SAT Solving with Conflict Driven Clause Learning

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CS 7240 Final Project

May 2, 2022

Overview and Project Goals

- Satisfiability is the canonical NP-complete problem.
- Much work has been devoted to building efficient SAT solvers over last decades.
- **Project Goal:** Implement a basic SAT solver based on *conflict driven clause learning* (CDCL), the dominant core technique used in modern solvers.
  - Gain a deeper understanding of the DPLL and CDCL based algorithms for SAT solving
  - Use as a platform for potentially exploring new SAT solving techniques
  - E.g. learning heuristics using a data-driven approach, extending methods of *CrystalBall* [SKM19]

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SAT, with $\{x_1 = 1, x_2 = 0, x_3 = 0\}$.

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CNF notation:

$$\{\{x_1, x_2\}, \{\neg x_3, \neg x_1\}\}$$

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- Basic idea of DPLL is to do a depth first, brute force search with backtracking along with some basic formula simplification as you go.
  - Also employs the *unit propagation rule*

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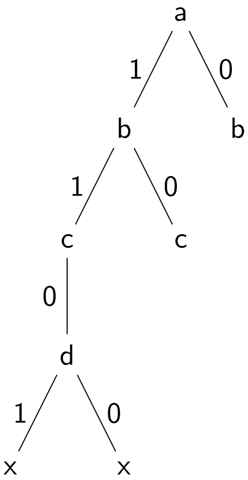
$$\{\} \quad \text{(SAT)}$$

DPLL: Example

$\{\neg a, b\}$

$\{\neg b, \neg c\}$

$\{c, \neg d\}$



DPLL: Example

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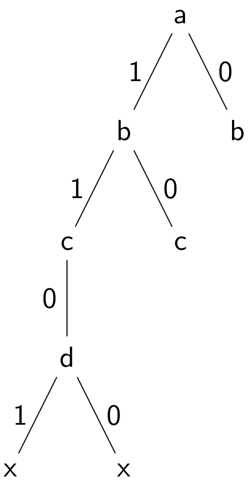

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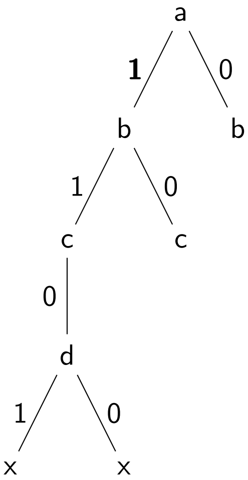

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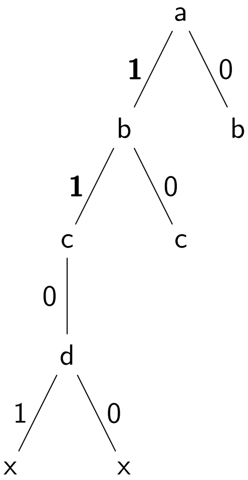
unit propagate b



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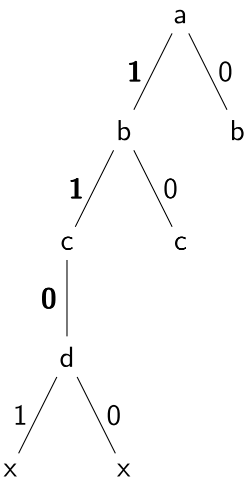

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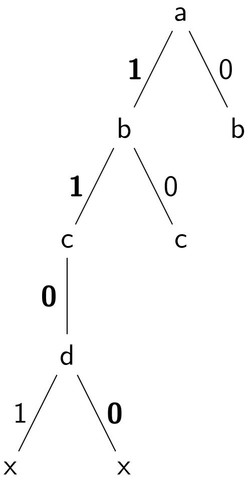

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- When you encounter a conflict in the search tree, *learn* a clause that prevents you from making the similar mistakes again
- This fundamental approach is known as *conflict-driven clause learning* (CDCL) and started being employed in SAT solvers around the late 90s and early 2000s.
- In addition, employ *non-chronological backtracking*

CDCL Example

c_1	$\{a, b\}$
c_2	$\{b, c\}$
c_3	$\{\neg a, \neg x, y\}$
c_4	$\{\neg a, x, z\}$
c_5	$\{\neg a, \neg y, z\}$
c_6	$\{\neg a, x, \neg z\}$
c_7	$\{\neg a, \neg y, \neg z\}$

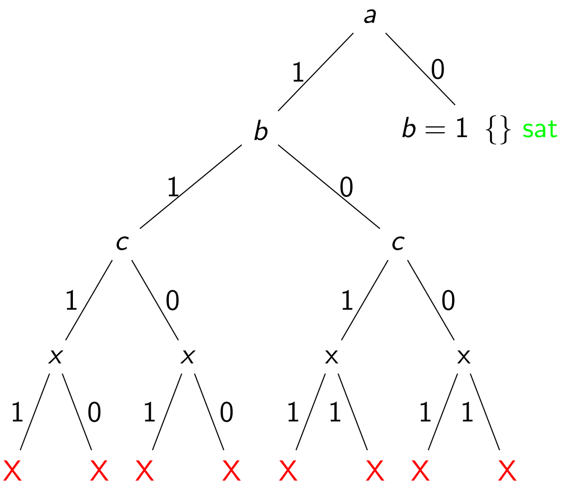

Figure: Basic DPLL termination tree. Explores large portion of left search tree.

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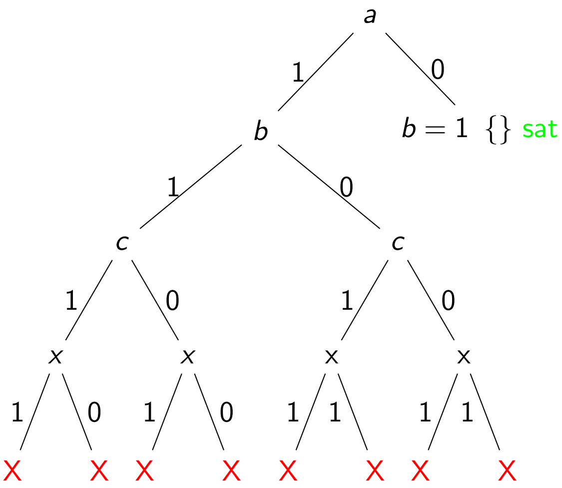

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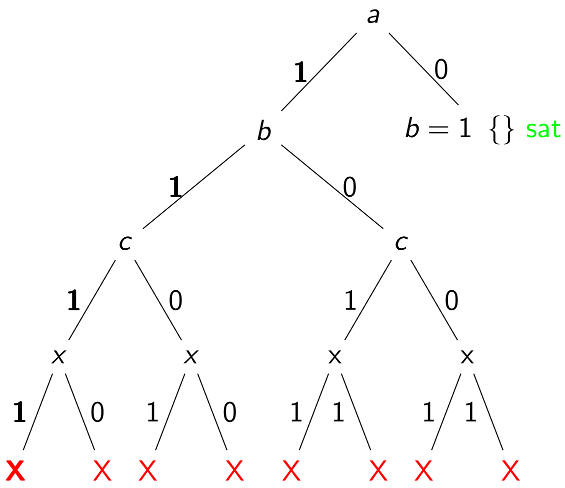

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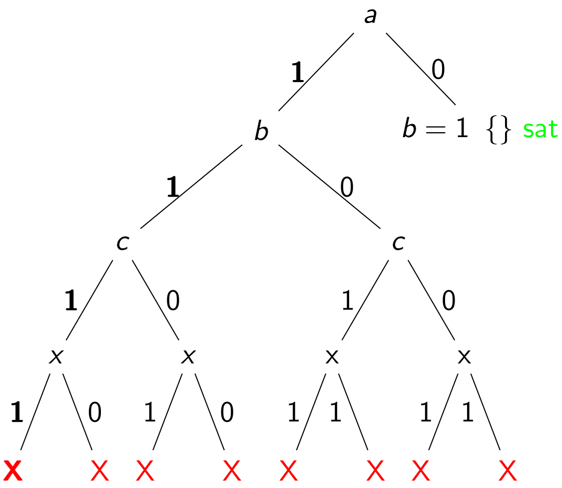
unit propagate y



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CDCL Example

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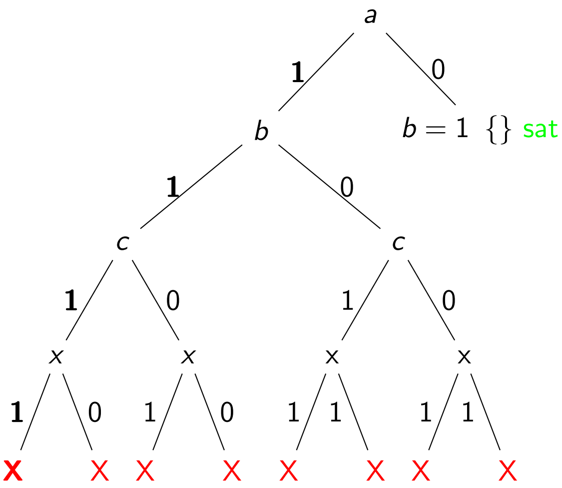
unit propagate z



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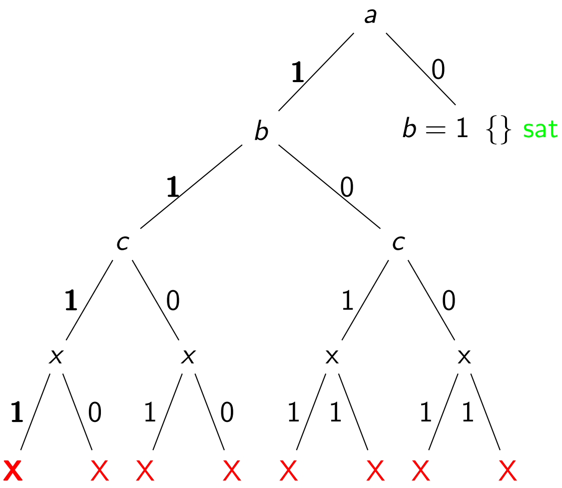
Note that b and c are irrelevant to the c_7 conflict. $(a \wedge x)$ or $(a \wedge y)$ are sufficient.



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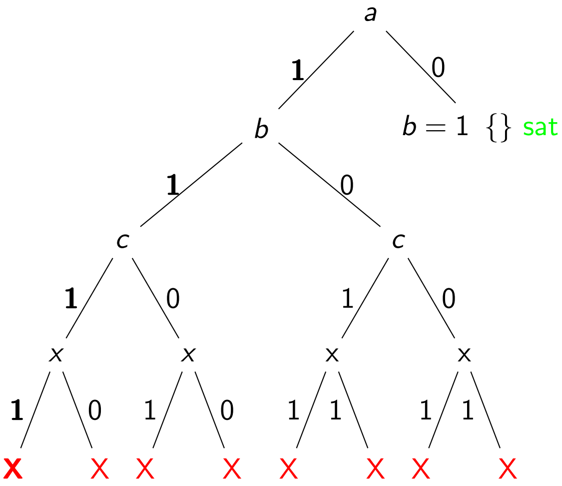
So, we can learn $\neg(a \wedge x) = (\neg a \vee \neg x)$ as a new constraint i.e. a *learned clause*.



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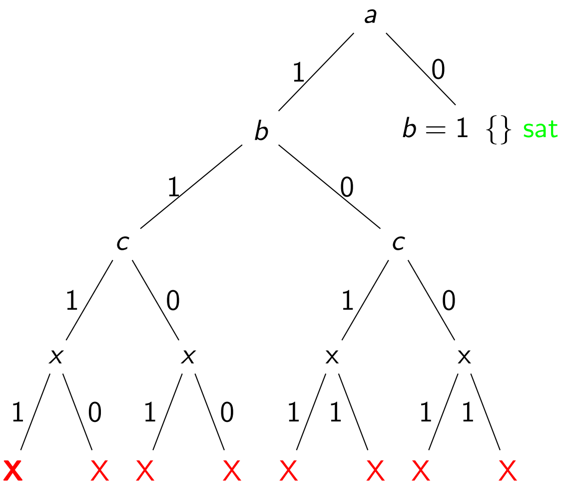

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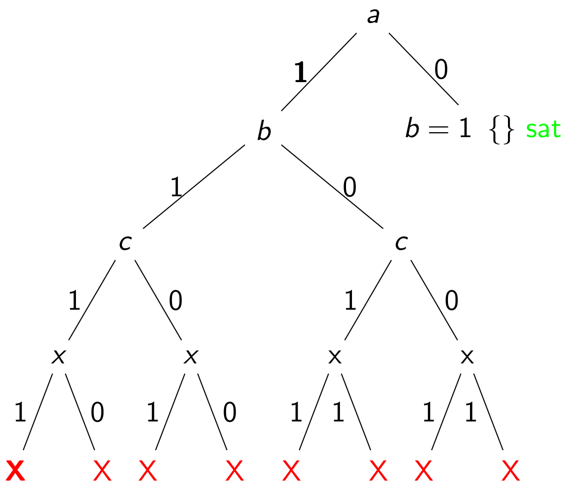

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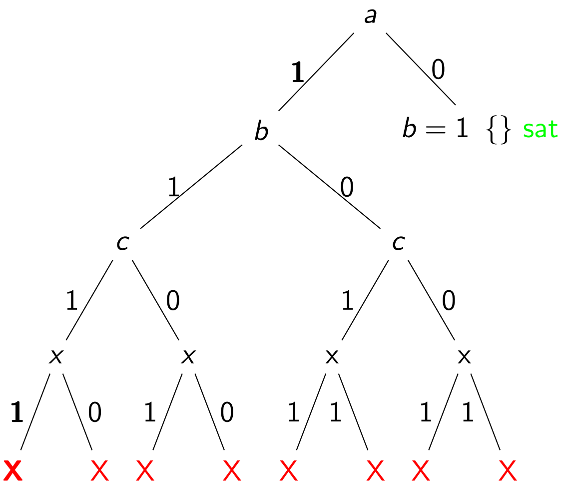

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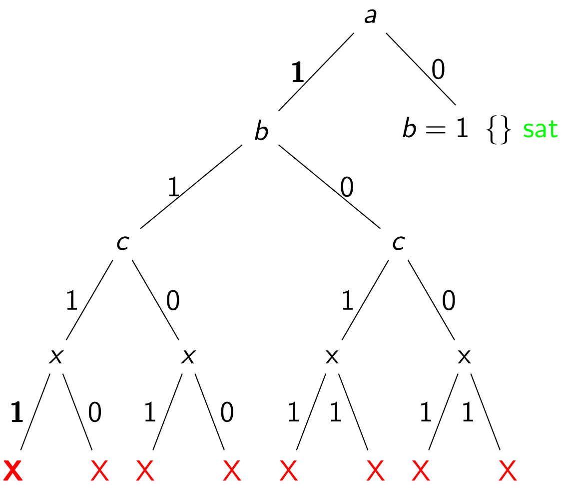

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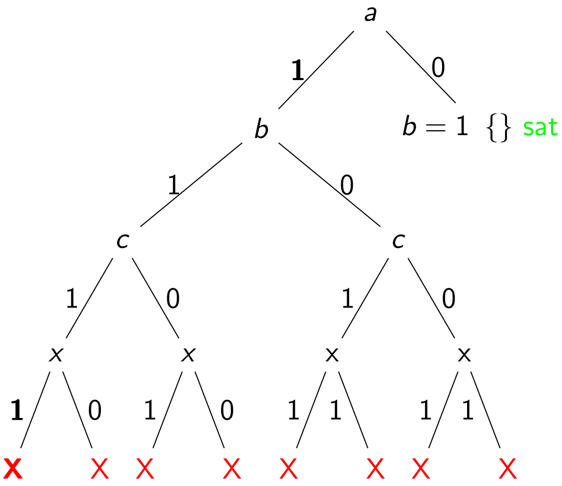
This time, a is sufficient to cause the conflict, so we learn $\neg a$.



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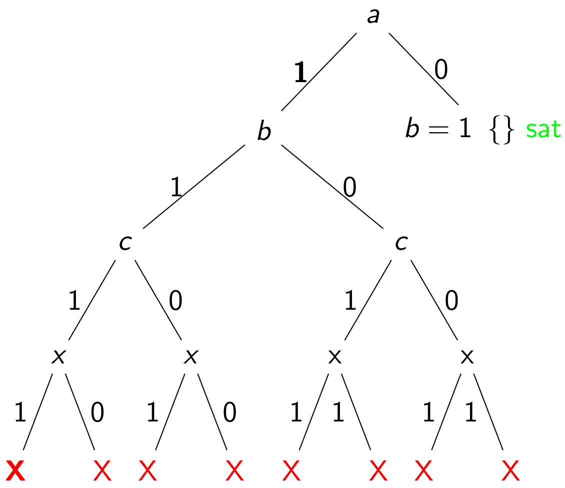
With $l_2 = \neg a$, we now get out of the unfruitful search space region.



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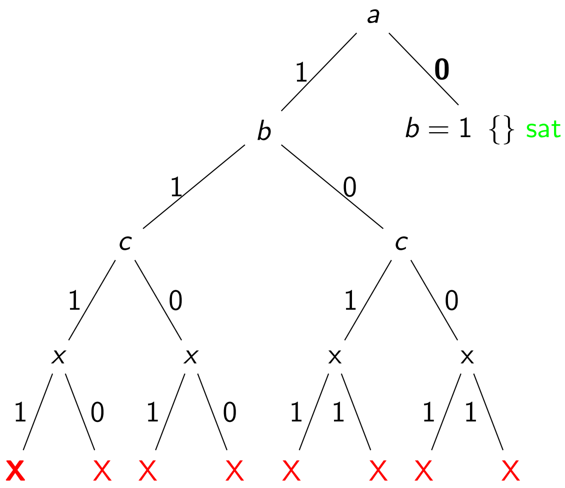

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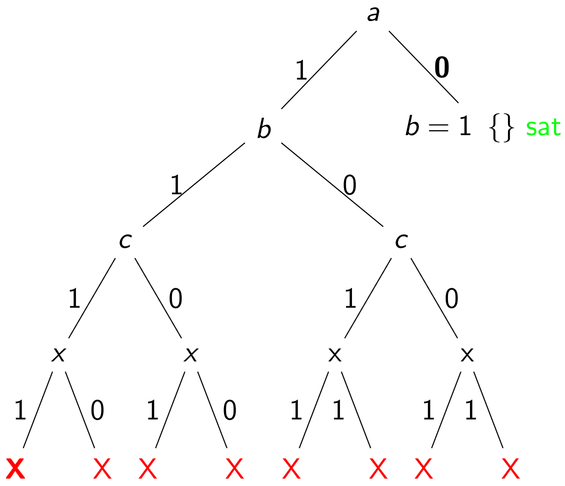

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  - Nodes of this graph represent variable assignments in the current search path. Edges to dependencies between these assignments.
  - Cuts in the graph correspond to a conflict set and, by negating it, a potential clause to learn

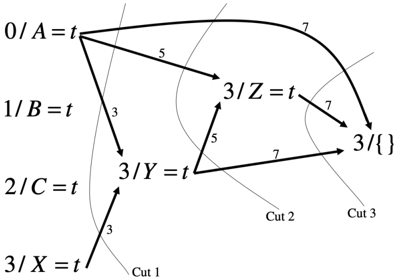

Figure 3.8. Three cuts in an implication graph, leading to three conflict sets.

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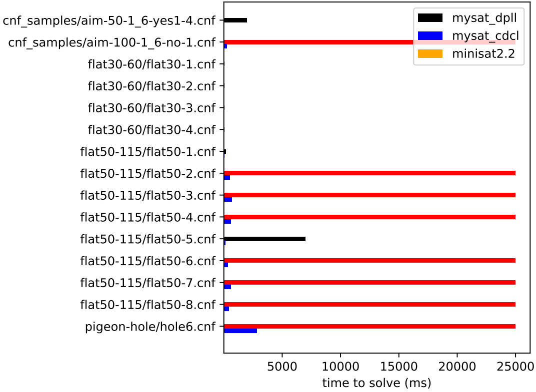
- e.g. runtime on some benchmarks with ≈ 50-200 variables, time budget of 25 seconds (red bar indicates timeout)



Future Extensions: Learning Heuristics

- Modern CDCL based SAT solvers employ many heuristics e.g. for determining:
  1. Variable ordering
  2. Learned clause deletion policies
  3. Random restarts
- Often these are "expertly tuned", based on experience/intuition

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- Learned clause deletion is an important heuristic for CDCL based SAT solvers
- Use data from SAT runs to train a model
- DRAT resolution proofs serve as a good source of data

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  - Proofs based on resolution inference rule i.e. if

$$C_1 = (x \lor a_1 \lor \cdots \lor a_n)$$
$$C_2 = (\neg x \lor b_1 \lor \cdots \lor b_m)$$

the clause

$$C = C_1 \bowtie C_2 = (a_1 \lor \cdots \lor a_n \lor b_1 \lor \cdots \lor b_m)$$

can be inferred.

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- Capture more statistics from solving runs as a basis for learning new heuristics e.g. clause activity

Questions?

Martin Davis, George Logemann, and Donald Loveland.
A machine program for theorem-proving.
Commun. ACM, 5(7):394–397, jul 1962.

Niklas Eén and Niklas Sörensson.
An extensible sat-solver.
In Enrico Giunchiglia and Armando Tacchella, editors, *Theory and Applications of Satisfiability Testing*, pages 502–518, Berlin, Heidelberg, 2004. Springer Berlin Heidelberg.

Mate Soos, Raghav Kulkarni, and Kuldeep S. Meel.
CrystalBall: Gazing in the Black Box of SAT Solving.
In Mikoláš Janota and Inês Lynce, editors, *Theory and Applications of Satisfiability Testing – SAT 2019*, pages 371–387, Cham, 2019. Springer International Publishing.