SAT Solving with Conflict Driven Clause Learning

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CS 7240 Final Project

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Overview and Project Goals

- Satisfiability is the canonical NP-complete problem.
- Much work has been devoted to building efficient SAT solvers over last decades.
- **Project Goal:** Implement a basic SAT solver based on *conflict driven clause learning* (CDCL), the dominant core technique used in modern solvers.
  - Gain a deeper understanding of the DPLL and CDCL based algorithms for SAT solving
  - Use as a platform for potentially exploring new SAT solving techniques
  - E.g. learning heuristics using a data-driven approach, extending methods of *CrystalBall* [SKM19]

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CNF notation:

$$\{\{x_1, x_2\}, \{\neg x_3, \neg x_1\}\}$$

DPLL: SAT as Search

- A basic approach to solving SAT is to view it as a search problem over possible assignments.
- This is the basis of the Davis–Putnam–Logemann–Loveland (DPLL) algorithm [DLL62]
- Basic idea of DPLL is to do a depth first, brute force search with backtracking along with some basic formula simplification as you go.
    - Also employs the *unit propagation rule*

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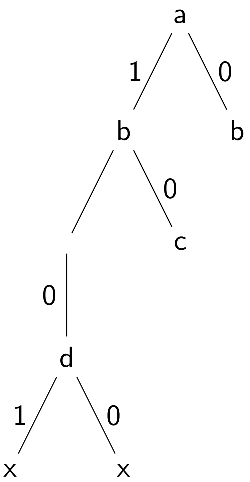
$$\{\} \quad \text{(SAT)}$$

DPLL: Example

$\{\neg a, b\}$

$\{\neg b, \neg c\}$

$\{\neg c, \neg d\}$



DPLL: Example

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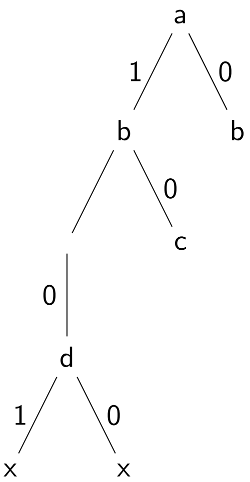

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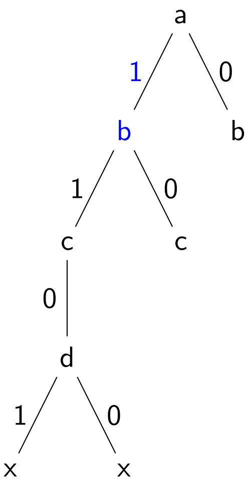

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- In addition, employ *non-chronological backtracking*

CDCL

- When using CDCL, if a conflict is encountered, we not only backtrack to the previous level, as in DPLL
- We try to learn a *conflict clause* along with a *backjump* level, which determines how far back in the search tree to unwind to.

CDCL Example

$\{a, b\}$

$\{b, c\}$

$\{\neg a, \neg x, y\}$

$\{\neg a, x, z\}$

$\{\neg a, \neg y, z\}$

$\{\neg a, x, \neg z\}$

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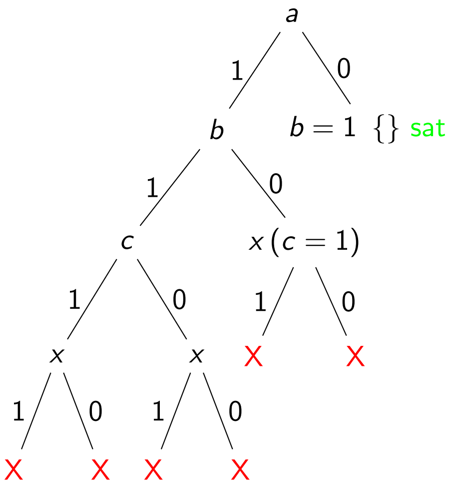

Figure: Termination tree using standard DPLL.

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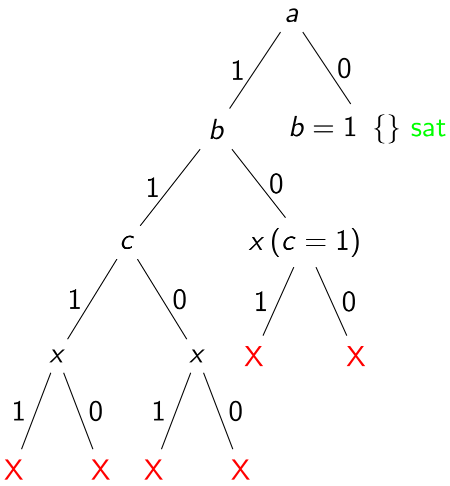

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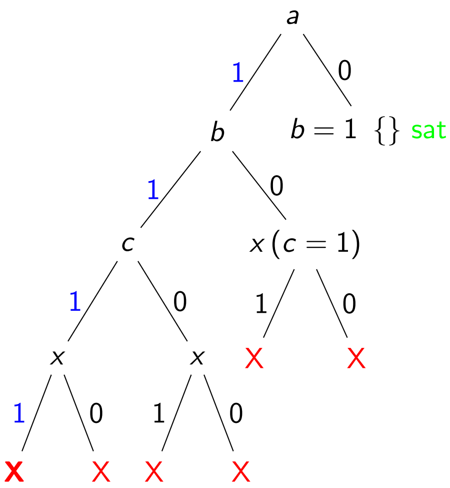

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CDCL Example

- From basic DPLL traversal we can see that there is no satisfying assignment where $a = 1$
- But, we could have learned earlier on that this was an unfruitful section of the search space
- Idea is to analyze the conflict the occurred from partial assignment $\{a = 1, b = 1, c = 1, x = 1\}$

CDCL: Implication Graph

- Can represent the propagation of variable assignments in an *implication graph*
- Nodes of this graph represent variable assignments made in the current search path
- Edges correspond to dependencies between these assignments.

SAT Solver Implementation

- Iimplementing my own CDCL SAT solver as a framework for exploring future potential SAT enhancements
- Around 1500 lines of C++, tested on a variety of easy to medium SAT benchmark problems
- `https://github.com/will62794/mysat`

Evaluation

- Some peformance results of my SAT solver against a performant, modern solver.

Future Extensions and Learning Heuristics

- Modern CDCL based SAT solvers employ many heuristics
  - Variable ordering
  - Clause deletion policies
- Often these are "expertly tuned" heuristics
- *CrystalBall* [SKM19]: Possible to learn better heuristics from data on SAT solver executions?

- CrystalBall

Questions?

Martin Davis, George Logemann, and Donald Loveland.
A machine program for theorem-proving.
Commun. ACM, 5(7):394–397, jul 1962.

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