Probabilistic Tools

William Schultz

September 16, 2022

See [1] for a reference on some probabilistic tools.

Union Bound

Let $E_1, \ldots, E_n$ be arbitrary events in some probability space. Then,

$$Pr\left[\bigcup_{i=1}^{n} E_i\right] \leq \sum_{i=1}^{n} Pr[E_i]$$

That is, for a set of events $E_1, \ldots, E_n$, the probability of at least one event occurring is less than or equal to the sum of the probabilities of the individual events.

Expectation and Variance

For a discrete random variable X taking values in some set $\Omega \subseteq \mathbb{R}$, its *expectation* is defined as

$$\mathbb{E}\left[X\right] = \sum_{\omega \in \Omega} \omega \cdot Pr\left[X = \omega\right]$$

and its *variance* is defined as

$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}\left[X\right])^2\right]$$

That is, expectation is essentially a weighted sum of the values that the random variable can take on, where each value is weighted by the probability of that event occurring, and variance is essentially a measure of the average deviation of the variable from its expectation/mean.

Markov's Inequality

Markov's inequality is an elementary large deviation bound valid for *all* non-negative random variables. Let X be a non-negative random variable with $\mathbb{E}\left[X\right] > 0$. Then, for all $\lambda > 0$

$$Pr(X \geq \lambda \mathbb{E}\left[X\right]) \leq \frac{1}{\lambda}$$

That is, this establishes a bound, for some given parameter λ, on how likely a random variable is to be far away from its mean. For example, if $\lambda = 10$, then this means that the probability that X is greater than 10 times its mean is $\leq \frac{1}{10}$. Note that this bound doesn't take into account anything about the actual distribution (only about its mean), so it may serve as only a rough estimate.

Chebyshev's Inequality

Let X be a random variable with $\mathrm{Var}(X) > 0$, and where $\sigma = \sqrt{\mathrm{Var}(X)}$ is its standard deviation. Then, for all $\lambda > 0$

$$Pr\left[|X - \mathbb{E}\left[X\right]| \geq \lambda\sigma\right] \leq \frac{1}{\lambda^2}$$

This bound tells us something about how far a random variable is from its expectation, in terms of the variance of the RV. That is, it puts a bound on the probability of how many (λ) standard deviations (σ) away from its mean X may be.

Chernoff Bound

TODO.

References

[1] Benjamin Doerr. Probabilistic tools for the analysis of randomized optimization heuristics. In *Natural Computing Series*, pages 1–87. Springer International Publishing, nov 2019.