Probability and Stats

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1 Random Variables

We can model probabilistic events based on *outcomes*, which are the outcome of an experiment or a trial (e.g., getting a "3" on the roll of a die), and a *sample space*, which is a set of possible outcomes, and *events*, which are a subset of the sample space.

A **random variable** is then, formally, simply a mapping from outcomes in the sample space to the set of real numbers. For example consider a 6-sided die with sides $\{1, 2, 3, 4, 5, 6\}$. Some possible outcomes are 1, 3, or 5, and the sample space is $\{1, 2, 3, 4, 5, 6\}$. The probability of each outcome for a fair die, deteremined by some random variable, is $1/6$. We can define an event A representing the case that the roll is odd i.e., $A = \{1, 3, 5\}$.

1.1 Expected Value

The **expected value** of a random variable X can be thought of as the "average" or "mean" value attained by that random variable. Formally, expected value is defined as

$$E[X] = \sum_x x * P(X = x)$$

for every outcome x in the sample space of X. For example, for a fair, 6-sided die, the expected value is

$$\frac{1}{6} * 1 + \frac{1}{6} * 2 + \frac{1}{6} * 3 + \frac{1}{6} * 4 + \frac{1}{6} * 5 + \frac{1}{6} * 6 = 3.5$$

Or if we have, say, a 4-sided die where each roll has some "payoff", e.g.

- $1 \mapsto 1$
- $2 \mapsto 2$
- $3 \mapsto 3$
- $4 \mapsto 10$

our expected payoff from rolling this die is

$$\frac{1}{4} * 1 + \frac{1}{4} * 2 + \frac{1}{4} * 3 + \frac{1}{4} * 10 = 4$$

Expected flips until k heads As another example, consider computing the expected number of flips of a fair coin until two heads appear in a row. We can think about this as a model where we sample by continuing to flip a coin until we get two heads, and then consider the length of the flip sequence we needed to get those two heads. Then, we want to consider the average number of flips it takes over all possible such flips sequences to get two heads in a row.

Start with simpler: number of expected flips to get 1 heads. Can consider all possible sequence prefixes of length k that end in a head. Probability of seeing that sequence is $\frac{1}{2}^k$, so expected number of flips to get 1 heads is

$$\sum_{k=1} k \cdot \frac{1}{2}^k = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots = 2$$

1.2 Variance

The **variance** of a random variable X is roughly a measure of how spread out it is from its mean. It is given as

$$Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

Some Examples

Coin Toss Game Assume there are two gamblers playing a coin toss game. Gambler A has $(n+1)$ fair coins, and B has n fair coins. What is the probability that A will have more heads than B if both flip all of their coins?

We cna look at this problem symmetrically if we imagine a slightly simplified scenario where A and B both have n coins. Then, we know that for the following 3 scenarios

- A flips more heads than B.
- A flips the same number of heads as B.
- A flips fewer heads than B.

the first and third are symmetric, and so must have equal probabilities. Then, we really only need to consider the second case, which is the case where A's $n+1$-th flip would actually make a difference.

Card Game A casino offers a card game from a deck of 52 cards with 4 cards each for $2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$. You pick up a card from the deck and the dealer picks another one with replacement. What is the probability that you picked a larger card than the dealer's?

Well, one way to analyze this is by considering all possible choices of two cards C_1 and C_2. The probability of pciking any particular suit for C_1 is $4/52 = 1/13$. We can choose to then simply analyze the probability of getting a smaller card for each possible suit case. Basically, for each suit, the probability of picking a lesser card for C_2 is $4/51 * k$, where k is 1 less than the rank of the suit. So, in general, we can compute the overall probability as

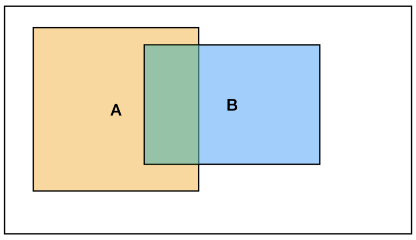
$$\left(\frac{1}{13} * \frac{4}{51} * 0\right) + \left(\frac{1}{13} * \frac{4}{51} * 1\right) + \left(\frac{1}{13} * \frac{4}{51} * 2\right) + \cdots + \left(\frac{1}{13} * \frac{4}{51} * 12\right) = 0.4705$$

2 Conditional Probability and Bayes' Formula

In standard probability notions, we may have some event A and want to know the probability of A occurring. If, however, we have two separate events, A and B, we can also consider the probability that one event occurs *given* that the other occurs. This is known as conditional probability i.e., for events A and B, the conditional probability of A *given* B is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

You can think of this formula visually:



That is, if we consider $P(A|B)$, we can imagine this is like saying our sample space is now shrunk just to B, and so the probability of A in this reduced space is dependent on the size of the intersection between A and B as a proportion of B, as represented in the formula.

2.1 Bayes' Rule

Bayes' rule provides an alternate expression for conditional probability. For two events A and B, it states:

$$P(A \mid B) = \frac{P(B|A) * P(A)}{P(B)}$$

You can think of this as deriving from the basic conditional probability statement above since $P(B \mid A)$ can be viewed as the probability of the intersection between A and B as a proportion to A as our sample space, and then we scale this up by factor of $P(A)$ to explicitly get $P(A \cap B)$ in the numerator.

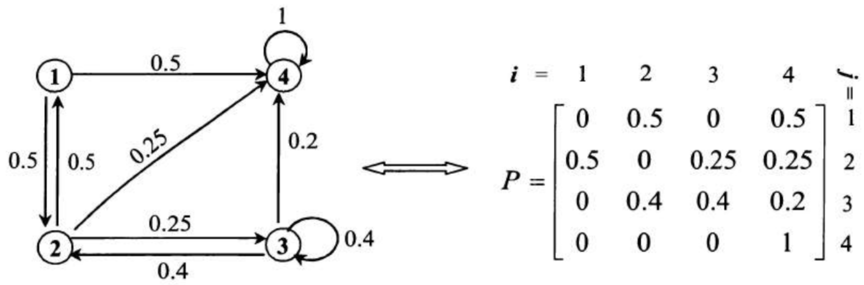
Examples

For example, if I roll a fair 6-sided die, then the probability of rolling a 4 is 1/6. But, if I tell you upfront that the die roll seen is >= 4, then the probability that the roll is a 4 is actually now 1/3, since there are now only 3 possible outcomes in the reduced/conditioned sample sample ($\{4, 5, 6\}$) and our outcome 4 is one of them.

3 Markov Chains

A **Markov chain** is a sequence of random variables $X_0, X_1, \ldots, X_n, \ldots$ with the Markov property that, given the present state, the future states and past states are independent. In other words, once the current state is known, past history has no bearing on the future. A Markov chain with M states can be completely described by an $M \times M$ transition matrix P and the initial probabilities $P(X_0)$, where p_{ij} is the probability of transitioning from state i to state j. The probability of a particular path of states then, is the product of the probabilities of taking each corresponding transition in that path.

A **transition graph** is more commonly used to graphically represent the transition matrix i.e.



$$\begin{array}{ccccccc} & i = & 1 & 2 & 3 & 4 & j = \\ P = & \left[\begin{array}{c} 0 \\ 0.5 \\ 0 \\ 0 \end{array} \right. & \begin{array}{c} 0.5 \\ 0 \\ 0.4 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0.25 \\ 0.4 \\ 0 \end{array} & \begin{array}{c} 0.5 \\ 0.25 \\ 0.2 \\ 1 \end{array} & \left. \right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

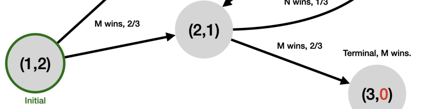
Figure 5.1 Transition graph and transition matrix of the Play

Note that a state i is called **absorbing** if it is impossible to leave this state.

Example: Gambler's Ruin As an example, consider the gambler's ruin problem. Say player M has \$1 and play N has \$2. Each game gives the winner of that game \$1 from the other. As a better player, M wins 2/3 of the games, and they play until one of them is bankrupt. What is the probability that M wins?

We can model this as a Markov chain where states are tuples of the account values of M and N, with initial state of $(1, 2)$. At each state, there are then two possible transitions: one where M wins and one where N wins, the former having probability 2/3 and the latter having probability 1/3. The M winning transition goes from a state (m, n) to $(m+1, n-1)$, and similarly for the N winning transition. A winning (losing) state is one where one player's account balance is ≤ 0. So, we want to determine the probability of reaching a state where M wins, starting from the initial state.

The Markov chain (i.e. probabilistic transition system) for this problem can be drawn as follows (where bankrupt values are shown in red):



4 Basic Calculus

Common Derivative Rules

- **Power Rule**: For $f(x) = ax^n$, we have

$$f'(x) = n \cdot ax^{n-1}$$

- **Product Rule**: For functions $f(x)$ and $g(x)$, we have

  $$(f \cdot g)' = f'g + fg'$$

  i.e., the derivative of the product of f and g can be remembered as “the derivative of f times g plus f times the derivative of g”.

- **Chain Rule**: For two functions $f(x)$ and $g(x)$, and their composition $h(x) = f(g(x))$, we have

  $$h'(x) = f'(g(x))g'(x)$$

  Alternately stated as:

  $$(f \circ g)' = (f' \circ g) \cdot g'$$

Derivatives of Common Functions

- $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- $\frac{d}{dx}(c^x) = c^x \ln c$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$