SAT Solving with Conflict Driven Clause Learning

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CS 7240 Final Project

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Overview and Project Goals

- Satisfiability is the canonical NP-complete problem.
- Much work has been devoted to building efficient SAT solvers over last decades.
- **Project Goal:** Implement a basic SAT solver based on *conflict driven clause learning* (CDCL), the dominant core technique used in modern solvers.
  - Gain a deeper understanding of the DPLL and CDCL based algorithms for SAT solving
  - Use as a platform for potentially exploring new SAT solving techniques
  - E.g. learning heuristics using a data-driven approach, extending methods of *CrystalBall* [SKM19]

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Given a boolean formula in conjunctive normal form (CNF), determine whether there exists an assignment to the variables of the formula that makes the overall formula true.

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$$(x_1 \lor x_2) \land (\neg x_3 \lor \neg x_1)$$

$$\text{SAT, with } \{x_1 = 1, x_2 = 0, x_3 = 0\}.$$

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CNF notation:

$$\{\{x_1, x_2\}, \{\neg x_3, \neg x_1\}\}$$

DPLL: SAT as Search

- A basic approach to solving SAT is to view it as a search problem over possible assignments.
- This is the basis of the Davis–Putnam–Logemann–Loveland (DPLL) algorithm [DLL62]
- Basic idea of DPLL is to do a depth first, brute force search with backtracking along with some basic formula simplification as you go.
    - Also employs the *unit propagation rule*

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$$\{\{{\color{green}b}\}, \{{\color{red}\neg b}, \neg c\}, \{c, \neg d\}\}$$

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\begin{aligned}
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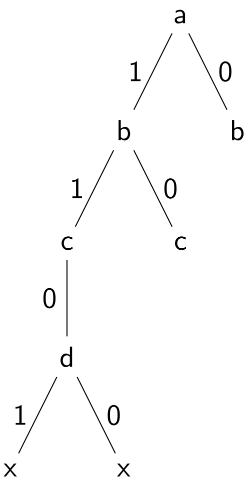
&\{\{\color{green}{b}\},\{\color{red}{\neg b},\neg c\},\{c,\neg d\}\}\\
&\{\{\neg c\},\{c,\neg d\}\}\\
&\{\{\color{green}{\neg c}\},\{\color{red}{c},\neg d\}\}\\
&\{\{\neg d\}\}\\
&\{\{\color{green}{\neg d}\}\}\\
&\{\}\quad \text{(SAT)}
\end{aligned}
$$

DPLL: Example

$\{\neg a, b\}$

$\{\neg b, \neg c\}$

$\{c, \neg d\}$



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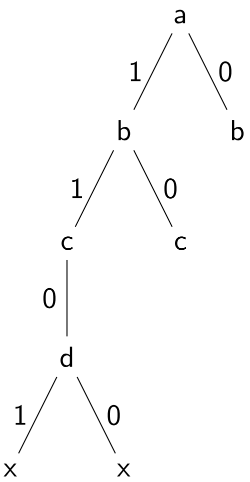

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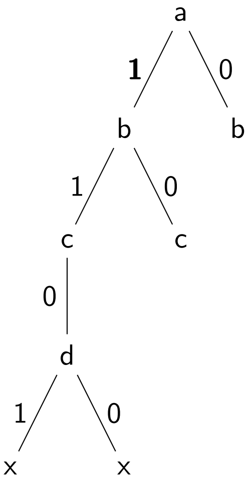

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$\{\neg a, b\}$

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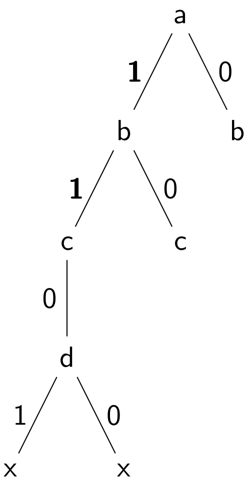
$\{c, \neg d\}$

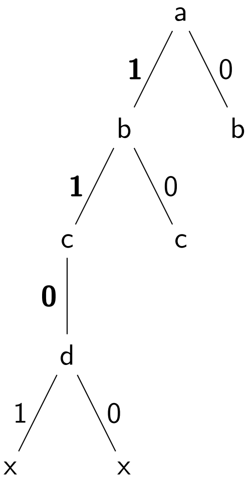
unit propagate b



DPLL: Example

$\{\neg a, b\}$
$\{\neg b, \neg c\}$
$\{c, \neg d\}$

unit propagate $\neg c$



DPLL: Example

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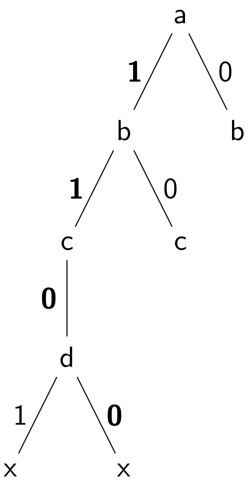
$\{c, \neg d\}$

unit propagate $\neg d$



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- When you encounter a conflict in the search tree, *learn* a clause that prevents you from making the similar mistakes again
- This fundamental approach is known as *conflict-driven clause learning* (CDCL) and started being employed in SAT solvers around the late 90s and early 2000s.
- In addition, employ *non-chronological backtracking*

CDCL Example

c_1 $\{a, b\}$

c_2 $\{b, c\}$

c_3 $\{\neg a, \neg x, y\}$

c_4 $\{\neg a, x, z\}$

c_5 $\{\neg a, \neg y, z\}$

c_6 $\{\neg a, x, \neg z\}$

c_7 $\{\neg a, \neg y, \neg z\}$

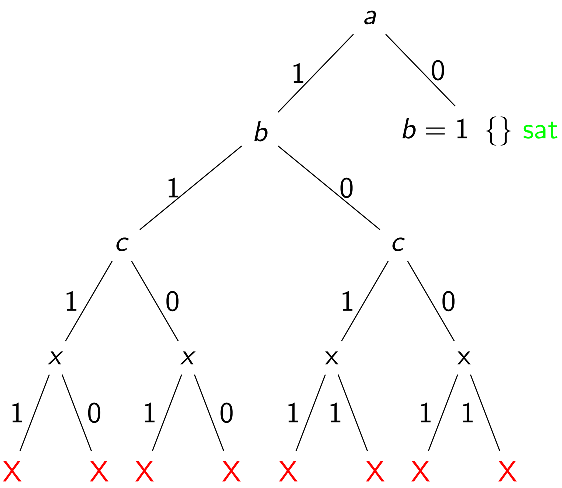

Figure: Termination tree from basic DPLL.

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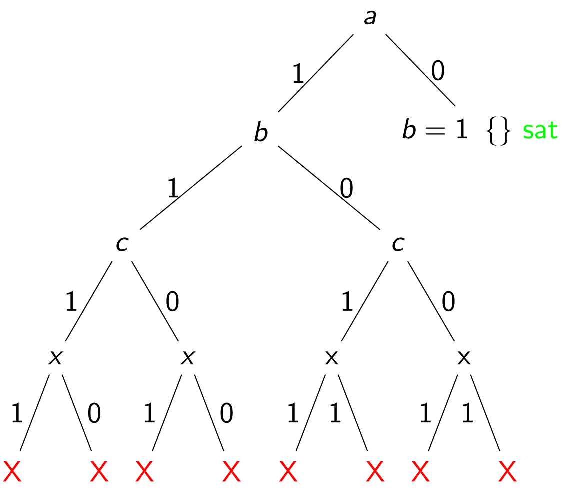

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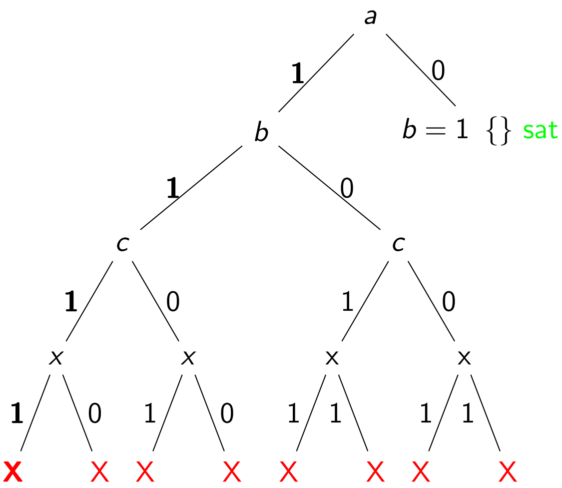

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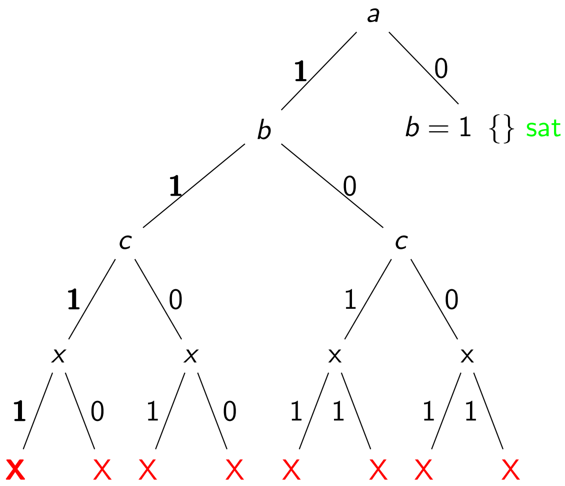
unit propagate y



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CDCL Example

c_1 $\{a, b\}$

c_2 $\{b, c\}$

c_3 $\{\neg a, \neg x, y\}$

c_4 $\{\neg a, x, z\}$

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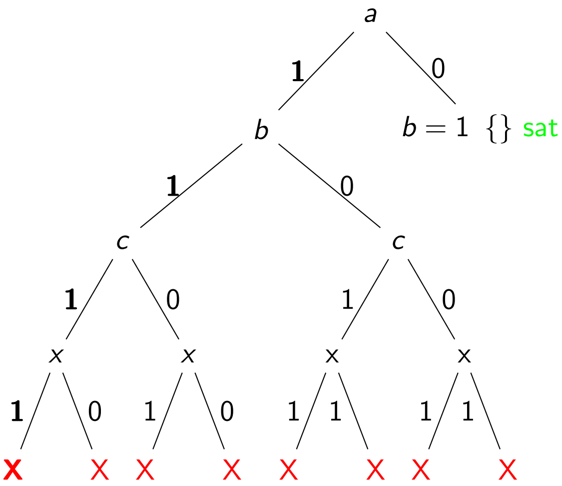
unit propagate z



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CDCL Example

c_1	$\{a, b\}$
c_2	$\{b, c\}$
c_3	$\{\neg a, \neg x, y\}$
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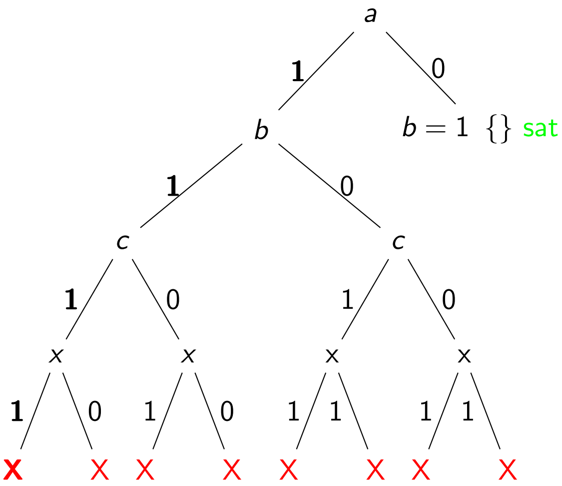
c_6	$\{\neg a, x, \neg z\}$
c_7	$\{\neg a, \neg y, \neg z\}$

Note that b and c are irrelevant to the c_7 conflict. $(a \wedge x)$ or $(a \wedge y)$ are sufficient.



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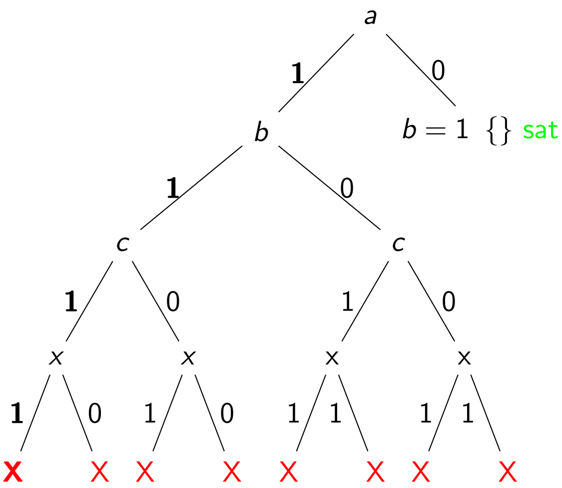
So, we can learn $\neg(a \wedge x) = (\neg a \vee \neg x)$ as a new constraint i.e. a *learned clause*.



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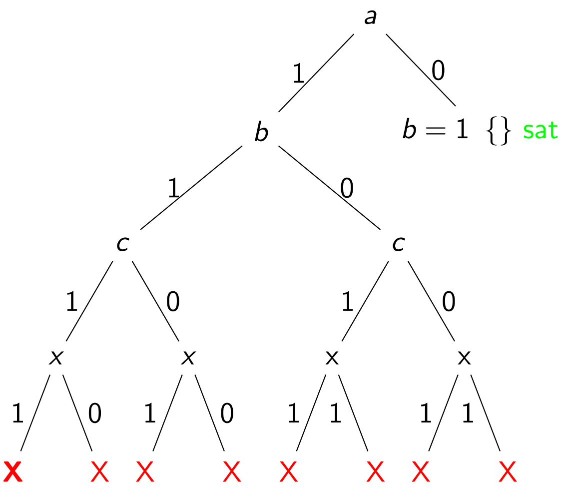

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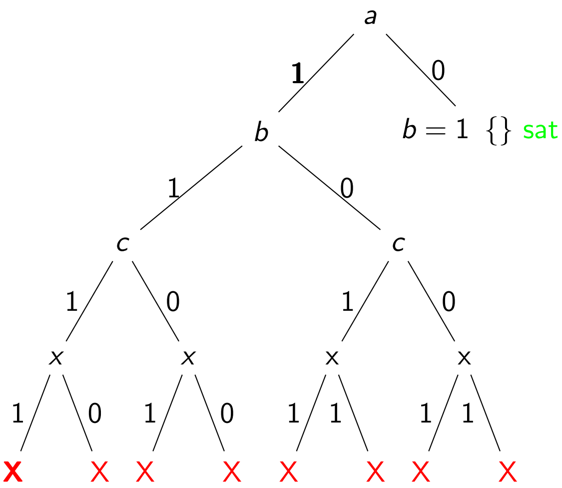

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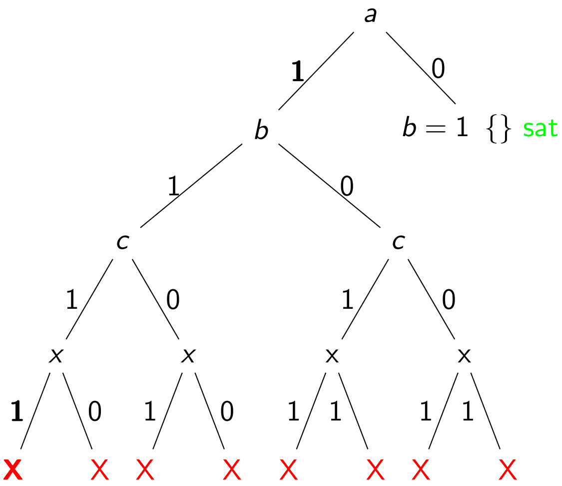
With the learned clause, we come to the conflict quickly.



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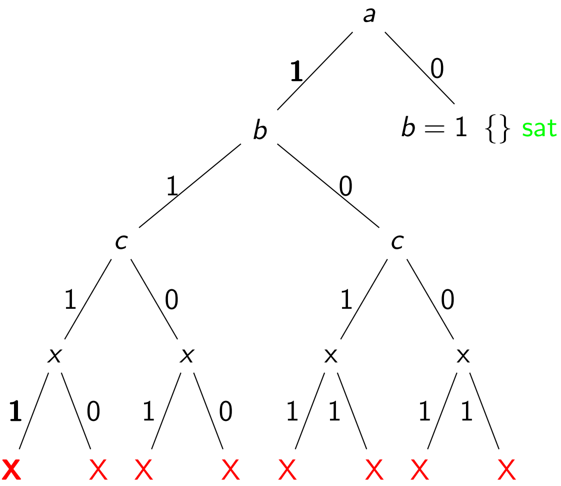

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CDCL Example

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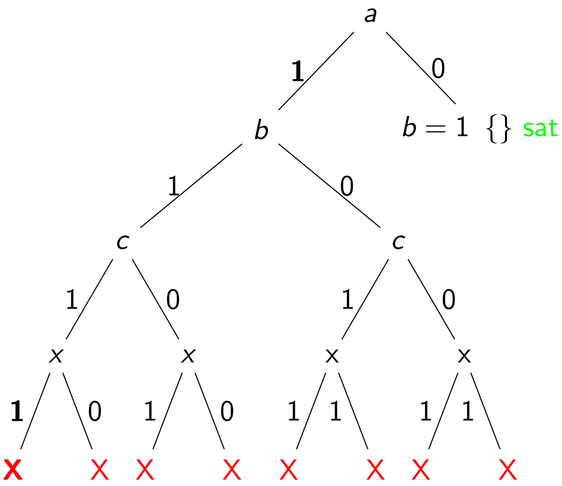
This time, a is sufficient to cause the conflict, so we learn $\neg a$.



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CDCL Example

l_2	$\{\neg a\}$
l_1	$\{\neg a, \neg x\}$
c_1	$\{a, b\}$
c_2	$\{b, c\}$
c_3	$\{\neg a, \neg x, y\}$
c_4	$\{\neg a, x, z\}$
c_5	$\{\neg a, \neg y, z\}$
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With $l_2 = \neg a$, we now get out of the unfruitful search space region.

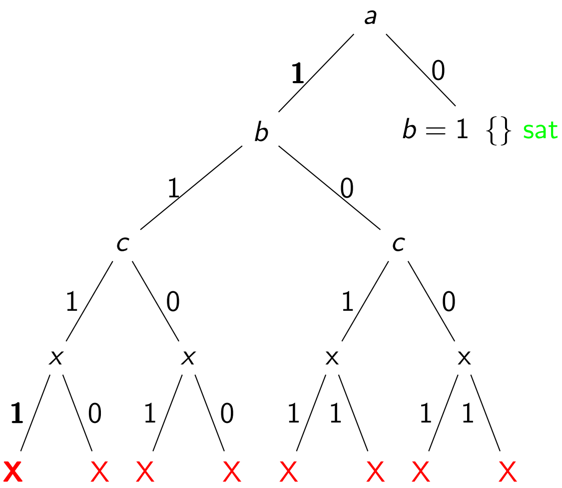

Figure: Termination tree from basic DPLL.

CDCL Example

- From basic DPLL traversal we can see that there is no satisfying assignment where $a = 1$
- But, we could have learned earlier on that this was an unfruitful section of the search space
- Idea is to analyze the conflict the occurred from partial assignment $\{a = 1, b = 1, c = 1, x = 1\}$

CDCL: Implication Graph

- Can represent the propagation of variable assignments in an *implication graph*
- Nodes of this graph represent variable assignments made in the current search path
- Edges correspond to dependencies between these assignments.

SAT Solver Implementation

- Iimplementing my own CDCL SAT solver as a framework for exploring future potential SAT enhancements
- Around 1500 lines of C++, tested on a variety of easy to medium SAT benchmark problems
- `https://github.com/will62794/mysat`

Evaluation

- Some peformance results of my SAT solver against a performant, modern solver.

Future Extensions and Learning Heuristics

- Modern CDCL based SAT solvers employ many heuristics
  - Variable ordering
  - Clause deletion policies
- Often these are "expertly tuned" heuristics
- *CrystalBall* [SKM19]: Possible to learn better heuristics from data on SAT solver executions?

- CrystalBall

Questions?

Martin Davis, George Logemann, and Donald Loveland.
A machine program for theorem-proving.
Commun. ACM, 5(7):394–397, jul 1962.

Mate Soos, Raghav Kulkarni, and Kuldeep S. Meel.
CrystalBall: Gazing in the Black Box of SAT Solving.
In Mikoláš Janota and Inês Lynce, editors, *Theory and Applications of Satisfiability Testing – SAT 2019*, pages 371–387, Cham, 2019. Springer International Publishing.