Probability and Stats

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December 2, 2023

1 Random Variables

We can model probabilistic events based on *outcomes*, which are the outcome of an experiment or a trial (e.g., getting a "3" on the roll of a die), and a *sample space*, which is a set of possible outcomes, and *events*, which are a subset of the sample space.

A **random variable** is then, formally, simply a mapping from outcomes in the sample space to the set of real numbers. For example consider a 6-sided die with sides $\{1, 2, 3, 4, 5, 6\}$. Some possible outcomes are 1, 3, or 5, and the sample space is $\{1, 2, 3, 4, 5, 6\}$. The probability of each outcome for a fair die, deteremined by some random variable, is $1/6$. We can define an event A representing the case that the roll is odd i.e., $A = \{1, 3, 5\}$.

1.1 Expected Value

The **expected value** of a random variable X can be thought of as the "average" or "mean" value attained by that random variable. Formally, expected value is defined as

$$E[X] = \sum_x x * P(X = x)$$

for every outcome x in the sample space of X. For example, for a fair, 6-sided die, the expected value is

$$\frac{1}{6} * 1 + \frac{1}{6} * 2 + \frac{1}{6} * 3 + \frac{1}{6} * 4 + \frac{1}{6} * 5 + \frac{1}{6} * 6 = 3.5$$

Or if we have, say, a 4-sided die where each roll has some "payoff", e.g.

- $1 \mapsto 1$
- $2 \mapsto 2$
- $3 \mapsto 3$
- $4 \mapsto 10$

our expected payoff from rolling this die is

$$\frac{1}{4} * 1 + \frac{1}{4} * 2 + \frac{1}{4} * 3 + \frac{1}{4} * 10 = 4$$

Some Examples

Coin Toss Game Assume there are two gamblers playing a coin toss game. Gambler A has $(n+1)$ fair coins, and B has n fair coins. What is the probability that A will have more heads than B if both flip all of their coins?

We cna look at this problem symmetrically if we imagine a slightly simplified scenario where A and B both have n coins. Then, we know that for the following 3 scenarios

- A flips more heads than B.
- A flips the same number of heads as B.
- A flips fewer heads than B.

the first and third are symmetric, and so must have equal probabilities. Then, we really only need to consider the second case, which is the case where A's $n + 1$-th flip would actually make a difference.

Card Game A casino offers a card game from a deck of 52 cards with 4 cards each for $2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$. You pick up a card from the deck and the dealer picks another one with replacement. What is the probability that you picked a larger card than the dealer's?

Well, one way to analyze this is by considering all possible choices of two cards C_1 and C_2. The probability of pciking any particular suit for C_1 is $4/52 = 1/13$. We can choose to then simply analyze the probability of getting a smaller card for each possible suit case. Basically, for each suit, the probability of picking a lesser card for C_2 is $4/51 * k$, where k is 1 less than the rank of the suit. So, in general, we can compute the overall probability as

$$\left(\frac{1}{13} * \frac{4}{51} * 0\right) + \left(\frac{1}{13} * \frac{4}{51} * 1\right) + \left(\frac{1}{13} * \frac{4}{51} * 2\right) + \cdots + \left(\frac{1}{13} * \frac{4}{51} * 12\right) = 0.4705$$