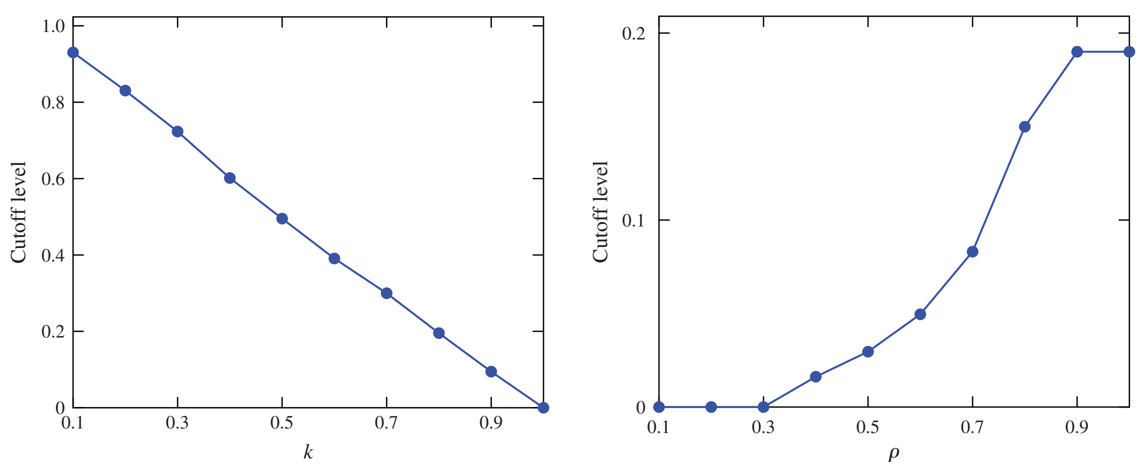


Figure 4. The optimal policy $z^*(x)$ and $y^{0*}(x)$ for $(p, c, \alpha, \rho) = (0.2, 0.2, 0.1, 1.0)$, where $\kappa = 0.2$ and $s = 0.091$ for the left figures, and $\kappa = 0.8$ and $s = 0.022$ for the right figures.

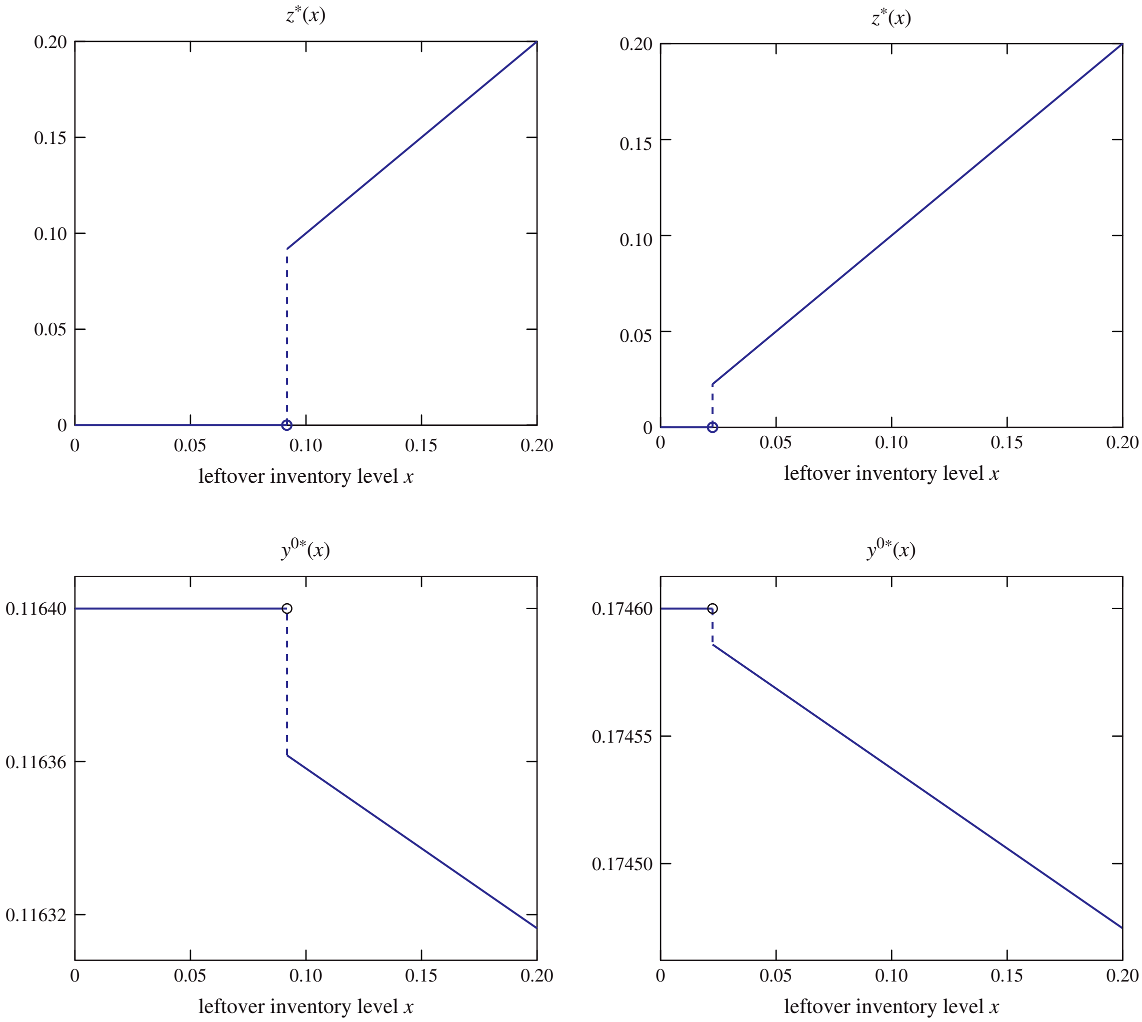
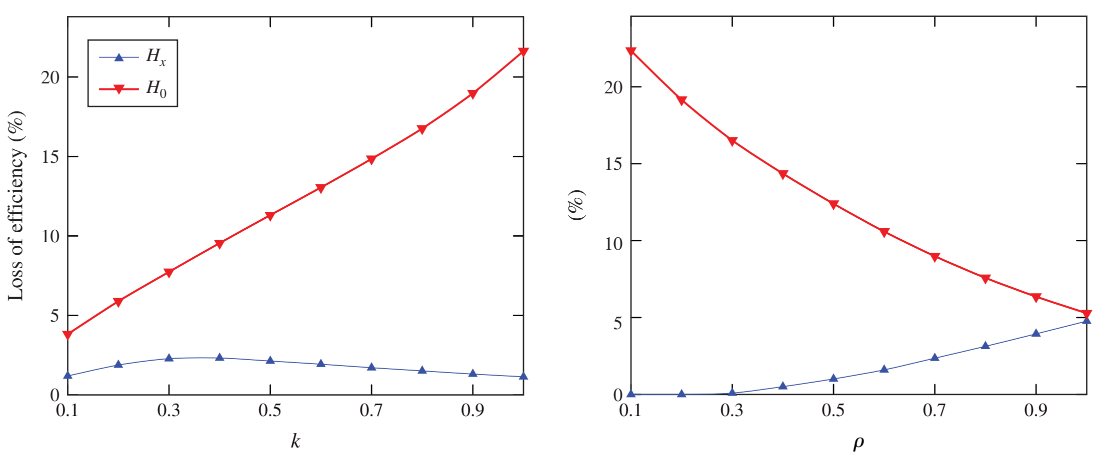


Figure 5. Average LoE_0 and LoE_x as functions of market-size uncertainty κ (left) and intertemporal substitution level ρ (right).



enhances the benefit of markdown. However, the performance of H_x is not always improved when κ increases. In particular, LoE_x first increases and then decreases in κ, meaning that uncertainty first worsens and then enhances the performance of H_x. The reason behind this is that while a larger uncertainty improves the profit advantage of markdown, all static policies suffer a deterioration of performance because of their nonresponsiveness to revealed market-size information.

While the former impact dominates when κ is large, the latter effect dominates when κ is small.

Meanwhile, as shown in the right panel of Figure 5, when the substitution level ρ increases, the performance of H_0 is improved (i.e., LoE_0 decreases), whereas that of H_x deteriorates (i.e., LoE_x increases). This is consistent with the aforementioned result that the firm is more inclined to refrain from markdown sales with a higher ρ.

Multipoint market-size distribution. Because of the analytical challenge arising from the quasiconvex function, it is very difficult to extend the multiperiod proof to incorporate more general market-size distributions. For multipoint distributions, we numerically test the 4,000 parameter combinations stated above when the aggregate demand M in each period follows the $(m+1)$-point distributions for $m = 4, 6, 8, 10$ with $\Pr\{M = 1-\kappa+2\kappa i/m\} = C_m^i q^i (1-q)^{m-i}$, $i = 0, \ldots, m$. We find that the optimal markdown strategy in each of the 4,000 instances, for each point distribution m, is of the bang-bang structure. This observation increases our confidence that the main results and insights shall remain valid for general market distributions.

5. Extensions

As alluded to in §3.1, so far we have focused on the aggregate effects of consumers' purchasing behavior on the intertemporal demand substitution. This allows us to focus on the firm's main trade-off between product spoilage and intertemporal demand substitution. In this section we extend the base model by incorporating a micro model for consumer choices and by incorporating endogenous markdown pricing and consumers' shopping costs. We show that our main result of the bang-bang solution is substantiated in all these extensions.

5.1. Consumer Choice Model

We start from describing the market and the consumers' preferences. Denote the size of the market (i.e., the total number of buyers if the product is priced at zero) in period n by M_n^0, and assume that M_n^0 in each period is random, stationary, bounded, and independent of each other. To describe the customer model, we differentiate aggregate demand M_n from market size M_n^0, where the market size M_n^0 represents all customers in the market and the aggregate demand M_n corresponds to customers who will buy in either phase. Thus, the aggregate demand M_n is a function of the firm's pricing decision, whereas the market size M_n^0 is not. Consumers have heterogeneous valuations v uniformly distributed between zero and one for the fresh product. They decide whether to purchase in the first phase or second phase, even though they only consume in the second phase. If they buy in the first phase, the valuation of the product will be discounted at a discount factor δ, since the product will be less fresh when they consume it. Nevertheless, the price in the first phase, p, is lower than the regular price r. Since there is only limited supply z_n in the first phase, not all consumers who want to buy the product can obtain it. Under the random allocation rule, denoted by $\lambda_1(z_n, p)$ the probability of a consumer obtaining the product in the first phase. Consumers who are rationed out in the first phase will join those who decide to purchase in the second phase. Demand in the second phase may also be higher or lower than the supply in that phase (i.e., the amount of new product ordered y_n^0). In the following two subsections, we present two scenarios on what happens with consumers who are rationed out in the second phase, and in both scenarios we show that the firm's problem reduces to our original model with the aggregate market demand.

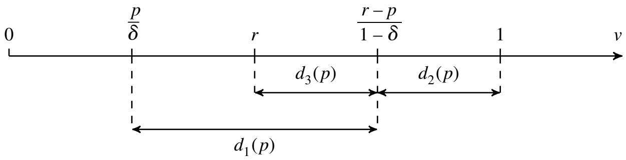
5.1.1. The Case with Outside Purchasing Option in the Second Phase. We first consider the case that consumers have outside sources of supply (e.g., a competing firm) from which they can buy the fresh product at price r if the firm stocks out in the second phase. Such outside purchasing options are not uncommon in the industries which motivate our study. For example, the bakery and grocery industries are highly competitive such that alternative sources of supply are readily available for consumers if their preferred seller runs out of stock at time of consumption. Consumers decide whether and when to purchase by evaluating the expected utilities from buying in either phase. For a consumer with valuation v, the expected utility from buying in the first phase is $\lambda_1(z_n, p)(\delta v - p) + [1-\lambda_1(z_n, p)](v-r)^+$, whereas that from buying in the second phase is $(v-r)^+$. Hence, the consumer attempts to buy in the first phase if and only if $\lambda_1(z_n, p)(\delta v - p) + [1-\lambda_1(z_n, p)](v-r)^+ \geqslant (v-r)^+$, or equivalently, $\delta v - p \geqslant (v-r)^+$, i.e.,

$$\frac{p}{\delta} \leqslant v \leqslant \frac{r-p}{1-\delta}. \tag{8}$$

On the other hand, the consumers with valuation $v > (r-p)/(1-\delta)$ skip the first phase and directly go to the second phase, whereas the consumers with valuation $v < p/\delta$ do not buy in either phase.[9] Note that by (8), it suffices to only consider the case where $p < \delta r$, since otherwise $p/\delta \geqslant (r-p)/(1-\delta)$, implying that the discount price is too high and no one will buy in the first phase.

The consumers' purchasing choices naturally lead to market segmentation in each period, which depends on the markdown price p. Specifically, for $p < \delta r$, define $d_1(p) = ((r-p)/(1-\delta)) \wedge 1 - p/\delta$ as the portion of consumers who attempt to make purchases in the first phase, $d_2(p) = 1 - ((r-p)/(1-\delta)) \wedge 1$ as the portion of consumers who have high valuation that they disdain to buy in the first phase and only buy the fresh product in the second phase, and $d_3(p) = ((r-p)/(1-\delta)) \wedge 1 - r$ as the portion of consumers who have a valuation higher than r but they are attracted by the promotion and try to make purchases in the first phase. If rationed in the first phase, this group of consumers will attempt to make purchase in the second phase. Observe that $d_1(p) + d_2(p) = 1 - p/\delta$, $d_2(p) + d_3(p) = 1 - r$, and $d_i(p) \geqslant 0$, $i = 1, 2, 3$. The market segments are illustrated in Figure 6.

Figure 6. Market segmentation when $\frac{r-p}{1-\delta} \leqslant 1$.



Mapping the micro model to the framework in our base model, we have $M_n(p) = M_n^0(1-p/\delta)$, $\alpha(p) = d_1(p)/(1-p/\delta)$ and $\rho(p) = d_3(p)/d_1(p)$, where the aggregate demand $M_n(p)$ equals to the total market size M_n^0 multiplied by the proportion of consumers who can afford at least the discount price (i.e., $v \geqslant p/\delta$); the aggregate demand $M_n(p)$ consists of two groups of consumers, $\alpha(p)$ portion for those who attempt to buy in the first phase, and $1-\alpha(p)$ portion for those only making purchases in the second phase. The fraction, $\alpha(p)$, equals to the first-phase demand $d_1(p)M_n^0$ divided by the aggregate demand. Furthermore, among the first-phase buyers, the proportion of those who can afford the full price, $\rho(p)$, is given by the corresponding segment size $d_3(p)M_n^0$ divided by the first-phase demand. After substituting the demand functions $M_n(p)$, $\alpha(p)$, and $\rho(p)$ into the dynamic program (1), the firm's problem under the micro model reduces to our original model with the aggregate market demand.

5.1.2. The Case Without Outside Purchasing Option. Thus far we have assumed in the micro model that outside options exist to ensure consumers of product availability in the second phase. If such outside options are absent, consumers' purchasing decisions need to factor in not only the fill rate in the first phase λ_1, but also the fill rate in the second phase λ_2. The lemma below characterizes the consumers' purchasing behavior for given fill rates and prices.

Lemma 1. *For given λ_1, λ_2, p, and r, the consumers' purchasing behavior is as follows.*

(a) *Suppose $\lambda_2 \leqslant \delta$. If $p \leqslant \delta r$, consumers with $v \geqslant p/\delta$ buy in the first phase and those with $v < p/\delta$ do not buy in either phase, i.e., $\alpha = 1$, $\rho = (1-r)/(1-p/\delta)$. If $p > \delta r$, consumers with $v \geqslant (p-\lambda_2 r)/(\delta-\lambda_2)$ buy in the first phase, those with $v \in [r, (p-\lambda_2 r)/(\delta-\lambda_2))$ buy in the second phase, and those with $v < r$ do not buy in either phase, i.e., $\alpha = (1/(1-r))(1-(p-\lambda_2 r)/(\delta-\lambda_2))^+$, $\rho = 1$.*

(b) *Suppose $\lambda_2 > \delta$. If $p \leqslant \delta r$, consumers with $v \in [p/\delta, (\lambda_2 r - p)/(\lambda_2-\delta)]$ buy in the first phase, those with $v > (\lambda_2 r-p)/(\lambda_2-\delta)$ buy in the second phase and those with $v < p/\delta$ do not buy in either phase, i.e., $\alpha = (1/(1-p/\delta))(1 \wedge ((\lambda_2 r-p)/(\lambda_2-\delta)) - p/\delta)$, $\rho = (1/(1 \wedge ((\lambda_2 r-p)/(\lambda_2-\delta)) - p/\delta))(1 \wedge ((\lambda_2 r-p)/(\lambda_2-\delta)) - r)$. If $p > \delta r$, consumers with $v \geqslant r$ buy in the second phase and those with $v < r$ do not buy in either phase, i.e., $\alpha = 0$, ρ is arbitrary.*

In both cases above, if $\alpha = 0$, then ρ is arbitrary.

Lemma 1 implies that the first-phase fill rate λ_1 does not play a role in consumer choices, whereas the second-phase fill rate λ_2 does. Denote the fractions, α and ρ, characterized in Lemma 1 by $\alpha^*(\lambda_2)$ and $\rho^*(\lambda_2)$, respectively. Note that the second-phase fill rate λ_2 is determined by the second-phase demand $(1-\alpha)M + \rho(\alpha M - z)^+$ and the inventory y^0, both of which depend on the initial inventory x. Nevertheless, in practice consumers can hardly observe or access information about markdown ration z, inventory y^0, or initial inventory x, as these are the firm's internal operating data. Similar to Su and Zhang (2008) and Su (2010), we adopt a rational expectation framework in the consumer model. The details of the framework are described in Appendix C1. Using this framework, we decouple this setting into a dynamic program for the firm and a purchasing problem for each consumer. In this case, the firm's problem also reduces to our original model with the aggregate market demand.

5.2. Endogenous Pricing Decision

So far we have assumed that the markdown price is exogenously given. This assumption is valid for chain bakery stores and grocery stores (including the one that motivates our study) because changing prices every day can lead to consumer confusion and management complexity. Since we show that the bang-bang strategy is optimal for any given discount price, this strategy remains optimal even if prices are determined at the beginning and remain unchanged for the whole planning horizon. On the other hand, there may be cases where the firm can change the markdown price every period when each period lasts for a longer duration (as opposed to only one day for bakery stores and grocery stores). We now consider the case when the firm's markdown pricing decision is endogenous. In particular, the discount price for the first phase is decided by the firm at the beginning of in a typical period and it is denoted as p. We focus on the case in which consumers have an outside option if the firm is out of stock in the second phase. With the micro model and the additional pricing decision, the Bellman equation (2) becomes

$$v(x) = \max_{y,z,p}\{\pi(y,z,p) + \gamma\, \mathbb{E}\, v(\tilde{x}):\; y \geqslant 0, z \in [0,x],\; p \in \mathscr{P}\}, \tag{9}$$

where $\tilde{x} = [D(y,z,p) - D(M^0,z,p)]^+$ with $D(y,z,p) = d_2(p)y + \{d_3(p)y - [d_3(p)/d_1(p)]z\}^+$, $\pi(y,z,p) = p\,\mathbb{E}\{z \wedge [d_1(p)M^0]\} - cD(y,z,p) + r\,\mathbb{E}\, D(y \wedge M^0, z, p)$, and $\mathscr{P}$ is the feasible set for markdown price. Let $[y^*(x), z^*(x), p^*(x)]$ be the optimal solution to problem (9).

Our intuition is that, since pricing and rationing are strategic substitutes (see, e.g., Maglaras and Meissner 2006), rationing is even less likely to happen. To confirm our intuition, we numerically verify that bang-bang solution for the markdown rationing is sustained in this extended model. Specifically, we fix the discount factor $\gamma = 0.9$ and focus on a two-point distribution of the market size M^0, given by $\Pr\{M^0 = 1-\kappa\} = \Pr\{M^0 = 1+\kappa\} = 0.5$. We examine a total of 2,250 problem instances by varying the following parameters: $\delta \in \{0.1, 0.2, \ldots, 0.9\}$, $\kappa \in \{0.1, 0.2, \ldots, 0.9, 1\}$, $r \in \{0.3, 0.4, \ldots, 0.7\}$, and $c \in \{0.05, 0.10, 0.15, 0.20, 0.25\}$. In searching for the optimal discount price p, we consider a discrete feasible region $\mathcal{P} = \{(i/10)\delta r\colon 1 \leqslant i \leqslant 9\}$. Also, similar to the setup in §4.3, the state space $\mathcal{X}$ is discretized into 200 levels, and value iterations for the dynamic programming stop once $\max_x |v_{n+1}(x) - v_n(x)| < 0.005$.

We confirm that the optimal markdown strategy in all of the 2,250 instances is of the bang-bang structure. Particularly, in each instance, the quasiconvexity of the profit function in z is preserved under the endogenous pricing, i.e., $\max_{y,\,p}\{\pi(y, z, p) + \gamma \mathbb{E}\, v(\tilde{x})\colon y \geqslant 0,\ p \in \mathcal{P}\}$ is quasiconvex in z. This observation suggests that the additional pricing decision does not alter the underlying trade-off in markdown. Furthermore, same as in the base model, the cutoff inventory level s decreases in the market-size uncertainty κ, implying that demand uncertainty amplifies the benefit of markdown sales. This pattern is observed in all the instances. We formally prove this result for the single-period problem with endogenous markdown pricing. Specifically, Proposition 8 in Appendix C2 confirms both the optimality of bang-bang policy and the monotonicity of the cutoff level in κ under some condition.

We do not consider endogenous pricing decision for the case when consumers do not have outside options in the second phase for two reasons. First, computing the equilibrium dynamically, whether analytically or numerically, is a technically challenging problem. Second, computing the dynamic equilibrium with endogenous pricing decision may not be very meaningful, since the pricing decision may have a signaling effect which will change the belief of customers regarding the fill rate and hence their purchasing behavior. This is different from the case in our original model with stationary prices where signaling is not an issue.

5.3. Effects of Consumers' Shopping Costs

In the consumer choice model presented in §5.1, we have assumed that consumers' purchases are effortless, i.e., they do not incur any cost for store visits in either phase. This corresponds to cases where the store is located at a very convenient location. In the local bakery chain example, the stores are usually located in or near metro stations where consumers pass by on the way to or after work. In such cases, consumers do not need to pay extra efforts visiting the stores. On the other hand, in the cases that the store is not in the vicinity of consumers, consumers may incur a fixed shopping cost (i.e., travel cost) for each visit to the store. We now incorporate such a shopping cost (denoted by ϕ) into the consumer choice model and evaluate its impact on consumers' strategic purchases and on the firm's markdown decision. We focus on the case when consumers do not have outside purchasing option and prices are exogenously given, as in §5.1.2.

Details of this extended consumer choice model are presented in Appendix C3. Similar to §5.1.2, we adopt a rational expectation framework where both the firm and the consumers make decisions based on conjectures about the other party's decisions and reach an equilibrium when the conjectures coincide with the actual actions. This framework allows us to decouple firm's dynamic program from the consumers' purchasing problem, and essentially reduces the firm's problem to our original model with aggregate market demand. This substantiates our main result of the bang-bang structure for markdowns.

We find that the fixed shopping cost plays an important role in determining consumers' strategic purchasing choices and hence the firm's optimal markdown strategy. Specifically, the shopping cost does not only increase consumers' cost of visiting the store during markdown, but also lowers their willingness to travel back to the regular sales if they are rationed out during markdown. Hence, it dis-incentivizes customers' strategic purchasing behavior and mitigates intertemporal demand substitution, which makes it more likely for the firm to sell under markdown. We formally prove that the cutoff level decreases in the shopping cost for the single-period problem under some conditions (see Proposition 11 in Appendix C3).

6. Conclusions

In this paper, we examine a firm's dynamic inventory and markdown policy for perishable products when consumers are strategic. Thus, sales at the discount price may cannibalize future demand at the full price. To the best of our knowledge, this is the first stochastic and dynamic inventory model facing strategic customer behavior with joint ordering and discounting decisions. The technical challenge is that the single-period profit is neither concave nor convex, but quasiconvex in the amount of leftover inventory to mark down. Despite the lack of existing results on the preservation of this kind of property in dynamic programming, we are able to characterize the firm's optimal policy with the use of a transformation technique. Our results suggest that, the firm should either put all leftover inventory on discount or dispose all leftover inventory without introducing any discount. In particular, the firm should introduce markdown if and only if the amount of leftover inventory is high. The firm is less likely to offer a discount if cannibalization is stronger. Furthermore, anticipating the cannibalization, the firm should reduce the quantity ordered for sales at the full price, whenever it offers a discount. This results in an interesting sudden drop in the optimal ordering quantity as the leftover inventory level increases. In addition, the loss of efficiency from adopting static policies in comparison to a dynamic policy can be very significant. Likewise, the loss in profit by ignoring intertemporal substitution can also be substantial. We show, in some cases through numerical studies, that our results and insights are robust in model extensions incorporating consumers' individual purchasing choices and endogenous markdown pricing.

A limitation in our model is that we assume that the lifetime of the product is only one period. Although this is true for bakery products and certain products in the grocery stores, there are other products that are perishable but yet have longer lifetimes. To extend our model to the products with lifetimes of more than one period, the dimension of state space has to be increased to specify the amounts of inventory that are going to expire at different points of time in the future. This definitely creates a technical challenge for the extension. Another issue is that, when product lifetime is longer than one period, the optimal policy depends on whether fresher products are sold first on a last-in-first-out basis or older products are sold first on a first-in-first-out basis (see Li et al. 2013 for a more detailed discussion). This in turn depends on the type of retail store and the format of inventory display (see, e.g., Yin et al. 2008 for a discussion of display format). Another possible extension is to incorporate a fixed operating cost for markdown sales. It! is straightforward to see that the bang-bang policy remains optimal with such a fixed cost in the single-period setting. Incorporating the fixed cost in the dynamic model, however, will introduce additional technical complexity to the dynamic program, and we leave it to future work.

This paper should only be taken as an initial attempt at studying dynamic inventory and markdown decisions with intertemporal substitution. As markdown becomes a popular strategy in the food retail industry to clear expiring products, and yet consumer purchasing behavior is increasingly sophisticated, the interaction between markdown and intertemporal demand presents fruitful directions for future research.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/opre.2015.1439.

Acknowledgments

The authors thank Chung-Piaw Teo, one anonymous associate editor, and two anonymous referees for their valuable comments. This research is partly supported by Hong Kong General Research Fund [Grants 11505114, 647312, and 16504014], and National Science Foundation of China [Grants 71201066 and 71131004].

Appendix A. Proofs

Proof of Proposition 2. With the two auxiliary functions $\theta(x)$ and $\lambda(x)$, the single-period expected profit given in (4) becomes

$$\pi(y,z)=\begin{cases}(\beta+\rho)\theta(y)-\rho\lambda(z), & \text{if } z\leqslant y,\\ \beta\theta(y)+\rho p_0\mathbb{E}[z\wedge(\alpha M)], & \text{if } z\geqslant y.\end{cases} \tag{10}$$

(a) Observe from (10) that $\pi(y,z)$ depends on y via either $(\beta+\rho)\theta(y)$ or $\beta\theta(y)$ depending on whether $z\leqslant y$ or not. Since τ maximizes the concave function $\theta(y)$, it implies that $\pi(y,z)$ is increasing in y when $y\leqslant\tau$ and decreasing in y when $y\geqslant\tau$.

(b) If $z\geqslant y$, then $\pi(y,z)$ depends on z via the term $\rho p_0\mathbb{E}[z\wedge(\alpha M)]$, which increases in z. If $z\leqslant y$, then $\pi(y,z)$ depends on z via the term $-\rho\lambda(z)=\rho[p_0-(r-c)]\mathbb{E}[z\wedge(\alpha M)]+\rho c\,\mathbb{E}(z-\alpha M)^+$. Since $-\lambda(z)$ is increasing if $p_0\geqslant r-c$ and it is convex otherwise, we conclude that $\pi(y,z)$ is increasing in z if $p_0\geqslant r-c$, and otherwise it is decreasing when $z\leqslant\sigma\wedge y$, and increasing when $z\geqslant\sigma\wedge y$.

Proof of Theorem 1. It is straightforward to see from Proposition 2 that $y^*(x)=\tau$ and $z^*(x)\in\{0,x\}$. In particular, if $p_0\geqslant r-c$, then $\pi(\tau,z)$ is increasing in z by Proposition 2(b), implying $\pi(\tau,x)\geqslant\pi(\tau,z)$ for all $0\leqslant z\leqslant x$, i.e., $z^*(x)=x$. Otherwise, $\pi(\tau,z)$ is quasiconvex in z. Observe from (10) that $\lim_{x\to\infty}\rho^{-1}[\pi(\tau,x)-\pi(\tau,0)]=p_0\mathbb{E}(\alpha M)-\theta(\tau)$. It implies that $\pi(\tau,x)\leqslant\pi(\tau,0)$ for all x if $p_0\leqslant[r\mathbb{E}(\tau\wedge\alpha M)-c\tau]/\mathbb{E}(\alpha M)$. Otherwise, there exists a cutoff level s such that $\pi(\tau,x)\leqslant\pi(\tau,0)$ if and only if $x<s$, i.e., $z^*(x)=0$ if $x<s$ and $z^*(x)=x$ if $x\geqslant s$.

Proof of Proposition 3. In preparation, we first derive an alternative expression for $\tilde{x}$. Observe from their definitions that $\tilde{x}=D(y\vee(\alpha M),z)-D(\alpha M,z)$ and $D(y,z)=\beta y+\rho(y\vee z)-\rho z$. Thus,

$$\tilde{x}=\beta(y-\alpha M)^+ +\begin{cases}\rho[(y-\alpha M)^+-(z-\alpha M)^+], & \text{if } z\leqslant y,\\ 0, & \text{if } z\geqslant y.\end{cases} \tag{11}$$

(a) The monotonicity of $v(x)$ is straightforward because in problem (2), the objective function $g(y,z)$ is independent of x, and the feasible set expands as x increases.

(b) When $z\geqslant y$, by (10) and (11), $g(y,z)=\beta\theta(y)+\rho p_0\mathbb{E}[z\wedge(\alpha M)]+\gamma\mathbb{E}\,v(\beta(y-\alpha M)^+)$, which increases in z. Furthermore, when $y\leqslant\tau$, since $\theta(y)$ is increasing in y, $g(y,z)$ is also increasing in y by monotonicity of $v(x)$ as proved in part (a).

(c) Reformulate $g(y,z)+\rho\lambda(z\vee\sigma)$ as $[\pi(y,z)+\rho\lambda(z\vee\sigma)]+\gamma\mathbb{E}\,v(\tilde{x})$, where $\mathbb{E}\,v(\tilde{x})$ is decreasing in z since $\tilde{x}$ is decreasing in z by (11) and $v(x)$ is increasing in x. We now show that the bracketed term is also decreasing in z.

(i) When $z\leqslant\sigma$, $z\leqslant y$ because $y\geqslant\tau\geqslant\sigma$, implying $\pi(y,z)+\rho\lambda(\sigma)=(\beta+\rho)\theta(y)-\rho\lambda(z)+\rho\lambda(\sigma)$ by (10). It is decreasing in z since $\lambda(z)$ is increasing when $z\leqslant\sigma$.

(ii) when $z\geqslant\sigma$, by (10) again, $\pi(y,z)+\rho\lambda(z)=\beta\theta(y)+\rho\theta(y\vee z)$. Because $(y\vee z)\geqslant y\geqslant\tau\geqslant\sigma$ with τ maximizing the concave function $\theta(x)$, we know that $\theta(y\vee z)$ is decreasing in z, implying that $\pi(y,z)+\rho\lambda(z)$ is also decreasing in z.

The above discussion ensures that $g(y,z)+\rho\lambda(z\vee\sigma)$ is decreasing in z. This implies that $u(z)+\rho\lambda(z\vee\sigma)$ also decreases in z, where $u(z):=\max_{y\geqslant\tau}g(y,z)$. Since $v(x)=\max_{0\leqslant z\leqslant x}u(z)$, the monotonicity of $v(x)+\rho\lambda(z\vee\sigma)$ immediately follows from Lemma 2 provided below. ☐

Lemma 2. *Given a real function $G(x)$ defined on $\Re^+$, suppose the function $F(x)=\max_{0\leqslant y\leqslant x}G(y)$ is well-defined for any $x\in\Re^+$. If there exists an increasing and convex function $H(x)$ such that $G(x)-H(x)$ is decreasing on $\Re^+$, then $F(x)-H(x)$ is also decreasing on $\Re^+$.*

Proof. We shall prove $F(x+\delta)\leqslant F(x)+H(x+\delta)-H(x)$ for any x, $\delta\geqslant 0$.

1. When $x=0$, since $G(y)-H(y)\leqslant G(0)-H(0)$ for any $y\geqslant 0$ as assumed, $F(\delta)=\max_{0\leqslant y\leqslant\delta}G(y)\leqslant\max_{0\leqslant y\leqslant\delta}[G(0)+H(y)-H(0)]$. Because $G(0)=F(0)$ by the definition of $F(x)$, and $H(y)$ is increasing, $F(\delta)\leqslant F(0)+\max_{0\leqslant y\leqslant\delta}[H(y)-H(0)]=F(0)-H(0)+H(\delta)$. Therefore $F(x+\delta)\leqslant F(x)+H(x+\delta)-H(x)$ at $x=0$.

2. When $x\geqslant\delta$, by the definition of $F(x)$, $F(x+\delta)=F(x)\vee\max_{x\leqslant y\leqslant x+\delta}G(y)=F(x)\vee\max_{x-\delta\leqslant y\leqslant x}G(y+\delta)$. Because $x-\delta\geqslant 0$ and $G(y+\delta)\leqslant G(y)-H(y)+H(y+\delta)$, by the above inequality, $F(x+\delta)\leqslant F(x)\vee\max_{0\leqslant y\leqslant x}[G(y)+H(y+\delta)-H(y)]$. By the convexity of $H(x)$, $H(y+\delta)-H(y)\leqslant H(x+\delta)-H(x)$ for any

$y \leqslant x$. It leads to $F(x+\delta) \leqslant F(x) \vee \max_{0\leqslant y\leqslant x}[G(y)+H(x+\delta)-H(x)] = F(x) \vee [F(x)+H(x+\delta)-H(x)]$. Since $H(x) \leqslant H(x+\delta)$, we have $F(x+\delta) \leqslant F(x)+H(x+\delta)-H(x)$ in this case, too.

3. When $0 < x < \delta$, there exists some integer n such that $nx \geqslant \delta$. If we let $\delta_m = (m/n)\delta$ for any $m \geqslant 0$, then clearly $x+\delta_m \geqslant \delta_1$. Therefore by what we proved in the previous step, $F(x+\delta_{m+1}) = F(x+\delta_m+\delta_1) \leqslant F(x+\delta_m)+H(x+\delta_m+\delta_1)-H(x+\delta_m)$. By the definition of δ_m, the above inequality implies that $\sum_{m=0}^{n-1}[F(x+\delta_{m+1})-F(x+\delta_m)] \leqslant \sum_{m=0}^{n-1}[H(x+\delta_{m+1})-H(x+\delta_m)]$. It together with $\delta_0 = 0$ and $\delta_n = \delta$ ensures that $F(x+\delta)-F(x) \leqslant H(x+\delta)-H(x)$. □

Proof of Theorem 2. Part (a) immediately follows from Proposition 3(b). We only need to focus on part (b).

When $p_0 \geqslant r-c$, $\lambda(x)$ is decreasing and hence its maximizer $\sigma = 0$. If $z \geqslant y$, then $g(y,z)$ depends on z via the term $\rho p_0 \mathbb{E}[z \wedge (\alpha M)]$ by (10) and (11). Thus, $g(y,z)$ is increasing in z. If $z \leqslant y$, then it depends on z via the term $-\rho[\lambda(z)+\gamma\lambda(\tilde{x})]+\gamma\mathbb{E}[v(\tilde{x})+\rho\lambda(\tilde{x})]$, where the leftover $\tilde{x}$ is decreasing in z with a rate no more than $\rho \leqslant 1$ by its expression given in (11). Since $\lambda(x)$ is decreasing in x, we know that $\lambda(z)+\gamma\lambda(\tilde{x})$ is decreasing in z. In addition, by $\sigma = 0$ and Proposition 3(c), $v(\tilde{x})+\rho\lambda(\tilde{x})$ is increasing in z. In summary, $g(y,z)$ is increasing in z, implying $z^*(x) = x$ in this case.

When $p_0 \mathbb{E} M \leqslant \theta(\tau)$, then by proof of Theorem 1, $\pi(x,\tau) \leqslant \pi(0,\tau)$ for all x. That is, $v_n(x)$ given by (6) is constant at $n = 1$. It then can be inductively proved that $v_n(x)$ is constant at each $n = 2, 3, \ldots$, implying that the limit $v(x)$ is also constant. Hence, $z^*(x) = 0$ for all x.

Proof of Lemma 1. A consumer with valuation v buys in the first phase if and only if $\lambda_1(\delta v-p)+(1-\lambda_1)\lambda_2(v-r)^+ \geqslant \lambda_2(v-r)^+$, or equivalently, $\delta v-p \geqslant \lambda_2(v-r)^+$. On the other hand, a consumer buys in the second phase if and only if $\lambda_2(v-r) \geqslant (\delta v-p)^+$, and buys in neither phase if $(\delta v-p) \vee [\lambda_2(v-r)] < 0$. To find the market segments, consider the following two cases:

1. If $\lambda_2 \leqslant \delta$, first-phase buyers have valuations $v \geqslant p/\delta \vee (p-\lambda_2 r)/(\delta-\lambda_2)$, and second-phase buyers have valuations v satisfying $v \geqslant r$ and $v \leqslant (p-\lambda_2 r)/(\delta-\lambda_2)$. Note that, if and only if $p \leqslant \delta r$, $p/\delta \geqslant (p-\lambda_2 r)/(\delta-\lambda_2)$ and $r \geqslant (p-\lambda_2 r)/(\delta-\lambda_2)$. Thus, we have two subcases. If $p \leqslant \delta r$, consumers with $v \geqslant p/\delta$ buy in the first phase, and those with $v < p/\delta$ do not buy in either phase, i.e., $\alpha = 1$ and $\rho = (1-r)/(1-p/\delta)$. If $p > \delta r$, consumers with $v \geqslant (p-\lambda_2 r)/(\delta-\lambda_2)$ buy in the first phase, those with $r \leqslant v < (p-\lambda_2 r)/(\delta-\lambda_2)$ buy in the second phase, and those with $v < r$ do not buy in either phase. In this case, $\alpha = (1/(1-r))(1-(p-\lambda_2 r)/(\delta-\lambda_2))^+$ and $\rho = 1$.

2. If $\lambda_2 > \delta$, first-phase buyers have valuations v satisfying $v \geqslant p/\delta$ and $v \leqslant (\lambda_2 r-p)/(\lambda_2-\delta)$, and second-phase buyers have valuations $v \geqslant r \vee (\lambda_2 r-p)/(\lambda_2-\delta)$. Meanwhile, if and only if $p \leqslant \delta r$, $p/\delta \leqslant (\lambda_2 r-p)/(\lambda_2-\delta)$ and $r \leqslant (\lambda_2 r-p)/(\lambda_2-\delta)$. Thus, we have two subcases. If $p \leqslant \delta r$, consumers with $v \in [p/\delta, (\lambda_2 r-p)/(\lambda_2-\delta)]$ buy in the first phase, those with $v > (\lambda_2 r-p)/(\lambda_2-\delta)$ buy in the second phase, and those with $v < p/\delta$ do not buy in either phase. In this case, $\alpha = [((\lambda_2 r-p)/(\lambda_2-\delta)) \wedge 1-p/\delta]/(1-p/\delta)$ and $\rho = [((\lambda_2 r-p)/(\lambda_2-\delta)) \wedge 1-r]/[((\lambda_2 r-p)/(\lambda_2-\delta)) \wedge 1-p/\delta]$. If $p > \delta r$, consumers with $v \geqslant r$ buy in the second phase, and those with $v < r$ do not buy in either phase, i.e., $\alpha = 0$ and ρ is arbitrary. □

Endnotes

1. See, e.g., "Coles Bakery Tip" discussion on https://www.ozbargain.com.au/node/41239, and http://forums.moneysavingexpert.com/showthread.php?t=419108.
2. See, e.g., "End of the day markdowns on grocery items" on http://avivahwerner.com/2013/02/14/end-of-the-day-markdowns-on-grocery-items.
3. See, e.g., http://www.yelp.com/biz/marukai-west-covina, http://www.yelp.com/biz/marukai-market-costa-mesa, http://www.yelp.com/biz/suruki-supermarket-san-mateo, http://www.yelp.ca/biz/galleria-supermarket-vaughan.
4. See, e.g., http://thishungrykitten.com/2013/04/22/st-lawrence-market-this-weekends-seafood-haul-and-resulting-feast/, http://www.yelp.com/biz/monterey-fish-company-inc-monterey-2.
5. Every day 12,800 kilograms of bread and other bakery products are discarded by Viennese bakeries because they remain unsold at the end of the day. Ref: *Sustainable Food Production and Ethics*, 2007, European Society for Agricultural and Food Ethics.
6. http://www.webmd.com/a-to-z-guides/features/do-food-expiration-dates-matter, http://www.eatbydate.com/sell-by-date-definition/.
7. http://www.tcpalm.com/news/what-happens-to-markets-unsold-food.
8. We label the time periods such that every period starts with the clearance phase. An equivalent alternative is to model the problem with the regular sales preceding the clearance. Our approach simplifies the formulation and is commonly used in literature, see e.g., Huggins and Olsen (2010), Li and Yu (2014).
9. It is worth noting that the consumers' purchasing choices do not depend on the first-phase fill rate λ_1, similar to the result in advance-selling literature (Xie and Shugan 2001, Yu et al. 2014). This is because consumers who fail to obtain a unit in the first phase have an option of buying in the second phase.

References

Adida E, Perakis G (2010) Dynamic pricing and inventory control: Uncertainty and competition. *Oper. Res.* 58(2):289–302.
Aviv Y, Vulcano G (2012) Dynamic list pricing. Ozer O, Phillips R, eds. *The Oxford Handbook of Pricing Management* (Oxford University Press, Oxford, UK), 522–584.
Barnhart C, Talluri K (1997) Airline operations research. ReVelle C, McGarrity A, eds. *Civil and Environ. Engrg. Systems* (John Wiley and Sons, New York), 435–469.
Baron O, Hu M, Najafi-Asadolahi S, Qian Q (2015) Newsvendor selling to loss-averse consumers with stochastic reference points. *Manufacturing Service Oper. Management* 17(4):456–469.
Besanko D, Winston W (1990) Optimal price skimming by a monopolist facing rational consumers. *Management Sci.* 36(5):555–567.
Chaneton J, Vulcano G (2011) Computing bid prices for revenue management under customer choice behavior. *Manufacturing Service Oper. Management* 13(4):452–470.
Chen X, Simchi-Levi D (2004) Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Oper. Res.* 52(6):887–896.
Chen X, Hu P, Shum S, Zhang Y (2015) Dynamic stochastic inventory management with reference price effects. Working paper, University of Illinois at Urbana–Champaign.
Chou M, Chua G, Teo C, Zheng H (2010) Design for process flexibility: Efficiency of the long chain and sparse structure. *Oper. Res.* 58(1):43–58.
Chou M, Chua G, Teo C, Zheng H (2011) Process flexibility revisited: The graph expander and its applications. *Oper. Res.* 59(5):1090–1105.
Federgruen A, Heching A (1999) Combined pricing and inventory control under uncertainty. *Oper. Res.* 47(3):454–475.

Gallego G, Iyengar G, Phillips R (2004) Managing flexible products on a network. Technical Report CORC TR=2004-01, Department of Industrial Engineering and Operations Research, Columbia University, New York.

Harrison JM, Keskin NB, Zeevi A (2012) Bayesian dynamic pricing policies: Learning and earning under a binary prior distribution. *Management Sci.* 58(3):570–586.

Honhon D, Gaur V, Seshadri S (2010) Assortment planning and inventory decisions under stockout-based substitution. *Oper. Res.* 58(5): 1364–1379.

Hu M, Wang Z (2014) Secrecy versus transparency in sales of network goods. Working paper, University of Toronto, Toronto.

Hu X, Duenyas I, Kapuscinski R (2008) Optimal joint inventory and transshipment control under uncertain capacity. *Oper. Res.* 56(4): 881–897.

Huggins E, Olsen T (2010) Inventory control with generalized expediting. *Oper. Res.* 58(5):1414–1426.

Karaesmen I, Scheller-Wolf A, Deniz B (2008) Managing perishable and aging inventories: Review and future research directions. Kempf K, Keskinocak P, Uzsoy R, eds. *Planning Production and Inventories in the Extended Enterprise* (Springer, New York), 393–436.

Lee H, Nahmias S (1993) Single product, single location models. Graves S, Kan AR, Zipkin P, eds. *Handbooks in Operations Research and Management Science*, Logistics of production and inventory, Vol. 4 (North-Holland, Amsterdam), 3–55.

Li Q, Yu P (2014) Multimodularity and structural properties of stochastic dynamic programs. *Manufacturing Service Oper. Management* 16(3):455–462.

Li Q, Yu P, Wu X (2013) Managing perishable inventories in retailing: Replenishment, clearance sales, and segregation. Working paper, School of Business and Management, Hong Kong University of Science and Technology, Hong Kong.

Littlewood K (1972) Forecasting and control of passengers. *Proc. 12th AGIFORS Sympos., Nathanya, Israel*, 95–128.

Liu Q, Shum S (2013) Pricing and capacity rationing with customer disappointment aversion. *Production Oper. Management* 22(5): 1269–1286.

Liu Q, van Ryzin G (2008) On the choice-based linear programming model for network revenue management. *Manufacturing Service Oper. Management* 10(2):288–310.

Maglaras C, Meissner J (2006) Dynamic pricing strategies for multiproduct revenue management problems. *Manufacturing Service Oper. Management* 8(2):136–148.

McGill J, van Ryzin G (1999) Revenue management: Research overview and prospects. *Transportation Sci.* 3(2):233–256.

Naddor E (1978) Sensitivity to distributions in inventory systems. *Management Sci.* 24(16):1769–1772.

Nahmias S (2011) *Perishable Inventory Systems* (Springer, New York).

Olsen T, Parker R (2008) Inventory management under market size dynamics. *Oper. Res.* 54(10):1805–1821.

Porteus E (1990) Stochastic inventory theory. Heyman D, Sobel M, eds. *Handbooks in Operations Research and Management Science*, Stochastic Models, Vol. 2 (North-Holland, Amsterdam), 605–652.

Shen Z, Su X (2007) Customer behavior modeling in revenue management and auctions: A review and new research opportunities. *Production Oper. Management* 16(6):713–728.

Shou B, Xiong H, Shen Z (2013) Consumer panic buying and quota policy under supply disruptions. Working paper, City University of Hong Kong, Hong Kong.

Su X (2007) Intertemporal pricing with strategic customer behavior. *Management Sci.* 53(5):726–741.

Su X (2010) Intertemporal pricing and consumer stockpiling. *Oper. Res.* 58(4):1133–1147.

Su X, Zhang F (2008) Strategic customer behavior, commitment, and supply chain performance. *Management Sci.* 54(10):1759–1773.

Talluri K (2012) Revenue management. Ozer O, Phillips R, eds. *The Oxford Handbook of Pricing Management* (Oxford University Press, Oxford, UK).

Talluri K, van Ryzin G (2004) Revenue management under a general discrete choice moel of consumer behavior. *Management Sci.* 50(1): 15–33.

Topkis D (1998) *Supermodularity and Complementarity* (Princeton University Press, Princeton, NJ).

Xie J, Shugan S (2001) Electronic tickets, mart cards, and online prepayments: When and how to advance sell. *Marketing Sci.* 20(3):219–243.

Xue Z, Ettl M, Yao D (2012) Managing freshness inventory: Optimal policy, bounds and heuristics. Working paper, IEOR Department, Columbia University, NY.

Yin R, Aviv Y, Pazgal A, Tang C (2008) Optimal markdown pricing: Implications of inventory display formats in the presence of strategic customers. *Management Sci.* 55(8):1391–1408.

Yu M, Ahn H, Kapuscinski R (2014) Advance selling with interdependent customer valuations. *Management Sci.* Forthcoming.

Zipkin P (2000) *Foundations of Inventory Management* (McGraw Hill, New York).

Peng Hu is an assistant professor in the School of Management at the Huazhong University of Science and Technology. His research interests include inventory and supply chain management, optimization, and the interface between operations and marketing.

Stephen Shum is an associate professor in the College of Business at the City University of Hong Kong. His research interests include supply chain management, revenue management, and operations-marketing interface.

Man Yu is an assistant professor in the Department of Information Systems, Business Statistics, and Operations Management at the Business School of the Hong Kong University of Science and Technology. Her research interests include pricing and revenue management, supply chain management, and consumer behavior in operations management.