SAT Solving with Conflict Driven Clause Learning

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CS 7240 Final Project

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Overview and Project Goals

- Satisfiability is the canonical NP-complete problem.
- Much work has been devoted to building efficient SAT solvers over last decades.
- **Project Goal:** Implement a basic SAT solver based on *conflict driven clause learning* (CDCL), the dominant core technique used in modern solvers.
  - Gain a deeper understanding of the DPLL and CDCL based algorithms for SAT solving
  - Use as a platform for potentially exploring new SAT solving techniques
  - E.g. learning heuristics using a data-driven approach, extending methods of *CrystalBall* [SKM19]

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CNF notation:

$$\{\{x_1, x_2\}, \{\neg x_3, \neg x_1\}\}$$

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- Basic idea of DPLL is to do a depth first, brute force search with backtracking along with some basic formula simplification as you go.
  - Also employs the *unit propagation rule*

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$$\{\} \quad \text{(SAT)}$$

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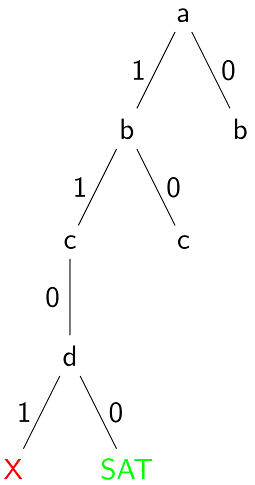

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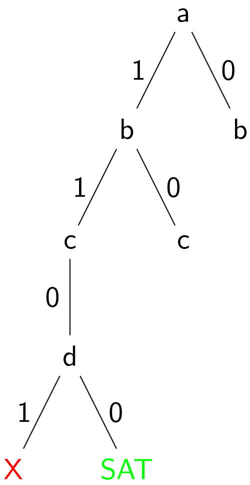

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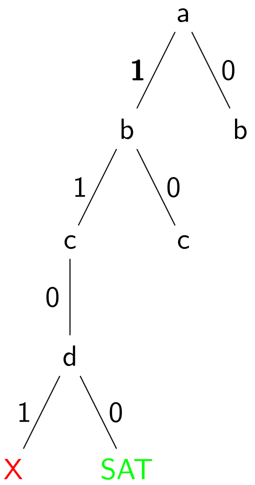

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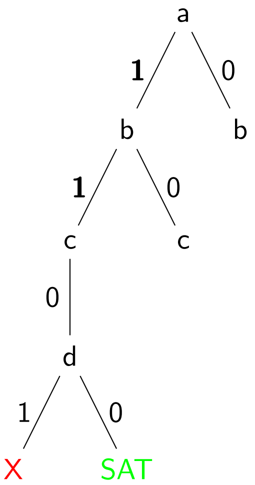

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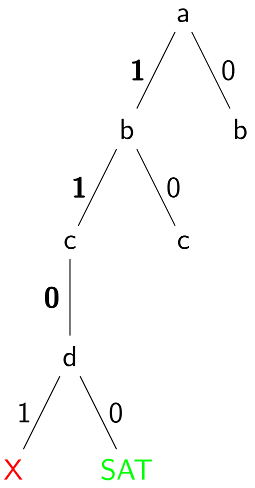

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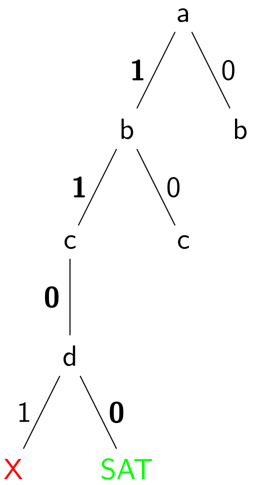

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- In addition, employ *non-chronological backtracking*

CDCL Example

$c_1 \quad \{a, b\}$

$c_2 \quad \{b, c\}$

$c_3 \quad \{\neg a, \neg x, y\}$

$c_4 \quad \{\neg a, x, z\}$

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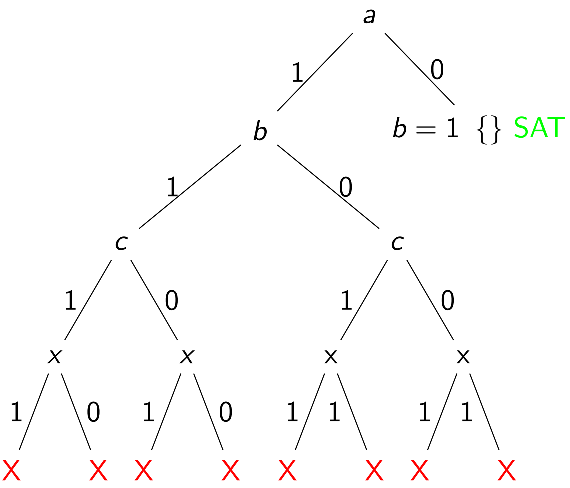

Figure: Basic DPLL termination tree. Explores large portion of left search tree.

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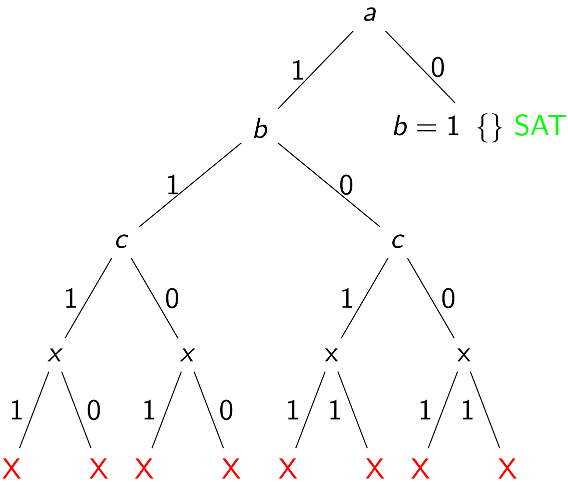

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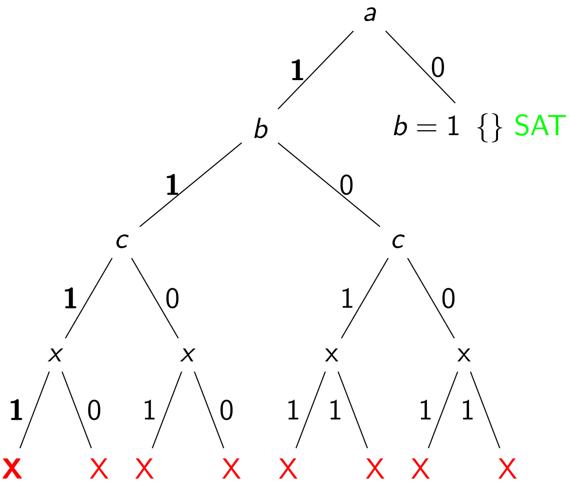

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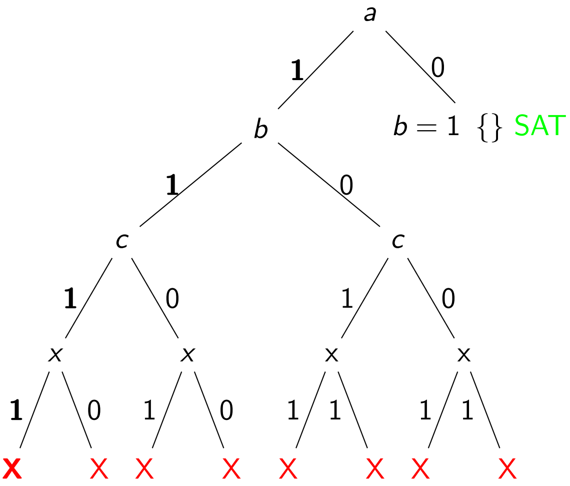
unit propagate y (c_3)



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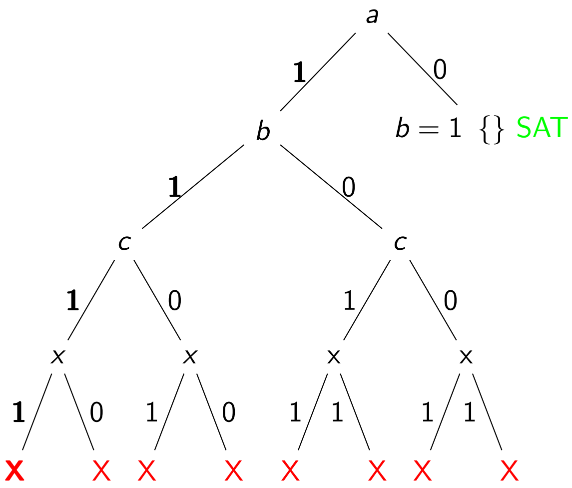
unit propagate z (c_5)
Conflict!



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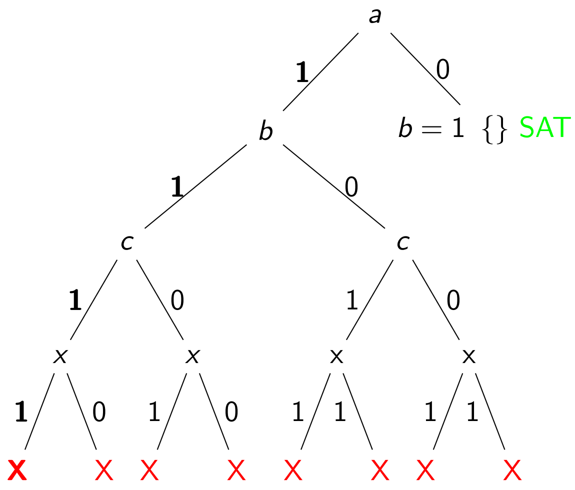
Note that b and c are irrelevant to the c_7 conflict. $(a \wedge y)$ (or $(a \wedge x)$) is sufficient.



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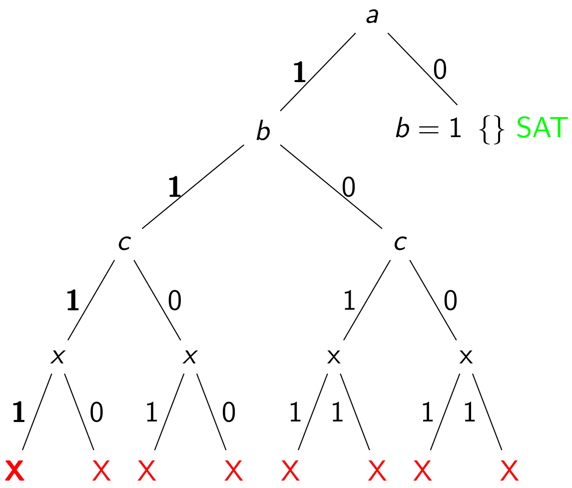
So, we can learn $\neg(a \wedge x) = (\neg a \vee \neg x)$ as a new constraint i.e. a *learned clause*.



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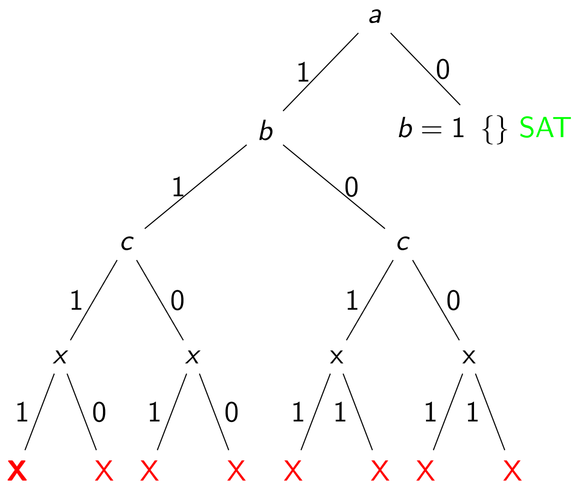

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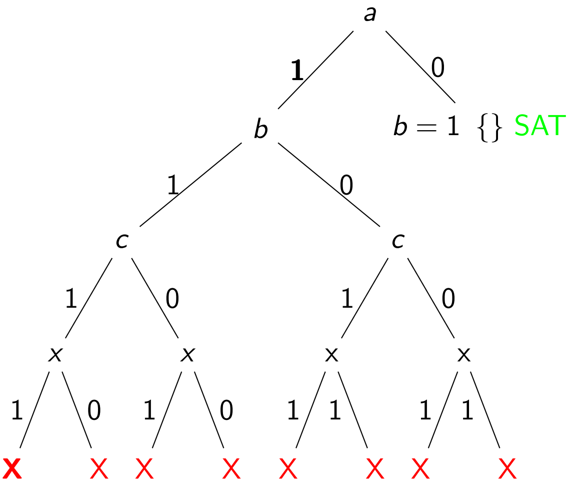

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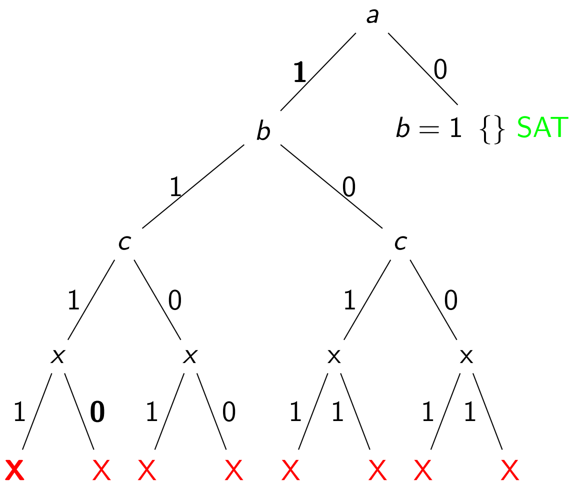

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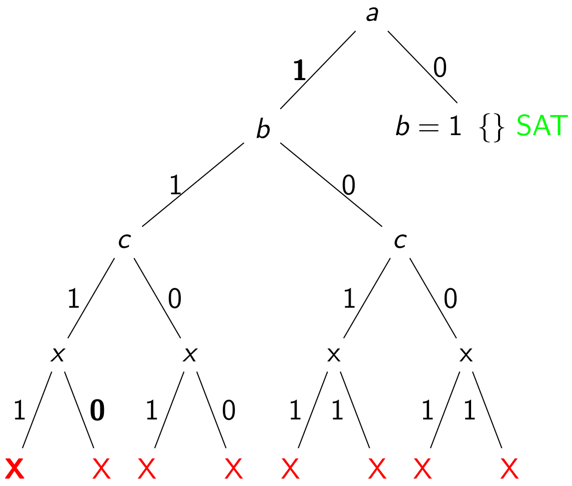

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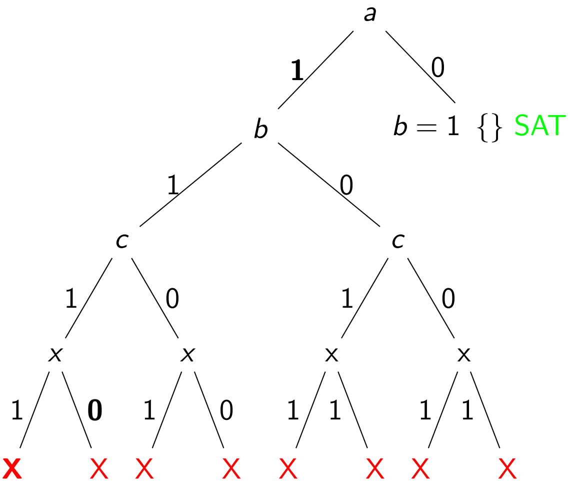
This time, a is sufficient to cause the conflict, so we learn $\neg a$.



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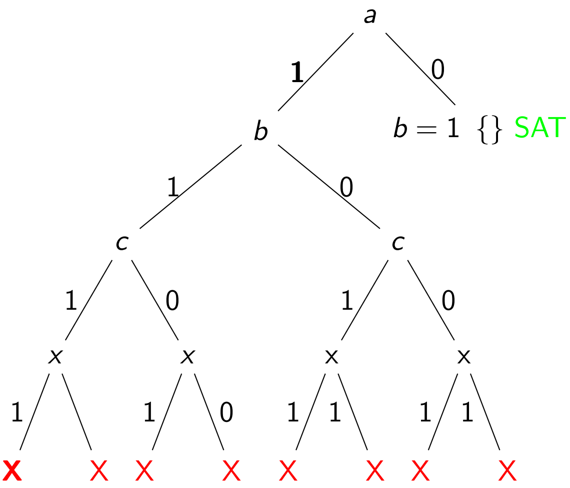
With $l_2 = \neg a$, we now get out of the unfruitful search space region.



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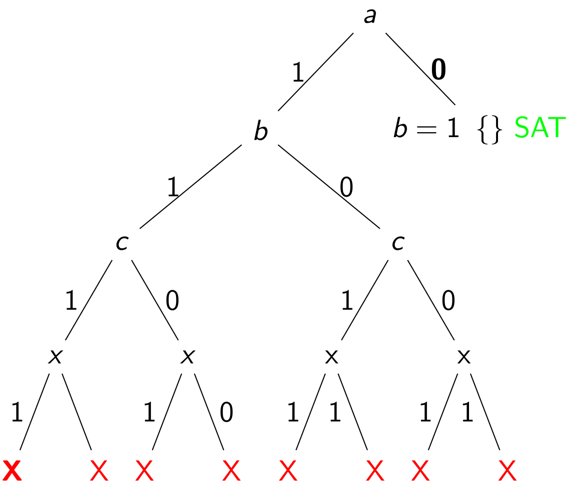

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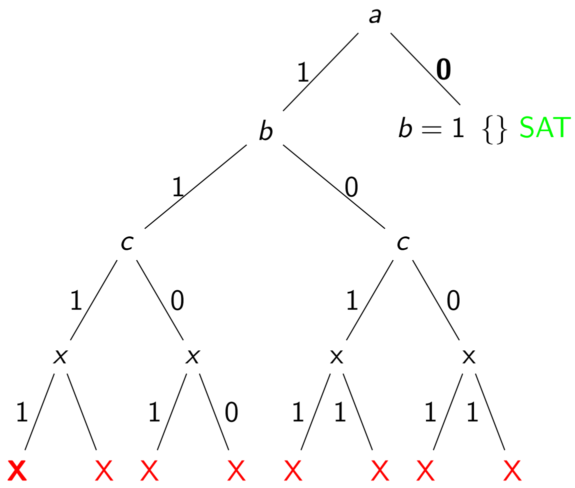

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  - Nodes of this graph represent variable assignments in the current search path. Edges to dependencies between these assignments.
  - Cuts in the graph correspond to a conflict set and, by negating it, a potential clause to learn

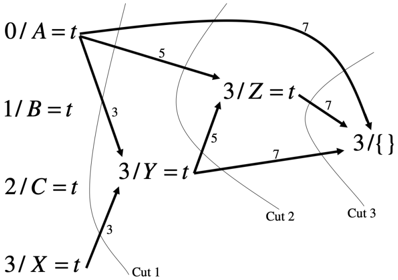

Figure 3.8. Three cuts in an implication graph, leading to three conflict sets.

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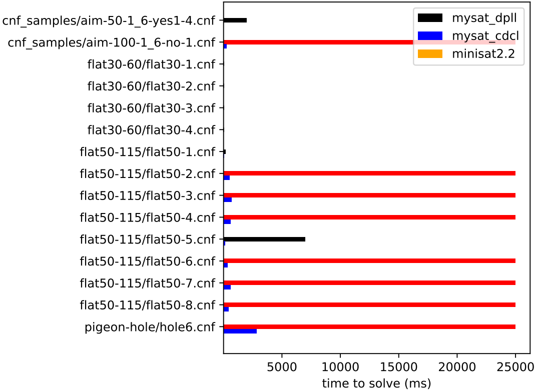

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- Often these are "expertly tuned", based on experience/intuition

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  - Proofs based on resolution inference rule i.e. if

$$C_1 = (x \vee a_1 \vee \cdots \vee a_n)$$
$$C_2 = (\neg x \vee b_1 \vee \cdots \vee b_m)$$

    the clause

$$C = C_1 \bowtie C_2 = (a_1 \vee \cdots \vee a_n \vee b_1 \vee \cdots \vee b_m)$$

    can be inferred.

Future Goals

- Implement support for resolution proof output in UNSAT cases

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- Capture more statistics from solving runs as a basis for learning new heuristics e.g. clause activity

Questions?

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