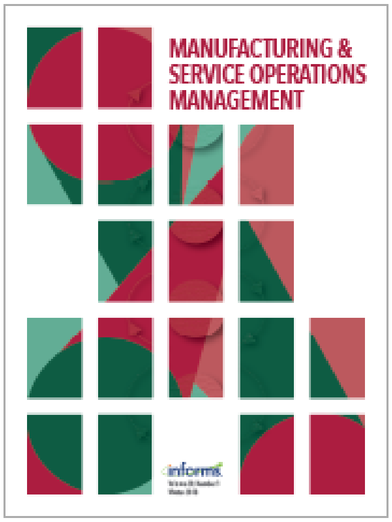

Manufacturing & Service Operations Management

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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MANUFACTURING & SERVICE OPERATIONS MANAGEMENT
Vol. 13, No. 4, Fall 2011, pp. 525–533
ISSN 1523-4614 | EISSN 1526-5498 | 11 | 1304 | 0525
http://dx.doi.org/10.1287/msom.1110.0340

A Censored-Data Multiperiod Inventory Problem with Newsvendor Demand Distributions

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We study the stochastic multiperiod inventory problem in which demand in excess of available inventory is lost and unobserved so that demand data are censored. A Bayesian scheme is employed to dynamically update the demand distribution for the problem with storable or perishable inventory and with exogenous or endogenous price. We show that the Weibull is the only newsvendor distribution for which the optimal solution can be expressed in scalable form. Moreover, for Weibull demand the cost function is not convex in general. Nevertheless, in all but the storable case, sufficient structure can be discerned so that the optimal solution can be easily computed. Specifically, for the perishable inventory case, the optimal policy can be found by solving simple recursions, whereas the perishable case with pricing requires solutions to more complex one-step look-ahead recursions. Interestingly, for the special case of exponential demand the cost function is convex, so that for the storable inventory case, the optimal policy can be found using simple one-step look-ahead recursions whereas for the perishable case the optimal policy can be expressed by exact closed-form formulas.

Key words: inventory; stochastic demand; lost sales; scalability; optimal policy
History: Received: August 4, 2010; accepted: April 15, 2011. Published online in *Articles in Advance* August 4, 2011.

1. Introduction

In a multiperiod inventory problem (MPIP), at the start of each period, the decision maker must choose the order quantity before the realization of demand in that period. Demand in excess of current stock is lost or backordered. If the stock in excess of demand perishes then the MPIP reduces to determining the target inventory level in a series of single-period inventory problems. If leftover stock is available to fill future demand, then leftover stock in each period becomes part of the initial inventory of the next period.

Veinott (1965) shows that when demand and cost parameters are known and stationary, the optimal solution for the MPIP can be myopic. However, this is not the case when the parameters that describe the uncertainty in demand are not known precisely. Typically, in such situations, distributions of parameters are updated dynamically using a Bayesian scheme, making the corresponding inventory decision problem nonstationary. Consequently, analytical and computational difficulties arise in finding optimal solutions because the state space that describes the dynamic program can grow rapidly. However, significant progress toward computational tractability can be made if the form of the optimal solution is scalable in the sense that it can be written as a product of two functions, one that depends on the scale parameter and the other that depends on the shape parameter. This form helps to reduce the dimensionality of a problem, as in Scarf (1959, 1960) and Azoury (1985), who assume that exact observations of demand are recorded in each period.

Whereas these *full demand information* scenarios arise most naturally when demand in excess of available inventory is backordered, this is not the case when demand in excess of inventory is lost and unobserved so that only sales are recorded, as in Harpaz et al. (1982), who were the first to consider the MPIP with price-dependent demand and censored data on demand observations. Because the optimal solution was not discernible, some of the insights in the paper are generated from a set of numerical examples.

An important step in elucidating the structure of the MPIP when demand is censored by the inventory level is due to Lariviere and Porteus (1999), who focus on the case when demand has a newsvendor distribution (see (1)). The newsvendor (newsboy) family, as noted by Braden and Freimer (1991), appears to be the only family whose updated distributions with censored demand information have conjugate priors under a Bayesian updating scheme. Subsequently, Lariviere and Porteus (1999) show that for the important subcase in which the demand distribution is Weibull with a gamma prior, the optimal solution for the finite-horizon MPIP is scalable. Furthermore, for the case of exponential demand (Weibull with shape parameter equal to 1), they also develop a set of recursive equations that characterize the optimal solution.

Whereas Lariviere and Porteus (1999) take price as given, Petruzzi and Dada (2001) consider the two-period case with Bayesian updating of censored demand in which the inventory decisions as well as prices are determined endogenously. In addition to structural results, a forward dynamic program is used to compute the optimal solutions. Athough significant progress continues to be made on elucidating the structure of the censored MPIP as in Petruzzi and Dada (2002), Ding et al. (2002), Lu et al. (2006, 2008), Bensoussan et al. (2007; 2008; 2009a, b, c), Bisi and Dada (2007), and Chen and Plambeck (2008), computing optimal solutions remains a difficult problem in general. As noted by Chen and Plambeck (2008) and Chen (2010), scalability, because it leads to dimensional reducibility, may hold the key to computing optimal solutions exactly.

To examine this dimensional reducibility closely, we streamline our study so that we can answer the following fundamental questions for the censored MPIP: Is dimension reduction (scalability) enough to guarantee exact or tractable optimal solutions? If not, what type of additional structures are needed to find the exact or tractable solutions? In which cases does scalability lead to exact/tractable optimal solutions? In those cases, how are optimal solutions computed?

We proceed by first identifying when optimal solutions are scalable. As in Lariviere and Porteus (1999), we focus on the exogenous price variants of the censored MPIPs whose demand distributions are from the family of newsvendor distributions. In Theorem 1, we complement their scalablity result by showing that Weibull is the only member of newsvendor distributions for which optimal solutions are scalable. Then using a decision-theoretic approach, we develop a structural result that relates the standardized cost with the prior distribution's shape parameter. This result is first used to show that for Weibull demand, the cost function need not be convex. It is then used to show for the case of exponential demand, that the cost function is strictly convex and that the optimal solution can be computed uniquely using simple recursions. Thus, not only do we provide theoretical justification for why the numerical computation used by Chen (2010) finds the unique optimal solution for the exponential demand case, we also present a simpler computational scheme for finding the optimal solution.

When inventory is perishable, the structural results are sharper. The resulting dynamic program is shown to have sufficient structure so that scalability yields tractable analytical solutions for the case of Weibull demand. Hence, it is possible to reduce finding the optimal solution to solving a series of one-step look-ahead recursive equations that can be solved efficiently by backward substitution for any finite horizon problem. Our method shows that each of these equations has a unique solution. The case of exponential demand is even simpler and yields explicit closed-form formulas for the optimal order quantity and cost. Moreover, taking limits appropriately results in a simple equation that easily yields the unique scalable optimal solution for the infinite horizon discounted cost problem.

Finally, we consider the general case in which price is also determined endogenously. We extend the scalable property of the optimal solutions to a multiplicative price-dependent demand model. This analysis also yields a recursive characterization of the optimal solutions for finite horizon problems. We also explain why the scalable property does not hold when demand is an additive function of price.

This paper is organized as follows. In §2 we describe the problem. The storable and perishable inventory models are discussed in §§3 and 4, respectively. The price-dependent demand models are studied in §5. All proofs are provided in the online appendix (available at http://msom.pubs.informs.org/ecompanion.html).

2. Problem Description and Related Properties

In our problem, the decision maker must choose the stocking quantity y_n at the beginning of each period n, $n = 1, 2, \ldots, N$. Then, the demand X_n is realized. We assume that the random demands X_ns are generated such that given a value for the unknown parameter θ, the conditional distributions of X_ns are independent and identically distributed with a known probability density $f(\cdot \mid \theta)$, $\theta \in \Theta$.

For each period denote the sales by $s_n = \min(X_n, y_n)$, where demand is exactly observed when sales are less than the stocking quantity, that is, when $X_n < y_n$; and the demand is *censored* at the stocking quantity when sales equal y_n, that is, when $X_n \geq y_n$. The procurement cost for the newsvendor model in each period is a variable ordering cost of c per unit. Because we will primarily formulate the problem in terms of minimizing cost, demand in excess of sales represents a loss of revenue (and other penalties) at the rate of p per unit. If the inventory is storable (nonperishable), a holding cost of h per unit is charged on leftover inventory at the end of each period. Otherwise, if the inventory is perishable, inventory in excess of demand, if any, is salvaged at a unit value of h. To rule out the trivial cases of ordering zero or holding infinite stock for speculative purposes, we assume $c < p$ when inventory is storable and $h < c < p$ when inventory is perishable.

Because the underlying demand parameter is unknown, we will use a Bayesian scheme to update its distribution over time. Let $\hat{\pi}_{n+1}(\theta \mid s_n)$ be the posterior density in period n that equals $\pi_{n+1}(\theta \mid x_n)$ if $s_n = x_n < y_n$, and $\pi^c_{n+1}(\theta \mid y_n)$ if $s_n = y_n$, where

$$\pi_{n+1}(\theta \mid x_n) = \frac{f(x_n \mid \theta)\hat{\pi}_n(\theta)}{\int_\Theta f(x_n \mid \theta')\hat{\pi}_n(\theta')\, d\theta'} \quad \text{and}$$

$$\pi^c_{n+1}(\theta \mid y_n) = \frac{\int_{y_n}^\infty f(x \mid \theta)\, dx\, \hat{\pi}_n(\theta)}{\int_\Theta \int_{y_n}^\infty f(x \mid \theta')\hat{\pi}_n(\theta')\, dx\, d\theta'}.$$

Let us denote $\psi_n(x \mid \hat{\pi}_n) = \int_\Theta f(x \mid \theta)\hat{\pi}_n(\theta \mid s_{n-1})\, d\theta$ and $\Psi_n(x \mid \hat{\pi}_n) = \int_0^x \psi_n(z \mid \hat{\pi}_n)\, dz$ for the updated probability density and distribution function of X_n, respectively.

With censored data on demand, the compact analytical form of using conjugate priors does not exist in general. However, when demand is described by a member of the newsvendor family, as shown by Braden and Freimer (1991), the gamma distribution remains a conjugate prior and therefore, it may be possible to mimic the analytical tractability of the conjugate approach under full demand information. As defined by Braden and Freimer (1991), a random variable X is a member of the newsvendor (newsboy) family of distributions if its density is given by

$$f(x \mid \theta) = \theta d'(x) e^{-\theta d(x)}, \tag{1}$$

for some function d: $(0, \infty) \to (0, \infty)$, where the prime represents its derivative. For $f(x \mid \theta)$ to be a valid density function on $[0, \infty)$, it is necessary and sufficient that

$$d'(x) \geq 0, \quad \lim_{x\to 0} d(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} d(x) = \infty. \tag{2}$$

This can be established by considering the cumulative distribution function $F(x \mid \theta) = 1 - e^{-\theta d(x)}$ that is required to be nondecreasing with $\lim_{x\to 0} F(x \mid \theta) = 0$ and $\lim_{x\to\infty} F(x \mid \theta) = 1$. We restrict attention to choices of $d(\cdot)$ for which $d'(x) > 0$ for all $x > 0$, so that $f(x \mid \theta)$ is positive on the whole of $(0, \infty)$. To ensure identifiability, we impose the condition $d(1) = 1$, because any newsvendor model with a given $d(\cdot)$ can be transformed into a newsvendor model with $d^*(1) = 1$ by rescaling the original $d(x)$ function to $d^*(x) = d(x)/d(1)$, and applying the inverse scale to change θ into $\theta^* = d(1)\theta$.

In (1), for $d(x) = x^l$ with a known constant $l > 0$, we get the Weibull distribution and in particular, if $l = 1$, we get the exponential distribution. The prior density of θ is given by a gamma density with shape and scale parameters a and S, respectively (denoted by Gamma(a, S) for later use). Setting a_1 and S_1 as the initial parameters in period 1, the sufficient statistics for a and S at the beginning of period n are given by $a_n = a_1 + m_n$ and $S_n = S_1 + \sum_{i=1}^{n-1} d(s_i)$, where m_n denotes the number of exact demand observations by the start of period n and s_i is the observed sales in period i (see Braden and Freimer 1991). Then the updated demand density and distribution function in period n are, respectively, given by

$$\psi_n(x \mid a_n, S_n) = \frac{a_n S_n^{a_n} d'(x)}{[S_n + d(x)]^{a_n+1}} \quad \text{and}$$

$$\Psi_n(x \mid a_n, S_n) = 1 - \left(\frac{S_n}{S_n + d(x)}\right)^{a_n}.$$

Now, following Lariviere and Porteus (1999), we call a decision variable *scalable* if it is of the form $y_n = q(S_n)g(a_n)$, so that y_n is separable into two terms, namely, $q(S_n)$ that depends on the scale parameter S_n and $g(a_n)$ that depends on the shape parameter a_n of period n. In this paper, we will examine when the optimal decision variable for the censored multiperiod inventory problem with demand from the newsvendor family is scalable. We will further examine when $g(a_n)$ has simple structure so that the optimal solution can either be expressed in closed/analytical form or be easily computed. To proceed, we will first discuss the storable inventory model.

3. The Storable Inventory Problem

In the storable inventory problem, the inventory manager incurs a holding cost of h per unit of leftover inventory at the end of each period. Let y_n denote the inventory level after ordering in period n. Then, the single-period Bayesian expected overage and underage cost with prior distribution $\hat{\pi}_n$ is given by

$$L(y_n \mid \hat{\pi}_n) = h\int_0^{y_n} (y_n - x)\psi_n(x \mid \hat{\pi}_n)\, dx + p\int_{y_n}^\infty (x - y_n)\psi_n(x \mid \hat{\pi}_n)\, dx.$$

To conveniently represent the optimality equations, with a discount factor $0 < \beta \leq 1$, let us denote

$$\begin{aligned} G_n(y_n \mid \hat{\pi}_n) &= cy_n + L(y_n \mid \hat{\pi}_n) \\ &\quad + \beta\int_0^{y_n} V_{n+1}(y_n - x \mid \pi_{n+1}(\cdot \mid x))\psi_n(x \mid \hat{\pi}_n)\, dx \\ &\quad + \beta V_{n+1}(0 \mid \pi^c_{n+1}(\cdot \mid y_n))[1 - \Psi_n(y_n \mid \hat{\pi}_n)]. \end{aligned}$$

Now, let $V_n(\bar{z}_n \mid \hat{\pi}_n)$ denote the optimal cost over periods $n, n+1, \ldots, N$, with initial inventory $\bar{z}_n$ and prior distribution $\hat{\pi}_n$ at the beginning of period n. Then the optimality equations can be written as

$$V_n(\bar{z}_n \mid \hat{\pi}_n) = -c\bar{z}_n + \min_{y_n \geq \bar{z}_n} G_n(y_n \mid \hat{\pi}_n), \tag{3}$$

for $n = 1, 2, \ldots, N$, with the boundary condition $V_{N+1}(\bar{z}_{N+1} \mid \hat{\pi}_{N+1}) = -c\bar{z}_{N+1}$ for all $\bar{z}_{N+1}$ and $\hat{\pi}_{N+1}$. We denote the optimal inventory level y_n^* in period n by y_n^{*c} if $s_{n-1} = y_{n-1}$, and $\max(y_n^{*e}, \bar{z}_n)$ if $s_{n-1} < y_{n-1}$, where y_n^{*c} and y_n^{*e} are the corresponding order-up-to levels. To proceed with our analysis, we first show the following result on scalability.

Theorem 1. *If the demand distribution is from the newsvendor family, then the optimal order-up-to levels are scalable only if the demand distribution is Weibull.*

Although the result that Weibull is sufficient for scalability is previously presented in Theorem 2 of Lariviere and Porteus (1999), the above necessity result is new and it shows that, when looking for scalable solutions, one need not look beyond the Weibull distribution. Because of this, we will now focus on the case of Weibull demand with a gamma prior. For this case, letting $\hat{\pi}_n = \text{Gamma}(a_n, S_n)$, by Theorem 2(b) of Lariviere and Porteus (1999), the optimal costs are scalable in the sense that we can write $V_n(\bar{z}_n \mid a_n, S_n) = \tilde{S}_n V_n(\bar{z}_n/\tilde{S}_n \mid a_n, 1)$, where $\tilde{S}_n = S_n^{1/l}$, with $S_n = (S_1 + \sum_{i=1}^{n-1} s_i^l)$. Therefore, (3) leads to

$$\begin{aligned} V_n(\bar{z}_n \mid a_n, S_n) &= \tilde{S}_n\Big[-c\bar{z}_n/\tilde{S}_n + \min_{q_n \ge \bar{z}_n/\tilde{S}_n} G_n(q_n \mid a_n, 1)\Big] \\ &= -c\bar{z}_n + \tilde{S}_n \min_{q_n \ge z_n'} G_n(q_n \mid a_n, 1), \end{aligned} \tag{4}$$

where $z_n' = \bar{z}_n/\tilde{S}_n$, and $G_n(q_n \mid a_n, 1)$ is given by

$$\begin{aligned} &G_n(q_n \mid a_n, 1) \\ &= E_{\psi_n(\cdot \mid a_n, 1)}\big[cq_n + h\max(0, q_n - \xi_n) + p\max(0, \xi_n - q_n) \\ &\qquad + \beta V_{n+1}(0 \mid a_n, 1 + q_n^l) I(\xi_n \ge q_n) \\ &\qquad + \beta V_{n+1}(q_n - \xi_n \mid a_n + 1, 1 + \xi_n^l) I(\xi_n < q_n)\big] \\ &= E_{\psi_n(\cdot \mid a_n, 1)}\big[cq_n + h\max(0, q_n - \xi_n) + p\max(0, \xi_n - q_n) \\ &\qquad + \beta V_{n+1}(0 \mid a_n, 1 + q_n^l) I(\xi_n \ge q_n) \\ &\qquad + \beta\{-c(q_n - \xi_n) + G_{n+1}(q_{n+1}^{*e} \mid a_n + 1, 1 + \xi_n^l)\} \\ &\qquad \cdot I(\xi_n < q_n, q_{n+1}^{*e} \ge q_n - \xi_n) \\ &\qquad + \beta\{-c(q_n - \xi_n) + G_{n+1}(q_n - \xi_n \mid a_n + 1, 1 + \xi_n^l)\} \\ &\qquad \cdot I(\xi_n < q_n, q_{n+1}^{*e} < q_n - \xi_n)\big]. \end{aligned} \tag{5}$$

The terms in (5) can be explained as follows. Suppose we have made the inventory decision y_n at the beginning of period n. Then, after the demand is realized for the period, one of the following three types of sample paths can be observed. For convenience, let c, e_L, and e_H represent, respectively, the sample paths with censored demand, the exactly observed demand sample paths with leftover inventory below the target order-up-to level so that an order is placed in the next period, and the exactly observed demand sample paths with leftover inventory above or equal to the target order-up-to level so that no order is placed in the next period (see Figure 1 in the online appendix). Now let us look at the second equation in (5). The first three terms correspond to the single-period cost in period n. The fourth term is the optimal discounted cost-to-go from period $(n+1)$ onward given that demand is censored in period n. The fifth and sixth terms are, respectively, the optimal discounted cost-to-go from period $(n+1)$ onward given that demands are exactly observed in period n resulting in e_L and e_H sample paths. The sample paths represented by e_H are the ones that make it difficult to compute the optimal solutions and costs for any storable inventory problem with time-varying order-up-to levels, as is the case here. Censoring of demand observations adds more complications because e_H sample paths can occur after censoring has occurred in an earlier period, and vice versa. The combined effect of censored demand observations on c sample paths and excess inventory carryover on e_H sample paths makes the storable inventory problem difficult to solve.

Nevertheless, we are able to establish the following result for the optimal cost of the Weibull demand model using a decision-theoretic approach.

Lemma 1. *Suppose the demand distribution is Weibull with a gamma prior on the unknown parameter θ. Let $\tilde{q}_n(a) = \arg\min_{q \ge 0} G_n(q \mid a, 1)$ and $\tilde{v}_n(a) = G_n(\tilde{q}_n(a) \mid a, 1)$. Then, for any period n and all $a > 1/l$,*

$$\left(a - \frac{1}{l}\right)\tilde{v}_n(a) \ge a\tilde{v}_n(a+1). \tag{6}$$

Lemma 1 has an interesting interpretation. With $B(\cdot, \cdot)$ denoting the beta function, because $aB(a - 1/l, 1 + 1/l)$ is the mean of the random variable ξ with density $\psi(\xi \mid a, 1) = al\xi^{l-1}/(1 + \xi^l)^{a+1}$, (6) can be rewritten as

$$\frac{\tilde{v}_n(a)}{E_{\psi(\cdot \mid a, 1)}[\xi]} \ge \frac{\tilde{v}_n(a+1)}{E_{\psi(\cdot \mid a+1, 1)}[\xi]}.$$

Therefore, Lemma 1 implies that for the storable inventory model with Weibull demand, the *standardized* cost (optimal cost per unit of mean demand) is higher with a lower value of the shape parameter a, that is, higher demand uncertainty. The special case of Lemma 1 for the perishable inventory model with exponential demand is presented in Theorem 3(c) of Lariviere and Porteus (1999).

One consequence of Lemma 1 is that it helps to characterize the structure of the cost function. In particular, for Weibull distribution with shape parameter $l > 1$, we show that the cost function is nonconvex in a two-period problem. Lemma 1 also has the consequence that it can be used to establish for the special case of exponential demand (Weibull with $l = 1$), that the cost function is indeed convex. The convexity is then sufficient to yield simple recursions to compute uniquely the optimal solution and cost. These are formalized as the following:

THEOREM 2. (i) *If the demand distribution is Weibull with a gamma prior distribution, then the cost function need not be convex.*

(ii) *If the demand distribution is exponential with a gamma prior on the unknown parameter θ, then for any period n, $n = 1, 2, \ldots, N$,*

(a) *$G_n(q \mid a, 1)$ is strictly convex and twice differentiable in q, for all $a > 1$, and satisfies,*

(b)

$$G_n(q \mid a, 1) = (c + h - \beta c)q + \frac{1}{a-1}\left[\frac{p + h - \beta c + \beta\{(a-1)\tilde{v}_{n+1}(a) - a\tilde{v}_{n+1}(a+1)\}}{(1+q)^{a-1}} - (h - \beta c)\right] + \beta T_{n,a}(q),$$

where $\tilde{v}_{n+1}(a) = G_{n+1}(\tilde{q}_{n+1}(a) \mid a, 1)$, and for $q < \tilde{q}_{n+1}(a+1)$,

$$T_{n,a}(q) = \frac{a}{a-1}\tilde{v}_{n+1}(a+1),$$

and for $q \geq \tilde{q}_{n+1}(a+1)$,

$$T_{n,a}(q) = \frac{a}{(1+q)^{a-1}}\left[\int_{\tilde{q}_{n+1}(a+1)}^{q} G_{n+1}(\xi \mid a+1, 1)(1+\xi)^{a-2}\,d\xi + \frac{\tilde{v}_{n+1}(a+1)}{a-1}(1 + \tilde{q}_{n+1}(a+1))^{a-1}\right].$$

(c) *With initial parameters a_1 and S_1 for the gamma prior, the optimal order-up-to level and cost in period n are given by $y^*_{n,k} = S_n\hat{q}_{n,k}$ and $V_{n,k}(\bar{z}_n \mid a_n, S_n) = -c\bar{z}_n + S_n\hat{v}_{n,k}$, for $k = 0, 1, 2, \ldots, n-1$, with $S_n = (S_1 + \sum_{i=1}^{n-1} s_i)$, where $\hat{q}_{n,k}$ and $\hat{v}_{n,k}$, the order-up-to level and cost for the normalized system with unit scale parameter, can be solved optimally and are unique.*

Because of the nonconvexity result of Theorem 2(i), it remains a challenge to devise an efficient computation scheme for the optimal solution of the storable inventory problem with Weibull demand. Nevertheless, Theorems 2(ii)(a) and (b) help to efficiently compute the optimal order-up-to levels and costs for the storable inventory model with exponential demand for any finite horizon problem; Theorem 2(ii)(c) establishes that these computations do really yield exact optimal solutions. As a demonstration of Theorem 2(ii)(c), the explicit solutions for the first period of a two-period and a three-period problem are shown in the online appendix. Note that, the recursion in our Theorem 2(ii)(b) requires the evaluation of one integral (for $q \geq \tilde{q}_{n+1}(a+1)$) in each period; it is different from the recursion presented in Chen (2010, p. 404) that requires the evaluation of one integral and the values of two incomplete beta functions in each period. Overall, Theorem 2 suggests that the storable inventory problem has a simpler structure when the shape parameter l of Weibull demand equals 1 to yield the exponential demand case. Now that we have analyzed the storable inventory model, we study the perishable model next.

4. The Perishable Inventory Problem

In the perishable inventory model, leftover inventory is salvaged at a value of h per unit at the end of each period. Thus, at the start of each period, there is no on-hand inventory, ensuring that an order is placed every period. Proceeding as in the storable case, using the superscript P for perishable, for the Weibull demand with prior $\hat{\pi}_n = \text{Gamma}(a_n, S_n)$, we can write the optimality equation as

$$V_n^P(a_n, S_n) = \tilde{S}_n \min_{q_n \geq 0} G_n^P(q_n \mid a_n, 1),$$

where $\tilde{S}_n = S_n^{1/l}$, with $S_n = (S_1 + \sum_{i=1}^{n-1} s_i^l)$, and $G_n^P(q_n \mid a_n, 1)$ can be written as

$$\begin{aligned} G_n^P&(q_n \mid a_n, 1)\\ &= E_{\psi_n(\cdot \mid a_n, 1)}\big[cq_n - h\max(0, q_n - \xi_n) + p\max(0, \xi_n - q_n) \\ &\qquad + \beta V_{n+1}^P(a_n, 1 + q_n)I(\xi_n \geq q_n) \\ &\qquad + \beta V_{n+1}^P(a_n + 1, 1 + \xi_n)I(\xi_n < q_n)\big] \\ &= E_{\psi_n(\cdot \mid a_n, 1)}\big[cq_n - h\max(0, q_n - \xi_n) + p\max(0, \xi_n - q_n) \\ &\qquad + \beta(1 + q_n)V_{n+1}^P(a_n, 1)I(\xi_n \geq q_n) \\ &\qquad + \beta(1 + \xi_n)V_{n+1}^P(a_n + 1, 1)I(\xi_n < q_n)\big] \\ &= E_{\psi_n(\cdot \mid a_n, 1)}[cq_n - h\max(0, q_n - \xi_n) + p\max(0, \xi_n - q_n)] \\ &\quad + \beta V_{n+1}^P(a_n, 1)\frac{1}{(1 + q_n^l)^{a_n - 1/l}} \\ &\quad + \beta V_{n+1}^P(a_n + 1, 1)\frac{a_n}{a_n - 1/l}\left(1 - \frac{1}{(1 + q_n^l)^{a_n - 1/l}}\right). \end{aligned} \tag{7}$$

The first three terms of (7) correspond to the single-period cost in period n. The fourth and fifth terms are the optimal discounted cost-to-go from period $(n+1)$ onward given that demands are, respectively, censored and exactly observed in period n (resulting in c and e sample paths of Figure 2 in the online appendix). Notice that, because the optimal cost is scalable, we are able to explicitly integrate out the last two terms. This allows us to derive the optimal order quantity and cost in simple analytical form for the case of Weibull demand as we will see in the following theorem. To guarantee finiteness of the mean of the updated distributions and optimal costs, we will assume $a_1 > 1/l$ for the Weibull demand model.

Theorem 3. *If the demand distribution is Weibull with an unknown parameter θ and the prior distribution on θ is gamma with initial parameters a_1 and S_1, then the following applies*:

(a) *The optimal order quantity in period n, $n = 1, 2, \ldots, N$, is given by*

$$y^{*P}_{n,k} = \tilde{S}_n q_{n,k}, \tag{8}$$

for $k = 0, 1, 2, \ldots, n-1$, where $q_{n,k}$ is uniquely obtained by solving

$$(c-h)(1+q^l)^{a_1+k-1/l+1} - (p-h)(1+q^l)^{1-1/l} + lq^{l-1}\tilde{v}_{n,k} = 0, \tag{9}$$

where $\tilde{v}_{n,k} = \beta[(a_1+k)v_{n+1,k+1} - (a_1+k-1/l)v_{n+1,k}]$, with $v_{n,k} \equiv V^P_n(a_1+k, 1)$.

(b) *The optimal cost in period n, $n = 1, 2, \ldots, N$, is given by*

$$V^P_{n,k}(a_n, S_n) = \tilde{S}_n v_{n,k}, \tag{10}$$

where $v_{n,k}$ is obtained from

$$v_{n,k} = p\mu_{a_1+k} + (c-h)q_{n,k} - (p-h)\bar{H}_{a_1+k}(q_{n,k}) + \beta\left[v_{n+1,k}\lambda_{n,k} + \frac{a_1+k}{a_1+k-1/l}v_{n+1,k+1}(1-\lambda_{n,k})\right], \tag{11}$$

for $k = 0, 1, 2, \ldots, n-1$, with the terminal conditions $v_{N+1,k} = 0$, for all $k = 0, 1, 2, \ldots, N$, where

$$\mu_{a_1+k} = (a_1+k)B\left(a_1+k-\frac{1}{l}, 1+\frac{1}{l}\right),$$

$$\bar{H}_{a_1+k}(q) = \frac{1}{l}B\left(a_1+k-\frac{1}{l}, \frac{1}{l}\right) \cdot \left\{1 - F_B\left(\frac{1}{1+q^l} \,\middle|\, a_1+k-\frac{1}{l}, \frac{1}{l}\right)\right\},$$

$$\lambda_{n,k} = (1+q^l_{n,k})^{-(a_1+k-1/l)},$$

where $B(p,q)$ denotes the beta function and $F_B(x \mid p, q)$ denotes the cumulative distribution function of the Beta(p,q) distribution.

Notice that (9) and (11) are only one-step look-ahead recursive equations of polynomial form so that to solve for period n we only need the solution for period $(n+1)$. To find the optimal solutions, we first solve (9) for period N. With this solution, we compute (11) for period N. Using this computed cost, we first solve (9) and then compute (11) for period $N-1$. Proceeding recursively this way we find the unique optimal solution for each period.

Thus, a consequence of Theorem 3 is that it shows that when inventory is perishable, the Weibull case, although not necessarily convex, has enough structure so that the optimal solution for the Weibull model is easily computable. Moreover, we can provide an interpretation for the optimal order quantity $y^{*P}_{n,k}$ given by (8). The same interpretation also holds for the optimal order quantity given by (12) for the exponential case. Because $E(X_n) = (a_1+k)B(a_1+k-1/l, 1+1/l)\tilde{S}_n$, from (8) we can write $y^{*P}_{n,k} = \tau^*_{n,k}E(X_n)$, where $\tau^*_{n,k} = q_{n,k}/(a_1+k)B(a_1+k-1/l, 1+1/l)$. Notice that, for any N-period problem, the values of $\tau^*_{n,k}$s can be precomputed (and assigned to all the nodes like those in Figure 2 in the online appendix) before observing any sales data. Thus, to find the optimal order quantity in period n, all we need to do is multiply the precomputed factor $\tau^*_{n,k}$ with the updated mean of X_n that is obtained based on the sales data. Thus, for the perishable inventory model, scalability of the optimal solution leads to precomputed factors that greatly simplifies computation. However, for the storable model, although scalability reduces the dimensionality of the problem, it does not seem to lead to precomputed factors because on e_H sample paths, the stocking levels are random variables because they depend on the sample observations (e.g., see nodes B_1 and C_1 in Figure 1 in the online appendix).

4.1. The Case of Exponential Demand Distribution

When demand is exponential, (9) and (11) become simpler and yield the following closed-form formulas.

Theorem 4. *If the demand distribution is exponential with an unknown parameter θ and the prior distribution on θ is gamma with initial parameters a_1 and S_1, then the following applies*:

(a) *The optimal order quantity in period n, $n = 1, 2, \ldots, N$, is given by*

$$y^{*P}_{n,k} = S_n(\alpha_{n,k} - 1), \tag{12}$$

where $S_n = (S_1 + \sum_{i=1}^{n-1} s_i)$ and for $k = 0, 1, 2, \ldots, n-1$,

$$\alpha_{n,k} = \left[\frac{1}{c-h}\left(p-h+\beta(c-h)\left[\sum_{j=k}^{N-(n+1)+k}\left\{(a_1+j)\alpha_{n+1+j-k,j} - (a_1+j+1)\alpha_{n+1+j-k,j+1}\right\} + N-n\right]\right)\right]^{1/(a_1+k)}, \tag{13}$$

with the terminal values $\alpha_{N,k} = ((p-h)/(c-h))^{1/(a_1+k)}$, for $k = 0, 1, 2, \ldots, N-1$.

(b) *The optimal cost in period n, $n = 1, 2, \ldots, N$, is given by*

$$V^P_{n,k}(a_n, S_n) = S_n\gamma_{n,k}, \tag{14}$$

where, for $k = 0, 1, 2, \ldots, n-1$,

$$\gamma_{n,k} = \frac{1}{a_1+k-1}\left[c + (c-h)(a_1+k)(\alpha_{n,k}-1) + \beta\left((N-n)h + (c-h)\sum_{j=k}^{N-(n+1)+k}\left\{(a_1+j+1)\cdot\alpha_{n+1+j-k,j+1} - (a_1+j)\right\}\right)\right]. \tag{15}$$

Observe that, in Theorem 4, we have written the exact closed-form formulas for the optimal solutions and costs. This shows that the perishable inventory problem has a very simple structure when demand is exponential. The main idea is that the uniqueness and a linear structure (for $l = 1$) combined with the scalability of the optimal solutions help to fold back the closed form of the optimal policies and costs by backward induction. Note that in (12) we have expressed the optimal order quantity for the normalized system with unit scale parameter in the form $(\alpha_{n,k} - 1)$ instead of $q_{n,k}$ as in (8). This representation helps us in writing the solutions in exact closed form as given by (13) and (15). Notice that in (13) and (15), both $\alpha_{n,k}$ and $\gamma_{n,k}$ are written explicitly only in terms of the αs that are already computed. In contrast, to compute the optimal solutions and costs from the recursive equations presented in Theorems 3(a) and (b) of Lariviere and Porteus (1999), one needs to use the two sets of equations alternatively.

Using Theorem 4 we now derive the optimal order quantity and cost in simple form for the infinite horizon problem with exponential demand.

Theorem 5. *If the demand distribution is exponential and the gamma prior has initial parameters a_1 and S_1, then the following applies*:

(a) *There exists a limit l_k (≥ 1) such that for any given period n, the sequence $\{\alpha_{\hat{n},k}\}_{\hat{n}>n}$ converges to l_k as $\hat{n} \downarrow n$ and $N \to \infty$ for all $k = 0, 1, 2, \ldots, n-1$.*

(b) *The optimal order quantity in period n, $n = 1, 2, \ldots$, is given by $y^{*\infty}_{n,k} = S_n(l_k - 1)$, where $S_n = (S_1 + \sum_{i=1}^{n-1} s_i)$, and for $k = 0, 1, 2, \ldots, n-1$, l_k satisfies*

$$l_k^{a_1+k} - \beta(a_1+k)l_k = \frac{p-h}{c-h} - \beta(a_1+k). \tag{16}$$

Moreover, l_k is unique, bounded, and decreases in k.

(c) *The optimal cost in period n, $n = 1, 2, \ldots$ is given by: $V^{\infty}_{n,k}(a_n, S_n) = S_n m_k$, where m_k satisfies*

$$m_k = \frac{1}{a_1+k-1}\left[\frac{c}{1-\beta} + (c-h)\sum_{j=k}^{\infty}\beta^{j-k}(a_1+j)(l_j-1)\right], \tag{17}$$

for $k = 0, 1, 2, \ldots, n-1$, and the above series converges.

Theorem 5 shows that the optimal order quantity and cost for the infinite horizon problem with exponential demand preserve the scalable property of the finite horizon counterparts. Moreover, the polynomial form of (16) is simpler than (9). Now that we have analyzed models with only inventory decisions, we next study the general model in which price is also a decision variable.

5. The Joint Pricing and Inventory Problem

We first consider the case of perishable inventory in which demand depends on price through a multiplicative isoelastic function as in Monahan et al. (2004). Thus, demand in period n, $n = 1, 2, \ldots, N$, is given by $D_n(r_n, X_n) = a r_n^{-b} X_n$ for $a > 0$, $b > 1$, where r_n is the per unit retail price and X_n represents the random component of the demand. The assumptions on X_n are the same as in §2. To derive the optimal solutions and examine their scalability for this endogenous pricing case, similarly to Petruzzi and Dada (1999) we denote the stocking factor by $z_n = y_n / a r_n^{-b}$, where y_n is the stock level in period n. Let us also denote $\hat{s}_n = \min(X_n, z_n)$. To incorporate price as a decision variable, we formulate the dynamic program as a profit maximization problem so that the optimality equations are given by

$$\begin{aligned} W_n(\hat{\pi}_n) = \max_{z_n \geq 0,\, r_n \geq 0} \bigg\{ & M(\hat{\pi}_n, z_n, r_n) \\ & + \beta \int_0^{z_n} W_{n+1}(\pi_{n+1}(\cdot \mid x)) \psi_n(x \mid \hat{\pi}_n)\, dx \\ & + \beta W_{n+1}(\pi^c_{n+1}(\cdot \mid z_n))[1 - \Psi_n(z_n \mid \hat{\pi}_n)] \bigg\}, \end{aligned} \tag{18}$$

for $n = 1, 2, \ldots, N$, with $W_{N+1}(\hat{\pi}_{N+1}) = 0$, for all $\hat{\pi}_{N+1}$, where the single-period expected profit is given by (after using the transformation $y_n = a r_n^{-b} z_n$)

$$\begin{aligned} & M(\hat{\pi}_n, z_n, r_n) \\ & = a r_n^{-b} \bigg[-c z_n + r_n \int_0^{z_n} x \psi_n(x \mid \hat{\pi}_n)\, dx \\ & \qquad + r_n z_n [1 - \Psi_n(z_n \mid \hat{\pi}_n)] + h \int_0^{z_n} (z_n - x) \psi_n(x \mid \hat{\pi}_n)\, dx \\ & \qquad - \hat{p} \int_{z_n}^{\infty} (x - z_n) \psi_n(x \mid \hat{\pi}_n)\, dx \bigg]. \end{aligned} \tag{19}$$

Denoting $\hat{S}_n = (S_1 + \sum_{i=1}^{n-1} \hat{s}_i^l)$, for the Weibull demand with prior $\hat{\pi}_n = \text{Gamma}(a_n, \hat{S}_n)$, using the variable transformation $z_n = \bar{S}_n \rho_n$, where $\bar{S}_n = \hat{S}_n^{1/l}$, we can write (19) as

$$\begin{aligned} & M(\hat{\pi}_n, \rho_n, r_n) \\ & \quad = \bar{S}_n a r_n^{-b} E_{\psi_n(\cdot \mid a_n, 1)}[-c\rho_n + r_n \xi_n I(\xi_n < \rho_n) + r_n \rho_n I(\xi_n \geq \rho_n) \\ & \qquad + h \max(0, \rho_n - \xi_n) - \hat{p} \max(0, \xi_n - \rho_n)]. \end{aligned} \tag{20}$$

Note that in the presence of revenue terms, the $\hat{p}$ in (19) or (20) represents the penalty for loss of goodwill; thus, $r_n + \hat{p}$ is analogous to the shortage penalty p in the problems considered previously.

With the above formulation, for Weibull and exponential models, we obtain the following:

Theorem 6. *If the demand distribution is Weibull or exponential* ($l=1$) *with an unknown parameter* θ *having a gamma prior with initial parameters* a_1 *and* S_1, *then for the multiplicative isoelastic model the following applies*:

(a) *For any given* $\rho > 0$, $r_{n,k}$ *is uniquely obtained from*

$$r(b-1)\bar{H}_{a_1+k}(\rho) - b[\hat{p}\mu_{a_1+k} + (c-h)\rho - (\hat{p}-h)\bar{H}_{a_1+k}(\rho)] = 0, \qquad (21)$$

and for any given $r > 0$, $\rho_{n,k}$ *is uniquely obtained from*

$$ar^{-b}\{-(c-h)(1+\rho^l)^{a_1+k-1/l+1} + (r+\hat{p}-h)(1+\rho^l)^{1-1/l}\} + l\rho^{l-1}\tilde{u}_{n,k} = 0, \qquad (22)$$

for $k = 0, 1, \ldots, n-1$, *where* $\tilde{u}_{n,k} = \beta[(a_1+k)u_{n+1,k+1} - (a_1+k-1/l)u_{n+1,k}]$ *and*, μ_{a_1+k} *and* $\bar{H}_{a_1+k}(\cdot)$ *are as defined in Theorem* 3. *Subsequently, the optimal price and order quantity in period* n, $n = 1, 2, \ldots, N$, *are given by*

$$r^*_{n,k} = \frac{b}{b-1}\left[c + \frac{\hat{p}\mu_{a_1+k} + (c-h)\rho_{n,k} - (c+\hat{p}-h)\bar{H}_{a_1+k}(\rho_{n,k})}{\bar{H}_{a_1+k}(\rho_{n,k})}\right],$$

$$\tilde{y}^*_{n,k} = a(r^*_{n,k})^{-b}z^*_{n,k} = \bar{S}_n a(r^*_{n,k})^{-b}\rho_{n,k}. \qquad (23)$$

(b) *The optimal profit in period* n, $n = 1, 2, \ldots, N$, *is given by*

$$W_{n,k}(a_n, \hat{S}_n) = \bar{S}_n u_{n,k},$$

where, for $k = 0, 1, 2, \ldots, n-1$, $u_{n,k}$ *is obtained from*

$$u_{n,k} = a(r^*_{n,k})^{-b}\{-\hat{p}\mu_{a_1+k} - (c-h)\rho_{n,k} + (r^*_{n,k} + \hat{p} - h)\bar{H}_{a_1+k}(\rho_{n,k})\} + \beta\left[u_{n+1,k}\eta_{n,k} + \frac{a_1+k}{a_1+k-1/l}u_{n+1,k+1}(1-\eta_{n,k})\right], \qquad (24)$$

with the terminal values $u_{N+1,k} = 0$, *for all* $k = 0, 1, 2, \ldots, N$, *where* $\eta_{n,k} = (1+\rho^l_{n,k})^{-(a_1+k-1/l)}$.

(c) *If the demand distribution is from the newsvendor family, then the optimal ordering decisions are scalable if and only if the demand distribution is Weibull. Moreover, the optimal pricing decisions depend on the shape parameter, but not on the scale parameter.*

From (22) we observe that each $\rho_{n,k}$ is obtained from an equation that involves the shape parameter $a_1 + k$ but not the scale parameter $\hat{S}_n$. Although (22) and (24) have structurally similar polynomial type recursive forms like (9) and (11), the difference is that they are now also influenced by the optimal pricing decisions. Because Theorem 6 is obtained for the profit maximization problem, by Lemma 1 the $\tilde{u}_{n,k}$ in (22) is nonnegative, as compared to the nonpositive $\tilde{v}_{n,k}$ in (9) for the cost minimization problem. Because (9) entails a search for just one decision $q_{n,k}$, in (22) the search for the analogous decision $\rho_{n,k}$ presupposes that the optimal $r_{n,k}$ is known. Because the revenue component in (18) is not concave and the cost component is not convex, the optimal $r_{n,k}$ requires a line search that can be executed effectively. The optimal solution is that pair of $(r_{n,k}, \rho_{n,k})$ that satisfies both (21) and (22) while yielding the highest profit using the easily computed function $W_{n,k}$. Theorem 6(c) shows that the scalability result of Theorem 1 extends to the multiplicative isoelastic price-dependent demand model with perishable inventory. Moreover, from (23) we can write $\tilde{y}^*_{n,k} = \sigma^*_{n,k}E(X_n)$, where $\sigma^*_{n,k} = a(r^*_{n,k})^{-b}(\rho_{n,k}/(a_1+k)B(a_1 + k - 1/l, 1 + 1/l))$. The implication is that, in an N-period problem, (for all nodes like those in Figure 2 in the online appendix) we can precompute the optimal prices $r^*_{n,k}$s and the factors $\sigma^*_{n,k}$s, thereby simplifying the computation of optimal solutions as we previously observed in Theorem 3.

For the case of storable inventory with multiplicative isoelastic demand function, results similar to Theorem 6(c) will hold for the optimal order-up-to levels and pricing decisions. However, because of nonconcavity of the profit function with Weibull demand, we can observe from the analysis of §3 that the optimal solutions will be tractable only for the exponential model. And in this case, because the results will be structurally similar to those in (23) and Theorem 2(ii), we omit their discussion.

We next consider the price-dependent additive demand model of the form $D_n(r_n, X_n) = a - br_n + X_n$, $a > 0$, $b > 0$. By analyzing a simple two-period problem with exponential demand and gamma prior, we show in the appendix that the structural results of Theorem 6 do not extend to the additive model. We show that, unlike the multiplicative model, for the additive case the optimal prices depend on both shape and scale parameters so that the optimal ordering decisions are not scalable. Consequently, precomputed simplification cannot be achieved because all optimal solutions now depend on sample observations.

Overall, our findings in this paper suggest that for the censored-data multiperiod inventory problem, the optimal solution that is scalable is unlikely to exist, except in special circumstances. And, additional structure, like that from precomputed factors, would be needed to exploit scalability to devise effective computational schemes. Thus, in the quest for methods that are computationally effective, researchers may want to devise scalable heuristics for this important but difficult problem.

Electronic Companion

An electronic companion to this paper is available on the *Manufacturing & Service Operations Management* website (http://msom.pubs.informs.org/ecompanion.html).

Acknowledgments

The authors sincerely thank the associate editor and the three anonymous referees for providing valuable comments and suggestions that improved the content and exposition of this paper. The research of the first author is partially supported by the Dauch Center for the Management of Manufacturing Enterprises and the Global Supply Chain Management Initiative of Purdue University.

References

Azoury, K. S. 1985. Bayes solution to dynamic inventory models under unknown demand distribution. *Management Sci.* **31**(9) 1150–1160.

Bensoussan, A., M. Çakanyıldırım, S. P. Sethi. 2007. A multiperiod newsvendor problem with partially observed demand. *Math. Oper. Res.* **32**(2) 322–344.

Bensoussan, A., M. Çakanyıldırım, S. P. Sethi. 2009a. A note on "The censored newsvendor and the optimal acquisition of information." *Oper. Res.* **57**(3) 791–794.

Bensoussan, A., M. Çakanyıldırım, S. P. Sethi. 2009b. Censored newsvendor model revisited with unnormalized probabilities. *J. Indust. Management Optim.* **5**(2) 391–402.

Bensoussan, A., M. Çakanyıldırım, A. Royal, S. P. Sethi. 2009c. Bayesian and adaptive controls for a newsvendor facing exponential demand. *J. Risk Decision Anal.* **1**(4) 197–210.

Bisi, A., M. Dada. 2007. Dynamic learning, pricing, and ordering by a censored newsvendor. *Naval Res. Logist.* **54**(4) 448–461.

Braden, D. J., M. Freimer. 1991. Informational dynamics of censored observations. *Management Sci.* **37**(11) 1390–1404.

Chen, L. 2010. Bounds and heuristics for optimal Bayesian inventory control with unobserved lost sales. *Oper. Res.* **58**(2) 396–413.

Chen, L., E. L. Plambeck. 2008. Dynamic inventory management with learning about the demand distribution and substitution probability. *Manufacturing Service Oper. Management* **10**(2) 236–256.

Ding, X., M. L. Puterman, A. Bisi. 2002. The censored newsvendor and the optimal acquisition of information. *Oper. Res.* **50**(3) 517–527.

Harpaz, G., W. Y. Lee, R. L. Winkler. 1982. Learning, experimentation, and the optimal output decisions of a competitive firm. *Management Sci.* **28**(6) 589–603.

Lariviere, M. A., E. L. Porteus. 1999. Stalking information: Bayesian inventory management with unobserved lost sales. *Management Sci.* **45**(3) 346–363.

Lu, X., J. S. Song, K. Zhu. 2006. Inventory control with unobservable lost sales and Bayesian updates. Working paper, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.

Lu, X., J. S. Song, K. Zhu. 2008. Analysis of perishable-inventory systems with censored demand data. *Oper. Res.* **56**(4) 1034–1038.

Monahan, G., N. Petruzzi, W. Zhao. 2004. The dynamic pricing problem from a newsvendor's perspective. *Manufacturing Service Oper. Management* **6**(1) 73–91.

Petruzzi, N. C., M. Dada. 1999. Pricing and the newsvendor problem: A review with extensions. *Oper. Res.* **47**(2) 183–194.

Petruzzi, N. C., M. Dada. 2001. Information and inventory recourse for a two-market, price-setting retailer. *Manufacturing Service Oper. Management* **3**(3) 242–263.

Petruzzi, N. C., M. Dada. 2002. Dynamic pricing and inventory control with learning. *Naval Res. Logist.* **49**(3) 303–325.

Scarf, H. 1959. Bayes solutions of the statistical inventory problem. *Ann. Math. Stat.* **30**(2) 490–508.

Scarf, H. 1960. Some remarks on Bayes solutions to the inventory problem. *Naval Res. Logist. Quart.* **7**(4) 591–596.

Veinott, A. F., Jr. 1965. Optimal policy for a multi-product, dynamic, nonstationary inventory problem. *Management Sci.* **12**(3) 206–222.