SAT Solving with Conflict Driven Clause Learning

William Schultz

CS 7240 Final Project

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Overview and Project Goals

- Satisfiability is the canonical NP-complete problem.
- Much work has been devoted to building efficient SAT solvers over last decades.
- **Project Goal:** Implement a basic SAT solver based on *conflict driven clause learning* (CDCL), the dominant core technique used in modern solvers.
    - Gain a deeper understanding of the DPLL and CDCL based algorithms for SAT solving
    - Use as a platform for potentially exploring new SAT solving techniques
    - E.g. learning heuristics using a data-driven approach, extending methods of *CrystalBall* [SKM19]

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CNF notation:

$$\{\{x_1, x_2\}, \{\neg x_3, \neg x_1\}\}$$

DPLL: SAT as Search

- A basic approach to solving SAT is to view it as a search problem over possible assignments.
- This is the basis of the Davis–Putnam–Logemann–Loveland (DPLL) algorithm [DLL62]
- Basic idea of DPLL is to do a depth first, brute force search with backtracking along with some basic formula simplification as you go.
  - Also employs the *unit propagation rule*

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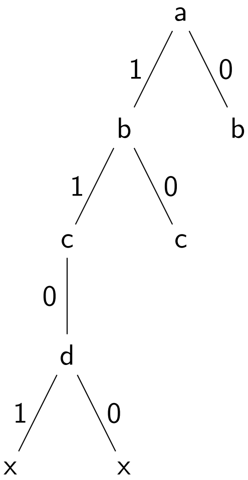
$$\{\} \quad \text{(SAT)}$$

DPLL: Example

$\{\neg a, b\}$

$\{\neg b, \neg c\}$

$\{c, \neg d\}$



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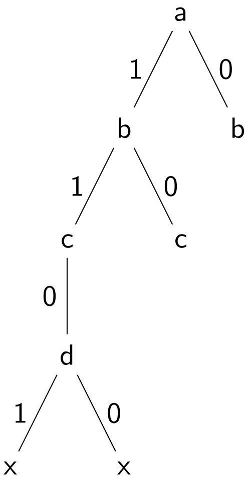

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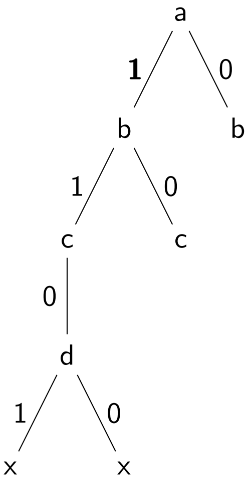

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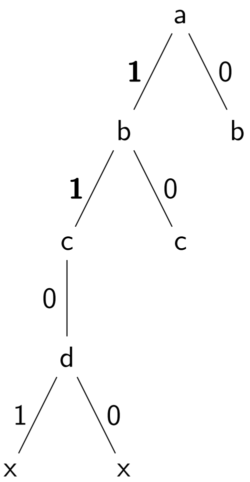
unit propagate b



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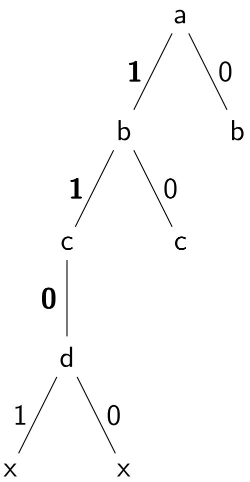

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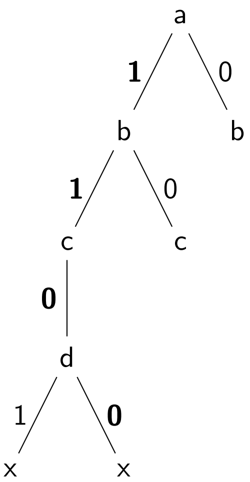

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- When you encounter a conflict in the search tree, *learn* a clause that prevents you from making the similar mistakes again
- This fundamental approach is known as *conflict-driven clause learning* (CDCL) and started being employed in SAT solvers around the late 90s and early 2000s.
- In addition, employ *non-chronological backtracking*

CDCL Example

c_1	$\{a, b\}$
c_2	$\{b, c\}$
c_3	$\{\neg a, \neg x, y\}$
c_4	$\{\neg a, x, z\}$
c_5	$\{\neg a, \neg y, z\}$
c_6	$\{\neg a, x, \neg z\}$
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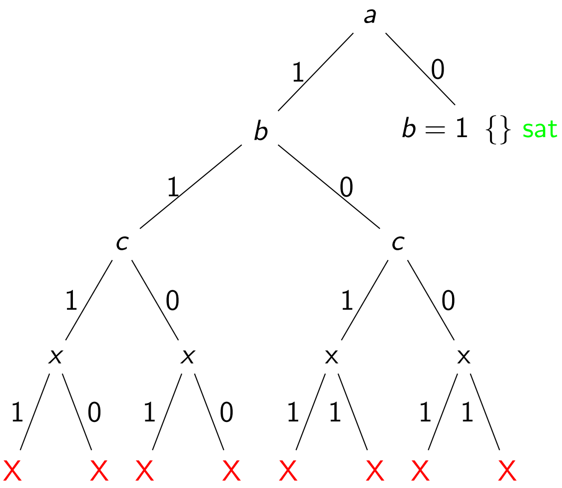

Figure: Basic DPLL termination tree. Explores large portion of left search tree.

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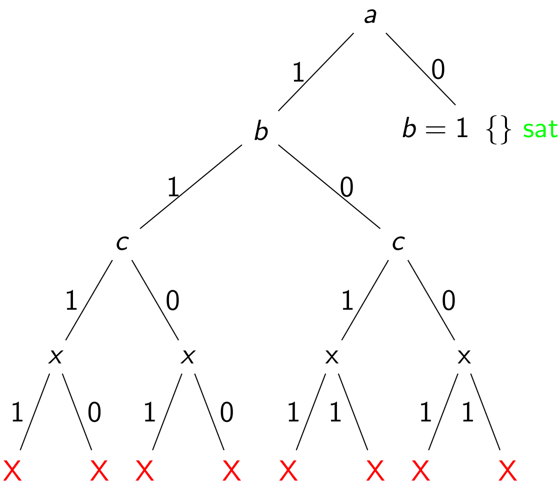

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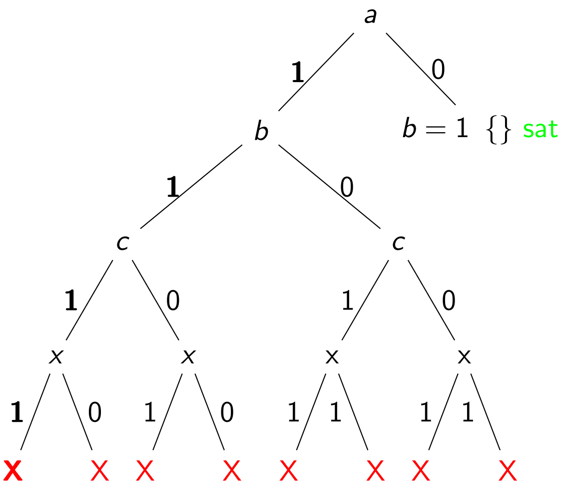

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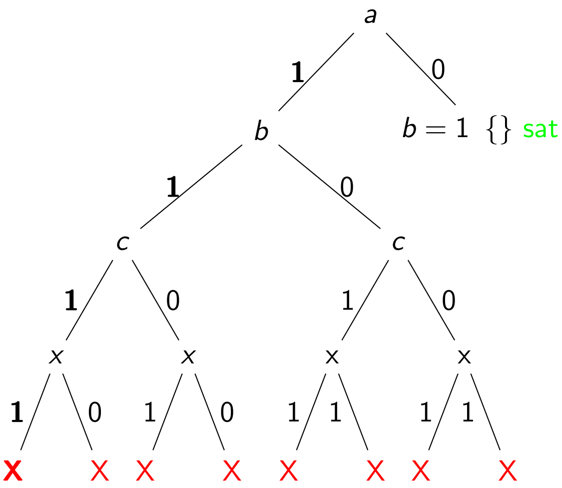
unit propagate y



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CDCL Example

$c_1 \quad \{a, b\}$

$c_2 \quad \{b, c\}$

$c_3 \quad \{\neg a, \neg x, y\}$

$c_4 \quad \{\neg a, x, z\}$

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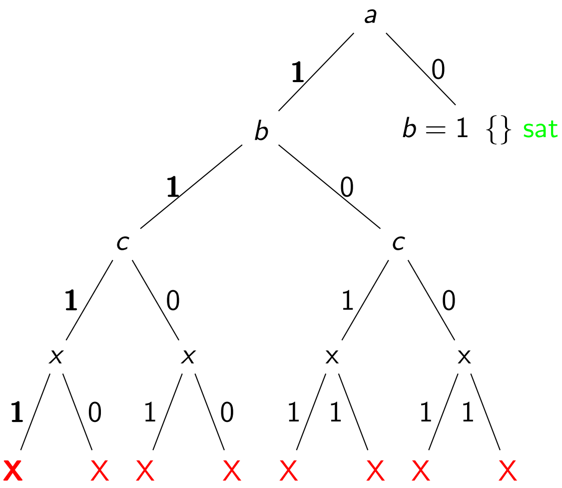
unit propagate z



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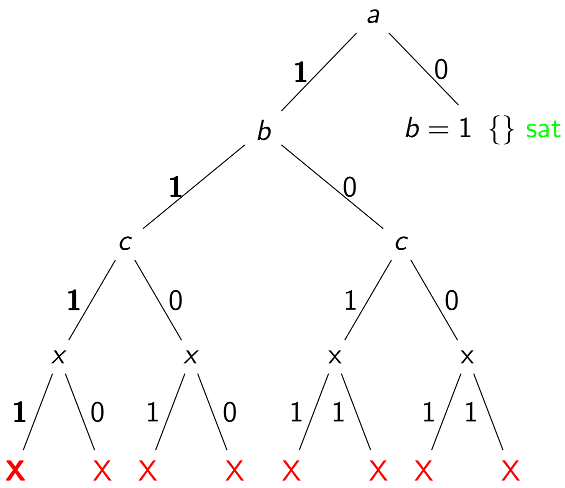
Note that b and c are irrelevant to the c_7 conflict. $(a \wedge x)$ or $(a \wedge y)$ are sufficient.



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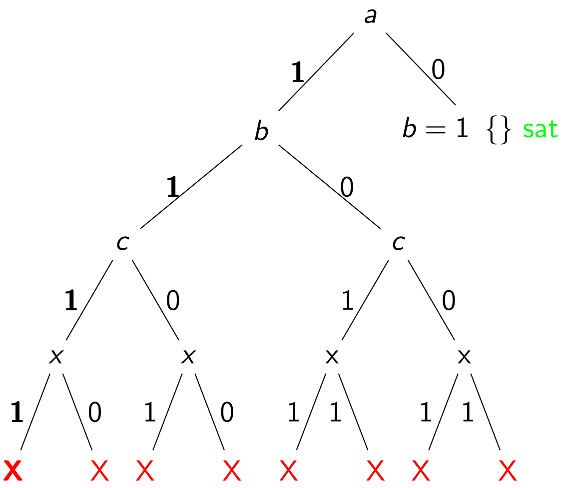
So, we can learn $\neg(a \wedge x) = (\neg a \vee \neg x)$ as a new constraint i.e. a *learned clause*.



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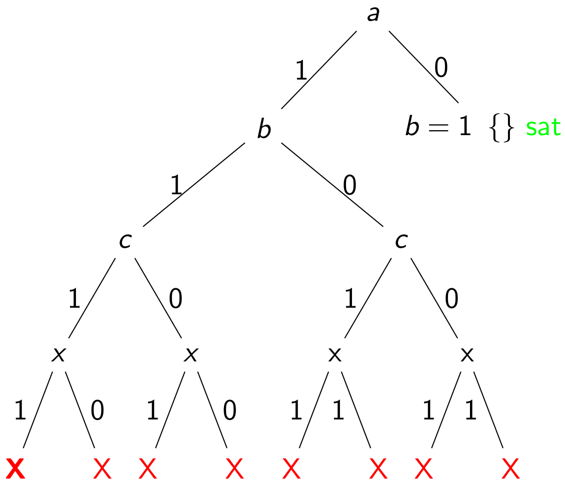

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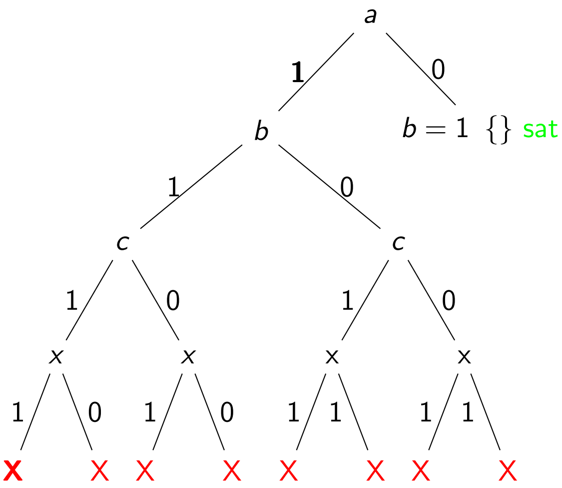

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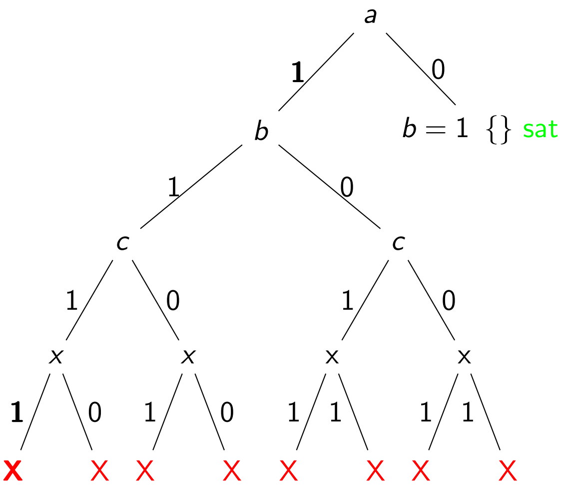
With the learned clause, we come to the conflict quickly.



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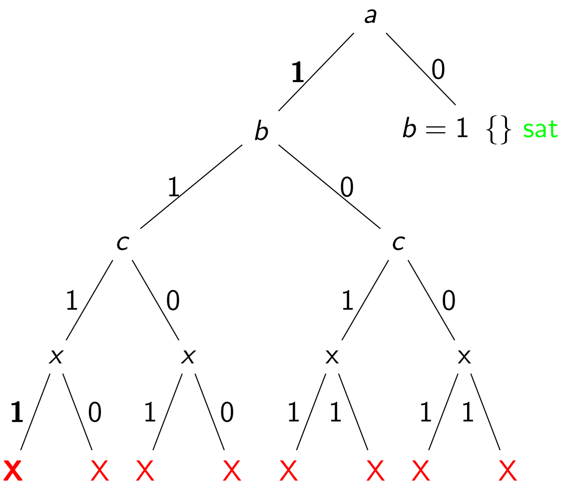

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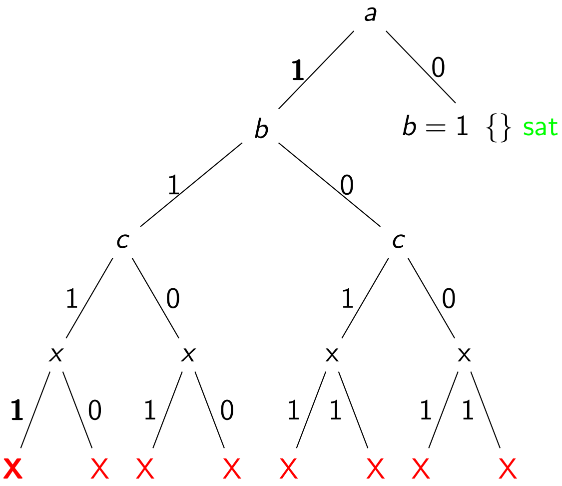
This time, a is sufficient to cause the conflict, so we learn $\neg a$.



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CDCL Example

l_2	$\{\neg a\}$
l_1	$\{\neg a, \neg x\}$
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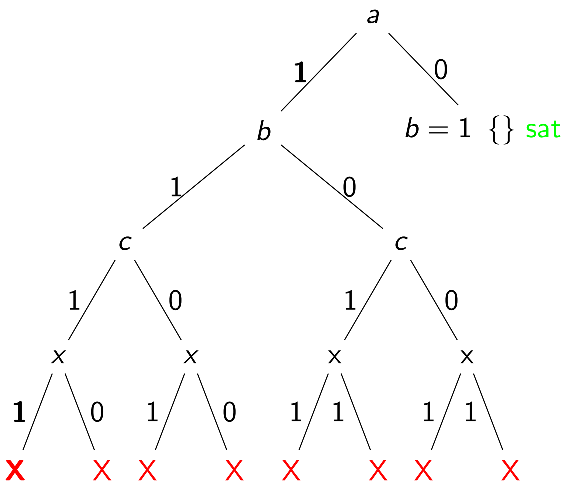
With $l_2 = \neg a$, we now get out of the unfruitful search space region.



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  - Nodes of this graph represent variable assignments in the current search path. Edges to dependencies between these assignments.
  - Cuts in the graph correspond to a conflict set and, by negating it, a potential clause to learn

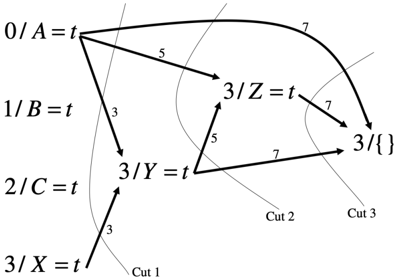

Figure 3.8. Three cuts in an implication graph, leading to three conflict sets.

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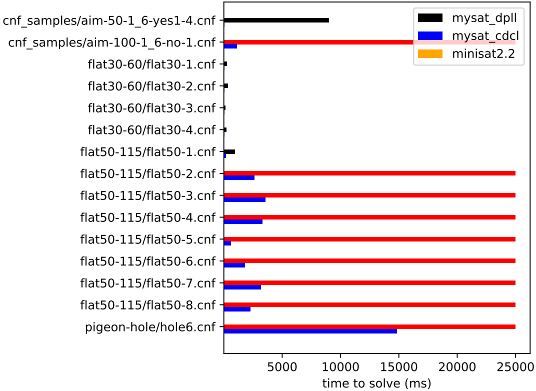
- e.g. runtime on some benchmarks with $\approx$ 50-200 variables, time budget of 25 seconds (red bar indicates timeout)



Future Extensions and Learning Heuristics

- Modern CDCL based SAT solvers employ many heuristics
  - Variable ordering
  - Learned clause deletion policies
- Often these are "expertly tuned" heuristics

CrystalBall

- *CrystalBall* [SKM19]: Possible to learn better heuristics from data on SAT solver executions?
- Learned clause deletion is an important heuristic for CDCL based SAT solvers
- Use data from SAT runs to train a model
- DRAT resolution proofs serve as a good source of data

Future Goals

- Implement support for resolution proof output in UNSAT cases
- Capture more statistics from solving runs as a basis for learning new heuristics

Questions?

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