SAT Solving with Conflict Driven Clause Learning

William Schultz

CS 7240 Final Project

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Overview and Project Goals

- Satisfiability is the canonical NP-complete problem.
- Much work has been devoted to building efficient SAT solvers over last decades.
- **Project Goal:** Implement a basic SAT solver based on *conflict driven clause learning* (CDCL), the dominant core technique used in modern solvers.
  - Gain a deeper understanding of the DPLL and CDCL based algorithms for SAT solving
  - Use as a platform for potentially exploring new SAT solving techniques
  - E.g. learning heuristics using a data-driven approach, extending methods of *CrystalBall* [SKM19]

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CNF notation:

$$\{\{x_1, x_2\}, \{\neg x_3, \neg x_1\}\}$$

DPLL: SAT as Search

- A basic approach to solving SAT is to view it as a search problem over possible assignments.
- This is the basis of the Davis–Putnam–Logemann–Loveland (DPLL) algorithm [DLL62]
- Basic idea of DPLL is to do a depth first, brute force search with backtracking along with some basic formula simplification as you go.
  - Also employs the *unit propagation rule*

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$$\{\} \quad (\text{SAT})$$

DPLL: Example

$\{\neg a, b\}$
$\{\neg b, \neg c\}$
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$$\{\neg a, b\}$$
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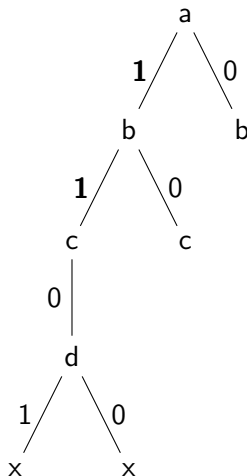$$\{c, \neg d\}$$

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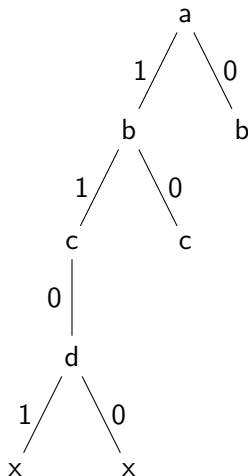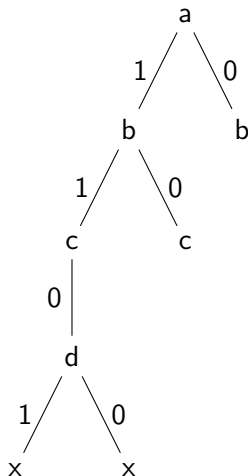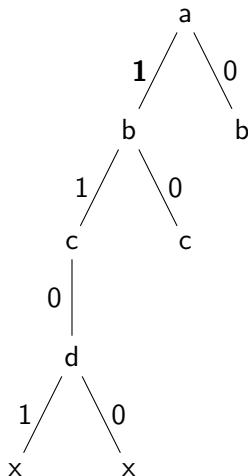
DPLL: Example

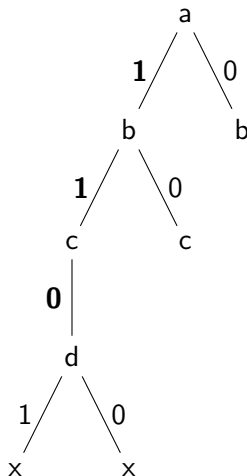$\{\neg a, b\}$
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unit propagate b

DPLL: Example

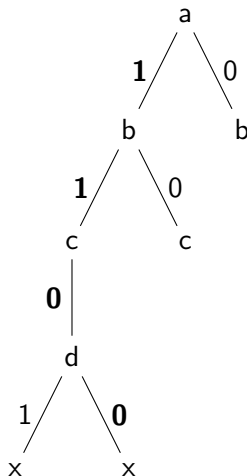$\{\neg a, b\}$
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$\{c, \neg d\}$

unit propagate $\neg c$

DPLL: Example

$\{\neg a, b\}$

$\{\neg b, \neg c\}$

$\{c, \neg d\}$

unit propagate $\neg d$

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- When you encounter a conflict in the search tree, *learn* a clause that prevents you from making the similar mistakes again
- This fundamental approach is known as *conflict-driven clause learning* (CDCL) and started being employed in SAT solvers around the late 90s and early 2000s.
- In addition, employ *non-chronological backtracking*

CDCL Example

c_1 $\{a, b\}$

c_2 $\{b, c\}$

c_3 $\{\neg a, \neg x, y\}$

c_4 $\{\neg a, x, z\}$

c_5 $\{\neg a, \neg y, z\}$

c_6 $\{\neg a, x, \neg z\}$

c_7 $\{\neg a, \neg y, \neg z\}$

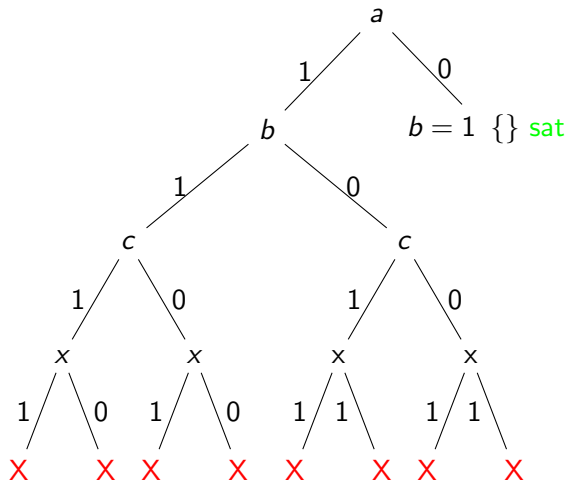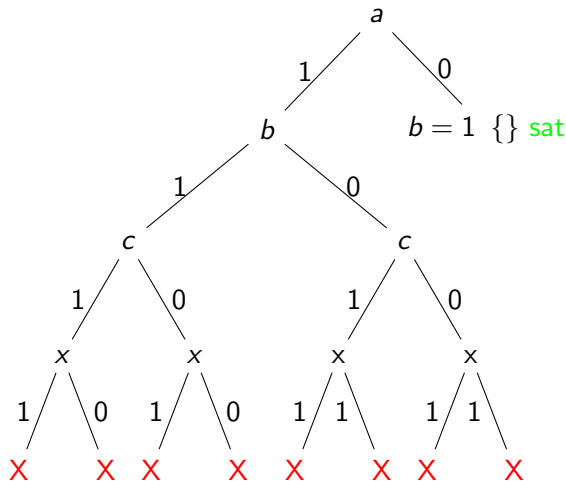
Figure: Basic DPLL termination tree. Explores large portion of left search tree.

CDCL Example

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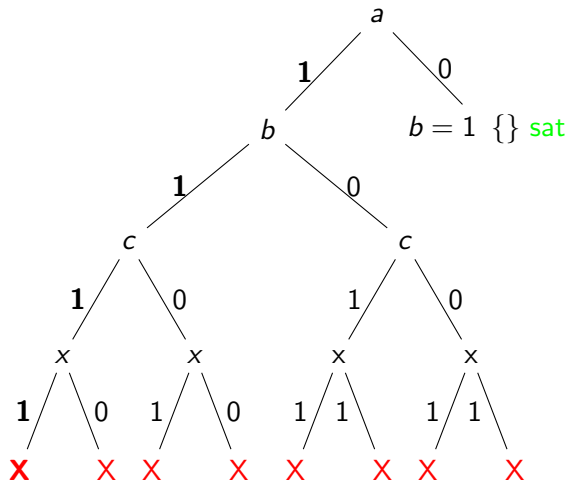c_6 $\{\neg a, x, \neg z\}$
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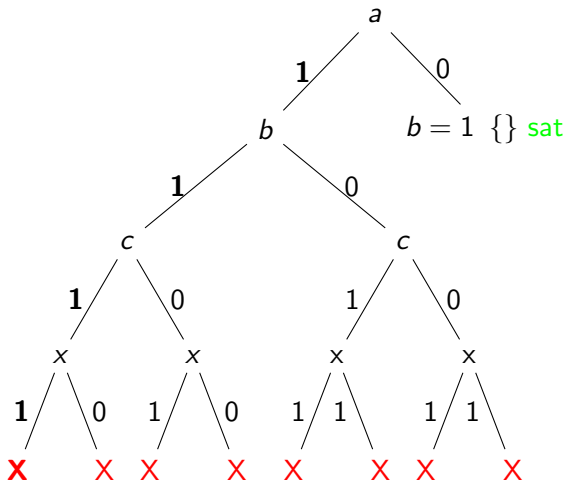c_6 $\{\neg a, x, \neg z\}$
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unit propagate y

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CDCL Example

c_1 $\{a, b\}$
c_2 $\{b, c\}$
c_3 $\{\neg a, \neg x, y\}$
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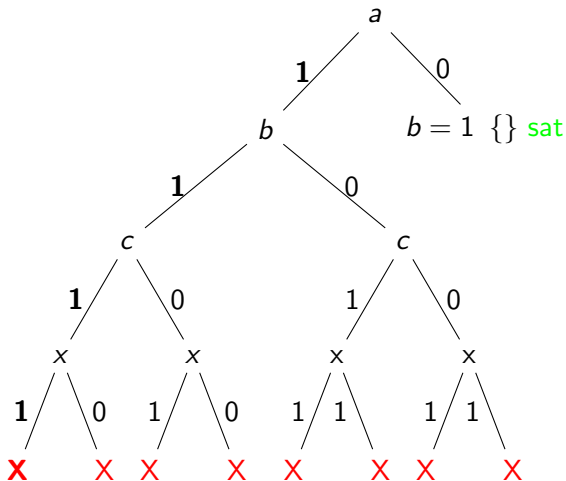
unit propagate z

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c_2 $\{b, c\}$

c_3 $\{\neg a, \neg x, y\}$

c_4 $\{\neg a, x, z\}$

c_5 $\{\neg a, \neg y, z\}$

c_6 $\{\neg a, x, \neg z\}$

c_7 $\{\neg a, \neg y, \neg z\}$

Note that b and c are irrelevant to the c_7 conflict. $(a \wedge x)$ or $(a \wedge y)$ are sufficient.
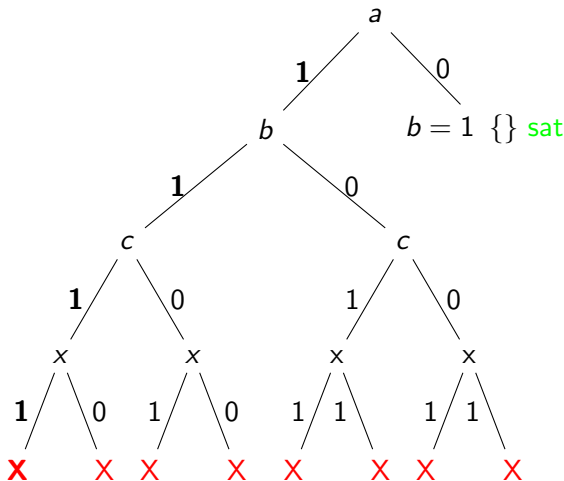

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So, we can learn $\neg(a \wedge x) = (\neg a \vee \neg x)$ as a new constraint i.e. a *learned clause*.
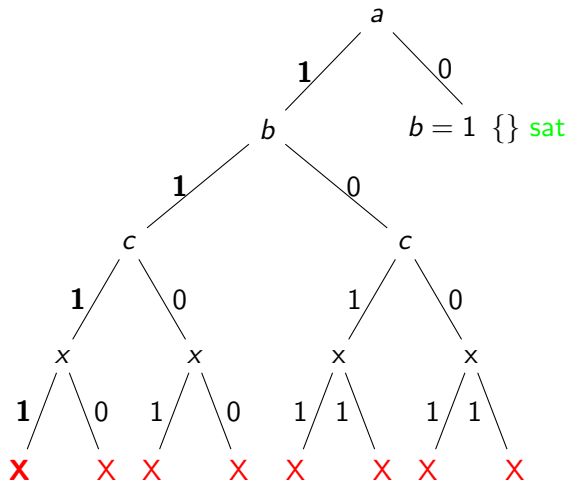
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c_1 $\{a, b\}$

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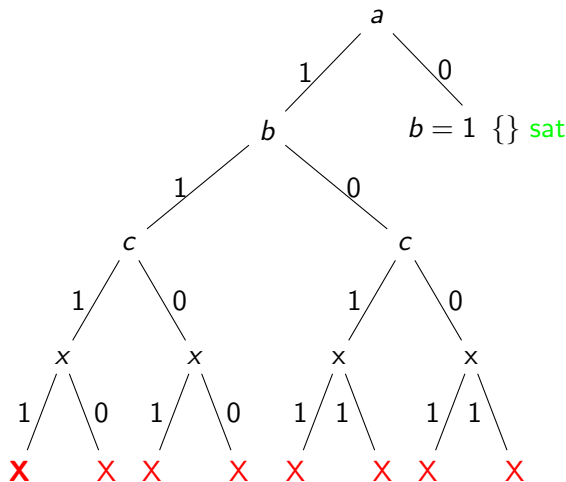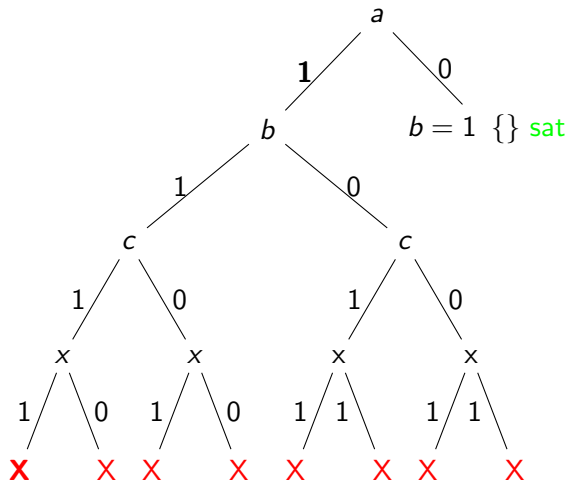
c_7 $\{\neg a, \neg y, \neg z\}$

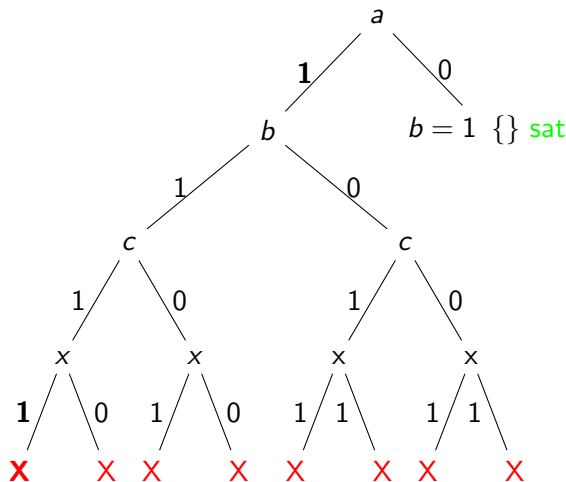Figure: Basic DPLL termination tree. Explores large portion of left search tree.

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c_7 $\{\neg a, \neg y, \neg z\}$

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CDCL Example

$$
\begin{array}{ll}
l_1 & \{\neg a, \neg x\} \\
\hline
c_1 & \{a, b\} \\
c_2 & \{b, c\} \\
c_3 & \{\neg a, \neg x, y\} \\
c_4 & \{\neg a, x, z\} \\
c_5 & \{\neg a, \neg y, z\} \\
c_6 & \{\neg a, x, \neg z\} \\
c_7 & \{\neg a, \neg y, \neg z\}
\end{array}
$$

With the learned clause, we come to the conflict quickly.

Figure: Basic DPLL termination tree. Explores large portion of left search tree.

CDCL Example

l_1	$\{\neg a, \neg x\}$
c_1	$\{a, b\}$
c_2	$\{b, c\}$
c_3	$\{\neg a, \neg x, y\}$
c_4	$\{\neg a, x, z\}$
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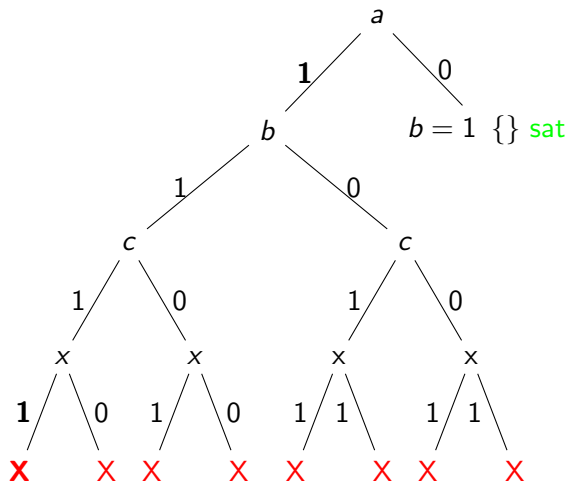

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CDCL Example

l_1 $\{\neg a, \neg x\}$

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c_1 $\{a, b\}$

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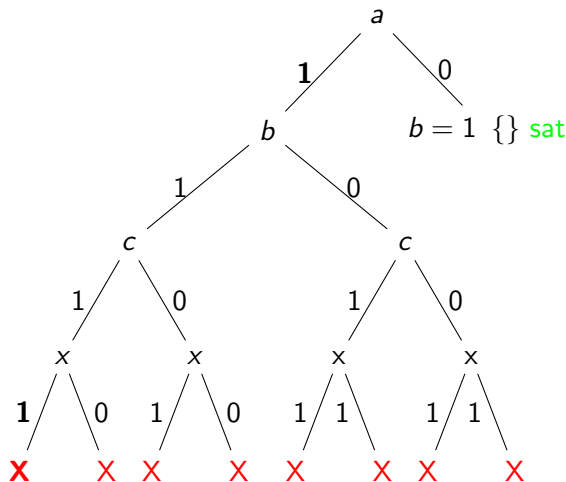This time, a is sufficient to cause the conflict, so we learn $\neg a$.

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CDCL Example

l_2 $\{\neg a\}$

l_1 $\{\neg a, \neg x\}$

c_1 $\{a, b\}$

c_2 $\{b, c\}$

c_3 $\{\neg a, \neg x, y\}$

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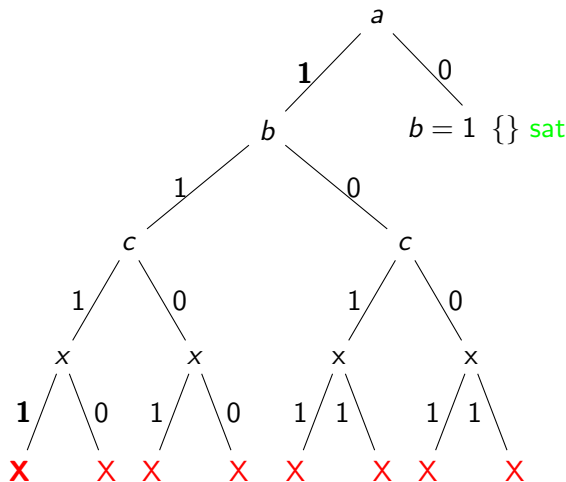With $l_2 = \neg a$, we now get out of the unfruitful search space region.

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CDCL Example

- From basic DPLL traversal we can see that there is no satisfying assignment where $a = 1$
- But, we could have learned earlier on that this was an unfruitful section of the search space
- Idea is to analyze the conflict the occurred from partial assignment $\{a = 1, b = 1, c = 1, x = 1\}$

CDCL: Implication Graph

- Can represent the propagation of variable assignments in an *implication graph*
- Nodes of this graph represent variable assignments made in the current search path
- Edges correspond to dependencies between these assignments.

SAT Solver Implementation

- Iimplementing my own CDCL SAT solver as a framework for exploring future potential SAT enhancements
- Around 1500 lines of C++, tested on a variety of easy to medium SAT benchmark problems
- https://github.com/will62794/mysat

Evaluation

- Some peformance results of my SAT solver against a performant, modern solver.

Future Extensions and Learning Heuristics

- Modern CDCL based SAT solvers employ many heuristics
  - Variable ordering
  - Clause deletion policies
- Often these are "expertly tuned" heuristics
- *CrystalBall* [SKM19]: Possible to learn better heuristics from data on SAT solver executions?

- CrystalBall

Questions?

Martin Davis, George Logemann, and Donald Loveland.
A machine program for theorem-proving.
Commun. ACM, 5(7):394–397, jul 1962.

Mate Soos, Raghav Kulkarni, and Kuldeep S. Meel.
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Springer International Publishing.