Coordinating Inventory Control and Pricing Strategies for Perishable Products

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Abstract: We analyze a joint pricing and inventory control problem for a perishable product with a fixed lifetime over a finite horizon. In each period, demand depends on the price of the current period plus an additive random term. Inventories can be intentionally disposed of and those that reach their lifetime have to be disposed of. The objective is to find a joint pricing, ordering, and disposal policy so as to maximize the total expected discounted profit over the planning horizon taking into account linear ordering cost, inventory holding and backlogging or lost-sales penalty cost and disposal cost. Employing the concept of $L^\natural$-concavity, we show some monotonicity properties of the optimal policies. Our results shed new light on perishable inventory management, and our approach provides a significantly simpler proof of a classical structural result in the literature. Moreover, we identify bounds on the optimal order-up-to levels and develop an effective heuristic policy. Numerical results show that our heuristic policy performs well in both stationary and non-stationary settings. Finally, we show that our approach also applies to models with random lifetimes and inventory rationing models with multiple demand classes.

Key words: Perishable inventory management, pricing strategies, $L^\natural$-convexity
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1. Introduction

The U.S. grocery industry is a very competitive market where more than two-thirds of industrial stores are supermarkets. The majority of sales revenue of grocery stores and supermarkets comes from perishables such as food items (e.g., meats and poultry, produce, dairy, and bakery products), pharmaceuticals (e.g., drugs and vitamins), and cut flowers. For instance, Food Market Institute

(2006) reports that perishables accounted for 50.12% of the total 2005 supermarket sales revenue of about $383 billion. The number was even higher in 2010 at 50.62% of $444 billion total sales revenue.

In grocery retailing, spoilage from perishables represents a major threat to the profitability of supermarkets. As quoted in a white paper of Power-ID, a radio frequency identification (RFID) technology company, "A survey by the National Supermarket Research Group found that a 300-store grocery chain loses about $34 million a year due to spoilage. On an industry-wide level, losses due to spoilage and shrinkage translate into $32 billion for chilled meats, seafood, and cheese; $34 billion for produce; and $5.4 billion for pharmaceutical and biomedical products (EPCGlobal)." [1] Thus, effective inventory management of perishables is crucial for the success in grocery retailing. The 2012 National Supermarket Shrink Survey reveals that "there exists a clear thread of practices associated with the control of inventory, inventory turnover and inventory management with resulting shrink loss levels. Top performing companies report having 26% lower inventory levels results in 30% more inventory turns and 15% lower shrink than companies without clear and consistently executed inventory control practices."[2]

Pricing is another important and effective lever to manage the profitability of perishables in retailing. Of course, it is a double-edged sword. A poor pricing strategy can also easily damage the profitability of a firm. For example, Tesco, the largest grocery retailer in the UK, failed to revive its sales despite spending £500 million on price cuts and as a result its CEO quit[3]. Compared to those simple pricing strategies that may change prices dramatically, it could be more appropriate to adjust the price in a dynamic fashion, which enables price changes according to the availability of inventory and products' residual shelf lives.

The advances of information technologies such as RFID tags allow retailers to accurately track product flows and make pricing and inventory decisions dynamically to better align supply with demand. The emergence of online grocery stores (e.g., AmazonFresh - a subsidiary of the Amazon.com) further facilitates the adoption of dynamic pricing strategies. A recent industry study sponsored by IBM (Webber et al. 2011) highlights that dynamic pricing can help retailers to effectively reduce food wastage by enabling the retailer to be more reactive to things like unexpected weather. This report also provides a case study which describes how a Dutch grocery retailer Albert Heijn experimented the integrated dynamic pricing and inventory control policies in one of its stores.

[1] http://www.power-id.com/Data/pdf/PowerTMP_White_Paper.pdf, accessed on Jan 20, 2012.

[2] http://www.retailcontrol.com/articles/time-to-get-your-stock-levels-in-line-and-overall-inventory-under-control/, accessed on 13 August, 2012

[3] http://www.bbc.co.uk/news/business-17378409, accessed on Aug 13, 2012

Similar issues are also faced by blood banks and pharmacies (Karaesmen et al. 2011, Pierskalla 2004). According to the 2009 national blood collection and utilization survey report, 4.7% of all components of blood processed for transfusion were outdated in 2008 in the United States. In particular, the outdated whole-blood-derived (WBD) platelets accounted for 24.4% of all WBD platelets processed in 2008 (AABB 2009). Karaesmen et al. (2011) point out that there is pressing need for research in the coordination of pricing and inventory management in blood supply chains, which further confirms the relevance of our work.

A key feature of perishable inventory systems is that a product has a finite shelf-life and hence the inventories of different ages for the same product may co-exist on the same shelf. Different from the durable inventory systems, the joint inventory and pricing strategies for a perishable product need to take into account the levels of the inventories of different ages and how inventories are issued.

The majority of the perishable inventory literature assumes first-in-first-out (FIFO) issuing policy (Nahmias 2011). This assumption is reasonable for blood inventory systems in which the blood banks have the luxury of determining inventory issuing policy. It also applies to some grocery retailers who only display the oldest items on the shelves to force customers to purchase oldest inventories first. Under the FIFO issuing policy, retailers often post a single price for all the units of a product at any point in time and may vary the price over time. It is natural for online grocery stores such as AmazonFresh to adopt this pricing strategy and inventory issuing policy. Intuitively, given the same total inventory level, the more aged inventory, the more likely the seller will set a lower price to turn the inventory over more quickly and reduce outdating (Nahmias 1982). Our model also assumes that the retailer can decide how to issue inventories and always post a single price at any point in time.

Nevertheless, some retailers would allow customers to choose items of different ages on the same shelves. In such a circumstance, if all the items are charged the same price, one can expect that customers will choose the freshest items, resulting last in first out (LIFO) sequence. In practice, retailers such as Bruegger's Bagels offer discounts for aged items while keeping regular prices for fresh ones. Some other retailers such as Chesapeake Bagel choose to dispose of old inventory as new inventory is available for sales. See Ferguson and Koenigsberg (2007) and Li et al. (2012) for more empirical evidences and discussions on different pricing and inventory control strategies.

In this paper, we analyze a joint inventory and pricing control problem for a retailer to manage a perishable product in a periodic-review inventory system. At the beginning of each period, the retailer decides how much to order and sets a single price for inventories of different ages. We assume that the retailer can decide how to issue inventories. At the end of each period, the retailer decides how much ending inventory to dispose of, including the inventory expiring in this period and possibly some of the inventory yet to expire. Demand in each period depends on the current price plus an additive random perturbation. The objective is to maximize the total expected discounted profit over the planning horizon taking into account linear ordering cost, inventory holding and backlogging or lost-sales penalty cost and disposal cost. We deal with both the backlogging and lost-sales cases and allow for positive replenishment lead time.

The problem is extremely complicated even when selling prices are fixed. Unlike standard inventory models with backlogging in which it suffices to use inventory position (total stock on hand and on order minus backorders) to describe the system state, one has to use a state vector to record the inventory levels of all ages and the outstanding orders in perishable inventory systems. Indeed, the structural analysis for perishable inventory models with zero lead time and exogenous demand in the literature has been long and intricate (see, e.g., Nahmias 1975 and Fries 1975). To highlight the complexity, note that "The main theorem requires 17 steps and is proven via a complex induction argument" (Nahmias 2011, page 10) and for models with discrete demand, a separate argument of using a sequence of continuous demand distributions to approximate the discrete demand distribution is needed (Nahmias and Schmidt 1986).

To deal with the complexity, we employ the concept of $L^\natural$-concavity to perform the structural analysis. Specifically, we prove that the optimal order quantity is nonincreasing in both outstanding and on-hand inventory levels and is most sensitive to the newly placed order and least sensitive to the oldest on-hand inventory with bounded sensitivity. On the contrary, the optimal price is most sensitive to the oldest on-hand inventory and least sensitive to the youngest order with bounded sensitivity.

Our analysis allows for both continuous and discrete decision variables and thus provides a unified approach for models with both continuous and discrete demand. Both backlogging and lost-sales cases are analyzed. In particular, in the lost-sales case, we propose a new regularity condition on demand models using the concept of $L^\natural$-concavity and identify sufficient conditions on demand models to ensure that the expected inventory-truncated revenue is $L^\natural$-concave, which in turn ensures the $L^\natural$-concavity of the optimal profit function. We further extend our analysis and results to the case with random lifetime and an inventory rationing model.

Note that the concept of $L^\natural$-convexity/concavity, developed by Murota (2003) in discrete convex analysis and first introduced into the inventory management literature by Lu and Song (2005), was used by Zipkin (2008) to establish the optimal structural policy of lost-sales inventory models with positive lead time. It was later extended by Huh and Janakiraman (2010) to serial inventory systems, and by Pang et al. (2012) to inventory-pricing models with positive lead time. Unlike these papers, the dynamics of the state variables in our perishable inventory system are much more complicated. Consequently, our analysis is significantly involved and requires the development of new preservation properties of $L^\natural$-concavity.

In addition to the structural analysis, we develop analytical bounds on the optimal order-up-to levels and propose a heuristic policy which applies to both stationary and non-stationary problems. Our numerical study consists of three parts. The first part compares the performance of the optimal policy and heuristic policies in the infinite-horizon setting. To assess the value of dynamic pricing and its role in reducing disposal wastes, we also compute the optimal fixed-price policy. The numerical results show that the heuristic policies perform well in both settings. To gain more insights, we examine the effects of several key modeling factors (which are product lifetime, variation of demand, unit disposal cost, and unit backlogging cost) on the optimal and heuristic policies and their performance, and the average disposal costs. In particular, we find that when the lifetime is longer, both the heuristic and the fixed-price policies perform better, the average disposal costs are lower, and the retailer tends to charge higher price and order more. The second part compares the performance of the optimal policy and heuristic policies in the finite-horizon setting with non-stationary demand. The observations are similar to those in infinite-horizon settings. But the value and the role of dynamic pricing with nonstationary demand are much more significant than those in the stationary setting. We also examine the effect of demand variability on the value of dynamic pricing. The third part examines the corresponding cost-minimization problems where the revenue (pricing) effects are removed. It appears that the heuristic policies perform better without the pricing decisions and revenue effects.

Contribution to Related Literature Our research is mostly related to two streams of literature: (1) Dynamic inventory control for perishable products, and (2) combined dynamic pricing and inventory control.

Dynamic inventory control for perishable products with fixed-lifetime was studied by Nahmias and Pierskalla (1973) in a two-period lifetime setting with zero lead time and demand uncertainty. Nahmias (1975) and Fries (1975) analyze the case with multi-period lifetime and zero lead time. They characterize the structure of the optimal policy and show that the optimal order quantity is decreasing in the levels of on-hand inventory of different ages and the sensitivity is bounded and monotone in the ages of on-hand inventory. Note that in their models only the excess inventory that expires at the end of the current period is disposed of. As we mentioned earlier, their analysis is lengthy and difficult to be generalized. Given the complexity, the literature thereafter focuses more on developing heuristics; see Nahmias (1982), Nahmias (2011) and Karaesmen et al. (2011) for excellent reviews of the early and recent developments. Recently, Xue et al. (2012) study a perishable inventory model with a secondary market where the excess inventory can be cleared with certain salvage value. They provide some structural properties and then propose a heuristic policy. Our paper generalizes this literature to allow for positive lead time and endogenous demand. As far as we know, it is the first attempt to perform the structural analysis for perishable inventory systems with positive lead time.

Our work is also closely related to the growing research stream on coordinated pricing and inventory management. Significant progress has been made in the past decade on nonperishable products (see, for example, Federgruen and Heching 1999; Chen and Simchi-Levi 2004a, 2004b, 2006; Huh and Janakiraman, 2008, Song et al. 2009). This literature demonstrates great benefits from pricing and inventory coordination and provides fundamental understanding of the structures of optimal policies. We refer to Yano and Gilbert (2003), Elmaghraby and Keskinocak (2003), Chan et al. (2004), and Chen and Simchi-Levi (2012) for some recent surveys.

Recognizing the importance of pricing decisions, Nahmias (1982) notes that determining pricing policies for perishable items under demand uncertainty was an open problem. Partially due to the complexity we discussed earlier, there are only a few papers that analyze perishable inventory models with joint pricing and inventory decisions (Karaesmen et al. 2011). Among them, Ferguson and Koenigsberg (2007) consider a two-period joint pricing and inventory control problem, addressing the impact of the competition between new inventory and leftover inventory in the second period on the first-period inventory and pricing decisions.

Li et al. (2009) consider a dynamic joint pricing and inventory control problem for a perishable product over an infinite horizon, assuming linear price-response demand model, backlogging and zero lead time. They characterize the structure of the optimal policy of a two-period lifetime problem and then develop a base-stock/list-price heuristic policy for stationary systems with multi-period lifetime. The infinite-horizon lost-sales case is analyzed in Li et al. (2012) in which the seller does not sell new and old inventory at the same time and at the end of a period the seller can decide whether to dispose of or carry all ending inventory until it expires. A replenishment decision is made at the beginning of a period if there is no inventory carried over to the current period. They propose a stationary structural policy consisting of an inventory order-up-to level, state-dependent price and inventory clearing decisions, and develop a fractional programming algorithm to compute the optimal policy amongst the class of proposed structural policies. It is not clear whether the proposed policy is optimal amongst the admissible policies. Our paper characterizes the structure of the optimal policy of perishable inventory systems, and is a significant generalization of these papers since we allow for positive lead time, arbitrary lifetime, both backlogging and lost-sales cases, and unrestricted ordering decisions.

In summary, our contribution to the literature is threefold. Firstly, we generalize the perishable inventory models to allowing coordinated pricing, inventory control and disposal decisions, and positive lead time. Secondly, we further develop the structural results of $L^\natural$-concavity which significantly simplify the structural analysis for perishable inventory systems. Thirdly, we generalize the regularity conditions of demand functions for lost-sales inventory-pricing models, using the concept of $L^\natural$-concavity.

Structure The remainder of this paper is organized as follows. In the next section, we summarize and develop some preliminary technical results. In Section 3, we describe the model and formulate the problem as a dynamic program. In Section 4, we perform the structural analysis for the backlogging case, followed by the lost-sales case in Section 5. In Section 6, we develop bounds on optimal replenishment decision, propose an effective heuristic policy, and perform a numerical study. In Section EC.2, we provide several important extensions.

Throughout this paper, we use decreasing, increasing, and monotonicity in a weak sense. Let $\Re$ denote the real numbers and $\Re^+$ the nonnegative reals, $\mathcal{Z}$ the integers, and $\mathcal{Z}_+$ nonnegative integers. In addition, we define $\bar{\Re} = \Re \cup \{\infty\}$, $\boldsymbol{e} \in \Re^n$ a vector whose components are all ones, and for $x, y \in \Re^n$, $x^+ = \max(x, 0)$, $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$ (all operations are taken componentwise).

2. Preliminaries

In this section, we summarize and develop some important technical results that will be used in our analysis. We first introduce the concept of $L^\natural$-convexity/concavity which can be defined on either real variables or integer variables. In the following, we use the notation $\mathcal{F}$ to denote either the real space $\Re$ or the set with all integers $\mathcal{Z}$, and notation $\mathcal{F}_+$ to denote the set of nonnegative elements in $\mathcal{F}$. Following Murota (2003,2009) and Simchi-Levi et al. (2014), $L^\natural$-convexity can be defined as follows.

Definition 1 ($L^\natural$-Convexity). A function $f : \mathcal{F}^n \to \bar{\Re}$ is $L^\natural$-convex if for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{F}$, $\alpha \in \mathcal{F}_+$

$$f(\boldsymbol{u}) + f(\boldsymbol{v}) \geq f((\boldsymbol{u} + \alpha \boldsymbol{e}) \wedge \boldsymbol{v}) + f(\boldsymbol{u} \vee (\boldsymbol{v} - \alpha \boldsymbol{e})),$$

where we note that $\boldsymbol{e} \in \Re^n$ is a vector whose components are all ones. A function f is $L^\natural$-concave if $-f$ is $L^\natural$-convex.

In the above definition, if $f(\boldsymbol{u}) = +\infty$ or $f(\boldsymbol{v}) = +\infty$, the inequality is assumed to hold automatically. Thus, for an $L^\natural$-convex function f, its effective domain $\mathcal{V} = \mathrm{dom}(f) = \{x \in \mathcal{F}^n | f(x) < +\infty\}$ is an $L^\natural$-convex set, i.e., it satisfies the following condition

$$\forall\ \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V} \text{ and } \alpha \in \mathcal{F}_+, (\boldsymbol{u} + \alpha \boldsymbol{e}) \wedge \boldsymbol{v} \in \mathcal{V} \text{ and } \boldsymbol{u} \vee (\boldsymbol{v} - \alpha \boldsymbol{e}) \in \mathcal{V}.$$

We sometimes say a function f is $L^\natural$-convex on a set $\mathcal{V}$ with the understanding that $\mathcal{V}$ is an $L^\natural$-convex set and the extension of f to the whole space by defining $f(\boldsymbol{v}) = +\infty$ for $v \notin \mathcal{V}$ is $L^\natural$-convex.

One can show that an $L^\natural$-convex function restricted to an $L^\natural$-convex set is also $L^\natural$-convex. In addition, the above definition is equivalent to saying that $g(\boldsymbol{v}, \xi) := f(\boldsymbol{v} - \xi \boldsymbol{e})$ is submodular in $(\boldsymbol{v}, \xi) \in \mathcal{F}^n \times \mathcal{S}$, where $\mathcal{S}$ is the intersection of $\mathcal{F}$ and any unbounded interval in $\Re$. We can also prove that the Hessian of a smooth $L^\natural$-convex function is diagonally dominant.

The next three lemmas are slight generalizations of those developed by Zipkin (2008).

Lemma 1. *If $f:\mathcal{F}^n \to \bar{\Re}$ is an $L^\natural$-convex function, then $g:\mathcal{F}^n\times\mathcal{F}\to\bar{\Re}$ defined by $g(x,\xi)=f(x-\xi e)$ is also $L^\natural$-convex.*

Lemma 2. *Assume that $\mathcal{A}$ is an $L^\natural$-convex set of $\mathcal{F}^n\times\mathcal{F}^m$ and $f(\cdot,\cdot):\mathcal{F}^n\times\mathcal{F}^m\to\bar{\Re}$ is an $L^\natural$-convex function. Then the function*

$$g(x)=\inf_{(x,y)\in\mathcal{A}} f(x,y)$$

is $L^\natural$-convex over $\mathcal{F}^n$ if $g(x)\neq-\infty$ for any $x\in\mathcal{F}^n$.

Lemma 3. *Let $g(x,\xi):\mathcal{F}^n\times\mathcal{F}\to\bar{\Re}$ be $L^\natural$-convex, and $\xi(x)$ be the largest optimal solution (assuming existence) of the optimization problem $f(x)=\min_{\xi\in\mathcal{F}} g(x,\xi)$ for any $x\in dom(f)$. Then $\xi(x)$ is nondecreasing in $x\in dom(f)$, but $\xi(x+\omega e)\le\xi(x)+\omega$ for any $\omega>0$ with $\omega\in\mathcal{F}$ and $\xi(x)+\omega\in dom(f)$.*

To address perishable inventory models, we need to develop some additional structural properties.

Let

$$\mathcal{V}_{n,k}=\{(s_1,\ldots,s_n)\in\mathcal{F}^n : s_1\le\ldots\le s_k\}$$

and

$$\mathcal{V}^+_{n,k}=\{(s_1,\ldots,s_n)\in\mathcal{F}^n : 0\le s_1\le\ldots\le s_k\}.$$

Note that both sets are $L^\natural$-convex.

The following lemma will be useful for the analysis of the backlogging case.

Lemma 4. *Assume that $f:\mathcal{F}^n\to\bar{\Re}$ is an $L^\natural$-convex function. If f is nondecreasing in its first k $(1\le k\le n)$ variables for $\boldsymbol{s}\in\mathcal{V}^+_{n,k}$, then the function*

$$\hat{f}(s_1,...,s_n,s_{n+1}):=f((s_1-s_{n+1})^+,...,(s_k-s_{n+1})^+,s_{k+1}-s_{n+1},...,s_n-s_{n+1})$$

is $L^\natural$-convex for $(s_1,...,s_n,s_{n+1})\in\mathcal{V}_{n+1,k}$.

Proof. Let

$$\tilde{f}(s_1,...,s_n):=f(s_1^+,...,s_k^+,s_{k+1},...,s_n).$$

Since f is nondecreasing in its first k $(1\le k\le n)$ variables for $\boldsymbol{s}\in\mathcal{V}^+_{n,k}$, we have that for any $\boldsymbol{s}\in\mathcal{V}_{n,k}$,

$$\tilde{f}(s_1,...,s_n)=\min_{v_i\ge s_i, i=1,\ldots,k, 0\le v_1\le\ldots\le v_k} f(v_1,\ldots,v_k,s_{k+1},\ldots,s_n).$$

In the above minimization problem, the set associated with the constraints is $L^\natural$-convex and the objective function is $L^\natural$-convex. Therefore, Lemma 2 implies that $\tilde{f}(s_1,...,s_n)$ is $L^\natural$-convex for $\boldsymbol{s}\in\mathcal{V}_{n,k}$. This, together with Lemma 1, implies that $\hat{f}(s_1,...,s_n,s_{n+1})=\tilde{f}(\boldsymbol{s}-s_{n+1}\boldsymbol{e})$ is $L^\natural$-convex for $(s_1,...,s_n,s_{n+1})\in\mathcal{V}_{n+1,k}$. Q.E.D.

The following lemma will be useful to address the lost-sales model with positive lead time.

Lemma 5. *Assume that $f: \mathcal{F}^n \to \bar{\Re}$ is an $L^\natural$-convex function. If $f(\boldsymbol{s})$ is nondecreasing in all variables for $\boldsymbol{s} \in \mathcal{V}^+_{n,n}$, then the function*

$$g(s_1,...,s_n,s_{n+1}) := f((s_1 - s_{n+1})^+, ..., (s_k - s_{n+1})^+, s_{k+1} - s_k \wedge s_{n+1}, ..., s_n - s_k \wedge s_{n+1})$$

is $L^\natural$-convex on the $L^\natural$-convex set $\mathcal{V}_{n+1,n}$.

Proof. Notice that

$$g(s_1,...,s_n,s_{n+1}) = \hat{f}(s_1,...,s_n,s_k \wedge s_{n+1}),$$

where $\tilde{f}$ is defined in Lemma 4. As we show in Lemma 4, $\tilde{f}$ is $L^\natural$-convex. Since $f(\boldsymbol{s})$ is nondecreasing in all variables, we have that $\hat{f}(s_1,...,s_n,s_{n+1})$ is nondecreasing in s_{n+1} and thus

$$g(s_1,...,s_n,s_{n+1}) = \min_{v \le s_k, v \le s_{n+1}} \hat{f}(s_1,...,s_n,v),$$

which is clearly $L^\natural$-convex on $\mathcal{V}_{n+1,n}$. Q.E.D.

3. The Model

Consider a periodic-review single-product inventory system over a finite horizon of T periods. The product is perishable and has a finite lifetime of exactly l periods. The replenishment lead time is k periods with $k < l$. In each period, a single price is charged for inventories of different ages, which are equally useful to fill consumer's price-sensitive demand. Demand is always met to the maximum extent with the on-hand inventory and we assume that unmet demand is either backlogged or lost. We assume that the retailer has the power or mechanism to determine how inventory is issued, and can also decide how much inventory to be carried over to the next period and how much inventory in addition to that at the end of its lifetime to be intentionally disposed of. The objective is to dynamically determine ordering, disposal and pricing decisions in all periods so as to maximize the total expected discounted profit over the planning horizon.

For convenience, we assume that the age of the inventory is counted from the period when the replenishment order is placed. If $l = 1, k = 0$, then the model reduces to a newsvendor model in the lost-sales case. If $l = \infty$, then it becomes a standard non-perishable inventory model. We assume for now that the costs and demand distributions are stationary. Nonstationary systems are discussed later.

The demand takes an additive form as is commonly used in the literature (see, e.g., Petruzzi and Dada 1999, Chen and Simchi-Levi 2004a,b). That is, the demand in period t is given as follows:

$$d_t := D(p) + \epsilon_t, \tag{1}$$

where $D(p)$ is the expected demand in period t and is strictly decreasing in the selling price p in this period, and ϵ_t is a random variable with zero mean. We assume that $\{\epsilon_t, t \ge 1\}$ are independently and identically distributed over time with a bounded support $[A,B]$, $(A \le 0 \le B)$. Let $F(\cdot)$ be the probability distribution function of ϵ_t. The selling price p is restricted to an interval $[\underline{p},\overline{p}]$. To ensure nonnegativity, we assume that $D(\overline{p})+A \ge 0$.

Note that the monotonicity of the expected demand function implies a one-to-one correspondence between the selling price p and the expected demand $d \in \mathcal{D} \equiv [\underline{d},\overline{d}]$, where $\underline{d}=D(\overline{p})$ and $\overline{d}=D(\underline{p})$. For convenience, we use the expected demand instead of the price as the decision variable in our analysis.

To provide a unified modeling framework for both backlogging and lost-sales cases, let $R(d,y)$ be the expected revenue for any given expected demand level d and on-hand inventory level y. In the backlogging case, we have $R(d,y)=P(d)d$, where $P(d)$ is the inverse function of $D(p)$. In the lost-sales case, we have $R(d,y)=P(d)E[\min(d+\epsilon_t,y)]$, where the sales are truncated by on-hand inventory level. We now introduce a unified regularity condition on the demand model for both the backlogging and lost-sales cases.

Assumption 1. *$R(d,y)$ is continuous and $L^\natural$-concave in $(d,y) \in \mathcal{D} \times \Re_+$.*

In the backlogging case, Assumption 1 is equivalent to the requirement that the expected revenue $P(d)d$ is concave in d, i.e., the expected revenue has a decreasing margin with respect to the expected demand level. This concavity assumption is commonly seen in the pricing literature (see, e.g., Chen and Simchi-Levi 2004a,b). However, in the lost-sales case, the concavity of the unconstrained revenue $P(d)d$ cannot even guarantee the joint concavity of the inventory-truncated revenue $R(d,y)$. In this case, Assumption 1 implies that the marginal value of the expected revenue is decreasing not only in the demand level but also in the on-hand inventory level. In addition, the higher the inventory level, the higher the marginal revenue of increasing demand level. In other words, the demand and on-hand inventory are complementary to each other. In fact, Assumption 1 requires stronger conditions, which will be discussed in more details in Section 5. Nevertheless, this condition generalizes the conventional concavity assumption on the expected revenue in the pricing literature.

The sequence of events in period t is as follows.

1. At the beginning of the period, the order placed k periods ago is received (if $k \ge 1$) and the inventory levels of different residual useful lifetimes are observed.
2. Based on the inventory levels of different residual useful lifetimes, an order is placed and will be delivered at the beginning of period $t+k$. When $k=0$, the order is delivered immediately. At the same time, the selling price p_t of period t is determined.
3. During period t, demand d_t arrives, which is stochastic and depends on the selling price p_t, and is satisfied by on-hand inventory.
4. Unsatisfied demand is either backlogged or lost and the remaining inventory with zero useful lifetime has to be discarded. Meanwhile, unused inventory with positive useful lifetimes can

be either intentionally discarded or carried over to the next period. In the latter case, their lifetimes decrease by one.

Each order incurs a variable cost c. Inventory carried over from one period to the next incurs a holding cost of h^+ per unit, and demand that is not satisfied from on-hand inventory incurs a cost of h^- per unit which represents the backlogging cost or the lost-sales penalty cost. Inventory that is disposed of incurs a disposal cost of θ per unit. Let $\gamma \in [0,1]$ be the discount factor.

The system state after receiving the order placed k periods ago but before placing an order can be represented by an $(l-1)$-dimensional vector $\boldsymbol{s} = [s_1, ..., s_{l-1}]$.

For the backlogging case, when $s_i \geq 0$, s_i represents the level of on-hand inventory with residual lifetime no more than i periods. When $s_i < 0$, $-s_i$ represents the shortfall of inventory with residual lifetime no more than i periods, defined as the additional units that should have been ordered $l-i$ periods ago to make s_i zero. In particular, s_{l-k} is the net inventory level and s_{l-1} is the inventory position of the system. Later on we will see that for $i < l-k$, the exact value of s_i will not affect our optimization model if $s_i < 0$; however, the way we specify $\widetilde{\boldsymbol{s}}$ is convenient for our analysis. The state variables satisfy the condition that $s_1 \leq s_2 \leq ... \leq s_{l-1}$ and the set of feasible states is given by

$$\mathcal{F}_b = \mathcal{V}_{l-1,l-1}.$$

Alternatively, the system state can also be represented by the vector $\boldsymbol{x} = [x_1, ..., x_{l-1}]$ where $x_1 = s_1$, $x_i = s_i - s_{i-1}$ for $i = 2, ..., l-1$. Here, x_1 represents the level for the on-hand inventory with residual lifetime no longer than one period (if $x_1 > 0$) or the shortfall for the inventory with residual lifetime no longer than one period (if $x_1 < 0$), and x_i represents the size of the order placed $l-i$ periods ago, $i = 2, ..., l-1$.

For the lost-sales case, s_i is the total amount of inventory with residual lifetimes no more than i periods, $i = 1, ..., l-1$. In particular, s_{l-1} is the system inventory position and s_{l-k} is the net on-hand inventory level. The state variables satisfy the condition that $0 \leq s_1 \leq s_2 \leq ... \leq s_{l-1}$ and the set of feasible states is given by

$$\mathcal{F}_l = \mathcal{V}^+_{l-1,l-1}.$$

In the literature, it is common to denote the system state by an $(l-1)$-dimensional vector $\boldsymbol{x} = [x_1, ..., x_{l-1}]$ where x_i is the amount of inventory on hand (for $i \leq l-k$ if $k \geq 1$ and $i \leq l-1$ if $k = 0$) or on order (for $i > l-k$ if $k \geq 1$) of age $l-i$. Clearly,

$$s_1 = x_1, s_2 = s_1 + x_2, ..., s_{l-1} = s_{l-1-1} + x_{l-1}.$$

It is straightforward to check that the feasible sets $\mathcal{F}_l$ and $\mathcal{F}_b$ are both $L^\natural$-convex. Hence, although both $\boldsymbol{s}$ and $\boldsymbol{x}$ can represent the system state, it is more convenient to use $\boldsymbol{s}$ to perform the structural analysis.

Let s_l be the order-up-to level and a be the amount of on-hand inventory to be depleted or the realized demand of period t, whichever is greater. We have that

$$s_1 \vee d_t \leq a \leq s_{l-k} \vee d_t.$$

When demand is greater than the total on-hand inventory level s_{l-k}, the above inequalities imply that $a = d_t$. Otherwise, they imply that $s_1 \vee d_t \leq a \leq s_{l-k}$, which in turn implies that (1) the firm satisfies the demand to the maximum extent, and (2) in addition to the inventory at the end of life, the firm could also intentionally dispose of some more inventory that will expire in later periods. We also note that since on-hand inventory can be intentionally disposed of, it is not difficult to show that it is optimal to deplete on-hand inventory sequentially with increasing useful lifetimes, i.e., FIFO depletion policy is optimal.

We next derive the system state at the beginning of the next period before ordering, denoted by $\widetilde{\boldsymbol{s}}$. We first consider the backlogging case. The dynamics of the state are expressed by

$$\widetilde{\boldsymbol{s}} = [s_2 - a, ..., s_{l-k} - a, s_{l-k+1} - a, ..., s_l - a]. \tag{2}$$

Note that s_{l-k} is the total on hand inventory if $s_{l-k} \geq 0$ and the amount of backorders if $s_{l-k} < 0$. When $s_{l-k} \geq d_t$, a is the amount of on-hand inventory that is depleted. When $s_{l-k} < d_t$, $a = d_t$.

For the lost-sales case, if $a \leq s_{l-k}$, then $\tilde{s}_i = (s_{i+1} - a)^+$ for $i = 1, \ldots, l-k-1$, and $\tilde{s}_j = s_{j+1} - a$ for $j = l-k, ..., l-1$; if $a \geq s_{l-k}$, then $\tilde{s}_i = 0$ for $i = 1, \ldots, l-k-1$, and $\tilde{s}_j = s_{j+1} - s_{l-k}$ for $j = l-k, ..., l-1$. Combining the two cases, the dynamics of system state for the lost-sales case evolve as follows:

$$\widetilde{\boldsymbol{s}} = [(s_2 - a)^+, ..., (s_{l-k} - a)^+, s_{l-k+1} - s_{l-k} \wedge a, ..., s_l - s_{l-k} \wedge a]. \tag{3}$$

We are now ready to present the unified model formulation for both the backlogging and lost-sales cases. Let $\hat{f}_t(\boldsymbol{s})$ be the profit-to-go function when the system state is specified by $\boldsymbol{s} \in \mathcal{S}$ ($= \mathcal{F}_b$ or $\mathcal{F}_l$ in the backlogging case and the lost sales case respectively) at the beginning of period t before ordering. We can write the optimality equation as follows:

$$\hat{f}_t(\boldsymbol{s}) = \max_{s_l \geq s_{l-1}, d \in \mathcal{D}} \left\{ R(d, s_{l-k}) + \mathbb{E}[\hat{g}_t(\boldsymbol{s}, s_l, d|\epsilon_t)] \right\},$$

where

$$\hat{g}_t(\boldsymbol{s}, s_l, d|\epsilon_t) = \max_{s_1 \vee d_t \leq a \leq s_{l-k} \vee d_t} \{-c(s_l - s_{l-1}) - \theta(a - d_t) - h^+(s_{l-k} - a)^+ - h^-(a - s_{l-k})^+ + \gamma \hat{f}_{t+1}(\widetilde{\boldsymbol{s}})\}.$$

Here the four terms in the maximization problem defining $\hat{g}_t$ represent the ordering cost, disposal cost, inventory holding cost, and backlogging cost or lost-sales penalty cost, respectively. Also recall that $d_t = d + \epsilon_t$. For simplicity, we assume that $\hat{f}_{T+1}(\boldsymbol{s}) = cs_{l-1}$, i.e., inventory (or unfilled orders) at the end of the planning horizon is salvaged (or filled) with a unit price (or cost) equal to the unit ordering cost. From the above formulation, one can see that for $i<l-k$, the exact value of s_i does not really matter for the optimization model if $s_i<0$. That is, $\hat f_t(s_1,...,s_{l-k-1},s_{l-k}...,s_{l-1})=\hat f_t(s_1^+,...,s_{l-k-1}^+,s_{l-k},...,s_{l-1})$.

It is more convenient to work with a slightly modified profit-to-go function. Define for $\boldsymbol{s}\notin\mathcal{S}$, $f_t(\boldsymbol{s})=+\infty$ and for $\boldsymbol{s}\in\mathcal{S}$, $f_t(\boldsymbol{s})=\hat f_t(\boldsymbol{s})-cs_{l-1}$ for all t. Then for $\boldsymbol{s}\in\mathcal{S}$, $f_{T+1}(\boldsymbol{s})=0$ and the optimality equation can be rewritten as follows:

$$f_t(\boldsymbol{s})=\max_{s_l\geq s_{l-1},d\in\mathcal{D}_t} G_t(\boldsymbol{s},s_l,d) \tag{4}$$

with

$$G_t(\boldsymbol{s},s_l,d)\,=\,R(d,s_{l-k})+\mathbb{E}\left[g_t(\boldsymbol{s},s_l,d|\epsilon_t)\right],$$

where

$$g_t(\boldsymbol{s},s_l,d|\epsilon_t)\,=\max_{s_1\vee d_t\leq a\leq s_{l-k}\vee d_t}\{\phi_t(\boldsymbol{s},s_l,d,a|\epsilon_t)\}, \tag{5}$$

$$\phi_t(\boldsymbol{s},s_l,d,a|\epsilon_t)\,=\gamma c\tilde s_{l-1}-cs_l-\theta(a-d_t)-h^+(s_{l-k}-a)^+-h^-(a-s_{l-k})^++\gamma f_{t+1}(\widetilde{\boldsymbol{s}}). \tag{6}$$

Denote by $s_{lt}(\boldsymbol{s}),d_t(\boldsymbol{s})$ the optimal order-up-to inventory position and demand level decisions and $a_t(\boldsymbol{s},s_l,d)$ the optimal inventory depletion solution for any given $(\boldsymbol{s},s_l,d)$.

The following monotonicity property applies to both the backlogging and lost sales cases.

Lemma 6. *For $t=1,...,T+1$ and any $\boldsymbol{s}\in\mathcal{S}$, $f_t(\mathbf{s})$ is nonincreasing in $s_1,...,s_{l-k-1}$, respectively.*

Proof. By induction. The statement is clearly true for $t=T+1$. Assume that it holds for $t+1$. Then, $\gamma f_{t+1}(\widetilde{\boldsymbol{s}})$ is nonincreasing in $s_2,...,s_{l-k-1}$ and so are $\phi_t(\boldsymbol{s},s_l,d,a|\epsilon_t)$ and $g_t(\boldsymbol{s},s_l,d_t|\epsilon_t)$. Note that the feasible set of a subject to the constraint $s_1\vee d_t\leq a\leq s_{l-k}\vee d_t$ becomes smaller as s_1 increases, which implies that $g_t(\boldsymbol{s},s_l,d|\epsilon_t)$ is nonincreasing in s_1. It is obvious that the monotonicity of $g_t(\boldsymbol{s},s_l,d|\epsilon_t)$ is preserved under the expectation over ϵ_t and the maximization operations in (4). Thus, the desired result holds. Q.E.D.

Remark 1. FIFO issuing rule is typically assumed in the literature without allowing disposals of inventory that is not outdated. For example, Nahmias (2011) argues that FIFO issuing rule is most cost-efficient and results in minimum outdating. However, we find that FIFO issuing rule may not always be optimal if inventory cannot be disposed of intentionally. To see this, consider the classic perishable inventory model in Nahmias (1975). Assume that the lifetime is 3, the replenishment lead time is zero, the discount factor is one, and the demand is one with probability q and zero with probability $1-q$. Consider a two-period planning horizon with initial state $\boldsymbol{s}=(1,2)$. Suppose the realized demand in period 1 is one. According to the FIFO rule, the old inventory is used to meet the demand and the fresh inventory is carried over to the next period, yielding the total expected cost $h^++(1-q)\theta$. Now consider a policy under which the demand is met with the fresh inventory and the old inventory is disposed of. Then under this policy the expected cost is $\theta + q(h^- + c)$ which is strictly smaller than $h^+ + (1-q)\theta$ if $h^+ > q(\theta + h^- + c)$. Therefore, when unit holding cost is sufficiently large, FIFO rule may not be optimal. It is appropriate to point out that when there is no holding cost, the firm has no incentive to intentionally dispose of any fresh inventory until it expires and FIFO rule is indeed optimal.

Remark 2 (Secondary Market). In practice, there may exist a secondary market where the firm can clear some inventories that have not expired with some salvage value (see, e.g., Xue et al. 2012). Our analysis can be readily extended to address this situation. Let π_s be the salvage value per unit of inventory. Assume that the firm decides how much inventory to dispose of and how much inventory to clear in the salvage market at the end of each period if there is excess on-hand inventory. Clearly, the firm will only dispose of the expired inventory. Using our notations, the amount of disposals is $(s_1 - d_t)^+$ and the amount of clearing inventory is $a - d_t - (s_1 - d_t)^+$. The term $-\theta(a-d_t)^+$ in (6) is then replaced by $\pi_s(a-d_t) - (\theta + \pi_s)(s_1 - d_t)^+$. The subsequent analysis applies immediately.

4. The Backlogging Case

In this section, we analyze the backlogging case. We first perform the structural analysis, and then present conditions under which it is optimal to dispose of to the minimum amount or the maximum amount at the end of each period.

Without loss of generality, we assume that $c \le h^-/(1-\gamma)$, which implies that it is cheaper to purchase a unit now than to carry this backorder and purchase a unit in the next period while experiencing a backorder. This eliminates the speculative motive for intentionally carrying the backorders. Recall that in the backlogging case we have $R(d, s_{l-k}) = P(d)d$ and

$$\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t) = -(1-\gamma)cs_l - \gamma ca - \theta(a - d_t) - h^+(s_{l-k} - a)^+ - h^-(a - s_{l-k})^+ + \gamma f_{t+1}(\widetilde{\boldsymbol{s}}).$$

We now employ the properties of $L^\natural$-concavity to characterize the structure of the optimal policy.

Theorem 1 (Monotonicity Properties of Backlog Model). *For $t = 1, ..., T+1$, the functions $f_t(\boldsymbol{s})$, $g_t(\boldsymbol{s}, s_l, d|\epsilon_t)$ and $\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t)$ are $L^\natural$-concave in $\boldsymbol{s}$, $(\boldsymbol{s}, s_l, d)$ and $(\boldsymbol{s}, s_l, d, a)$, respectively. Thus, the optimal order-up-to level $s_{lt}(\boldsymbol{s})$ and the optimal demand level $d_t(\boldsymbol{s})$ are nondecreasing in $\boldsymbol{s}$ (i.e., the optimal price $p_t(\boldsymbol{s})$ is nonincreasing in $\boldsymbol{s}$), and for any $\omega \ge 0$*

$$s_{lt}(\boldsymbol{s} + \omega \boldsymbol{e}) \le s_{lt}(\boldsymbol{s}) + \omega, \text{ and } d_t(\boldsymbol{s} + \omega \boldsymbol{e}) \le d_t(\boldsymbol{s}) + \omega. \tag{7}$$

Given the realized demand d_t, the optimal depletion decision $a_t(\boldsymbol{s}, s_l, d|\epsilon_t)$ is nondecreasing in $(\boldsymbol{s}, s_l, d)$ and for any $\omega \ge 0$

$$a_t(\boldsymbol{s} + \omega \boldsymbol{e}, s_l + \omega, d + \omega|\epsilon_t) \le a_t(\boldsymbol{s}, s_l, d|\epsilon_t) + \omega. \tag{8}$$

Proof. It can be shown by induction that $L^\natural$-concavity is preserved in the dynamic programming recursion. The statement is obviously true for $t = T+1$. Assume that it holds for $t+1$.

We first show that $\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t)$ is $L^\natural$-concave in $(\boldsymbol{s}, s_l, d, a)$. The system dynamics (2) together with Lemma 1 immediately imply that $f_{t+1}(\tilde{\boldsymbol{s}})$ is $L^\natural$-concave for $(\boldsymbol{s}, s_l, a) \in \mathcal{V}_{l+1,l}$. The other terms in $\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t)$ are all $L^\natural$-concave in their variables. Thus, $\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t)$ is $L^\natural$-concave for$(\boldsymbol{s}, s_l, d, a) \in \mathcal{V}_{l+2,l}$.

We next show that $g_t(\boldsymbol{s}, s_l, d|\epsilon_t)$ is $L^\natural$-concave in $(\boldsymbol{s}, s_l, d) \in \mathcal{V}_{l+1,l}$. The idea is to use Lemma 2 to show that $L^\natural$-concavity can be preserved under the optimization problem (5). Unfortunately, the constraint in (5), $s_1 \vee d_t \le a \le s_{l-k} \vee d_t$, is not $L^\natural$-convex. Interestingly, we can prove that this constraint $a \le s_{l-k} \vee d_t$ can be removed, i.e., $g_t(\boldsymbol{s}, s_l, d|\epsilon_t) = \tilde{g}_t(\boldsymbol{s}, s_l, d|\epsilon_t)$, where

$$\tilde{g}_t(\boldsymbol{s}, s_l, d|\epsilon_t) = \max_{a \ge s_1 \vee d_t} \{\phi_t(\boldsymbol{s}, s_l, a, d|\epsilon_t)\}.$$

Although $a > s_{l-k} \vee d_t$ does not have any physical meaning, it is well defined mathematically. We show $\phi_t(\boldsymbol{s}, s_l, a, d)$ is decreasing in a for $a \ge s_{l-k} \vee d_t$.

Suppose that $a' > a \ge s_{l-k} \vee d_t$. Let $\delta = a' - a$. At the beginning of period $t+1$, we observe that the state $\widetilde{\boldsymbol{s}} = (s_2 - a, ..., s_l - a)$ and the state $\widetilde{\boldsymbol{s}}' = (s_2 - a', ..., s_l - a')$ have the same on-order inventory profiles while the later has δ more units of backorders. We now consider two systems starting with states $\widetilde{\boldsymbol{s}}$ and $\widetilde{\boldsymbol{s}}'$ at the beginning of period $t+1$, referred to systems 1 and 2 respectively. For system 1, construct a policy that mimics exactly the optimal policy of system 2 except that the first δ units of deliveries to system 1 are intentionally disposed of (instead of being used to satisfy the backorders). Then, the two systems end up with the same state. Clearly, under the constructed policy, the additional cost incurred in system 1 is no more than $\theta\delta$. Due to the suboptimality of the constructed policy, we have

$$\hat{f}_{t+1}(\widetilde{\boldsymbol{s}}') - \theta\delta \le \hat{f}_{t+1}(\widetilde{\boldsymbol{s}}), \text{or, equivalently, } f_{t+1}(\widetilde{\boldsymbol{s}}') - f_{t+1}(\widetilde{\boldsymbol{s}}) \le (\theta + c)\delta.$$

Here we can actually replace the right hand side with a smaller number when we also take into account the difference of backorder costs. However, the above inequality suffices for our purpose and implies that

$$\begin{aligned}\phi_t(\boldsymbol{s}, s_l, a', d|\epsilon_t) - \phi_t(\boldsymbol{s}, s_l, a, d|\epsilon_t) &= -(\gamma c + \theta + h^-)\delta + \gamma[f_{t+1}(\widetilde{\boldsymbol{s}}') - f_{t+1}(\widetilde{\boldsymbol{s}})] \\ &\le -(\gamma c + \theta + h^-)\delta + \gamma(\theta + c)\delta \\ &< 0. \end{aligned} \tag{9}$$

That is, $\phi_t(\boldsymbol{s}, s_l, a, d)$ is strictly decreasing in a as $a \ge s_{l-k} \vee d_t$. Thus, it is never optimal to set $a > s_{l-k} \vee d_t$ and hence $g_t(\boldsymbol{s}, s_l, d|\epsilon_t) = \tilde{g}_t(\boldsymbol{s}, s_l, d|\epsilon_t)$.

It is easy to check that the set $\{(s_1, d, a) : a \ge s_1 \vee d_t\} = \{(s_1, d, a) : a \ge s_1, a \ge d_t\}$ is $L^\natural$-convex. From Lemma 2, we have $\tilde{g}_t(\boldsymbol{s}, s_l, d|\epsilon_t)$ and hence $g_t(\boldsymbol{s}, s_l, d|\epsilon_t)$ are $L^\natural$-concave in $(\boldsymbol{s}, s_l, d)$. Since $L^\natural$-concavity is preserved under expectation and a single variable function is $L^\natural$-concave, the objective function in the optimization problem (4) is $L^\natural$-concave. This, together with $L^\natural$-convexity of the set associated with the constraints in the optimization problem (4) and Lemma 2, implies that $f_t(\boldsymbol{s})$ is $L^\natural$-concave in $\boldsymbol{s}$.

By Lemma 3, we know that the optimal order-up-to level $s_{lt}(\boldsymbol{s})$ and the optimal demand level $d_t(\boldsymbol{s})$ are nondecreasing in $\boldsymbol{s}$ and the inequalities in (7) hold (the existence of optimal solutions is straightforward to check and is thus omitted). The monotonicity of $P(d)$ implies that the optimal price $p_t(\boldsymbol{s})$ is nonincreasing in $\boldsymbol{s}$. The desired results hold. Q.E.D.

The inequalities in (7) imply that the optimal pricing and ordering decisions have bounded sensitivity. That is, a unit increase in some or all of the state variables will increase the order-up-to inventory position level $s_{lt}(\boldsymbol{s})$ and the optimal demand level $d_t(\boldsymbol{s})$ by at most one unit. The rates of the increase in decision variables are slower than that of the increase in state variables. These inequalities also provide insight into how the freshness of the inventory affects the inventory and pricing decisions. Comparing the states $\boldsymbol{s}$ and $\boldsymbol{s}+\boldsymbol{e}_i$, the latter has one more unit of inventory with residual lifetime of i periods but one less unit of inventory with residual lifetime of $i+1$ periods, $i=1,...,l-2$. These monotonicity properties imply that the fresher the inventory in the system, the less inventory to order and the higher price to charge.

The inequalities of (8) imply that the optimal depletion decisions also have bounded sensitivity. Furthermore, since $s_{lt}(\boldsymbol{s})$ and $d_t(\boldsymbol{s})$ are increasing in $\boldsymbol{s}$, the optimal depletion decision under the optimal policy, $a_t(\boldsymbol{s}, s_{lt}(\boldsymbol{s}), d_t(\boldsymbol{s})|\epsilon_t)$, is increasing in $\boldsymbol{s}$ and satisfies $a_t(\boldsymbol{s}+\omega\boldsymbol{e}, s_{lt}(\boldsymbol{s}+\omega\boldsymbol{e}), d_t(\boldsymbol{s}+\omega\boldsymbol{e})|\epsilon_t) \le a_t(\boldsymbol{s}, s_{lt}(\boldsymbol{s}+\omega\boldsymbol{e}), d_t(\boldsymbol{s}+\omega\boldsymbol{e})|\epsilon_t)+\omega$ for any ϵ_t and $\omega>0$. That is, the higher the total on-hand inventory level or the more aged the inventory in the system, the more inventory to deplete and to dispose of.

We now translate the structural properties of the optimal decisions with respect to $\boldsymbol{s}$ back to that with respect to $\boldsymbol{x}$. Let $\hat{x}_{lt}(\boldsymbol{x})$ and $\hat{d}_t(\boldsymbol{x})$ be the optimal order quantity and demand level with respect to $\boldsymbol{x}$, respectively. The following corollary is implied by Theorem 1. The proof is identical to Theorem 2 of Pang et al. (2012) and it is therefore skipped.

Corollary 1 (Monotone Sensitivity). *For $t=1,...,T+1$ and for any $\omega \ge 0$, the following inequalities hold:*

$$-\omega \le \hat{x}_{lt}(\boldsymbol{x}+\omega\boldsymbol{e}_{l-1}) - \hat{x}_{lt}(\boldsymbol{x}) \le ... \le \hat{x}_{lt}(\boldsymbol{x}+\omega\boldsymbol{e}_1) - \hat{x}_{lt}(\boldsymbol{x}) \le 0, \tag{10}$$

$$0 \le \hat{d}_{lt}(\boldsymbol{x}+\omega\boldsymbol{e}_{l-1}) - \hat{d}_{lt}(\boldsymbol{x}) \le ... \le \hat{d}_{lt}(\boldsymbol{x}+\omega\boldsymbol{e}_1) - \hat{d}_{lt}(\boldsymbol{x}) \le \omega. \tag{11}$$

Corollary 1 reveals that the optimal order quantity and demand level have bounded and monotone sensitivity. In particular, the inequalities in (10) imply that the optimal order quantity decreases in the inventory level of each age and the sensitivity decreases in age. That is, it is more sensitive to the younger inventory or outstanding order and least sensitive to the oldest order. The inequalities in (11) show that the optimal demand level increases in the level of inventory of each age and the sensitivity increases in age as well. In particular, it is most sensitive to the inventory close to expiration. That is, the more inventory to expire the more discount the seller should offer to induce more sales and avoid the disposals.

Remark 3 (Non-stationary Backlog Systems). So far we have restricted our attention to the stationary systems. In fact, when the system is non-stationary and the system parameters are time varying, indexed with the subscript t, under a very mild condition that $\gamma c_t + \theta_t + h_t^- \geq \gamma(c_{t+1} + \theta_{t+1})$, we still have that $\phi_t(\boldsymbol{s}, s_l, a', d) < \phi_t(\boldsymbol{s}, s_l, a, d)$ for $a' > a \geq s_{l-k} \vee d_t$. Therefore, the above analysis and results hold.

The previous analysis addresses the general case where the disposal decision is endogenous, i.e, the seller can dispose of any on hand inventory which is yet to expire. However, the majority of perishable inventory models assume that only the excess inventory expiring in the current period is disposed of. In other words, the inventory is disposed of to the minimum extent. In the following, we identify two sufficient conditions under which it is optimal to dispose of the inventory to the minimum extent or to the maximum extent.

The following proposition shows that when the disposal cost is sufficiently high, it is optimal to deplete/dispose of to the minimum extent.

Proposition 1 (Minimum Disposal). *Suppose that $\theta \geq h^+/(1-\gamma)$. For all t, the optimal depletion decision is always equal to $d_t \vee s_1$.*

Proof. The proposition is straightforward since the disposal cost is greater than the net present value of holding the unit forever, i.e., $h^+/(1-\gamma) = \sum_{i=1}^{\infty} \gamma^{i-1} h^+$. Therefore the seller has no incentive to intentionally dispose of the inventory before it expires. Q.E.D.

The next proposition shows that when the disposal cost is sufficiently low, it is optimal to deplete/dispose of to the maximum extent.

Proposition 2 (Maximum Disposal). *Suppose the replenishment lead time is zero, i.e., $k = 0$. If $\gamma c + \theta \leq h^+$, then for all t, the optimal depletion decision is always equal to $d_t \vee s_l$.*

Proof. Recall that when lead time is zero, $f_t(\boldsymbol{s})$ is nonincreasing in $\boldsymbol{s}$, which implies that $f_{t+1}(\tilde{\boldsymbol{s}})$ is nondecreasing in a. It is straightforward to verify that under the condition $\gamma c + \theta \leq h^+$, $\phi_t(\boldsymbol{s}, s_l, a, d|\epsilon_t)$ is increasing in a when $d_t < s_l$ and $d_t \vee s_1 \leq a \leq s_l$. The desired result holds. Q.E.D.

This proposition states that it is economical to deplete all the on-hand inventory (i.e., dispose of all the excess inventory) instead of leaving it until it expires when holding one unit inventory costs more than disposing it now and ordering one new unit in the next period.

5. The Lost-Sales Case

This section addresses the lost-sales case. For simplicity, we restrict our attention to the setting with zero lead time ($k=0$). The extension to the positive lead time, which requires some stronger conditions, will be presented in the online appendix EC.2.

When unmet demand is lost, the sales are truncated by the on-hand inventory. Recall that in the lost-sales case $R(d,s_l)=P(d)E[\min(d+\epsilon_t,s_l)]$ and

$$\begin{aligned}\phi_t(\boldsymbol{s},s_l,d,a|\epsilon_t) &= -(1-\gamma)cs_l-\gamma ca-\theta(a-d_t)-h^+(s_l-a)^+-(h^--\gamma c)(a-s_l)^++\gamma f_{t+1}(\widetilde{\boldsymbol{s}}),\\ &= -cs_l+h^-(s_l-a)-(h^++h^--\gamma c)(s_l-a)^+-\theta(a-d_t)+\gamma f_{t+1}(\widetilde{\boldsymbol{s}}),\end{aligned}$$

where

$$\widetilde{\boldsymbol{s}}=[(s_2-a)^+,...,(s_{l-1}-a)^+,(s_l-a)^+].$$

We assume that $h^++h^-\geq\gamma c$ which implies that the cost of carrying a unit of inventory to the next period while facing lost sales is larger than the potential salvage value of the inventory.

To analyze the structure of the optimal policy, we need the concavity property of the revenue function $R(d,y)$. Due to the lost sales, the sales are truncated by the on-hand inventory level. Thus, the concavity of the unconstrained revenue in the backlogging model cannot ensure the joint concavity of $R(d,y)$ in the lost-sales case. A stronger condition is required to ensure the joint concavity.

Recently, Kocabiyikoglu and Popescu (2011) introduced the concept of *lost-sales rate (LSR) elasticity*, defined as $\varrho(d,y)=-\frac{P(d)}{P'(d)}\frac{F'(y-d)}{\bar F(y-d)}$, to measure the relative sensitivity of lost-sales probability ($Pr(d+\epsilon_t\geq y)=\bar F(y-d)$) with respect to inventory level y and price p, where $\bar F(\cdot)=1-F(\cdot)$. In the newsvendor model, they show that when the unconstrained revenue $pd(p,\epsilon_t)$ is concave in price and the lost-sales rate elasticity is greater than one, then the expected truncated revenue is jointly concave and submodular in price and inventory decisions. However, Pang (2011) points out that Kocabiyikoglu and Popescu's (2011) analysis cannot be readily extended to the dynamic setting and it is more convenient to work on the inverse demand function using the expected demand level as the decision variable. He identifies two sufficient conditions which ensure that $R(d,s_l)$ is jointly concave and supermodular in (d,s_l):

(C1) $P''(d)d+P'(d)\leq 0$ for all $d\in\mathcal{D}$; and

(C2) $\varrho_t(d,y)\geq 1$ for all $d\in\mathcal{D}$ and $y\geq 0$.

Condition (C1) is slightly stronger than concavity. Table 1 below summarizes some common demand models. One can observe that, except iso-elasticity demand, all the others satisfy (C1).

Condition (C2) was first developed by Kocabiyikoglu and Popescu (2011). Note that $\varrho(d,y)=d\times\frac{-P(d)}{dP'(d)}\times\frac{F'(y-d)}{\bar F(y-d)}$ where $\frac{-P(d)}{dP'(d)}$ is the price elasticity of demand and $\frac{F'(y-d)}{\bar F(y-d)}$ is the hazard rate of the

Table 1 Demand models ($a>0, b>0$)

Demand model	$d_t(p)$	$P_t(d)$	$P''(d)d+P'(d)$	$P''(d)d+2P'(d)$
linear	$a-bp$	$(a-d)/b$	$-1/b$	$-2/b$
log	$\ln(a-bp)$	$\frac{a-e^d}{b}$	$-\frac{1+d}{b}e^d$	$-\frac{2+d}{b}e^d$
logit	$\frac{e^{a-bp}}{1+e^{a-bp}}$	$\frac{a+\ln(1/d-1)}{b}$	$-\frac{1}{b(1-d)^2}$	$-\frac{2-d}{b(1-d)^2}-\frac{1}{bd}$
exponential	e^{a-bp}	$\frac{a-\ln(d)}{b}$	0	$-\frac{1}{bd}$
iso-elasticity	ap^{-b}	$(d/a)^{-1/b}$	$a^{1/b}(1/b)^2 d^{-1-1/b}$	$a^{1/b}\frac{1-b}{b^2}d^{-1-1/b}$

random shock. Thus, (C2) essentially requires that price elasticity and hazard rate are sufficiently large for any given $d \in \mathcal{D}$. In particular, if $\underline{d} \geq 1$, and both price elasticity and hazard rate are greater than one, then (C2) holds. We refer to Kocabiyikoglu and Popescu (2011) for a detailed justification for the bounded LSR elasticity.

The following proposition shows that conditions (C1) and (C2) are sufficient for Assumption 1. The proof is provided in the online appendix EC.1.

Proposition 3. *Suppose (C1) and (C2) hold. For $t=1,\cdots,T$, $R(d,y)$ is $L^\natural$-concave in (d,y).*

Now we are ready to characterize the structure of the optimal policy for the lost-sales model.

Theorem 2 (Monotonicity Properties of Lost-sales Model). *Suppose that (C1) and (C2) hold and the replenishment lead time is zero ($k=0$). For $t=1,...,T$, the functions $f_t(\boldsymbol{s})$, $g_t(\boldsymbol{s},s_l,d)$ and $\phi_t(\boldsymbol{s},s_l,d,a|\epsilon_t)$ are $L^\natural$-concave in $\boldsymbol{s}$, $(\boldsymbol{s},s_l,d)$ and $(\boldsymbol{s},s_l,d,a)$, respectively. The joint pricing, inventory replenishment and depletion policy has the same monotonicity properties as those shown in Theorem 1.*

Proof. First, applying Lemma 6, we know that $f_t(\boldsymbol{s})$ is nonincreasing in $\boldsymbol{s}$ for all t. Then, by Lemma 4, the $L^\natural$-concavity of f_{t+1} implies that $f_{t+1}(\tilde{\boldsymbol{s}})$ is $L^\natural$-concave in $(\boldsymbol{s},s_l,d,a)$. Furthermore, by Proposition 3, $R(d,s_l)$ is $L^\natural$-concave in (d,s_l) under conditions (C1) and (C2). Following the induction arguments of the proof of Theorem 1, the desired results hold. Q.E.D.

Lemma 2 shows that under conditions (C1) and (C2) the lost-sales model could have the same optimal policy structure as that of the backlogging model. Correspondingly, the monotone sensitivity property of the optimal policy as that characterized in Corollary 1 also holds in the lost-sales case.

Note that it is not difficult to show that the results of Propositions 1 and 2 hold for the lost-sales case under the same conditions. That is, when the disposal cost is sufficiently high such that $\theta \geq \frac{h^+}{1-\gamma}$, it is optimal to deplete to the minimum extent; and when the lead time is zero, the depletion cost is sufficiently low and the holding cost is sufficiently high such that $\gamma c+\theta<h^+$, it is optimal to deplete all the excess on-hand inventory.

Remark 4 (Lost-sales Models with Positive Lead time). Our analysis can be further extended to the case with positive lead time using the preservation property provided by Lemma 4 or Lemma 5. To this end, some monotone structural properties are required. We identify a sufficient condition which helps us construct a monotone function to prove the preservation of $L^\natural$-concavity and the corresponding monotonicity properties of the optimal policies using Lemma 4. The detailed analysis is presented in the last section of the online appendix EC.2.

Remark 5 (Depleting Decision Made Before Demand Is Realized). So far we assume that the depleting decision is made after demand is realized. It may also be interesting to consider the case that the depleting decision is made before demand is realized which at least can provide a lower bound to the profit function of the lost-sales model studied above. Let a be the amount of inventory to be depleted that is determined jointly with pricing and replenishment decisions at the beginning of a period. Assume that $s_1 \le a \le s_{l-k}$. If a is greater than the demand, $a - d_t$ units of inventory are disposed of. Otherwise, $d_t - a$ units of demand are lost. The amount of inventory to be carried over to the next period is $s_{l-k} - a$. Then, the recursive optimality equation can be written as

$$\begin{aligned}\tilde{f}_t(\mathbf{s}) = \max_{s_l \ge s_{l-1}, d \in \mathcal{D}_t, s_1 \le a \le s_{l-k}} &\{R(d,a) - h^+(s_{l-k} - a) - h^- E[(d_t - a)^+] - \theta E[(a - d_t)^+] \\ &-c(s_l - s_{l-1}) + \gamma \tilde{f}_{t+1}[(s_2 - a)^+, (s_3 - a)^+, ..., (s_{l-k} - a)^+, s_{l-k+1} - a, ..., s_l - a]\}.\end{aligned}$$

Using the following facts: (1) $R(d,a)$ is $L^\natural$-concave under conditions (C1) and (C2), (2) all the other cost-related terms are $L^\natural$-concave, (3) if $\tilde{f}_{t+1}$ is $L^\natural$-concave and decreasing in $s_1, ..., s_{l-k-1}$, then the last term is also $L^\natural$-concave, and (4) the constraint set forms an $L^\natural$-convex set, we can show that $\tilde{f}_t(\mathbf{s})$ is $L^\natural$-concave and decreasing in $s_1, ..., s_{l-k-1}$, and the optimal policy structure is similar to that characterized by Theorem 1.

6. Bounds, Heuristics and Numerical Study

One of the most significant features of $L^\natural$-convexity is that it ensures the local optimum to be the global optimum and a steepest descent-type polynomial-time algorithm can be used to find the optimal solution (Murota 2005). Nevertheless, the computation for the perishable model still suffers from the curse of dimensionality. In this section, we first derive analytical bounds on the optimal inventory and demand level decisions and propose a heuristic to the perishable inventory management problem. We then perform a numerical study to assess the performance of the heuristic policies against the optimal policy. We restrict our discussions to the case with zero lead time.

6.1. Bounds

We first find bounds on the optimal order-up-to levels. For simplicity, we assume that $\theta \ge h^+/(1-\gamma)$ which ensures that the inventory is always depleted to the minimum extent. This is in line with the perishable inventory literature and it allows us to focus on the replenishment and pricing decisions.

When $a_t = d_t \vee s_1$, we have

$$\begin{aligned}\phi_t(\boldsymbol{s}, s_l, d_t \vee s_1, d|\epsilon_t) &= \gamma c\tilde{s}_{l-1} - cs_l - \theta(d_t \vee s_1 - d_t) - h^+(s_l - d_t \vee s_1)^+ - h^-(d_t \vee s_1 - s_l)^+ + \gamma f_{t+1}(\widetilde{\boldsymbol{s}})\\ &= v_t(s_l, d_t) - \tilde{\theta}(s_1 - d_t)^+ + \gamma f_{t+1}(\widetilde{\boldsymbol{s}}),\end{aligned}$$

where for the backlogging case

$$v_t(s_l, d_t) = -(1-\gamma)cs_l - \gamma c d_t - h^+(s_l - d_t)^+ - h^-(d_t - s_l)^+,$$

and for the lost-sales case

$$v_t(s_l, d_t) = -cs_l - (h^+ - \gamma c)(s_l - d_t)^+ - h^-(d_t - s_l)^+,$$

with $\tilde{\theta} = \theta + \gamma c - h^+$.

Define

$$\overline{S}_t(d) = \arg\max_{s_l \geq 0}\{\Pi_t(s_l, d) = R(d, s_l) + E[v_t(s_l, d + \epsilon_t)]\},$$

and

$$\begin{aligned}\underline{S}_t(d) = \arg\max_{s_l \geq 0}\{&\tilde{\Pi}_t(s_l, d) = R(d, s_l) + E[v_t(s_l, d + \epsilon_t)] - \gamma c\tilde{s}_{l-1}\\ &-(\gamma h^+ + \gamma^2 h^+ + ... + \gamma^{l-2}h^+ + \gamma^{l-1}\tilde{\theta})E[(s_l - d_t)^+]\}.\end{aligned}$$

Note that $\overline{S}_t(d)$ represents a myopic solution considering the expected disposal cost to the minimum extent, whereas $\underline{S}_t(d)$ is a myopic solution assuming that all the excess inventory is intentionally held until it expires l periods later. The next theorem shows that $\overline{S}_t(d)$ and $\underline{S}_t(d)$ provide upper and lower bounds on the inventory decisions for any given d. Note that $\overline{S}_t(d)$ is the ordering decision without taking into account the possible disposal and holding costs associated with the ordered inventory, which encourages ordering more, while $\underline{S}_t(d)$ is obtained by charging the maximum possible holding and disposal costs, which overly penalizes the ordering decision and hence leads to a lower order quantity.

THEOREM 3 (BOUNDS). *Suppose that the starting inventory level of the planning horizon is zero. For all t, for any given demand level d,*

$$\underline{S}_t(d) \vee s_{l-1} \leq s_{lt}(\boldsymbol{s}, d) \leq \overline{S}_t(d) \vee s_{l-1}.$$

Proof. We first prove the upper bound. To this end, consider two inventory decisions $s'_l > s_l$. Let $\tilde{\boldsymbol{s}}$ and $\tilde{\boldsymbol{s}}'$ be the corresponding starting states of period $t+1$. Note that $G_t(\boldsymbol{s}, s_l, d) = \Pi_t(s_l, d) + \gamma E[f_{t+1}(\tilde{\boldsymbol{s}})]$. The monotonicity of f_{t+1} implies that

$$G_t(\boldsymbol{s}, s'_l, d) - G_t(\boldsymbol{s}, s_l, d) \leq \Pi_t(s'_l, d) - \Pi_t(s_l, d).$$

Thus, $\Pi_t(s_l, d)$ increases in s_l when $G_t(\boldsymbol{s}, s_l, d)$ increases in s_l. By the concavity of $\Pi_t(s_l, d)$ and $G_t(\boldsymbol{s}, s_l, d)$ in s_l, we know that $s_{lt}(\boldsymbol{s}, d) \le \bar{S}_t(d) \vee s_{l-1}$.

We next turn to the lower bound. Consider two inventory decisions $s_l' > s_l$. Let $\tilde{\boldsymbol{s}}$ and $\tilde{\boldsymbol{s}}'$ be the corresponding starting states of period $t+1$. At the beginning of period $t+1$, the state $\tilde{\boldsymbol{s}}'$ has $\delta = \tilde{s}_{l-1}' - \tilde{s}_{l-1}$ more units of inventory of age 1 (or fewer units of backorders) than $\tilde{\boldsymbol{s}}$ does. Construct a policy (for state $\boldsymbol{s}'$) such that the δ units of excessive inventory (if exist) are intentionally held until they expire at the end of period $t+l-1$, incurring holding costs in periods $t+1, ..., t+l-2$ and disposal cost in period $t+l-1$, while the rest of the system is operated as if the starting state of period $t+1$ were $\tilde{\boldsymbol{s}}$. The total profit incurred by the policy from period t+1 to the end of the planning horizon is exactly $\hat{f}_{t+1}(\tilde{s}) - (h^+ + \gamma h^+ + ... + \gamma^{t+l-3}h^+ + \gamma^{t+1-2}\tilde{\theta})\delta$. Thus

$$\hat{f}_{t+1}(\tilde{\boldsymbol{s}}) - (h^+ + \gamma h^+ + ... + \gamma^{t+l-3}h^+ + \gamma^{t+1-2}\tilde{\theta})\delta \le \hat{f}_{t+1}(\tilde{\boldsymbol{s}}'),$$

which implies that

$$f_{t+1}(\tilde{\boldsymbol{s}}) + c\tilde{s}_{l-1} - (h^+ + \gamma h^+ + ... + \gamma^{t+l-3}h^+ + \gamma^{t+1-2}\tilde{\theta})\delta \le f_{t+1}(\tilde{\boldsymbol{s}}') + c\tilde{s}_{l-1}'.$$

Therefore, we have

$$\begin{aligned} & G_t(\boldsymbol{s}, s_l', d) - G_t(\boldsymbol{s}, s_l, d) \\ \ge\ & \Pi_t(s_l', d) - \Pi_t(s_l, d) - \gamma c(\tilde{s}_{l-1}' - \tilde{s}_{l-1}) + \gamma[-(h^+ + \gamma h^+ + ... + \gamma^{t+l-3}h^+ + \gamma^{t+1-2}\tilde{\theta})\delta] \\ =\ & \tilde{\Pi}_t(s_l', d) - \tilde{\Pi}_t(s_l, d), \end{aligned}$$

which implies that $G_t(\boldsymbol{s}, s_l, d)$ is increasing in s_l as $\tilde{\Pi}_t(s_l, d)$ increases in s_l. Thus, $s_{lt}(\boldsymbol{s}, d) \ge \underline{S}_t(d) \vee s_{l-1}$. The desired results hold. Q.E.D.

Remark 6. The preceding analysis is based on stationary model parameters and independent and identically distributed demands. Thus, the bounds are in fact independent of time. When the system is non-stationary, we still have $\underline{S}_t(d) \vee s_{l-1} \le s_{lt}(\boldsymbol{s}, d) \le \overline{S}_t(d) \vee s_{l-1}$, where $\underline{S}_t$ and $\overline{S}_t$ are both time-dependent.

6.2. Heuristics

We now develop a one-dimensional approximation where the expected disposal cost associated with each order is discounted to the period the order is placed. Since the future demand depends on the future prices, we need to approximate the future demand levels or pricing decisions.

Federgruen and Heching (2002) point out that the optimal price path can be closely approximated by the optimal price path under deterministic models. Using this observation, we approximate the expected demand levels (and the corresponding prices) in the future periods with the optimal demand levels derived from the corresponding deterministic models, which are also called the optimal risk-less prices. That is, in any period t, the demand levels of next $l-1$ periods are approximated by $\bar{d}_{t+i}, i=1,...,l-1$ where

$$\bar{d}_{t+i} = \arg\max_{d\in\mathcal{D}}\{(P_{t+i}(d)-c)d\}.$$

The corresponding demands are therefore approximated by $\bar{d}_{t+i}+\epsilon_{t+i}$, $i=1,...,l-1$

Let x and y be the inventory levels before and after ordering. Define

$$B_t(y,d) = E\left[\left(y-d-\bar{d}_{t+1}-...-\bar{d}_{t+l-1}-\sum_{i=0}^{l-1}\epsilon_{t+i}\right)^+\right].$$

Given the demand approximation developed above, the expected amount of inventory to be disposed of at the end of period $t+l-1$, denoted by $O_t(x,y,d)$ satisfies

$$B_t(y-x,d) \le O_t(x,y,d) \le B_t(y,d) - B_t(x,d) \le B_t(y,d).$$

The expected disposal can be approximated by its upper bound $B_t(y,d)$. That is, all the inventory after ordering is treated as new inventory that can serve the demand in the next $l-1$ periods. Then, the optimality equations for the approximate model can be expressed by the following: Starting from $v_{T+1}(x)=0$,

$$v_t(x) = \max_{y\ge x, d\in\mathcal{D}} \tilde{G}_t(y,d) \tag{12}$$

where

$$\tilde{G}_t(y,d) = \Pi_t(y,d) - \gamma^{l-1}\tilde{\theta}B_t(y,d) + \gamma E[v_{t+1}(\tilde{y}_t)].$$

For the backlogging case, $\tilde{y}_t = y-d-\epsilon_t$. For the lost sales case, $\tilde{y}=(y-d-\epsilon_t)^+$

Using the standard induction argument and applying Assumption 1, one can easily show that $\tilde{G}_t(y,d)$ is jointly concave and supermodular in (y,d) and $\tilde{f}_t$ preserves the nonincreasing and concave properties. Therefore, the base-stock/list-price policy is optimal.

When the system is stationary, one may expect that the optimal price is stable as well. Thus, we can use the current demand level to approximate the future demand levels, i.e., $d_{t+i}=d+\epsilon_t, i=1,...,l-1$ where d is the current demand level (see, e.g., Federgruen and Heching 1999). Then, expected inventory disposal can be replaced by

$$B_t(y,d) = E\left[\left(y-l\times d-\sum_{i=0}^{l-1}\epsilon_{t+i}\right)^+\right].$$

Using this approximation, one can show that a myopic base-stock list-price policy is optimal for the stationary approximate model and the optimal solution is obtained by solving the following problem:

$$\max_{y\ge x, d\in\mathcal{D}}\left\{\Pi_t(y,d) - \gamma^{l-1}\tilde{\theta}B_t(y,d)\right\}. \tag{13}$$

For convenience, we call the above heuristic policy H1. It is easy to show that for any given d, the optimal order-up-to level, denoted by $y^{H1}(d)$, is in $[\underline{S}_t, \overline{S}_t]$.

Li et al. (2009) follow Nahmias (1976) to approximate the expected disposal with a tighter upper bound: $B_t(y,d) - B_t(x,d)$. They suggest a myopic stationary policy by solving

$$\max_{y \geq x, d \in \mathcal{D}} \left\{ \Pi_t(y,d) - \gamma^{l-1}\tilde{\theta}[B_t(y,d) - E[B_t(y-d-\epsilon_t,d)]] \right\}. \tag{14}$$

We call this heuristic policy $H2$. It is easy to show that for any given d the optimal order-up-to level of H2, denoted by $y^{H2}(d)$, is greater than $y^{H1}(d)$. It is notable that if we use $B_t(y,d) - B_t(x,d)$ to approximate the expected disposal (to replace $B_t(y,d)$ in (12) by $B_t(y,d) - B_t(x,d)$) then the base-stock list-price policy is not optimal (since the optimal order up to level must be dependent on the inventory level). This is why we choose to approximate the disposal cost by $B_t(y,d)$.

6.3. Numerical Study

Our numerical study aims to evaluate the performance of the heuristics in comparison with that of optimal policies and assess the value of dynamic pricing in both infinite-horizon and finite-horizon settings. The study is restricted to the backlogging cases with zero lead time. It is notable that although the preceding analysis is focused on finite-horizon settings, the results can be readily extended to infinite-horizon settings.

In the following numerical study, we first consider an infinite-horizon setting under the long-run average profit criterion. Compared to the total expected discounted profit criterion, the long-run average profit criterion provides a single performance indicator (i.e., long-run average profit) which is independent of the initial system states and is therefore commonly used in the literature to compare the performance of different policies. We then consider a finite-horizon setting with both stationary and nonstationary demands. To compare the performance of different policies, the performance indicator is chosen as the total expected discounted profits over the planning horizon with a common starting state at zero. Finally, we examine the performance of the heuristics for cost-minimization problems under the long-run average cost criterion. For simplicity, we require that only the expired inventory is disposed of under different policies. Note that such a treatment is optimal under the discounted profit criterion when $\theta \geq h^+/(1-\gamma)$. Under the long-run average profit (or cost) criterion, although it may be optimal to dispose of some inventory that is yet to expire at some states, our numerical study shows that the impact of such a restriction on the long-run average profits (or costs) is negligible for the instances we analyze.

6.3.1. Infinite-Horizon Setting We first compare the performance of different policies under the infinite-horizon setting with the long-run average profit criterion.

Following Li et al. (2009), we adopt the test case parameters from Federgruen and Heching (1999). The demand function is specified as an additive linear model $d_t = \alpha - \beta p + \epsilon_t$, where $p \in [\underline{p}, \overline{p}]$, and

ϵ_t follows a truncated normal distribution with zero mean and satisfies the nonnegativity condition $\alpha - \beta\overline{p} + \epsilon_t \geq 0$. The expected demand level is chosen from $[\underline{d}, \overline{d}]$, where $\underline{d} = \alpha - \beta\overline{p}$ and $\overline{d} = \alpha - \beta\underline{p}$. Note that an up-tail truncated normal distribution has a positive mean $\sigma\frac{\phi(A/\sigma)}{1-\Phi(A/\sigma)}$ for any given truncating point A, where Φ and ϕ are standard normal distribution and density functions. For the normally distributed random variable Z with zero mean and standard deviation σ, we find the smallest point $a \geq -\underline{d}$ such that $A - \sigma\frac{\phi(A/\sigma)}{1-\Phi(A/\sigma)} \geq -\underline{d}$. Let $\epsilon_t = \{Z|Z \geq A\} - \sigma\frac{\phi(A/\sigma)}{1-\Phi(A/\sigma)}$, where $\{Z|Z \geq A\}$ represents the up-tail truncation of Z from A. Clearly, ϵ_t has a zero mean and satisfies the inequality $\epsilon_t \geq -\underline{d}$, which ensures the nonnegativity condition of the demand.

The system parameters are specified as follows. Three lifetimes are considered: $l = 2, 3, 4$. For any l, we first set the base case as

$$[\alpha, \beta, c.v., \theta, c, h^+, h^-, \underline{p}, \overline{p}] = [174, 3, 1, 10, 22.15, 0.22, 10.78, 25, 44].$$

We then vary the parameters $c.v.$, h^- and θ, respectively, such that $c.v. \in \{0.6, 0.8, 1.0, 1.2, 1.5\}$, $h^- \in \{1.98, 4.18, 10.78, 21.78\}$ (correspondingly, $h^-/(h^- + h^+) \in \{90\%, 95\%, 98\%, 99\%\}$), and $\theta \in \{5, 10, 20\}$. In total, 33 instances are reported.

For each instance, we compute the optimal policy, the fixed-price policy and the heuristic policies H1 and H2, and the corresponding long-run average profits. In particular, when computing the fixed-price policy, we first compute the inventory policy for each given expected demand level d, and then find the optimal demand level that leads to the maximum average profit.

The system state is discretized with step size 1. We apply the standard value iteration approach to compute the long-run average profit per period and the optimal policies. The value iteration algorithm is terminated when three-digit accuracy is obtained. See Bertsekas (1995) for the detailed introduction to the value iteration approach under the long-run average cost or reward criterion. When computing the optimal policies, for each iteration, given any state, we conduct linear search for the optimal order-up-to level and the demand level. It is worthwhile to mention that the properties of $L^\natural$-concavity allow us to further reduce the search space. In particular, $L^\natural$-concavity ensures that the local optimum is globally optimal and the steepest accent method can be used (instead of searching the whole decision space). The monotonicity properties of the state-dependent policies ensure that the optimal solutions of smaller state can serve as lower bounds of the solutions of larger states, which helps further reduce the computational effort.

Let d^{FP} be the optimal expected demand level under fixe-price policy, and (s_l^{Hi}, d^{Hi}) be the optimal myopic base stock and demand levels under the heuristic policy Hi, $i = 1, 2$. The performance of different policies is measured by the percentage profit loss $\rho^u = \frac{V^* - V^u}{V^*} \times 100\%, u \in \{\text{FP, H1, H2}\}$, where V^*, V^{FP}, V^{H1} and V^{H2} are the long-run average profits under the optimal policy, fixed-price policy, H1 and H2 heuristics, respectively. To gain insight into the role of dynamic pricing in reducing the wastes arising from the disposals, we also compute the average disposal costs per period under different policies, denoted by $DC^u, u \in \{*, \mathrm{FP}, \mathrm{H1}, \mathrm{H2}\}$. The percentage of the average disposal cost in the average profit under policy u is denoted by $\delta^u = \frac{DC^u}{V^u} \times 100\%$.

Table EC.1 below reports the performance of the optimal, fixed-price and heuristic policies, and the corresponding static policy parameters for the 33 instances, indexed from 1 to 33. Table EC.2 reports the corresponding disposal costs and their percentage ratios in profits. For ease of reading, we use '-' to represent the same value of the corresponding parameter in the base case. The observations are summarized as follows.

- Table EC.1 shows that the long-run average profit under the optimal policy decreases in the coefficient of variation, the unit backlogging cost, and the unit disposal cost, respectively, while it increases in the product lifetime. The percentage profit losses of the fixed-price policy and the heuristics have the same monotone patterns as the optimal policy, which implies that dynamic pricing policy becomes more advantageous to the fixed-price policy and the heuristics when the coefficient of variation, the unit backlogging, or the unit disposal cost becomes larger. In the extreme case when the lifetime is infinite, our model reduces to the durable inventory model and the heuristic policies become the optimal base-stock list-price policies (see, e.g., Federgruen and Heching 1999). A natural conjecture arising from these observations is that when the lifetime becomes sufficiently long, the performance of fixed-price and heuristic policies will converge to that of the optimal policy.

- The heuristic policies H1 and H2 perform pretty well against the optimal policy. The percentage profit losses of most of the instances are below 1%, which is in line with the observations of Li et al. (2009). In particular, when the lifetime is 4 periods and the coefficient of variation is 0.6, the percentage profit losses of heuristics are only 0.01%. Comparing the two heuristics, the average profits and the optimal policy parameters (when taking integer values) are the same for most instances with only a few exceptions (e.g., instances 3, 8, 9) in which H2 performs slightly better than H1.

- The expected demand levels under the fixed-price policy or the heuristics tend to be smaller as the lifetime increases. On the other hand, the order-up-to levels under H1 and H2 policies tend to be larger as the lifetime increases. This implies that the retailer should offer lower prices and order less when the product lifetime is shorter. That is, perishability reduces profit margins and service levels.

- Table EC.2 compares the average disposals under different policies. One can observe that both the average disposal cost and the percentage of the disposal cost in profit under the optimal policy are lower than that under the fixed-price policy, which implies that dynamic pricing indeed reduces the disposal wastes. The heuristic policies yield lower average disposal costs than those of the optimal policy and fixed-price policy for most of the instances. For all the policies, it appears that both the average disposal cost and the percentage of the disposal cost in profit increase in the coefficient of variation, the backlogging cost, and the unit disposal cost. But when the lifetime increases the average disposal cost decreases. This is because the likelihood to dispose of the inventory becomes smaller when the product lifetime becomes longer.

- It is also worthwhile to mention that when the lifetime becomes longer the time to compute the optimal policy increases exponentially. For example, for the base cases with $l = 2, 3, 4$, the computing times for the optimal policies are 5 seconds, 283 seconds and 28 minutes, respectively, using Matlab 7.0 on a personal computer with an Intel Core Duo 3.00GHz CPU.

In summary, these numerical results show that in the infinite-horizon setting, both the heuristics perform very well. The errors of the approximations are ignorable in comparison with the possible errors in calibrating the price-responsive demand functions.

6.3.2. Finite-Horizon Setting We next examine the performance of the heuristic policy H1 in a finite-horizon setting with nonstationary demand. Note that under H1 the base-stock/list-price policy is optimal and the optimal policy parameters can be obtained by solving the optimality equations (12).

Specifically, we consider a five-period horizon, $T = 5$. The time-varying demand functions are specified as $d_t = \kappa_t\alpha - \beta p, t = 1, ..., 5$, where κ_t is a time-varying seasonality factor such that $\kappa_t = 1.2 - 0.1 \times (t-1)$, i.e., the market size falls over time from 1.2α to 0.8α with a slope of 0.1α. This slope indicates how fast the demand falls over time. The smaller the slope, the more stable the demand process. The average market size per period is α.

With all the other parameters being equal to those in the infinite-horizon setting, we vary $c.v.$, h^- and θ, respectively. As the discounted profits are state-dependent, for the purpose of comparison, we assume that the decision criterion is to maximize the total discounted profit over the planning horizon with the initial inventory level being zero, denoted by the l-dimensional vector $\mathbf{0}$. The discount factor is $\gamma = 0.95$. Clearly, for all the tested instances, we have $\theta > h^+/(1-\gamma)$, which ensures that it is optimal to only dispose of the expired inventory at the end of each period.

Similar to the infinite-horizon setting, the performance indicators of the heuristic policies can be defined as the percentage profit losses at the common starting state $\mathbf{0}$,

$$\rho^u = \frac{\hat{f}_t(\mathbf{0}) - \hat{f}_t^u(\mathbf{0})}{\hat{f}_t(\mathbf{0})} \times 100\%, u \in \{FP, H1\},$$

where $\hat{f}_t$, $\hat{f}_t^{FP}$ and $\hat{f}_t^{H1}$ are the discounted profit-to-go functions under the optimal, fixed-price, and heuristic policies respectively. Let DC^*, DC^{FP} and DC^{H1} be the corresponding discounted disposal costs with the starting state $\mathbf{0}$ under different policies.

The numerical results are reported in Table EC.3. To examine the effect of demand variability on the value of dynamic pricing, we further consider two scenarios: (1) the market size falls from 1.1α to 0.9α with a slope 0.05α while keeping the average market size per period at α, and (2) the market size remains constant at α. The corresponding percentage profit losses under the FP policy are denoted by ρ_I^{FP} and ρ_{II}^{FP} respectively. They are presented in the last two columns of Table EC.3. We have the following observations.

- Consistent with the observations in the infinite-horizon setting, the discounted profit of the optimal policy decreases in the coefficient of variation, the unit backlogging cost, and the unit disposal cost respectively while it increases in the product lifetime. The magnitudes of the discounted disposal costs in the finite-horizon setting with nonstationary demand are much larger than those in the infinite-horizon setting.
- For the fixed-price policy, as the coefficient of variation, the unit backlogging cost, or the unit disposal cost increases, or the lifetime becomes shorter, the performance becomes worse (i.e., dynamic pricing becomes more valuable) and the discounted disposal cost becomes larger. The magnitudes of the percentage losses and the increased disposal costs (against the optimal policy) are much larger than those in the infinite-horizon setting. This implies that the value of dynamic pricing and its role in reducing wastes are much more significant when the demand is nonstationary.
- For the H1 heuristic policy, the percentage profit loss and the discounted disposal cost also increase in the unit backlogging cost, or the unit disposal cost, and decreases in the lifetime. But, when coefficient of variation increases the percentage profit loss of H1 becomes smaller. That is, in the finite-horizon setting with nonstationary demand, H1 may perform better as the demand variation increases, which is different from the infinite-horizon setting.
- When the slope of the seasonality factor κ_t falls from 0.1 to 0 while keeping the average market size per period at α, the percentage profit losses fall quickly; see columns ρ^{FP}, ρ_I^{FP} and ρ_{II}^{FP}. As a smaller slope implies a more stationary demand process and a smaller the percentage profit loss implies a higher value of dynamic pricing, our results imply that the more variable the demand process, the higher the value of dynamic pricing.

In summary, the value of dynamic pricing becomes much more significant in the finite-horizon setting with nonstationary demand. Compared with the fixed-price policy, the heuristic policy H1 still performs well while keeping a simple policy structure.

6.3.3. Performance of Heuristics for Cost-Minimization Problems We now evaluate the performance of the heuristic policies for the cost minimization problem with only replenishment decisions. We restrict our study to the infinite-horizon setting with the long-run average cost as the decision criterion, ignoring the revenue term $P_t(d)d$. The expected demand level, which is exogenously given, is set as $d = 54$. For consistency, we still call the optimal policy with the fixed expected demand level the optimal fixed-price policy. All the other parameters are the same as those of the above numerical examples in the stationary setting.

The results are reported in Table EC.4 where C^u and DC^u denote the long-run average cost under policy u, $u \in \{FP, H1, H2\}$, and $\rho^u = \frac{C^u - C^{FP}}{C^{FP}} \times 100\%$ denote the percentage cost increase of the heuristic policy $u \in \{H1, H2\}$ against the FP policy.

From Table EC.4, we can observe that the percentage cost increases of the heuristics are less than one percent for all the instances. Recall that the percentage profit losses of the heuristics in Table EC.1 could be greater than 3%. This suggests that the heuristics perform better when there are no pricing decisions and the revenue effects are removed. The sensitivity of the costs and relative performance with respect to the system parameters are similar to those in Table EC.1 and Table EC.2. But the average disposal costs of the cost-minimization problems are consistently higher than those in Table EC.2, which implies that the optimization of the pricing decisions can indeed reduce the disposal costs. The order-up-to levels under H1 are slightly lower than that under H2 and the performance of H2 is slightly better, which implies that under H1 the replenishment decisions may be more conservative.

7. Concluding Remark

In this paper, we address a joint pricing and inventory control problem for stochastic perishable inventory systems with positive lead time. Both the backlogging and lost-sales cases are studied. We employ the concept of $L^\natural$-convexity and develop new preservation properties to show some monotonicity properties of the optimal policies. Unlike previous work which uses a sequence of models with continuous demand distributions to approximate the discrete demand distributions, our analysis allows for both continuous and discrete decision variables and thus provides a unified approach to deal with models with both continuous and discrete demand distributions. We also develop analytical bounds and propose a heuristic policy. Numerical study shows that the heuristic policy performs very well. Finally, we show that our analysis can be further extended to the case with random lifetime and the inventory rationing model with multiple demand classes.

Our models assume that inventories of any age can be disposed of purposely and customers are not sensitive to the ages of inventories, which ensures that FIFO inventory issuing rule is optimal. The FIFO assumption is particularly applicable to the contexts where organizations (e.g., online retailers, blood banks and pharmacies) themselves fully control which inventory units are used to meet the demand. Nevertheless, there are contexts in which other inventory issuing policies are more plausible. For example, when customers are very sensitive to the ages of inventories (e.g., a self-service grocery store where customers can make choices among units of different ages on the same shelf) or when the freshest units are committed to be supplied first (e.g., some blood centres and hospitals may have such service level agreements, Nahmias 2011), the last-in-first-out (LIFO) inventory issuing policy is appropriate. When different prices are charged for inventories of different ages, it is then critical to model the consumer choice behavior. Taking into account these issues in perishable inventory models remains a significant challenge which deserves further exploration.

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E-companion to
"Coordinating Inventory Control and Pricing Strategies for Perishable Products"

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This e-companion contains the technical proof of Theorem 3, two extensions, and four tables of the numerical study.

EC.1. Proof of Theorem 3

It suffices to show that $R(d-\zeta, y-\zeta)$ is supermodular in (d, y, ζ) for $\zeta \in \{\zeta : d-\zeta \in \mathcal{D}_t, 0 \le \zeta \le y\}$.

Note that $R(d-\zeta, y-\zeta) = P(d-\zeta)(d-\zeta) + P(d-\zeta)E[\min(\epsilon_t, y-d)]$. Taking second-order cross-partial derivatives with respect to (d, y) yields

$$\frac{\partial^2 R}{\partial d \partial y} = P'(d-\zeta)\bar{F}(y-d) + P(d-\zeta)F'(y-d) = -P'(d-\zeta)\bar{F}(y-d)[\varrho(d,y)-1] \ge 0,$$

where the inequality is by condition (C2).

Taking second-order cross-partial derivatives with respect to (y, ζ) yields

$$\frac{\partial^2 R}{\partial \zeta \partial y} = -P'(d-\zeta)\bar{F}(y-d) \ge 0,$$

where the inequality is due to $P' < 0$.

Taking second-order cross-partial derivatives with respect to (d, ζ) yields

$$\frac{\partial^2 R}{\partial d \partial \zeta} = -2P'(d-\zeta) - P^{''}(d-\zeta)(d-\zeta) - P^{''}(d-\zeta)E[\min(\epsilon_t, y-d)] + P^{'}(d-\zeta)\bar{F}(y-d).$$

If $P^{''}(d) \le 0$, then

$$\begin{aligned}\frac{\partial^2 R}{\partial d \partial \zeta} &= -2P'(d-\zeta) - P^{''}(d-\zeta)E[\min(d-\zeta+\epsilon_t, y-\zeta)] + P^{'}(d-\zeta)\bar{F}(y-d)\\ &\ge -2P'(d-\zeta) + P^{'}(d-\zeta)\bar{F}(y-d)\\ &\ge 0,\end{aligned}$$

where the first inequality is by the nonnegativity of $E[\min(d-\zeta+\epsilon_t, y-\zeta)]$ and the second by the fact that $-P'(d-\zeta)+P^{'}(d-\zeta)\bar{F}(y-d) = -P'(d-\zeta)F(y-d) \geq 0$ and $P'(d) \leq 0$.

If, otherwise, $P^{''}(d) > 0$, then

$$\begin{aligned}
\frac{\partial^2 R}{\partial d \partial \zeta} &= -2P'(d-\zeta) - P^{''}(d-\zeta)(d-\zeta) - P^{''}(d-\zeta)E[\min(\epsilon_t, y-d)] + P^{'}(d-\zeta)\bar{F}(y-d) \\
&\geq -2P'(d-\zeta) - P^{''}(d-\zeta)(d-\zeta) + P^{'}(d-\zeta)\bar{F}(y-d) \\
&\geq -[P'(d-\zeta) + P^{''}(d-\zeta)(d-\zeta)] \\
&\geq 0,
\end{aligned}$$

where the first inequality is from $E[\min(\epsilon_t, y-d)] \leq E[\epsilon_t] = 0$, the second from $P'(d-\zeta) \leq 0$ and the third from (C1). This completes the proof.

EC.2. Extensions

EC.2.1. Lost-Sales Model with Positive Lead Time

We now address the lost-sales model with positive lead time. Recall that in the lost-sales case $R(d, s_{l-k}) = P(d)E[\min(d+\epsilon_t, s_{l-k})]$ and

$$\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t) = -(1-\gamma)cs_l - \gamma ca - \theta(a-d_t) - h^+(s_{l-k}-a)^+ - (h^- - \gamma c)(a - s_{l-k})^+ + \gamma f_{t+1}(\widetilde{\boldsymbol{s}}),$$

where

$$\widetilde{\boldsymbol{s}} = [(s_2-a)^+, ..., (s_{l-k}-a)^+, s_{l-k+1} - a \wedge s_{l-k}, ..., s_l - a \wedge s_{l-k}].$$

Note that the analysis for the lost-sales case with zero lead time relies on the monotonicity of $f_t(\boldsymbol{s})$. However, when the lead time is positive ($k \geq 1$), the partial monotone structure of the profit function under the dynamics (3) is not sufficient to ensure the preservation of $L^\natural$-concavity. The application of Lemma 5 requires that the underlying function to be monotone in all its components. However, in the lost-sales case with positive lead time, the profit function f_t may not always be monotone in all its components. To show that $L^\natural$-concavity can be preserved, we need to impose the following sufficient condition.

(C3) $h^+ + h^- - \gamma c \geq \gamma(h^- - \gamma c + \bar{p})$.

Condition (C3) states that the inventory cost incurring in the current period is greater than the maximum potential benefit of carrying the inventory to the next period while holding on-hand inventory and facing unmet demand simultaneously. This condition implies that it is always more beneficial to meet the demand to the maximum extent. Note that the left-hand side of above inequality does not take into account of the revenue (price) impact of inventory whereas the right-hand side relaxes the price level to its upper bound, which implies that this condition can be potentially further relaxed.

The next theorem shows that under conditions (C1)-(C3) the desired structural properties hold.

Theorem EC.1 (Monotonicity Properties of Optimal Policies). *Suppose (C1)-(C3) hold and $k \geq 1$. For $t = 1, ..., T$, the functions $f_t(\boldsymbol{s})$, $g_t(\boldsymbol{s}, s_l, d)$ and $\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t)$ are $L^\natural$-concave in $\boldsymbol{s}$, $(\boldsymbol{s}, s_l, d)$ and $(\boldsymbol{s}, s_l, d, a)$, respectively. The joint pricing, inventory replenishment and depletion policy has the same monotonicity properties as shown in Theorem 1.*

Proof. The proof is by induction as the proof of Theorem 1. It suffices to show that $\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t)$ is $L^\natural$-concave in $(\boldsymbol{s}, s_l, d, a)$ if f_{t+1} is $L^\natural$-concave.

For any $\delta > 0, u > 0$ and $u + \delta \leq a \wedge s_{l-k}$, comparing the system starting with state $\tilde{\boldsymbol{s}} = ((s_2 - u)^+, ..., (s_{l-k} - u)^+, s_{l-k+1} - u, ..., s_l - u)$ and the system with state $\tilde{\boldsymbol{s}}' = ((s_2 - u - \delta)^+, ..., (s_{l-k} - u - \delta)^+, s_{l-k+1} - u - \delta, ..., s_l - u - \delta)$ at the beginning of period $t + 1$, the latter has δ units less on-hand inventory. Then, compared to the first system, the second system has at most δ units of shortage in the following $l - k - 1$ periods, incurring at most δ units of lost sales, which implies

$$f_{t+1}(\boldsymbol{s}') - f_{t+1}(\boldsymbol{s}) \geq -(h^- - \gamma c + \bar{p})\delta.$$

Define $\psi_t(\boldsymbol{s}, s_l, u) = (h^+ + h^- - \gamma c)u + \gamma f_{t+1}(\tilde{\boldsymbol{s}})$. We have

$$\psi_t(\boldsymbol{s}, s_l, u + \delta) - \psi_t(\boldsymbol{s}, s_l, u) \geq \delta[(h^+ + h^- - \gamma c) - \gamma(h^- - \gamma c + \bar{p})] \geq 0,$$

where the first inequality is from above analysis and the second is from (C3). That is, $\psi_t(\boldsymbol{s}, s_l, u)$ is monotone increasing in u, which implies that

$$\psi_t(\boldsymbol{s}, s_l, a \wedge s_{l-k}) = \max_{u \leq a \wedge s_{l-k}} \psi_t(\boldsymbol{s}, s_l, u).$$

By Lemma 6, we know that $f_{t+1}(\boldsymbol{s})$ is nonincreasing in $(s_1, ..., s_{l-k-1})$. Then, by Lemma 4, we know that $f_{t+1}(\tilde{\boldsymbol{s}})$ is $L^\natural$-concave in $(\boldsymbol{s}, s_l, u)$. Clearly, $\phi_t(\boldsymbol{s}, s_l, u)$ is also $L^\natural$-concave in $(\boldsymbol{s}, s_l, u)$. Note that the constraint set $\{u : 0 \leq u \leq a \wedge s_{l-k}\}$ forms a lattice. By Lemma 2, we know that $\phi_t(\boldsymbol{s}, a \wedge s_{l-k})$ is $L^\natural$-concave in $(\boldsymbol{s}, s_l, a)$.

Note that $(s_i - a)^+ = (s_i - s_{l-k} \wedge a)^+$ for all $i < l - k$. The dynamics of the system state can be expressed as

$$\widetilde{\boldsymbol{s}} = [(s_2 - s_{l-k} \wedge a)^+, ..., (s_{l-k} - s_{l-k} \wedge a)^+, s_{l-k+1} - s_{l-k} \wedge a, ..., s_l - s_{l-k} \wedge a],$$

Then, for any $a \leq s_{l-k} \wedge d_t$,

$$\begin{aligned}
&\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t) \\
&= -(1-\gamma)cs_l - \gamma ca - \theta(a - d_t)^+ - h^+(s_{l-k} - a)^+ - (h^- - \gamma c)(a - s_{l-k})^+ + \gamma f_{t+1}(\widetilde{\boldsymbol{s}}) \\
&= -(1-\gamma)cs_l - \gamma ca - \theta(a - d_t)^+ - h^+[s_{l-k} - a \wedge s_{l-k}] - (h^- - \gamma c)[a - a \wedge s_{l-k}] + \gamma f_{t+1}(\widetilde{\boldsymbol{s}}) \\
&= -(1-\gamma)cs_l - \gamma ca - \theta(a - d_t)^+ - h^+ s_{l-k} - (h^- - \gamma c)a + \psi_t(\boldsymbol{s}, a \wedge s_{l-k}).
\end{aligned}$$

Clearly, all the terms of the right-hand side of last equation are $L^\natural$-concave. Thus, $\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t)$ is $L^\natural$-concave in $(\boldsymbol{s}, s_l, a)$. It is clear that f_t is $L^\natural$-concave under conditions (C1) and (C2). By induction, the desired structural results characterized in Theorem 1 hold. Q.E.D.

Theorem EC.1 shows that the desired structural properties could still hold in the lost-sales case with positive lead time. Its proof shows that (C3) supplies some monotonicity to a transformed profit-to-go functions, in addition to the partial monotonicity of the original profit-to-go function, which allows us to apply Lemma 4.

Nevertheless, one may also be interested in applying Lemma 5, given the fact that the profit-to-go function of the starting state of next period, $f_{t+1}(\widetilde{\boldsymbol{s}})$, has the same form as that of the function in Lemma 5. To this end, we need to impose new conditions under which some transformation of the profit-to-go function is monotone. Note that the key tradeoff is between the lost-sales cost and revenue in the future. It is possible to construct a transformed profit function with the lost-sales cost when the lost-sales cost is high. We now present the idea without providing the proof: Assume that $(h^- - \gamma c)\rho \geq \gamma\theta + \bar{p}$, where $\rho = \frac{1-\gamma}{\gamma(1-\gamma^k)}$. The transformed profit-go-to function $\hat{f}_t(\boldsymbol{s}) = f_t(\boldsymbol{s}) - \rho(h^- - \gamma c)[s_{l-k} + \gamma s_{l-k+1} + ... + \gamma^{k-1} s_{l-1}]$ is monotone decreasing in $\boldsymbol{s}$. By Lemma 5, we can show that $L^\natural$-concavity can be preserved.

Remark EC.1 (Lost-Sales Inventory Models with Exogenous Price). When the price p is exogenously given and constant over time, the model can be reduced to the standard lost-sales perishable inventory model with the equivalent lost-sales cost $h^- + p$. Then, condition (C3) is replaced by $(h^+ + h^- + p - \gamma c) \geq \gamma(h^- + p - \gamma c)$, which holds automatically. Thus, our structural analysis applies directly to the lost-sales perishable inventory model with positive leatimes, which implies that our model generalizes Nahmias' (2011) model to the case with positive lead time.

Remark EC.2 (Non-Stationary Lost-Sales Systems). Similar to the non-stationary backlogging systems, our analysis can be easily extended to the non-stationary case under the conditions $\theta_t \geq \gamma\theta_{t+1}$ and $h_t^+ + h_t^- - \gamma c_{t+1} \geq \gamma(h_{t+1}^- - \gamma c_{t+2} + \bar{p}_{t+1})$.

EC.2.2. Random Lifetime

We next extend our analysis to the case with random lifetime. For notational convenience, we restrict our attention to the backlogging case with zero lead time.

As summarized by Nahmias (1977), the useful lifetime of many products (e.g., fresh produce, meat, fowl, and fish) cannot be predicted in advance. Following Nahmias (1977), we assume that the inventories outdate in the same order in which they enter the system. For each period t, let K_t be a nonnegative integer random variable defined on the set $\{1, 2, ..., l\}$. Assume that $K_1, ..., K_T$ are independent and identeically distributed. For a realization of K_t in period t, all on-hand inventory that is at least K_t periods old at the end of period t will expire. Let $\pi_i = Pr(K_t = i)$. The dynamics of the system state depend on the realization of K_t, denoted by $\tilde{\boldsymbol{s}}^{(i)}$. Let $a^{(i)}$ be the inventory depletion level when $K_t = i$ such that $a^{(i)} \in [s_{l-i+1} \vee d_t, s_l \vee d_t]$. Then,

$$
\begin{aligned}
\tilde{\boldsymbol{s}}^{(l)} &= (s_2 - a^{(l)}, s_3 - a^{(l)}, ..., s_{l-1} - a^{(l)}, s_l - a^{(l)}), \\
\tilde{\boldsymbol{s}}^{(l-1)} &= (0, s_3 - a^{(l-1)}, ..., s_{l-1} - a^{(l-1)}, s_l - a^{(l-1)}), \\
&\vdots \\
\tilde{\boldsymbol{s}}^{(1)} &= (0, 0, ..., 0, 0).
\end{aligned}
$$

Then, letting $f_{T+1}(\boldsymbol{s}) = 0$, the optimal profit-go-to function f_t satisfies

$$
f_t(\boldsymbol{s}) = \max_{s_l \geq s_{l-1}, d \in \mathcal{D}_t} R(d, s_l) + \mathbb{E}\left[g_t(\boldsymbol{s}, s_l, d|\epsilon_t)\right],
$$

where

$$
\begin{aligned}
g_t(\boldsymbol{s}, s_l, d|\epsilon_t) &= \pi_l \max_{s_1 \vee d_t \leq a^{(l)} \leq s_l \vee d_t} \{-v_t(s_l, a^{(l)}, d_t) + \gamma f_{t+1}(\tilde{\boldsymbol{s}}^{(l)})\} \\
&+ \pi_{l-1} \max_{s_2 \vee d_t \leq a^{(l-1)} \leq s_l \vee d_t} \{-v_t(s_l, a^{(l-1)}, d_t) + \gamma f_{t+1}(\tilde{\boldsymbol{s}}^{(l-1)})\} \\
&\vdots \\
&+ \pi_1[-v_t(s_l, s_l \vee d_t, d_t) + \gamma f_{t+1}(0, 0, ..., 0, 0)],
\end{aligned}
$$

and

$$
v_t(s_l, a, d_t) = (1-\gamma)cs_l + \gamma ca + \theta(a - d_t) + h^+(s_l - a)^+ + h^-(a - s_l)^+.
$$

Analogous to the analysis in the proof of Theorem 1, one can show that f_t is nondecreasing for $a^{(i)} \geq s_l \vee d_t$, the constraints $a^{(i)} \leq s_l \vee d_t$ are redundant, and the $L^\natural$-concavity is preserved.

EC.2.3. Inventory Rationing with Multiple Demand Classes

Parallel to the pricing management, inventory rationing in the presence of multiple demand classes is also an important strategy to leverage the supply and demand. Consider a lost-sales inventory system with N classes of customers with different unit payments and lost-sales penalty costs. Assume that the unit payments (prices) are fixed throughout the planning horizon. At the beginning of each period, an order is placed and at the end of each period, after observing the realized demands the system operator decides which customers' demands to fulfill. In addition to satisfying demands, we assume that the system operator can intentionally dispose of some aged inventory to reduce the holding cost. The replenishment lead time of each order is of k periods.

Let n index the demand class. Let p_n be the unit payment, h_n^- be the unit penalty cost for class n, and d_{nt} be the demand of class n in period t. Let $\boldsymbol{d}_t = (d_{1t}, ..., d_{Nt})$. These demands could be correlated in each period but they are independent across periods. Without loss of generality, we assume that $p_1 + h_1^- > p_2 + h_2^- > ... > p_n + h_n^-$, which implies that the class-1 demand has the highest priority, then the class-2, and so on. Denote by a_n the amount of inventory allocated to class n's demands, a_0 the amount of additional inventory to be disposed of and $a = \sum_{n=0}^{N} a_n$. Note that $0 \le a_n \le d_{nt}$ and $s_1 \le a \le s_{l-k}$, the dynamics of the inventory are expressed as

$$\widetilde{\boldsymbol{s}} = ((s_2 - a)^+, ..., (s_{l-k} - a)^+, s_{l-k+1} - a, ..., s_l - a).$$

Letting $f_{T+1}(\boldsymbol{s}) = 0$, the optimality equation can be expressed as:

$$f_t(\boldsymbol{s}) = \max_{s_l \ge s_{l-1}} E[g_t(\boldsymbol{s}, s_l | \boldsymbol{d}_t)], \tag{EC.1}$$

where

$$g_t(\boldsymbol{s}, s_l|\boldsymbol{d}_t)$$
$$= \max_{0 \le a_n \le d_{nt}, s_1 \le \sum_{n=0}^{N} a_n \le s_{l-k}} \left\{ \tilde{R}(a_0, a_1, ..., a_N|\boldsymbol{d}_t) - (1-\gamma)cs_l - \gamma c \sum_{n=0}^{N} a_n - h^+(s_{l-k} - \sum_{n=0}^{N} a_n) + \gamma f_{t+1}(\widetilde{\boldsymbol{s}}) \right\},$$

where

$$\tilde{R}(a_0, a_1, ..., a_N|\boldsymbol{d}_t) = \sum_{n=1}^{N} p_n a_n - \sum_{n=1}^{N} h_n^-(d_{nt} - a_n) - \theta a_0.$$

Clearly, for any given total amount allocation a, it is always optimal to meet the demands to the maximum extent from the highest priority to the lowest priority and the payoff function $\tilde{R}(a_1, ..., a_N|\boldsymbol{d}_t)$ is entirely determined by a (see, e.g., Zipkin (2008) for a similar treatment). Let $R(a|\boldsymbol{d}_t) = \max\{\tilde{R}(a_0, a_1, ..., a_N|\boldsymbol{d}_t) | 0 \le a_n \le d_{nt}, \sum_{n=0}^{N} a_n = a\}$. It is clear that $R(a|\boldsymbol{d}_t)$ is concave in a. Then, we can represent g_t as

$$g_t(\boldsymbol{s}, s_l|\boldsymbol{d}_t) = \max_{s_1 \le a \le s_{l-k}} \{R(a|\boldsymbol{d}_t) - (1-\gamma)cs_l - \gamma ca - h^+(s_{l-k} - a) + \gamma f_{t+1}(\widetilde{\boldsymbol{s}})\}.$$

Applying the previous analysis, we can show by induction that f_t is nonincreasing in $(s_1, ..., s_{l-k-1})$. By Lemma 4, $f_{t+1}(\widetilde{\boldsymbol{s}})$ is $L^\natural$-concave. Similar to preceding analysis, we can show that $f_t(\mathbf{s})$ is also $L^\natural$-concave. Hence the optimal inventory replenishment, rationing and disposal policy has a similar structure to that characterized in Theorem 1.

Remark EC.3 (Backlogging Model with Multiple Demand Classes). Our analysis can also be easily extended to the backlogging case with N demand classes. Let b_n denote the number of class-n backorders and h_n^- denote the unit backlogging cost per period. Assume that $h_1^- > .. > h_N^-$. Define $\mathbf{z} = (z_1, ..., z_N)$ where $z_n = b_1 + ... + b_n$ represents the partial sum of backorders from class 1 to class n, $n = 1, ..., N$. Then the system state can be represented by $(\mathbf{z}, \mathbf{s})$. Again, let a be the total amount of inventory to be depleted such that $s_1 \le a \le s_{l-k}$. Let $\hat{d}_{nt} = d_{1t} + ... + d_{nt}$ be the partial sum of demands from class 1 to class n. The dynamics of the system state can be expressed as $(\tilde{\mathbf{z}}, \tilde{\mathbf{s}}) = ((z_1 + \hat{d}_{1t} - a)^+, ..., (z_N + \hat{d}_{Nt} - a)^+, s_2 - a, ..., s_l - a)$. Let $f_{T+1}(\tilde{\mathbf{z}}, \tilde{\mathbf{s}}) = 0$. The optimal value functions satisfy the following optimality equations.

$$f_t(\mathbf{z}, \mathbf{s}) = \max_{s_l \ge s_{l-1}} E[g_t(\mathbf{z}, \mathbf{s}, s_l|\mathbf{d_t})],$$

where

$$g_t(\mathbf{z},\mathbf{s},s_l|\mathbf{d}_t)=\max_{s_1\le a\le s_{l-k}}\{R(a|\mathbf{d}_t,\mathbf{z}_t)-(1-\gamma)cs_l-\gamma ca-h^+(s_{l-k}-a)^+ +\gamma f_{t+1}(\tilde{\mathbf{z}},\tilde{\mathbf{s}})\},$$

with $R(a|\mathbf{d}_t,\mathbf{z}_t)=\sum_{n=1}^N p_n d_{nt}-\sum_{n=1}^N(h_n^- - h_{n+1}^-)(z_n+\hat{d}_{nt}-a)^+ -\theta(a-z_N-\hat{d}_{Nt})^+$ and $h_{N+1}^-=0$. Clearly, $R(a,\mathbf{z}_t|\mathbf{d}_t)$ is $L^\natural$-concave in $(a,\mathbf{z}_t)$, and f_t is decreasing in $z_n, n=1,...,N$. By Lemma 4, $f_{t+1}(\tilde{\mathbf{z}},\tilde{\mathbf{s}})$ is $L^\natural$-concave. Similar to preceding analysis, we can obtain the desired results.

EC.3. Tables of Numerical Study

References

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Table EC.1 Long-Run Average Profits Per Period

id	l	$c.v.$	$\frac{h^-}{h^++h^-}$	θ	V^*	ρ^{FP}	d^{FP}	ρ^{H1}	(d^{H1}, s_l^{H1})	ρ^{H2}	(d^{H2}, s_l^{H2})
1	2	1	0.98	10	846.13	1.06%	58	1.39%	$(59, 69)$	1.39%	$(59, 69)$
2	–	0.6	–	–	899.46	0.68%	57	0.85%	$(58, 73)$	0.85%	$(58, 73)$
3	–	0.8	–	–	868.43	0.92%	58	1.25%	$(59, 71)$	1.16%	$(58, 70)$
4	–	1.2	–	–	830.30	1.15%	58	1.52%	$(59, 67)$	1.52%	$(59, 67)$
5	–	1.5	–	–	814.31	1.21%	58	1.62%	$(59, 65)$	1.62%	$(59, 65)$
6	–	–	0.90	–	926.19	0.24%	55	0.28%	$(55, 55)$	0.28%	$(55, 55)$
7	–	–	0.95	–	899.38	0.51%	56	0.64%	$(57, 55)$	0.60%	$(56, 54)$
8	–	–	0.99	–	790.80	1.58%	60	2.41%	$(61, 80)$	2.23%	$(61, 81)$
9	–	–	–	5	851.49	0.85%	58	1.31%	$(59, 70)$	1.21%	$(58, 69)$
10	–	–	–	20	838.05	1.45%	59	1.76%	$(59, 66)$	1.76%	$(59, 66)$
11	–	1.5	0.99	20	717.00	2.78%	62	3.82%	$(63, 77)$	3.82%	$(63, 77)$
12	3	1	0.98	10	921.66	0.63%	57	0.74%	$(57, 92)$	0.74%	$(57, 92)$
13	–	0.6	–	–	945.22	0.20%	55	0.22%	$(55, 90)$	0.22%	$(55, 90)$
14	–	0.8	–	–	933.05	0.44%	56	0.56%	$(57, 92)$	0.53%	$(57, 93)$
15	–	1.2	–	–	912.09	0.76%	57	0.98%	$(58, 92)$	0.98%	$(58, 92)$
16	–	1.5	–	–	901.01	0.90%	58	1.12%	$(58, 91)$	1.12%	$(58, 91)$
17	–	–	0.90	–	945.60	0.15%	55	0.17%	$(55, 71)$	0.17%	$(55, 71)$
18	–	–	0.95	–	936.97	0.32%	56	0.37%	$(56, 80)$	0.37%	$(56, 80)$
19	–	–	0.99	–	906.71	0.94%	58	1.26%	$(59, 102)$	1.19%	$(59, 103)$
20	–	–	–	5	922.92	0.53%	57	0.65%	$(57, 93)$	0.65%	$(57, 93)$
21	–	–	–	20	919.77	0.83%	57	1.00%	$(58, 91)$	1.00%	$(58, 91)$
22	–	1.5	0.99	20	869.66	1.87%	60	2.32%	$(61, 103)$	2.32%	$(61, 103)$
23	4	1	0.98	10	940.93	0.21%	55	0.20%	$(55, 107)$	0.20%	$(55, 109)$
24	–	0.6	–	–	950.80	0.01%	54	0.01%	$(54, 101)$	0.01%	$(54, 101)$
25	–	0.8	–	–	946.04	0.10%	55	0.11%	$(55, 107)$	0.11%	$(55, 107)$
26	–	1.2	–	–	936.03	0.32%	56	0.38%	$(56, 110)$	0.38%	$(56, 110)$
27	–	1.5	–	–	929.69	0.43%	56	0.54%	$(57, 113)$	0.54%	$(57, 113)$
28	–	–	0.90	–	950.36	0.03%	54	0.03%	$(54, 87)$	0.03%	$(54, 87)$
29	–	–	0.95	–	946.64	0.09%	54	0.10%	$(55, 98)$	0.10%	$(55, 98)$
30	–	–	0.99	–	935.80	0.34%	56	0.40%	$(56, 117)$	0.38%	$(56, 118)$
31	–	–	–	5	941.18	0.17%	55	0.20%	$(55, 109)$	0.20%	$(55, 109)$
32	–	–	–	20	940.55	0.28%	59	0.33%	$(56, 108)$	0.32%	$(56, 109)$
33	–	1.5	0.99	20	917.86	0.93%	57	1.09%	$(58, 121)$	1.09%	$(58, 121)$

Table EC.2 Average Disposal Costs Per Period

id	l	$c.v.$	$\frac{h^-}{h^++h^-}$	θ	DC^*	δ^*	DC^{FP}	δ^{FP}	DC^{H1}	δ^{H1}	DC^{H2}	δ^{H2}
1	2	1	0.98	10	9.67	1.14%	13.24	1.58%	9.65	1.16%	9.65	1.16%
2	–	0.6	–	–	6.60	0.73%	8.85	0.99%	7.04	0.79%	7.04	0.79%
3	–	0.8	–	–	8.78	1.01%	11.66	1.36%	8.50	0.99%	9.23	1.08%
4	–	1.2	–	–	10.16	1.22%	13.71	1.67%	9.80	1.20%	9.80	1.20%
5	–	1.5	–	–	8.78	1.01%	11.66	1.36%	8.50	0.99%	9.23	1.08%
6	–	–	0.90	–	1.50	0.16%	2.23	0.24%	1.57	0.17%	1.57	0.17%
7	–	–	0.95	–	3.64	0.40%	5.33	0.60%	3.59	0.40%	4.04	0.45%
8	–	–	0.99	–	17.74	2.24%	22.95	2.95%	16.24	2.10%	17.37	2.24%
9	–	–	–	5	5.99	0.70%	7.68	0.91%	5.23	0.62%	5.66	1.67%
10	–	–	–	20	13.62	1.63%	19.39	2.35%	14.87	1.81%	14.87	1.81%
11	–	1.5	0.99	20	28.18	3.93%	39.64	5.69%	27.10	3.93%	27.10	3.93%
12	3	1	0.98	10	2.30	0.25%	4.14	0.45%	3.26	0.36%	3.26	0.36%
13	–	0.6	–	–	0.81	0.09%	1.53	0.16%	1.35	0.14%	1.35	0.14%
14	–	0.8	–	–	1.63	0.17%	3.14	0.34%	2.11	0.23%	2.31	0.25%
15	–	1.2	–	–	2.73	0.30%	5.04	0.56%	3.31	0.37%	3.31	0.37%
16	–	1.5	–	–	3.07	0.34%	5.53	0.62%	3.89	0.44%	3.89	0.44%
17	–	–	0.90	–	0.47	0.05%	0.86	0.09%	0.72	0.08%	0.72	0.08%
18	–	–	0.95	–	1.05	0.11%	1.86	0.20%	1.42	0.15%	1.42	0.15%
19	–	–	0.99	–	3.71	0.41%	6.80	0.76%	4.45	0.50%	4.79	0.53%
20	–	–	–	5	1.42	0.15%	2.38	0.26%	1.77	0.19%	1.77	0.19%
21	–	–	–	20	3.11	0.34%	6.97	0.76%	4.62	0.51%	4.62	0.51%
22	–	1.5	0.99	20	7.30	0.84%	14.87	1.74%	9.86	1.16%	9.86	1.16%
23	4	1	0.98	10	0.45	0.05%	1.14	0.12%	0.83	0.09%	0.91	0.10%
24	–	0.6	–	–	0.05	0.01%	0.11	0.01%	0.11	0.01%	0.11	0.01%
25	–	0.8	–	–	0.23	0.03%	0.52	0.05%	0.46	0.05%	0.46	0.05%
26	–	1.2	–	–	0.65	0.07%	1.50	0.16%	1.10	0.12%	1.10	0.12%
27	–	1.5	–	–	0.86	0.09%	2.11	0.23%	1.39	0.15%	1.39	0.15%
28	–	–	0.90	–	0.07	0.01%	0.17	0.02%	0.15	0.02%	0.15	0.02%
29	–	–	0.95	–	0.18	0.02%	0.49	0.05%	0.34	0.04%	0.34	0.04%
30	–	–	0.99	–	0.75	0.08%	1.74	0.19%	1.38	0.15%	1.49	0.16%
31	–	–	–	5	0.27	0.03%	0.62	0.07%	0.50	0.05%	0.50	0.05%
32	–	–	–	20	0.63	0.07%	1.94	0.21%	1.27	0.14%	1.39	0.15%
33	–	1.5	0.99	20	1.99	0.22%	6.04	0.66%	3.84	0.42%	3.84	0.42%

Table EC.3 Finite-Horizon Models with Non-Stationary Demand

id	l	$c.v.$	$\frac{h^-}{h^++h^-}$	θ	$\hat{f}_1(\mathbf{0})$	DC^*	ρ^{FP}	DC^{FP}	ρ^{H1}	DC^{H1}	ρ_I^{FP}	ρ_{II}^{FP}
1	2	1	0.98	10	4030.01	36.35	7.06%	52.83	5.25%	11.91	2.88%	1.14%
2	–	0.6	–	–	4254.17	26.84	6.32%	39.32	13.10%	2.29	2.37%	0.65%
3	–	0.8	–	–	4120.29	34.40	6.79%	48.46	10.21%	4.86	2.69%	0.96%
4	–	1.2	–	–	3968.62	37.51	7.24%	54.21	0.92%	34.57	2.99%	1.26%
5	–	1.5	–	–	3908.39	36.76	7.39%	53.83	0.35%	60.80	3.09%	1.35%
6	–	–	0.90	–	4580.56	1.13	4.77%	1.52	0.44%	1.41	1.34%	0.02%
7	–	–	0.95	–	4385.61	8.73	5.36%	13.07	1.81%	4.55	1.78%	0.26%
8	–	–	0.99	–	3671.74	77.92	9.26%	107.76	8.72%	28.19	4.29%	2.43%
9	–	–	–	5	4050.30	22.43	6.82%	30.31	4.75%	6.83	2.67%	0.94%
10	–	–	–	20	399.49	51.30	7.51%	81.66	6.27%	19.48	3.26%	1.54%
11	–	1.5	0.99	20	3352.01	122.58	11.40%	186.14	2.81%	276.49	5.84%	4.27%
12	3	1	0.98	10	4275.98	7.95	6.37%	19.03	8.10%	0.08	2.10%	0.26%
13	–	0.6	–	–	4419.89	3.71	5.49%	11.30	12.58%	0.01	1.55%	0.04%
14	–	0.8	–	–	4338.89	6.52	6.05%	18.67	11.56%	0.03	1.86%	0.15%
15	–	1.2	–	–	4228.68	8.91	6.57%	21.50	4.15%	0.39	2.27%	0.36%
16	–	1.5	–	–	4178.09	9.45	6.67%	23.45	2.65%	0.64	2.38%	0.45%
17	–	–	0.90	–	4591.46	0.13	4.76%	0.15	0.29%	0.00	1.32%	0.00%
18	–	–	0.95	–	4460.42	1.78	5.11%	4.58	2.33%	0.02	1.45%	0.03%
19	–	–	0.99	–	4112.85	16.89	7.70%	43.24	14.34%	0.32	2.75%	0.65%
20	–	–	–	5	4280.44	5.07	6.23%	10.51	7.47%	0.05	1.98%	0.21%
21	–	–	–	20	4269.36	10.96	6.62%	31.10	9.02%	0.15	2.30%	0.37%
22	–	1.5	0.99	20	3918.44	29.29	9.52%	78.71	3.73%	7.90	3.92%	1.57%
23	4	1	0.98	10	4340.39	1.22	5.76%	8.24	7.70%	0.00	1.54%	0.01%
24	–	0.6	–	–	4445.33	0.25	5.09%	2.21	9.54%	0.00	1.37%	0.00%
25	–	0.8	–	–	4387.17	0.77	5.46%	5.63	9.56%	0.00	1.45%	0.00%
26	–	1.2	–	–	4303.57	1.58	5.98%	9.87	5.29%	0.00	1.62%	0.02%
27	–	1.5	–	–	4262.57	1.90	6.21%	11.23	4.54%	0.00	1.73%	0.04%
28	–	–	0.90	–	4592.30	0.00	4.76%	0.00	0.18%	0.00	1.32%	0.00%
29	–	–	0.95	–	4475.38	0.14	4.96%	1.03	1.95%	0.00	1.36%	0.00%
30	–	–	0.99	–	4237.33	3.01	6.77%	15.33	14.02%	0.00	1.86%	0.05%
31	–	–	–	5	4341.07	0.77	5.67%	4.57	7.29%	0.00	1.51%	0.01%
32	–	–	–	20	4339.40	1.62	5.91%	13.87	8.36%	0.00	1.59%	0.02%
33	–	1.5	0.99	20	4107.71	6.06	8.07%	33.36	7.73%	0.00	2.56%	0.26%

Table EC.4 Average Costs and Disposal Costs Per Period for Cost-Minimization Problems

id	l	$c.v.$	$\frac{h^-}{h^++h^-}$	θ	C^{FP}	DC^{FP}	ρ^{H1}	DC^{H1}	s_l^{H1}	ρ^{H2}	DC^{H2}	s_l^{H2}
1	2	1	0.98	10	1329.39	14.62	0.24%	11.32	61	0.24%	11.32	61
2	–	0.6	–	–	1270.30	9.79	0.12%	8.34	67	0.12%	8.34	67
3	–	0.8	–	–	1305.02	13.49	0.24%	10.00	63	0.17%	10.81	64
4	–	1.2	–	–	1346.42	15.74	0.34%	10.64	58	0.26%	11.53	59
5	–	1.5	–	–	1363.36	15.98	0.27%	11.58	57	0.27%	11.58	57
6	–	–	0.90	–	1236.55	2.25	0.02%	1.84	44	0.02%	1.84	44
7	–	–	0.95	–	1267.01	5.44	0.09%	4.04	50	0.09%	4.04	50
8	–	–	0.99	–	1395.38	27.52	0.61%	19.72	69	0.48%	20.97	70
9	–	–	–	5	1321.39	8.80	0.32%	6.11	62	0.25%	6.58	63
10	–	–	–	20	1342.43	22.94	0.22%	17.74	57	0.22%	17.74	57
11	–	1.5	0.99	20	1484.95	49.12	0.71%	33.67	62	0.57%	36.04	63
12	3	1	0.98	10	1247.11	5.20	0.13%	3.82	85	0.13%	3.82	85
13	–	0.6	–	–	1217.16	1.70	0.02%	1.48	88	0.02%	1.48	88
14	–	0.8	–	–	1232.76	3.70	0.06%	2.97	87	0.06%	2.97	87
15	–	1.2	–	–	1258.90	6.24	0.21%	4.22	83	0.16%	4.55	84
16	–	1.5	–	–	1272.20	7.27	0.22%	4.93	82	0.17%	5.31	83
17	–	–	0.90	–	1216.03	0.97	0.01%	0.81	69	0.01%	0.81	69
18	–	–	0.95	–	1226.78	2.25	0.06%	1.58	75	0.04%	1.74	76
19	–	–	0.99	–	1267.95	8.91	0.29%	6.32	92	0.22%	6.75	93
20	–	–	–	5	1244.36	2.97	0.15%	2.06	86	0.11%	2.23	87
21	–	–	–	20	1251.81	8.42	0.16%	6.00	82	0.12%	6.52	83
22	–	1.5	0.99	20	1320.79	21.60	0.51%	14.06	87	0.42%	15.02	88
23	4	1	0.98	10	1221.47	1.34	0.03%	0.99	105	0.03%	0.99	105
24	–	0.6	–	–	1209.33	0.11	0.00%	0.11	101	0.00%	0.11	101
25	–	0.8	–	–	1214.96	0.63	0.01%	0.50	104	0.01%	0.50	104
26	–	1.2	–	–	1227.78	1.92	0.06%	1.41	105	0.04%	1.52	106
27	–	1.5	–	–	1235.90	2.63	0.08%	1.92	105	0.08%	1.92	105
28	–	–	0.90	–	1209.91	0.17	0.00%	0.17	88	0.00%	0.17	88
29	–	–	0.95	–	1214.22	0.49	0.01%	0.38	95	0.01%	0.43	96
30	–	–	0.99	–	1228.51	2.30	0.05%	1.87	113	0.05%	1.87	113
31	–	–	–	5	1220.78	0.72	0.03%	0.54	106	0.02%	0.59	107
32	–	–	–	20	1222.67	2.28	0.03%	1.82	104	0.03%	1.82	104
33	–	1.5	0.99	20	1255.48	8.03	0.20%	5.51	110	0.16%	5.90	111