Unrecognized yet amazing world of origami

Woo Jung Lee

woojung422@yahoo.ca
Tel: 12505190654 (H)
12505077350 (C)

Introduction

Origami is a traditional Japanese art in which a folder uses a square piece of paper to fold it to a model without cutting or pasting. It has been practiced in Japan since 17th century. From about 19th century, it dispersed throughout the world. At first, origami was considered as an alternative form of art that creates relatively simple form such as animals or plants. However, from 20th century, researchers around the world have started to recognize the beneficial and unique advantages of origami; their studies showed that origami can be applied in many subjects. Objects are folded based on physics and mathematics laws. Hence, often times, the most efficient way of storing certain materials can be found with origami calculations.

Aside from storing technology, origami can be used in many different mathematics subjects. Specifically, origami has a potential of being utilized in school classes to teach students certain mathematical concepts. Numerous mathematical studies uncovered beautiful underlying relationships between this art and mathematics, which can be employed in class curriculum. There are teachers around the world who already implement origami into their classes while teaching students. In Israel, origami is used in schools around the country; the Israeli Origami Center (IOC) developed a program called Origametria in which students learn geometric concepts with origami. Origami can not only teach simple geometric shapes, as the program Origametria does, but also teach more complex mathematics such as differential calculus.

Moreover, origami involves precise finger manipulation, which consequently aids children's brain developments. Studies have shown that moving fingers helps adolescents develop coordination skill and critical thinking. Teaching with origami lets children learn mathematics and develop their brain at the same time. For instance, a program called Theragami™ in New York uses origami as educational, therapeutic, and recreational method of teaching children. Personally, I have been surprised by the effect of origami on children. I have seen children who love origami in an origami convention in Korea. Unlike other children who love video games and act raucously, these children were calm and orderly. Folding origami models causes folders to be patient, focused, and investigative; the children who fold origami models were able to acquire these beneficial effects of origami. Also, other children, who did not know origami, exhibited great interests when I showed them some complex origami models. Hence, implementing origami into mathematical education can attract students to math by arousing interests from children and letting them study mathematics interactively with educational and psychological benefits. I wish to continue study of relationship between origami and mathematics to advance this field and promote use of origami in school curriculum to teach students.

This research paper explains origami application in the field of science, technology, and mathematics. The former part illustrates recent discoveries of potential and current application of origami in the modern science community. The latter part explains how origami can be used to teach geometric similarities, Pythagorean Theorem, geometric construction, equation solving, and exponential decay, mainly to high school students.

Part 1: the application of origami in the field of science and technology

When people hear the word origami, they usually think of the simple paper crane that they might have folded in elementary school or do not even know what it is. However, there has been a huge progress in origami study in 20th century because origami artists started to discover different techniques and styles of origami. These discoveries led researchers to delve into different topic of origami: the application of origami on other topics. Although regarded just as a type of art until 19th century, researchers have found out that origami has many possible applications from simple shopping bag to complicated airbag.

Normally, origami is done with paper, which can bend easily. Some folds require the folding material to bend. For example, if with a metal with hinges on the creases, some folds will not be feasible since metals do not bend easily. There is a subject of origami that studies such properties called rigid origami. Rigid origami can have many applications in engineering. By studying which folds require the material to bend, one can make a tool with rigid materials so that it does not return to its normal form easily, because the rigidness of the material will prevent the model from unfolding. Scientists have found folding schemes that can be used in many areas. One simple example is the grocery shopping bag. When it is being stored or transported, it is flattened, but when it is used by the shoppers, it is unfolded into the opened state. It is folded based on rigid origami calculation: because the material is somewhat rigid, this type of shopping bag will tend to stay in its original state. When it is transported, it tends to stay closed; when it is used, it tends to stay open. This rigidness makes grocery bags more efficient.

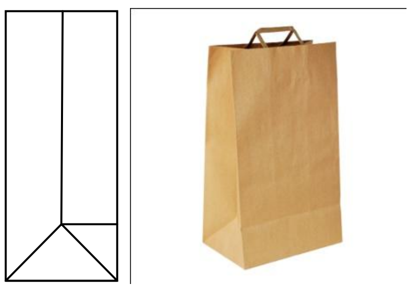


Figure 1. Grocery shopping bag and its folding scheme

In this sense, rigidness is beneficial, but sometimes it can be a problem. For example, a standard cardboard box must have open top and bottom in order to be folded flat. Every time it is received from transportation, the bottom part needs to be reconstructed in order to put objects in it. The box cannot be folded without opening the bottom part because there is no simple folding scheme that allows the box to be completely flat without distorting the bottom and top part of cardboard.

In March 2011, Zhong You and Weina Wu developed a new box that can solve this problem. They created a box with closed bottom that can be folded flat without having to open the bottom plane. They tweaked the crease pattern and added hinges so that the bottom face of the box will not disrupt the folding. This will enable more efficient transportation in manufacturing industry.

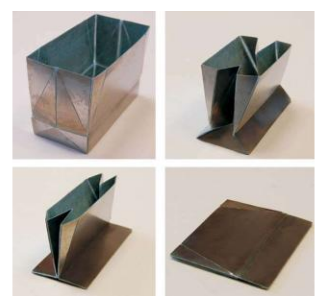


Figure 2. Steel foldable bag developed by Zhong You and Weina Wu

Not only simple boxes, but also complex satellites are built with origami. In 1995, Japanese scientists incorporated ‘Miura-ori’ pattern onto the solar power array of their satellite called Space Flight Unit (SFU). ‘Miura-ori’ is an origami folding that enables one to easily deploy a flat sheet of material into a compact and much smaller form. As figure 3 shows, one can deploy the paper vertically and horizontally at the same time.

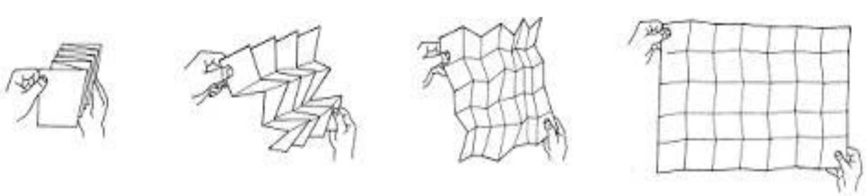


Figure 3. Miura-ori folding scheme

This allows high compression rate relative to small force required to deploy the paper. Also, Miura fold is the solution to the following problem: “determine the deformation of a perfectly flat, infinite, elastic plate subject to uniform terminal compression or shortening.”, proposed by Koryo Miura. This implies that Miura fold is an excellent folding scheme for compressing certain plate into a smaller plate. After realizing the benefits of Miura-ori, Japanese engineers implemented the Miura folding to their satellite. Before it was launched to the space, the solar panels were folded, which later would open up into a full size and function properly.

Similar efforts were also done in America. Origami artist Robert J. Lang assisted scientists at Lawrence Livermore National Laboratory with making a folding pattern that would corrugate the lens of the telescope into a smaller form. In order to view farther, astronomers need to launch telescope with large lens, but such large lens would be difficult and inefficient to launch. Lang developed a crease pattern that, similar to the Miura-ori, can be easily folded to a smaller size. In 2002, this pattern was used in their lens, which was three meters long; with origami pattern, it reduced to 1.2 meters long.

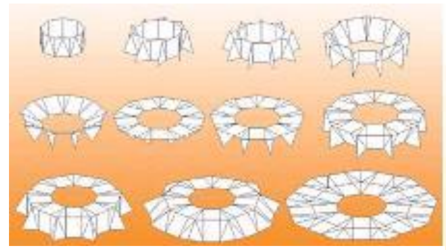


Figure 4. Lens corrugation pattern developed by Robert J. Lang

Miura-ori is not just an artificial creation. It is also present in nature. A species of tree called hornbeam tree has a special type of leaf. Its leaf has corrugated shapes that resemble Miura-ori. When the leaf is packed inside the bud, it is folded according to its crease pattern. As the leaf blooms, it unfolds according to Miura-ori scheme: expanding both vertically and horizontally. This is another example that shows that Miura fold is probably one of the most efficient folding patterns for storing a thin material.

Figure 5. Hornbeam tree leaf

Although not as widely used as Miura fold, waterbomb fold is also another powerful folding pattern that has many useful applications. It is a basic folding scheme that changes a square piece of paper into a triangle, which has size that is quarter to that of original paper. In 2003, Zhong You and Kaori Kuribayashi developed a stent that contracts and expands based on the basic mechanism of waterbomb fold. Stent is a tube that is inserted to human's arteries or veins in order to open up the clogged spot. Thus, it requires easy contraction and expansion. The wall of the cylinder is a series of multiple waterbomb folds. Waterbomb scheme contracts a paper into a smaller size easily, so this waterbomb based stent is able to contract while travelling in veins, and expand when arrived at the clogged spot very easily.

Figure 6. Medical stent based on waterbomb scheme

Origami is also useful at saving people in traffic accidents. Nowadays, almost all the cars have airbags installed. Airbags must unfold out from the folded state quickly and safely. EASi Engineering, a German company, in an effort to create better airbags worked with Robert J. Lang to develop an airbag folding scheme. They found an efficient scheme, and tested it with origami programming algorithm. By using origami, they were able to develop better airbags and save money by testing it with programs, which were derived from origami calculation. Aside from the airbags, 'crumple zones' are also based on folding. These are zones of the car that crumples in a certain way when the car crashes. The crushing motion absorbs the impact and alleviates the damage to the people inside the car. The crumple zones also received some help from origami; Ichiro Hagiwara utilized origami knowledge to create crumple zones with Nissan Company.

Figure 7. Airbag programming developed by Robert J. Lang

Superficially, origami does not seem to have much scientific aspects. It merely appears to be a type of art for children. Nevertheless, origami can be used in many places where compact, efficient, and quick method of storing certain objects is required. Furthermore, origami is already used everywhere around our lives by nature; from our genes and proteins to the immense universe and galaxies, many things fold and unfold according to laws of mathematics.

Origamics

Paper folding is a unique form of geometric manipulation that is different from many widely known geometric actions such as graphing. It can sometimes easily create some beautiful equations or important numbers in mathematics that are difficult to create with other geometric maneuvers. Developed primarily by Kazuo Haga, origamics is a subject of paper folding mathematics that deals primarily with how certain folds create certain numbers. There are three primary theorems that Haga created. Each theorem deals with a specific fold that divides the sides of the paper into certain fractions that normally takes some complex work to create with, for example, compass and straight edge construction. Students who like mathematics will be surprised by the inconspicuous yet simple mathematical rules that are beneath Haga's theorems.

Proofs of origamics involve many triangle congruency and Pythagorean Theorem. As the paper is folded, the edges of the papers combine to form different triangles; using these triangles, one proves that certain edges are divided into certain fractions. Hence, teaching Haga's theorem and their proofs can help students grasp the congruency and Pythagorean Theorem concepts. Moreover, learning Haga's theorems, trying it out with a piece of paper, and proving it can be an introduction to proofs. These three steps teach students how to prove a mathematical problem. Thus, teachers can teach geometric

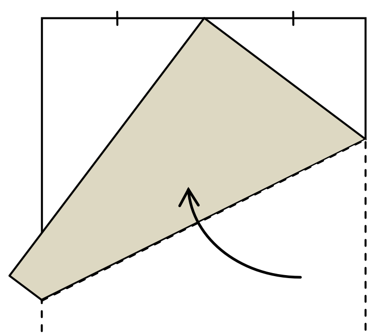


Figure 1. Haga's First Theorem

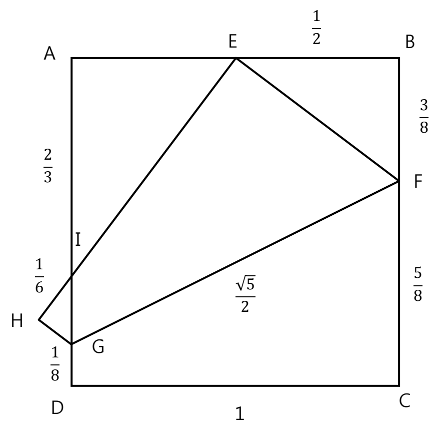


Figure 2. Result of Haga's First Theorem

calculations and proofs simultaneously with Haga's theorems.

The first theory is illustrated in the diagrams above. Figure 1 shows Haga's First Theorem. First, divide the top edge of the paper into halves. Then, fold one of the two bottom corners to the midpoint of the top edge. Consequently, as shown in figure 2, the sides of the paper are divided into certain fractions. The edges are divided into 3rd, 6th, and 8th. Each division can be proved by rigorous application of similar triangles and Pythagorean Theorem.

First, we need to prove that $\overline{\mathrm{BC}}$ is divided into $3/8$ and $5/8$ by using triangle ΔBEF. Let $\overline{BF} = a$. Then $\overline{CF} = \overline{\mathrm{EF}} = 1 - a$, and $\overline{\mathrm{BE}} = \frac{1}{2}$. Then, by Pythagorean Theorem,

$$a^2 + \left(\frac{1}{2}\right)^2 = (1 - a)^2$$

$$a^2 + \frac{1}{4} = 1 - 2a + a^2$$

$$2a = \frac{3}{4}$$
$$a = \frac{3}{8}$$

Then, we prove that $\overline{AD}$ is divided into 3rd. Since $\angle HEF$ is a right angle, ΔAEI and ΔBFE are similar triangles. So,

$$\overline{AI}:\overline{AE} = \overline{BE}:\overline{BF} \rightarrow \overline{AI}:\frac{1}{2} = \frac{1}{2}:\frac{3}{8} \rightarrow \overline{AI} = \frac{2}{3}$$

Subsequently, we prove that $\overline{EH}$ is divided into 6th. Using the two similar triangles ΔAEI and ΔBFE, we get

$$\overline{EI}:\overline{AI} = \overline{FE}:\overline{BE} \rightarrow \overline{EI}:\frac{2}{3} = \frac{5}{8}:\frac{1}{2} \rightarrow \overline{EI} = \frac{5}{6}$$

Finally, we prove that $\overline{DG}$ is $1/8$. Since $\angle AIE = \angle GIH$ and both ΔAIE and ΔGIH are right triangles, they are similar triangles. Using these two triangles, we get

$$\overline{HG}:\overline{IH} = \overline{AE}:\overline{IA} \rightarrow \overline{HG}:\frac{1}{6} = \frac{1}{2}:\frac{2}{3} \rightarrow \overline{HG} = \frac{1}{8}$$

Now, all of the divisions have been proven. With normal construction, it takes longer time to divide lines into 3rd, 6th, and 8th. However, with origamics, it takes only two folds to achieve all the division. Furthermore, the triangles formed by this fold are the famous Egyptian triangle, also known as the 3-4-5 triangle. It is known that Egyptians took a long time to create this triangle with their rulers and compasses; with origami, it takes only two folds.

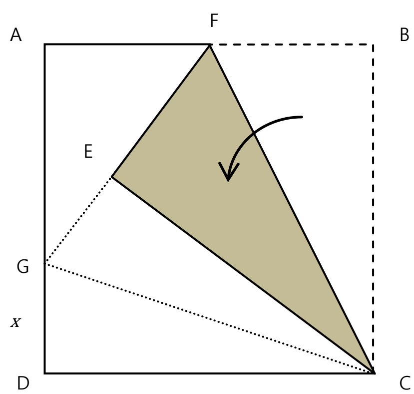


Figure 3. Haga's Second Theorem

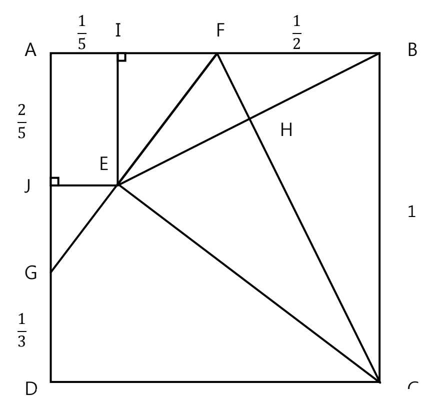


Figure 4. Result of Haga's Second Theorem

Haga's Second theorem is shown in the diagrams above. The crease begins from the midpoint of the top edge to the right bottom corner. This is shown on the left diagram. (ignore the dotted lines for now; they are used for proofs later) In figure 4, $\overline{EG}$ is the extension of $\overline{EF}$, and point I and J are ends of perpendicular lines from point E to side $\overline{AB}$ and $\overline{AD}$, respectively. Figure 4 shows the divisions that are created with Haga's second theorem. Edge $\overline{AD}$ is divided into 3rd, and edge $\overline{AB}$ is divided into 5th. There are many ways of proving the 3rd division; a clever and simple way will be presented here. Proving 5th division requires use of similar triangles.

In figure 3, there are three types of triangles, ΔBCF, ΔCDG, and ΔAFG. The other two triangles are similar to one of the three triangles. If we let the side length of the square be 1, and let $\overline{DG} = x$, we get

$$2\Delta BCF + 2\Delta CDG + \Delta AFG = 1$$

$$2\left(\frac{1}{2}\cdot 1\cdot\frac{1}{2}\right)+2\left(x\cdot 1\cdot\frac{1}{2}\right)+\left((1-x)\cdot\frac{1}{2}\cdot\frac{1}{2}\right)=1$$

Simplifying this, we find that $x=\overline{DG}=1/3$, and that G divides $\overline{AD}$ into third.

Fifth division proof is little more complex. First, we find the length of $\overline{CF}$ by using Pythagorean Theorem.

$$\overline{BF}^2+\overline{BC}^2=\overline{CF}^2 \rightarrow \frac{1}{4}+1=\overline{CF}^2 \rightarrow \overline{CF}=\frac{\sqrt{5}}{2}$$

Next, we use similar triangles. ΔBCF, ΔHBF, and ΔIBE are similar triangles because they are all right triangles and share an angle with each other. From ΔBCF and ΔHBF, we get

$$\overline{BC}:\overline{CF}=\overline{HB}:\overline{BF} \rightarrow 1:\frac{\sqrt{5}}{2}=\overline{HB}:\frac{1}{2} \rightarrow \overline{HB}=\frac{1}{\sqrt{5}}$$

And furthermore, since ΔBCF and ΔECF are same triangles with same heights, $\overline{BE}=2/\sqrt{5}$. Then, we use ΔBFH and ΔBEI to get

$$\overline{BF}:\overline{BH}=\overline{BE}:\overline{BI} \rightarrow \frac{1}{2}:\frac{1}{\sqrt{5}}=\frac{2}{\sqrt{5}}:\overline{BI} \rightarrow \overline{BI}=\frac{4}{5}$$

Thus, the point I divides edge $\overline{AB}$ into fifth. Also, the length $\overline{AJ}$ can be found by applying Pythagorean Theorem on ΔBEI.

The amazing aspect of Haga's Second Theorem is the simplicity of dividing a length into fifth. Normally, it takes much more effort than two simple folds in order to create fifth division point with compass and straightedge construction.

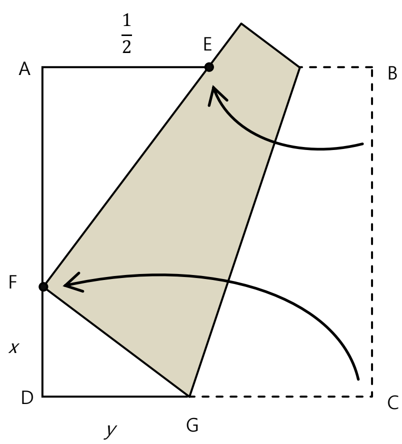


Figure 5. Haga's Third Theorem

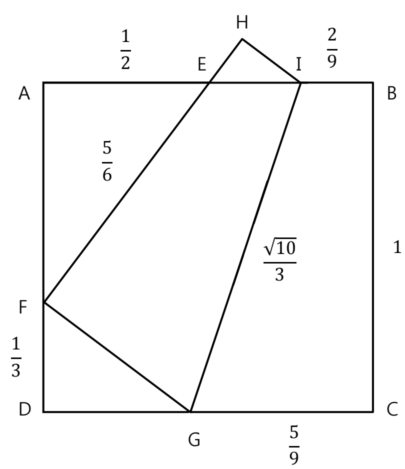


Figure 6. Result of Haga's Third Theorem

Haga's Third Theorem is a little more complex to fold and prove. It is folded by using the axiom 6, which is mentioned in the origami construction section. Haga's Third Theorem is shown above. Let the midpoint of $\overline{AB}$ be p_1, edge $\overline{BC}$ be l_1, point C be p_2, and edge $\overline{AD}$ be l_2. The paper needs to be folded so that p_1 lies on l_1 and p_2 lies on l_2. Then, we get 3rd division, 6th division, and 9th division. In order to prove the sixth and ninth division, we need to prove third division first. It requires one to use two variables and solve a system of equations with substitution; thus, it is a little harder proof.

First, let $\overline{DF}=x$ and $\overline{DG}=y$. Then, we let the side length be 1 so that $\overline{FG}=1-y$. Then we use Pythagorean Theorem on ΔDFG.

$$x^2+y^2=(1-y)^2$$

$$\therefore y=\frac{1-x^2}{2}\cdots①$$

Also, since ΔAEF and ΔDFG are similar triangles,

$$\overline{AE}:\overline{AF} = \overline{DF}:\overline{DG} \ \rightarrow \ \frac{1}{2}:(1-x) = x:y \ \rightarrow \ \frac{y}{2} = x - x^2$$

Substituting in the result from ①,

$$\frac{1-x^2}{4} = x - x^2$$
$$3x^2 - 4x + 1 = 0$$
$$\therefore x = \frac{1}{3} \ \ (x = 1 \text{ is excluded because } \ 0 < x < 1) \ \text{ and } \ y = \frac{4}{9}$$

So $\overline{AF} = 2/3$ and $\overline{FG} = 5/9$.

Next, we use Pythagorean Theorem on ΔAEF to prove the sixth division.

$$\left(\frac{1}{2}\right)^2 + \left(\frac{2}{3}\right)^2 = \overline{EF}^2 \ \rightarrow \ \overline{EF} = \frac{5}{6}$$

Finally, we prove the second ninth division on the top edge by using the similar triangles ΔAEF and ΔHEI.

$$\overline{AE}:\overline{AF} = \overline{HE}:\overline{HI} \ \rightarrow \ \frac{1}{2}:\frac{2}{3} = \frac{1}{6}:\overline{HI} \ \rightarrow \ \overline{HI} = \frac{2}{9}$$

Normally, one would have to make a third division after another in order to make a ninth division. However, with Haga's Third Theorem, two different ninth divisions can be created, in addition to sixth division and third division.

Haga's theorems can teach geometric similarity and mathematical proofs. Teachers can show students the three theorems and encourage students to find out what types of divisions are present and to prove them. The Haga's Theorems can serve as exercises of similar triangles calculations and introduction to proofs. Moreover, they can show the power of origami to create intricate divisions in few simple folds. Also, folding papers with fingers increases brain activity, and allows easier understanding of mathematical concepts; the interactions makes the students absorb the idea of similar triangles and Pythagorean Theorem more easily.

Origami Construction

Compass and straightedge construction – a technique that can draw geometric shapes easily and precisely – is taught in high school to help students develop senses of geometric plane and shapes. Although it is widely accepted and praised, it still has some shortcomings. There are certain maneuvers that cannot be done with this technique; for example, problems such as trisecting an angle or doubling a cube have been proven to be impossible.

Origami construction is more adept than the compass and straightedge construction. Origami construction can do all the moves that can be done with latter technique, with the addition of some moves that cannot be done. The trisecting an angle and doubling a cube can be done with origami construction with few simple folds.

Origami construction consists of six main axioms. These are basic moves, which can be done by folding a paper once, that can build up to construct geometric elements. The list of the axioms is shown below. All of the following axioms are done on a piece of paper.

- Axiom 1: With two points p_1 and p_2, one can make a fold that passes through both points.
- Axiom 2: With two points p_1 and p_2, one can make a unique fold so that p_1 lies on p_2.
- Axiom 3: With two lines l_1 and l_2, one can make a fold that places l_1 on l_2.
- Axiom 4: With a point p_1 and a line l_1, one can make a unique fold through p_1 that is perpendicular to l_1.
- Axiom 5: With two points p_1 and p_2 and a line l_1, one can make a fold through p_2 that places p_1 on l_1.
- Axiom 6: With two points p_1 and p_2 and two lines l_1 and l_2, one can fold paper so that p_1 lies on l_1 and p_2 lies on l_2.

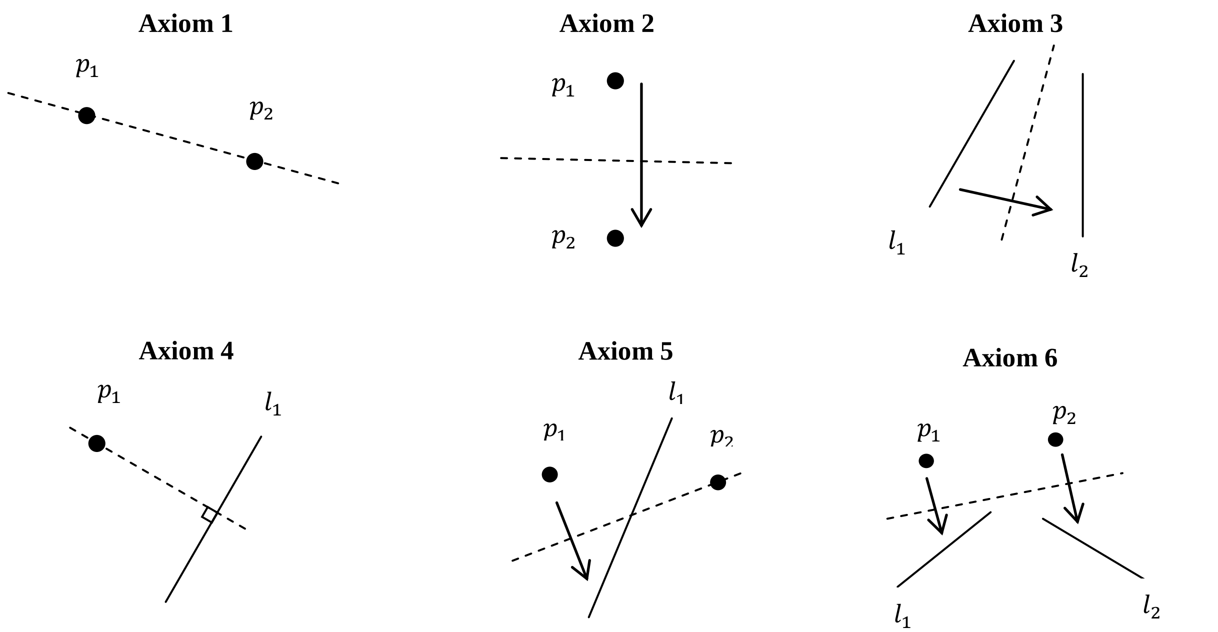


Figure 1. Six basic axioms of origami construction

With these six axioms, one can construct geometric elements, even the unfeasible ones by compass and straightedge construction, with only few simple folds. Using only the first five axioms makes origami construction less useful than compass and straight edge construction; however, with the axiom 6, origami construction becomes more useful and sophisticated. Mathematically speaking, axiom 6 is equivalent to finding a common tangent to two different parabolas. This is equivalent to solving a cubic equation, in which there are three possible answers, and this ability makes origami construction more useful than compass and straightedge construction. To understand why axiom 6 is dealing with tangents of parabola, we must look at axiom 5 first.

Axiom 5 is essentially finding a tangent line of a parabola with p_1 as the focus and l_1 as the directrix. One can even use this axiom to draw a parabola on a paper. On a piece of paper, set p_1 as a point anywhere on the paper, l_1 as the bottom edge of the paper, and p_2 as a moving point on the edge of the paper, starting from one end and ending at the other end. Then repeat axiom 5 for every p_2. The creases will add up to form a parabola, each crease representing a tangent line on a specific point, as shown in figure 2. (In a way, performing axiom 5 is equivalent to doing calculus to find tangent lines.) As shown in figure 3, drawing a perpendicular line from $p_1$and when the paper is folded creates two lines figure 4. It is evident from figure 4 that p_1 is the focus and l_1 is the directrix of the parabola because the point a, where the tangent line touches the parabola, is at equal distance from the directrix and the focus. Thus, axiom 5 is equivalent to finding a certain point on a parabola, which means that axiom 5 can solve quadratic equations. So far, origami construction is not stronger than the compass and straightedge construction, as the latter one can also solve quadratic equation.

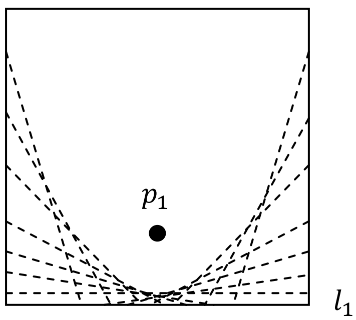


Figure 2. Parabola drawn with axiom 5

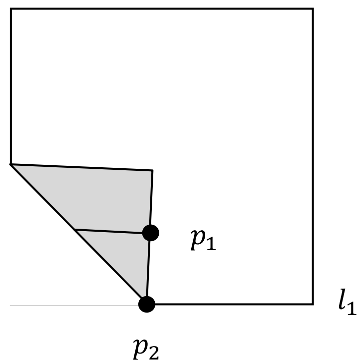


Figure 3. Creating a tangent line to the parabola

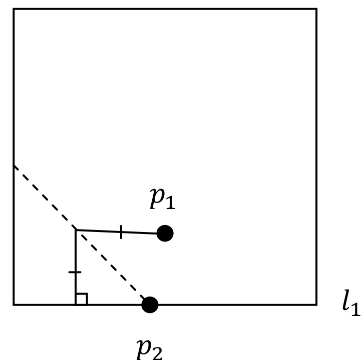


Figure 4. A tangent line to the parabola

Nevertheless, axiom 6 improves the power of origami construction, making it more powerful than compass and straightedge construction. Axiom 6 is similar to doing two axiom 5 simultaneously. From this, one can make a conjecture that axiom 6 finds a line that is tangent to two parabolas, as it is similar to doing axiom 5 twice. Axiom 6 indeed is finding a common tangent line to two parabolas with the foci p_1 and p_2 and directrices l_1 and l_2. Axiom 6 can have three possible answers according to the positions of the points and the lines: no line, one line, or two lines that are tangent to both parabolas. With its ability to solve a cubic equation, axiom 6 extends the breadth of the origami construction. It allows origami construction to trisect an angle or measure $\sqrt[3]{2}$. Hence, it can solve millennium-long problems with few simple folds.

Hisashi Abe developed a simple three step folding technique that allows trisecting an acute angle. First, draw a line with angle θ from the bottom edge of the paper. Then fold a crease that is parallel to the bottom edge. Next, fold the bottom edge to the crease so that the rectangle under the crease is bifurcated. Lastly, do an axiom 6, as shown in figure 5 and 6, so that p_1 is placed on l_1 and p_2 is placed on l_2, ending with fold shown in figure 6. Then extend l_1 and draw another line on the paper. This line trisects the angle θ. To finish, unfold the paper and bisect the angle between bottom edge and newly created line.

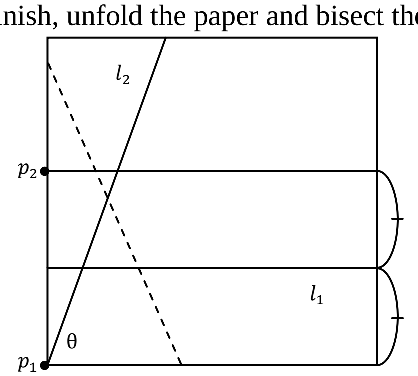


Figure 5. Trisecting an angle

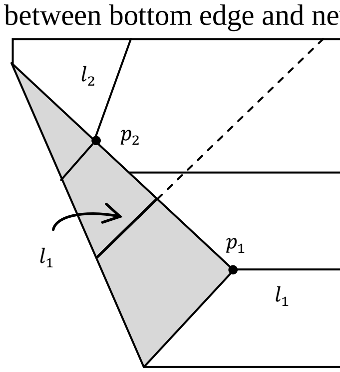


Figure 6. Folding that trisects an angle

Teaching origami construction and solving problems with origami construction to students teaches not only drawing geometric shapes but also understanding the concepts of geometric proof and congruency. The proof of above trisection involves working with similar triangles.

As shown in two figures below, let A, B, and C be the points of intersection of the folded layer and the three lines that trisect the angle. Point D is a perpendicular drop from point C. After the paper is unfolded, there are three triangles: ΔABp_1, ΔBCp_1, and ΔCDp_1. Then, $\overline{AB} = \overline{BC} = \overline{CD}$; $\overline{BC} = \overline{CD}$ because they are both side lengths of one rectangle, and $\overline{AB} = \overline{BC}$ because the horizontal creases are equally spaced. Also, $\overline{AC}$ is perpendicular to $\overline{Bp_1}$. Since ΔABp_1 and ΔBCp_1 both have $\overline{Bp_1}$, right triangle, and same length ($\overline{AB} = \overline{BC}$), these two triangles are congruent triangles with SAS congruence. Because $\overline{Cp_1}$ is created by bisecting $\angle Bp_1D$, $\angle C p_1 B$ is equal to $\angle Cp_1D$. Thus, ΔBCp_1 and ΔDCp_1 are congruent triangles. Also folding bottom edge to $\overline{Bp_1}$ shows that these two triangles fit with each other exactly. Since all three triangles are congruent, their angles at p_1 are the same, so angle θ is trisected.

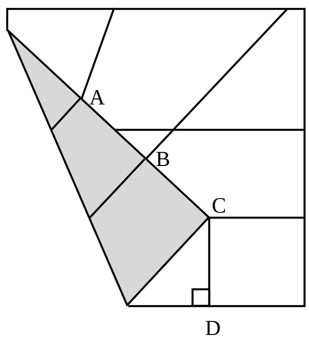


Figure 7. Angle trisecting fold

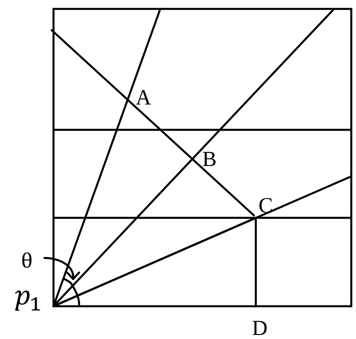


Figure 8. Trisected angle

Doubling a cube also shows the power of axiom 6. The technique developed by Peter Messer allows one to measure $\sqrt[3]{2}$ with only three folds. First, divide the paper into three identical rectangles; one can do this by either using Fujimoto approximation, which is mentioned in later part of the paper, or by exact measuring. Then, with points and lines shown in figure 9, do an axiom 6 so that the paper folds like figure 10. The intersection point C divides side of the paper into two sections, x and y. Then, $x/y = \sqrt[3]{2}$, so one can measure $\sqrt[3]{2}$.

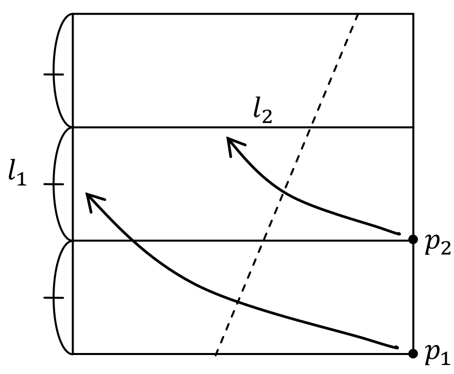


Figure 9. Preparing axiom 6

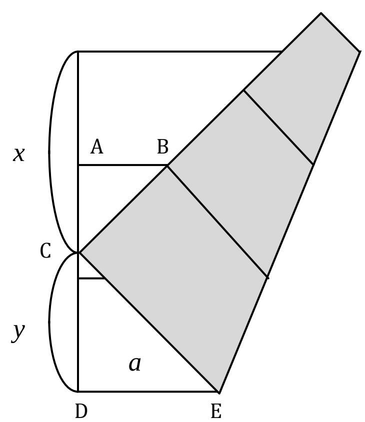


Figure 10. Folding axiom 6

Proving this requires calculating with similar triangles and Pythagorean Theorem. First, let $y = 1$, so that we only need to prove that $x = \sqrt[3]{2}$. Also, let $\overline{DE} = a$. In ΔCDE, $\overline{CE} = 1 + x - a$, and according to Pythagorean Theorem, we get

$$(1 + x - a)^2 = 1^2 + a^2$$

$$a = \frac{x^2 + 2x}{2x + 2}$$

In ΔABC, $\overline{BC} = (x+1)/3$ and $\overline{CA} = x - (x+1)/3 = (2x-1)/3$. Also, since ΔABC and ΔCDE are similar triangles, we can create another equation:

$$\overline{BC}:\overline{CA} = \overline{CE}:\overline{ED}$$
$$\frac{x+1}{3}:\frac{2x-1}{3} = 1 + x - a : a$$
$$2x^2 + x - 1 = 3ax$$

Plugging in the a value from above,

$$2x^2 + x - 1 = \frac{x^2+2x}{2x+2} \cdot 3x$$
$$x^3 = 2$$
$$x = \sqrt[3]{2}$$

Although compass and straightedge construction is more widely recognized, origami construction is also a powerful technique that can be taught in high schools. Paper folding has strong mathematical abilities such as axiom 6; this axiom involves movements that cannot be done with compass and straightedge construction. Teaching geometric construction with origami can be an excellent alternative as it is more powerful than compass and straightedge construction.

Quadratic and Cubic Equations

At first glance, origami seems to have only geometric properties to it. However, relating origami with geometric analysis could lead to applications in graphing and finding the roots of an equation. Although one can manually find the roots by drawing a graph by folding and measuring, there is more efficient method of solving a generic quadratic and cubic equation with origami. As mentioned in Origami Construction section, one can use axiom 5 to solve quadratic equation, and axiom 6 to solve cubic equation.

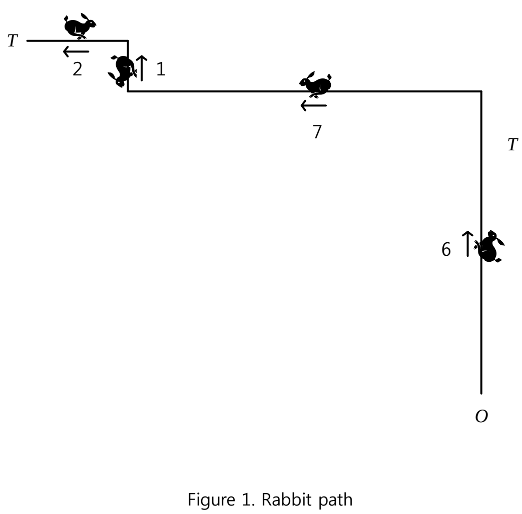


Figure 1. Rabbit path

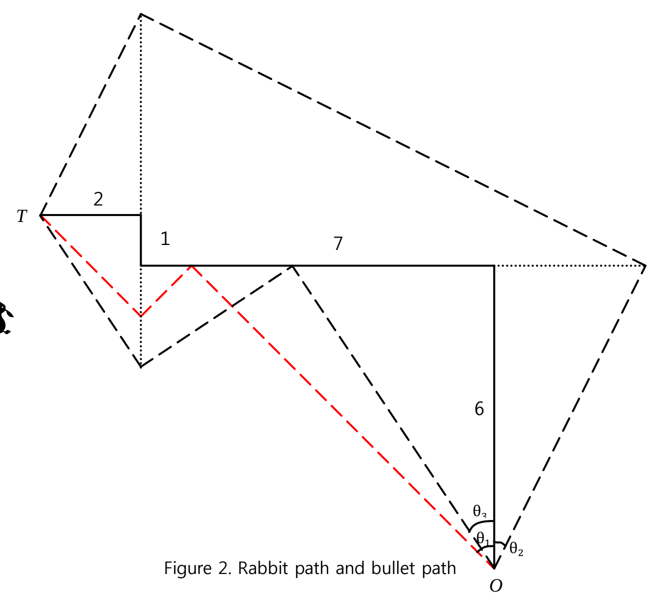


Figure 2. Rabbit path and bullet path

Axiom 5 and 6 can be combined with Lill's method, a geometric method of solving equations in the form of $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$, to solve quadratic and cubic equations for real roots.

Lill's method, developed by Eduard Lill in the end of 19th century, involves drawing a 'path' starting from a point *O* and ending at a point *T*. An easy way of understanding the 'path' is using the example of rabbit and hunter. Rabbit begins running at origin *O*. First, it runs north by the coefficient of term with highest degree. Then, the rabbit turns 90° counterclockwise, then runs the distance of the coefficient of the next highest degree term. If a coefficient is negative, the rabbit runs the opposite way, keeping the direction it is facing. (It is running backwards by an absolute value of the negative coefficient) After processing through all the coefficients, rabbit stops at point *T*. Then, the hunter, who is at origin *O*, tries to find the angle between the rabbit's path and the path of the bullet so that the bullet reflects and refracts off each segment of the path and hits the rabbit. The bullet bounces off anything at an angle of 90°. The bullet can also refract off of an extension of a segment of path. The following diagram shows the path of rabbit with equation $6x^3 + 7x^2 - x - 2 = 0$ and the bullet angles θ_n. (In figures 1 and 2, thinly dotted lines are extensions of path.)

Each angle, measured to the left of the rabbit's path, is used to find distinct roots of the equation. For each angle θ_n, $-\tan\theta_n$ is a root of the equation. In the diagram above, there are three angles.

$$\theta_1 = 45°, \theta_2 \approx -26.6°, \theta_3 \approx 33.7°$$

So the roots of this equation are

$$-\tan(45) = -1, -\tan(-26.6) = 0.5, -\tan(33.7) = -0.67$$

Plugging in each root, one can see that they are indeed roots of the equation above.

Lill's method can be proved with trigonometry. As the bullet bounces off, it creates similar triangles. These similar triangles are used to prove Lill's method. For example, in the case above, we pick an angle, $\theta_1 = 45°$. The red line denotes the path of the bullet with this angle. We can see that there are three similar triangles that are created by this path. In the first triangle, the biggest one, the horizontal leg can be written as $6\tan\theta_1$. Then, the horizontal leg of the next triangle, the smallest one, can be written as $7 - 6\tan\theta_1$. Subsequently, the vertical leg of the same triangle can be written as $\tan\theta_1(7 - 6\tan\theta_1)$. Similarly, the next triangle has vertical leg $1 + \tan\theta_1(7 - 6\tan\theta_1)$ and horizontal leg $\tan\theta_1(1 + \tan\theta_1(7 - 6\tan\theta_1))$. This length of horizontal leg is 2. So, we get the equation

$$\tan\theta_1\left(1 + \tan\theta_1(7 - 6\tan\theta_1)\right) = 2$$

$$-6\tan^3\theta_1 + 7\tan^2\theta_1 + \tan\theta_1 - 2 = 0$$

The original equation was

$$6x^3 + 7x^2 - x - 2 = 0$$

Hence, we can conclude that

$$x = -\tan\theta_1$$

Lill's method cannot be easily done with manual tools such as compass and ruler. However, origami axiom can be used to solve a quadratic or cubic equation with Lill's method.

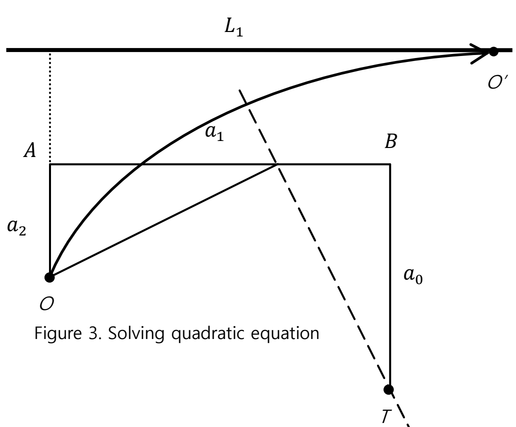


Figure 3. Solving quadratic equation

Axiom 5 is used to solve quadratic equation in the form of $a_2x^2 + a_1x + a_0 = 0$. First step is to draw the rabbit path on a paper. Graphing paper is ideal as the grids provide guidelines. Also, to facilitate folding, the lengths can be scaled; since we are trying to find angles, as long as the ratios are the same, the result will not be affected. Let first and second turning points be *A* and *B*, respectively. Then, as shown in figure 3, draw a line that is parallel to $\overline{AB}$ and distance of $\overline{OA}$ away from point *A*. Call this line L_1. Then, use axiom 5 to find a crease that passes through T and places O on L_1. Draw a perpendicular line from O to the crease, and this line and the crease combine to form the bullet path. Finally, simply measure the angles and solve for the roots. The number of possible creases will be equal to number of real roots of the equation.

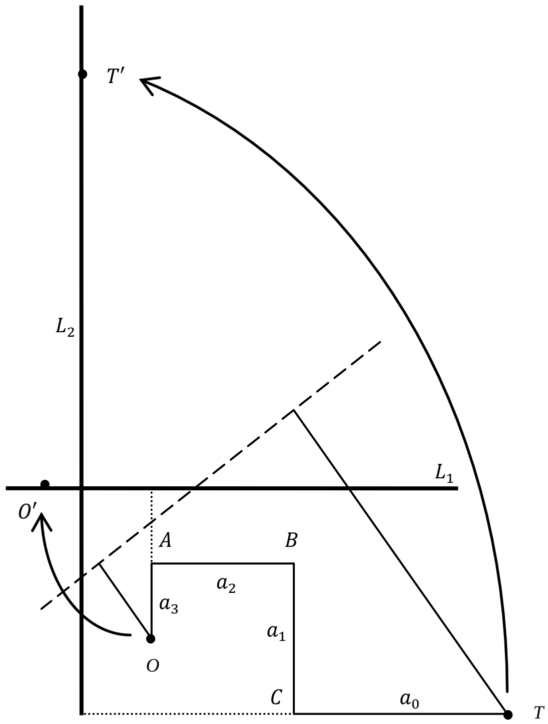


Figure 4. Solving cubic equation

Solving cubic equation requires axiom 6. Similar to solving quadratic equations, we first draw the rabbit path. Then, label each turn as A, B, and C. Draw the line L_1, parallel to $\overline{AB}$ and distance of $\overline{OA}$ away from point A. Also, draw the line L_2, parallel to $\overline{BC}$ and distance of $\overline{TC}$ away from point C. Then, use axiom 6 to find a crease that places O on L_1 and T on L_2. Finally, draw two perpendicular lines form the crease to point O and T. These two perpendicular lines and the crease combine to form the bullet path. Finally, measure the angles to find the roots. As mentioned above, the number of possible creases will be equal to number of real roots of the equation. Axiom 6 can be tricky sometimes, so it might require some practice folds before one can easily find the crease.

Lill's method combined with origami axioms can be a surprising method of solving quadratic and cubic equation. Students can learn that equations can be solved not only by algebra and factoring but also by trigonometry and paper folding. Moreover, although Lill's method has been forgotten because of lack of tools to carry it out, it can be used more widely as origami provides a method of doing Lill's method to solve quadratic and cubic equations. Unless one has graphing calculator, it is complex to solve cubic equations, especially if they do not factor. However, with only a piece of paper, pencil, and a protractor, one can easily solve any cubic equations by using Lill's method and origami, even if the equations do not factor.

Exponential Decay

The exponential growth and decay are subjects that are taught often in high school. The common application problems deal with bacteria growth or money investment. However, these examples are not easy to demonstrate in a math classes. Also, students are not always familiar with these examples to understand exponential concepts clearly. There is a relationship between origami and mathematics, specifically exponential decay, that can be easily shown in classroom to give an example of exponential decay. It is called Fujimoto Approximation.

Fujimoto Approximation is a technique that enables the folder to mark a $1/n$th division crease on a paper with relatively small error. It begins with an approximate value with error, but each step decreases the error exponentially. For example, if the folder wants to fold a $1/5$th mark on a paper, the folder first makes an approximate crease for $1/5$ from the left side of the paper. Using the first crease, the folder makes an approximate $3/5$ mark on the paper by folding the right side of the paper to the first crease. Then, the folder folds the right end of paper to the approximate $3/5$ mark to make an approximate $4/5$ mark. Similarly, the folder makes $2/5$ pinch. Lastly, by folding the left side to the

$2/5$ crease, the folder makes a final pinch. This final pinch will be very close to the actual $1/5$ mark of the paper.

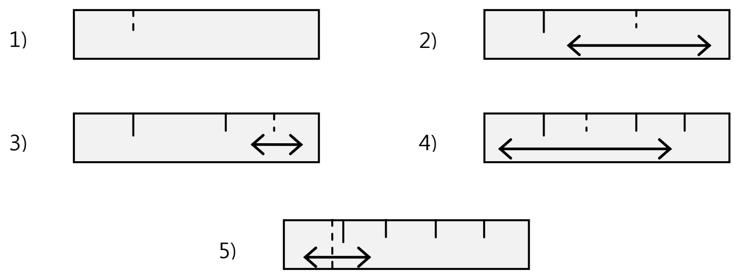


Figure 1. Fujimoto Approximation – taken from 'Project Origami' by T. Hull

When origami artists design their model, it is important that they make very accurate division marks because these marks will determine if the work will be perfectly folded or not. Many origami artists use Fujimoto approximation because they can easily create an accurate $1/\mathrm{nth}$ marks. There are other ways to make theoretically perfect $1/\mathrm{nth}$ marks, but Fujimoto approximation creates a crease with relatively same error; even if other methods are theoretically perfect, there are still some errors caused by folder. There are many ways Fujimoto approximation can be mathematically analyzed, such as using binary decimals or number theory. However, using exponential decay is the simplest method, and most helpful to high school students.

The theoretically perfect method of making $1/\mathrm{nth}$ marks tend to make errors after each fold because it is almost impossible for folders to fold a paper perfectly with absolutely no error. Thus, after every fold, the errors accumulate. Nonetheless, with Fujimoto approximation, folders decrease error with each fold. Figure 2 illustrates the decreasing error. E is the initial error made from making a guess pinch.

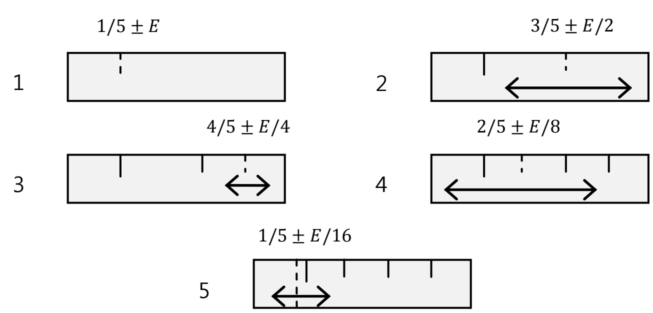


Figure 2. Decreasing errors after each fold – taken from 'Project Origami' by T. Hull

As each fold divides a section into two, the error is also divided into halves. In other words, each fold is exponentially decreasing the error, making the error very close to 0 in the end. This can be mathematically written as $E_f = E_i/2^n$, where n is the number of folds, E_f is the final error, and E_i is the initial error. Since E_i is determined by the first fold, it is a constant. Hence, a function f can be created: $E_f = f(n) = E_i/2^n$. Figure 3 shows the graph of this function. As n increases, the final error approaches 0. Mathematically, $\lim_{n\to\infty} E_i/2^n = 0$. In other words, the final division line created by Fujimoto approximation will have error very close to zero.

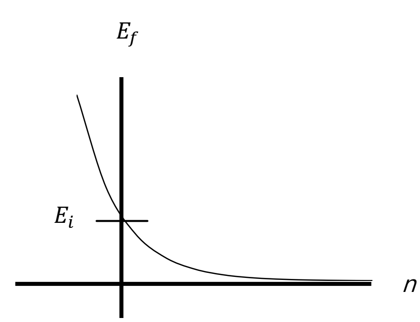


Figure 3. Graph of $f(n)$

Surprising aspect of Fujimoto approximation is that the error is zero in the end even though it starts with a big error. Similar to other origami manipulations, Fujimoto approximation is based on mathematical equations. Teaching these equations to students can be a surprising and interactive method of teaching exponential decay. Unlike the classic examples of exponential decay such as bacteria growth or money investment, Fujimoto approximation is an example that can be used immediately in classroom with just a piece of paper to help students understand exponential decay.

Bibliography

Hull, Thomas, C. "Solving Cubics With Creases: The Work of Beloch and Lill." *The American Mathematical Monthly*, 118.4 (2011): 307-315.

Hull, Thomas. *Project Origami: activities for exploring mathematics*. Wellesley: AK Peters, Ltd., 2006

Kim, Hyang Sook. *Understanding geometry with origami.* 2nd edition. Goyang: Kyung Moon Sa, 2010

Lang, Robert, James. *Origami^4: Fourth International Meeting of Origami, Science, Mathematics, and Education.* Natick: A K Peters, Ltd., 2009.

Lang, Robert. "Origami and Geometric Constructions." *Robert J. Lang Origami*. 2010.

"Origami Science." *Origami Resource Center*. 28 August 2011. <http://www.origami-resource-center.com/origami-science.html>

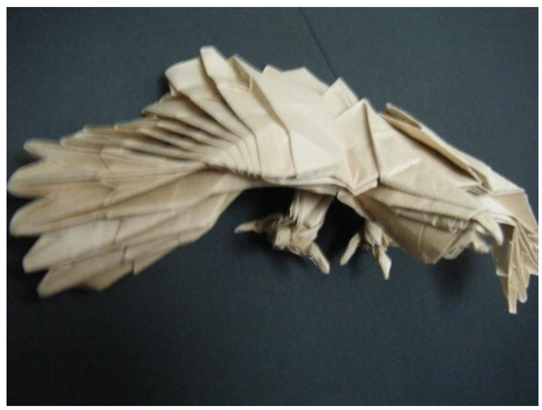
O'Rourke, Joseph. *How To Fold It.* New York: Cambridge University Press, 2011

<Origami models>

Divine Dragon
Designed by Kamiya Satoshi
Folded by Woo Jung Lee

Bull Moose, opus 413
Designed by Robert Lang
Folded by Woo Jung Lee

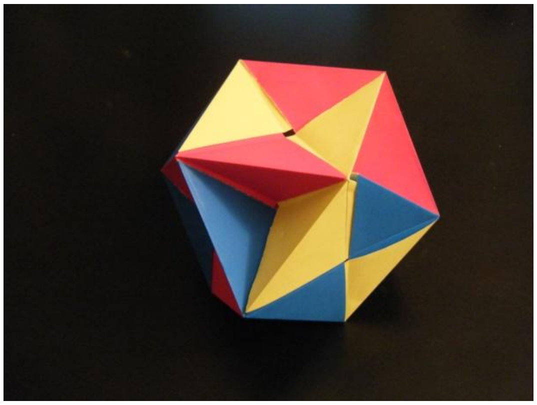
Wizard
Designed by Kamiya Satoshi
Folded by Woo Jung Lee



Eagle
Designed by Nguyen Hung Cuong
Folded by Woo Jung Lee

Kirin
Designed by Kamiya Satoshi
Folded by Woo Jung Lee

Unicorn
Designed by Kamiya Satoshi
Folded by Woo Jung Lee

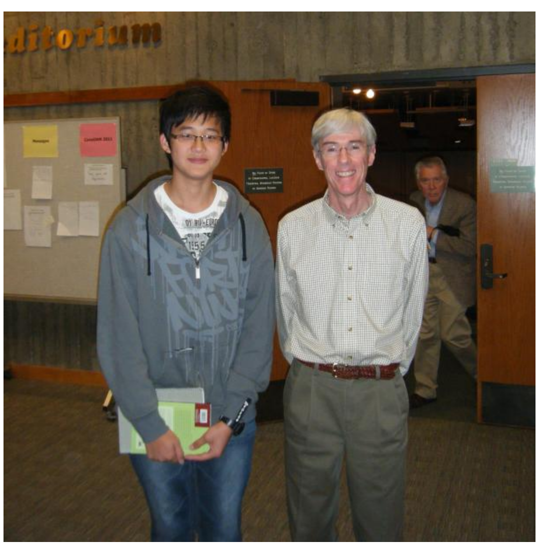


Butterfly Bomb
Designed by Kenneth Kawamura
Folded by Woo Jung Lee

Spinosaurus
Designed by Kamiya Satoshi
Folded by Woo Jung Lee

<Origami professionals>

I would like to thank Professor Joseph O'Rourke and Doctor Robert J. Lang for assisting me with this research.



Professor O'Rourke at Smith College

Dr. Robert Lang, former researcher at NASA