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Anca Ralescu

Fuzzy Sets Methods in Image Processing and Understanding

Medical Imaging Applications

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To Dan and Stephan
Anca Ralescu

Acknowledgments

Isabelle Bloch's work on fuzzy sets started in the early 1990s with fuzzy mathematical morphology, which was a completely new topic both in image analysis and in fuzzy sets theory, and with the fusion of image information. She warmly thanks Henri Maître for his support and encouragement to investigate research tracks out of the mainstream since then.

She is also grateful to Didier Dubois and Henri Prade for their positive feedback and enlightening discussions during many workshops and conferences. Seeing her work appreciated by these two prominent researchers was always a great motivation to continue exploring new ideas.

Reading the papers by Lotfi Zadeh was a great source of inspiration, in particular to really see fuzzy sets as sets in the spatial domain. Then, with the aim of exploiting expert knowledge to help guiding segmentation and recognition in images, in particular in the field of medical imaging, she investigated the field of knowledge-based image understanding, and it became soon obvious that spatial relations had to play an important role. However, expert knowledge is often expressed in a linguistic way, and for several relations, no satisfactory mathematical definitions existed. This led her to work on proposing fuzzy models of such relations, mostly based on mathematical morphology. She benefitted from a sabbatical stay at Berkeley (October–November 1995 and January 1997) to work in this direction with Mori Anvari. Enlightening discussions during seminars chaired by Lotfi Zadeh on various subjects related to the theory of fuzzy sets and the applications thereof, and other nice discussions with several persons of his team, certainly encouraged her to continue to develop this research line. Although no one was working in the field of image understanding and spatial reasoning, she could get inspiration from these seminars and discussions. She thanks Lotfi Zadeh and Mori Anvari for their warm hosting during her sabbatical period at Berkeley, and the "Fonds France-Berkeley" for the financial support of this sabbatical.

She would like to thank all the PhD candidates and colleagues with whom part of the work presented in this book was done, as acknowledged by the joint papers mentioned in the bibliography.

Anca Ralescu's encounter with fuzzy sets and fuzzy logic goes back to 1972 when she was a student in Mathematics at the University of Bucharest Romania, and her classmate and husband, Dan Ralescu, was invited to collaborate on a fuzzy sets project. The result of that project was to be the classical book on fuzzy systems and applications coauthored by C.V. Negoita and D.A. Ralescu. Although students in mathematics in Romania had heard of logics other than the classical Boolean logic (thanks to the highly inspirational professor Grigore Moisil), it was very exciting to hear a new theory of uncertainty.

Anca's first work in the field of fuzzy sets came much later, after obtaining her PhD in Mathematics, with the specialty in Probability Theory, at Indiana University Bloomington. In 1983, she was appointed as Assistant Professor in the Department of Mathematical Sciences at the University of Cincinnati.

She was greatly influenced by Professor Lotfi Zadeh whom she had met in 1976, especially by his work on linguistic, imprecise quantifiers. Her first works were on the representation of rules involving such quantifiers. Frequent discussions with professor Zadeh, whom she visited often at the University of California, Berkeley, and who also visited the University of Cincinnati, contributed a great deal to the direction she took in her research.

Following a stay of five years in Japan, first at the Graduate School of the Tokyo Institute of Technology as guest of Professor Michio Sugeno (famous for his work on fuzzy control), and then at the Laboratory for International Fuzzy Engineering Research (LIFE), mentored by Professor Toshio Terano, her interests crystalized around the problem of image understanding. In particular, she was interested in defining image understanding as the ability to verbally describe the image contents. In charge of the Image Understanding Group at LIFE, with her colleagues there, she contributed to a new definition of spatial relations, to their high-level description compatible with the way humans describe them. Having read some of the papers written by Dr. Isabelle Bloch, she contacted her, and they met in 1994. Since then, they have collaborated or exchanged ideas in the domain of image understanding and applications, most recently in connection with the project of writing this book.

The two authors had many discussions and joint works, in particular during Anca's visits to Paris, almost every May for several years, which led to the writing of this book. Anca is grateful to Isabelle for this collaboration. They both express their warm thanks to Springer and the editors for their support.

Contents

Chapter 1
Introduction

Since its introduction [160], fuzzy sets theory was rapidly exploited and further developed for image processing and image understanding problems. One reason for this development is that fuzzy sets provide a consistent mathematical framework for dealing with imprecision in knowledge representation, information modeling at different levels, fusion of heterogeneous information, reasoning, and decision making. In this book, we present the main research works on fuzzy techniques for image understanding, with emphasis on recent ones and emerging topics. It includes several works not only by the authors but also by many other authors. It is assumed that the reader is familiar with basics in image processing, analysis, and understanding (that can be found, e.g., in [104]), although the book is rather self-contained. Applicative examples illustrate each addressed topic, with emphasis on medical imaging.

The remainder of this introduction is based on the review paper in [33].

1.1 Fuzzy Sets and Image Understanding Under Imprecision

1.1.1 Sources of Imprecision

Imprecision is often inherent to images. Its causes can be found at several levels, including observed phenomena (imprecise boundaries between structures or objects), acquisition process (limited resolution, numerical reconstruction methods, potential artifacts), image processing steps (e.g., filtering imprecision). Imprecision also occurs in the available knowledge, about the acquisition process, the properties of the obtained images, the domain, etc.

Sources of imprecision in images will be detailed in Chap. 2.

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1.1.2 Advantages and Usefulness of Fuzzy Sets

Fuzzy sets have several advantages as they provide a unified framework for representing and processing both numerical and symbolic information, along with its imprecisions, as in other domains of information processing [70].

Basic definitions on fuzzy sets theory will be recalled in Chap. 2.

First, fuzzy sets are able to represent several types of imprecision in images, as, for instance, imprecision in spatial location of objects, or imprecision in membership of an object to a class. For instance, partial volume effect finds a consistent representation in fuzzy sets (membership degrees of a pixel or voxel to objects directly represent partial membership to the different objects mixed up in this pixel or voxel, leading to a modeling consistent with respect to reality). Secondly, image information can be represented at different levels with fuzzy sets (local, regional, or global), as well as under different forms (numerical, or symbolic). For instance, classification based only on gray levels involves very local information (at the pixel level); introducing spatial coherence in the classification, or relations between features, involves regional information; and introducing relations between objects or regions for scene interpretation involves more global information and is related to the field of spatial reasoning. Thirdly, the fuzzy set framework allows for the representation of very heterogeneous information and is able to deal with information extracted directly from the images, as well as with information derived from some external knowledge, such as expert knowledge. This is exploited in particular in model-based pattern recognition, where fuzzy information extracted from the images is compared and matched to a model representing knowledge expressed in fuzzy terms.

Therefore this theory can support tasks at several levels, from low level (e.g., gray-level based classification) to high level (e.g., model-based structural recognition and scene interpretation). It provides a flexible framework for information fusion as well as powerful tools for reasoning and decision making. From a mathematical point of view, fuzzy sets can be equipped with a complete lattice structure, which is suitable for its association with other theories of information processing based on such structures, such as mathematical morphology or logics. While first applications mainly addressed reasoning at low level for classification, edge detection or filtering, higher level information modeling and processing are now more widely developed and still topics of current research. This includes dealing with spatial information at intermediate or higher level, via mathematical morphology, spatial reasoning, ontologies, graphs, or knowledge-based systems, as well as advances in machine learning, higher level descriptions of image content, handling different levels of granularity, to name but a few.

1.1.3 Semantic Gap

One important problem when reasoning at higher level, for instance based on symbolic models, is the semantic gap [149]. The question to be addressed is as follows: how to link visual percepts from the images to symbolic descriptions? This is also known as the anchoring or the symbol grounding problem in artificial intelligence [57, 83]. The usefulness of fuzzy sets to answer this question relies in their capability to model linguistic as well as related quantitative knowledge and information. A typical example is the notion of linguistic variable [161], where symbolic values have semantics defined by membership functions on a concrete domain.

This approach will be used for spatial relations, knowledge representation, and spatial reasoning in Chaps. 6, 8, and 9.

1.1.4 A Short Review of Existing Books

Let us now summarize the evolution of the domain, based on published textbooks or collections of more or less independent papers. The first works dealt with classification and pattern recognition and textbooks on these topics were published in the late 1990s [17, 18, 53, 121]. Indices of fuzziness and cluster validity were also introduced, with the main drawback of providing better results for crisp classifications, which contradicts the initial claimed need for fuzzy classes. The book by Tizhoosh [151] also mentions some work in geometry and mathematical morphology applied on fuzzy sets. It mainly focuses on low-level processing, for edge detection and image quality improvement, using rules applied on local neighborhoods of pixels, or minimizing a fuzziness index. The evolution towards rule-based systems and neuro-fuzzy approaches is acknowledged by the book edited by Bezdek et al. [18]. All these approaches have been rapidly used in different application domains, such as medical imaging [150]. The Ghent team has edited several books, providing an overview of the advances in fuzzy mathematical morphology, filtering using local approaches, control and rule-based methods, with the associated applications in [92], and more specifically in image filtering in [113]. New applications in remote sensing, image retrieval, video analysis, medical imaging are included in [114], as well as some theoretical advances. Let us also mention the review paper by Karmakar et al. [91], covering techniques for fuzzy segmentation, clustering, IF-THEN rules, mostly adapted to bi-modal images (or images with a known number of classes or clusters). Additional applications in image compression are included in [125], as well as various soft computing approaches, still mostly for low-level processing.

Higher level methods were then progressively introduced. As an evidence of this evolution, let us mention the book edited by Matsakis et al. [107] and the review paper [25], dealing with spatial relations, such as topological, metric, directional relations. This domain has then evolved towards more complex relations. Note that the usefulness of fuzzy sets for spatial relations had been recognized already in 1975 [77], but formal models for them appeared much later. These models are now exploited for model-based segmentation and recognition of structures in images. The book on granular computing by Pedrycz et al. [129] deals with fuzzy sets, as well as with other models for imprecision modeling such as interval analysis and rough sets. Some chapters deal with images and spatial reasoning. Noticeably, the preface mentions images as a typical domain where the methods described in the book are useful (for instance, for reasoning from pixel level to object level). Advances in databases, data mining, machine learning, case-based reasoning are described in [78]. A few chapters deal with spatial data, as part of the larger domain of fuzzy information processing.

More recently, the book by Chaira et al. [51] deals again with low-level processing, with some mentions of image retrieval and applications in remote sensing. Interestingly enough, the book also includes Matlab® examples, thus highlighting the concrete practical use of all these methods.

Finally, let us mention the now large number of special sessions in conferences, dedicated to fuzzy approaches for image processing and understanding (e.g., IPMU, WILF, FUZZ-IEEE, ICIP, EUSFLAT, i.e., both conferences focusing on fuzzy sets and conferences in image processing and computer vision). Several of these sessions are organized by the Working Group on Soft Computing in Image Processing.[1] This group is a follow-up of the SCIP group (Soft Computing in Image Processing)[2] and was recognized as a working group of the European Society for Fuzzy Logic and Technology (EUSFLAT).[3]

1.2 Representations

Fuzzy sets can be used to represent both image information, along with its imprecision, and domain and expert knowledge.

Fuzzy sets representing image information can be considered from two points of view. First, a membership function can be a function from the space on which the image is defined into [0, 1], representing the membership degree of each point to a spatial fuzzy object. Such models may represent different types of imprecision, either on the boundary of the objects (due, for instance, to partial volume effect, or to the spatial resolution), on the variability of these objects, on the potential ambiguity between classes, etc. Secondly, a membership function can be a function from a space of attributes into [0, 1]. At numerical level, such attributes are typically the gray levels. The membership value then represents the degree to which a gray level

[1] http://graphicwg.irafm.osu.cz/index.php/Main_Page.

[2] http://www.fuzzy.ugent.be/SCIP/index.html.

[3] http://www.eusflat.org/.

supports the membership to an object or a class, described in vague terms, such as bright, dark, etc. At intermediate level, attributes can refer, for instance, to the shape of image regions. Membership functions then allow determining the degree to which an image region is elongated, regular, etc.

As for knowledge representation, fuzzy sets are typically used to model, in a semi-quantitative way, symbolic or qualitative knowledge describing the expected content of the images (appearance and shape of the objects, spatial relations, type of objects, etc.). The concept of linguistic variable is then often used. These representations contribute to reduce the semantic gap, by associating a symbolic or qualitative value with a representation in a concrete domain (spatial domain or attribute domain). This applies to different types of knowledge useful in image understanding: generic knowledge on the type of observed scene and on the type of image, specific knowledge related to images, used for extracting meaningful information from these images, and knowledge linking image and model.

1.3 Low Level—Clustering, Enhancement, Filtering, Edge Detection

Low-level processing relies on a representation of image information at pixel or voxel level (or in a small neighborhood around them). One of the most common tasks addressed at this level is clustering and classification. The idea is to model classes with imprecise boundaries as fuzzy classes and to find the best partition optimizing some criteria. Among these methods, the most used is the fuzzy C-means algorithm (FCM) [16], often applied only to the gray level information. The main drawback of this method is that, due to a normalization constraint, the membership functions are not decreasing with respect to the distance to the class centers. An alternative solution to FCM, which avoids the normalization drawbacks, is given by possibilistic C-means (PCM) [94]. More recently, approaches have been proposed, modifying the objective function to increase the robustness of FCM to noise and to incorporate spatial information, by defining membership functions that depend on a local neighborhood around each point, e.g., [1, 75, 93, 99, 102, 134, 145]. Fuzzy classification has been associated with other formalisms, such as Markov random fields [144] or belief functions [106].

Recent advances in this domain are described in Chap. 7.

Representations based on fuzzy transforms [131] have also been proved to be useful for low-level image processing, such as image compression or coding [64, 132].

Another important task in local processing concerns filtering, enhancement, and edge detection. The main approaches can be grouped into two classes: techniques based on functional optimization, on the one hand, and rule-based techniques, on the other hand. These aspects have been largely developed in the literature (see, e.g., [6, 7, 18, 19, 49, 90, 97, 101, 152]). Functional approaches consist in minimizing or maximizing a functional, which can be interpreted as an analytical representation of some objective. For instance, enhancing the contrast of an image according to this technique amounts to reduce the fuzziness of the image. This can be performed by a simple modification of membership functions (for instance, using intensification operators), by minimizing a fuzziness index such as entropy, or even by determining an optimal threshold value (for instance, optimal in the sense of minimizing a fuzziness index) which provides an extreme enhancement (until binarization) [122, 124]. Other methods consist in modifying classical filters (median filter for instance) by incorporating fuzzy weighting functions [98]. Rule-based techniques rely on ideal models (of filters, contours, etc.). These ideal cases being rare, variations and differences with respect to these models are permitted through fuzzy representations of the models, as fuzzy rules [42, 96, 141, 142]. Note that rules are sometimes only a different representation of functional approaches. Their main advantage is that they are easy to design (in particular for adaptive operators) and to interpret, and they facilitate the communication with the user.

These low-level approaches are the most widely developed (and it is not possible to review all of them here) and are already covered in the existing literature. Therefore, they will not be considered in this book (except for advances in classification and clustering in Chap. 7). Moreover, they remain limited, and the impact of the fuzzy sets is not always the most prominent one. It increases when working at intermediate or higher level, as summarized in the next sections.

1.4 Intermediate Level

Work at intermediate level is performed via geometrical, topological, or metrical models and operations. Some geometrical objects such as points, disks, rectangles, or lines have been extended to make them fuzzy [46, 80, 138]. Several operations have been defined in the literature on fuzzy objects, in particular spatial fuzzy objects, starting with the early work of Zadeh [161] on set operations, and of Rosenfeld on geometrical operations [138]. Typical examples of geometrical operations are the area and perimeter of a fuzzy object. They can be defined as crisp numbers, where the computation involves each point up to its degree of membership. But when objects are not well defined, it is convenient to consider that their measurements are imprecise too. This point of view leads to definitions of these measures as fuzzy numbers [67]. It has been shown in [147, 148] that using fuzzy representations of digital objects allows deriving more robust measures than using crisp representations, and in particular, dealing properly with the imprecision induced by the digitization process. Such geometrical measures can typically be used in shape recognition, where geometrical attributes of the objects are taken into account (e.g., in mammography images [43, 133]), or as descriptors for indexing and data mining applications.

Let us now consider fuzzy connectivity as an example of a topological feature. It was initially defined in [138], and then exploited in fuzzy connectedness notions [153], now widely used, for instance in medical image segmentation, and incorporated in freely available software such as ITK.[4] More general classes of fuzzy connectivity have later been developed, with, again, applications in medical imaging [45, 118, 126]. Using both topology and metrics, the notion of skeleton and medial axis was also extended to fuzzy sets [85, 103, 120, 123].

Thanks to the strong algebraic structure of fuzzy sets, extension of mathematical morphology to the fuzzy case was very natural. Initial developments can be found in the definition of fuzzy Minkowski addition [68]. Then this problem has been addressed by several authors independently, e.g., [12, 20, 35, 58, 59, 61, 105, 112, 135, 146]. These works can be divided into two main approaches. In the first one [35, 59, 60], an important property that is put to the fore is the duality between erosion and dilation, the two core operations of mathematical morphology. The second type of approach is based on the notions of adjunction and fuzzy implication and was formalized in [61]. The links between both approaches have been clarified in [29], with conditions for their equivalence. It was also proved that the definitions of dilation and erosion in these approaches are the most general ones if we want them to share a set of classical properties with standard mathematical morphology. The general setting of complete lattices has been further investigated in [31], with extensions to deal also with bipolar information in [32, 108, 115, 116]. Besides the classical applications of mathematical morphology for filtering, enhancement, segmentation, its fuzzy version was used to model spatial imprecision (e.g., [37]), to define fuzzy spatial relations, to define median fuzzy sets and series of interpolating fuzzy sets [28], etc.

Fuzzy spatial objects and operations on them will be detailed in Chap. 3, while fuzzy mathematical morphology is addressed in Chap. 4.

1.5 Higher Level

1.5.1 Representations of Structural Information

The main information contained in images consists of properties of the objects and of relations between objects, both being used for pattern recognition and scene interpretation purposes. Relations between objects are particularly important since they carry structural information about the scene, by specifying the spatial arrangement of objects. These relations highly support structural recognition based on models. These models can be of iconic type, as an atlas, or of symbolic type, as linguistic descriptions, conceptual or semantic graphs, or ontologies.

Spatial relations are strongly involved in linguistic descriptions of visual scenes. They constitute a very important information to guide the recognition of structures embedded in a complex environment, and are more stable and less prone to

[4] http://www.itk.org.

variability (even in pathological cases) than object characteristics such as shape or size. Mathematical models of several spatial relations (adjacency, distances, directional relations, symmetry, betweenness, parallelism...) have been proposed in the framework of fuzzy sets theory, strongly relying on mathematical morphology operators [22, 23, 25, 36, 38, 41, 54, 154]. For instance, the semantics of a relation such as *close to, to the right of* can be modeled as a fuzzy structuring element, and the dilation of a reference object by this structuring element provides the fuzzy region of space where the corresponding relation is satisfied.

These fuzzy representations can enrich ontologies and contribute to reduce the semantic gap between symbolic concepts, as expressed in the ontology, and visual percepts, as extracted from the images. These ideas were used in particular in the segmentation and recognition methods described in [8, 39, 40, 55, 86, 119, 155]: a concept of the ontology is used for guiding the recognition by expressing its semantics as a fuzzy set, for instance, in the image domain or in an attribute domain, which can therefore be directly linked to image information.

Similarly, such spatial relations are useful attributes in graphs and fuzzy graphs, and endow recognition and mining methods based on similarity between graphs with structural information [5, 15, 50, 130], benefiting from the huge literature on fuzzy comparison tools (see, e.g., [44]). Spatial relations can also be embedded in conceptual graphs and their fuzzy extensions, as in [155].

Some important spatial relations and their modeling in a fuzzy set framework will be the topic of Chap. 6. Their use in structural representations (including linguistic representations, graphs, logics, ontologies, etc.) and spatial reasoning will be addressed in Chaps. 8 and 9.

1.5.2 Fusion

A lot of approaches for image processing and understanding, whatever their level, involve fusion steps. Information fusion becomes increasingly important due to the increasing number of imaging techniques. The information to be combined may come from several images, or from one image only, combining, for instance, several relations between objects, or several features of the objects, or images and a model, such as an anatomical atlas or a conceptual graph, or knowledge expressed in linguistic form or as ontologies. The advantages of fuzzy sets and possibilities rely in the variety of combination operators, offering a lot of flexibility in their choice, that can be adapted to any situation at hand, and which may deal with heterogeneous information [69, 158]. A classification of these operators was proposed in [21], with respect to their behavior (in terms of conjunctive, disjunctive, compromise [69]), the possible control of this behavior, their properties and their decisiveness, which proved to be useful for several applications in image processing. The fusion process can be done at several levels of information representation, from pixel level to higher level. Local fusion is often limited because spatial information is not really taken into account, and working at intermediate or higher level (for instance, combining several spatial relations to guide the understanding process) is more interesting and powerful. Examples can be found in various domains [40, 55, 109, 119, 127, 139, 155].

Fuzzy fusion will be the topic of Chap. 5.

1.5.3 Scene Understanding

Scene understanding using fuzzy approaches mostly belongs to the domain of spatial reasoning, which can be defined as the domain of spatial knowledge representation, in particular spatial relations between spatial entities, and reasoning on these entities and relations. This field has been largely developed in artificial intelligence, in particular using qualitative representations based on logical formalisms [3]. In image interpretation and computer vision it is much less developed and is mainly based on quantitative representations. However, to cope with the inherent imprecision of data and knowledge, semi-quantitative or semi-qualitative approaches would be preferable, hence highlighting the usefulness of fuzzy models for image interpretation.

A typical example in this domain concerns model-based structure recognition in images, where the model represents spatial entities and relations between them. Two main components of this domain are spatial knowledge representation and reasoning. In particular, spatial relations constitute an important part of the knowledge we have to handle. Imprecision is often attached to spatial reasoning in images and can occur at different levels, from knowledge to the type of question we want to answer. The reasoning component includes fusion of heterogeneous spatial knowledge, decision making, inference, recognition.

Scene understanding, including knowledge representation and reasoning, will be further detailed in Chaps. 8 and 9, with several examples and references. This includes graph matching and inexact graph matching, possibly using spatial relations, sequential segmentation and recognition of image structures, recognition framed as a constraint satisfaction problem, ontological reasoning, abductive reasoning...

1.6 Emerging Topics

1.6.1 Mining and Retrieval

Mining and retrieval are already quite old topics in imaging, justified by the large (and ever increasing) size and number of images. Little has been done in this domain using fuzzy sets, despite their potential [95]. A few works use low-level features such as color for image retrieval (e.g., [52, 82, 89]), and even fewer use structural information (e.g., [5]). Machine learning approaches have also been extended to the fuzzy case (e.g., [5, 110, 111]). This is clearly still an open research direction, including the associated questions of symbol grounding and semantic gap.

1.6.2 Towards Bipolarity

A recent trend in contemporary information processing focuses on bipolar information, both from a knowledge representation point of view, and from a processing and reasoning one. Bipolarity is important to distinguish between (i) positive information, which represents what is guaranteed to be possible, for instance, because it has already been observed or experienced, and (ii) negative information, which represents what is impossible or forbidden, or surely false [71, 74]. This domain has recently motivated work in several directions. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed [13, 14, 72, 74]. Three types of bipolarity are distinguished in [72]: (i) symmetric univariate, (ii) symmetric bivariate, (iii) asymmetric or heterogeneous, where two types of information are not necessarily linked together and may come from different sources. This last type is particularly interesting in image interpretation and spatial reasoning and can benefit from mathematical morphology [27, 30–32, 34, 108, 115, 116]. Further work in this direction, in association with reasoning based on different types of logics (see a first step, e.g., in [87]), is appealing.

1.6.3 Towards More Interactions Between Knowledge and Image Information

Such interactions should be understood in both directions and are likely to give rise to a lot of research work in the next years. As mentioned in Sect. 1.5, knowledge, in particular expressed in a fuzzy form, is useful to guide image understanding and spatial reasoning. However, reasoning aspects deserve to be more developed. For instance, image understanding could be expressed as an abduction process [9, 11], or information updating based on temporal information as a revision process. Merging temporal and spatial information is also an important direction to pursue, for applications such as video analysis, change detection, etc. Several frameworks could be involved, such as modal logics [24], description logics [65, 88], formal concept analysis [4, 10], etc. Conversely, results obtained from images could be further exploited to provide a linguistic description of the observed scene, e.g., in the expert domain language, considering the process of image understanding as the generation of a verbal description of the image content [136]. Recent work on fusion of multi-granularity linguistic terms could be helpful [84].

1.6.4 Deep Neuro-Fuzzy Systems

The combination of neural networks and fuzzy systems, called neuro-fuzzy systems, was developed in the early 1990s with the motivation to achieve the best trade-off between the precision capabilities of neural networks and the interpretability power of fuzzy systems. Basically, such systems can be seen as systems of fuzzy rules, with potentially fuzzy inputs and outputs, that are trained by a neural network. With the huge development of deep learning approaches, neuro-fuzzy systems naturally evolved towards deep neuro-fuzzy systems and are more and more developed, as acknowledged by the number of special sessions in journals and conferences. Applications in computer vision and image understanding are now emerging. Some examples are illustrated in Chap. 7.

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Chapter 2
Preliminaries

This chapter introduces some concepts, notations, and terminology, which will be useful subsequently in this book. Section 2.1 summarizes the main sources of imprecision in the context of image processing and understanding. They concern both data and related knowledge. The various types of imprecision encountered constitute a compelling reason for exploiting fuzzy sets theory for image processing, analysis, and understanding. In order to make this book as self-contained as possible, Sect. 2.2 reviews selected definitions of the fuzzy sets theory. Only the important notions that are useful for the topics of the next chapters are given. The reader may refer to existing texts for further insight on fuzzy set theory (e.g., [7]). The main operators are then given in Sect. 2.3. The concept of linguistic variable is the subject of Sect. 2.4. A core notion in fuzzy set theory, this is used to capture both the conceptual and concrete levels. General methods to extend an operation applying on crisp sets to the corresponding operation applying on fuzzy sets are given in Sect. 2.5. Finally the main notations used in the book are summarized in Sect. 2.6.

2.1 Imprecision in Images and Related Knowledge

Imprecision is often inherent to images, and its causes can be found at several levels:

- Observed phenomenon: imprecise limits between structures or objects that exist in reality (for instance, between healthy and pathological tissues when the pathology diffuses inside the normal tissues) will induce similar imprecise limits in observed images;
- Acquisition process (limited resolution, numerical reconstruction methods);
- Image processing steps (imprecision induced by a filtering for instance);

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Similarly, imprecision occurs in the descriptions of available knowledge. For instance, when describing the organization of brain structures, textbooks often include linguistic descriptions that are inherently imprecise (e.g., "structure A is anterior to structure B").

Moreover, the aim of an image understanding process can be expressed in an imprecise way, which is sometimes even preferable to a statement which is precise, but likely not sufficiently accurate.

Several examples illustrating the above considerations will be provided in the different chapters of this book.

Fuzzy sets have several advantages for representing such imprecision, as explained in Chap. 1. In particular, fuzzy set theory is of great interest to provide a rich collection of tools in a consistent mathematical framework, for all the issues described in Chap. 1. It allows representing imprecision of objects, relations, knowledge, and aims, at different levels of representation. It provides an unified framework for representing and processing related numerical and symbolic information, as well as structural information (e.g., spatial relations between objects in an image). Therefore this theory can be employed for tasks at several levels, from low level (e.g., gray-level based classification) to high level (e.g., model-based structural recognition and scene interpretation). At the same time, it provides a flexible framework for information fusion as well as powerful tools for reasoning and decision making.

Let us provide a simple example to illustrate the usefulness of fuzzy models to explicitly represent imprecision in the information provided by the images, as well as possible ambiguity between classes. For instance, the problem of partial volume effect finds a consistent representation in this model. A pixel or voxel suffering from partial volume effect is characterized by the fact that it belongs partially to two (or more) different tissues or classes. Using fuzzy sets, this translates immediately into non-zero membership values to more than one class. Figure 2.1 shows an example of an MR image of the brain of a patient suffering from adrenoleukodystrophy, and where the slice thickness induces a high partial volume effect. The grey levels on the right figure represent the membership values to the pathology. The pathology is then considered as a fuzzy object, represented by a membership function defined on the spatial domain.

More generally, a spatial fuzzy object may represent different types of imprecision, either on the boundary of the objects (due, for instance, to partial volume effect, or to the spatial resolution) or on the individual variability of these structures, etc.

As another example, fuzzy measures have been used in [25] for detecting masses in digital breast tomosynthesis. The measures are performed on detected fuzzy regions that are considered as candidate particles (Fig. 2.2). A decision concerning their recognition is performed by combining fuzzy attributes. Fuzzy decision trees can be used to this aim [4, 23] (see Sect. 8.7).

All this will be further developed in the next chapters.

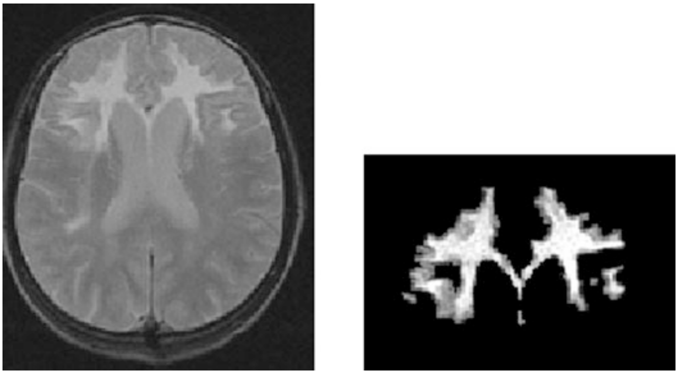

Fig. 2.1 MR image of the brain (left) (courtesy Prof. C. Adamsbaum, Kremlin-Bicêtre hospital, France), and estimation of the partial membership to the pathology (right) in the pathological area (white means that there is only pathological tissue in the considering voxel, black means no pathological tissue, and intermediate values represent the partial volume effect, i.e., voxels that have also a non-zero membership value to the white matter class)

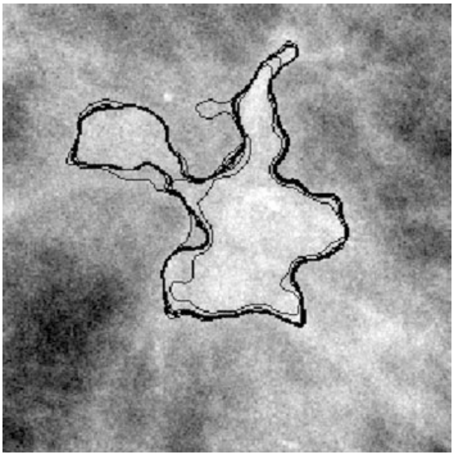

Fig. 2.2 Fuzzy particle (black lines represent the contours of the α-cuts of the fuzzy object) extracted from a digital mammography. Computing fuzzy attributes on this fuzzy object leads to a decision concerning this region (From [24])

2.2 Basic Definitions of Fuzzy Sets Theory

2.2.1 Fuzzy Sets

Let $\mathcal{U}$ be the universe of discourse, i.e., the space of objects of interest. It is a classical (or crisp) set. We denote by x, y, etc. its elements (or points). In image processing, $\mathcal{U}$ can typically be the space on which the image is defined (usually $\mathbb{Z}^n$ or $\mathbb{R}^n$, with $n = 2, 3, \ldots$) and will then be denoted by $\mathcal{S}$. Then the elements of $\mathcal{U} = \mathcal{S}$ are the points of the image (pixels, voxels). The universe can also be a set of values taken by some image characteristics, for instance, the scale of gray levels. Then an element x is a value (a gray level). The set $\mathcal{U}$ can also be a set of features, primitives, or objects extracted from the images (e.g., segments, regions, objects), leading to a higher level representation of the image content.

A subset X of $\mathcal{U}$ is defined by its characteristic function μ_X, such that $\mu_X(x) = 1$ if $x \in X$ and $\mu_X(x) = 0$ if $x \notin X$. The characteristic function μ_X is a binary function, specifying the crisp membership of each point of $\mathcal{U}$ to X.

Fuzzy set theory aims at dealing with gradual membership, accomplished by a rather modest extension of the definition of μ to take values in [0, 1] rather than {0, 1}. A fuzzy subset of $\mathcal{U}$ is then defined through its membership function μ from $\mathcal{U}$ into [0, 1].[1] For each x of $\mathcal{U}$, $\mu(x) \in [0, 1]$ represents the membership degree of x to the fuzzy subset, i.e., to which extent x belongs to it. Although the correct terminology would be to speak of "fuzzy subset," commonly, the simpler term "fuzzy set" is used (just as in the case of crisp subsets). We keep this term in the following, for the sake of simplicity.

Various notations are used to designate a fuzzy set. A fuzzy set is completely defined by the set $\{(x, \mu(x)), x \in \mathcal{U}\}$, which can be noted as $\int_{\mathcal{U}} \mu(x)/x$ or in the discrete finite case $\sum_{i=1}^{N} \mu(x_i)/x_i$ where N denotes the cardinality of $\mathcal{U}$.

Since the set of all couples $(x, \mu(x))$ is completely equivalent to the definition of the function μ, we have chosen here to simplify notations and to always use the functional notation μ, a function of $\mathcal{U}$ into [0, 1], and μ will alternatively denote a fuzzy set or its membership function.

The support of a fuzzy set μ is the set of points that have a strictly positive membership to μ (it is a crisp set):

$$Supp(\mu) = \{x \in \mathcal{U} \mid \mu(x) > 0\}. \tag{2.1}$$

The core of a fuzzy set μ is the set of points that belong completely to μ (it is a crisp set):

$$Core(\mu) = \{x \in \mathcal{U} \mid \mu(x) = 1\}. \tag{2.2}$$

A normalized fuzzy set μ is such that at least one point belongs completely to μ (i.e., $Core(\mu) \neq \emptyset$):

$$\exists x \in \mathcal{U},\ \ \mu(x) = 1. \tag{2.3}$$

A unimodal fuzzy set μ is such that there exists a unique point x such that $\mu(x) = 1$. A less constraining definition allows the core of μ to be a compact set and not only one point.

[1] The interval [0, 1] is the most used. However, any other interval, or other set (typically a lattice) could be used. This also allows for extensions such as L-fuzzy sets [15], where L is a lattice defining the co-domain of the membership functions.

2.2.2 Set Theoretical Operations: Original Definitions of L. Zadeh [34]

Since fuzzy sets have been introduced by L. Zadeh in [34] in order to generalize sets, the first operations that have been proposed are set theoretical (algebraic) operations. We recall here the original definitions proposed by L. Zadeh. Further operations are defined later, in Sect. 2.3.

The equality of two fuzzy sets is defined by the equality of their membership functions:

$$\mu = \nu \Leftrightarrow \forall x \in \mathcal{U}, \mu(x) = \nu(x). \tag{2.4}$$

The inclusion of a fuzzy set in another one is defined as an inequality on their membership functions:

$$\mu \subseteq \nu \Leftrightarrow \forall x \in \mathcal{U}, \mu(x) \leq \nu(x). \tag{2.5}$$

The intersection (respectively, union) between two fuzzy sets is defined as the pointwise minimum (respectively, maximum) of their membership values:

$$\forall x \in \mathcal{U}, (\mu \cap \nu)(x) = \min[\mu(x), \nu(x)], \tag{2.6}$$

$$\forall x \in \mathcal{U}, (\mu \cup \nu)(x) = \max[\mu(x), \nu(x)]. \tag{2.7}$$

The complement of a fuzzy set μ, $\overline{\mu}$, is defined as:

$$\forall x \in \mathcal{U}, \overline{\mu}(x) = 1 - \mu(x). \tag{2.8}$$

The main properties of these definitions are the following:

- They are all consistent with crisp set operations, that is, in the particular case where the membership functions only take values 0 and 1 (i.e., they are crisp sets), these definitions reduce to the classical definitions; note that this property is important since it is the least we can ask to the fuzzy extension of an operation on sets.
- $\mu = \nu \Leftrightarrow \mu \subseteq \nu$ and $\nu \subseteq \mu$.
- The fuzzy complementation is involutive, that is $\overline{(\overline{\mu})} = \mu$.
- Intersection and union are commutative and associative.
- Intersection and union are idempotent and mutually distributive.
- Intersection and union are dual with respect to the complementation: $\overline{(\mu \cap \nu)} = \overline{\mu} \cup \overline{\nu}$, $\overline{(\mu \cup \nu)} = \overline{\mu} \cap \overline{\nu}$.
- If we consider the empty set $\emptyset$ as a fuzzy set having membership values all equal to 0, then we have $\mu \cap \emptyset = \emptyset$ and $\mu \cup \emptyset = \mu$, for any fuzzy set μ defined on $\mathcal{U}$.

- If we consider the universe as a fuzzy set having membership values all equal to 1, then we have $\mu \cap \mathcal{U} = \mu$ and $\mu \cup \mathcal{U} = \mathcal{U}$, for any fuzzy set μ defined on $\mathcal{U}$.

These properties are the same as the corresponding crisp operations. However, some properties do not hold, such as the excluded-middle and non-contradiction laws, since:

$$\mu \cup \overline{\mu} \neq \mathcal{U}, \tag{2.9}$$

$$\mu \cap \overline{\mu} \neq \emptyset. \tag{2.10}$$

2.2.3 Structure and Types of Fuzzy Sets

Let us denote by $\mathcal{C}$ the set of all crisp subsets of $\mathcal{U}$, and by $\mathcal{F}$ the set of all fuzzy subsets of $\mathcal{U}$. The set $\mathcal{C}$ is a Boolean lattice with respect to the intersection and union (i.e., a complemented distributive lattice). It can be considered as the lattice induced by the structure of $\{0, 1\}$. The interval $[0, 1]$ is a pseudo-complemented distributive lattice (in the lattice terminology, the complementation to 1 is a pseudo-complementation), which induces a pseudo-complemented distributive lattice structure on $\mathcal{F}$. This structure can be enhanced with connectives (in particular a t-norm and its residual implications, see Sect. 2.3.6), leading to a residuated lattice (see Chap. 4).

Several types of fuzzy sets can be considered. The membership functions considered so far take values that are numbers. These are called type-1 fuzzy sets. But membership values are not necessarily numbers. They can also be fuzzy sets, by making use of the lattice structure of the set of fuzzy sets. A type-2 fuzzy set is a fuzzy set whose membership values are type-1 fuzzy sets. More generally a type-m fuzzy set is a fuzzy set whose membership values are type-$(m-1)$ fuzzy sets (for $m > 1$). Such extensions of fuzzy sets are particularly useful when the membership value to be attached to an element is imprecisely defined. In the sequel, mainly type-1 fuzzy sets are considered. The operations defined below can be generalized to type-m fuzzy sets, using the extension principle (see Sect. 2.5.1).

2.2.4 α-Cuts

The α-cut (or level set) of a fuzzy set μ is the crisp set defined as:

$$\mu_\alpha = \{x \in \mathcal{U} \mid \mu(x) \geq \alpha\}. \tag{2.11}$$

Strict (or strong) α-cuts are defined as:

$$\mu_\alpha = \{x \in \mathcal{U} \mid \mu(x) > \alpha\}. \tag{2.12}$$

A fuzzy set can be considered as a "stack" of its α-cuts. It can be reconstructed from them using different formulas, the main ones being:

$$\mu(x) = \int_0^1 \mu_\alpha(x) d\alpha, \tag{2.13}$$

$$\mu(x) = \sup_{\alpha \in]0,1]} \min(\alpha, \mu_\alpha(x)), \tag{2.14}$$

$$\mu(x) = \sup_{\alpha \in]0,1]} (\alpha \mu_\alpha(x)). \tag{2.15}$$

Let us now look at the links with Zadeh's operators. The following relationships hold:

$$\forall (\mu, \nu) \in \mathcal{F}^2, \ \mu = \nu \Leftrightarrow \forall \alpha \in]0, 1], \mu_\alpha = \nu_\alpha, \tag{2.16}$$

$$\forall (\mu, \nu) \in \mathcal{F}^2, \ \mu \subseteq \nu \Leftrightarrow \forall \alpha \in]0, 1], \mu_\alpha \subseteq \nu_\alpha, \tag{2.17}$$

$$\forall (\mu, \nu) \in \mathcal{F}^2, \ \forall \alpha \in [0, 1], (\mu \cap \nu)_\alpha = \mu_\alpha \cap \nu_\alpha, \tag{2.18}$$

$$\forall (\mu, \nu) \in \mathcal{F}^2, \ \forall \alpha \in [0, 1], (\mu \cup \nu)_\alpha = \mu_\alpha \cup \nu_\alpha, \tag{2.19}$$

$$\forall \mu \in \mathcal{F}, \ \forall \alpha \in [0, 1], \overline{\mu}_\alpha = \overline{(\mu_{1-\alpha})}. \tag{2.20}$$

Note that the last equation is not as straightforward as the previous ones.

Taking the α-cut of a fuzzy set, for some given value of α, amounts to select the elements of $\mathcal{U}$ that belong to the fuzzy set with degree α at least. It is therefore a kind of thresholding process on the membership function. It can also be seen as a "defuzzification" process, i.e., a mean to return in the domain of crisp sets, which is used, for instance, to make a decision in a fuzzy system.

2.2.5 Cardinality

In this section, we consider only fuzzy sets that are defined over a finite universe, or that have a finite support. This is not restrictive when applying fuzzy sets theory to image processing, since in this domain, we are working mainly with finite (discrete) universes.

The cardinality of such a fuzzy set μ can be defined as:

$$|\mu| = \sum_{x \in \mathcal{U}} \mu(x), \tag{2.21}$$

or, if $\mathcal{U}$ is not finite but the support of μ is finite:

$$|\mu| = \sum_{x \in Supp(\mu)} \mu(x). \tag{2.22}$$

Again this definition is consistent with the cardinality of a crisp set. It can be interpreted as counting each point for an amount corresponding to its membership to the fuzzy set. It is also called the power of the fuzzy set (e.g., in [21]).

This definition can be extended to the case where $\mathcal{U}$ is not finite but measurable. Let M be a measure on $\mathcal{U}$ (such that $\int_{\mathcal{U}} dM(x) = 1$). The cardinality of μ is defined as:

$$|\mu| = \int_{\mathcal{U}} \mu(x) dM(x). \tag{2.23}$$

Note that all these definitions provide a numeric result which, however, is not necessarily an integer. Extensions will be mentioned in Sect. 2.2.7.

2.2.6 *Convexity*

In this section, the universe $\mathcal{U}$ is a real Euclidean space (of any dimension).

The convexity of a fuzzy set is defined from its α-cuts as follows: a fuzzy set μ is convex iff its α-cuts are convex (for all α in [0, 1]). This definition is not equivalent to the convexity of the membership function in an analytical sense.[2] The analytical equivalent expression for fuzzy convexity is as follows: μ is convex iff

$$\forall (x, y) \in \mathcal{U}^2,\ \forall \lambda \in [0, 1],\ \min(\mu(x), \mu(y)) \leq \mu(\lambda x + (1 - \lambda) y). \tag{2.24}$$

2.2.7 *Fuzzy Number*

In this section, we set $\mathcal{U} = \mathbb{R}$.

[2] The convexity of a function f is defined as $\forall (x, y),\ f(\lambda x + (1 - \lambda) y) \geq \lambda f(x) + (1 - \lambda) f(y)$.

A fuzzy quantity is a fuzzy set μ on $\mathbb{R}$. A fuzzy interval is a fuzzy quantity which is convex (i.e., its α-cuts are intervals). There is an equivalence between the upper-semi-continuity of μ and the fact that its α-cuts are closed intervals.

A fuzzy number is a fuzzy upper-semi-continuous (u.s.c.) interval with compact support and which is unimodal. Less strict definitions are also considered in the literature, in particular by accepting not only one single modal value, but an interval of modal values, i.e., there exist four reals a, b, c, d, with $a \leq b \leq c \leq d$, such that $\mu(x) = 0$ outside the interval $[a, d]$, increasing on $[a, b]$, decreasing on $[c, d]$ and equal to 1 on $[b, c]$ [13, 14].

A fuzzy number can be interpreted as a flexible representation of an imprecise quantity, which is more general than a crisp interval.

Fuzzy Cardinality Let us return to the definition of the cardinality of a fuzzy set. It has been defined previously as a number. However, when considering a fuzzy set, it can be interesting to define it as a fuzzy number, since the cardinality of an imprecisely defined set may be considered as imprecise too. The cardinality of a fuzzy set as a fuzzy number (see [7]) is defined as:

$$|\mu|_f(n) = \sup\{\alpha \in [0, 1],\ |\mu_\alpha| = n\}. \tag{2.25}$$

This constitutes a first example of fuzzy number. Several others will appear in the next chapters.

Let us detail another approach to fuzzy cardinality. In addition to the fact that the definition as a crisp number does not guarantee a whole number, as the cardinality should be, it has other drawbacks as discussed in [28]. Other definitions, including [1, 26–28], discuss the cardinality of a fuzzy set as a fuzzy set over a set of whole numbers. Such a representation of the cardinality of a fuzzy set is especially useful in the connection with quantifiers, as it will be seen later in Sect. 8.2.2.

According to [26, 28], for the fuzzy set $A = \{\mu_1 \dots \mu_n\}$ defined on the domain $\{x_1 \dots x_n\}$, where, without loss of generality $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, the cardinality of A is the fuzzy set, $CardA$ with membership function $\mu_{CardA} : \{0, \dots, n\} \to [0, 1]$ defined as:

$$\mu_{CardA}(k) = \mu_k \wedge (1 - \mu_{k+1}), \tag{2.26}$$

where $\mu_0 = 1$ and $\mu_{n+1} = 0$ are introduced for convenience. When A is a crisp set with n elements, then Equation (2.26) produces an indicator function (i.e., valued in $\{0, 1\}$), defined as:

$$\mu_{CardA}(k) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

A crisp value can be obtained from the fuzzy cardinality, by defuzzification either by using the center of gravity (COG), which, however, not being guaranteed to be a whole value, is not really satisfactory, or as defined in [28]:

$$nCardA = \begin{cases} 0 & \text{if } A = \emptyset \\ j & \text{if } A \neq \emptyset \text{ and } \mu_j \geq 0.5 \\ j-1 & \text{if } A \neq \emptyset \text{ and } \mu_j < 0.5, \end{cases} \quad (2.27)$$

where $j = \max\{1 \leq s \leq n \mid \mu_{s-1} + \mu_s > 1\}$.

It has been shown in [28] that if in the fuzzy set A exactly k elements have degree greater than or equal to 0.5, $nCardA = k$, that is, $nCard$ is the usual cardinality of the (crisp) 0.5-cut (level set at level 0.5) of A. This, again, points to the fact that a better defuzzification may actually be a crisp set.

A simple example illustrates the concepts of fuzzy cardinality discussed above. Consider the fuzzy set

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 1 & 0.9 & 0.8 & 0.4 & 0.3 & 0.1 \end{pmatrix}$$

Using Eq. 2.21 the cardinality of A is $|A| = 3.6$, which, in terms of whole numbers could be interpreted as "*Between 3 and 4*," or "*Approximately Half* of the cardinality (crisp) of the support of A." Using Eq. 2.26 yields:

$$Card_A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & .1 & .2 & .6 & .4 & .3 & .1 \end{pmatrix}$$

and from Eq. 2.27, $nCard = 3$.

***L*-*R* Fuzzy Numbers** A widely used type of fuzzy numbers is called L-R fuzzy numbers. They are defined through a parametric representation of their membership function. A L-R fuzzy number μ is defined as:

$$\forall x \in \mathbb{R},\ \mu(x) = \begin{cases} L(\frac{m-x)}{\alpha}) & \text{for } x \leq m \\ R(\frac{x-m}{\beta}) & \text{for } x \geq m, \end{cases} \quad (2.28)$$

where α and β are strictly positive numbers called left and right spreads, m is a number called mean value of the fuzzy number and L and R are functions (referred to as the reference functions of the fuzzy number) having the following properties:

- $\forall x \in \mathbb{R}$, $L(x) = L(-x)$.
- $L(0) = 1$.
- L is non-increasing on $[0, +\infty[$

and similar properties for R.

One of the main advantages of such fuzzy numbers is that they have a compact representation, and computations on fuzzy numbers can be performed easily.

2.3 Main Operators on Fuzzy Sets

After the early work of Zadeh [34], many more operators have been proposed in the fuzzy set theory, to combine membership functions, or possibility distributions.[3] These operators include connectives, aggregation, and combination (or fusion) operators. Several review papers summarize the main classes of operators (e.g., [2, 8, 33]). The presentation in this section is based on these reviews.

In the following, definitions of the main classes of operators are provided, as well as examples of the most used forms in each class, and simple interpretations (according to [8, 9]). Further interpretations in terms of set operations and in terms of data fusion are postponed to the corresponding chapters.

Since most operators work pointwise (i.e., combine membership degrees or possibility degrees at the same point of $\mathcal{U}$), it is sufficient to define the operators on the values taken by membership functions or possibility distributions. They are therefore defined as functions from [0, 1] or from $[0, 1] \times [0, 1]$ into [0, 1].

2.3.1 Fuzzy Complementation

A fuzzy complementation is a function c from [0, 1] into [0, 1] such that:

1. $c(0) = 1$.
2. $c(1) = 0$.
3. c is involutive, i.e., $\forall x \in [0, 1], c(c(x)) = x$.
4. c is strictly decreasing.

The most obvious example is the one introduced in Sect. 2.2.2 as:

$$\forall x \in [0, 1],\ c(x) = 1 - x. \tag{2.29}$$

The general form of any continuous complementation is:

$$\forall x \in [0, 1],\ c(x) = \varphi^{-1}[1 - \varphi(x)], \tag{2.30}$$

where φ is any function from [0, 1] into [0, 1] such that $\varphi(0) = 0$, $\varphi(1) = 1$, φ is strictly increasing.

[3] We draw the reader's attention to the fact that since a possibility distribution and a membership functions have similar mathematical expression and given the links that exist between them, the same operators can apply to both of them. However, they have different meanings and origins, and this should not be under-estimated. We do not further consider possibilistic interpretations in this book.

If, for some n, φ takes the form $\forall x \in [0,1],\ \varphi(x) = x^n$, then the derived complementation is:

$$\forall x \in [0,1],\ c(x) = (1 - x^n)^{1/n}. \tag{2.31}$$

This form becomes closer to a crisp complementation when n increases (for $n > 1$) or when n decreases (for $n < 1$). In the first case, almost all values (but those close to 1) have a complement close to 1, and in the second one, almost all values (but those close to 0) have a complement close to 0.

If, for some real a in $]0, 1]$, φ takes the form:

$$\forall x \in [0,1],\ \varphi(x) = \frac{ax}{(1-a)x + 1}, \tag{2.32}$$

then the derived complementation is:

$$\forall x \in [0,1],\ c(x) = \frac{1-x}{1+a^2 x}. \tag{2.33}$$

Another example, depending on four parameters a, b, c (such that $0 \le a < b < c \le 1$) and n, is:

$$\forall x \in [0,1],\ c(x) = \begin{cases} 1 & \text{if } 0 \le x \le a \\ 1 - \frac{1}{2}[\frac{x-a}{b-a}]^n & \text{if } a \le x \le b \\ \frac{1}{2}[\frac{c-x}{c-b}]^n & \text{if } b \le x \le c \\ 0 & \text{if } c \le x \le 1 \end{cases} \tag{2.34}$$

2.3.2 *Triangular Norms and Conorms*

In the context of stochastic geometry (e.g., [22, 30]), a t-norm t, modeling fuzzy conjunction, is defined as a function of two variables from $[0,1] \times [0,1]$ to $[0,1]$ satisfying the following properties:[4]

1. t is commutative, i.e., $\forall (x, y) \in [0,1]^2,\ t(x, y) = t(y, x)$.
2. t is associative, i.e., $\forall (x, y, z) \in [0,1]^3,\ t[t(x, y), z] = t[x, t(y, z)]$.
3. 1 is unit element, i.e., $\forall x \in [0,1],\ t(x, 1) = t(1, x) = x$.
4. t is increasing in its two arguments:

$$\forall (x, x', y, y') \in [0,1]^4,\ (x \le x' \text{ and } y \le y') \Rightarrow t(x, y) \le t(x', y').$$

[4] Note that weaker forms of conjunctions can be used, not detailed here.

From these properties, limit conditions can be derived: $t(0,1) = t(0,0) = t(1,0) = 0$ and $t(1,1) = 1$, and it can be easily shown that 0 is null element ($\forall x \in [0,1], t(x,0) = 0$).

A continuity property is often added to these properties.

The t-norm operators generalize intersection to fuzzy sets, as well as logical "and." Examples of t-norms are $\min(x,y)$, xy, $\max(0, x+y-1)$.

It is easy to prove the following result: for any t-norm t, the following inequality holds:

$$\forall (x,y) \in [0,1]^2, t(x,y) \leq \min(x,y). \tag{2.35}$$

This shows that the "min" is the largest t-norm and that any t-norm has a conjunctive behavior (an operator is said conjunctive if the result of the combination is less than each of the combined values).

On the opposite, any t-norm is always greater than t_0, which is the smallest t-norm, defined as:

$$\forall (x,y) \in [0,1]^2,\ t_0(x,y) = \begin{cases} x \text{ if} & y = 1 \\ y \text{ if} & x = 1 \\ 0 \text{ otherwise.} & \end{cases} \tag{2.36}$$

Moreover, the following inequalities hold between the previous examples of t-norms:

$$\forall (x,y) \in [0,1]^2,\ t_0(x,y) \leq \max(0, x+y-1) \leq xy \leq \min(x,y). \tag{2.37}$$

Parametrized definitions of these operators provide variations between some of these common operators. For instance, the t-norm defined in [31] as:

$$\forall (x,y) \in [0,1]^2,\ t(x,y) = 1 - \min[1, [(1-x)^p + (1-y)^p]^{1/p}] \tag{2.38}$$

varies from Lukasiewicz t-norm $\max(0, x+y-1)$ for $p = 1$ to the min for $p = +\infty$.

Given a t-norm t and a complementation c, another operator T can be defined by duality, called t-conorm, and modeling fuzzy disjunction:

$$\forall (x,y) \in [0,1]^2,\ T(x,y) = c[t(c(x), c(y))]. \tag{2.39}$$

A t-conorm T satisfies following properties:

1. T is commutative, i.e., $\forall (x,y) \in [0,1]^2,\ T(x,y) = T(y,x)$.
2. T associative, i.e., $\forall (x,y,z) \in [0,1]^3,\ T[T(x,y),z] = T[x, T(y,z)]$.
3. 0 is unit element, i.e., $\forall x \in [0,1],\ T(x,0) = T(0,x) = x$.
4. T is increasing in its two arguments:

$$\forall (x,x',y,y') \in [0,1]^4,\ (x \leq x' \text{ and } y \leq y') \Rightarrow T(x,y) \leq T(x',y').$$

5. Limit conditions: $T(0,1) = T(1,1) = T(1,0) = 1$ and $T(0,0) = 0$.
6. 1 is null element ($\forall x \in [0,1]$, $T(x,1) = 1$).

The t-conorm operators generalize union to fuzzy sets, as well as logical "or." Examples of t-conorms are $\max(x, y)$, $x + y - xy$, $\min(1, x + y)$. For any t-conorm T, the following inequality holds:

$$\forall (x, y) \in [0,1]^2, T(x, y) \geq \max(x, y). \tag{2.40}$$

This shows that the "max" is the smallest t-conorm and that any t-conorm has a disjunctive behavior (an operator is said disjunctive if the result of the combination is greater than each of the combined values).

On the opposite, any t-conorm is always less than T_0, which is the largest t-conorm, defined as:

$$\forall (x, y) \in [0,1]^2,\ T_0(x, y) = \begin{cases} x \text{ if} & y = 0 \\ y \text{ if} & x = 0 \\ 1 \text{ otherwise.} & \end{cases} \tag{2.41}$$

Moreover, the following inequalities hold between the previous examples of t-conorms:

$$\forall (x, y) \in [0,1]^2,\ T_0(x, y) \geq \min(1, x + y) \geq x + y - xy \geq \max(x, y). \tag{2.42}$$

Further useful properties of t-norms and t-conorms are:

- Any t-norm or t-conorm is distributive with respect to min and max:

$$\forall (x, y, z) \in [0,1]^3,\ t[x, \min(y, z)] = \min[t(x, y), t(x, z)], \tag{2.43}$$

  and a similar equality for t-conorms.
- The only t-norms and t-conorms that are mutually distributive are min and max.
- The only t-norm that is idempotent is the min, and the only t-conorm that is idempotent is the max.
- Given any t-norm t and any continuous strictly increasing function h from $[0,1]$ into $[0,1]$ such that $h(0) = 0$ and $h(1) = 1$, the following function t' is a t-norm [29]:

$$\forall (x, y) \in [0,1]^2,\ t'(x, y) = h^{-1}[t(h(x), h(y))]. \tag{2.44}$$

  This shows how families of t-norms can be generated from a given one.

Several works have been dedicated to the precise description of specific families of t-norms and t-conorms, for which generic definitions could be found [8]. Two of these are particularly useful: Archimedean t-norms and nilpotent t-norms.

An Archimedean strictly monotonous t-norm is a t-norm t such that:

$$\forall x \in]0, 1[, \ t(x, x) < x, \tag{2.45}$$

and

$$\forall (x, y, y') \in [0, 1]^3, x \neq 0, \ y < y' \Rightarrow t(x, y) < t(x, y'). \tag{2.46}$$

Similarly an Archimedean strictly monotonous T-conorm T satisfies the following two properties:

$$\forall x \in]0, 1[, \ T(x, x) > x, \tag{2.47}$$

$$\forall (x, y, y') \in [0, 1]^3, x \neq 1, \ y < y' \Rightarrow T(x, y) < T(x, y'). \tag{2.48}$$

Any Archimedean strictly monotonous t-norm t can be expressed in the following form:

$$\forall (x, y) \in [0, 1]^2, t(x, y) = f^{-1}[f(x) + f(y)], \tag{2.49}$$

where f, referred to as "generating function," is a continuous decreasing bijection from $[0, 1]$ into $[0, +\infty]$ such that $f(0) = +\infty$ and $f(1) = 0$.

The associated t-conorms take the form:

$$\forall (x, y) \in [0, 1]^2, T(x, y) = \varphi^{-1}[\varphi(x) + \varphi(y)], \tag{2.50}$$

where the generating function φ is a continuous increasing bijection from $[0, 1]$ into $[0, +\infty]$ such that $\varphi(0) = 0$ and $\varphi(1) = +\infty$.

Such t-norms and t-conorms never satisfy the non-contradiction and excluded-middle laws. These laws are expressed as:

$$\forall x \in [0, 1], \ t[x, c(x)] = 0, \tag{2.51}$$

and:

$$\forall x \in [0, 1], \ T[x, c(x)] = 1. \tag{2.52}$$

These two statements do not hold for Archimedean strictly monotonous t-norms and t-conorms.

An Archimedean strictly monotonous t-norm (respectively, t-conorm) can be defined by a multiplicative generating function as well, both expressions (the additive one and the multiplicative one) being equivalent [5]. It is then expressed as:

$$\forall (x, y) \in [0, 1]^2, \ t(x, y) = h^{-1}[h(x)h(y)], \tag{2.53}$$

where h is a strictly increasing function from [0, 1] into [0, 1] such that $h(0) = 0$ and $h(1) = 1$. The equivalence with the additive form is simply obtained by setting:

$$h = e^{-f}, \tag{2.54}$$

where f is the additive generating function introduced previously.

The most used t-norms and t-conorms of this class are the product and the algebraic sum:

$$\forall (x, y) \in [0, 1]^2,\ t(x, y) = xy, \quad T(x, y) = x + y - xy. \tag{2.55}$$

The only rational t-norms of this class are the Hamacher's t-norms, defined as [18]:

$$\forall (x, y) \in [0, 1]^2,\ \frac{xy}{\gamma + (1 - \gamma)(x + y - xy)}, \tag{2.56}$$

where γ is a positive parameter (for $\gamma = 1$ the operator coincides with the t-norm product).

Another parametric family of this class is made of Frank's functions, defined as [11]:

$$\forall (x, y) \in [0, 1]^2,\ t(x, y) = \log_s[1 + \frac{(s^x - 1)(s^y - 1)}{s - 1}], \tag{2.57}$$

where s is a strictly positive parameter. These t-norms and their dual t-conorms satisfy the following remarkable equality (and they are the only t-norms and t-conorms satisfying this relation):

$$\forall (x, y) \in [0, 1]^2,\ t(x, y) + T(x, y) = x + y. \tag{2.58}$$

If s is small and tends towards 0, the t-norm tends towards the minimum. If s tends towards $+\infty$, the t-norm tends towards the Lukasiewicz t-norm. If $s = 1$, the t-norm is equal to the product.

Another useful family of t-norms and t-conorms is constituted by the nilpotent ones, taking the following general form:

$$\forall (x, y) \in [0, 1]^2,\ t(x, y) = f^*[f(x) + f(y)], \tag{2.59}$$

where f is a decreasing bijection from [0,1] into [0,1], such that $f(0) = 1$, $f(1) = 0$, and $f^*(x) = f^{-1}(x)$ if $x \in [0, 1]$. $f^*(x) = 0$ if $x \geq 1$. The general form of nilpotent t-conorms can be derived by duality. They satisfy the excluded-middle and non-contradiction laws.

The most used t-norm and t-conorm of this class are the Lukasiewicz operators:

$$\forall (x, y) \in [0, 1]^2, \ t(x, y) = \max(0, x + y - 1), \quad T(x, y) = \min(1, x + y). \tag{2.60}$$

Examples of generating f functions have been proposed, e.g., by Schweizer and Sklar [29] or by Yager [31].

Finally, combinations of t-norms and t-conorms are also useful. For instance, compensation operators have been introduced in [39] and take the form:

$$\forall (x, y) \in [0, 1]^2, \ C_\gamma(x, y) = t(x, y)^{1-\gamma} T(x, y)^\gamma, \tag{2.61}$$

where γ is a parameter in [0, 1].

2.3.3 Mean Operators

A mean operator is defined as a function m from $[0, 1] \times [0, 1]$ into $[0, 1]$ such that:

1. $m \neq \min$, $m \neq \max$.
2. The result of the combination is a compromise between the smallest and the largest value: $\forall (x, y) \in [0, 1]^2, \ \min(x, y) \leq m(x, y) \leq \max(x, y)$.
3. m is commutative: $\forall (x, y) \in [0, 1]^2, \ m(x, y) = m(y, x)$.
4. m is increasing with respect to both arguments:

$$\forall (x, x', y, y') \in [0, 1]^4, \ (x \leq x' \text{ and } y \leq y') \Rightarrow m(x, y) \leq m(x', y').$$

A consequence of this definition (and more precisely of the second property) is that any mean operator m is idempotent, i.e.:

$$\forall x \in [0, 1], \ m(x, x) = x.$$

Note that associativity is generally not satisfied by mean operators. The only associative mean operators are the median operators defined as:

$$\forall (x, y) \in [0, 1]^2, \ m(x, y) = med(x, y, \alpha) = \begin{cases} x \text{ if } y \leq x \leq \alpha \text{ or } \alpha \leq x \leq y \\ y \text{ if } x \leq y \leq \alpha \text{ or } \alpha \leq y \leq x \\ \alpha \text{ if } y \leq \alpha \leq x \text{ or } x \leq \alpha \leq y \end{cases} \tag{2.62}$$

where α is a parameter in [0, 1].

Several mean operators satisfy an additional property, called bisymmetry, that can be considered as a counter-part of associativity:

$$\forall (x, y, z, t) \in [0, 1]^4, \ m[m(x, y), m(z, t)] = m[m(x, z), m(y, t)]. \tag{2.63}$$

Table 2.1 Examples of bisymmetric, continuous, and strictly increasing mean operators. For the harmonic mean, we use the convention $m(0, 0) = 0$

α	$m(x, y)$	Comment
$-\infty$	$\min(x, y)$	Limit value
-1	$\frac{2xy}{x+y}$	Harmonic mean
0	$(xy)^{-1/2}$	Geometrical mean
$+1$	$\frac{x+y}{2}$	Arithmetical mean
$+2$	$\sqrt{\frac{x^2+y^2}{2}}$	Quadratic mean
$+\infty$	$\max(x, y)$	Limit value

The general form of bisymmetric, continuous, and strictly increasing mean operators is as follows:

$$\forall (x, y) \in [0, 1]^2, \; m(x, y) = k^{-1}[\frac{k(x) + k(y)}{2}], \tag{2.64}$$

where k is a continuous and strictly monotonous function from [0, 1] into [0, 1].

Classical mean operators are found in this class, for k defined as:

$$\forall x \in [0, 1], k(x) = x^{\alpha},$$

where $\alpha \in \mathbb{R}$. In particular, harmonic, geometrical, arithmetical, and quadratic means are obtained for α equal to -1, 0, 1, and 2, respectively. Table 2.1 summarizes these results.

Other mean operators involve weights. This amounts to take the values to be combined into account at different levels. One particularly interesting weighted operator has been proposed by Yager [32], as an ordered weighted average operator (OWA). Weights are defined according to the ranking of the values to be combined. If these values are denoted by $a_1, a_2, \ldots a_n$, they are ordered in a sequence $a_{j_1}, a_{j_2}, \ldots a_{j_n}$ such that:

$$a_{j_1} \leq a_{j_2} \leq \ldots \leq a_{j_n}.$$

Then, for a set of weights w_i such that:

$$\sum_{i=1}^{n} w_i = 1, \quad \forall i, 1 \leq i \leq n, w_i \in [0, 1],$$

the OWA operator is defined by the expression:

$$OWA(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} w_i a_{j_i}. \tag{2.65}$$

Fuzzy integrals (not detailed in this book) also belong to the class of mean operators [16]. Indeed, both Choquet's and Sugeno's integrals are idempotent, continuous, increasing and take values between the minimum and the maximum. They include as particular cases order statistics, and therefore minimum, maximum, and median. Choquet's integral defined with respect to an additive measure μ is equivalent to a weighted arithmetical mean, where the weight w_i of the value x_i is equal to $\mu(\{x_i\})$.

The OWA operators can also be seen as a particular class of Choquet's fuzzy integrals, where the fuzzy measure is defined as:

$$\forall A, |A| = i, \quad \mu(A) = \sum_{j=0}^{i-1} w_{n-j}.$$

Conversely, any commutative Choquet's integral is such that $\mu(A)$ only depends on $|A|$ and is equal to an OWA operator, the weights of which are:

$$w_1 = 1 - \sum_{i=2}^{n} w_i,$$

$$\forall i \geq 2, \; w_i = \mu(A_{n-i+1}) - \mu(A_{n-i}),$$

where A_i denotes any subset such that $|A_i| = i$.

These properties among other have been extensively studied in [16, 17].

2.3.4 Symmetric Sums

A symmetric sum is defined as an operator σ from $[0, 1] \times [0, 1]$ into $[0, 1]$ such that:

1. $\sigma(0, 0) = 0$.
2. σ is commutative.
3. σ is increasing with respect to both arguments.
4. σ is continuous.
5. $\forall (x, y) \in [0, 1]^2, 1 - \sigma(x, y) = \sigma(1 - x, 1 - y)$.

The last property means that the scale of values can be reversed without changing the conclusions that may be drawn from their combination. This has been used, for instance, for expert opinion pooling, where saying that 0 means "bad" and 1 means "good" or saying the reverse should not be important [8]. This property can also be expressed using other fuzzy complementations.

Other properties hold that can be derived from the definition:

- $\sigma(1, 1) = 1$.
- $\forall x \in [0, 1], \sigma(x, 1 - x) = \frac{1}{2}$.

Table 2.2 Examples of symmetric sums, defined using t-norms and t-conorms as generating functions

$g(x, y)$	$\sigma(x, y)$	Property
xy	$\sigma_0(x, y) = \frac{xy}{1-x-y+2xy}$	Associative
$x + y - xy$	$\sigma_+(x, y) = \frac{x+y-xy}{1+x+y-2xy}$	Not associative
$\min(x, y)$	$\sigma_{\min}(x, y) = \frac{\min(x,y)}{1-\lvert x-y\rvert}$	Mean operator
$\max(x, y)$	$\sigma_{\max}(x, y) = \frac{\max(x,y)}{1+\lvert x-y\rvert}$	Mean operator

- the only symmetric sum that is both associative and a mean operator is the median with parameter $\alpha = \frac{1}{2}$.

The general form of symmetric sums depends on an increasing, positive, continuous function g such that $g(0, 0) = 0$:

$$\forall (x, y) \in [0, 1]^2, \sigma(x, y) = \frac{g(x, y)}{g(x, y) + g(1 - x, 1 - y)}. \tag{2.66}$$

It is easy to verify that this form satisfies all properties of a symmetrical sum (if $g(0, 1) = 0$, we set $\sigma(0, 1) = \frac{1}{2}$).

The general form of associative, strictly increasing symmetric sums is:

$$\forall (x, y) \in [0, 1]^2, \sigma(x, y) = \psi^{-1}[\psi(x) + \psi(y)], \tag{2.67}$$

where ψ is a strictly monotonous function such that $\psi(0)$ and $\psi(1)$ are non-bounded and $\forall x \in [0, 1]$, $\psi(1 - x) + \psi(x) = 0$. It follows that the values 0 and 1 are null elements, while $\frac{1}{2}$ is the unit element.

Typical examples of symmetric sums are given in Table 2.2. They are obtained by taking t-norms or t-conorms for the generating function g.

These operators satisfy the following order property:

$$\forall (x, y) \in [0, 1]^2, x + y \leq 1 \Rightarrow \sigma_0(x, y) \leq \sigma_{\min}(x, y) \leq \sigma_{\max}(x, y) \leq \sigma_+(x, y), \tag{2.68}$$

$$\forall (x, y) \in [0, 1]^2, x + y \geq 1 \Rightarrow \sigma_0(x, y) \geq \sigma_{\min}(x, y) \geq \sigma_{\max}(x, y) \geq \sigma_+(x, y). \tag{2.69}$$

2.3.5 Adaptive Operators

Many other operators have been proposed in the literature. Among them, operators that are adaptive with respect to some contextual information are of particular interest. This contextual information can be, for instance, a conflict measure between pieces of information coming from different sources or a reliability attached to each of the pieces of information to be combined. Several examples are given in [10] in the framework of possibility theory. We will come back to this question in Chap. 5.

2.3.6 *Logical Connectives*

In a logical setting, the same operators are used. The logical conjunction is represented as a t-norm in fuzzy logic, the disjunction as a t-conorm, the negation as a complementation, and the implication as a fuzzy implication I that can be defined either from a t-conorm and a complementation ($I(x, y) = T(c(x), y)$) or from a t-norm by residuation ($I(x, y) = \sup\{z \in [0, 1] \mid t(x, z) \leq y\}$). See also Chap. 8.

2.4 Linguistic Variable

In many situations, numerical representations are not sufficient to describe one situation. For instance, if a variable has a large range of variation, we can hardly attach one value to any specific situation and we may prefer to use a qualifier, usually from the natural language, to group coarsely some typical subsets of interest. For instance, to describe the size of an object, it may be more convenient to use only a few terms having rough boundaries, such as large, medium, or small. This corresponds to a granulation of the information. According to [37], the concept of granule is the starting point in "computing with words," and it is defined as "a fuzzy set of points having the form of a clump of elements drawn together by similarity" (quoted from [37]). A word is then a label of a granule. When it comes to compute with these representations, specific tools are needed. Such representations are particularly useful in approximate reasoning.

Such representations are called linguistic variables. They are variables whose values are words or sentences, which in turn may be represented as fuzzy sets [35]. The advantage of such representations is that linguistic characterizations may be less specific than numerical ones (and therefore need less information) [35]. Moreover, as mentioned in Chap. 1, they are very useful to solve the semantic gap issue. This feature will be exploited, for example, to model spatial relations in Chap. 6, and to reason on them in Chaps. 8 and 9.

2.4.1 *Definition*

The concept of linguistic variable has been introduced by Zadeh [35] and is described by several authors, e.g., [7, 38]. Formally, a linguistic variable is defined as a quintuple $(v, T(v), \mathcal{U}, G, M)$, where v is the name of the variable, $T(v)$ the set of values of v (called "terms"), $\mathcal{U}$ is the universe of discourse on which the values of v are defined (as fuzzy variables), G is a syntactic rule for generating the name X of values of v, and M is a semantic rule, $M(X)$ being a fuzzy set of $\mathcal{U}$ representing the meaning of X.

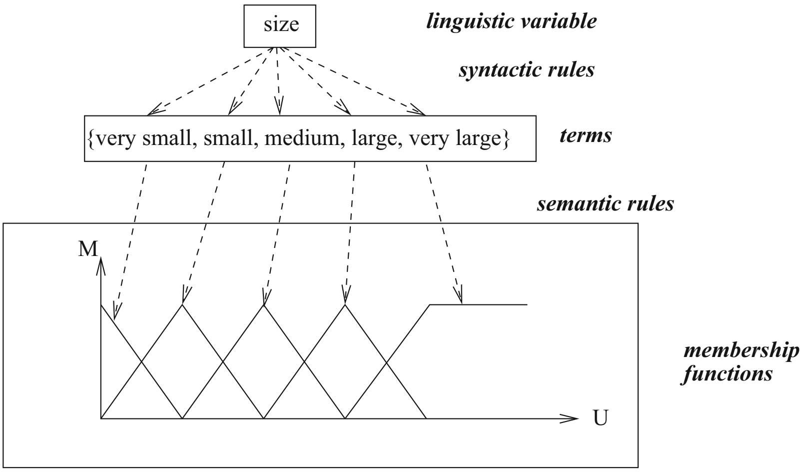

Fig. 2.3 Illustration of the linguistic variable “size,” its terms and the associated fuzzy sets. The arrows from the linguistic variable to the set of terms represent the syntactic rules, while the second set of arrows represents the semantic rules that translate terms into membership functions

Such a definition represents a symbolic-numerical conversion and establishes links between a language and a numerical scale, or more generally a concrete domain.

2.4.2 Example of Linguistic Variable

Let us consider the example of the size of an object. In numerical terms, this size can be expressed by a value ranging over a domain $\mathcal{U}$ (typically $\mathcal{U}$ is a subset of $\mathbb{R}^+$). In linguistic terms, the size can be expressed with a few terms: very small, small, medium, large, very large. Figure 2.3 illustrates the linguistic variable “size.”

2.4.3 Modifiers

The meaning of a term of a linguistic variable can be adapted by operators called “hedges” or “modifiers.” If A is a fuzzy set, then the hedge or modifier h generates a composite term $h(A)$ which is a fuzzy set on the same universe of discourse $\mathcal{U}$. The most common operators that are used for defining modifiers are the following:

- Normalization:

$$\mu_{norm(A)}(x) = \frac{\mu_A(x)}{\sup_{y \in \mathcal{U}} \mu_A(y)},$$

  where μ_A denotes the membership function of A and x any value in $\mathcal{U}$.
- Concentration: $\mu_{con(A)}(x) = (\mu_A(x))^2$.
- Dilation:[5] $\mu_{dil(A)}(x) = (\mu_A(x))^{0.5}$.
- Contrast intensification:

$$\mu_{int(A)}(x) = \begin{cases} 2(\mu_A(x))^2 & \text{if } \mu_A(x) \in [0, 0.5] \\ 1 - 2(1 - \mu_A(x))^2 & \text{otherwise.} \end{cases}$$

These functions are illustrated in Fig. 2.4, for a simple triangular-shaped membership function.

Typical hedges derived from these operators are [7]:

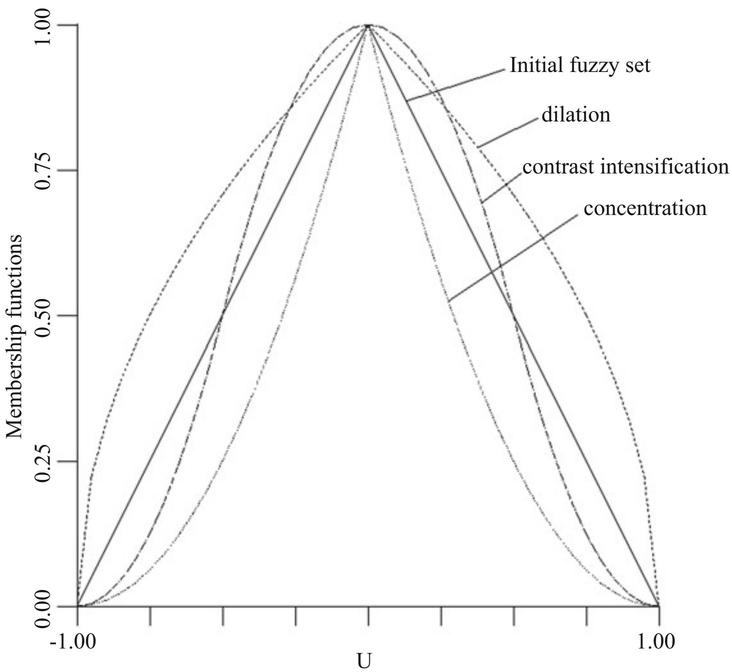

Fig. 2.4 Illustration of some operations used for defining hedges, on a triangular-shaped initial fuzzy set. Dotted: concentration, dashed: dilation, dot-dashed: contrast intensification

[5] Note that it is not a dilation in the morphological sense.

- $very\ A = con(A)$,
- $more\ A\ or\ less\ A = dil(A)$,
- $plus\ A = A^{1.25}$,
- $slightly\ A = int(norm(plus\ A\ and\ not(very\ A)))$ where "and" and "not" are defined as a t-norm and a complementation, respectively.

Let us mention, as a funny application of modifiers, the explanation of the Golden Ratio using fuzzy logic, as proposed in [19]. The author claims that the concept of something being pleasing often depends on an optimum of some variable. For instance, using a little bit of perfume may increase someone's attractiveness. But a person using too much perfume creates the opposite effect. In this example, and in several others, it appears that the degree x to which something is pleasing follows the following rule: if it increase too much, the opposite effect is obtained. Therefore we have, in linguistic terms:

$$very(x) = not(x).$$

Since $very(x)$ is expressed as x^2 and $not(x)$ as $1 - x$, x happens to be the solution of:

$$x^2 = 1 - x,$$

the solution of which is the golden ratio, i.e., $\frac{\sqrt{5}-1}{2}$.

More on modifiers and quantifiers will be detailed in Sect. 8.2.2 since they are much involved in linguistic descriptions (e.g., of image content).

2.5 Translating a Crisp Operation into a Fuzzy Operation

In this section, we present some common and generic methods that can be used for defining fuzzy operators or relationship from their equivalent binary ones. These methods can be categorized into three main classes. The first method relies on the "extension principle," as introduced by Zadeh [35]. The second method relies on computation on α-cuts. These two classes of definitions explicitly involve the operations or relations on crisp sets. The third class of methods consists in providing directly fuzzy definitions of the operations or of the relationships, by substituting all crisp expressions by their fuzzy equivalents.

2.5.1 Extension Principle

Definition

The first generic method for extending binary operators to fuzzy operators is due to Zadeh [35] and known as the extension principle.

Let us first consider a function f from $\mathcal{U}$ to $\mathcal{V}$. Let μ be a fuzzy set defined on $\mathcal{U}$. The extension of f to a fuzzy set is a fuzzy set μ' defined on $\mathcal{V}$. It is constructed as follows:

$$\forall y \in \mathcal{V},\ \ \mu'(y) = \begin{cases} 0 & \text{if } f^{-1}(y) = \emptyset, \\ \sup_{x \in \mathcal{U} | y = f(x)} \mu(x) & \text{otherwise.} \end{cases} \tag{2.70}$$

For an injective function, this equation reduces to:

$$\forall y \in \mathcal{V},\ \ \mu'(y) = \begin{cases} 0 & \text{if } f^{-1}(y) = \emptyset, \\ \mu[f^{-1}(y)] & \text{otherwise.} \end{cases} \tag{2.71}$$

Let us now consider the more general case where f is defined on a product space $\mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_n$. Let $\mu_1, \ldots \mu_n$ be n fuzzy sets defined, respectively, on $\mathcal{U}_1, \ldots \mathcal{U}_n$. The extension of f on the μ_i provides a fuzzy set of $\mathcal{V}$ defined as: $\forall y \in \mathcal{V}$,

$$\mu'(y) = \begin{cases} 0 & \text{if } f^{-1}(y) = \emptyset, \\ \sup_{(x_1, \ldots x_n) \in \mathcal{U}_1 \times \ldots \times \mathcal{U}_n | y = f(x_1, \ldots x_n)} \min[\mu_1(x_1), \ldots \mu_n(x_n)] & \text{otherwise.} \end{cases} \tag{2.72}$$

The min operator is used in this expression for the Cartesian product of fuzzy sets.

If the supremum is reached, i.e., if:

$$\forall y \in \mathcal{V}, \exists (x_1, \ldots x_n) \in \mathcal{U}_1 \times \ldots \times \mathcal{U}_n \mid \mu'(y) = \min[\mu_1(x_1), \ldots \mu_n(x_n)], \tag{2.73}$$

then the fuzzy extension of a function commutes with the α-cuts, i.e.:

$$\forall \alpha \in [0, 1], [f(\mu_1, \ldots \mu_n)]_\alpha = f[(\mu_1)_\alpha, \ldots (\mu_n)_\alpha]. \tag{2.74}$$

Other extensions principles can be defined, using, for instance, the product instead of the minimum [7].

The extension principle is illustrated in Fig. 2.5 for an injective function defined on a 1D space.

Application to the Compatibility of Two Fuzzy Sets

A typical example of application of the extension principle is the compatibility between two fuzzy sets, as defined by Zadeh [36]. Let us consider a fuzzy set μ on $\mathcal{U}$. The value $\mu(x)$ may be interpreted as a degree of compatibility of x with the fuzzy set μ [7] (μ being, for instance, a fuzzy value, or a linguistic variable). The compatibility of a fuzzy set μ' of $\mathcal{U}$ with μ can be evaluated using the extension principle as a fuzzy set μ_{comp} on [0, 1]:

$$\forall t \in [0, 1],\ \ \mu_{comp}(t) = \begin{cases} 0 & \text{if } \mu^{-1}(t) = \emptyset, \\ \sup_{x \in \mathcal{U} | t = \mu(x)} \mu'(x) & \text{otherwise.} \end{cases} \tag{2.75}$$

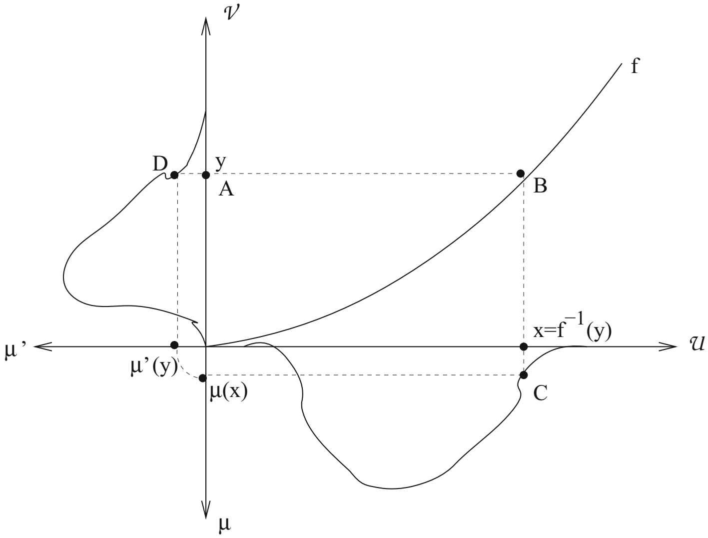

Fig. 2.5 Extension principle. Starting from any point A with y as coordinate, B is obtained on the graph of f. Point B provides $x = f^{-1}(y)$, the abscissa of the point C, on the graph of $\mu(x)$ at position x. With y as abscissa and $\mu(x)$ as coordinate, point D of the curve defining $\mu'(y)$ is obtained

The construction of the fuzzy compatibility set is illustrated in Fig. 2.6. Note that compatibility is not symmetric in μ and μ'.

Application to Fuzzy Numbers

One of the most important applications of the extension principle is for the operations on fuzzy numbers, such as addition, multiplication, etc. [7]. If μ_A and μ_B are two fuzzy numbers (convex, normalized fuzzy sets on $\mathbb{R}$), and $*$ any operation on numbers, the extension of $*$ is defined as:

$$\forall z \in \mathbb{R},\ \ \mu_{A*B}(z) = \sup_{(x,y)\in\mathbb{R}^2 \mid x*y=z} \min[\mu_A(x), \mu_B(y)]. \tag{2.76}$$

The extension of a commutative (respectively, associative) operation is commutative (respectively, associative).

The computation of such extended operations can be performed using simple algorithms especially in the case of L-R fuzzy numbers, as shown in [7]. For instance, let μ_1 and μ_2 be two fuzzy numbers defined as:

$$\forall x \in \mathbb{R},\ \ \mu_1(x) = \begin{cases} L(\frac{m_1 - x}{\alpha_1}) \text{ for } x \leq m_1, \\ R(\frac{x - m_1}{\beta_1}) \text{ for } x \geq m_1. \end{cases} \tag{2.77}$$

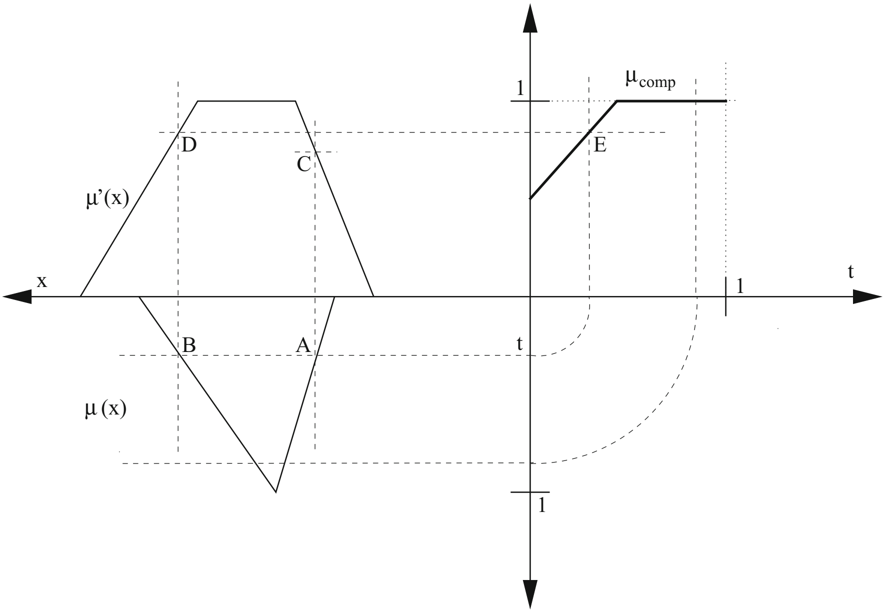

Fig. 2.6 Construction of the compatibility function between two fuzzy sets μ and μ'. Starting from any level t, we determine on the graph of μ (which has been drawn up-side down for the sake of interpretation), and the two points A and B so that $\mu(x) = t$. The two points C and D are obtained on the graph of $\mu'(x)$, and D is chosen such that $\mu(D) \geq \mu(C)$. The final point E with coordinates t and $\mu(D)$ is drawn, belonging to the graph of $\mu_{comp}(t)$

$$\forall x \in \mathbb{R},\ \ \mu_2(x) = \begin{cases} L(\frac{m_2 - x}{\alpha_2}) \text{ for } x \leq m_2, \\ R(\frac{x - m_2}{\beta_2}) \text{ for } x \geq m_2. \end{cases} \tag{2.78}$$

The sum of μ_1 and μ_2 is a L-R fuzzy number with parameters $m_1 + m_2$, $\alpha_1 + \alpha_2$ and $\beta_1 + \beta_2$.

Figure 2.7 illustrates the difference and the sum of two fuzzy numbers obtained by the extension principle.

2.5.2 *Combination of Results on α-Cuts*

Reconstruction from α-Cuts

One way to define crisp sets from a fuzzy set consists in taking the α-cuts of this set. Conversely, a fuzzy set can be reconstructed from its α-cuts, as seen in Sect. 2.2.4. Therefore a class of methods for defining fuzzy operations from crisp ones relies on

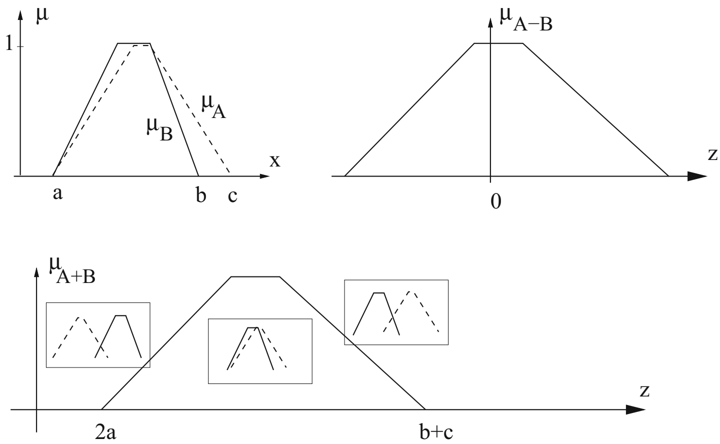

Fig. 2.7 Application of the extension principle to the computation of two fuzzy numbers A and B. In the upper right corner, the difference $A - B$; in the lower row, the addition $A + B$. The three boxes present three different configurations of $\mu_A(x)$ and $\mu_B(z - x)$ for low, average, and high values of z. Note that the shapes of the two resulting functions are the same, but translated by $Supp(A) + Supp(B)$

the application of the crisp operation on each α-cut and then combining the results to reconstruct a fuzzy operation by stacking up the α-cuts.

Let us denote by μ the membership functions of a fuzzy set defined on the space $\mathcal{U}$. Let us consider a crisp set function R_B (or operation on sets) taking values in a space $\mathcal{V}$ (e.g., $\mathbb{R}$). The fuzzy equivalent R of R_B is then defined as a function from $\mathcal{F}$ in $\mathcal{V}$ (see, e.g., [3, 6, 20]):

$$R(\mu) = \int_0^1 R_B(\mu_\alpha) d\alpha. \tag{2.79}$$

Other fuzzification equations are possible, such as in [3, 12]:

$$R(\mu) = \sup_{\alpha \in [0,1]} \min(\alpha, R_B(\mu_\alpha)), \tag{2.80}$$

if the relation takes values in [0, 1], or:

$$R(\mu) = \sup_{\alpha \in [0,1]} (\alpha R_B(\mu_\alpha)). \tag{2.81}$$

These equations may provide different results. But there are also some links between them as shown later in this section.

Let us now consider a crisp operation R_B having two arguments (typically a relation between sets). The fuzzy equivalent R of R_B, applied to two fuzzy sets μ and ν of $\mathcal{U}$, is defined as a generalization of the previous equations:

$$R(\mu, \nu) = \int_0^1 R_B(\mu_\alpha, \nu_\alpha) d\alpha, \tag{2.82}$$

or, in this case, by a double integration as:

$$R(\mu, \nu) = \int_0^1 \int_0^1 R_B(\mu_\alpha, \nu_\beta) d\alpha d\beta. \tag{2.83}$$

The other fuzzification equations (2.80 and 2.81) can also be directly extended to operations on more than one fuzzy set.

The extension to n-ary operators is straightforward.

Extension Principle Based on α-Cuts

Another method for combining the results on α-cuts is similar to the extension principle [35]. In general it leads to a fuzzy set on $\mathcal{V}$. For instance, if $\mathcal{V} = \mathbb{R}$, the crisp relation provides real values, the corresponding fuzzy relation using previous equations also provides numbers, while the following one provides fuzzy numbers. For a binary relation, we have:

$$\forall n \in \mathcal{V}, R(\mu, \nu)(n) = \sup_{R_B(\mu_\alpha, \nu_\alpha) = n} \alpha. \tag{2.84}$$

We have similar equations for unary or n-ary relations.

If the relationship to be extended only takes binary values (0/1, or true/false), then this equation reduces to:

$$R(\mu, \nu) = \sup_{R_B(\mu_\alpha, \nu_\alpha) = 1} \alpha. \tag{2.85}$$

The extension principle based on the α-cuts can be interpreted as follows: taking an α-cut of the sets corresponds to taking a binary decision on the boundaries of the objects. Then we look at the obtained value of the crisp relationship for this decision. If the same value of the relation can be obtained from different values of α, then the highest value of α is retained.

2.5.3 Translating Binary Terms into Functional Ones

A last class of methods consists in translating binary equations into their fuzzy equivalent. This approach completely differs from the two previous ones in the sense that it does not use explicitly the crisp relation or operation. Indeed, in the extension principle as well as in α-cuts based approaches, the definition of a fuzzy operation

Table 2.3 Translation of crisp concepts in their fuzzy equivalents

Crisp concept	Equivalent fuzzy concept
Set X	Fuzzy set/membership function μ
Complement of a set	Fuzzy complementation c
Intersection $\cap$	t-norm t
Union $\cup$	t-conorm T
Existence $\exists$	Supremum
Universal symbol $\forall$	Infimum

is a function of the corresponding crisp operation. Here, a fuzzy operation is given directly by an equation involving fuzzy terms that just mimics crisp equation.

This translation is generally done term by term. For instance, intersection is replaced by a t-norm, union by a t-conorm, sets by fuzzy set membership functions, etc. This translation is particularly straightforward if the binary relationship can be expressed in set theoretical and logical terms. Table 2.3 summarizes these main crisp concepts involved in set equations, and their fuzzy equivalent.

The many possibilities to translate, for instance, set union using a t-conorm means that many definitions can be obtained from this method, depending on the choice of the fuzzy operators used for translating the crisp corresponding ones.

Let us take a simple example to illustrate this method. According to Zadeh's original definition, a fuzzy set μ is said to be included in another fuzzy set ν if:

$$\forall x \in \mathcal{U},\ \ \mu(x) \leq \nu(x).$$

This is a crisp definition of inclusion of fuzzy sets. We may also suggest that if two sets are imprecisely defined, their inclusion relationship may be imprecise too. Therefore inclusion of fuzzy sets becomes a matter of degree. This degree of inclusion can be obtained using the translation principle.

In the crisp case, the set equation expressing inclusion of a set X in a set Y can be written as follows:

$$X \subseteq Y \Leftrightarrow X^C \cup Y = \mathcal{U} \tag{2.86}$$

$$\Leftrightarrow \forall x \in \mathcal{U},\ \ x \in X^C \cup Y, \tag{2.87}$$

where X^C denotes the set complement of X in $\mathcal{U}$. Using the equivalence of Table 2.3 for each term, we have:

$$\forall x \in \mathcal{U} \leftrightarrow \inf_{x \in \mathcal{U}}, \tag{2.88}$$

$$x \in X^C \leftrightarrow c[\mu(x)], \tag{2.89}$$

$$x \in Y \leftrightarrow \nu(x), \tag{2.90}$$

$$X^C \cup Y \leftrightarrow T[c(\mu), \nu]. \tag{2.91}$$

Finally, the degree of inclusion of μ in ν is defined as:

$$\Delta_{\subseteq}(\mu, \nu) = \mu_{Inc}(\mu, \nu) = \inf_{x \in \mathcal{U}} T[c(\mu(x)), \nu(x)], \tag{2.92}$$

where T is a t-conorm and c a fuzzy complementation. This corresponds also to the infimum of a fuzzy implications. The implication (here built from a t-conorm) can also be the residuated implication of a t-norm (see Sect. 2.3.6), and more generally we can write $\mu_{Inc}(\mu, \nu) = \inf_{x \in \mathcal{U}} I[\mu(x), \nu(x)]$. Further developments on fuzzy inclusion can be found in the next chapter.

A similar approach can be adopted for the degree of intersection between two fuzzy sets, see next chapter.

Such translations will be used extensively in Chaps. 3, 4 and 6.

2.5.4 Comparison

The extension principle has been originally defined for functions. The approaches presented in Sect. 2.5.2 deal mainly with operators (set operators, relationships between sets, etc.). Links between extension principle and combination of α-cuts using Eq. 2.80 have been established in [12]. Let f be a function from $\mathcal{U}_1 \times \ldots \times \mathcal{U}_n$ to $\mathcal{V}$, and R_f a set operator defined as:

$$R_f(X_1, X_2, \ldots X_n) = \{f(x_1, x_2, \ldots x_n) \mid x_1 \in X_1, \ldots x_n \in X_n\}, \tag{2.93}$$

where $X_1, \ldots X_n$ are subsets of $\mathcal{U}_1, \ldots \mathcal{U}_n$. Then the extension of R_f using Eq. 2.80 coincides with Zadeh's extension of f (Eq. 2.72).

Other links exist between definitions of Sect. 2.5.2. For instance, if R_B is a crisp relation taking values in $\{0, 1\}$, its extension using Eq. 2.85 is a value in $[0, 1]$ and is equivalent to the two fuzzification procedures given by Eqs. 2.80 and 2.81.

Let us take the example of extending union between sets, using the methods of Sect. 2.5.2. Let μ and ν be two fuzzy sets on $\mathcal{U}$. Using the integration over α-cuts, we get:

$$\begin{aligned}(\mu \cup \nu)(x) &= \int_0^1 (\mu_\alpha \cup \nu_\alpha)(x) d\alpha \\ &= \int_0^{\max[\mu(x), \nu(x)]} 1 d\alpha \\ &= \max[\mu(x), \nu(x)],\end{aligned}$$

since $(\mu_\alpha \cup \nu_\alpha)(x) = 1$ iff $\mu(x) \geq \alpha$ or $\nu(x) \geq \alpha$. This extension leads exactly to the fuzzy union as defined originally in [34]. Exactly the same result is obtained using other fuzzification methods, e.g., with Eq. 2.80 or Eq. 2.85.

Using formal translation of equations as described in Sect. 2.5.3, we may obtain the same results as using some combination of α-cuts. Examples will be seen later, e.g., for fuzzy connectivity.

The question of which extension should be used does not have a definite answer until now. The extension principle is well adapted for translating analytical expressions, while formal translation is well adapted if the operators to be extended can be expressed using set theoretical and logical terms. The properties of the obtained extended operators have to play an important role in the choice of a method, since they may vary depending on the method. For instance, as will be seen later, using simple integration on the α-cuts (Eq. 2.82) for extending a distance between sets to a distance between fuzzy sets endows the fuzzy distance with the same properties as the crisp distance, while using a double integration (Eq. 2.83), some properties may be lost.

2.6 Summary of the Main Notations

The main notations used in this book are summarized in Table 2.4.

Table 2.4 Main notations used in this book

Universe of discourse	$\mathcal{U}$
Spatial domain	$\mathcal{S}$
Ground distance	$d_{\mathcal{S}}$
Feature space (grey levels…)	E
Point of space	$x, y \ldots$
Fuzzy set and membership function	$\mu, \nu \ldots$
Set of fuzzy sets	$\mathcal{F}$
α-cut	μ_α
Support	$Supp(\mu)$
Core	$Core(\mu)$
Connectives and aggregation operators:	
Conjunction	t (t-norm) or $\wedge$
Disjunction	T (t-conorm) or $\vee$
Complementation/negation	c or $\bar{\mu}$
Implication	I or $\rightarrow$
Degree of intersection	$\Delta_\cap(\mu, \nu)$, $\mu_{int}(\mu, \nu)$
Degree of inclusion	$\Delta_\subseteq(\mu, \nu)$, $\mu_{Inc}(\mu, \nu)$
Cardinality	$\|\mu\|$ or μ_{card}
Similarity measure	$s(\mu, \nu)$ or $sim(\mu, \nu)$
Quantifiers	$\mu_{most} \ldots$
Morphological operators	δ (dilation), ε (erosion)…

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Chapter 3
Fuzzy Spatial Objects

In this chapter, we consider the spatial interpretation of fuzzy sets. We define first spatial fuzzy objects, i.e., fuzzy sets in the spatial domain (Sect. 3.1). Throughout this chapter, the underlying space $\mathcal{S}$ is the spatial domain, typically $\mathbb{Z}^2$ or $\mathbb{Z}^3$ for digital 2D or 3D images, or, in the continuous case, $\mathbb{R}^2$ or $\mathbb{R}^3$. Basic operations on these objects are set theoretical (algebraic) operations. Some of these have been defined in Chap. 2, where the adopted point of view was to construct a fuzzy set which is the result of the combination of two fuzzy sets by a set operation, using connectives such as conjunction, disjunction, etc. Now, we take another point of view, which addresses the question: are two fuzzy sets satisfying some set relationships? For instance: is μ included in ν? In the crisp case, such questions receive binary answers. When the objects are fuzzy, imprecisely defined, the answer to such questions becomes a matter of degree, and amounts to define a degree to which the relation is satisfied. To achieve such a result, degrees of inclusion, intersection, union, and equality are defined. This question is addressed in Sect. 3.2. Topological notions such as boundaries and connectedness are introduced in Sect. 3.3. Geometry is then addressed in Sect. 3.4. The notations used in this chapter are those of Table 2.4.

Generally speaking, there are three interpretations of fuzziness, in terms of plausibilities, similarities, preferences, as shown in [9, 10]. The same interpretations apply in applications of fuzzy sets for image analysis and understanding problems. The interpretation in terms of plausibilities is, for instance, used for the membership to a class, in the definition of a fuzzy spatial object (an object with imprecise limits). The interpretation in terms of similarities is that used for the definition of a fuzzy class in a characteristic space as a function of the distance to a prototype, for example, of linguistic variables representing information or knowledge about spatial objects, or also of degrees of satisfaction of a relation, a constraint. Finally, preferences are used in the expression of choice criteria (for example, for planning applications in robotics), which are often related to constraints or knowledge outside the image.

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3.1 Fuzzy Sets in the Spatial Domain

We consider objects or structures in images that we may describe as fuzzy sets. Thus we often call them fuzzy image objects. A fuzzy image object is a fuzzy set defined on $\mathcal{S}$, i.e., a spatial fuzzy set. Its membership function μ represents the imprecision in the spatial extent of the object. For any point x of $\mathcal{S}$ (pixel or voxel), $\mu(x)$ is the degree to which x belongs to the fuzzy object.

As an illustrative example, a slice of a brain image is shown in Fig. 3.1. It is obtained using a T1 weighted acquisition in magnetic resonance imaging (MRI). A few internal structures are represented in Fig. 3.2 as spatial fuzzy sets, where membership degrees are represented using gray levels. The use of fuzzy sets may represent different types of imprecision, either on the boundary of the objects (due, for instance, to partial volume effect, or to the spatial resolution) or on the individual variability of these structures, etc.

Fig. 3.1 MR image of a brain (one axial slice of a 3D volume)

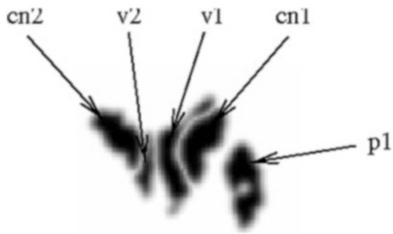

Fig. 3.2 Fuzzy objects representing internal brain structures of the image shown in Fig.3.1 (membership values rank between 0 and 1, from white to black). From left to right: right caudate nucleus (cn2), right lateral ventricle (v2), left lateral ventricle (v1), left caudate nucleus (cn1), left putamen (p1). The usual convention "left is right" in medical imaging is adopted here

3.2 Set Theoretical Operations

In this section we consider set theoretical (or algebraic) relations between fuzzy objects. For each type of relation, we start from the crisp setting and extend it to the fuzzy case, while keeping the consistency with the crisp case. Note that these operations can also be considered as spatial relations, as further developed in Chap. 6.

3.2.1 *Degree of Intersection*

Crisp Case

Saying that two sets X and Y intersect means:

$$X \cap Y \neq \emptyset, \tag{3.1}$$

or equivalently:

$$\exists x \in \mathcal{S}, \; x \in X \cap Y. \tag{3.2}$$

On the other hand, the fact that X and Y do not intersect is expressed by the non-satisfaction of this equation. These two possible states in the crisp case correspond to a binary "degree" of intersection, which is equal to 1 if the equation is satisfied and to 0 if it is not (considering the algebraic notion of intersection, not the size of the intersection).

Direct Extension

In the fuzzy domain, this binary degree becomes a degree in [0, 1], denoted by $\mu_{int}(\mu, \nu)$ or $\Delta_\cap(\mu, \nu)$, expressing the degree to which two fuzzy sets μ and ν intersect, or the degree of satisfaction of the intersection relation. The simplest fuzzy extension (see Table 2.3) is provided by:

$$\mu_{int}(\mu, \nu) = \sup_{x \in \mathcal{S}} t[\mu(x), \nu(x)], \tag{3.3}$$

where t is a conjunction, often taken as a t-norm. This expression may vary from 0, which corresponds to no intersection at all (typically if μ and ν have disjoint supports) to 1 if at least one point x belongs completely to both μ and ν.

Note that in this case, the other fuzzification methods (combination of α-cuts, extension principle, as seen in Chap. 2) provide the same result, or a particular case for $t = \min$. For instance, the integration over the α-cuts produces:

$$\mu_{int}(\mu, \nu) = \int_0^1 1_{\mu_\alpha \cap \nu_\alpha \neq \emptyset} d\alpha, \tag{3.4}$$

where $1_{\mu_\alpha \cap \nu_\alpha \neq \emptyset}$ is the binary function expressing intersection, equals 1 if $\mu_\alpha \cap \nu_\alpha \neq \emptyset$ and 0 otherwise. Since

$$\forall (\alpha, \alpha') \in [0, 1]^2, \; \alpha \leq \alpha' \Rightarrow \mu_{\alpha'} \subseteq \mu_\alpha \text{ and } \nu_{\alpha'} \subseteq \nu_\alpha, \tag{3.5}$$

it follows that:

$$\mu_{\alpha'} \cap \nu_{\alpha'} \neq \emptyset \Rightarrow \mu_\alpha \cap \nu_\alpha \neq \emptyset. \tag{3.6}$$

Therefore, it is sufficient to consider only the supremum of the values of α such that $\mu_\alpha \cap \nu_\alpha \neq \emptyset$, and the following equalities hold:

$$\mu_{int}(\mu, \nu) = \int_0^1 1_{\mu_\alpha \cap \nu_\alpha \neq \emptyset} d\alpha \tag{3.7}$$

$$= \sup_{\mu_\alpha \cap \nu_\alpha \neq \emptyset} \alpha \tag{3.8}$$

$$= \sup_\alpha \min[\alpha, 1_{\mu_\alpha \cap \nu_\alpha \neq \emptyset}] \tag{3.9}$$

$$= \sup_\alpha [\alpha 1_{\mu_\alpha \cap \nu_\alpha \neq \emptyset}] \tag{3.10}$$

$$= \sup_{x \in \mathcal{S}} \min[\mu(x), \nu(x)]. \tag{3.11}$$

This formulation is equivalent to Eq. 3.3 for $t = \min$.

From the degree of intersection, a degree of non-intersection (or of disjunction) is then derived as:

$$\mu_{\neg int}(\mu, \nu) = c[\mu_{int}(\mu, \nu)], \tag{3.12}$$

where c is a fuzzy complementation (for instance, defined as $\forall a \in [0, 1], c(a) = 1 - a$).

This form has already been widely used in the fuzzy set literature. In particular, it is often interpreted as a degree of conflict between two fuzzy sets or two possibility distributions [8].

Introducing the Volume of the Overlapping Domain

However, this form is not always adequate for image processing purposes since it does not include any spatial information. This may even lead to counter-intuitive results, as the expression $\sup_{x \in \mathcal{S}} t[\mu(x), \nu(x)]$ only represents the maximum height of the intersection. Although it is generally low for fuzzy sets that have almost

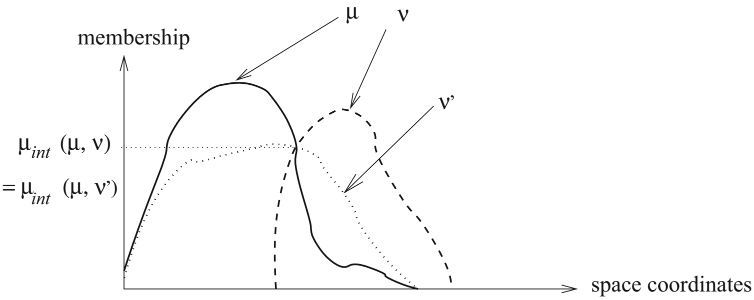

Fig. 3.3 Low discrimination power of the definition of degree of intersection between two fuzzy sets using the maximum of intersection: $\mu_{int}(\mu, \nu') = \mu_{int}(\mu, \nu)$, although μ and ν' strongly overlap and should be considered as more intersecting than μ and ν

disjoint supports, its value does not account for different overlapping situations, as illustrated in Fig. 3.3 (for sake of clarity, $\mathcal{S}$ is represented in 1D only).

The degree of intersection and of non-intersection can therefore be reformulated in order to better represent the notion of spatial overlapping. Another solution, which may be better for some applications, consists in defining a degree of intersection by considering the fuzzy hypervolume V_n (in a space of dimension n) of the intersection. This also corresponds to a translation process, in the sense that we have:

$$X \cap Y = \emptyset \;\Leftrightarrow\; V_n(X \cap Y) = 0. \tag{3.13}$$

The hypervolume of a fuzzy set is simply defined using the classical fuzzy cardinality. This provides for a fuzzy set μ (having bounded support) in the discrete case:

$$V_n(\mu) = \sum_{x \in \mathcal{S}} \mu(x), \tag{3.14}$$

and in the continuous case:

$$V_n(\mu) = \int_{x \in \mathcal{S}} \mu(x) dx. \tag{3.15}$$

From the hypervolume of $t(\mu, \nu)$, we can derive a degree of intersection in $[0, 1]$. It should be equal to 0 if μ and ν have completely disjoint supports, it should be large if one set is included in the other, and it should be increasing with respect to the hypervolume of the intersection. The following definition satisfies these requirements:[1]

[1] Other definitions leading to similar properties are possible.

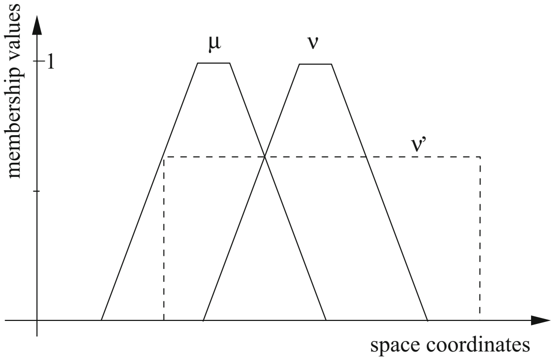

Fig. 3.4 μ and ν have the same degree of intersection as μ and ν' using the maximum of the intersection, while they have a lower one using the fuzzy hypervolume

$$\mu_{int}(\mu, \nu) = \frac{V_n[t(\mu, \nu)]}{\min[V_n(\mu), V_n(\nu)]}. \tag{3.16}$$

Again a degree of non-intersection can be derived from this expression using Eq. 3.12.

In the example shown in Fig. 3.4, μ and ν have the same degree of intersection according to Eq. 3.3 (maximum of the intersection, equal to 0.63 in this case) as μ and ν'. However, using the fuzzy volume and Eq. 3.16, we obtain $\mu_{int}(\mu, \nu) = 0.31$ and $\mu_{int}(\mu, \nu') = 0.66$. This corresponds well to the fact that μ and ν' have a larger overlap than μ and ν.

Properties

An important property of these various definitions is that the intersection degrees defined by Eqs. 3.3 and 3.16 are both consistent with the binary definition. Moreover, they satisfy the following properties:

- Symmetry: $\forall(\mu, \nu) \in \mathcal{F}^2,\ \mu_{int}(\mu, \nu) = \mu_{int}(\nu, \mu)$.
- Reflexivity (Eq. 3.3) if the fuzzy sets are normalized: $\exists x \in \mathcal{S}, \mu(x) = 1 \Rightarrow \mu_{int}(\mu, \mu) = 1$; for Eq. 3.16, reflexivity holds if $t = \min$.
- If one of the sets is empty ($\forall x \in \mathcal{S}, \nu(x) = 0$), then the degree of intersection is always 0.
- If one of the sets is equal to $\mathcal{S}$ ($\forall x \in \mathcal{S}, \nu(x) = 1$), the degree of intersection is always equal to 1 using Eq. 3.16 and to 1 for normalized fuzzy sets using Eq. 3.3.
- Invariance with respect to geometrical rigid transformations (translation, rotation).

Application to the Non-contradiction Principle

In the crisp case, we have for any set X:

$$X \cap X^C = \emptyset \tag{3.17}$$

which is a set equation equivalent to the logical principle of non-contradiction.

In the fuzzy case, the degree of intersection between any fuzzy set μ and its complement, based on Eq. 3.3, is:

$$\mu_{int}(\mu, c(\mu)) = \sup_{x\in\mathcal{S}} t[\mu(x), 1-\mu(x)], \tag{3.18}$$

if we take $c(a) = 1 - a$ for the fuzzy complementation.

In general this degree is non-zero. However, for the Lukasiewicz t-norm (see Chap. 2), we have:

$$\mu_{int}(\mu, c(\mu)) = \sup_{x\in\mathcal{S}} \max[0, \mu(x) + 1 - \mu(x) - 1] = 0, \tag{3.19}$$

and the non-contradiction principle is then satisfied.

3.2.2 Degree of Union and Covering

When considering the union, the problem is posed in different terms from what is done for intersection. Here, it is not interesting to look to which degree two sets have a non-empty union, for this property is true whenever at least one of the two fuzzy sets is not empty. Instead, what is useful is the degree to which the union of two sets covers the whole space. This will be useful, for instance, for looking at the law of excluded-middle.

In the crisp case, we have for two sets X and Y:

$$X \cup Y = \mathcal{S} \Leftrightarrow \forall x \in \mathcal{S},\ x \in X \cup Y. \tag{3.20}$$

In the fuzzy sets case, this equation is replaced (again using the formal translation principle) by:

$$\mu_{union}(\mu, \nu) = \inf_{x\in\mathcal{S}} T[\mu(x), \nu(x)], \tag{3.21}$$

where T is a disjunction (often a t-conorm), yielding a degree in [0, 1]. The same result (and in particular with $T = \max$) is obtained using the other fuzzification principles.

The properties of this degree of union are:

- Consistency with the binary definition.
- Symmetry: $\forall(\mu, \nu) \in \mathcal{F}^2,\ \mu_{union}(\mu, \nu) = \mu_{union}(\nu, \mu)$.
- If one of the sets is empty ($\forall x \in \mathcal{S}, \nu(x) = 0$), then μ_{union} is always 0 for bounded support fuzzy sets.
- If one of the sets is equal to $\mathcal{S}$ ($\forall x \in \mathcal{S}, \nu(x) = 1$), μ_{union} is always equal to 1.

- Invariance with respect to rigid geometrical transformations (translation, rotation).

Let us now look at the excluded-middle law. In the crisp case, we have for any set X:

$$X \cup X^C = \mathcal{S}, \tag{3.22}$$

which is a set equation equivalent to the logical principle of excluded-middle (in the sense, for instance, that the truth can be in X or in X^c, but there is no other possibility). In the fuzzy case, the degree of union between any fuzzy set μ and its complement is:

$$\mu_{union}(\mu, c(\mu)) = \inf_{x \in \mathcal{S}} T[\mu(x), 1 - \mu(x)], \tag{3.23}$$

if we take $c(a) = 1 - a$ for the fuzzy complementation. In general this degree is not equal to 1. However, for the Lukasiewicz t-conorm, we have:

$$\mu_{union}(\mu, c(\mu)) = \inf_{x \in} \min[1, \mu(x) + 1 - \mu(x)] = 1, \tag{3.24}$$

and the excluded-middle law is then satisfied.

3.2.3 Degree of Inclusion

In this section, we further develop the degree of inclusion introduced in Chap. 2.

Inclusion from Other Set Operations

In the crisp case, for two sets X and Y, the inclusion is defined as

$$X \subseteq Y \Leftrightarrow X \cap Y^C = \emptyset \tag{3.25}$$

$$\Leftrightarrow X^C \cup Y = \mathcal{S}. \tag{3.26}$$

These two equivalences show that inclusion is simply expressed either by an intersection or by a union. The extension to fuzzy sets can therefore directly use the degrees of intersection and union as defined previously.

As already explained in Chap. 2, using the union leads to:

$$\mu_{Inc}(\mu, \nu) = \mu_{union}(c(\mu), \nu) = \inf_{x \in \mathcal{S}} T[c(\mu(x)), \nu(x)], \tag{3.27}$$

where T is a t-conorm.

Using the degree of intersection yields the following degree of inclusion of μ in ν:

$$\mu_{Inc}(\mu, \nu) = c[\mu_{int}(\mu, c(\nu))] = c[\sup_{x \in \mathcal{S}} t[\mu(x), c(\nu(x))]], \tag{3.28}$$

where t is a t-norm and c a fuzzy complementation.

Actually, due to the duality between t-norms and t-conorms, these two definitions are equivalent. Let t and T be a pair of dual t-norm and t-conorm according to the complementation c. Then we have:

$$\inf_{x \in \mathcal{S}} T[c(\mu(x)), \nu(x)] = \inf_{x \in \mathcal{S}} c[t[\mu(x), c(\nu(x))]] = c[\sup_{x \in \mathcal{S}} t[\mu(x), c(\nu(x))]], \tag{3.29}$$

which proves the equivalence between both formulas.

This definition has the same drawback as those for the degrees of intersection and union, which may, basically, depend on one point only. Here again, the spatial coverage between both fuzzy sets could be taken into account.

The properties of the degree of inclusion are directly derived from those of intersection and union:

- Consistency with the binary definition.
- If μ is empty ($\forall x \in \mathcal{S}, \mu(x) = 0$), then $\mu_{Inc}(\mu, \nu)$ always equals 1.
- If ν is empty, then $\mu_{Inc}(\mu, \nu)$ is equal to 0 for normalized fuzzy sets.
- If μ is equal to $\mathcal{S}$ ($\forall x \in \mathcal{S}, \mu(x) = 1$), $\mu_{Inc}(\mu, \nu)$ is equal to 0 for bounded support fuzzy sets.
- If ν is equal to $\mathcal{S}$, $\mu_{Inc}(\mu, \nu)$ is always equal to 1.
- Invariance with respect to geometrical transformations (translation, rotation).

Inclusion from Fuzzy Implication

Finally, inclusion can be defined from implication [1, 29, 30], as:

$$\mu_{Inc}(\mu, \nu) = \inf_{x \in \mathcal{S}} \mathcal{I}(\mu(x), \nu(x)), \tag{3.30}$$

where $\mathcal{I}(\mu(x), \nu(x))$ denotes the degree to which $\mu(x)$ implies $\nu(x)$. Recall that a fuzzy implication $\mathcal{I}$ is a mapping from $[0, 1] \times [0, 1]$ into $[0, 1]$ which is decreasing in the first argument, increasing in the second one and satisfies $\mathcal{I}(0, 0) = \mathcal{I}(0, 1) = \mathcal{I}(1, 1) = 1$ and $\mathcal{I}(1, 0) = 0$.

A common way to define an implication $\mathcal{I}$ (called S-implication) consists in deriving it from a t-conorm T and a negation (or complementation c) [7]:

$$\forall(\alpha, \beta) \in [0, 1]^2, \mathcal{I}(\alpha, \beta) = T(c(\alpha), \beta)), \tag{3.31}$$

this equation translating directly the crisp logical equivalence between $(\varphi \Rightarrow \psi)$ and $(\psi \vee \neg\varphi)$, and we recover exactly the same definition of degree of inclusion as the one obtained from a t-conorm (Eq. 3.27).

Another interesting and usual approach to derive an implication $\mathcal{I}$ (called R-implication) is to apply the residuation principle, from a t-norm t:

$$\forall(\alpha, \beta) \in [0, 1]^2, \mathcal{I}(\alpha, \beta) = \sup\{\gamma \in [0, 1] \mid t(\alpha, \gamma) \leq \beta\}. \tag{3.32}$$

We then have the adjunction property between t and $\mathcal{I}$, expressed as $\forall(a, b, c) \in [0, 1]^3, t(a, b) \leq c \Leftrightarrow a \leq \mathcal{I}(b, c)$, which is very important when considering operations expressed using algebraic lattice properties. This definition coincides with the previous one for particular forms of T, typically the Lukasiewicz t-norm.

Other Axiomatic Definitions for the Fuzzy Inclusion

Other methods for defining a degree of inclusion rely on a set of axioms and on the determination of functions satisfying these axioms, as adopted, for instance, in [25] and in [30].

The axioms of [25] are the following:

1. $\mu_{Inc}(\mu, \nu) = 1$ iff $\mu \subseteq \nu$ in Zadeh's sense, i.e., $\forall x \in \mathcal{S},\ \mu(x) \leq \nu(x)$.
2. $\mu_{Inc}(\mu, \nu) = 0$ iff $Core(\mu) \cap \overline{Supp(\mu)} \neq \emptyset$.
3. μ_{Inc} is increasing in ν: if $\nu_1 \subseteq \nu_2$, then $\mu_{Inc}(\mu, \nu_1) \leq \mu_{Inc}(\mu, \nu_2)$.
4. μ_{Inc} is decreasing in μ: if $\mu_1 \subseteq \mu_2$, then $\mu_{Inc}(\mu_1, \nu) \geq \mu_{Inc}(\mu_2, \nu)$.
5. μ_{Inc} is invariant under geometric transformations like translation, rotation, etc.
6. $\mu_{Inc}(\mu, \nu) = \mu_{Inc}(c(\nu), c(\mu))$.
7. $\mu_{Inc}(\mu \cup \mu', \nu) = \min[\mu_{Inc}(\mu, \nu), \mu_{Inc}(\mu', \nu)]$.
8. $\mu_{Inc}(\mu, \nu \cap \nu') = \min[\mu_{Inc}(\mu, \nu), \mu_{Inc}(\mu, \nu')]$.
9. $\mu_{Inc}(\mu, \nu \cup \nu') \geq \max[\mu_{Inc}(\mu, \nu), \mu_{Inc}(\mu, \nu')]$.

Additional optional properties are suggested in [25]:

10. $\mu_{Inc}(\mu, \nu) + \mu_{Inc}(\mu, c(\nu)) \geq 1$.
11. $\mu_{Inc}(\mu \cup c(\mu), \mu \cap c(\mu)) \leq \mu_{Inc}(\nu \cup c(\nu), \nu \cap c(\nu))$ if μ is a refinement of ν.
12. $\mu_{Inc}(\mu, \nu) \geq 0.5$ if μ is weakly included in ν (i.e., $\forall x \in \mathcal{S}$ either $\mu(x) \leq 0.5$ or $\nu(x) > 0, 5$).

A degree of inclusion satisfying the axioms is defined in [25] as:

$$\forall(\mu, \nu) \in \mathcal{F}^2,\ \mu_{Inc}(\mu, \nu) = \inf_{x \in \mathcal{S}} \min[1, \lambda(\mu(x)) + \lambda(1 - \nu(x))], \tag{3.33}$$

where λ is a function from $[0, 1]$ into $[0, 1]$ such that: λ is non-increasing, $\lambda(0) = 1$, $\lambda(1) = 0$, the equation $\lambda(x) = 0$ has a single solution, $\forall\alpha \in [0.5, 1]$, the equation $\lambda(x) = \alpha$ has a single solution, $\forall a \in [0, 1],\ \lambda(a) + \lambda(1 - a) \geq 1$.

A typical example for λ is: $\lambda(a) = 1 - a^n$, with $n \geq 1$. In particular, for $n = 1$, the degree of inclusion becomes:

$$\mu_{Inc}(\mu, \nu) = \inf_{x \in \mathcal{S}} \min[1, 1 - \mu(x) + \nu(x)] \tag{3.34}$$

which is exactly the inclusion obtained from intersection or union (Eqs. 3.28 and 3.27) for the complementation $c(a) = 1 - a$ and for the Lukasiewicz t-norm and t-conorm (or equivalently from Lukasiewicz implication).

Despite the apparent similarity between Eqs. 3.33 and 3.27, they are not equivalent. Indeed, the function defined as:

$$\min[1, \lambda(1 - a) + \lambda(1 - b)], \tag{3.35}$$

which plays in Eq. 3.33 the same role as the t-conorm T in Eq. 3.27, is actually not a t-conorm, since it is not associative and it does not admit 0 as unit element, except for $\lambda(a) = 1 - a$ [2]. This induces the loss of properties of the degree of inclusion by comparison to those of inclusion derived from a true t-conorm. This may be a major drawback when algebraic properties are very important, as, for instance, for fuzzy mathematical morphology (Chap. 4).

Another axiomatization has been proposed in [30], as follows:

1. $\mu_{Inc}(\mu, \nu) = 1$ iff $\mu \subseteq \nu$ in Zadeh's sense, i.e., $\forall x \in \mathcal{S},\ \mu(x) \leq \nu(x)$; this is the same as the first axiom of [25].
2. Let ν be such that $\forall x \in \mathcal{S},\ \nu(x) = 0.5$. If $\nu \subseteq \mu$, then $\mu_{Inc}(\mu, c(\mu)) = 0$ iff $\mu = \mathcal{S}$ (i.e., $\forall x \in \mathcal{S},\ \mu(x) = 1$); this contrasts with the second axiom of [25].
3. if $\nu \subseteq \mu_1 \subseteq \mu_2$, then $\mu_{Inc}(\mu_1, \nu) \geq \mu_{Inc}(\mu_2, \nu)$, which is weaker than axiom 4 of [25]; if $\nu_1 \subseteq \nu_2$, then $\mu_{Inc}(\mu, \nu_1) \leq \mu_{Inc}(\mu, \nu_2)$, which is axiom 3 of [25].

These axioms are weaker than the ones of [25]. They lead to weaker properties of the degree of inclusion and also to weaker properties than those of the degree of inclusion derived from t-norms and t-conorms.

Inclusion and Fuzzy Entropy

Connections between the degree of inclusion and fuzzy entropy have been studied by several authors, including [14, 30], as shown by the following equation:

$$\forall \mu \in \mathcal{F},\ \mu_{Inc}(\mu \cup c(\mu), \mu \cap c(\mu)) = E(\mu), \tag{3.36}$$

where $E(\mu)$ denotes the fuzzy entropy of μ [15] (i.e., a function that takes value 0 iff μ is crisp, is maximal iff μ is constantly equal to 0.5, and decreases when μ gets sharper, in the sense that values larger than 0.5 increase while the others decrease). Several inclusion measures and corresponding entropies have been defined (e.g., [11–14, 30]). They are not detailed here, because they share weaker properties than the algebraic definitions.

Conclusion on Inclusion The previous review of existing definitions of fuzzy set inclusion shows that best properties are obtained for definitions based on t-norms and t-conorms, and definitions based on residual implications. Hence these are the ones that will be used in the next chapters. Note that they are equivalent in particular for Lukasiewicz' operators. This will be further detailed in Chap. 4.

3.2.4 Degree of Equality

The degree of equality between two fuzzy sets μ and ν can be simply defined from the inclusion degree, according to the definition of equality between two crisp sets X and Y:

$$X = Y \Leftrightarrow X \subseteq Y \text{ and } Y \subseteq X. \tag{3.37}$$

Extending this definition to fuzzy sets leads to the following definition of the degree of equality between μ and ν :

$$\mu_{Equa}(\mu, \nu) = \min[\mu_{Inc}(\mu, \nu), \mu_{Inc}(\nu, \mu)], \tag{3.38}$$

or more generally, for any t-norm t:

$$\mu_{Equa}(\mu, \nu) = t[\mu_{Inc}(\mu, \nu), \mu_{Inc}(\nu, \mu)]. \tag{3.39}$$

The properties of the degree of equality are the following:

- Consistency with the binary definition.
- Reflexivity if T is such that the excluded-middle law holds (e.g., for the Lukasiewicz t-conorm).
- Symmetry: $\mu_{Equa}(\mu, \nu) = \mu_{Equa}(\nu, \mu)$.
- If μ is empty ($\forall x \in \mathcal{S}, \mu(x) = 0$), then $\mu_{Equa}(\mu, \nu) = \inf_{x \in \mathcal{S}} c(\nu(x))$, which is equal to 0 for normalized fuzzy sets.
- If μ is equal to $\mathcal{S}$ ($\forall x \in \mathcal{S}, \mu(x) = 1$), $\mu_{Equa}(\mu, \nu)$ is equal to 0 for bounded support fuzzy sets.
- Invariance with respect to rigid geometrical transformations (translation, rotation).

3.3 Topology: Neighborhood, Boundary, and Connectedness of a Fuzzy Set

3.3.1 Fuzzy Neighborhood

In this section, we examine several possibilities for defining a degree of neighborhood n_{xy} between two points x and y in $\mathcal{S}$, endowed with a discrete connectivity if $\mathcal{S} = \mathbb{Z}^n$, or with a usual topology if $\mathcal{S} = \mathbb{R}^n$. They will be used in the definitions of fuzzy adjacency in Chap. 6.

Let us first consider binary definitions of n_{xy}. In the continuous case, we set $n_{xy} = 1$ if $x \in V(y)$, where $V(y)$ is a neighborhood of y, $n_{xy} = 0$ otherwise. In the discrete case, we set $n_{xy} = 1$ if x and y are neighbors in the sense of the considered discrete connectivity, and $n_{xy} = 0$ otherwise (i.e., $n_{xy} = n_c(x, y)$). With these definitions, the consistency with the binary case is guaranteed.

We now concentrate on fuzzy versions of n_{xy}, to express the degree to which two points x and y are neighbors. A first definition of the degree of neighborhood between any two points x and y has been proposed in [6], which depends on the distance between x and y, denoted by $d_{\mathcal{S}}(x, y)$ (Euclidean distance on $\mathcal{S}$):

$$n_{xy} = \frac{1}{1 + d_{\mathcal{S}}(x, y)}. \tag{3.40}$$

This definition has the drawback to provide a degree of 0.5 for two points being at distance 1 (i.e., two neighbors in the discrete case), leading therefore to a definition which is not consistent with the binary one. This could be avoided in the discrete case by considering this formula only for $x \neq y$ (then $d_{\mathcal{S}}(x, y) \geq 1$) and by multiplying all values by a factor 2.

Another definition was proposed in [4], based on a parameterized function, which allows for flexibility in the spatial extent of the fuzzy neighborhood. Of course, other functions sharing these properties could be used (see, e.g., the S-function in [18]). In the discrete case, the degree of neighborhood between any two points x and y is defined as:

$$n_{xy} = f^d_{b,S}(d_{\mathcal{S}}(x, y)) = \frac{1 + \exp(-b)}{1 + \exp b(\frac{d_{\mathcal{S}}(x,y)-1}{S} - 1)}, \tag{3.41}$$

and in the continuous case as:

$$n_{xy} = f^c_{b,S}(d_{\mathcal{S}}(x, y)) = \frac{1 + \exp(-b)}{1 + \exp b(\frac{d_{\mathcal{S}}(x,y)}{S} - 1)}, \tag{3.42}$$

where b and S are two positive parameters controlling, respectively, the flattening (or slope) and the width of the curve representing n_{xy} as a function of $d_{\mathcal{S}}(x, y)$.

These definitions of fuzzy neighborhood depend on local information and are therefore not invariant with respect to scaling. If this property is mandatory for the application at hand, it can be satisfied by simply normalizing the distance $d_{\mathcal{S}}(x, y)$ by the scaling factor. Invariance with respect to translation and rotation is always satisfied.

In the discrete case, the Euclidean distance can be replaced by any discrete distance, for instance, using a chamfer algorithm. This way the distance can be adapted to the type of space digitization, to the considered discrete connectivity, and to possible anisotropy of the data (see, e.g., [16]).

Equations 3.40–3.42 rely only on spatial information. Another point of view could be to combine this spatial information with the membership of x and y to the considered fuzzy sets. We suggest to do this in one of the two following ways:

1. Define a fuzzy neighborhood as a fuzzy structuring element, i.e., a fuzzy set on $\mathcal{S}$, and use the result of the fuzzy dilation (see Chap. 4 for details on fuzzy mathematical morphology) as the degree of neighborhood (this has been used for introducing spatial imprecision in fuzzy sets in [3]).
2. Combine one of the previous definitions with the degree of connectivity (or connectedness, see Sect. 3.3.3) defined in [21], using a t-norm.

3.3.2 Boundary of a Fuzzy Set

The boundary b_μ of a fuzzy set μ defined on $\mathcal{S}$ is expected to be a fuzzy set too and was defined in [4] as:

$$\forall x \in \mathcal{S}, b_\mu(x) = t[\mu(x), \sup_{z \in \mathcal{S}} t[c(\mu)(z), n_{xz}]], \tag{3.43}$$

where t is a t-norm, c a fuzzy complementation, and n_{xz} the degree to which x and z are neighbors. It expresses the degree to which a point x belongs to the fuzzy set and has in its neighborhood a point belonging to the complement of the fuzzy set, with the highest possible degree.

This definition is consistent with the crisp case, and invariant with respect to geometrical transformations such as translations and rotations (for scaling, only if n_{xy} is itself invariant).

Another definition of boundary can be derived from morphological dilation and erosion (see Chap. 4).

3.3.3 Connectedness

A lot of work related to connectedness has led to the notion of degree of connectivity between two points in a fuzzy set and of fuzzy connectedness, expressing to which extent points hold together to build an object (see, e.g., [21, 28]). Let μ denote the membership function of a fuzzy subset of $\mathcal{S}$ (considered as a discrete space). The degree of connectedness in μ between two points x and y is defined as:

$$c_\mu(x, y) = \max_{p \in L_{xy}} \min_{z \in p} \mu(z), \tag{3.44}$$

where L_{xy} denotes the set of paths from x to y in $\mathcal{S}$, in a sense of the ground connectivity defined in $\mathcal{S}$. This equation means that two points are highly connected in μ if there is a path between these points that stays as much as possible in μ.

This notion has been applied as a criterion to drive image segmentation, privileging segmented regions that are highly connected, in particular in medical imaging (see, e.g., [23]).

Connectedness can be extended to the more general notion of connection (or connectivity class), defined in the crisp case as a family of subsets of a given space that are said connected. This more general notion has been extended in [17, 19], leading to a nested family of hyperconnections associated with a tolerance parameter, with associated efficient computation algorithms. It is out of the scope of this book to go into details. Interestingly enough, this extension can be used to define filters of images, based on attributes of hyperconnected components (see an example in Chap. 4).

3.4 Fuzzy Geometry

Among the features of importance in image processing and understanding, in particular for shape recognition, are the geometrical features. If the objects in the images are represented as fuzzy sets, geometrical properties have to be extended to deal with fuzzy objects. This is the scope of this section. We first introduce some simple fuzzy geometric objects (points, lines, rectangles) and then present the main geometric measures and transformations to describe fuzzy objects.

3.4.1 Fuzzy Points and Lines

Fuzzy points are an extension in the n-dimensional space of fuzzy numbers. Indeed, a fuzzy number can be considered as a fuzzy point of the real line [31]. Two methods can be used for defining a fuzzy point from a fuzzy number [5]:

- The first approach consists in defining a fuzzy number on each axis, let say $\mu_1, \dots \mu_n$. A fuzzy point is then the set $(\mu_1, \dots \mu_n)$, where μ_i is a fuzzy number on the i-th axis.
- The second method consists in defining a fuzzy point as a fuzzy set on $\mathcal{S}$, through its membership function μ, that has to satisfy a set of properties extending the properties of fuzzy numbers:

  1. μ is upper semi-continuous.
  2. $\exists! x_0 \in \mathcal{S},\ \forall x \in \mathcal{S},\ \mu(x) = 1 \Leftrightarrow x = x_0$, which expresses the unicity of the modal value of μ; the fuzzy number is said to be centered at x_0.
  3. $\forall \alpha \in [0, 1],\ \ \mu_\alpha$ is a compact and convex subset of $\mathcal{S}$; note that this is equivalent to say that μ is a convex fuzzy set.

This second definition is more convenient since a fuzzy point is a fuzzy object in the image space, which is in accordance with the notion of fuzzy image object defined in Sect. 3.1. An example of fuzzy point, defined according to the second method, is shown in Fig. 3.5. Note that the first definition could lead to a fuzzy set in $\mathcal{S}$ as well, by considering the Cartesian product $\mu_1 \times \dots \times \mu_n$.

Several definitions of fuzzy lines have been proposed in [5], and then in subsequent work. They are summarized below for a 2D space (extensions to higher dimension are straightforward):

1. Let μ_A, μ_B, and μ_C be the membership functions of three fuzzy numbers. A fuzzy line is defined as the set of all fuzzy numbers μ_1 and μ_2 that are solutions of the following equation:

$$\mu_A \mu_1 + \mu_B \mu_2 = \mu_C. \tag{3.45}$$

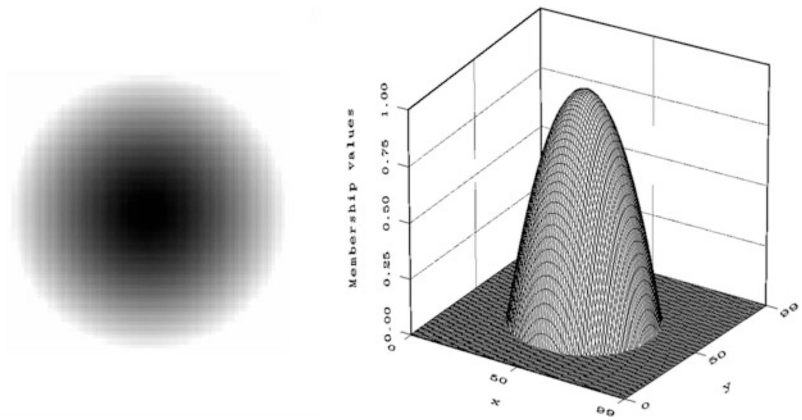

Fig. 3.5 Example of fuzzy point in the 2D space represented using gray levels (white for a zero membership value), and 3D representation of the membership values (membership is represented on the vertical axis)

To solve such equations we need the tools of the fuzzy arithmetic [24]. However, they do not always have solutions. Therefore, this definition is not further considered.

2. Let μ_A and μ_B be the membership functions of two fuzzy numbers. A fuzzy line is the set (μ_1, μ_2) where μ_1 is any fuzzy number and μ_2 is obtained by:

$$\mu_2 = \mu_A \mu_1 + \mu_B, \tag{3.46}$$

where the sum and product of fuzzy numbers is computed using the extension principle. However, this definition does not lead to a fuzzy object in $\mathcal{S}$. One of the consequences mentioned in [5] is that it cannot be visualized (as pictures or graphs as in Fig. 3.5).

3. Let μ_A, μ_B, and μ_C be the membership functions of three fuzzy numbers, such that the modal values of μ_A and μ_B are not both zero. A fuzzy line is defined as a fuzzy set on $\mathcal{S}$ through its membership function μ_L as:

$$\forall (x, y) \in \mathcal{S},\ \ \mu_L(x, y) =$$
$$\sup\{\alpha \in [0, 1] \mid ax + by = c,\ \ a \in \mu_{A\alpha}, b \in \mu_{B\alpha}, c \in \mu_{C\alpha}\}. \tag{3.47}$$

This definition corresponds to the extension principle applied on the definition of a crisp line, and extending it by considering that the coefficient defining the line are fuzzy.

4. Another definition is obtained using the same method as for the third definition, but using the equation $y = ax + b$, which is another form of the equation of a crisp line.
5. Similarly, another definition is obtained using the slope form of a line: $y - b = c(x - a)$.
6. The last definition uses two fuzzy points μ_A and μ_B of $\mathcal{S}$ and uses again the extension principle:

$$\forall (x, y) \in \mathcal{S},\ \ \mu_L(x, y) =$$
$$\sup\{\alpha \in [0, 1] \mid \frac{y - v_1}{x - u_1} = \frac{v_2 - v_1}{u_2 - u_1},\ \ (u_1, v_1) \in \mu_{A\alpha}, (u_2, v_2) \in \mu_{B\alpha}\}. \tag{3.48}$$

Definitions 3–6 are in agreement with the concept of fuzzy image object. For reasons already indicated, Definitions 1 and 2 are not considered further in this book. Note that Definitions 3–6 all rely on the same principle and just use different analytical expressions of a crisp line. Any other expression could be used and would lead to a definition of fuzzy line.

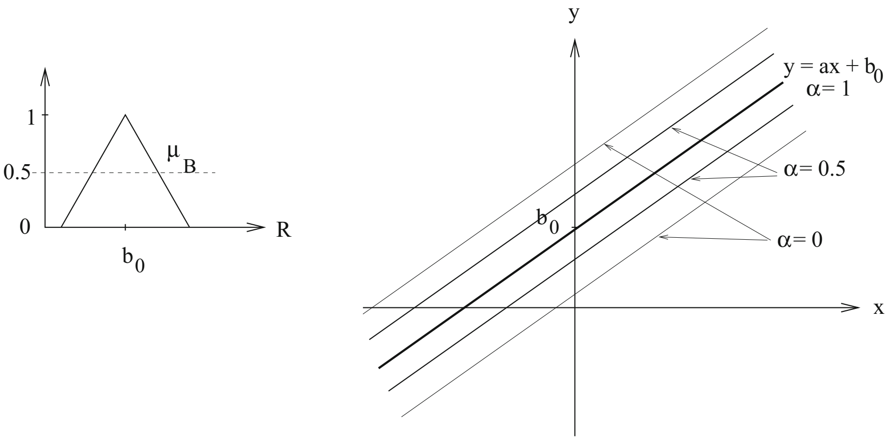

Fig. 3.6 Example of fuzzy line in the 2D space, where only b is fuzzy in the equation $y = ax + b$. Three α-cuts are shown (the one indicated with $\alpha = 0$ is actually the support). The fuzzy number μ_B is represented on the left

It is shown in [5] that, under certain conditions (e.g., 0 does not belong to the support of μ_B in Definition 3), the fuzzy lines resulting from Definitions 3 to 6 may coincide.

The properties of μ_L according to Definitions 3–6 are:

- The α-cuts of μ_L are closed, connected, and arc-wise connected, but not necessarily convex.
- μ_L is normalized, and there is at least one crisp line in μ_{L1}.
- μ_L is upper semi-continuous.

As an example, consider the equation $y = ax + b$, where only b is extended to a fuzzy number μ_B, while a remains a crisp number. Then the α-cuts of the fuzzy line are bands in the direction defined by a, whose respective widths are equal to the width of μ_B at level α, as illustrated in Fig. 3.6.

3.4.2 Fuzzy Rectangles and Fuzzy Convex Polygons

Another example of a fuzzy geometrical structure extends the notion of rectangle, and, more generally, of convex polygon, as defined in [21].

Here, we consider again a 2D space $\mathcal{S}$. A fuzzy set μ on $\mathcal{S}$ is separable if there exists a frame of coordinates (x, y) such that μ can be expressed as the conjunction of two fuzzy sets μ_x and μ_y defined on the x-axis and y-axis, respectively:

$$\forall (x, y) \in \mathcal{S}^2,\ \mu(x, y) = \min(\mu_x(x), \mu_y(y)). \tag{3.49}$$

Note that, for a more general definition, instead of the minimum, any t-norm could be used. Moreover, using the product t-norm, the classical definition of a separable function, as defined in functional analysis, is obtained.

From this notion of separable fuzzy set, the one of fuzzy rectangle is derived. A fuzzy set μ on $\mathcal{S}$ is a fuzzy rectangle if it is separable and fuzzy convex. It is proved in [21] that for a separable fuzzy set, the following three properties are equivalent:

1. μ is fuzzy connected, i.e., $\forall(P, Q) \in \mathcal{S}^2$, there exists an arc from P to Q such that for each point R in the arc the following inequality holds:

$$\mu(R) \geq \min(\mu(P), \mu(Q)).$$

2. μ is fuzzy convex, i.e., for each point R on the segment joining P to Q, $\mu(R)$ satisfies:

$$\mu(R) \geq \min(\mu(P), \mu(Q)).$$

3. μ is fuzzy orthoconvex, i.e., the same equation as for convexity holds whenever P and Q have the same x-coordinate or the same y-coordinate (when they form a vertical or horizontal segment).

A fuzzy rectangle can be characterized as follows: μ is a fuzzy rectangle if and only if there exist two fuzzy convex sets in 1D, μ_x and μ_y, such that:

$$\forall(x, y) \in \mathcal{S}^2, \;\; \mu(x, y) = \min[\mu_x(x), \mu_y(y)]. \tag{3.50}$$

Another characterization is that μ is a fuzzy rectangle if and only if all its α-cuts are crisp rectangles. Therefore a fuzzy rectangle appears as a "stack" of nested crisp rectangles, as illustrated in Fig. 3.7.

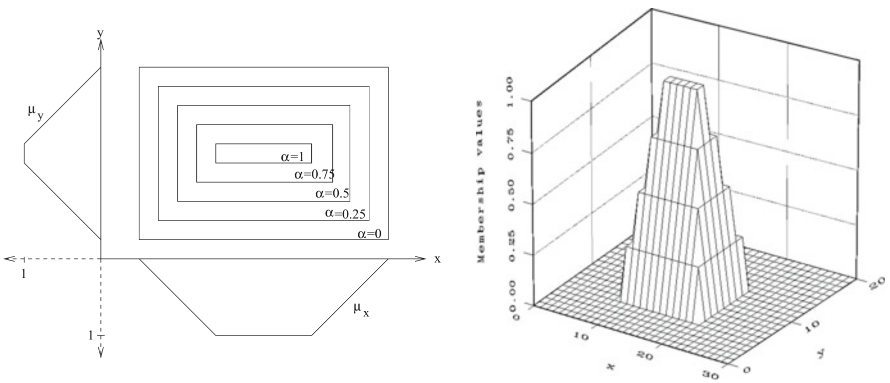

Fig. 3.7 Example of fuzzy rectangle. Left: μ_x and μ_y and a few α-cuts of μ. Right: 3D representation of μ (the third axis represents the membership values)

In a more general way, fuzzy convex polygons are defined as intersections of fuzzy half-planes. A fuzzy set μ is a fuzzy half-plane if there exists a direction x and a 1D fuzzy set ν such that:

$$\forall (x, y) \in \mathcal{S}^2, \ \mu(x, y) = \nu(x), \tag{3.51}$$

and ν is monotonically non-increasing:

$$x_1 \leq x_2 \Rightarrow \nu(x_1) \geq \nu(x_2). \tag{3.52}$$

A fuzzy set μ of $\mathcal{S}$ is a fuzzy halfplane if and only if the α-cuts of μ are (possibly degenerate) nested half-planes. An important property is that a fuzzy half-plane is fuzzy convex.

Let us now consider k fuzzy half-planes $\mu_1, \dots \mu_k$ of $\mathcal{S}$, whose associated directions are denoted by $x_1, \dots x_k$ and are given in cyclic order modulo 2π. If any two successive directions (for $i < k$, (x_i, x_{i+1}) and (x_k, x_1)) differ by less than π, then the fuzzy set μ defined on $\mathcal{S}$ as:

$$\mu = \min(\mu_1, \mu_2, \dots \mu_k) \tag{3.53}$$

is a fuzzy convex polygon, which follows from the fact that it is defined as the intersection of fuzzy convex sets. Again a characteristic property is that μ is a fuzzy convex polygon if and only if its α-cuts are nested convex polygons.

It can be proved that a fuzzy rectangle is a fuzzy convex polygon.

3.4.3 Fuzzy Disks

A fuzzy set μ of $\mathcal{S}$ is a fuzzy disk if there exists a point P of $\mathcal{S}$ such that the membership of any point only depends on its distance to P. The point P is called the center of the fuzzy disk. In polar coordinates in a 2D space, say (r, θ), a fuzzy disk has a membership function that is a function of r alone.

A fuzzy disk is convex if and only if $\mu(r)$ is non-increasing in r. Moreover, we generally assume that μ has a bounded support. The α-cuts of a convex fuzzy disk are nested crisp disks. If a fuzzy disk has a unique modal value at the center P ($\mu(P) = 1$), then it is a fuzzy point.

Two examples of fuzzy disks are shown in Fig. 3.8, one being convex and the other not.

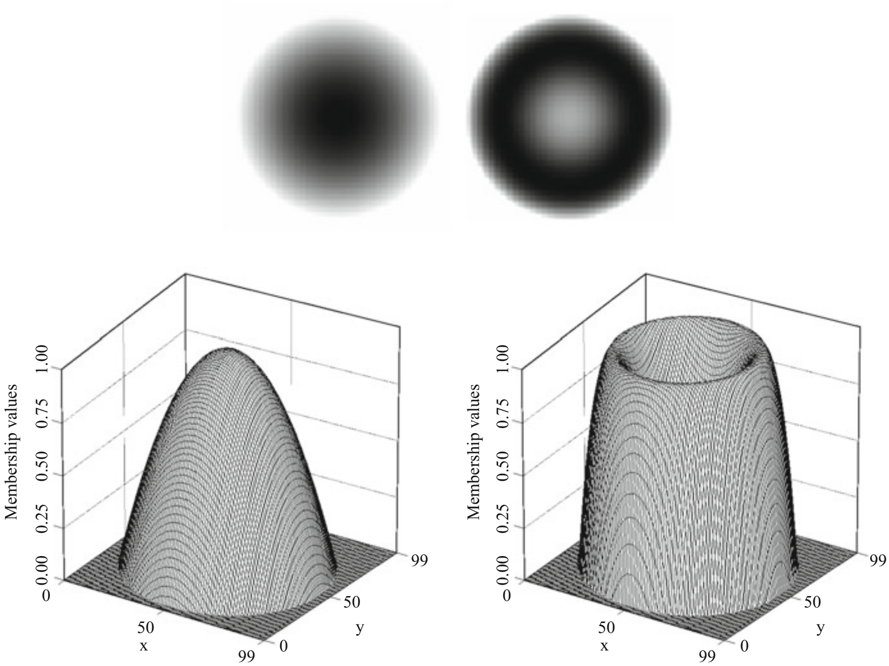

Fig. 3.8 Two examples of fuzzy disks. On the left, top, and bottom row, a convex fuzzy convex disk and its 3D membership function are shown; on the right, top, and bottom rows, a fuzzy non-convex disk and its 3D membership function are shown. Membership degrees are along the vertical axis

3.4.4 Fuzzy Geometrical Measures

Several geometrical measures are useful for describing, characterizing, and recognizing characteristics of geometrical shapes, such as perimeter, area, compactness, as well as more complex measures such as symmetry, parallelism, etc. This section presents how to compute such measures for fuzzy objects.

Area of a Fuzzy Set

The area of a fuzzy set μ defined on the space $\mathcal{S}$ is defined as [21]:

$$a(\mu) = \int_{\mathcal{S}} \mu(x)dx, \tag{3.54}$$

and in the case where $\mathcal{S}$ is finite:

Table 3.1 Some examples of fuzzy areas (in pixels), for the objects shown in Fig. 3.2. For each fuzzy object, the cardinality of its support (i.e., the number of points having a strictly positive membership value) and its fuzzy area are given

Fuzzy object	Size of support	Fuzzy area
cn1	512	239.40
cn2	451	192.55
p1	535	244.10
v1	636	224.86
v2	295	77.98

$$a(\mu) = \sum_{x \in \mathcal{S}} \mu(x). \tag{3.55}$$

Note that this is nothing else but the crisp cardinality of the fuzzy set μ (see Chap. 2).[2]

For the two examples of fuzzy disks shown in Fig. 3.8, this definition yields $a(\mu) = 3168.62$ for the convex one, and it has 6325 points in its support, and $a(\mu) = 4567.99$ for the non-convex one (which has 6349 in its support, i.e., approximately the same as for the first one).

More generally for a fuzzy disk, where μ is a function of r only in polar coordinates, the area is equal to:

$$a(\mu) = \int_{(x,y)\in\mathcal{S}} \mu(x, y)dxdy = \int_0^{2\pi} \int_0^{+\infty} r\mu(r)drd\theta = 2\pi \int_0^{+\infty} r\mu(r)dr. \tag{3.56}$$

Further examples are shown in Table 3.1, for the brain structures shown in Fig. 3.2.

Perimeter of a Fuzzy Set

The perimeter of a fuzzy set has been defined in [22], again in the 2D case. Let us first consider the case where $\mathcal{S}$ is finite, and where the fuzzy set μ is piecewise constant. Such a case is necessary in image processing, when a quantization of the membership values is carried out. Then the membership functions can take a finite number of values μ_i $(i = 1 \dots n)$ in $[0, 1]$ (i.e., only a finite number of α-cuts are different). In such cases, there exists a partition of $\mathcal{S}$ in regions R_i on which μ is constant and equal to μ_i. Let us denote by B_{ij} the common boundary of R_i and R_j, that is written as:

[2] See the discussion in Chap. 2 on the fact that this formula of cardinality does not necessarily provide a number in $\mathbb{N}$.

$$B_{ij} = \overline{R_i} \cap \overline{R_j}, \tag{3.57}$$

where $\overline{R_i}$ denotes the closure of R_i. It is composed of a set of arcs, denoted by $A_{ijk}, k = 1 \dots n_{ij}$, that are rectifiable, and the length of which is denoted by $l(A_{ijk})$. The perimeter of μ is then defined as:

$$p(\mu) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=1}^{n_{ij}} |\mu_i - \mu_j| l(A_{ijk}). \tag{3.58}$$

If μ is crisp, this definition is equivalent to the classical perimeter of a crisp set.

Let us now consider the continuous case, and assume that the fuzzy object has a membership function μ that is differentiable. The gradient of μ is then defined, and its magnitude is written as:

$$|\nabla \mu(x, y)| = \sqrt{(\frac{\partial \mu(x, y)}{\partial x})^2 + (\frac{\partial \mu(x, y)}{\partial y})^2}. \tag{3.59}$$

Then the perimeter of μ is defined as:

$$p(\mu) = \int_{(x,y)\in \mathcal{S}} |\nabla \mu(x, y)| dxdy, \tag{3.60}$$

if this integral exists.

It is shown in [22] that the definition in the continuous case (Eq. 3.60) and the definition in the piecewise constant case (Eq. 3.58) coincide in the limit. Moreover, they are both special cases of a more general formulation in the framework of generalized functions.

In the convex case, the α-cuts of μ are nested convex crisp sets, and therefore simply connected. The boundaries B_{ij} are then just curves. The perimeter is then a weighted sum of the lengths of these curves (see Fig. 3.9).

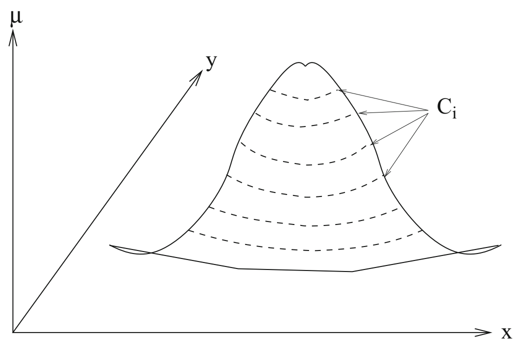

Fig. 3.9 Computation of the perimeter of a convex fuzzy set: it is a weighted sum of the lengths of the curves C_i delimitating the α-cuts of μ

An important property of the perimeter for piecewise constant fuzzy convex subsets μ and ν of $\mathcal{S}$ states monotony of the perimeter with respect to the inclusion:

$$\mu \leq \nu \Rightarrow p(\mu) \leq p(\nu).$$

In the case where μ is a fuzzy disk, then, with $\mu'(r)$ denoting the derivative of μ with respect to r, we have:

$$p(\mu) = 2\pi \int_0^{+\infty} r|\mu'(r)|dr. \tag{3.61}$$

This means that a fuzzy disk may have an arbitrarily large perimeter (typically if μ oscillates rapidly), while having a small area. It is interesting to note that this phenomenon, that is not surprising for crisp sets of complex shape, can also be observed for disks in the fuzzy case.

In the case of a convex fuzzy disk, we have:

$$|\mu'(r)| = -\mu'(r), \tag{3.62}$$

since μ is decreasing in r. Then the perimeter becomes:

$$p(\mu) = 2\pi \int_0^{+\infty} -r\mu'(r)dr = 2\pi \left(\int_0^{+\infty} (-r\mu(r))'dr + \int_0^{+\infty} \mu(r)dr \right). \tag{3.63}$$

If we further assume that μ decreases fast enough so that:

$$\lim_{r \to +\infty} r\mu(r) = 0, \tag{3.64}$$

then we have:

$$p(\mu) = 2\pi \int_0^{+\infty} \mu(r)dr. \tag{3.65}$$

Table 3.2 shows the perimeter of a few fuzzy 2D objects. The computation is performed on a discrete grid, and the neighbors are determined using both 4- and 8-connectivities (the standard types of discrete connectivity used in image processing on a square grid). These results show, for instance, that for the two fuzzy disks of Fig. 3.8, the non-convex disk has a much larger perimeter, although the size of its support is about the same as for the convex disk. This is due to the non-convexity of its membership function.

Compactness of a Fuzzy Set

The compactness of a fuzzy set is defined as [21]:

Table 3.2 Some examples of fuzzy perimeters (in pixels), for the two fuzzy disks of Fig. 3.8 and for the objects shown in Fig. 3.2. For each fuzzy object, the perimeter in 4-connectivity and in 8-connectivity are given

Fuzzy object	Fuzzy perimeter (4c)	Fuzzy perimeter (8c)
Convex fuzzy disk	144.35	168.93
Non-convex fuzzy disk	243.86	285.39
cn1	54.25	60.72
cn2	44.20	51.29
p1	60.54	67.46
v1	70.35	78.42
v2	32.48	35.06

Table 3.3 Some examples of fuzzy compactness, for the two fuzzy disks of Fig. 3.8 and for the objects shown in Fig. 3.2. For each fuzzy object, the perimeter in 4-connectivity and in 8-connectivity is used for computing the compactness. Comparing the values obtained with $\frac{1}{4\pi} \simeq 0.08$ provides a measure of how far from being a true crisp disk each shape is

Fuzzy object	Fuzzy compactness (4c)	Fuzzy compactness (8c)
Convex fuzzy disk	0.15	0.11
Non-convex fuzzy disk	0.08	0.06
cn1	0.08	0.06
cn2	0.10	0.07
p1	0.07	0.05
v1	0.05	0.04
v2	0.07	0.06

$$c(\mu) = \frac{a(\mu)}{p(\mu)^2}. \tag{3.66}$$

In the crisp case, the compactness is the largest for disks, where it is equal to $\frac{1}{4\pi}$ (isoperimetric inequality), which means that the perimeter cannot be small while the area is large. In the fuzzy case, we do not have the same inequality. However, it can be shown that if μ is a convex fuzzy disk, then:

$$\frac{a(\mu)}{p(\mu)^2} \geq \frac{1}{4\pi}. \tag{3.67}$$

It means that among all possible convex fuzzy disks, the compactness is smallest for a crisp disk.

Table 3.3 shows the values of compactness obtained for the fuzzy objects of Figs. 3.8 and 3.2.

Height, Width, and Diameter of a Fuzzy Set

The height and width of a fuzzy object μ in a 2D space are defined, respectively, by [21]:

$$h(\mu) = \int_y (\sup_x \mu(x, y))dy, \tag{3.68}$$

$$w(\mu) = \int_x (\sup_y \mu(x, y))dx. \tag{3.69}$$

These equations correspond to the integral of the maximum membership value on horizontal (respectively, vertical) lines (x and y range over the coordinate values). In the finite case, these equations become:

$$h(\mu) = \sum_y \max_x \mu(x, y), \tag{3.70}$$

$$w(\mu) = \sum_x \max_y \mu(x, y). \tag{3.71}$$

The following relationship holds with the area:

$$a(\mu^2) \leq h(\mu)w(\mu). \tag{3.72}$$

The extrinsic diameter of a fuzzy object μ is defined as the supremum of the integrals of its projections [21]:

$$e(\mu) = \sup_{\theta \in [0,\pi]} \int_u (\sup_v \mu(u, v))du, \tag{3.73}$$

where u and v are orthogonal directions, defined by a rotation of angle θ with respect to the original coordinate frame.

The extrinsic diameter is related to the area by the following inequality:

$$a(\mu^2) \leq (e(\mu))^2. \tag{3.74}$$

Consider now a connected fuzzy object μ, i.e., such that all its α-cuts are connected in the crisp sense. Then the intrinsic diameter of μ (in the finite case) is defined as:

$$i(\mu) = \max_{P,Q} \left(\min_{\mathcal{P}_{PQ}} \int_{\mathcal{P}_{PQ}} \mu(x, y)dxdy \right), \tag{3.75}$$

where the max is taken over all points P and Q in $\mathcal{S}$, and the min is taken over all paths $\mathcal{P}_{PQ}$ between P and Q such that:

$$\forall(x, y) \in \mathcal{P}_{PQ}, \ \mu(x, y) \geq \min(\mu(P), \mu(Q)).$$

The connection of μ guarantees that such paths always exist. If μ is crisp, $i(\mu)$ is exactly the intrinsic crisp diameter, i.e., the greatest possible distance between two points in μ, computed along paths that are completely included in μ. In the crisp case, $e(\mu) \leq i(\mu)$. In the convex crisp case, $e(\mu) = i(\mu)$. However, in the fuzzy case, we can have $e(\mu) > i(\mu)$, even in the convex case. Still, if μ is convex, the two following inequalities hold:

$$e(\mu) \geq i(\mu), \tag{3.76}$$

$$i(\mu) \leq \frac{p(\mu)}{2}. \tag{3.77}$$

Intersection and Parallelism Between Fuzzy Lines

These notions have been introduced in [5]. Let μ_{L_1} and μ_{L_2} be two fuzzy lines. A measure of parallelism between these two fuzzy lines if defined by $\rho(\mu_{L_1}, \mu_{L_2}) = 1 - \lambda(\mu_{L_1}, \mu_{L_2})$, where:

$$\lambda(\mu_{L_1}, \mu_{L_2}) = \sup_{(x,y)\in\mathcal{S}} (\min(\mu_{L_1}(x, y), \mu_{L_2}(x, y))). \tag{3.78}$$

This is nothing but the height of the intersection between μ_{L_1} and μ_{L_2}. If the supports of the two fuzzy lines do not intersect, then the lines are completely parallel. In the crisp case, the degree of parallelism is equal to 1 iff the lines are parallel in the standard sense. Other definitions of approximate parallelism between fuzzy objects (not necessarily lines) will be given in Chap. 6.

Geometrical Measures as Fuzzy Numbers

In the previous definitions, the geometrical measures of fuzzy objects were defined as numbers. If the objects are imprecisely defined, it is also expected that their geometrical measures are imprecise too, i.e., to be fuzzy numbers instead of numbers. The extension principle can be used for this purpose.

Let M be any geometrical measure (area, perimeter, etc.). The extension principle applied on α-cuts leads to:

$$\forall\lambda \in \mathbb{R}^+, \ M(\mu)(\lambda) = \sup_{M(\mu_\alpha)=\lambda} \alpha. \tag{3.79}$$

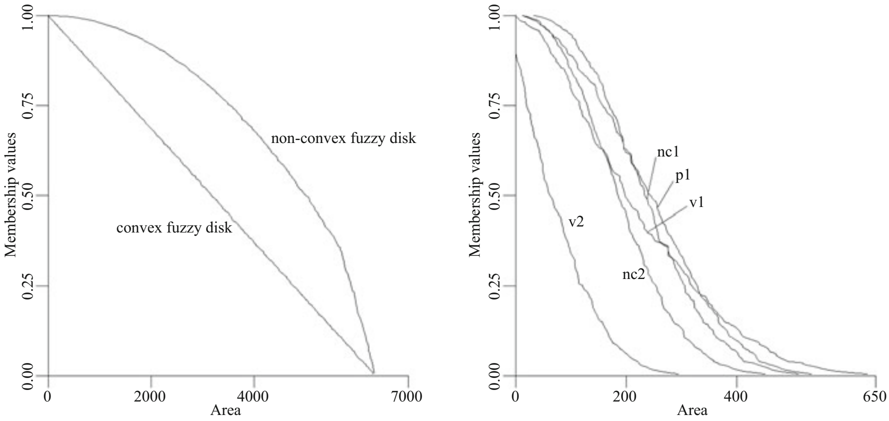

Fig. 3.10 Area of a fuzzy set as a fuzzy number, illustrated on the two fuzzy disks shown in Fig. 3.8 (left) and on the brain structures shown in Fig. 3.2 (right)

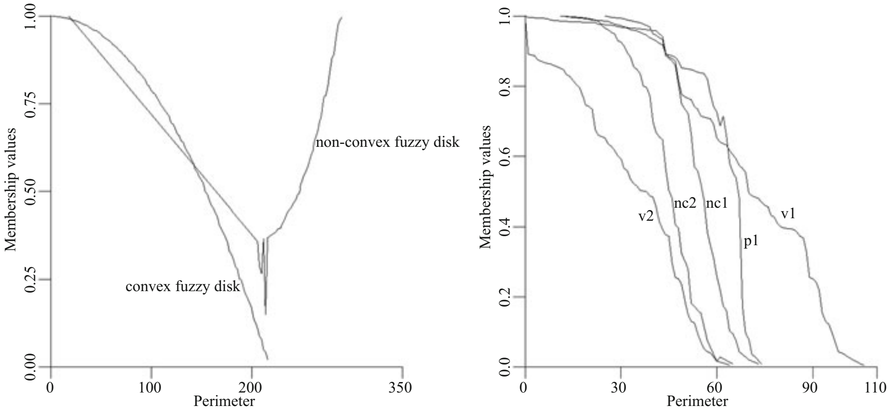

Fig. 3.11 Perimeter of fuzzy sets as fuzzy numbers, illustrated on the two fuzzy disks shown in Fig. 3.8 (left) and on the brain structures shown in Fig. 3.2 (right)

For instance, for the area, the obtained definition is equivalent to the cardinality of a fuzzy set defined as a fuzzy number [7].

Since the α-cuts of μ are nested, their area decreases when α increases. It follows that the fuzzy area has a decreasing membership function.

The definition of the fuzzy area as a fuzzy number is illustrated in Fig. 3.10 for the two fuzzy disks shown in Fig. 3.8 and in the brain structures shown in Fig. 3.2.

This way of defining a fuzzy geometric measure applies for any measure. It is illustrated for the perimeter in Fig. 3.11. In this case, the variation of the perimeter of the α-cuts with respect to α is no longer monotonic, and therefore the resulting fuzzy set is not necessarily convex.

This form can be useful for further processing, like evaluating propositions such as "the measure is large, small, etc.," where "large" or "small" are values of linguistic variables. Such an evaluation can be performed by comparing the membership functions of the geometrical measure and of the linguistic variable, using pattern matching, or compatibility fuzzy set.

Further definitions and examples can be found in [26, 27].

3.5 Fuzzy Geometric Transformations

This section describes geometric transformations of fuzzy sets, including translation and rotation. One can distinguish two cases, for well-defined transformations, and for fuzzy transformations, respectively.

3.5.1 Transformation of a Fuzzy Set by a Crisp Operation

Let $\mathcal{T}$ be any geometric transformation (translation, rotation, symmetry, scaling...). It is assumed that $\mathcal{T}$ is well defined, i.e., its parameters are crisp. Let μ be a fuzzy object defined in $\mathcal{S}$ (in any dimension). The transformation of μ by $\mathcal{T}$ is then simply defined as:

$$\forall x \in \mathcal{S},\ \mathcal{T}(\mu)[\mathcal{T}(x)] = \mu(x), \tag{3.80}$$

or equivalently:

$$\forall x \in \mathcal{S},\ \mathcal{T}(\mu)(x) = \mu[\mathcal{T}^{-1}(x)]. \tag{3.81}$$

In the discrete case, $\mathcal{T}(x)$ may not belong to $\mathcal{S}$ (i.e., may not coincide with one of the points of the space discretization). The problem is similar to the one encountered in classical image processing, and similar interpolation methods can be used [20]. The transformed fuzzy set is then computed as:

$$\forall x \in \mathcal{S},\ \mathcal{T}(\mu)(x) = Interpol\{\mu(y),\, y \in V(\mathcal{T}^{-1}(x))\}, \tag{3.82}$$

where $Interpol$ denotes any interpolation function, and $V(\mathcal{T}^{-1}(x))$ denotes a neighborhood of $\mathcal{T}^{-1}(x)$, composed of points of $\mathcal{S}$ (i.e., points of the discretization that are close to the, possibly not on the discrete grid, point $\mathcal{T}^{-1}(x)$). Typically, $V(\mathcal{T}^{-1}(x))$ can be the nearest neighbor of $\mathcal{T}^{-1}(x)$, or the 4 closest grid points in 2D (or 8 closest points in 3D), etc.

3.5.2 Transformation of a Fuzzy Set by a Fuzzy Operation

Assume now that the transformation $\mathcal{T}$ is a fuzzy transformation, depending on a set of fuzzy parameters $p_1, \ldots p_n$. Let μ_{p_i} be the membership function associated with the parameter p_i. Denote by $\mathcal{T}_{p_1,\ldots p_n}$ the crisp transformation obtained for precise values of the parameters.

Let us first consider a point y in $\mathcal{S}$, whose transformation by $\mathcal{T}$ is a fuzzy set with membership function defined using the extension principle [3]:

$$\forall z \in \mathcal{S},\ \mu_{\mathcal{T}(y)}(z) = \sup_{p_1,\ldots p_n | \mathcal{T}_{p_1,\ldots p_n}(y)=z} t[\mu_{p_1}(p_1), \ldots \mu_{p_n}(p_n)], \tag{3.83}$$

where t is a t-norm.[3]

This is interpreted as follows: for a given set of parameter values, the transformation of y by $\mathcal{T}_{p_1,\ldots p_n}$ gives rise to a point $\mathcal{T}_{p_1,\ldots p_n}(y)$ with membership value

$$t[\mu_{p_1}(p_1), \ldots \mu_{p_n}(p_n)].$$

When several parameter values yield the same result, the final result corresponds to the supremum of the membership values.

To extend the definition for one point to a fuzzy set μ, the membership values of $\mu(y)$ have to be combined with the ones of $\mu_{\mathcal{T}(y)}$, which leads to the following definition of the transformation of μ [3]:

$$\forall x \in \mathcal{S},\ \mathcal{T}(\mu)(x) = \sup_{y \in \mathcal{S}} t[\mu(y), \mu_{\mathcal{T}(y)}(x)]. \tag{3.84}$$

In practice, the computation of such transformations is not always straightforward. One solution consists in directly applying the extension principle, which might be computationally heavy. Another solution could be to derive some analytical expressions using parametric representations of the fuzzy parameters defining the transformations, such as L-R fuzzy numbers. However, such a derivation is not straightforward for all types of operations, and for chained operations where the same variable can be involved several times.

In the discrete case, another solution can be developed [3]: the definition domain of each parameter can be discretized, and for each possible value of the parameters, the corresponding point can be computed according to Eq. 3.84, along with its membership degree.

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[3] In such expressions $t(a, b, c)$ stands for $t[t(a, b), c]$. This notation is adopted for sake of simplicity and justified since any t-norm is commutative and associative. This extends to the combination of n values.

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Chapter 4
Fuzzy Mathematical Morphology

In this chapter, we present the theory of fuzzy mathematical morphology, starting from the algebraic notions, the basic operators, and providing some examples of derived operators. Extending mathematical morphology to fuzzy sets was addressed by several authors during recent years. Some definitions just consider gray levels as membership degrees, or use binary structuring elements. Here we restrict ourselves to really fuzzy approaches, where fuzzy sets have to be transformed according to fuzzy structuring elements. Initial developments can be found in the definition of fuzzy Minkowski addition [48]. Subsequently, this problem has been addressed by several authors independently, e.g., [8, 13, 27, 42, 43, 45, 66, 70, 79, 92]. These works can be divided into two main approaches, compared in [22]. In the first approach [27, 43, 44], an important property that is put to the fore is the duality between erosion and dilation. A second type of approach is based on the notions of adjunction and fuzzy implication and was formalized in [45]. This goes with the underlying structure of fuzzy sets, forming a lattice. Lattice theory has become a popular mathematical framework in different domains of information processing, such as mathematical morphology, fuzzy sets, formal concept analysis, among others. Extending mathematical morphology to fuzzy sets in this framework allows one to deal with imprecision and vagueness in knowledge representation and information processing, while benefiting from the strong properties induced by the algebraic framework.

4.1 Lattice Structure of $\mathcal{F}$

Mathematical morphology [89] usually relies on the algebraic framework of complete lattices [80]. However, it has also been extended to complete semi-lattices and general posets [62], based on the notion of adjunction [57] (see also [31] for a

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general description of the algebraic framework). Here, within the scope of this book, we only consider the case of complete lattices.

Let $\mathcal{F}$ be the set of fuzzy sets defined over a domain $\mathcal{S}$ and with membership values in [0, 1] (any lattice L could be used as well[1]). Recall that throughout this book, we identify a fuzzy set with its membership function. Let $\leq$ denote the classical ordering on fuzzy sets, defined as:

$$\forall(\mu, \nu) \in \mathcal{F}^2, \mu \leq \nu \Leftrightarrow \forall x \in \mathcal{S}, \mu(x) \leq \nu(x). \tag{4.1}$$

Note that the same symbol is used here for the ordering on [0, 1] and on $\mathcal{F}$ (without ambiguity).

It is easy to see that $(\mathcal{F}, \leq)$ is a complete lattice. The least element is the "empty" fuzzy set $\mu_\emptyset$ ($\forall x \in \mathcal{S}, \mu_\emptyset(x) = 0$). The greatest element is the "full" fuzzy set $\mu_\mathcal{S}$ ($\forall x \in \mathcal{S}, \mu_\mathcal{S}(x) = 1$). The infimum and supremum of a family $(\mu_i)_{i\in\Xi}$, where Ξ is an index set, are:

$$(\inf_{i\in\Xi} \mu_i)(x) = \inf_{i\in\Xi}(\mu_i(x)),$$

$$(\sup_{i\in\Xi} \mu_i)(x) = \sup_{i\in\Xi}(\mu_i(x)).$$

Note that again the same notations are used over [0, 1] and over $\mathcal{F}$. The inf and sup operators become min and max in particular on finite families.

Moreover, $(\mathcal{F}, \leq, \inf, \sup, t, I)$ where t and I are adjoint conjunction and implication (i.e., I is the residuated implication derived from t) is a residuated lattice (see Chap. 2 for the definition of connectives and their properties).

4.2 Algebraic Operators

Let us first recall the general algebraic framework of mathematical morphology. Let $(\mathcal{L}, \preceq)$ and $(\mathcal{L}', \preceq')$ be two complete lattices (which do not need to be equal). The following definitions and results are common to the general algebraic framework of mathematical morphology in complete lattices [12, 31, 56, 57, 73, 83, 90]. Note that different terminologies can be found in different contexts related to lattice theory (refer to [82] for equivalence tables).

An operator $\delta : \mathcal{L} \to \mathcal{L}'$ is an *algebraic dilation* if it commutes with the supremum (sup-preserving mapping):

[1] This refers to the notion of L-fuzzy sets [54], which is useful to model different types of semantics, such as interval-valued fuzzy sets, bipolar fuzzy sets, etc.

$$\forall (x_i) \in \mathcal{L},\ \delta(\vee_i x_i) = \vee'_i \delta(x_i), \tag{4.2}$$

where $\vee$ denotes the supremum associated with $\preceq$ and $\vee'$ the one associated with $\preceq'$, and i ranges over any index set, finite or not.

An operator $\varepsilon \colon \mathcal{L}' \to \mathcal{L}$ is an *algebraic erosion* if it commutes with the infimum (inf-preserving mapping):

$$\forall (x_i) \in \mathcal{L}',\ \varepsilon(\wedge'_i x_i) = \wedge_i \varepsilon(x_i), \tag{4.3}$$

where $\wedge$ and $\wedge'$ denote the infimum associated with $\preceq$ and $\preceq'$, respectively.

This general definition allows defining mathematical morphology operators such as dilations and erosions in many types of settings, such as sets, functions, fuzzy sets, rough sets, graphs, hypergraphs, various logics, etc., based on their corresponding lattice structures.

Algebraic dilations δ and erosions ε are increasing operators; moreover δ preserves the smallest element and ε preserves the largest element.

Two other fundamental operators are opening and closing. An *algebraic opening* on a lattice is an increasing, idempotent, and anti-extensive operators. An *algebraic closing* is an increasing, idempotent, and extensive operator.

Fundamental in this algebraic framework is the notion of adjunction.

A pair of operators (ε, δ), $\delta \colon \mathcal{L} \to \mathcal{L}'$, $\varepsilon \colon \mathcal{L}' \to \mathcal{L}$, defines an *adjunction* if

$$\forall x \in \mathcal{L}, \forall y \in \mathcal{L}', \delta(x) \preceq' y \iff x \preceq \varepsilon(y). \tag{4.4}$$

The main properties that will be used in the following are summarized as follows (see, e.g., [57, 83] for more details). If a pair of operators (ε, δ) defines an adjunction, then the following results hold:

- δ preserves the smallest element and ε preserves the largest element.
- δ is a dilation and ε is an erosion (in the sense of the above definition).
- $\delta\varepsilon$ is anti-extensive: $\delta\varepsilon \preceq' Id_{\mathcal{L}'}$, where $Id_{\mathcal{L}'}$ denotes the identity mapping on $\mathcal{L}'$, and $\varepsilon\delta$ is extensive: $Id_{\mathcal{L}} \preceq \varepsilon\delta$. The compositions $\delta\varepsilon$ and $\varepsilon\delta$ are called morphological opening and morphological closing, respectively.
- $\varepsilon\delta\varepsilon = \varepsilon$, $\delta\varepsilon\delta = \delta$, $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$ and $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$, i.e., morphological opening and closing are idempotent operators.
- if $\mathcal{L} = \mathcal{L}'$ then the following statements are equivalent:
  - δ is a closing (i.e., increasing, extensive, and idempotent),
  - ε is an opening (i.e., increasing, anti-extensive, and idempotent),
  - $\delta\varepsilon = \varepsilon$,
  - $\varepsilon\delta = \delta$.

Let δ and ε be two increasing operators such that $\delta\varepsilon$ is anti-extensive and $\varepsilon\delta$ is extensive. Then (ε, δ) is an adjunction.

The following representation result also holds. If ε is an increasing operator, it is an algebraic erosion if and only if there exists δ such that (ε, δ) is an adjunction.

The operator δ is then an algebraic dilation and can be expressed as $\delta(x) = \wedge'\{y \in \mathcal{L}' \mid x \preceq \varepsilon(y)\}$. A similar representation result holds for erosion.

Now, setting $(\mathcal{L}, \preceq) = (\mathcal{L}', \preceq') = (\mathcal{F}, \leq)$, we obtain directly algebraic dilation and erosion on fuzzy sets from this very general framework, as operators that commute with the supremum and the infimum, respectively. Algebraic closing and opening are also defined as in the general framework. As particular cases, morphological opening and closing are derived by composition, as in the general case. All properties listed above are directly inherited.

4.3 Structuring Elements and Basic Morphological Operators

Specific and usual forms of the basic morphological operators involve the notion of structuring element. In the following, $\mathcal{S}$ is the spatial domain $\mathbb{R}^n$ or $\mathbb{Z}^n$. In the particular case of the lattice of subparts of $\mathcal{S}$, endowed with inclusion as partial inclusion, adding a property of invariance under translation leads to the following particular forms (called morphological dilations and erosions):

$$\forall X \subseteq \mathcal{S},\ \delta_B(X) = \{x \in \mathcal{S} \mid \check{B}_x \cap X \neq \emptyset\}, \quad \varepsilon_B(X) = \{x \in \mathcal{S} \mid B_x \subseteq X\}, \tag{4.5}$$

where B is a subset of $\mathcal{S}$ called structuring element, B_x denotes its translation at point x and $\check{B}$ its symmetrical with respect to the origin of space. These operations are dual with respect to the set complementation, i.e., $\delta_B(\mathcal{S} \setminus X) = \mathcal{S} \setminus \varepsilon_B(X)$. Opening and closing are defined by composition (using the same structuring element). These are the usual forms of the basic mathematical morphology operators, used typically in image processing (extending these definitions to functions) and in spatial reasoning.

A simple illustrative example is given in Fig. 4.1.

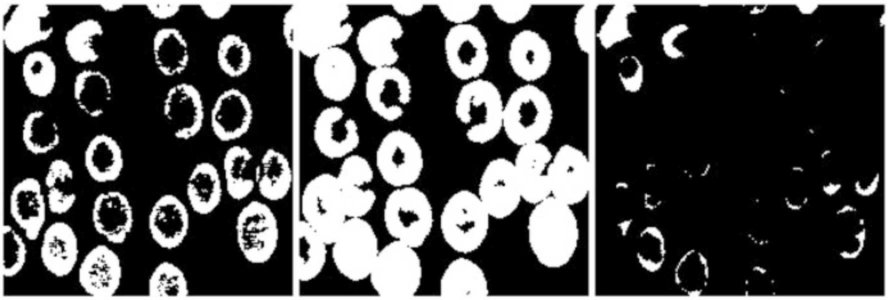

Fig. 4.1 Binary image, dilation, and erosion using a small disk as structuring element

Note that B can be considered in a more general way as a binary relation between two points of $\mathcal{S}$ (i.e., y is in relation with x if and only if $y \in B_x$). This allows establishing interesting links with several other domains, such as rough sets [17], graphs, hypergraphs [25, 32], logics [18], etc., and, in the more general case where the morphological operations are defined from one set to another one, with Galois connections and formal concept analysis, as shown, for example, in [7, 31].

Let us now extend these notions to fuzzy sets. Let ν be a structuring element, defined as a binary relation on $\mathcal{S} \times \mathcal{S}$, and taking values in [0, 1], i.e., for any $x \in \mathcal{S}$, $y \in \mathcal{S}$, $\nu(x, y)$ represents the degree to which the relation holds between x and y. In the spatial domain, the relation is typically a neighborhood relation, and $\nu(x, y)$ is the degree to which y belongs to the neighborhood of x.

The first attempts to build fuzzy mathematical morphology were based on translating binary equations into fuzzy ones, as developed in [13, 27]. This translation is done term by term, by substituting all crisp expressions by their fuzzy equivalents. For instance, intersection is replaced by a t-norm, union by a t-conorm, sets by fuzzy set membership functions, etc. (see Chap. 2). This allows expressing erosion as a degree of inclusion and dilation as a degree of intersection (see Chap. 3). An important property that is put forward in this approach is the duality between erosion and dilation. Translating Eq. 4.5 using their equivalent fuzzy notions yields the following definition of dilation and erosion of a fuzzy set μ by a fuzzy structuring element ν:

$$\forall x \in \mathcal{S},\ \delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} t[\nu(x, y), \mu(y)], \tag{4.6}$$

where t is a t-norm, and

$$\forall x \in \mathcal{S},\ \varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} T[c(\nu(y, x)), \mu(y)], \tag{4.7}$$

where T is a T-conorm and c a complementation.

Equation 4.6 corresponds to a degree of intersection between μ and ν at x, while Eq. 4.7 corresponds to a degree of inclusion of ν in μ at x. These definitions provide dual operators, by construction.

These forms of fuzzy dilation and fuzzy erosion are very general, and several definitions found in the literature are, in fact, particular cases, such as [8, 43, 44, 92] (for comparison, see, for example, [15, 27, 96]). It is interesting to note that similar ideas have been developed independently at about the same time by different teams [13, 43, 92].

Note that these forms are different from the ones on functions (and used for gray-level images, for instance), where the conjunction would be replaced by a sum for the dilation. This corresponds to a different algebra. The one adopted here is in accordance with the semantics of fuzzy sets.

Fuzzy opening (respectively, fuzzy closing) is simply defined as the combination $\delta\varepsilon$ (respectively, $\varepsilon\delta$) of a fuzzy erosion followed by a fuzzy dilation (respectively, a fuzzy dilation followed by a fuzzy erosion), by using dual t-norms and t-conorms.

Detailed properties of these definitions can be found in [27]. Most properties of classical morphology are satisfied for any choice of t and T. But in order to get true closing and opening, i.e., which are extensive (respectively, anti-extensive) and idempotent, a necessary and sufficient condition on t and T is [27, 42]:

$$\forall \alpha \in [0,1], \forall \beta \in [0,1], t(\beta, T(c(\beta), \alpha)) \leq \alpha, \tag{4.8}$$

which is satisfied, for instance, for Lukasiewicz t-norm and t-conorm (i.e., $t(\alpha, \beta) = \max(0, \alpha + \beta - 1)$ and $T(\alpha, \beta) = \min(1, \alpha + \beta)$) with the standard complementation ($c(\alpha) = 1 - \alpha$).

A second type of approach is based on the notions of adjunction and fuzzy implication. Here the algebraic framework is the main guideline, which contrasts with the previous approach where duality was imposed in first place. This approach is consistent with the general algebraic framework presented in Sect. 4.2. The derivation of fuzzy morphological operators from residual implication has first been proposed in [40, 41] and then developed, e.g., in [42, 70]. One of its main advantages is that it leads to idempotent fuzzy closing and opening. This approach was formalized from the algebraic point of view of adjunction in [45]. It has then been used by other authors, e.g., [66]. This leads to general algebraic fuzzy erosions and dilations (i.e., operations that commute with the infimum and the supremum of the lattice, respectively). Let us detail this approach.

Let t be a conjunction, and I its residuated implication. Fuzzy dilation and erosion are then defined as:

$$\forall x \in \mathcal{S}, \;\; \delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} t(\nu(x, y), \mu(y)), \tag{4.9}$$

$$\forall x \in \mathcal{S}, \;\; \varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} I(\nu(y, x), \mu(y)). \tag{4.10}$$

Note that (I, t) is an adjunction if and only if $(\varepsilon_\nu, \delta_\nu)$ is an adjunction for any ν (i.e., $\delta_\nu(\mu) \leq \mu' \Leftrightarrow \mu \leq \varepsilon_\nu(\mu')$) [22].

Opening and closing derived from these operations by combination have all required properties, for any choice of adjoint t and I. Some properties of dilation, such as iterativity, require t to be associative and commutative, i.e., a t-norm.

Note that here the classical definition of adjunction is used. A fuzzification of this notion has been proposed in [71], based on the equality between the degree of inclusion of $\delta(\mu)$ in μ' and the degree of inclusion of μ in $\varepsilon(\mu')$.

The two approaches are not always equivalent, since pairs of dual operators are not identical to pairs of adjoint operators (deriving a disjunction from an implication as $T(\alpha, \beta) = I(c(\alpha), \beta)$). However, the equivalence holds for Lukasiewicz operators, up to a bijection on membership values. As shown in [22], the condition for dual t-norms and t-conorms leading to idempotent opening and closing, given by Eq. 4.8 (i.e., $t(\beta, T(c(\beta), \alpha)) \leq \alpha$) is equivalent to the adjunction property between t and I (for $T(\alpha, \beta) = I(c(\alpha), \beta)$). This result can also be derived, via the morphological operators, from the fact that if ε and δ are increasing operators such that $\varepsilon\delta$ is extensive and $\delta\varepsilon$ is anti-extensive, then (ε, δ) is an adjunction. Conversely, if (ε, δ) is an adjunction, then the compositions $\varepsilon\delta$ and $\delta\varepsilon$ are extensive (respectively, anti-extensive) and idempotent, hence morphological closing and opening. This provides an algebraic and morphological proof of the same result, since Eq. 4.8 is a necessary and sufficient condition to have $\varepsilon\delta$ and $\delta\varepsilon$ idempotent and extensive (respectively, anti-extensive), and the adjunction property on (ε, δ) is equivalent to the adjunction property on (I, t). Results in [42, 65] also show that a t-norm t is left continuous if and only if Eq. 4.8 holds, for T derived from the residual implication I of the t-norm, and if and only if (I, t) is an adjunction, which shows the equivalence in the case of a residual implication derived from a left-continuous t-norm

Let us now comment on the choice of the connectives. It has a twofold influence, on theoretical and practical sides, and the understanding of this influence can guide the choice of a specific form of fuzzy mathematical morphology. From a theoretical point of view, when using adjoint connectives, opening is a "true" opening (i.e., increasing, anti-extensive and idempotent) and closing is a true closing. The algebraic framework of mathematical morphology [83, 90] can then be entirely applied in the fuzzy case, and a whole theory of morphological fuzzy filtering can then be directly derived. However, for other connectives, some other properties of erosion and dilation can be lost, in particular if the selected conjunction does not have all the properties of a t-norm. When using connectives that are dual but not adjoint, opening is not anti-extensive, nor idempotent (but it is increasing). Furthermore, it has been proved in [22] that the definitions of dilation and erosion using a t-norm and its residuated implication are the most general ones in order to satisfy all classical properties of mathematical morphology.

From a practical point of view, two aspects can be considered: the properties that are actually needed for a specific application, and the spatial extent of the applied transformations (typically for image processing applications). For instance, duality may be of prime importance to be in agreement with set theoretical interpretations. For example, this is the case when considering a spatial object as a set (actually a fuzzy set) and the background as its complement. The iterativity property of dilation and erosion is also important to guarantee that a sequence of operations has the desired behavior. It is also directly used when decomposing a complex structuring element into simpler parts, having then a direct impact on the algorithms and their computational complexity. Extensivity of dilation and anti-extensivity of erosion for a structuring element containing the origin of space is also an often required property. By contrast, we may accept to lose some properties. For instance, if opening and closing are simply used to regularize the membership values and remove noisy points, even non adjoint operators can be used (see, for instance, the examples in [27]), at the price of losing the control on the properties of the results. Concerning the spatial extent of the transformations, it is expected that it reflects the size of the structuring element. The increasingness property of the main

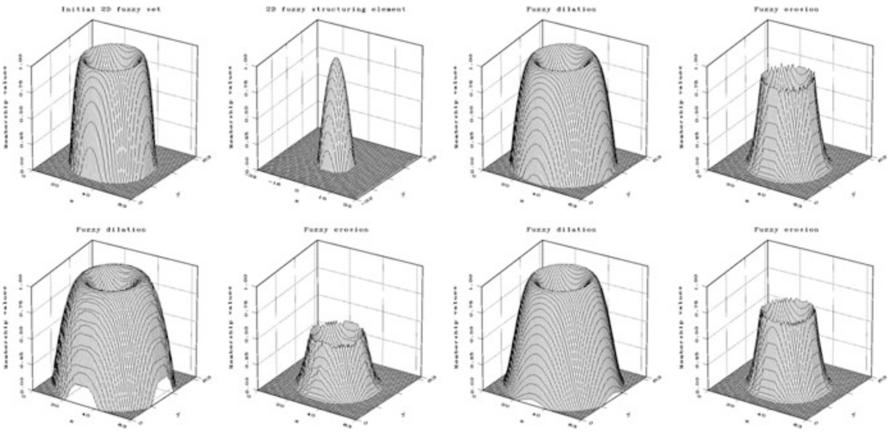

Fig. 4.2 Fuzzy dilations and erosions of a fuzzy set defined on a 2D domain. Top: fuzzy set and fuzzy structuring element (the third (vertical) axis represents the membership values), dilation and erosion using Lukasiewicz operators (best from a theoretical point of view). Bottom: dilations and erosion using min and max, dilation and erosion using product and algebraic sum

morphological operations guarantees that using a larger structuring element will have a larger effect on the transformed fuzzy set, thus preserving the order in some sense. But the extent of a transformation also depends on the choice of the operator: for instance, using smaller t-norms for dilation leads to smaller results as well. Using the Lukasiewicz t-norm will lead to smaller results than using the minimum (see the examples in [20, 27]). Further examples can be found in [27, 45, 66, 70], among others.

An illustrative example is displayed in Fig. 4.2, showing the effect of dilation and erosion using different connectives on a simple example.

4.4 An Example in Medical Imaging

An issue for concrete applications of mathematical morphology operators is the choice of the structuring element. In this section, we summarize a real application, where the structuring elements are directly derived from the images and from the way they are processed, and fuzzy dilation is used to introduce imprecision explicitly. The application aims at 3D vessel reconstruction, by fusion of angiographic and echographic images [28]. Figure 4.3a–d illustrates the data to be combined for the reconstruction.

The fusion of these two modalities require to register them. This process is prone to imprecision and may lead to inconsistencies in the reconstructed volumes from the two modalities. Instead of considering precise numbers for the parameters of the geometric transformations involved in the registration, they are modeled as fuzzy

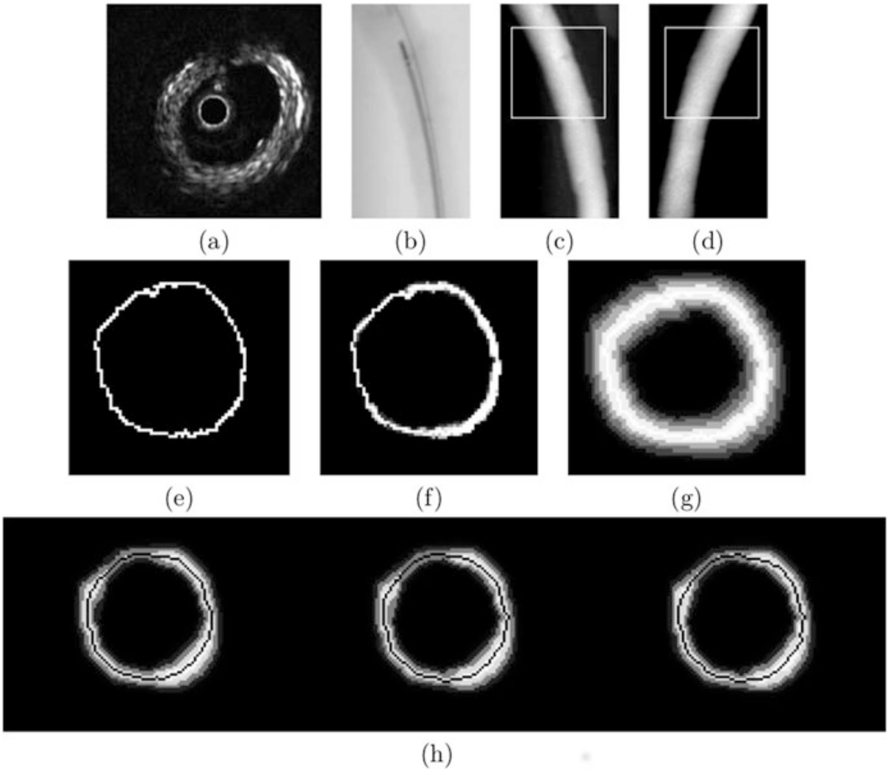

Fig. 4.3 First row: images used for 3D reconstruction. (**a**) Echographic slice. (b) Control radiography (which allows determining the orientation and position of echographies). (**c**,**d**) Left and right angiographies. The frame indicates the region in which the reconstruction will be performed. Second row: axial slice of the reconstructed vessel from echographic data. (**e**) Binary reconstruction. (**f**) Fuzzy reconstruction including all imprecisions on rotation parameters. (**g**) Fuzzy reconstruction including all imprecisions (rotation and translation), computed as fuzzy dilations. (**h**) Fuzzy volume after fusion and watersheds (in black) on a few slices of the vascular segment

numbers, in order to cope with these imprecisions. Imprecisions on the translation and rotation parameters make possible defining fuzzy structuring elements, around each point of the vessel wall. Fuzzy dilation appears then as a useful tool for introducing these imprecisions in a controlled manner, and by preserving good properties. Figure 4.3 represents an axial slice of the reconstructed vessel, at the different steps of the method (which is actually applied in 3D).

Once this process is applied on both modalities, the obtained fuzzy reconstructions include explicitly all imprecisions and are no longer inconsistent. A conjunctive fusion (using a min operator, for instance) can then be performed (see Chap. 5) and leads to a reconstructed consistent volume with reduced imprecision. The final binary decision is obtained by the watershed of this fuzzy volume, leading to a crisp reconstruction with the required topology (here a cylinder) and which goes through the points with maximal membership to the fused volume (i.e., on the crest line). Inconsistencies between the two modalities are solved in a satisfactory way: in regions where the two binary reconstructions are consistent, then the same result is obtained, while in regions where there was some conflict, an intermediary position of the vessel wall is obtained.

Further applications of fuzzy dilations and erosions for defining spatial relations will be presented in Chap. 6.

4.5 Towards a Fuzzy Mathematical Morphology Toolbox

In this section we detail a few other morphological operations, derived from the basic operations. We still consider $\mathcal{S}$ as a spatial domain ($\mathbb{R}^n$ or $\mathbb{Z}^n$).

4.5.1 Neighborhood and Boundary from Fuzzy Dilation and Erosion

Obviously, the neighborhood $\mathcal{V}(x)$ of a point x of $\mathcal{S}$, according to a structuring element ν, is directly related to the dilation of $\{x\}$, and it is fuzzy if ν is fuzzy:

$$\forall x \in \mathcal{S}, \forall y \in \mathcal{S}, (\mathcal{V}(x))(y) = \nu(y, x) = \delta_\nu(\{x\})(y). \tag{4.11}$$

If ν is symmetrical, then $(\mathcal{V}(x))(y) = (\mathcal{V}(y))(x)$.

In the crisp case, the notion of boundary can be derived from morphological dilation or erosion. Indeed, in the discrete case, the interior boundary of a set X can be defined as:

$$\partial^i X = X \setminus \varepsilon_{B_c}(X), \tag{4.12}$$

where $\varepsilon_{B_c}(X)$ denotes the morphological erosion of X by the structuring element B_c of size 1 defined according to the chosen discrete connectivity [89] (i.e., B_c is composed of the origin and its neighbors according to this connectivity). Similarly the external boundary can be defined as:

$$\partial^e X = \delta_{B_c}(X) \setminus X, \tag{4.13}$$

where $\delta_{B_c}(X)$ denotes the morphological dilation of X by the structuring element B_c.

The fuzzy boundary of a fuzzy set can then be defined using similar expressions [29]. The internal fuzzy boundary b^i_μ of a fuzzy set μ is defined from fuzzy erosion, by translating Eq. 4.12, as:

$$b^i_\mu(x) = t(\mu(x), c(\varepsilon_{B_c}(\mu))(x)), \tag{4.14}$$

where t is a t-norm and c a complementation. Note that this expression is equivalent to

$$b^i_\mu(x) = t(\mu(x), \delta_{B_c}(c(\mu))(x)), \tag{4.15}$$

hence involving now dilation, if δ and ε are chosen as dual operators (see Sect. 4.3).

The external fuzzy boundary b^e_μ of a fuzzy set μ is defined from fuzzy dilation, by translating Eq. 4.13 as:

$$b^e_\mu(x) = t(\delta_{B_c}(\mu)(x), c(\mu)(x)). \tag{4.16}$$

Using duality, we have $b^e_\mu(x) = t(c(\varepsilon_{B_c}(c(\mu)))(x), c(\mu)(x)) = b^i_{c(\mu)}(x)$. Hence, as in the crisp case, the external boundary of a fuzzy set is equal to the internal boundary of its complementation.

Instead of a crisp structuring element B_c, a fuzzy structuring element can be used as well, representing, for instance, the local imprecision related to the digitization of the image, to partial volume effect, or to any other source of imprecision.

4.5.2 Fuzzy Morphological Filters

The meaning of filter in image processing covers a large range of operators, with different types of properties depending on the underlying mathematical framework. In mathematical morphology, a filter is defined algebraically as an operator ψ on a lattice $(\mathcal{L}, \preceq)$ that is [89, 90]:

- increasing, that is $\forall (x, y) \in \mathcal{L}^2, x \preceq y \Rightarrow \psi(x) \preceq \psi(y)$,
- and idempotent, that is $\forall x \in \mathcal{L}, \psi\psi(x) = \psi(x)$.

This algebraic framework applies in any lattice, hence in particular on the lattice $(\mathcal{F}, \leq)$ of fuzzy sets defined on a domain $\mathcal{S}$. Some examples of such filters include:

- Any algebraic opening γ (i.e., an increasing, idempotent, and anti-extensive operator) is a filter, a particular case being the openings defined by composition of adjoint erosion and dilation ($\gamma = \delta\varepsilon$) mentioned above.

- Any algebraic closing φ (i.e., an increasing, idempotent, and extensive operator) is a filter, a particular case being the closings defined by composition of adjoint erosion and dilation ($\varphi = \varepsilon\delta$) mentioned above.[2]
- The supremum $\vee_i \gamma_i$ of a family of openings γ_i is an opening and hence a filter.
- The infimum $\wedge_i \varphi_i$ of a family of closings φ_i is a closing and hence a filter.

These particular examples have several useful properties. Let $Inv(\psi) = \{x \in \mathcal{L} \mid \psi(x) = x\}$ the set of invariants elements by ψ. Then for any opening γ: $\gamma(x) = \bigvee\{y \in Inv(\gamma) \mid y \leq x\}$, and similarly for any closing φ: $\varphi(x) = \bigwedge\{y \in Inv(\varphi) \mid x \leq y\}$. If γ_1 and γ_2 are two openings, then the following three statements are equivalent: (i) $\gamma_1 \leq \gamma_2$; (ii) $\gamma_1\gamma_2 = \gamma_2\gamma_1 = \gamma_1$; (iii) $Inv(\gamma_1) \subseteq Inv(\gamma_2)$. A similar result holds for closings.

Another important property is given by the filter composition theorem [90], expressed as follows. Let ξ and ψ two filters such that $\xi \geq \psi$, Then we have:

- $\xi \geq \xi\psi\xi \geq \xi\psi \vee \psi\xi \geq \xi\psi \wedge \psi\xi \geq \psi\xi\psi \geq \psi$,
- $\xi\psi$, $\psi\xi$, $\xi\psi\xi$, and $\psi\xi\psi$ are filters,
- $Inv(\xi\psi\xi) = Inv(\xi\psi)$ and $Inv(\psi\xi\psi) = Inv(\psi\xi)$,
- $\xi\psi\xi$ is the smallest filter that is larger than $\xi\psi \vee \psi\xi$.

Based on this result, very common and useful filters in mathematical morphology are the so-called alternate sequential filters [89], defined as follows: let $(\gamma_i)_{i\in\mathbb{N}}$ be a family of openings and $(\varphi_i)_{i\in\mathbb{N}}$ a family of closings such that:

$$i \leq j \Rightarrow \gamma_j \leq \gamma_i \leq Id \leq \varphi_i \leq \varphi_j. \tag{4.17}$$

From the filter composition theorem, the compositions $m_i = \gamma_i\varphi_i$, $n_i = \varphi_i\gamma_i$, $r_i = \varphi_i\gamma_i\varphi_i$ and $s_i = \gamma_i\varphi_i\gamma_i$ are filters, and

$$\begin{aligned}
M_i &= m_i m_{i-1} \ldots m_2 m_1 \\
N_i &= n_i n_{i-1} \ldots n_2 n_1 \\
R_i &= r_i r_{i-1} \ldots r_2 r_1 \\
S_i &= s_i s_{i-1} \ldots s_2 s_1
\end{aligned}$$

are also filters, called alternate sequential filters. An important property of these filters is

$$i \leq j \Rightarrow M_j M_i = M_j,$$

[2] In the literature, most applications of fuzzy mathematical morphology for filtering use only openings and closings. It is out of the scope of this book to review them all. Here we go one step further, by presenting more elaborate filters.

and similar results for the other filters.

Again these very general algebraic definitions and properties hold in the particular case of $(\mathcal{F}, \leq)$.

Let $(\nu_i)_{i\in\mathbb{N}}$ be an increasing sequence of fuzzy structuring elements (in practice a finite sequence is used), and define $\gamma_i = \delta_{\nu_i}\varepsilon_{\nu_i}$, $\varphi_i = \varepsilon_{\nu_i}\delta_{\nu_i}$, which are openings and closings satisfying Condition 4.17. Then a typical concrete example of fuzzy alternate sequential filter is

$$\gamma_n\varphi_n \cdots \gamma_2\varphi_2\gamma_1\varphi_1$$

or a similar expression, exchanging the order of openings and closings.

The effect of these filters is to reduce noise from the images or membership functions, to produce images with flat zones, where small bright or dark regions are suppressed, and to preserve sharp contours of the regions that are not suppressed.

Several other filters exist in mathematical morphology [73, 89, 90], in particular based on attributes computed on connected components and level-sets, and can be directly extended to fuzzy sets. They are not detailed here. Examples based on fuzzy connectivity and hyperconnection can be found in [74, 77]. Figure 4.4 illustrates one such a filter, applied to the lateral ventricles in a brain MRI image.

Let us only mention the potential usefulness of combining filters with reconstruction (see Sect. 4.5.3), in order to retrieve small details of objects that have been overall kept by the filter. In these filters, a reconstruction by dilation is applied after each opening and a reconstruction by erosion is applied after each closing.

Figure 4.5 illustrates the result of an alternate sequential filter with and without reconstruction on a retina image. High gray levels correspond to high membership degrees to the vessels, but the representation is noisy. The noise suppression effect is clearly visible on the results after applying the filter. Only a few traces of vessels is visible on the result of the filter without reconstruction, while their details are recovered by the reconstruction steps. Note also that the contours of the preserved structures are not smoothed.

4.5.3 Conditioning and Fuzzy Geodesic Operators

Geodesic operators are useful when the result of a transformation has to be constrained to remain in a given set, or below a given function. They are also called geodesic operators, since they strongly rely on the geodesic distance in this set. Structuring elements are hence balls of this geodesic distance.

In the crisp case, the geodesic distance d_X between two points, conditionally to a set X, is the length of a shortest path between these two points which is included in X. The geodesic ball of center x and radius r conditionally to X is defined as:

$$B_X(x, r) = \{y \in X \mid d_X(x, y) \leq r\}. \tag{4.18}$$

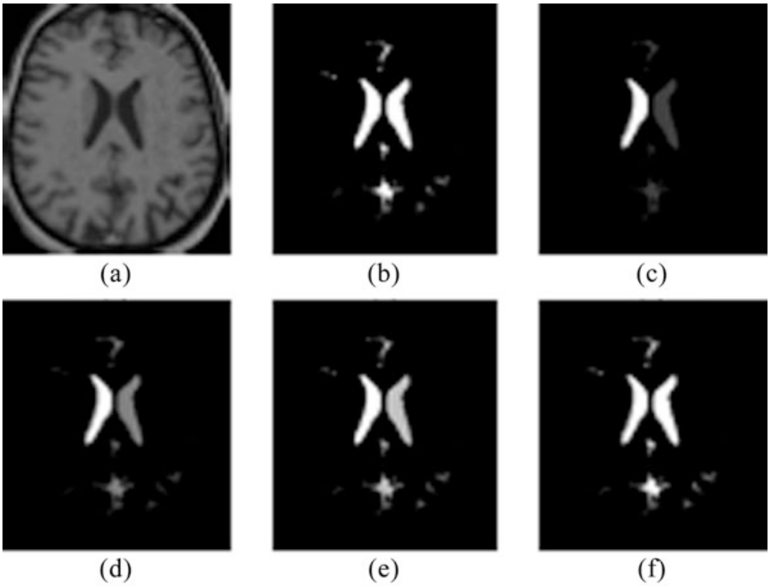

Fig. 4.4 Filtering based on hyperconnection [74]. (**a**) Slice of a brain MRI volume. (**b**) Upper estimation of the lateral ventricles based on a conjunctive fusion (see Chap. 5) of gray level and location information. (**c–f**) Filtering based on hyperconnection and a marker defined as one point in the right ventricle (on the left in the figure) with decreasing membership value. In (**c**) the right lateral ventricle is well distinguished from the rest of the image, as the region which is best connected to the marker, and the filter is less and less selective when the membership degree of the marker decreases

The geodesic dilation of Y in X (or dilation of Y conditionally to X) by a ball of radius r is defined as:

$$\delta_X(Y, B_r) = \{x \in \mathcal{S} \mid B_X(x, r) \cap Y \neq \emptyset\} = \{x \in \mathcal{S} \mid d_X(x, Y) \leq r\}. \tag{4.19}$$

The erosion is defined in a similar way, and opening and closing are defined by composition, as in the classical case.

In practice if $\mathcal{S}$ is a discrete space, Eq. 4.19 can be computed very easily from the classical dilation (using balls of the Euclidean distance as structuring elements) by the elementary structuring element B_1 (i.e., of radius 1):

$$\delta_X(Y, B_r) = [\delta(Y, B_1) \cap X]^r, \tag{4.20}$$

where the exponent means that the operation is iterated r times. If r tends towards infinity (in practice the sequence convergences in a finite number of iterations), then the result is the reconstruction of X from the markers Y. This operation is particularly useful for filtering operations, in order to preserve all details of the objects that are not completely suppressed by the filter. Similarly the reconstruction

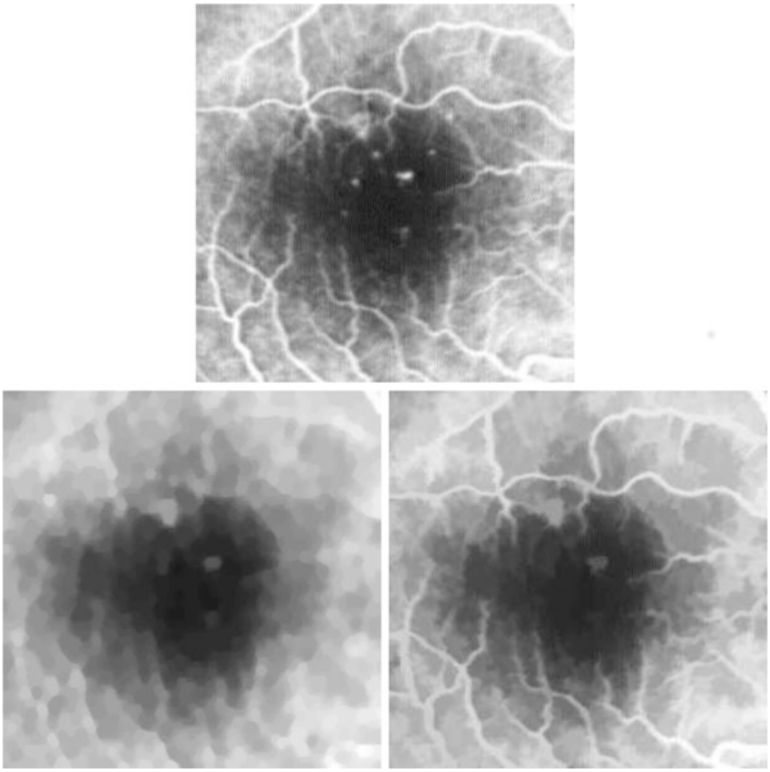

Fig. 4.5 Image of retina vessels on top (high gray levels represent high membership values to the vessels), and below its filtering using alternate sequential filter without reconstruction (left) and with reconstruction (right)

by erosion is defined as $\varepsilon_X(Y, b_r) = [\varepsilon(Y, B_1)) \cup X]^r$ for $r \to \infty$. These operations extend to functions.

The extension of these operations to fuzzy sets can be done in two ways. The first way consists in extending Eq. 4.20, using fuzzy dilation as defined in Sect. 4.3, and fuzzy intersection. An illustrative example is shown in Fig. 4.6 where the result of an opening of a retina image is used as a marker to reconstruct the image (using iterated geodesic dilations until convergence). The vessels are very well recovered after the reconstruction, while the filtering effect is preserved (more homogeneous regions and suppressions of small white objects). See also the example in Fig. 4.5, where a reconstruction step was applied after each opening and after each closing in the alternate sequential filter.

The second way is to fuzzify the geodesic distance itself. This is the approach proposed in [16]. The geodesic distance between two points x and y conditionally to a fuzzy set μ is defined as a crisp number, from the degree of connectedness between x and y, as defined in Eq. 3.44 (Chap. 3):

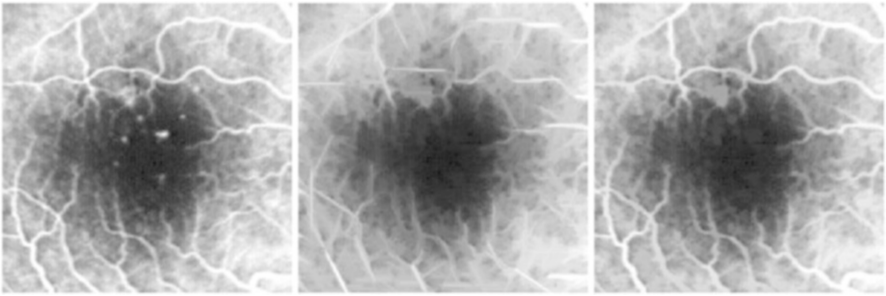

Fig. 4.6 Retina image, union of openings by segments in different directions, geodesic reconstruction

$$\frac{l(p^*(x, y))}{c_\mu(x, y)},$$

where $p*(x, y)$ denotes a path between x and y leading to the optimum in Eq. 3.44, and $l(.)$ denotes the length of a path. This is equivalent to the weighted geodesic distance (in the classical sense) computed in the α-cut of μ at level $\alpha = c_\mu(x, y)$ (in this cut, x and y belong to the same connected component). The geodesic distance can also be defined as a fuzzy number, taking into account the fact that, if the set is imprecisely defined, then geodesic distances in this set are imprecise too. One solution to achieve this aim is to use the extension principle, based on a combination of the geodesic distances computed on each α-cut of μ. Let us denote by $d_{\mu_\alpha}(x, y)$ the geodesic distance between x and y in the crisp set μ_α. Using the extension principle, we define the degree to which the geodesic distance between x and y in μ is equal to d as:

$$\forall d \in \mathbb{R}+,\ d_\mu(x, y)(d) = \sup\{\alpha \in [0, 1],\ d_{\mu_\alpha}(x, y) = d\}. \tag{4.21}$$

It is now possible to define a fuzzy geodesic ball $\beta_\mu(x, \rho)$ of center x and radius ρ, conditionally to μ, for a given geodesic distance. Intuitively, given that x is in μ to some degree, for each point y the value $\beta_\mu(x, \rho)(y)$ represents the fact that y belongs to μ to some degree and that it is at a geodesic distance in μ from x less than ρ. For that, $\beta_\mu(x, \rho)(y)$ is defined as a conjunction of three terms: the degree to which x belongs to μ, the degree to which y belongs to μ, and the degree $\delta(d_\mu(x, y) \leq \rho)$ to which $d_\mu(x, y) \leq \rho$, i.e.:

$$\forall y \in \mathcal{S},\ \beta_\mu(x, \rho)(y) = t(\mu(x), \mu(y), \delta(d_\mu(x, y) \leq \rho)), \tag{4.22}$$

where t is a t-norm. The term $\delta(d_\mu(x, y)$ can be defined as a crisp number or as a fuzzy number, using comparison of fuzzy numbers. See [16] for details on the properties of these definitions. Such fuzzy geodesic balls can then be used as structuring elements on fuzzy morphological operators, to define fuzzy geodesic morphological operators. Their properties are similar to the properties in the crisp case, as detailed in [16].

4.5.4 Fuzzy Skeleton and Skeleton by Influence Zones

This section is reproduced to a large extent from [24]. Representing an object by its skeleton is a widely addressed topic. It allows simplifying a shape and its description, while preserving its most relevant features. When shapes and objects are imprecisely defined, it is convenient to represent them as fuzzy sets, instead of crisp ones. Simplifying fuzzy objects then requires to extend definitions designed for the crisp case to the fuzzy case. Similarly, skeleton by influence zones (SKIZ), which structures the background of a set of objects, has to be extended when objects are imprecisely defined. This section contains a review of the main approaches for defining skeleton and skeleton by influence zones of fuzzy sets.

One of the main approaches to extend an operation to fuzzy sets is to apply this operation to α-cuts and then reconstruct the result by a combination of these α-cuts (see Chap. 2). This approach is well suited for increasing operations, assuring that their application on α-cuts provides sets that can be considered as α-cuts of a resulting fuzzy set. Unfortunately skeleton and SKIZ are not increasing, and therefore this approach cannot be applied directly. Other approaches have thus been developed.

Several problems have been addressed in the literature, exploiting fuzzy representations:

- Define the skeleton (as a crisp set) of a fuzzy set.
- Define the fuzzy skeleton (i.e., as a fuzzy set) of a fuzzy set.
- Define the fuzzy SKIZ of a set of fuzzy sets.

They will be reviewed in this section and categorized according to the type of approach they use. Note that fuzzy sets can also be used to represent some additional information on a classical skeleton. This will not be addressed here, since we focus only on fuzzy inputs, but as an example let us mention the work in [104], where a fuzzy measure of significance is computed on each branch of a classical skeleton, with applications to skeleton pruning.

Defining and computing the skeleton in the crisp case is a widely addressed topic. While there is a consensus on continuous approaches that provide the required properties (in particular representing semi-continuously a shape using thin and centered structures, respecting the topology of the initial shape, and allowing for its reconstruction),[3] several approaches have been defined in the discrete case, based on different principles, in order to address important issues directly caused by the digitization [68]. It is out of the scope of this book to review them, and the reader

[3] A detailed analysis of the properties of the skeleton of an open set in $\mathbb{R}^n$ can be found in [67].

may refer to [94], and recently [88], for details. The main approaches can be grouped as follows:

- Distance-based approaches, with the important notion of center of maximal balls (CMB), and related methods such as wavefront (or grass fire) propagation, ridges of the distance function, and minimal paths
- Morphological approaches for the computation of centers of maximal balls
- Morphological approaches based on homotopic thinning

The first two approaches correspond to the intuitive idea proposed initially by Blum [34],[4] where the grass-fire principle led to the definition of the skeleton as shock points at which wave-fronts intersect. These points are those that are equidistant to at least two boundary points, or equivalently the centers of maximal disks. The extensions to the fuzzy case of these two approaches are then equivalent to Blum's approach when restricted to crisp sets.

Approximate approaches in the continuous case rely on the Voronoï diagram of points sampled on the object contour or surface. The quality of this approximation can be estimated in the case of regular objects in the sense of mathematical morphology (i.e., sets equal to both their opening and closing by a disk or sphere of given radius [89]), see, e.g., [37]. Regularization methods [75] and surface reconstruction methods [4] have been proposed based on this approach. Here, we do not consider such approximations. However, the extended Voronoï diagram of shapes will be considered for defining the fuzzy SKIZ.

In the fuzzy case, we also expect the skeleton to well represent the input fuzzy shape, to be centered in the shape, to be thin enough (this notion being more complicated to define than in the crisp case). The function associating a skeleton to a fuzzy set should also be anti-extensive and idempotent, as in the crisp case. The question of homotopy is also a difficult one in the case of fuzzy sets. Finally as an extension, it is expected that the fuzzy definitions boil down to the crisp ones in the particular case where the membership functions take only values 0 and 1 (i.e., defining crisp sets).

The review in this section is organized according to the type of approach used to define the skeleton or SKIZ of fuzzy sets, extending the above mentioned approaches. Distance-based approaches are first reviewed, with results being either crisp sets or fuzzy ones. While several approaches define a crisp skeleton of a fuzzy object, fuzzy versions are interesting too, for, if an object is imprecisely defined, it is expected its skeleton is imprecise too. Approaches based on mathematical morphology are presented next, for defining centers of maximal balls, for extending the notion of thinning, mostly based on extension of thinning to gray scale images, and finally for defining fuzzy SKIZ. Surprisingly enough, the question of thinning of gray scale images has been addressed a long time ago, in the seventies (maybe one of the first papers is [63]), with applications to fuzzy skeleton in the early eighties.

[4] Blum's work was first presented in a conference in the 1960's, and formalized by Calabi also in the 1960s.

Very recently, a renewed interest in fuzzy skeleton using various approaches can be observed.

Distance-Based Approaches

In the crisp case, this approach leads to several almost equivalent definitions. The idea is to define a distance function inside the shape, the ridges of which build the skeleton. The skeleton points are also defined as the points through which no minimal path from any point to the shape boundary goes. According to the underlying distance on the spatial domain, maximal balls included in the shape are defined (i.e., which cannot be included in any other ball included in the shape), the centers of which build the skeleton (also called medial axis in this case). The set of centers of maximal balls is denoted $CMB(.)$ in the following. This approach can be computed efficiently according to various discrete distances [5, 35].

The ridge approach was used for fuzzy sets in [58], where the fuzzy sets were defined using a S-shape transformation of the original image. Note that this implicitly assumes that higher gray levels correspond to the expected skeleton. A ridge following algorithm was proposed, using weighted neighborhood, along with a post-processing to get a one pixel width skeleton. This approach provides a crisp result, which is thin, connected, and passes though the center of the regions with the highest membership values (which is not necessarily the same as the center of the support or of any α-cut).

The notion of maximal fuzzy disks (or more generally fuzzy balls) for a distance was proposed by several authors. In [76], the space $\mathcal{S}$ is endowed with a metric d. A fuzzy disk included in a fuzzy set μ and centered at x is defined as the following fuzzy set:

$$\forall y \in \mathcal{S}, g_x^\mu(y) = \inf_{z \mid d(x,z)=d(x,y)} \mu(z). \tag{4.23}$$

Note that the standard inclusion of fuzzy sets is used, expressed here as $\forall y \in \mathcal{S}, g_x^\mu(y) \leq \mu(y)$. The medial axis of μ is then defined as the set $CMB(\mu)$ of local maxima of g_x^μ (hence a crisp set), and the fuzzy medial axis transform is the set of fuzzy sets $\{g_x^\mu \mid x \in CMB(\mu)\}$.

Another definition of centers of fuzzy maximal balls was given in [64], based on the distance of a point to the thresholded fuzzy boundary of the initial fuzzy set.

An extension of the distance transform for fuzzy sets was defined in [87]. The idea is to weight the local distance between points by the membership of these points to the considered fuzzy set. For instance, the distance between two neighboring points x and y can be defined as $\max(\mu(x), \mu(y))||x - y||$, or $\frac{\mu(x)+\mu(y)}{2}||x - y||$. The length of a path is then classically defined by summing local weighted distances along the path, and the distance between two points is the length of a shortest path between these two points. Axial points, i.e., center of maximal balls, are then derived from this weighted distance in a usual way [86]. Fuzziness is then taken into account only as weighting factors, leading to distance values defined as classical (crisp) numbers, and then a classical approach is used for the next steps. A similar approach using a discrete distance was developed in [97]. Then, based on the reverse distance transform, the centers of maximal fuzzy balls can be obtained to define the skeleton [98], which is crisp. A similar approach was developed in [60, 61] with further selection, filtering, and refinement steps to obtain a thin crisp skeleton.

Another weighted approach was proposed in the early work published in [63], for gray-level images (but could be used for fuzzy sets as well). The length of an arc is defined by a weighted sum (or integral in the continuous case). Minimum paths from each point to the contour of the support of the initial function are then computed. The skeleton is then formed by the points that do not belong to the minimum path of another point.

All these approaches are equivalent to the crisp CMB approach if the input is crisp. However, the final result ignores the imprecision of the input set, providing a crisp result. One may consider that an important information is then lost, and that the result is an over-simplified representation. When transposed to the discrete case, CMB approaches suffer from the same limitations in terms of topology preservation as their crisp versions.

Morphological Approaches to Compute the Centers of Maximal Balls

Centers of maximal balls can be obtained using mathematical morphology operations [89, 90], in particular erosion (denoted by ε) and opening (denoted by γ). The idea is that the center of a maximal ball of radius ρ is given by the set difference between the erosion of the set by a structuring element of size ρ and the opening (by the smallest possible structuring element) of this erosion. Formally, for a crisp set A, the skeleton is given by:

$$r(A) = \bigcup_{\rho > 0} s_\rho(A), \tag{4.24}$$

where $s_\rho(A)$ is the set of centers of maximal balls of radius ρ included in A, given by:

$$s_\rho(A) = \bigcap_{\mu > 0} (\varepsilon_{B_\rho}(A) \setminus \gamma_{\bar{B}_\mu}(\varepsilon_{B_\rho}(A))),$$

where B_ρ (respectively, $\bar{B}_\rho$) denotes the open (respectively, closed) ball of radius ρ. This definition is shown to have good properties for open sets [67]. In particular, the original set can be reconstructed from the skeleton by dilating all the s_ρ:

$$A = \bigcup_{\rho > 0} \delta_{B_\rho}(s_\rho(A)).$$

In the discrete case, this definition of the skeleton is transposed as:

$$S(A) = \bigcup_{n \in \mathbb{N}} (\varepsilon_{B_n}(A) \setminus \gamma_B(\varepsilon_{B_n}(A))), \tag{4.25}$$

where B is the elementary structuring element on the digital grid (i.e., of radius one), and ε_{B_0} (erosion by a structuring element of radius 0) is the identity mapping. The set difference is computed as usual as $X \setminus Y = X \cap Y^C$ with extensions of intersection and complementation to fuzzy sets as defined in Chap. 2. While the reconstruction property is preserved, the homotopy property is not, and therefore a disconnected skeleton of a connected object may be obtained [68].

This type of construction was used in [78] for gray scale images and could also be used for fuzzy sets. The idea of shrinking and expanding used in the construction of the skeleton is replaced by local min and max operators (hence corresponding to erosions and dilations). This was used in [102] for approximating an image.

Based on a similar idea, we propose to extend Eq. 4.25 to fuzzy sets using fuzzy mathematical morphology, leading to the definition of the fuzzy skeleton of a fuzzy set in the discrete case as:

$$FS(\mu) = \bigvee_{n \in \mathbb{N}} (\varepsilon_{B_n}(\mu) \setminus \gamma_B(\varepsilon_{B_n}(\mu))), \tag{4.26}$$

where the supremum $\bigvee$ is the fuzzy union. The continuous formulation extends in a similar way, and the same definition applies regardless of the dimension of the space, as in the crisp case. Note that instead of a crisp structuring element, a fuzzy one could be used, for instance, representing the smallest spatial unit, given the intrinsic imprecision in the image. Figure 4.7 illustrates this new definition and shows that the skeleton is fuzzy, representing the fact that regions, even on crest lines of the membership function, may belong to the fuzzy set with a low degree, as also do the corresponding parts of the skeleton.

Morphological Thinning

In the crisp case, a common way to overcome the homotopy problems raised by a direct computation of centers of maximal balls on digital images consists in applying iterative thinning to the input shape, while guaranteeing the preservation of topology at each step, until convergence, leading to an homotopic skeleton [89]. The structuring elements used in this process to delete the so-called simple points (i.e., whose deletion does not change the topology) are illustrated in Fig. 4.8: simple points are the centers of these structuring elements and are removed from the object by the thinning operation. The 3D case was also addressed, e.g., in [10, 85].

This approach was extended to gray-level images in several ways. Note that these extensions directly apply to fuzzy sets if gray levels are bounded and can be matched monotonically to membership degrees such that high gray levels correspond to high

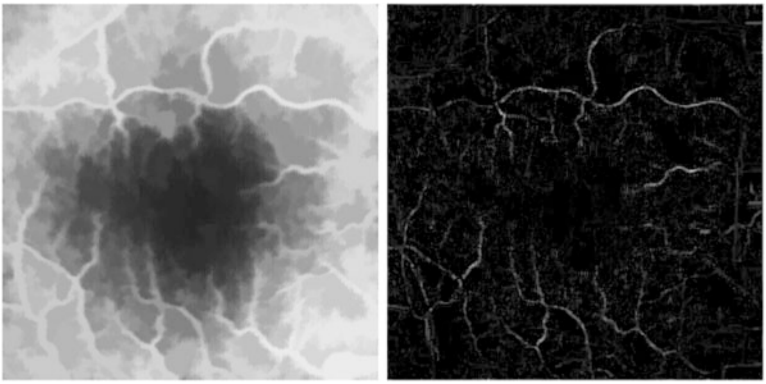

Fig. 4.7 Left: original fuzzy set, derived from an eye vessel images. High gray levels indicate high membership degrees to the blood vessels. Right: fuzzy skeleton obtained using Eq. 4.26, where membership degrees range from 0 (black) to 1 (white)

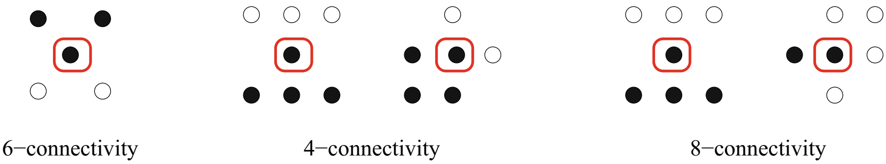

Fig. 4.8 Structuring elements used for homotopic thinning on a hexagonal grid (6-connectivity), and on a square grid (4-connectivity or 8-connectivity). Black points represent object points while white points correspond to background points. The center of the structuring elements is circled. All rotations of these structuring elements have to be used sequentially and iteratively to get the skeleton

membership degrees (or the reverse). Historically, a first approach was introduced in [52], based on a weighted connectedness defined as follows: two points x and y are connected if there exists a path from x to y that does not go through a point with higher value. Then thinning of a function f is performed by replacing $f(x)$ by $\min_{y \in V(x)} f(x)$, where $V(x)$ denotes a neighborhood of x (i.e., this corresponds to an erosion), only if this change does not disconnect any pair of points in $V(x)$.

Still among the early works, a similar approach was proposed in [53], were local min and max operators are used, referring to fuzzy logic, with structuring elements adapted to the type of thinning (for instance, a 3×3 neighborhood except the center to suppress pepper and salt noise, or oriented neighborhoods to compute the skeleton, similarly as in Fig. 4.8).

While thinning is usually defined as the set difference between the original image and the result of a hit-or-miss transformation with appropriate structuring elements, an interesting formulation in [94] allows for a direct extension to functions. Let B_1 be the part of the structuring element composed of object points (the black points in Fig. 4.8) and B_2 the other part of the structuring element. For thinning, the origin belongs to B_1. The thinning of a function f by $B = (B_1, B_2)$ is expressed as:

$$\forall x \in \mathcal{S}, (f \circ B)(x) = \begin{cases} \delta_{B_2}(f)(x) & \text{if } \varepsilon_{B_1}(f)(x) = f(x) \text{ and } \delta_{B_2}(f)(x) < f(x) \\ f(x) & \text{otherwise .} \end{cases} \quad (4.27)$$

If f is a binary image, then this definition reduces to the classical one.

Another extension was proposed in [38], where topological operators for gray-level images were defined. Thinning and skeleton are based on the notion of destructible point: a point x is destructible for a function f if it is simple (according to the binary case) for the threshold $f_k = \{y \mid f(y) \geq k\}$ with $k = f(x)$. Then the topology is not modified if $f(x)$ is replaced by $f(x) - 1$ (assuming that f takes values in a subset of $\mathbb{N}$). The skeleton of f is then obtained by reducing iteratively the value of destructible points that are not end points (x is an end-point if $CC(V(x) \setminus \{x\} \cap f_k) = 1$, with $k = f(x)$ and $CC(A)$ the number of connected components of A, for a given discrete connectivity).

Since thinning is often derived from hit-or-miss transformation (HMT), let us finally mention some extensions of this transformation to gray scale images. In [81], a function is expressed as the supremum of the impulse functions below it, and a similar construction is used to define HMT, based on interval operators (an interval operator by (A, B) is equivalent to the HMT by (A, B^c) in the crisp case). A related approach, although somewhat different, is based on probing, as introduced in [9]. In [95], the cardinality of the set of gray levels such that a point belongs to the HMT of thresholds is considered. All these works have been nicely unified in [72] in the framework of interval operators. The idea is to decompose the HMT process into a fitting step and an evaluation step (which can be done using a supremum as in [81], a sum as in [95], any other set measure, or using a newly proposed binary valuation). The authors show the links between the different approaches and the power of the unified framework. All definitions reduce to the classical ones if the functions are binary (i.e., sets).

Fuzzy Skeleton of Influence Zones

The notions of Voronoï diagram, generalized Voronoï diagram or skeleton by influence zones (SKIZ), define regions of space which are closer to a region or object than to the other ones and have important properties and applications [89, 94]. If knowledge or information is modeled using fuzzy sets, it is natural to see the influence zones of these sets as fuzzy sets too. The extension of these notions to the fuzzy case is therefore important, for applications such as partitioning the space where fuzzy sets are defined, implementing the notion of separation, reasoning on fuzzy sets (fusion, interpolation, negotiations, spatial reasoning on fuzzy regions of space, etc.). Despite these potential applications, surprisingly enough, so far such an extension has been little developed. This section summarizes the approach in [21], based on mathematical morphology.

In the crisp case, for a set X composed of several connected components ($X = \bigcup_i X_i$, with $X_i \cap X_j = \emptyset$ for $i \neq j$), the influence zone of X_i, denoted by $IZ(X_i)$, is defined as the set of points which are strictly closer to X_i than to X_j for $j \neq i$, according to a distance d defined on $\mathcal{S}$ (usually the Euclidean distance or a discrete version of it on digital spaces), that is:

$$IZ(X_i) = \{x \in \mathcal{S} \mid d(x, X_i) < d(x, X \setminus X_i)\}. \tag{4.28}$$

The SKIZ of X, denoted by $SKIZ(X)$, is the set of points which belong to none of the influence zones, i.e., which are equidistant of at least two components X_i:

$$SKIZ(X) = (\bigcup_i IZ(X_i))^c. \tag{4.29}$$

The SKIZ is also called generalized Voronoï diagram. Note that the SKIZ is a subset of the morphological skeleton of X^c (i.e., the set of centers of maximal balls included in X^c where X^c denotes the complement of X in $\mathcal{S}$) [89, 94]. It is not necessarily connected and contains in general less branches than the skeleton of X^c (this may be exploited in a number of applications).

An important and useful property of this definition based on distance (Eq. 4.28) is that it can be expressed in terms of morphological operations as well. Indeed, denoting by δ_λ the dilation by a ball of radius λ, and ε_λ the erosion by a ball of radius λ, the influence zones can be expressed as [11]:

$$IZ(X_i) = \bigcup_\lambda \left(\delta_\lambda(X_i) \cap \varepsilon_\lambda((\cup_{j\neq i} X_j)^c)\right) = \bigcup_\lambda \left(\delta_\lambda(X_i) \setminus \delta_\lambda(\cup_{j\neq i} X_j)\right). \tag{4.30}$$

Another link between SKIZ and distance can be expressed, by involving the watersheds (WS) [94]:

$$SKIZ(X) = WS(d(y, X), y \in X^c). \tag{4.31}$$

Let us now extend these definitions and properties to fuzzy sets. For the sake of clarity, we assume two fuzzy sets, with membership functions μ_1 and μ_2 defined on $\mathcal{S}$. The extension to an arbitrary number of fuzzy sets is straightforward. Fuzzy dilations and erosions are defined using a t-norm t and a fuzzy implication I, respectively, and we choose here the dual definitions of these operations, with respect to a complementation c (see Sect. 4.3).

Let us first consider the expression of influence zone using morphological dilations (Eq. 4.30). This expression can be extended to fuzzy sets by using fuzzy intersection and union, and fuzzy mathematical morphology. For a given elementary structuring element ν, the influence zone of μ_1 is defined as:

$$IZ_{dil}(\mu_1) = \bigcup_\lambda \left(\delta_{\lambda\nu}(\mu_1) \cap \varepsilon_{\lambda\nu}(\mu_2^c)\right) = \bigcup_\lambda \left(\delta_{\lambda\nu}(\mu_1) \setminus \delta_{\lambda\nu}(\mu_2)\right). \tag{4.32}$$

The influence zone for μ_2 is defined in a similar way. The extension to any number of fuzzy sets μ_i is straightforward:

$$IZ_{dil}(\mu_i) = \bigcup_{\lambda} \left(\delta_{\lambda\nu}(\mu_i) \cap \varepsilon_{\lambda\nu}((\cup_{j\neq i}\mu_j)^c)\right). \tag{4.33}$$

In these equations, intersection and union of fuzzy sets are implemented as t-norms t and t-conorms T (min and max, for instance). The fuzzy complementation used in the following is always $c(a) = 1 - a$, but other forms could be employed as well. Equation 4.32 then becomes:

$$IZ_{dil}(\mu_1) = \sup_{\lambda} t(\delta_{\lambda\nu}(\mu_1), 1 - \delta_{\lambda\nu}(\mu_2)). \tag{4.34}$$

In the continuous case, if ν denotes the elementary structuring element of size 1, then $\lambda\nu$ denotes the corresponding structuring element of size λ (for instance, is ν is a ball of some distance of radius 1, then $\lambda\nu$ is the ball of radius λ). In the digital case, the operations performed using $\lambda\nu$ as structuring elements (λ being an integer in this case) are simply the iterations of λ operations performed with ν (iterativity property of fuzzy erosion and dilation [27]). Note that the number of dilations to be performed to compute influence zones in a digital bounded space $\mathcal{S}$ is always finite (and bounded by the length of the largest diagonal of $\mathcal{S}$).

The fuzzy SKIZ is then defined as:

$$\text{SKIZ}(\cup_i \mu_i) = (\bigcup_i IZ(\mu_i))^c. \tag{4.35}$$

This expression also defines a fuzzy (generalized) Voronoï diagram.

Another approach consists in extending the definition in terms of distances (Eq. 4.28) and defining a degree to which the distance to one of the sets is lower than the distance to the other sets. Several definitions of the distance of a point to a fuzzy set have been proposed in the literature. Some of them provide real numbers and Eq. 4.28 can then be applied directly. But then the information on the imprecision in the object definition is lost (the problem is the same as when using a weighted distance for computing CMB, as mentioned before). Definitions providing fuzzy numbers are therefore more interesting, since if the sets are imprecise, it may be expected that distances are imprecise too, as also underlined, e.g., in [14, 49, 84]. In particular, as it will be seen next, it may be meaningful to use the distance proposed in [14], based on fuzzy dilation (see also Chap. 6 for distances between fuzzy sets):

$$d(x, \mu)(n) = t(\delta_{n\nu}(\mu)(x), 1 - \delta_{(n-1)\nu}(\mu)(x)). \tag{4.36}$$

It expresses, in the digital case, the degree to which x is at a distance n of μ (t is a t-norm, and $n \in \mathbb{N}^*$). For $n = 0$, the degree becomes $d(x, \mu)(0) = \mu(x)$. This expression can be generalized to the continuous case as:

$$d(x, \mu)(\lambda) = \inf_{\lambda' < \lambda} t(\delta_{\lambda\nu}(\mu)(x), 1 - \delta_{\lambda'\nu}(\mu)(x)), \tag{4.37}$$

where $\lambda \in \mathbb{R}^{+*}$, and $d(x, \mu)(0) = \mu(x)$.

When distances are fuzzy numbers, the fact that $d(x, \mu_1)$ is lower than $d(x, \mu_2)$ becomes a matter of degree. The degree to which this relation is satisfied is obtained using methods for comparing fuzzy numbers (see, e.g., [103]). For example, based on [47], the degree $\mu(d_1 < d_2)$ to which $d_1 < d_2$, d_1 and d_2 being two fuzzy numbers, using the extension principle [105] is defined as:

$$\mu(d_1 < d_2) = \sup_{a<b} \min(d_1(a), d_2(b)). \tag{4.38}$$

The influence zone of μ_1 based on the comparison of fuzzy numbers (using Eq. 4.38) is defined as:

$$\begin{aligned} IZ_{dist1}(\mu_1)(x) &= \mu(d(x, \mu_1) < d(x, \mu_2)) \\ &= \sup_{n<n'} \min(d(x, \mu_1)(n), d(x, \mu_2)(n')). \end{aligned} \tag{4.39}$$

Note that this approach can be applied regardless of the definition of fuzzy distances chosen.

When distances are more specifically derived from a dilation, as the ones in Eqs. 4.36 and 4.37, a more direct approach takes into account explicitly this link between distances and dilations. Indeed, in the binary case, the following equivalences hold:

$$\begin{aligned} (d(x, X_1) \leq d(x, X_2)) &\Leftrightarrow (\forall\lambda, x \in \delta_\lambda(X_2) \Rightarrow x \in \delta_\lambda(X_1)) \\ &\Leftrightarrow (\forall\lambda, x \in \delta_\lambda(X_1) \vee x \notin \delta_\lambda(X_2)). \end{aligned} \tag{4.40}$$

This means that if x is closer to X_1 than to X_2, x is reached faster by dilating X_1 than by dilating X_2. This expression extends to the fuzzy case as follows. The degree $\mu(d(x, \mu_1) \leq d(x, \mu_2))$ to which $d(x, \mu_1)$ is less than $d(x, \mu_2)$ is defined as:

$$\mu(d(x, \mu_1) \leq d(x, \mu_2)) = \inf_{\lambda} T(\delta_{\lambda\nu}(\mu_1)(x), 1 - \delta_{\lambda\nu}(\mu_2)(x)), \tag{4.41}$$

where T is a t-conorm (fuzzy disjunction). A residual implication can be used as well. This equation also defines a way to compare fuzzy numbers representing distances.

Defining influence zones requires a strict inequality between distances, which is derived by complementation:

$$\mu(d(x, \mu_1) < d(x, \mu_2)) = 1 - \mu(d(x, \mu_2) \leq d(x, \mu_1)). \tag{4.42}$$

The influence zone of μ_1 is then defined as:

$$IZ_{dist2}(\mu_1)(x) = 1 - \inf_{\lambda} T(\delta_{\lambda\nu}(\mu_2)(x), 1 - \delta_{\lambda\nu}(\mu_1)(x)). \tag{4.43}$$

Regardless of the definition of IZ, the SKIZ is always defined by Eq. 4.35.

Comparison and properties are detailed in [21]. In particular the definitions derived from the dilation approach and from the direct distance approach coincide: $IZ_{dil}(\mu_1) = IZ_{dist2}(\mu_1)$. However, the two distance-based approaches are not equivalent, since they rely on different orderings between fuzzy sets. Actually the direct approach always provides a larger result: $\forall x \in \mathcal{S}, IZ_{dist1}(\mu_1)(x) \leq IZ_{dist2}(\mu_1)(x)$. In terms of complexity, the direct approach is computationally less expensive. Another important property is the consistency with the crisp case, as generally required when extending an operation on crisp sets to fuzzy sets. Finally, the SKIZ is symmetrical with respect to the μ_i, hence independent of their order.

Illustrative examples can be found in [21].

Discussion

In this section, the main approaches for fuzzy skeleton and fuzzy SKIZ have been reviewed. Some were designed specifically for fuzzy sets, while others were developed for gray-level images, but can be used for fuzzy sets as well, as soon as the gray-level scale is bounded (and then isomorphic to [0, 1]). Semantics has to be considered with care to guarantee that the transformation from gray levels to membership degrees has a suitable interpretation in terms of fuzzy sets and gradual membership.

Among these definitions, some provide crisp results, while other provide fuzzy results, which may be more convenient to keep track of the spatial imprecision of the input. This may be even more intuitive. For instance, if an object has imprecise boundaries, then it is expected that points at equal distance of two or more boundary points would not be precisely located either. Similarly, thinning methods that require to define both a shape and its complement have to account for the fact that the transition is not crisply defined. Concerning semantics, this may depend on the semantics of the fuzziness in the initial object, which may represent the observation of an intrinsically imprecise object, an imprecise object representation due to sensor limitations, an object suffering from imprecision during a detection or segmentation process, a preferred region of space, etc.

Therefore a hint for future work would be to extend distance-based approaches to provide fuzzy skeletons. This could be done by replacing the weighted distance transforms by distances taking values defined as fuzzy numbers (see, e.g., [14, 19] for such distances).

Another aspect that deserves to be further explored is a deeper analysis of the properties of the various definitions, in particular fuzzy homotopy and its preservation (up to some degree if fuzziness if kept), measurement of the thinness of a fuzzy skeleton and of how central it is in the input fuzzy set, and reconstruction capabilities. At theoretical level, this would allow for an easier comparison between different approaches.

At a more practical level, besides efficient implementations, real world applications would deserve to be more developed, to demonstrate the importance of fuzzy versions of skeleton and SKIZ.

4.5.5 Fuzzy Median, Application to Interpolation Between Fuzzy Sets

In the mathematical morphology community, two types of approaches have been considered to define the median set of two crisp sets, or to interpolate between two sets. The first approach relies on the SKIZ [11, 100], while the second approach relies on the notion of geodesics of some distance [36, 55, 59, 91, 93]. The first approach is extended here to the case of fuzzy sets, based on the definitions of the fuzzy SKIZ described in Sect. 4.5.4. This section is based on [21].

The morphological definition of the median set from the SKIZ applies originally to sets X and Y having a non-empty intersection. The SKIZ is computed for $X_1 = X \cap Y$ and $X_2 = (X \cup Y)^c$. The median set of X and Y is defined as the influence zone of $X_1 = X \cap Y$, i.e., as the set of points which are closer to $X \cap Y$ than to the complement of $X \cup Y$.

Let us consider two fuzzy objects with membership functions μ_1 and μ_2 and with non-empty intersection of their supports. Depending on the definition adopted for the influence zones, two definitions can be given for the fuzzy median set.

Based on the definition of influence zones from dilations, or equivalently the direct approach from distances, the median fuzzy set of μ_1 and μ_2 is defined as the influence zone of $\mu_1 \cap \mu_2$ with respect to $c(\mu_1 \cup \mu_2)$ (intersection is still defined by a t-norm t and union by a t-conorm T):

$$\begin{aligned}\forall x \in \mathcal{S}, M(\mu_1, \mu_2)(x) &= \sup_\lambda t(\delta_{\lambda\nu}(\mu_1 \cap \mu_2)(x), 1 - \delta_{\lambda\nu}(c(\mu_1 \cup \mu_2))(x)) \\ &= \sup_\lambda t(\delta_{\lambda\nu}(\mu_1 \cap \mu_2)(x), \varepsilon_{\lambda\nu}(\mu_1 \cup \mu_2)(x)). \qquad (4.44)\end{aligned}$$

By using the definition of influence zones based on comparison of fuzzy distances, the median set is defined as:

$$\forall x \in \mathcal{S}, M'(\mu_1, \mu_2)(x) = \sup_{n<n'} \min(d(x, \mu_1 \cap \mu_2)(n), d(x, c(\mu_1 \cup \mu_2))(n')). \qquad (4.45)$$

As shown in [21], for any two fuzzy sets μ_1 and μ_2, the following inequality holds:

$$\forall x \in \mathcal{S}, M'(\mu_1, \mu_2)(x) \leq M(\mu_1, \mu_2)(x). \tag{4.46}$$

Several other properties are detailed in [21]. In particular,

- if the origin belongs to the structuring element ν with a membership value equal to 1 or if ν represents a reflexive relation (which is the condition to have extensive dilations), then the median set is included in the union of the two objects;
- under the same condition ($\nu(0) = 1$), the cores verify the following inclusion relations:

$$Core(\mu_1 \cap \mu_2) \subseteq Core(M(\mu_1, \mu_2)) \subseteq Core(\mu_1 \cup \mu_2); \tag{4.47}$$

- if additionally the origin is the only modal value of ν ($\nu(0) = 1$ and $\forall x \in \mathcal{S} \setminus \{0\}, \nu(x) < 1$), then the median set and the union of the two sets have the same support:

$$Supp(M(\mu_1, \mu_2)) = Supp(\mu_1 \cup \mu_2), \tag{4.48}$$

  and the cores of the median set and of the intersection are equal:

$$Core(\mu_1 \cap \mu_2) = Core(M(\mu_1, \mu_2)). \tag{4.49}$$

The median set can be exploited to derive a series of interpolating sets between μ_1 and μ_2, by applying recursively the median computation in a dichotomy process. More precisely, for μ_1 and μ_2 two fuzzy sets, a series of interpolating sets is defined by recursive application of the median computation:

$$\begin{aligned}
Interp(\mu_1, \mu_2)_{1/2} &= M(\mu_1, \mu_2) \\
Interp(\mu_1, \mu_2)_{1/4} &= M(Interp(\mu_1, \mu_2)_{1/2}, \mu_1) \\
Interp(\mu_1, \mu_2)_{1/8} &= M(Interp(\mu_1, \mu_2)_{1/4}, \mu_1) \\
Interp(\mu_1, \mu_2)_{3/8} &= M(Interp(\mu_1, \mu_2)_{1/4}, Interp(\mu_1, \mu_2)_{1/2}) \\
&\dots \\
Interp(\mu_1, \mu_2)_{3/4} &= M(Interp(\mu_1, \mu_2)_{1/2}, \mu_2) \\
&\dots
\end{aligned}$$

This process can be written in a general recursive form as:

$$Interp_0 = \mu_1$$
$$Interp_1 = \mu_2$$
$$Interp_{\frac{i+j}{2}} = M(Interp_i, Interp_j) \text{ for } 0 \le i \le 1, 0 \le j \le 1.$$

This sequence allows transforming progressively μ_1 into μ_2. These two fuzzy sets can represent spatial objects, different situations, sets of constraints or preferences, etc. For instance, the sequence allows building intermediate estimates between distant observations or pieces of information.

Let us provide an example on real objects, from medical images. We consider the putamen (a brain structure) in different subjects, obtained from the IBSR database.[5] The images are registered, which guarantees a good correspondence between the different instances. Fuzziness at the boundary of the objects is introduced to represent spatial imprecision due to partial volume effect or imprecise segmentation, using a fuzzy dilation. Four examples of the resulting fuzzy objects are illustrated in Fig. 4.9. The fuzzy median set has been computed between the two first instances, then between this result and the third instance, etc. Using this iterative approach, the fuzzy median set between the 18 instances of this structure has been computed (corresponding to the 18 normal subjects of the IBSR database). Finally, the fuzzy median sets for four structures (thalamus and putamen in both hemispheres) on the 18 normal subjects of the database have been computed. Such results could be used, for instance, for representing the inter-individual variability, or to build anatomical atlases.

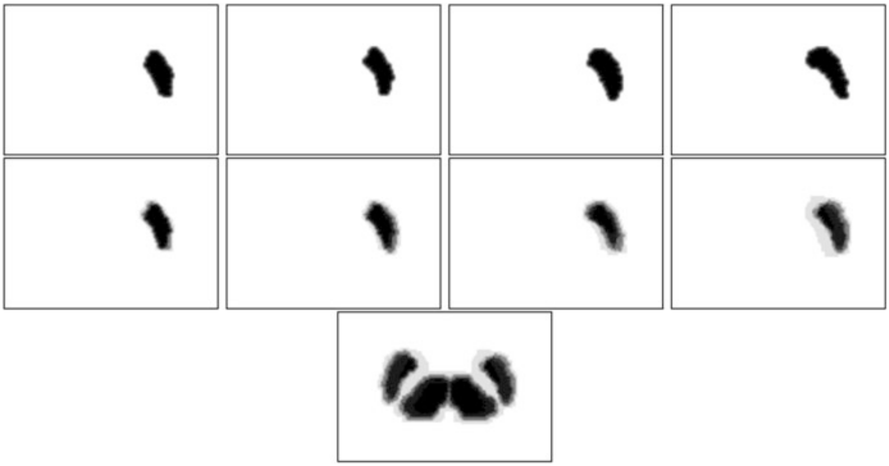

Fig. 4.9 Illustration of the fuzzy median set. First row: four instance of a brain structure. Second row: median set between two, three, four of the instances (shown in the first row) and between the 18 instances of the IBSR database. Third row: median sets between the 18 instances of the IBSR database for four structures (thalamus and putamen in both hemispheres)

[5] https://www.nitrc.org/projects/ibsr.

Let us briefly comment possible extensions of this approach to non-intersecting objects. In the spatial domain, the approach proposed in [100] in the case of crisp objects can be easily extended. It consists in performing a translation of the objects such that the translated objects intersect each other. The median set is then computed on the resulting translated sets, leading to a result that is a median in terms of shape and located midway between the two original objects. This approach is intuitively very satisfactory and can be extended directly to the case of fuzzy objects. It can be applied as soon as the underlying space $\mathcal{S}$ is endowed with an affine structure (as the usual spatial domain), in order to define translations.

4.5.6 Extensions

Since mathematical morphology is a powerful framework for image analysis, image understanding and spatial reasoning, its extension to fuzzy sets allows dealing with imprecision, as shown in this chapter. Moreover, it can be defined on different types of complete lattices, such as L-fuzzy sets [54] and bipolar fuzzy sets to handle both positive and negative information [23, 50, 51], graphs [39, 69, 99, 101] and hypergraphs [25, 32] to deal with structural information, formal concept analysis [6, 7, 46], logics [1–3, 18, 26, 30, 33], thus extending the scope of the potential formalizations and applications.

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Chapter 5
Fusion

Fusion has emerged as an important aspect of information processing in several very different fields. This chapter addresses the question of information fusion under imprecision, in the domain of image processing and understanding, using fuzzy methods. The chapter is largely inspired by the relevant chapters of [4]. A more general presentation of information fusion can be found in [6].

5.1 Definitions

In this book, the word "information" is used in a broad sense. In particular, it covers both data (for example, measurements, images, signals, etc.) and knowledge (regarding the data, the subject, the constraints, etc.) that can be either generic or specific.

In the context of this book, fusion is defined as follows: *fusion of information consists in combining information originating from several sources in order to improve decision making.* For each type of problem and application, this definition can be made more specific by answering a number of questions, such as: what is the objective of the fusion? what is the information we wish to fuse? where does it come from? what are its characteristics (uncertainty, relation between the different pieces of information, generic or factual, static or dynamic, etc.)? what methodology should be used? how can methods and results be evaluated? what are the major difficulties, the limits? etc. Note that in the case of images, fusion goes far beyond a simple image registration or alignment, which may be only preliminary steps to achieve a good spatial correspondence between the pieces of information to be combined. In the case of images, decision can cover various objectives, at different levels, such as detection of an object, segmentation, recognition or classification, change detection, updating or revising knowledge about a phenomenon based on

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several observations, etc. The need for fusion can occur in different types of situations:

- **Fusion of several images from the same sensor.** This consists, for example, of several channels of the same satellite acquisition device, or multi-echo images in magnetic resonance imaging (MRI), or also of image sequences for scenes in motion. In each of these cases, the data are relatively homogeneous because they correspond to similar physical measurements.
- **Fusion of several images from different sensors.** This is the most common case, in which the different physical principles of each sensor allow the user to have complementary perspectives of the scene. They can consist of ERS and SPOT images, MRI or ultrasound images, etc. The heterogeneity is then much greater, since the various sensors do not capture the same aspects of the phenomenon. Each image gives a partial observation with no information on the characteristics they are not meant to observe (for example, an anatomical MRI yields no functional information and the definition of a PET scan is too low for a precise view of the anatomy).
- **Fusion of several elements of information extracted from a same image.** In this situation, different types of information are extracted from an image using several types of processing, operators, classifiers, etc. that rely on different characteristics of the data and attempt to extract different objects, often leading to very heterogeneous elements of information to fuse. The extracted information can involve the same object (e.g., fusion of contour detectors) or different objects and the goal is then to find an overall interpretation of the scene and consistency between the objects. The elements of information can be on different levels (very local, or more structural when studying spatial relations between objects).
- **Fusion of images and other sources of information.** Other sources of information about the content of an image may include a model, which may be particular like a map, or generic like an anatomy atlas, a knowledge base, rules, information provided by experts, etc. The elements of information are again in very different formats, both in nature and in their initial representation (images in the case of a map or a digital atlas, but also linguistic descriptions, databases, etc.).

Let us now briefly describe the general characteristics of the information we wish to fuse, characteristics that have to be taken into account in a fusion process.

A first characteristic involves the type of information we wish to fuse. It can consist of direct observations, results obtained after processing these observations, more generic knowledge, expressed in the form of rules, for example, or opinions of experts. This information can be expressed either in numerical or symbolic form, and the fuzzy set setting is appropriate to manage both forms. The different levels of the elements of information we wish to fuse are also a very important aspect. Usually, the lower level (typically the original measurements) is distinguished from a higher level requiring preliminary steps, such as processing, extracting primitives or structuring the information. Depending on the level, the constraints as well as the difficulties of fusion can vary, especially in image fusion.

One of the important characteristics of information in fusion is its imperfection, which is always present (fusion would otherwise not be necessary). It can take different forms, which are briefly described below. Let us note that there is not always a consensus on the definition of these concepts in other works.

- **Uncertainty** is related to the truth of an element of information and characterizes the degree to which it conforms with reality [8]. It refers to the nature of the object or fact involved, to its quality, its essence, or its occurrence.
- **Imprecision** involves the content of the information and therefore is a measurement of a quantitative lack of knowledge on a measurement [8]. It involves the lack of accuracy in quantity, size, time, the lack of definition of a concept which is open to different interpretations or with vague and ill-defined contours. This concept is often confused with uncertainty because both these imperfections can be present at the same time and one can cause the other. It is important to be able to tell the difference between these two terms because they are often antagonistic, even if they can be included in a broader meaning for uncertainty. Other classifications with a larger number of categories have been suggested [11].
- **Incompleteness** characterizes the absence of information. The information provided by each source is usually partial, i.e., it only provides one vision of the world or the phenomenon we are observing, by only pointing out certain characteristics. Incompleteness of the information originating from each source is the main reason for fusion.
- **Ambiguity** expresses the possibility for a piece of information to fit more than one interpretation. It can be caused by previous imperfections, for example, an imprecise measure that does not make it possible to distinguish two situations, or the incompleteness that causes possible confusion between objects and situations that cannot be separated based on the characteristics exposed by the source. One of the objectives of fusion is to reduce or even eliminate the ambiguities of a source using the information provided by the other sources or additional knowledge.
- **Conflict** characterizes two or more elements of information leading to contradictory and therefore incompatible interpretations. Conflict situations are common in fusion problems and are often difficult to solve. First of all, detecting conflicts is not always simple. They can easily be confused with other types of imperfections, or even with the complementarity of sources. Furthermore, identifying and classifying them are questions that often arise, but in different ways depending on the field. Finally, solutions come in different forms, including the elimination of unreliable sources, taking into account additional information, etc. In some cases, it may be preferable to delay the combination and wait for other elements of information that might solve the conflicts, or even not go through with the fusion at all.

There are other, more positive characteristics of information that can be used to limit the imperfections.

- **Redundancy** is the quality of a source that provides the same information several times. Redundancy among sources is often observed, since the sources provide information about the same phenomenon. Ideally, redundancy is used to reduce uncertainties and imprecisions.
- **Complementarity** is the property of sources that provide information on different variables. It comes from the fact that they usually do not provide information about the same characteristics of the observed phenomenon. It is directly used in the fusion process in order to obtain more complete overall information and to erase ambiguities.

Finally, the spatial nature of information, specific to images, deserves particular attention. Its introduction in fusion methods, often inspired by other fields lacking this spatial nature, is not immediate and yet necessary in order to ensure the spatial consistency of the results. Imprecision is also often present in spatial information. On a low level, it consists of problems of realignment or partial volume, for example. At a higher level, it consists, for example, of relations between objects that can be intrinsically vague or poorly defined (such as a relation like "to the left of"). Chapter 6 deals specifically with spatial relations.

Figure 5.1 illustrates the need for information on an example of thorax CT and PET images. The same part of the body is imaged, however, with two modalities relying on completely different principles, hence the complementarity. While CT provides mostly anatomical information, PET provides information in function and metabolism. Both are useful for diagnosis and therapy planning. The type of decision in such a situation would be the localization of the tumor, its spatial extension, its position with respect to organ at risks, etc. For all these tasks, merging information extracted from the two images is of great help. Another example on brain imaging is given in Fig. 5.2 at the end of this chapter, and examples at higher level, where image information and knowledge are combined can be found in Chap. 9.

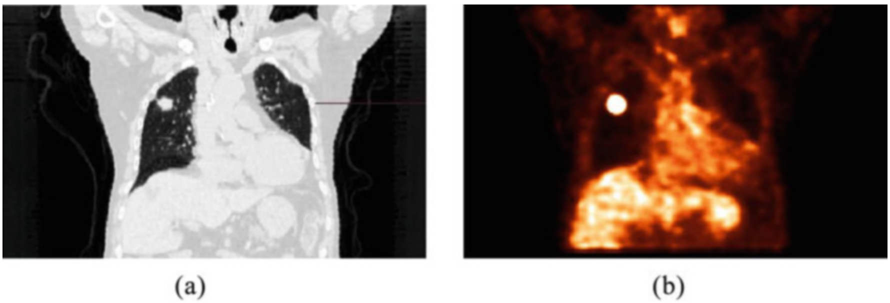

Fig. 5.1 Usefulness of fusion in medical imaging. (**a**) Slice of a CT image. (**b**) Slice of a PET image

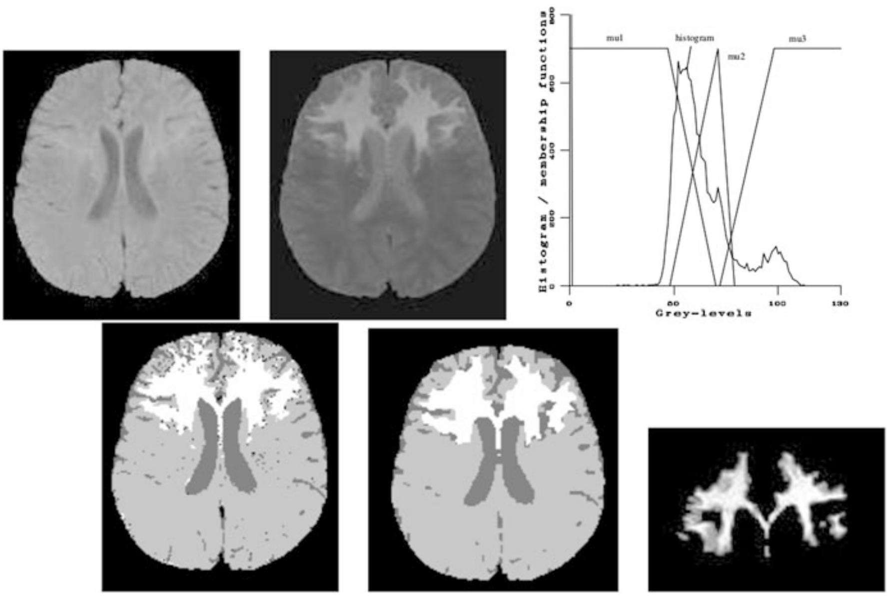

Fig. 5.2 First line: dual-echo MR image of the brain (one slide is displayed), showing three main classes: brain, ventricles and pathology (the white area on the second image); right: membership function in the second image. Second line: final decision after fuzzy combination at voxel level, final decision after spatial regularization, and membership function to the pathology class, illustrating the partial volume effect

5.2 Fusion Systems and Architectures Types

Simplifying, fusion can be divided into several tasks. A general fusion problem consists of m sources $S_1, S_2, ..., S_m$, where the objective is to make a decision among n possible decisions (or hypotheses) $d_1, d_2, ..., d_n$. The main steps in order to build the fusion process are as follows:

1. **Modeling:** this step includes choosing a formalism and expressions for the elements of information we wish to fuse within this formalism. This modeling can be guided by additional information (regarding the information and the context or the field). Let us assume, to give the reader a better idea, that each source S_j provides an element of information represented by M_i^j regarding the decision d_i. The form of M_i^j depends, of course, on the chosen mathematical formalism. It can, for example, be a distribution in a numerical formalism, or a formula in a logical formalism.
2. **Estimation:** most models require an estimation phase (for example, all of the methods that use distributions). Again, the additional information can come into play.

3. **Combination:** this step involves the choice of an operator, compatible with the modeling formalism that was chosen and guided by the additional information.
4. **Decision making:** this is the final step of fusion, which allows us to go from information provided by the sources to the choice of a decision d_i.

These steps will be detailed in the next sections, for the specific case of fuzzy fusion.

The way these steps are interlocked defines the fusion system and its architecture. In the ideal case, the decision is made based on all of the M_i^j, for all of the sources and all of the decisions. This is referred to as global fusion. In the global model, no information is overlooked. The complexity of this model and of its implementation leads to the development of simplified systems, but with more limited performances. A second model thus consists in first making local decisions for each source separately. In this case, a decision $d(j)$ is made based on all of the information originating from the source S_j only. This is known as a decentralized decision. Then, in a second step, these local decisions are fused into a global decision. This model is the obvious choice when the sources are not available simultaneously. It provides answers rapidly because procedures are specific to each source and can easily be adapted to the addition of new sources. This type of model can benefit from the use of distributed architectures. It is also referred to as decision fusion. Its main drawback comes from the fact that it poorly describes relations between sensors, as well as the possible correlations or dependences between sources. Furthermore, this model very easily leads to contradictory local decisions ($d(j) \neq d(k)$ for $j \neq k$) and solving these conflicts implies arbitration on a higher level, which is difficult to optimize, since the original information is no longer available. Models of this type are often implemented for real-time applications. A third model, "orthogonal" to the previous one, consists in combining all of the M_i^j related to the same decision d_i using an operation F, in order to obtain a fused form $M_i = F(M_i^1, M_i^2, ..., M_i^m)$. A decision is then made based on the result of this combination. In this case, no intermediate decision is made and the information is handled within the chosen formalism up until the last step, thus reducing contradictions and conflicts. This model, just like the global model, is a centralized model that requires for all of the sources to be available simultaneously. Simpler than the global model, it is not as flexible as the distributed model, making the possible addition of sources of information more difficult. Finally, an intermediate, hybrid model consists in choosing adaptively which information is necessary for a given problem based on the specificities of the sources. This type of model often copies the human expert and involves symbolic knowledge of the sources and objects. It is therefore often used in rule-based systems. Multi-agent architectures are well suited for this model.

5.3 Fuzzy Modeling in Fusion

Among the non-probabilistic techniques that have appeared over the past 20 years in fusion, fuzzy sets theory provides a very efficient tool for explicitly representing imprecise information, in the form of membership functions. As a result, the measure $M_i^j(x)$, where x denotes the element for which a decision has to be taken (e.g., pixel, voxel, region, object) introduced in the previous section is written in the form $M_i^j(x) = \mu_i^j(x)$, where $\mu_i^j(x)$, refers, for example, to the degree of membership of x to the class C_i according to the source S_j, or the translation of a symbolic element of information expressed by a linguistic variable. In a classification problem this model means that classes are considered as fuzzy sets. The reader is referred to Chap. 2 for the main definitions related to fuzzy sets. The loose constraints imposed in the definition of membership functions make them suitable to model a large spectrum of situations, in a flexible way. Other models rely on possibility theory and allow modeling both imprecision and uncertainty. They will not be detailed here.

5.4 Defining and Estimating Membership Functions

Constructing membership functions can be done in several ways. In most applications, this construction is done either by taking ideas directly from probabilistic learning methods, from heuristics, from neuromimetic methods used for learning the parameters of particular forms of membership functions or finally by minimizing classification criteria [1].

A first method consists in defining a fuzzy class membership function based on the image intensity function I (typically the gray levels): $\mu_i(x) = f_i(I(x))$ (the superscript j indicating the source is omitted here), where f_i is a function that is determined according to the problem. The most commonly used are normalization functions or S functions (which is equivalent to considering that the lighter parts of the image have a high membership to the class, which is obviously a limitation), Π functions (monomodal, they associate the class with a range of gray levels with imprecise limits), or also multi-modal functions. These functions are often determined in a supervised way, but can also be learned, for example, using automatic classification algorithms such as fuzzy C-means (FCM) [1] or possibilistic C-means (PCM) [12] (see, for example, [2] for an overview of fuzzy classification algorithms). The main drawback of fuzzy C-means is that the membership functions have counter-intuitive forms: the class membership values are not decreasing with respect to the distance to the center of the class. This problem is avoided with possibilistic C-means, for instance.

Other characteristics can be used to achieve this goal. For example, the set of contours in an image can be defined by a spatial fuzzy set whose membership function is a function of the image gradient: $\mu_i(x) = f_i(||\nabla I(x)||)$, where f_i is a decreasing function.

If specific object detectors are available, the membership functions of these objects can be defined as functions of the response to these detectors (the case of contours falls into this category). For example, in a satellite image, a road detector can provide a response whose amplitude increases with the membership to the road.

In the case of linguistic variables, the forms of membership functions and their parameters are often defined by the user, but can also be learned from sets of examples or questionnaires.

The spatial imprecision on the definition of the limits between classes (if the membership functions are defined in the image space) can be introduced based on a preliminary binary detection of the classes. A membership function is constructed as equal to 1 inside the binary region at a certain distance from the edges, as equal to 0 outside this region at a certain distance from the edges and as decreasing between these two limits. For example, an imprecision zone on the edge of the class can be modeled as the zone included between the erosion and the dilation of this object, the size of these operations depending on the spatial extension of the imprecision we wish to represent (see Chap. 4 for the definitions of mathematical morphology operators). If R is the binary region we start with, $\varepsilon^n(R)$ its erosion of size n and $\delta^m(R)$ its dilation of size m, the fuzzy class membership function can be defined by:

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \varepsilon^n(R), \\ 0 & \text{if } x \notin \delta^m(R), \\ f(d(x, \varepsilon^n(R))) & \text{otherwise,} \end{cases}$$

where f is a decreasing function of the distance from x to $\varepsilon^n(R)$.

Some other methods rely on probabilistic learning, followed by a transformation of probability into membership function (or possibility distribution) [7]. The main advantage in image processing is that statistical information is often available, particularly the intensity histogram, which is well suited for applying statistical learning methods. Other methods try to directly estimate the membership functions based on the histogram, in order to optimize entropy criteria or minimal specificity and consistency criteria. In any case, these methods attempt to find a similarity between the histogram and the membership functions or the possibility distributions and do not take into consideration interpretations that are specific to fuzziness because they invalidate some of these similarities. For example, the tails of the histogram correspond to classes with little representation in the image, hence with values that can be very low, even if the points involved belong strongly to the corresponding classes. The method suggested in [5] provides a way of avoiding this problem with the help of a criterion that combines the similarities of membership functions and the histogram where they have meaning, with an a priori shape of the functions that correspond to the desired interpretation. The parameters of the membership functions are then estimated in order to optimize this criterion by using a simulated annealing method.

5.5 Fuzzy Combination

One of the advantages of fuzzy sets theory, beyond the fact that it imposes few constraints on modeling, is that it offers a wide variety of combination operators. The main operators have been presented in Chap. 2 (t-norms and t-conorms, mean operators, symmetric sums, adaptive operators). In this chapter we discuss their properties with respect to fusion problems and proceed to indicate how to choose a fusion operator according to its properties and its behavior.

An important feature is that these combination operators provide a result of the same nature as the functions we started with (closure property) and therefore with the same interpretation in terms of imprecision and uncertainty. Therefore, they make it possible not to make any partial crisp decision before the combination takes place, which could lead to inconsistencies that would be difficult to eliminate. The decision is only made at the very end, based on the result of the combination.

From here on, the letters x, y, etc. refer to the values we wish to combine, i.e., values in [0, 1] representing values of μ_i^j. The choice of a fusion operator is made according to several criteria presented in [3]. A first criterion is the operator behavior, expressed as conjunction, disjunction, or compromise. Let x and y be two real numbers (in [0, 1]) representing the degrees to combine. The combination of x and y by an operator F is described as:

- Conjunctive if $F(x, y) \leq \min(x, y)$ (corresponding to a strict behavior).
- Disjunctive if $F(x, y) \geq \max(x, y)$ (lenient behavior).
- Compromise if $x \leq F(x, y) \leq y$ if $x \leq y$ and $y \leq F(x, y) \leq x$ otherwise (cautious behavior).

This distinction is not sufficient to categorize operators whose behaviors are not always the same. This is why the classification defined in [3] describes operators not only as conjunctive and disjunctive but also depending on their behavior according to the values of the information to combine. The three classes suggested correspond to:

- Constant behavior, independent operators (CBI): the result depends only on the values to combine (the calculation involves no other information) and the behavior is the same regardless of what those values are.
- Variable behavior, independent operators (VBI): the behavior depends on the numerical values of the information to fuse.
- Context dependent (CD) operators, for example, of more comprehensive knowledge such as the reliability of the sensors, or the conflict between the sources.

Fuzzy fusion operators fall into three categories (see Chap. 2 for definitions and properties). T-norms, which generalize set intersection to fuzzy sets, are conjunctive CBI operators, since any t-norm is smaller than the minimum. On the other hand, t-conorms which generalize union, are disjunctive CBI operators, since any t-conorm is larger than the maximum. Mean operators are also CBI operators and have a compromise behavior, since the result is always between the minimum and the maximum of the values to be combined. In the VBI operator class we have, for example, some symmetrical sums. Generally speaking, any associative symmetric sum σ (except for medians) is conjunctive if $\max(x, y) < 1/2$, disjunctive if $\min(x, y) > 1/2$, and a compromise if $x \leq 1/2 \leq y$ or $y \leq 1/2 \leq x$ [8]. Non-associative symmetric sums also have a variable behavior, but according to less simple rules (see, e.g., [3] for details). Examples of CD operators are those depending on an overall measure of the conflict between two sources of information [9], which are applicable to cases where one of the two elements of information is reliable, but it is not known which one. Therefore, they are conjunctive if the sources are consonant (low conflict), disjunctive if the sources are dissonant (high conflict) and have a compromise behavior in the case of partial conflict. The difficulty is then to find a good conflict measure. Other examples are CD operators that include the reliability of each source.

The advantage of CD operators for image processing is undeniable, since they allow us to take into consideration a wider variety of situations, several of which occur simultaneously in image processing. Here are a few examples:

- Sources can be in conflict when they provide information regarding one type of event (a class, for example) and consonant for another class.
- Sources can have different overall reliabilities.
- A source can be reliable for one class and less reliable for another, etc. Unfortunately, these operators still have not, in our opinion, been developed far enough in image processing and would deserve specific research.

This classification, which includes all of the commonly used operators, constitutes a first criterion for choosing an operator for a specific application.

The second criterion is given by the properties of operators and their interpretations in terms of uncertain, imprecise, incomplete, or ambiguous data fusion. The commutativity and associativity properties reflect the fact that the result of the combination does not depend on the order according to which the elements of information are arranged when they are combined. Whereas commutativity is satisfied by all of the commonly used operators, this is not necessarily true for associativity (means and symmetric sums usually are not associative). These two properties are often laid out as the minimum properties that fusion operators have to satisfy. However, human reasoning does not always comply with them. For example, a photo interpreter often starts by constructing a primary interpretation of the scene based on a single image, then improves this interpretation by using the other images, according to a process that clearly is not commutative.

The existence of an identity element means that a source yielding this value will have no influence on the result of the combination and represents some sort of indifference on the part of the source towards the information sought, or even a complete absence of knowledge regarding it. Such an element exists for t-norms and t-conorms (1 for t-norms, 0 for t-conorms, see Chap. 2).

Another distinctive element, the zero element, means that a source yielding this value has complete determination over the result of the fusion. Such elements also exist for t-norms and t-conorms (0 for t-norms, 1 for t-conorms, see Chap. 2).

The property of monotonic increase is usually imposed on operators and matches the intuition.

Boundary conditions, which define the behavior of the operators when the information to combine has extreme values (typically 0 or 1 here), guarantee compatibility with the crisp case, where all of the propositions are simply either true or false (1 or 0).

The continuity property satisfied by most usual operators guarantees the robustness of the fusion. However, this property is not always necessary, since natural phenomena (particularly time phenomena) are not always continuous.

Idempotence means that providing information that is already available will not change the fusion result. This property is not systematically imposed. It holds for means, the t-norm min and the t-conorm max (and those are the only ones among t-norms and t-conorms). We might want, by contrast, to have the combination of two identical values reinforce or weaken the overall result. Let us consider the example of identical simultaneous testimonies. On the one hand, if the witnesses are plotting together, it is not surprising if they state the same thing and the associated degrees of confidence will therefore be combined in an idempotent way. On the other hand, if they are independent, the credibility of what they are saying will be reinforced if they are trusted, or weakened if they are not. Let us note that the combination rules modeling these behaviors have been known since Bernoulli. Generally speaking, it is considered that if sources are dependent (in the cognitive sense), idempotence can be imposed, whereas if they are independent, reinforcement effects may be needed.

Along the same idea, the nilpotence property will be imposed, for example, to combine consecutive testimonies, in order to model the deterioration of information along a chain of witnesses that are not completely reliable. For example, for certain t-conorms (e.g., Lukasiewicz), satisfying this property will help achieve a result equal to 1 by combining a certain number of measures, which are not all equal to zero. This type of behavior may be useful when the information is the result of a long processing chain. The excluded middle and non-contradiction properties, satisfied only for certain operators (e.g., Lukasiewicz), have an accepted interpretation in reasoning terms, in the field of artificial intelligence and fuzzy reasoning. There are examples in image processing where the excluded middle is not advisable, whenever there is a need for introducing absence of knowledge regarding an event and its complements, and therefore to relax the exhaustivity constraint applied, for example, in probabilities.

The generalization of all of the above to the combination of two elements of information poses no particular difficulty (in particular, the same types of behavior are found in VBI operators with rules that are a little more complicated), except for non-associative operators. The main question for these operators is to know in which order these pieces of information should be combined. Several situations can occur:

- In some applications, each piece of information has to be combined with the others as soon as it becomes available (for example, in order to make partial decisions based on the data available at every instant): the order is then set by the order in which the elements of information arrive.
- The order can be imposed by priorities on the information to take into account, and operators have been designed to respond to these needs (for example, in order to combine database requests).
- In other situations, criteria have to be determined for finding an order adapted to the application, particularly when the elements of information are in conflict, since the results can be very different depending on whether the consonant or the conflicting elements of information are combined first.

Finally, the study of the behavior of operators in terms of the quality of the decision they lead to and of their reactions when faced with conflicting situations leads to a final criterion for choice. An important point, however, involves the discriminating power of the operators. Highly conjunctive or disjunctive operators (for example, Lukasiewicz t-norm and t-conorm, see the ordering among some operators in Chap. 2) quickly saturate at 0 or 1 and therefore often discriminate poorly. For example, with the Lukasiewicz t-conorm $F(a, b) = \min(a + b, 1)$, we have$F(0.5, 0.5) = 1$, $F(0.1, 0.9) = 1$, and also $F(0.8, 0.8) = 1$, whereas these three situations may have quite different interpretations.

The ability of operators to combine information that is quantitative (numerical) or qualitative (for which only the order is known) can also be a criterion for choice. For example, the min, the max and any rank filter are useful in this regard since they can combine both types of information. This is because the calculation of $\min(x, y)$, for example, only requires knowing an order between x and y, but does not require their numerical values to be known. Additionally, ordinal operations are imposed if we want them to remain invariant by an increasing transformation of the membership degrees [10].

5.6 Decision in Fuzzy Fusion

The main rule used in fuzzy fusion is the maximum degree of membership. In a classification problem, it is expressed as:

$$x \in C_i \text{ if } \mu_i(x) = \max_{k \in \{1 \ldots n\}} \mu_k(x),$$

where μ_k refers to the membership function to the class k resulting from the combination (i.e., $\mu_k = F(\mu_k^1, \ldots \mu_k^m)$).

The quality of the decision is evaluated basically according to two criteria:

- The first involves the "crispness" of the decision: the maximum degree of membership (or more generally the one that corresponds to the decision) is compared to a threshold, which is chosen depending on the applications (and possibly depending on the chosen combination operator).

- The second involves the "discriminating" power of the decision, which is evaluated by comparing the two highest values.

If these criteria are not met for an element x, then this element is placed in a rejection class, or reclassified according to other criteria, such as spatial criteria, for example (see Sect. 5.7).

5.7 Exploiting Spatial Information

As mentioned in Sect. 5.1, spatial information is fundamental in image processing. Including it in fusion methods is crucial and often requires specific developments to adapt the methods originating in other fields of information fusion. One of the most common objectives of these developments it to ensure that the decision is spatially consistent. For example, in multi-source classification, the goal is to avoid assigned to a different class isolated points, or points scattered in a homogeneous class. Let us give a few hints for introducing spatial information at different steps of the fusion process.

At the Modeling Level Spatial information at the modeling level is generally implicit depending on the selected representation level. If reasoning is done at a pixel level, the information contained in a pixel does not include any spatial information, so this information will have to be added explicitly. The spatial context considered is most often the local neighborhood of each pixel (or voxel). A simple way of taking it into account is to define the membership functions based on the characteristics of the neighbor of each pixel. This type of approach can be seen as a spatial filtering problem. In this direction, several methods have been proposed to introduce the local spatial context in the objective function of FCM or PCM. If working at the level of primitives (segments, contours, regions) or at the level of objects or structures in the scene, the local spatial information is implicitly accounted for in the representation. If the detection of these elements or their localization are not precise (for example, because of the imperfection of the registration), it is often advisable to explicitly include this spatial imprecision in the representation, before the fusion. Fuzzy dilation is an operation well suited for this purpose (see Chap. 4). This allows the conflict to be reduced to the moment when the fusion takes place and hence to choose a conjunctive combination mode simply and without risk. This is the approach adopted in the example illustrated in Fig. 4.3. In a less local fashion, the spatial relations between primitives constitute important information regarding the structure of the scene and they can be used in fusion, as a source of additional information (see Chaps. 6, 8, 9). In this case, the spatial context of an element x is a set of primitives or objects whose spatial relations with respect to x are known.

At the Decision Level The easiest way to include spatial information in fusion is at the decision level. The most common method consists in first establishing a rejection rule (depending on the crispness and the discriminating nature of the decision) then to reclassify the rejected elements according to their spatial context. For example, reclassification can be performed according to the absolute majority rule, expressing that at least half of the elements of the neighborhood have to be in C_i in order to put x in C_i. This rule does not always allow x to be assigned to a class. A less sever rule only considers the most represented class in the neighborhood (majority rule). These rules apply regardless of the level of representation of the elements considered for the fusion and the decision.

At the Combination Level The inclusion of spatial information on the combination is less common and more difficult. The idea is to consider the spatial context as an additional source of information and to introduce it in the combination after a suitable modeling. Again this can be done either at pixel or voxel level or at higher level (primitives or objects), taking spatial relations into account. For instance, recognizing an object can be the result of the fusion of information regarding that object and of information regarding the relations it has to have with respect to other objects. The fuzzy set framework allows both the representation and the fusion of such information. Examples will be given in Chaps. 8 and 9.

5.8 Illustrative Examples

A first example has been given in Chap. 4 for a 3D reconstruction problem from multi-modal medical images. In this example, illustrated in Fig. 4.3, the imprecision inherent to each modality and each processing step was explicitly taken into account via fuzzy dilations with appropriate structuring elements, modeling this imprecision. Hence at the fusion step, the data to be combined were consistent and a conjunctive fusion could be applied. Moreover, the dilations include explicitly spatial information.

Let us now illustrate another example, where a conjunctive fusion is not adapted. We consider a typical application of fusion in multi-source classification. We show an example of image fusion problem in brain imaging, where we combine dual-echo brain MR images in order to provide a classification of the brain into three classes: brain, ventricles, and cerebro-spinal fluid (CSF), and pathology. These images are shown in Fig. 5.2. The fusion is mandatory here to achieve a good classification based on gray levels. Indeed, the slices are quite thick, and each voxel may contain parts belonging to several tissues. This is particularly visible in the pathological region (bright area in the second image), where, due to its very irregular shape, a strong partial volume effect appears. This leads to intermediate gray levels, between those of pure white matter and pure pathology, which happen to be similar to those of the ventricles. A tissue classification only from this image would then lead to a wrong assignment of these partial volume voxels. In contrast, the first image does not allow us to distinguish between pathology and normal white matter, while the ventricles are well delineated. The membership functions for these classes have been estimated in each image separately using an unsupervised classification method from the histogram, with a normalization adapted to the semantics of fuzzy sets. In particular the extreme classes (darkest and brightest ones) may be represented by only a few points (hence low values in the histogram), but which completely belong to these classes. This is illustrated in Fig. 5.2 for the second image. We then use these membership functions in a fuzzy fusion scheme [5]. Since both images provide similar information about the ventricles, we use a mean operator to combine the membership functions obtained in both images for this class. Brain and pathology cannot be distinguished in the first echo and we obtain only one class for this image, denoted by μ_c^1. In the second image, we obtain two classes denoted by μ_c^2 and μ_{path}^2, respectively. We combine μ_c^1 and μ_c^2 using an arithmetical mean again. As for the pathology, we combine μ_c^1 and μ_{path}^2 using a symmetrical sum defined as: $\sigma(a, b) = \frac{ab}{1-a-b+2ab}$. This guarantees that no pathology is detected in the areas where $\mu_{path}^2 = 0$, and this reinforces the membership to that class otherwise, in order to include the partial volume effect areas in the pathology (this corresponds to what radiologists do). After the combination, the decision is made according to the maximum of membership values. This decision is taken at each voxel. The result is shown in Fig. 5.2. A spatial regularization is then applied, by reclassifying each voxel according to its neighborhood. Finally, the resulting membership function in the pathological area is also displayed, resulting in a good representation of the strong partial volume effect in this region: a voxel belonging completely to the pathology has a high membership value (white) while the value decreases when the voxel also includes parts from other tissues (here white matter).

Further examples, involving higher level representations, in particular structural ones using spatial relations, will be illustrated in Chaps. 8 and 9.

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Chapter 6
Spatial Relations

In this chapter we move to more structural information, expressed in terms of spatial relationships between objects or fuzzy objects.

Some relations are well defined when the objects are crisp, and need to be extended to the fuzzy case when objects are defined as fuzzy sets. This is the case for adjacency, for instance. Note that even in this case, a fuzzy formulation can lead to more robust definitions, even for crisp objects (for instance, adjacency may be satisfied or not depending on only one point of one of the two objects). Another class of relations calls directly for fuzzy definitions. This is the case for directional relations, or more complex relations such as between, along, etc. The usefulness of fuzzy representations of spatial relations was already advocated in the 1970s [47]. Different classes of spatial relations were identified, e.g., in [65], in particular the two important classes of topological and metric relations. Here more complex relations are considered as well. A part of this chapter refers to the work in [13].

6.1 Set Theoretical and Topological Relations

Set theoretical (or algebraic) relations have already been defined in Chap. 3, including degrees of intersection, union, inclusion. Such relations have already been used for defining morphological operators in Chap. 4. In this section, fuzzy adjacency is defined, as an important topological relation. Next, an extension of region connection calculus to the fuzzy case is presented.

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6.1.1 Adjacency

Adjacency is often used and it is important in vision and image processing, both locally in the neighborhood of a point and globally when spatial relationships between objects have to be assessed. Fuzzy neighborhoods have already been defined in Chap. 3.

Rosenfeld and Klette [95] defined a degree of adjacency between two crisp sets, using a geometrical approach based on the notion of "visibility" of a set from another one. This definition is then extended to degree of adjacency between two fuzzy sets. However, this definition is not symmetrical, and probably not easy to transpose to higher dimensions. Another approach consists in using the notion of contours, frontiers, and neighborhood [20] and is presented here. The space $\mathcal{S}$ is endowed with a digital connectivity c. Note that only the discrete case is considered here (see [20] for more details, including the continuous case).

In the crisp digital case, two image regions X and Y are adjacent if:

$$X \cap Y = \emptyset \text{ and } \exists x \in X, \exists y \in Y : n_c(x, y), \tag{6.1}$$

where $n_c(x, y)$ is the Boolean variable stating that x and y are neighbors in the sense of the digital c-connectivity. The extension of this definition, as detailed in [20], involves a degree of intersection $\mu_{int}(\mu, \nu)$ between two fuzzy sets μ and ν defined on $\mathcal{S}$, as well as a degree of non-intersection $\mu_{\neg int}(\mu, \nu)$ (which can be defined as $1 - \mu_{int}(\mu, \nu)$), and a degree of neighborhood n_{xy} between two points x and y of $\mathcal{S}$. All these notions have been introduced previously (see Chap. 3). This leads to the following definition for fuzzy adjacency between μ and ν:

$$\mu_{adj}(\mu, \nu) = t(\mu_{\neg int}(\mu, \nu), \sup_{x \in \mathcal{S}} \sup_{y \in \mathcal{S}} t(\mu(x), \nu(y), n_{xy})), \tag{6.2}$$

where t is a t-norm realizing the conjunction between several conditions.[1]

This definition is symmetric, consistent with the discrete binary definition (i.e., in the case where μ and ν are crisp and $n_{xy} = n_c(x, y)$), and decreasing with respect to the distance between the two fuzzy sets. It is invariant with respect to geometrical transformations (for scaling, only if n_{xy} is itself invariant). It should be noted that the condition in Eq. 6.1 is achieved for x and y belonging to the boundary of X and Y, respectively. This constraint could also be added in the fuzzy extension, as shown in [20].

[1] Since any t-norm is associative, for the sake of simplicity we denote the $t(a, b, c)$ the expression $t(t(a, b), c)$.

Equation 6.1 can also be expressed equivalently in terms of morphological dilation (see Chap. 4), as:

$$X \cap Y = \emptyset \text{ and } \delta_{B_c}(X) \cap Y \neq \emptyset, \quad \delta_{B_c}(Y) \cap X \neq \emptyset, \tag{6.3}$$

where $\delta_{B_c}(X)$ denotes the dilation of X by the structuring element B_c.

The degree of adjacency between μ and ν involving fuzzy dilation is then defined as:

$$\mu_{adj}(\mu, \nu) = t\left(\mu_{\neg int}(\mu, \nu), \mu_{int}(\delta_{B_c}(\mu), \nu), \mu_{int}(\delta_{B_c}(\nu), \mu)\right). \tag{6.4}$$

This definition represents a conjunctive combination of a degree of non-intersection between μ and ν and a degree of intersection between one fuzzy set and the dilation of the other. Again the same properties are satisfied. The structuring element B_c can be taken as the elementary structuring element related to the considered connectivity (i.e., a central point and its neighbors as defined by the c-connectivity). It can also be a fuzzy structuring element, representing, for instance, spatial imprecision (e.g., the possibility distribution of the location of each point).

Let us now compute the fuzzy adjacency on a few objects extracted from a real medical image. As an illustrative example, we consider the fuzzy structures illustrated in Fig. 3.2, extracted from the brain MRI image shown in Fig. 3.1. The adjacency degrees between some of these fuzzy objects are given in Table 6.1. The results are in agreement with what can be expected from a brain model (an anatomical atlas, for instance, where objects and adjacency are defined in a crisp way). Two classes of values are obtained: very low values which correspond to non-adjacency in the model and a set of higher values corresponding to adjacency in the model. In this case, crisp adjacency would provide completely different results in the model and in the image, preventing its use for recognition. Note that the absolute values of the degree of adjacency are not really significant and depend in particular on the choice of the operators in the intersection degree. The ranking of these values is more important. These results suggest that, combined with other spatial

Table 6.1 Results obtained for fuzzy adjacency between the structures displayed in Fig. 3.2. Higher degrees are obtained between structures where adjacency is expected (1 in the model), while very low degrees are obtained in the opposite case

Fuzzy object 1	Fuzzy object 2	Degree of adjacency	Adjacency in the model (crisp)
v1	v2	0.368	1
v1	cn1	0.463	1
v1	p1	0.000	0
v1	cn2	0.035	0
v2	cn2	0.427	1
cn1	p1	0.035	0

relationships, fuzzy adjacency degrees can indeed be used for pattern recognition purposes (as will be demonstrated in Chap. 9).

6.1.2 Fuzzy Region Connection Calculus

Region connection calculus (RCC) is an abstract representation of topological relations between arbitrary regions, expressed in first order logic[2] [93]. Starting from a connection predicate C, basic relations are defined, as summarized in Table 6.2. A set of eight relations forms a basis, hence the usual terminology of RCC-8. Note that these definitions are based on regions directly, without referring to points. This contrasts with the definitions of the previous sections. Figure 6.1 illustrates these relations.

These definitions have been extended to the fuzzy case in [97], starting with a fuzzy reflexive and symmetric relation C (i.e., for any two regions x and y,

Table 6.2 Definition of RCC relations between any two regions x and y, based on a connection predicate C

$DC(x, y)$	x is disconnected from y	$\neg C(x, y)$
$P(x, y)$	x is a part of y	$\forall z, C(z, x) \rightarrow C(z, y)$
$PP(x, y)$	x is a proper part of y	$P(x, y) \wedge \neg P(y, x)$
$EQ(x, y)$	x is identical with y	$P(x, y) \wedge P(y, x)$
$O(x, y)$	x overlaps y	$\exists z, P(z, x) \wedge P(z, y)$
$DR(x, y)$	x is discrete from y	$\neg O(x, y)$
$PO(x, y)$	x partially overlaps y	$O(x, y) \wedge \neg P(x, y) \wedge \neg P(y, x)$
$EC(x, y)$	x is externally connected to y	$C(x, y) \wedge \neg O(x, y)$
$NTP(x, y)$	x is a non-tangential part of y	$P(x, y) \wedge \neg(\exists z, EC(z, x) \wedge EC(z, y))$
$TPP(x, y)$	x is a tangential proper part of y	$PP(x, y) \wedge \neg NTP(x, y)$
$NTPP(x, y)$	x is a non-tangential proper part of y	$\neg P(y, x) \wedge NTP(x, y)$

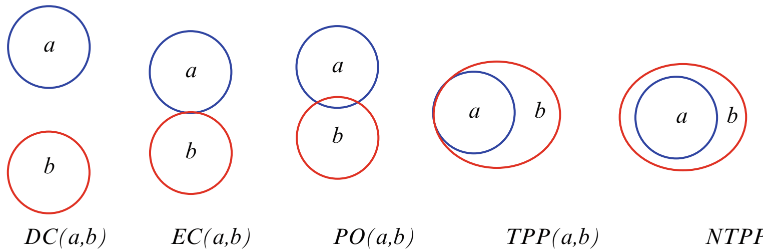


Fig. 6.1 Illustration of some RCC relations

[2] Note that this theory can also be formalized in modal logic.

Table 6.3 Definition of fuzzy RCC relations from [97] (NTP is generalized using an equivalent expression of the crisp version of this relation)

$DC(x, y)$	$1 - C(x, y)$
$P(x, y)$	$\inf_z I_t(C(z, x), C(z, y))$
$PP(x, y)$	$\min(P(x, y), 1 - P(y, x))$
$EQ(x, y)$	$\min(P(x, y), P(y, x))$
$O(x, y)$	$\sup_z t(P(z, x), P(z, y))$
$DR(x, y)$	$1 - O(x, y)$
$PO(x, y)$	$\min(O(x, y), 1 - P(x, y), 1 - P(y, x))$
$EC(x, y)$	$\min(C(x, y), 1 - O(x, y))$
$NTP(x, y)$	$\inf_z I_t(C(z, x), O(z, y))$
$TPP(x, y)$	$\min(PP(x, y), 1 - NTP(x, y))$
$NTPP(x, y)$	$\min(1 - P(x, y), NTP(x, y))$

$C(x, y) \in [0, 1]$, $C(x, x) = 1$ and $C(x, y) = C(y, x)$), and using the usual extensions of connectives, universal, and existential symbols. Denoting t a left-continuous t-norm and I_t the associated residual implication, the definitions of fuzzy RCC given in [97] are summarized in Table 6.3. There is no assumption on how the two (fuzzy) regions x and y are defined.

Properties of these relations, such as reflexivity or irreflexivity, ordering between some of them, etc., are detailed in [97]. Composition tables are also given, leading to useful tools for spatial reasoning. Spatial reasoning is further extended in [98], and reasoning tasks such as satisfiability of fuzzy RCC relations, entailment checking, inconsistency repairing are proposed. Furthermore, links are established with egg-yolk models [34], by considering the nested sets defining the general egg-yolk representation of an imprecise region as the α-cuts of a fuzzy set.

Another way to consider RCC relations and their extensions to fuzzy sets is to frame them within mathematical morphology. Indeed, as seen for adjacency, RCC relations can be expressed in a simple way based on morphological operations, leading to fuzzy extensions using fuzzy mathematical morphology (see Chap. 4). Details on this approach can be found in [14, 67].

Another extension of RCC was proposed in [48]. Instead of using a fuzzy relation C, the neighborhood structure of the relations is exploited. Indeed, the RCC relations can be organized in a conceptual neighborhood graph, where two relations between regions x and y are neighbors if x or y can be continuously deformed such that one relation is transformed in the other one, without passing through the third relation. For instance, EC has two neighbors: DC and PO, PO has four neighbors, etc. On this graph, a conceptual distance Δ is defined (as the minimal number of steps between two relation concepts in the graph). Now, satisfaction degrees are attached to the relations as follows. Let α_i be numbers such that $1 \geq \alpha_0 \geq \alpha_1 \geq \ldots \geq 0$. The fuzzy RCC-8 relation $\tilde{R}$ generalizing R is given by: $\tilde{R} = \{(R', \mu_{\tilde{R}}(R')) \mid R' \text{ RCC-8 relation}\}$, where $\mu_{\tilde{R}}(R')) = \alpha_{\Delta(R,R')}$, i.e., the membership degree of R' depends on its conceptual distance to R.

6.2 Distances Between Image Regions or Objects

The importance of distances in image processing and image interpretation is well established. Their extensions to fuzzy sets can be useful in several problems in image processing under imprecision. Several definitions can be found in the literature for distances between fuzzy sets (which is the main addressed problem[3]). They can be roughly divided in two classes: distances that take only membership functions into account and that compare them point-wise, and distances that additionally include spatial distances. The wide literature on fuzzy similarities, dissimilarities, and distances is rather silent on methods dealing with spatial information, and, unfortunately, not all approaches are suitable to this purpose. The presentation given below is directly inspired by the classification proposed in [117] but adapted to image and computer vision purposes, by underlining for each definition its properties and the type of image information on which it relies. A complete review can be found in [12].

6.2.1 Representations

Before reviewing the main definitions, we address some representation issues. They are given here in the case of distances, but these representations are also used for the other spatial relations.

The most used representation of a distance between two fuzzy sets is as a number d, taking values in $\mathbb{R}^+$ (or more specifically in $[0, 1]$ for some of them). However, since we consider fuzzy sets, i.e., objects that are imprecisely defined, we may expect that the distance between them is imprecise too [42, 94]. Therefore, the distance is better represented as a fuzzy set, and more precisely as a fuzzy number. Somewhat simpler representations of imprecision are intervals, with bounds representing, for instance, necessity and possibility degrees.

In [94], Rosenfeld defines two concepts that will be used in the following. One is distance density, denoted by $\delta(\mu, \nu)$, and the other distance distribution, denoted by $\Delta(\mu, \nu)$, both being fuzzy sets on $\mathbb{R}^+$. They are linked together by the following relation:

$$\Delta(\mu, \nu)(n) = \int_0^n \delta(\mu, \nu)(n')dn'. \tag{6.5}$$

[3] Distance from a point to a fuzzy set can be defined using mathematical morphology and was used for defining the SKIZ in Chap. 4. Geodesic distances in a fuzzy set were also defined in Chap. 4.

While the distance distribution value $\Delta(\mu, \nu)(n)$ represents the degree to which the distance between μ and ν is less than n, the distance density value $\delta(\mu, \nu)(n)$ represents the degree to which the distance is equal to n.

Histograms of distances inspired from angle histograms (see Sect. 6.4.3), carrying a complete information about distance relationships but at the price of a heavier representation, have been introduced in [12]. In the discrete case, the histogram of distances between two sets X and Y is defined as:

$$H_d(X, Y)(n) = |\{(x, y) \mid x \in X, y \in Y, d_{\mathcal{S}}(x, y) = n\}|,$$

or in normalized form:

$$H_d^N(X, Y)(n) = \frac{H_d(X, Y)(n)}{\max_{m \in \mathbb{N}} H_d(X, Y)(m)},$$

where $d_{\mathcal{S}}$ denotes the distance in the spatial domain $\mathcal{S}$ (Euclidean distance or a discrete approximation of it).

The concept of distance can be represented as a linguistic variable. This assumes a granulation [116] of the set of possible distance values into symbolic classes such as "near," "far," etc., each of these classes being defined as a fuzzy set. This approach has been drawn, e.g., in [3, 5, 64]. For instance, in [64], the relation "far" is defined as an increasing function of the average distance between both sets.

Finally, spatial representations are useful to define the regions of the space where some distance constraint to a reference object is satisfied (see, e.g., [12, 21]). Such constraints are often expressed as imprecise statements or in linguistic terms, thus reinforcing the usefulness of fuzzy modeling.

In the following, methods providing numbers or fuzzy numbers are mostly presented.

6.2.2 Comparison of Membership Functions

This section reviews the main distances proposed in the literature that aim at comparing membership functions. They have generally been proposed in a general fuzzy set framework, and not specifically in the context of image processing. They do not really include information about spatial distances. The classification chosen here is inspired from the one in [117]. Similar classifications can be found in [32, 56, 82]. More details and properties are given in [12].

The **functional approach** is probably the most popular one. It relies on a L_p norm between μ and ν, leading to the following generic definition [42, 59, 70]:

$$d_p(\mu, \nu) = \left(\int_{x \in \mathcal{S}} |\mu(x) - \nu(x)|^p \right)^{1/p}, \tag{6.6}$$

$$d_\infty(\mu, \nu) = \sup_{x \in \mathcal{S}} |\mu(x) - \nu(x)|. \tag{6.7}$$

In the discrete finite case, these definitions use discrete sums and max, respectively. A noticeable property of d_p is that it takes a constant value if the supports of μ and ν are disjoint, thus ignoring how far the supports are from each other in $\mathcal{S}$.

Among the **information theoretic approaches**, definitions based on fuzzy entropy or fuzzy divergence have been proposed [4, 24, 71, 113]. But one main drawback of most of these approaches is that the obtained distance is always equal to 0 for crisp sets.

In the **set theoretic approach**, distance between two fuzzy sets is seen as a dissimilarity function, based on fuzzy union and intersection. Examples are given in [117]. Most of the proposed measures are inspired from the work by Tversky [106] who proposed two parametric similarity measures between two sets A and B:

$$\theta f(A \cap B) - \alpha f(A - B) - \beta f(B - A), \tag{6.8}$$

and in a rational form:

$$\frac{f(A \cap B)}{f(A \cap B) + \alpha f(A \cap B^C) + \beta f(B \cap A^C)}, \tag{6.9}$$

where $f(X)$ is typically the cardinality of X, and α, β and θ are parameters leading to different kinds of measures.

Let us mention a few examples (they are given here in the finite discrete case). A measure being derived from the second Tversky measure by setting $\alpha = \beta = 1$ has been used by several authors (see, e.g., [32, 41, 82, 112, 117], among others):

$$d(\mu, \nu) = 1 - \frac{\sum_{x \in \mathcal{S}} \min[\mu(x), \nu(x)]}{\sum_{x \in \mathcal{S}} \max[\mu(x), \nu(x)]}. \tag{6.10}$$

It does not satisfy the triangle inequality and always takes the constant value 1 as soon as the two fuzzy sets have disjoint supports. It also corresponds to the Jaccard index [37]. With respect to the typology presented in [25], this distance is a comparison measure, and more precisely a dissimilarity measure. Moreover, $1 - d$ is a resemblance measure. Applications in image processing can be found, e.g., in [110], where it is used on fuzzy sets representing objects features (and not directly spatial image objects) for structural pattern recognition on polygonal 2D objects.

Note that slightly different formulas have been proposed in [35, 111] in order to achieve better properties, in particular by normalizing by the size of the union of the supports of the two fuzzy sets.

Another measure takes into account only the intersection of the two fuzzy sets [32, 117]:

$$d(\mu, \nu) = 1 - \max_{x \in \mathcal{S}} \min(\mu(x), \nu(x)). \tag{6.11}$$

Again it is a dissimilarity measure, always equal to 1 when the supports of μ and ν are disjoint, and $1 - d$ is a resemblance measure.

If we set $(\mu \Box \nu)(x) = \max(\min(\mu(x), 1-\nu(x)), \min(1-\mu(x), \nu(x)))$, two other distances can be derived, as in [117]:

$$d(\mu, \nu) = \sup_{x \in \mathcal{S}} (\mu \Box \nu)(x), \tag{6.12}$$

$$d(\mu, \nu) = \sum_{x \in \mathcal{S}} (\mu \Box \nu)(x). \tag{6.13}$$

These two distances are symmetrical measures. They are separable only for binary sets. Also $d(\mu, \mu) = 0$ holds only for binary sets. They are dissimilarity measures. The first one is equal to 1 if μ and ν have disjoint supports and are normalized (if they are not normalized, then this constant value is equal to the maximum membership value of μ and ν). The second measure is always equal to $|\mu| + |\nu|$ is μ and ν have disjoint supports.

These measures actually rely on measures of inclusion of each fuzzy set in the other (see Sect. 3.2). Indeed, the distance should be small if the two sets have a small degree of equality (the equality between μ and ν can be expressed by "μ included in ν and ν included in μ", which leads to an easy transposition to fuzzy equality, see Chap. 3). Other inclusion indices can be defined, e.g., from Tversky measure by setting $\alpha = 1$ and $\beta = 0$, leading to $\frac{f(A \cap B)}{f(A)}$ [37].

The last definitions given by Eqs. 6.11 and 6.12 are, respectively, equivalent to $1 - \Pi(\mu; \nu)$ and $1 - \max(N(\mu; \nu), N(\nu; \mu))$ (where Π and N are possibility and necessity functions) used in fuzzy pattern matching [29, 44], which has several application domains, including image processing and vision (see, e.g., [57]).

Finally, the **pattern recognition approach** consists in first expressing each fuzzy set in a feature space (for instance, cardinality, moments, skewness) and to compute the Euclidean distance between two feature vectors [117] or attribute vectors [101]. These approaches are not in the main focus of this book since they do not directly use the fuzzy sets in the spatial domains but features extracted from them.

Application to Symmetry The distances presented in this section are mostly based on a point-wise comparison of membership function. They can be applied to define another spatial relation, namely symmetry, either between two objects with respect to a line or a plane or between two parts of an object. For instance, minimizing

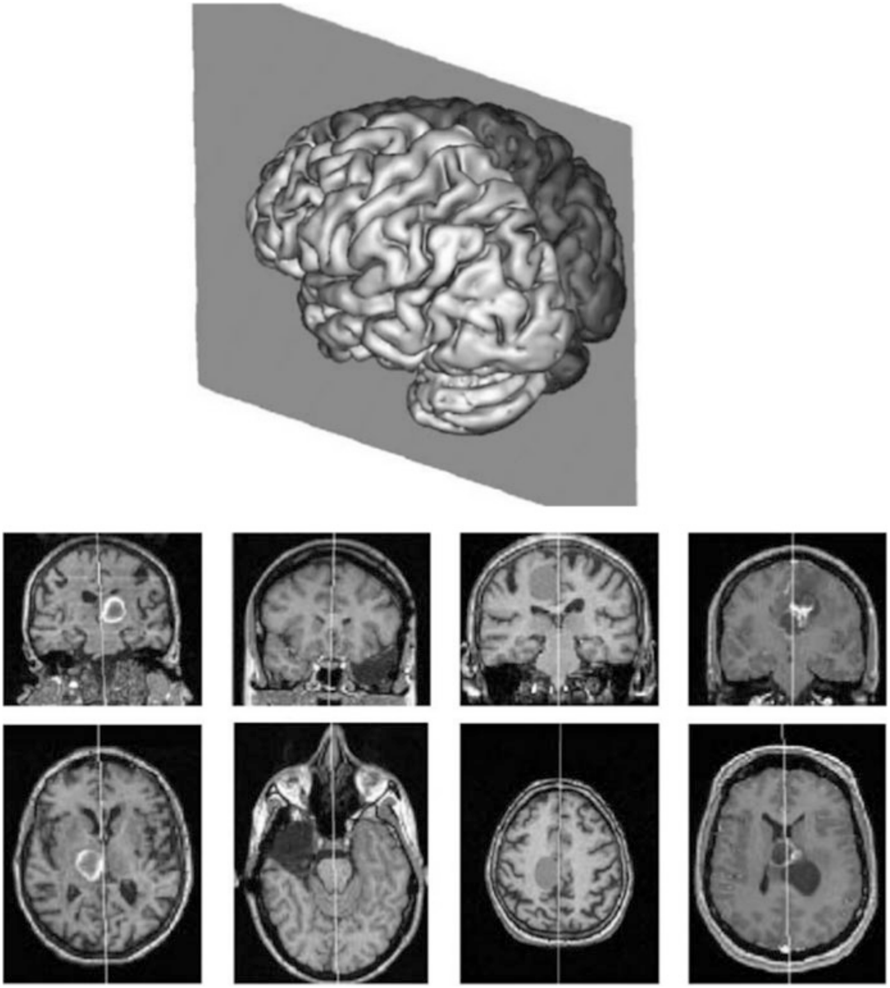


Fig. 6.2 Best symmetry plane in brain images. Top: 3D view. Bottom: a few pathological cases illustrated on axial slices [105]

the distance between an object and its reflection with respect to a plane, over all possible planes, provides the best symmetry plane of an object. This has been proposed in [105] to fine the inter-hemispheric plane of the brain from MRI images. An example is shown in Fig. 6.2. The fuzzy set framework allows us to deal with approximate symmetry, which is useful in pathological cases.

6.2.3 *Combination of Spatial and Membership Comparisons*

In the second class of methods, the spatial distance $d_{\mathcal{S}}$ (Euclidean distance in $\mathcal{S}$, for instance) is taken into account in the distance between μ and ν. In contrast to the definitions given above, in this second class the membership values at different points of $\mathcal{S}$ are linked using some formal computation, making the introduction of $d_{\mathcal{S}}$ possible. This leads to definitions that do not share the drawbacks of previous approaches, for instance, when the supports of the two fuzzy sets are disjoint.

The **geometric approach** consists in generalizing distances between crisp sets. This has been done for nearest point distance [42, 94], mean distance [94], Hausdorff distance [42] and could easily be extended to other distances (see, e.g., [19] for a review of crisp set distances). These generalizations can be obtained according to a fuzzification by integration over α-cuts (see Sect. 2.5) [40, 117].

Another method consists in weighting distances by membership values. For the mean distance this leads, for instance, to [94]:

$$d(\mu, \nu) = \frac{\sum_{x\in\mathcal{S}} \sum_{y\in\mathcal{S}} d_{\mathcal{S}}(x, y) \min(\mu(x), \nu(y))}{\sum_{x\in\mathcal{S}} \sum_{y\in\mathcal{S}} \min(\mu(x), \nu(y))}. \tag{6.14}$$

The third approach consists in defining a fuzzy distance as a fuzzy set on $\mathbb{R}^+$ instead of as a crisp number using the extension principle (see Sect. 2.5). For the nearest point distance this leads to [94]:

$$d(\mu, \nu)(r) = \sup_{x,y,d_{\mathcal{S}}(x,y)\leq r} \min(\mu(x), \nu(y)). \tag{6.15}$$

A similar approach has been used in [74], and the corresponding distance density is expressed as:

$$d(\mu, \nu)(r) = \sup_{x,y,d_{\mathcal{S}}(x,y)= r} \min(\mu(x), \nu(y)). \tag{6.16}$$

The fuzzy extension of the Hausdorff distance is probably the most widely studied of all the fuzzy extensions of distances between sets, with the first occurrence in [86]. One reason for this may be that it is a true metric in the crisp case, while other set distances like minimum or average distances have weaker properties. Another reason is that it has been used to determine a degree of similarity between two objects, or between an object and a model [51]. Extensions of this distance have been defined using fuzzification over the α-cuts and using the extension principle [26, 31, 38, 87, 117]. One potential problem with these approaches occurs in the case of empty α-cuts [27, 46]. Boxer [26] proposed to add a crisp set to every set, but the result is highly dependent on this additional set and does not reduce to the classical Hausdorff distance when applied on crisp sets. The solution proposed in [46] consists in clipping the distance at some maximum distance, but similar problems arise. Other authors use the Hausdorff distance between the endographs or umbras of the two membership functions [38] (but the additional dimension has not the same meaning as the spatial dimension). Several generalizations of the Hausdorff distance have also been proposed under the form of fuzzy numbers [42]. Extensions of the Hausdorff distance based on fuzzy mathematical morphology have also been developed [6] and are presented below.

These distances share most of the advantages and drawbacks of the underlying crisp distance [19]: the computational cost can be high (it is already high for several crisp distances); moreover, interpretation and robustness strongly depend on the chosen distance (for instance, Hausdorff distance may be noise sensitive, whereas average distance is less but has weaker properties). Extensions of these definitions may be obtained by using other weighting functions, for instance, by using t-norms instead of min.

A **morphological approach** has been proposed in [6, 10]. Examples of nearest point distance and Hausdorff distance are described here, in the discrete case (similar equations hold for the continuous case). The main idea is to use links between distances and morphological dilations to derive algebraic expressions of distances (instead of classical analytical ones), which are then easy to translate into fuzzy expressions.

In the binary case, and in a digital space $\mathcal{S}$, for $n > 0$, the nearest point distance d_N can be expressed in morphological terms, based on dilations δ (δ^n denoting the dilation of size n), as:

$$d_N(X, Y) = n \Leftrightarrow \delta^n(X) \cap Y \neq \emptyset \text{ and } \delta^{n-1}(X) \cap Y = \emptyset \tag{6.17}$$

or equivalently the symmetrical expression. For $n = 0$ we have: $d_N(X, Y) = 0 \Leftrightarrow X \cap Y \neq \emptyset$. The translation of these equivalences provides, for $n > 0$, the following distance density:

$$\delta_N(\mu, \mu')(n) = t(\sup_{x \in \mathcal{S}} t(\mu'(x), \delta_\nu^n(\mu)(x)), c(\sup_{x \in \mathcal{S}} t(\mu'(x), \delta_\nu^{n-1}(\mu)(x)))) \tag{6.18}$$

or a symmetrical expression derived from this one, and $\delta_N(\mu, \mu')(0) = \sup_{x \in \mathcal{S}} t(\mu(x), \mu'(x))$ (i.e., a degree of intersection as in Eq. 3.3).

In a similar way, the Hausdorff distance is extended by translating directly the binary equation defining the Hausdorff distance:

$$d_H(X, Y) = \max(\sup_{x \in X} d_{\mathcal{S}}(x, Y), \sup_{y \in Y} d_{\mathcal{S}}(y, X)). \tag{6.19}$$

This distance can be expressed in morphological terms as:

$$d_H(X, Y) = \inf\{n, X \subseteq \delta^n(Y) \text{ and } Y \subseteq \delta^n(X)\}. \tag{6.20}$$

From Eq. 6.20, a distance distribution can be defined, involving fuzzy dilation:

$$\Delta_H(\mu,\mu')(n) = t(\inf_{x\in\mathcal{S}} T(\delta_\nu^n(\mu)(x), c(\mu'(x))), \inf_{x\in\mathcal{S}} T(\delta_\nu^n(\mu')(x), c(\mu(x)))), \tag{6.21}$$

where c is a complementation, t a t-norm, and T a t-conorm. A distance density can be derived implicitly from this distance distribution.

A direct definition of a distance density can be obtained from: $d_H(X,Y)=0 \Leftrightarrow X=Y$, and for $n>0$:

$$\begin{aligned} d_H(X,Y)=n \Leftrightarrow \quad & X \subseteq \delta^n(Y) \text{ and } Y \subseteq \delta^n(X) \\ & \text{and } \left(X \not\subseteq \delta^{n-1}(Y) \text{ or } Y \not\subseteq \delta^{n-1}(X)\right). \end{aligned} \tag{6.22}$$

Translating these equations leads to a definition of the Hausdorff distance between two fuzzy sets μ and μ' as a fuzzy number:

$$\delta_H(\mu,\mu')(0) = t(\inf_{x\in\mathcal{S}} T(\mu(x), c(\mu'(x))), \inf_{x\in\mathcal{S}} T(\mu'(x), c(\mu(x)))), \tag{6.23}$$

$$\begin{aligned} \delta_H(\mu,\mu')(n) = t(&\inf_{x\in\mathcal{S}} T(\delta_\nu^n(\mu)(x), c(\mu'(x))), \inf_{x\in\mathcal{S}} T(\delta_\nu^n(\mu')(x), c(\mu(x))), \\ &T(\sup_{x\in\mathcal{S}} t(\mu(x), c(\delta_\nu^{n-1}(\mu')(x))), \sup_{x\in\mathcal{S}} t(\mu'(x), c(\delta_\nu^{n-1}(\mu)(x)))). \end{aligned} \tag{6.24}$$

Note that the inclusion is extended here using a t-conorm and a complementation. Residual implications from a t-norm can be used as well.

The above definitions of fuzzy nearest point and Hausdorff distances (defined as fuzzy numbers) between two fuzzy sets do not necessarily share the same properties as their crisp equivalents. All distances are positive, in the sense that the defined fuzzy numbers have always a support included in $\mathbb{R}^+$. By construction, all defined distances are symmetrical with respect to μ and μ'. The separability property is not always satisfied. However, if μ is a normal fuzzy set (i.e., $\exists x \in \mathcal{S}, \mu(x) = 1$), then for the nearest point distance the following holds: $\delta_N(\mu,\mu)(0) = 1$ and $\delta_N(\mu,\mu)(n) = 0$ for $n > 1$. For the Hausdorff distance, $\delta_H(\mu,\mu')(0) = 1$ implies $\mu = \mu'$ for T being the bounded sum ($T(a,b) = \min(1, a+b)$, Lukasiewicz t-conorm), while it implies μ and μ' crisp and equal for $T = \max$. Also the triangle inequality is not satisfied in general.

Another morphological approach has been suggested in [102], based on links between the minimum distance and the Minkowski difference $\ominus$. In the crisp case, if X and Y are non-intersecting crisp sets, then $d_N(X,Y) = \inf\{|z|,\ z \in Y \ominus X\}$. In order to account for possible intersection between the two sets, the authors introduce also the notion of penetration distance, defined along a direction σ as the maximum translation of X along σ such that X still meets Y: $\delta(\sigma; X, Y) = \max\{k, (X + k\sigma) \cap Y \neq \emptyset\}$. The extension to fuzzy sets is done by assuming fuzzy numbers on each axis. This leads to reasonable computation times, but can unfortunately not be directly extended to any fuzzy objects because of this assumption.

A **tolerance-based approach** has been developed in [70]. The basic idea is to combine spatial information and membership values by assuming a tolerance value τ, indicating the differences that can occur between objects considered as similar. Note that this approach has strong links with morphological approaches, since the neighborhood considered around each point can be considered as a structuring element.

According to a **graph theoretic approach**, a similarity function between fuzzy graphs may also induce a distance between fuzzy sets. This approach contrasts with the previous ones, since the objects are no more represented directly as fuzzy sets on $\mathcal{S}$ or as vectors of attributes, but as higher level structures. Fuzzy graphs in structural recognition can be used for representing objects, as in [73], or a scene, as in [61]. In the first case, nodes are parts of the objects and edges are links between these parts. In the second case, nodes are objects of the scene and edges are relationships between these objects. These two examples use different ways to consider distances (or similarity) between fuzzy graphs. In [73], the distance is defined from a similarity between nodes and between edges (both being fuzzy sets), given a correspondence between nodes (respectively, between edges). The similarity used compares only membership functions, using a set theoretic approach (see Sect. 6.2.2) and corresponds to Eq. 6.10. Although it has not been considered in this reference, spatial distance can then be taken into account by including it in the attribute set. In a similar way, several distances between graphs have been proposed as an objective function to find the correspondence between graphs. This function compares attributes of nodes of the two graphs to be matched, and attributes of arcs. One of the main difficulties is to deal with non-bijective matching. This has been addressed, for instance, in [2, 30, 85], where a formalism for defining fuzzy morphisms between graphs is proposed, as well as optimization methods for finding the best morphism according to an objective function including spatial distance information as an edge attribute. Fuzzy graphs are further detailed in Chap. 8, and their application to image understanding in Chap. 9. Another way to consider distances between objects is in terms of cost of deformations to bring one set in correspondence with the other. Such approaches are particularly powerful in graph-based methods. The distance can then be expressed as the cost of the matching of two graphs, as done in [61] for image processing applications, or as the Levenshtein distance accounting for the necessary transformations (insertions, substitutions, deletions) for going from the structural representation of one shape to the representation of the other [36]. In [61], the fuzzy aspect is taken into account as weighting factors, therefore the method is quite close to the weighted Levenshtein distance of [36]. Spatial distances could also be introduced as one of the relationships between objects in these approaches. A distance between conceptual graphs is defined in [72], as an interval $[N, \Pi]$ where N represents the necessity and Π the possibility, obtained by a fuzzy pattern matching approach. Although these examples are still far from the main focus of this book, it is worth mentioning them, since they bring an interesting structural aspect (see also Chaps. 8 and 9).

6.2.4 Discussion and Examples

In the first class of methods (Sect. 6.2.2), the only way μ and ν are combined is by computation linking $\mu(x)$ and $\nu(x)$, i.e., only the membership values at the same point of $\mathcal{S}$. No spatial information is taken into account. A positive consequence is that the corresponding distances are easy to compute, with a linear complexity in the cardinality of $\mathcal{S}$. For image processing and vision applications, we suggest that the first class of methods, comparing membership functions only, be restricted to cases when the two fuzzy sets to be compared represent the same structure or a structure and a model. Applications in model-based or case-based pattern recognition are foreseeable, as well as for symmetry relations, as mentioned above.

On the other hand, the definitions which combine spatial distance and fuzzy membership comparison (Sect. 6.2.3) allow for a more general analysis of structures in images, for applications where the topological and spatial arrangement of the structures of interest is important, i.e., segmentation, classification, scene interpretation (see Chap. 9). This is enabled by the fact that these distances combine membership values at different points in the space, therefore taking into account their proximity in $\mathcal{S}$. The price to pay is an increased complexity, generally quadratic in the cardinality of $\mathcal{S}$.

When facing the problem of choosing a distance, several criteria can be used. First, the type of application at hand plays an important role. While both classes of methods can be used for comparing an object and a model object, only the second class can be used for evaluating distances between objects in the same image. Among the distances of the first class, the results we obtained show that entropy and divergence based approaches are not satisfactory. Also normalized distances should be avoided in most cases. The choice among the remaining distances can be done by looking at the properties of the distances (for instance, is $d(\mu, \mu) = 0$ needed for the application at hand?), and at the computation time. Among the distances of the second class, similar choice criteria can be used. Although one may speak about distances between image objects in a very general way, this way of speaking does not make necessarily the assumption that we are dealing with true metrics. For several applications in image processing, it is not sure that all properties are needed. For instance, the triangle inequality is not always a requirement.

In order to illustrate the differences between various definitions, the distances from all structures shown in Fig. 3.2 to v2 have been computed, using most of the definitions reviewed in this section (see [13] for details on all these results). The results obtained with the distances of the first class do not provide satisfactory results, for the reasons explained above, in particular a poor differentiation between

Table 6.4 Distances between fuzzy sets of Fig. 3.2 using the geometric approach: weighted average distance using three different t-norms, fuzzification (using integral over α-cuts) of mean, min, and Hausdorff distances

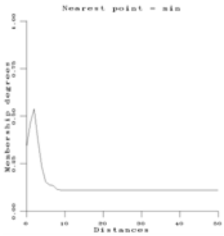
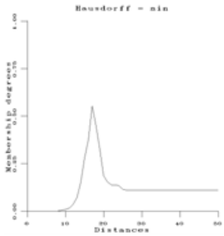
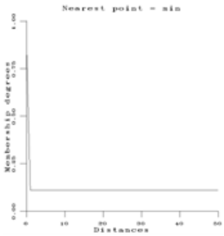
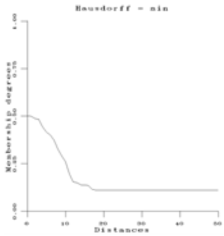
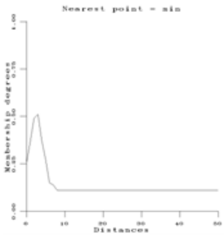
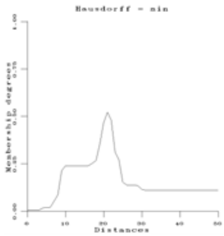
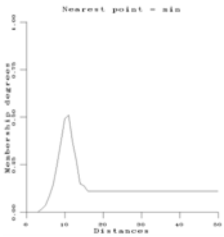
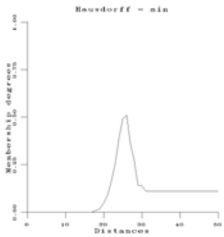
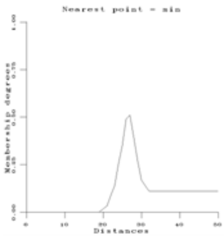
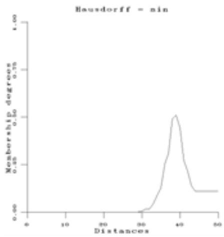
Distance between v2 and					
Distance	nc2	v2	v1	nc1	t1
Weighted mean dist (min)	16.296	8.165	16.402	22.820	36.762
Weighted mean dist (prod)	16.174	7.501	15.402	22.668	36.589
Weighted mean dist (Lukasiewicz)	16.096	5.855	13.574	22.502	36.299
Integral of mean dist	14.536	5.145	12.897	20.298	32.453
Integral of min dist	1.696	0.000	2.071	8.937	23.204
Integral of Hausdorff dist	19.068	0.000	21.944	25.373	35.952

structures. The results using distances of the second class are given in Table 6.4 for the geometrical approach, and in Table 6.5 for the morphological approach.

Using distances taking into account spatial information, more satisfactory results are obtained. Using the geometric approach (Table 6.4), the lowest value is always obtained for v2. A null value is obtained only using the minimum and the Hausdorff distances, since they are the only ones which satisfy $d(\mu, \mu) = 0$. Objects nc2 and v1 have similar distances to v2, as it appears in Fig. 3.2. Then nc1 is found farther, followed by t1. These results fit well the intuition. Using different t-norms in the weighted average distance changes the absolute values that are obtained, but not the ranking. Since the following inequalities hold: $\forall (a, b) \in [0, 1]^2,\ \max(0, a + b - 1) \leq ab \leq \min(a, b)$, similar inequalities between the derived distances are obtained. For this distance, the choice of the t-norm is not really important, since it does not change the properties of the distance, and for image processing purposes, the ranking between distance values is often more important that their absolute value.

All previous examples provide results as numbers. When using the morphological approach, the results take the form of fuzzy numbers as seen in Table 6.5. The curves in this table show the degrees to which the distance is equal to n as a function of n. Again the results fit well the intuition. The distributions obtained for v2 are concentrated on the low distance values. Then, when the structures become farther away from v2, the curves are shifted towards higher distance values. Here again the choice of a specific t-norm is not crucial as it changes mainly the absolute values and not the ranking. Lower membership degrees are obtained when using a smaller t-norm (see [13] for more results). The fact that the Hausdorff distance provides higher values than the nearest point distance corresponds to the fact that the size of the dilation applied to one set needed to reach the other is less than the size of the dilation needed to completely include the other set. This is the case for crisp sets, and the same property holds in the fuzzy case.

Table 6.5 Distances between fuzzy sets of Fig. 3.2 using the morphological approach, for the nearest point distance and the Hausdorff distance, using the min t-norm

Distance between v2 and:		
	δ_N	δ_H
nc2		
v2		
v1		
nc1		
t1		

6.3 Fuzzy Hamming Distance

This section is based on [52–55] and the references therein.

A short paragraph in Richard Hamming's 1950 paper [49], on error detection correction codes, introduced a distance metric between binary vectors. Subsequently, due to its successful use in many applications, this metric received the name of its inventor, becoming the *Hamming Distance* (henceforth denoted as HD). In a nutshell, for two binary vectors of the same size, HD counts the number of components in which these vectors differ. Thus, at least at first glance HD appears to be different than the usual distance metrics most of which can be expressed as an aggregation of vectors difference along individual components.

The Fuzzy Hamming Distance (*FHD*) described in this section was introduced in [90] as an extension of the HD to real-valued vectors, to count the number of components along which its arguments are different. It was further developed and applied in [52–54]. With respect to the analysis and classification of fuzzy distances (in image processing and understanding) of [10] (summarized in the previous section), this distance falls into the fourth category, namely when the data is crisp (not fuzzy) and the distance is fuzzy set valued. *FHD* can be used on heterogenous data (records whose components are in different spaces), mixtures of crisp and fuzzy data, and moreover, it can also be expressed as an aggregation of differences along individual components.

Recalling that *HD* simply counts the components along which the vectors are *different*, when the vectors are binary, it is clear that along each component, the vectors can be either equal or, if not, their difference is always equal to 1. By contrast, in the case of the real-valued vectors, when the vectors are different along a component, their differences can take any value, and therefore, the *magnitude of the difference* is important. This is then the point of departure for defining FHD. For computer vision and image processing applications, real-valued vectors are more natural to consider, since they are obtained from applying some image processing operator. For example, in a simple case, the vectors are obtained from some function of the pixel values (e.g., [0, 255]) in an image, or portion of thereof.

To begin with, the degree of *agreement* between two real values is defined as follows. Given the real values x and y, the degree of the agreement between x and y, modulated by $\alpha > 0$, denoted by $a_\alpha(x, y)$, is defined as:

$$a_\alpha(x, y) = e^{-\alpha(x-y)^2}. \tag{6.25}$$

As a function of x and y, Eq. 6.25 is a good choice for its good mathematical properties, which may be useful (e.g., differentiability). According to [80], it captures the agreement between stimulus similarity and psychological distance. Such an agreement is useful when compatibility with human evaluation of proximity is desired. The parameter $\alpha \geq 0$ is a small addition to what is otherwise a Gaussian decay, in which case it can be linked to the variance σ^2. However, its impact can be very important in modulating the degree of agreement between x and y, in the sense that for the same value of $|x - y|$, different values of α will result in different values of $a_\alpha(x, y)$. For defining a distance-like measure, the *degree of difference* between x and y is needed, which is defined as

$$d_\alpha(x, y) = f(a_\alpha(x, y)), \tag{6.26}$$

where $f : (0, 1] \to [0, 1)$ is a decreasing function, hence an increasing function of the difference $(x - y)^2$. For the discussion in this section, f is defined as $f(x) = 1 - x$. Therefore, from this point on, we set:

$$d_\alpha(x, y) = 1 - a_\alpha(x, y) = 1 - e^{-\alpha(x-y)^2}. \tag{6.27}$$

This degree of difference d_α has the following properties:

1. $0 \le d_\alpha(x, y) < 1$, with $d_\alpha(x, y) = 0 \iff x = y$;
2. $d_\alpha(x, y) = d_\alpha(y, x)$;
3. for $x = a \pm c$, $d_\alpha(x, a) = 1 - e^{-\alpha c^2}$;
4. $d_\alpha(x, y) = d_\alpha(0, (x - y))$.

Using the notion of degree of difference defined above, the *difference fuzzy set*, $D_\alpha(x, y)$, on the components $\{1, 2, \ldots, n\}$ of two vectors x and y is now defined by the membership function $\mu_{D_\alpha(x,y)} : \{1, \ldots, n\} \to [0, 1]$ given by Ralescu [90]:

$$\mu_{D_\alpha(x,y)}(i) = d_\alpha(x_i, y_i), \tag{6.28}$$

where $d_\alpha(x_i, y_i)$ is the degree of difference between x and y, along the ith components x_i, y_i. The properties of $\mu_{D_\alpha(x,y)}$ are determined from the properties of $d_\alpha(x_i, y_i)$.

Since HD is the *number* of components along which two binary vectors are different (which is also in fact, the power p of any L_p norm of these vectors), a fuzzy extension of this concept is necessarily based on the concept of *cardinality* of a (discrete) fuzzy set, discussed in Chap. 2. Recall here that the cardinality of a fuzzy set A, with membership function $\mu_A(x_i) = \mu_i$, for $i = 1, \ldots, n$, is the fuzzy set $CardA$, with membership function $\mu_{CardA}(i)$ given by

$$\mu_{CardA}(i) = \mu_{(i)} \wedge (1 - \mu_{(i+1)}), \tag{6.29}$$

where $\mu_{(i)}$ denotes the ith largest value of μ_i; the values $\mu_{(0)} = 1$ and $\mu_{(n+1)} = 0$ are introduced for convenience, and $\wedge$ denotes the min operation. By analogy with the definition of the classical Hamming distance, and using the cardinality of a fuzzy set, FHD is then defined as follows. Given two real-valued n dimensional vectors, x and y, and their difference fuzzy set $D_\alpha(x, y)$, with membership function $\mu_{D_\alpha(x,y)}$ defined in Eq. 6.28, FHD between x and y, denoted by $FHD_\alpha(x, y)$, is the *fuzzy cardinality of the difference fuzzy set*, $D_\alpha(x, y)$, denoted by $CardD_\alpha(x, y)$, with membership function $\mu_{FHD(x,y)}(\cdot\,;\,\alpha) : \{0, \ldots, n\} \to [0, 1]$ defined by:

$$\mu_{FHD(x,y)}(k;\alpha) = \mu_{CardD_\alpha(x,y)}(k) \tag{6.30}$$

for $k \in \{0,\ldots,n\}$ where $n = |Support D_\alpha(x,y)|$. Equation 6.30 means that for a given value k, $\mu_{FHD(x,y)}(k\,;\,\alpha)$ is the degree to which the vectors x and y are different along *exactly* k components (with the modulation constant α).

The modulating parameter α can be tuned to include context dependence (with respect to other components of the two vectors), or to capture only local, current component, information [52]. One way to set the value for α is to impose a lower bound on the membership function subject to constraints on the difference between vector components, e.g., as: $|x_i - y_i| \geq M \Rightarrow \mu_{D_\alpha(x,y)}(i) \geq 1-\varepsilon$. That is, for arbitrary $\varepsilon > 0$, and for (x_i, y_i), a generic component pair, it is desired that the membership degree $\mu_{D_\alpha(x,y)}(i)$ be as close to 1 as possible (i.e., greater than $1-\varepsilon$) as soon as the actual difference $|x_i - y_i| \geq M$ for some positive constant M. For example, M can be set as $M = \beta MAX$ where $\beta \in [0,1]$ and MAX denotes the maximum value in the context selected (e.g., column domain), which leads to $1 - e^{-\alpha(x_i-y_i)^2} \geq 1-\varepsilon$, from which it follows that

$$\alpha \geq \frac{1}{(x_i - y_i)^2} \ln\left(\frac{1}{\varepsilon}\right), \tag{6.31}$$

and therefore, to a formula for defining a global value for the parameter α:

$$\alpha = \frac{\ln\left(\frac{1}{\varepsilon}\right)}{MAX^2}\frac{1}{\beta^2}, \tag{6.32}$$

where MAX and β have the meaning stated above. The parameter β can be viewed as the percentage from MAX that is considered as a change for that column. For example, for pixel values, for $\beta = 0.1$ (10%), $MAX = 255$ and $\varepsilon = 0.5$, if the difference between compared components of the vectors is greater or equal to $\beta MAX = 25$, then the degree of change will be greater than $1 - \varepsilon = 0.5$, and therefore it will be counted in defuzzification of FHD. Since β can be different for each column, FHD sensitivity can be controlled differently for each column (feature) of the image (data set). This also suggests that FHD can be used effectively in evaluating the distance between collections of data of different types, that is, for *heterogeneous data*, as already shown in [55]. In image applications, such data may be obtained, for example, by using extended pixel representations [81]. Table 6.6 shows other ways in which α can be used to adapt FHD.

The formulas in Table 6.6 determine the following behavior for the degree of difference: for values x_i, x_j, such that $|x_i - x_j| = c$, the degree to which they are different decreases as their magnitude increases (1st row), the range of possible values for x increases (2nd row), or both the magnitude and range increase (3rd row). That is, the degree to which two values, say 1 and 2, are different is larger than the degree to which 100 and 101 are different; and it is smaller when range is 10 than when the range is 100. The function $f(m,r) = w_m m + w_r r$, with $w_m, w_r \geq 0$,

Table 6.6 Adding context dependence to FHD (f is a nondecreasing function in each of its arguments)

Context for x	Context equation	α	Degree of difference $d_\alpha(x_i, x_j)$
Magnitude of x	$m = \|x_i\| + \|x_j\|$	$1/m$	$1 - e^{-\frac{(x_i - x_j)^2}{m}}$
Range of x	$r = \max(x) - \min(x)$	$1/r$	$1 - e^{-\frac{(x_i - x_j)^2}{r}}$
Magnitude and Range	$f(m, r)$	$1/f(m, r)$	$1 - e^{-\frac{(x_i - x_j)^2}{f(m,r)}}$

and $w_m + w_r = 1$ can specify the relative importance of the magnitude and range to determine the degree of difference. Obviously, this choice of f generalizes the first two rows of Table 6.6.

With Eq. 6.30, the FHD computation as a fuzzy set is finished. However, as it is often the case with many fuzzy concepts, a non-fuzzy (crisp) approximation for them is useful. Defuzzification is a post-hoc operation, which, strictly speaking, does not belong to the fuzzy approach. Its aim is to extract a crisp value of a fuzzy result. Although, traditionally, defuzzification maps a fuzzy set into a single value, this may be a too drastic reduction in imprecision to be done in a single step. Rather, a two-step defuzzification procedure, as proposed in [89], can be used to extract first a crisp set, and then, a single crisp value, which represents this set. As shown in [88] the operation which always preserves the intended meaning of FHD as a count (i.e., whole number) is given by:

$$cFHD = |L_{D_\alpha;0.5}|, \tag{6.33}$$

where $L_{\mu;a}$ denotes the α-level of the fuzzy set with membership function μ, and $|\cdot|$ denotes cardinality. This defuzzification procedure is, in fact, a two-step procedure: (i) extract a crisp set from a fuzzy set: usually, this is a level set of the membership function; (ii) extract a crisp value based on the set obtained at the previous step.

The fact that $cFHD$ turns out to be related to the 0.5 level set of D_α is interesting since the defuzzification of the (fuzzy) cardinality of a fuzzy set [88] has been derived independently of level sets. On the other hand, it supports a long established, intuitive rule of thumb, according to which a fuzzy set is approximated by its (crisp) 0.5-level set. Note that the fuzzy cardinality, i.e., FHD, is always convex [88] which means that its level sets are closed intervals. Thus, the two-step defuzzification procedure provides the user with a very flexible mechanism which provides access to various descriptions of the Hamming distance between real-valued vectors, as a:

1. fuzzy set, FHD;
2. crisp set, the 0.5-level set of the difference fuzzy set $[a_{D_\alpha;0.5}, b_{D_\alpha;0.5}]$;
3. crisp value, $cFHD$.

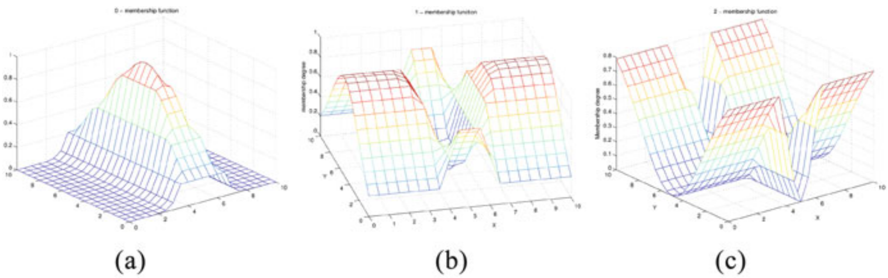


Fig. 6.3 FHD membership function for the distance 0 (**a**), 1 (**b**), and 2 (**c**)

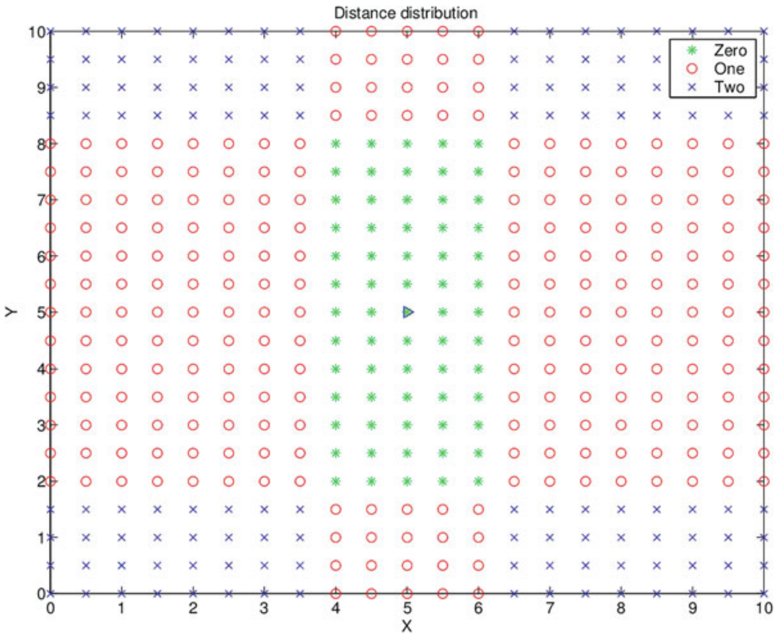


Fig. 6.4 FHD distribution over $(x, y) \in \{0, \ldots, 10\} \times \{0, \ldots, 10\}$

Figures 6.3a, b, c and 6.4 show the membership function of the FHD distance from the point (5, 5) and a generic point $(x, y) \in \{0 \ldots 10\} \times \{0 \ldots 10\}$, when $\beta_x = 0.1$ and $\beta_y = 0.3$, for all possible values of the actual distance—0, 1, or 2. Figure 6.4 shows the distance distribution over the 2D space (x, y). It can be seen from these figures that μ_{FHD} measures how many components are changed with a certain degree. In the middle, for points close to the center, (5, 5), the distance is zero with high degree, while for points close to the margins (determined by the values for β_x and β_y), the degree for 0 decreases and the degree of for 1 is increases (passing $1 - \varepsilon$

level). Beyond this margin, along both x and y, the degree for the distance to be equal to 2 is increasing passing $\geq 1-\varepsilon$ and the degrees for 0 and 1 are again low.

The relation between FHD and other distances deserves more attention. For binary vectors, when $\alpha \geq \ln 2$, $cFHD$, the defuzzified FHD, coincides with HD that is, $cFHD(x, y) = HD(x, y)$. To investigate the relation between FHD and other distances, it is useful to also note that using the fuzzy set operations and the degree of equality along the component i, *the degree of equality along all components* is obtained by taking the t-norm of the degrees of equality along component i, for $i = 1, \ldots, n$. Using as t-norm operators (hence a particular case of aggregation operators) product and minimum, one obtains, respectively:

$$\prod_{i=1}^{n} e^{-\alpha(x_i-y_i)^2} = e^{-\alpha\sum_{i=1}^{n}(x_i-y_i)^2} = e^{-\alpha l_2^2(x,y)} \tag{6.34}$$

$$\min_{i=1\ldots n}\{e^{-\alpha(x_i-y_i)^2}\} = e^{-\alpha \max_{i=1\ldots n}\{(x_i-y_i)^2\}} = e^{-\alpha l_\infty^2(x,y)}. \tag{6.35}$$

The following remarks can be made at this point:

1. Using a different β for each feature means, in effect, weighing each feature thereby allowing a larger or smaller variation between feature values to be considered as a change.
2. If α_i are used instead of α, Eq. 6.34 obtains $e^{-\sum_{i=1}^{n}\alpha_i(x_i-y_i)^2}$, where the exponent is the weighted Euclidean distance or weighted l_2.
3. If α_i are used instead of α, Eq. 6.35 remains the same with $\alpha = \max_{i\in\{1,\ldots,n\}}\alpha_i$.
4. If vectors x, y are heterogeneous, i.e., $x_i, y_i \in D_i$, where the D_i denote different domains, then upon defining a domain specific *discrepancy or difference* measure Δ_i, the definition of FHD holds with $x_i - y_i$ replaced by $\Delta(x_i, y_i)$.

According to Eqs. 6.34 and 6.35, the degree to which vectors x and y are equal increases as the distance (Euclidean in Eq. 6.34, Manhattan in Eq. 6.35) between x and y decreases. The equations further suggest that the choice of $1 - e^{-\alpha(x_i-y_i)^2}$ for describing the degree of the difference is justified. Further comparison between FHD with Euclidean Distance (l_2) shows that FHD is more sensitive than the Euclidean distance. More precisely, the relation between l_2 and FHD is given in the following statements whose proofs can be found in [52]:

1. For any dimension n, there exist vectors x, y, z such that $l_2(x, y) = l_2(x, z)$ and $FHD(x, y) \neq FHD(x, z)$.
2. For all vectors x, y, z, if $FHD(x, y) = FHD(x, z)$ then $l_2(x, y) = l_2(x, z)$.

6.4 Directional Relations

A completely different type of relationships, directional relative position, as a typical example of relations that defy precise definitions, is considered next. Although such relations are very important and explicitly mentioned by Freeman [47] and by Kuipers [65], crisp definitions are clearly not appropriate. The usual way to consider such relations in common language is to consider that they hold to some degree and are non-exclusive, as several relations between two given objects can be satisfied to some degree. Fuzzy set theory is then an appropriate tool for such modeling even for crisp sets. The main fuzzy approaches are reviewed here. More details can be found in [18].

Similar to distances, directional relations and their degree of satisfaction can be expressed as a number, an interval, a fuzzy number (or a distribution). Spatial representations are useful as well, to model the fuzzy region of space where a relation is satisfied, to some degree, with respect to a reference object.

6.4.1 Fuzzy Relations Describing Relative Position

In [60, 79], the angle between the segment joining two points a and b and the x-axis of the coordinate frame (in 2D) is computed. This angle, denoted by $\theta(a, b)$, takes values in $[-\pi, \pi]$, which constitutes the domain on which primitive directional relations are defined.

The four such relations "left," "right," "above," and "below" are defined in [79] as $\cos^2\theta$ and $\sin^2\theta$ functions. Other functions are possible, e.g., trapezoidal shaped membership functions. Whatever the equations, the membership functions for these relations are denoted by μ_{left}, μ_{right}, μ_{above}, and μ_{below} and are mappings from $[-\pi, \pi]$ into [0,1]. The equations are chosen according to simplicity (e.g., cos or sin functions), to the fact that they define a fuzzy partition of $[-\pi, \pi]$, and to their invariance properties with respect to rotation (i.e., a rotation should correspond to a translation of the membership functions).

In the work relying on these definitions, only these four basic directions are used, other relations being expressed in terms of these. However, a straightforward extension to any direction can be given. In 2D, a direction is defined by an angle α with respect to the x-axis. Using this convention, the relationship "right" corresponds to $\alpha = 0$. From $\mu_{right} = \mu_0$, we derive μ_α, representing the relationship "in direction α," for any α, as follows:

$$\forall\theta \in [-\pi, \pi], \mu_\alpha(\theta) = \mu_0(\theta - \alpha), \tag{6.36}$$

with, for instance:

$$\mu_0(\theta) = \begin{cases} \cos^2(\theta) & \text{if } \theta \in [-\frac{\pi}{2}, +\frac{\pi}{2}], \\ 0 & \text{otherwise.} \end{cases} \tag{6.37}$$

This makes the definitions based on angle computation more general. Moreover, it guarantees geometric invariance.

Another solution for defining relations intermediate between the four basic ones can be based on logical combinations of these four basic relations. For instance, "oblique right" is defined by "(above and right of) or (below and right of)." The advantage of this approach is that only four membership functions have to be defined, which is consistent with the usual way of speaking about relative position. The drawback is that, by contrast to the definition proposed in Eq. 6.36, we cannot achieve a great precision in direction using this approach. Also, the shape of the membership function will vary depending on the considered direction, leading to a high anisotropy and therefore a loss of rotation invariance, while it remains the same using Eq. 6.36.

In 3D, direct extensions are obtained by considering six elementary directions and defining a direction by two angles.

6.4.2 Centroid Method

The first simple solution to evaluate a fuzzy relationship between two objects consists in representing each object by a characteristic point. This point is chosen as the object centroid in [64]. Let c_R and c_A denote the centroids of objects R and A, and $\theta(c_R, c_A)$ the angle between the line joining the two points and the x-axis. The degree of satisfaction of the proposition "A is to the right of R" is then defined as:

$$\mu^R_{right}(A) = \mu_{right}(\theta(c_R, c_A)), \tag{6.38}$$

where the membership function μ_{right} is defined as in Sect. 6.4.1. Extension to fuzzy objects can be done in two ways. One way consists in computing a weighted centroid, where the contribution of each object point is equal to its membership value. The second way consists in applying the definition for binary objects on each α-cut and then aggregating the results using a summation [40], or the extension principle [115]. However, this second method may be computationally expensive, depending on the quantization of the object membership values.

6.4.3 Histogram of Angles: Compatibility Method

The method proposed in [79] consists in computing the normalized histogram of angles and in defining a fuzzy set in [0,1] representing the compatibility between this histogram and the fuzzy relation. More precisely, the angle histogram is computed from the angle between any two points in both objects as defined before and normalized by the maximum frequency. Let us denote by $H^R(A)$ this normalized histogram, where R is the reference object and A the object the position of which with respect to R is evaluated:

$$H^R(A)(\theta) = \frac{h^R(A)(\theta)}{\max_\xi h^R(A)(\xi)} \tag{6.39}$$

with

$$h^R(a)(\theta) = \sum_{a,b,\theta(a,b)=\theta} \min(\mu_R(a), \mu_A(b)). \tag{6.40}$$

The function $H^R(A)$ represents the spatial directional relations of the object A with respect to the reference object R.

Expressing $H^R(A)$ in terms of the basic relations can be performed using compatibility (see Sect. 2.5) of the two fuzzy sets μ_α (for some direction α of interest) and $H^R(A)$. The center of gravity of the compatibility fuzzy set provides then a global evaluation.

Another solution is to use a fuzzy pattern matching approach [29, 44] (between μ_α and $H^R(A)$) by computing both the degree of inclusion of μ_α in $H^R(A)$ and their degree of intersection using the definitions of Sect. 3.2, as suggested in [7]. Then the global evaluation is given in the form of a pair necessity/possibility.

The fuzzy extension of this method is based on a weighted histogram [79] which is equivalent to computing a histogram on each α-cut and to combine the obtained results by summation as in [40].

6.4.4 Aggregation Method

An aggregation method has been proposed in [64], which uses all points of both objects instead of only one characteristic point. For any pair of points i in R and j in A, the angle $\theta(i, j)$ is computed, and the corresponding membership value for a direction α (being one of the 4 considered relations) is computed as previously: $\mu_{ij} = \mu_\alpha(\theta(i, j))$. All these values are then aggregated, e.g., using a weighted mean.

6.4.5 Histogram of Forces

Instead of considering pairs of points as in angle histogram approaches, pairs of longitudinal sections are considered in [77], where the concept of F-histogram is introduced. The degree to which an object A is in the direction α with respect to a reference object R is computed using successively points, segments, and longitudinal sections. This leads to a so-called histogram of forces which computes the weight supporting a proposition like "object A is in direction α from object R." Basically, this approach amounts to considering a weighted angle histogram:

$$H^R(A)(\theta) = \sum_{a,b,\theta(a,b)=\theta} \varphi(||\vec{ab}||), \tag{6.41}$$

where φ is a decreasing function. Typically, $\varphi(x) = \frac{1}{x^r}$. For $r = 0$, the weighted histogram is equal to the angle histogram, and for $r \geq 1$, it gives more importance to points of A that are close to some points of R. This is effective in situations where A and R have very different partial extents, and to account only for the closest parts of them.

6.4.6 Projection Based Approach

The approach proposed in [62] is very different from the previous ones since it does not use any histogram. It is based on a projection of the considered object on the axis related to the direction to be assessed (e.g., the x-axis for evaluating the relations "left to" and "right to"). Let us detail the computation for the relation "A is left from R." The same construction applies for any direction. Denoting by $R^f(x)$ the normalized projection of the set R on the x-axis, the degree for a point x to be left to R is defined as:

$$R^{\leftarrow}(x) = \frac{\int_x^{+\infty} R^f(y)dy}{\int_{-\infty}^{+\infty} R^f(y)dy}. \tag{6.42}$$

This definition provides a degree of 1 for points that are completely on the left of the support of R^f and a degree of 0 for points that are completely on the right of the support of R^f, and the degree decreases in-between. The degree $(A \leftarrow R)^f(x)$ to which x is in the projection of another set A and to the left of R is expressed as a conjunction of $A^f(x)$ and $R^{\leftarrow}(x)$. The conjunction is taken as a product in [62]. The degree to which A is left from R is then deduced as the ratio of the areas below $(A \leftarrow R)^f$ and A^f:

$$\mu_\alpha^R(A) = \frac{\int_{-\infty}^{+\infty} A^f(x) \int_x^{+\infty} R^f(y)dydx}{\int_{-\infty}^{+\infty} A^f(x)dx \int_{-\infty}^{+\infty} R^f(y)dy}. \tag{6.43}$$

This approach can be generalized to fuzzy sets [62] by taking each point into account in the projection to the amount of its membership function, leading to similar properties as in the crisp case.

6.4.7 Morphological Approach

In [7–9] a morphological approach has been proposed in order to evaluate the degree to which an object A is in some direction with respect to a reference object R, consisting of two steps (note that this approach applies directly in 3D and on fuzzy objects):

1. A *fuzzy landscape* is first defined around the reference object R as a fuzzy set such that the membership value of each point corresponds to the degree of satisfaction of the spatial relation under examination. This makes use here of a spatial representation of the relation. Therefore the fuzzy landscape $\mu_\alpha(R)$ is directly defined in the same space as the considered objects, in contrast to the projection method [62], where the fuzzy region is defined on a one-dimensional axis.
2. Then the object A is compared to the fuzzy landscape $\mu_\alpha(R)$, in order to evaluate how well the object matches with the regions having high membership values (i.e., regions that are in the desired direction). This is done using a fuzzy pattern matching approach, which provides an evaluation as an interval or a pair of numbers instead of one number only. An average value can be computed as well.

The key point relies in the definition of $\mu_\alpha(R)$ in the first step. The requirements stated above for this fuzzy set are not strong and leave room for a large spectrum of possibilities. This flexibility allows the user to define any membership function according to the application at hand and the context requirements. The definition proposed in [8] looks precisely at the domains of space that are visible from a reference object point in the desired direction. This applies to any kind of objects, including those having strong concavities. This amounts to dilate μ_R by a fuzzy structuring element defined on $\mathcal{S}$ as:

$$\forall P \in \mathcal{S},\ \nu(P) = \max\left(0, 1 - \frac{2}{\pi} \arccos \frac{\vec{OP} \cdot \vec{u}_\alpha}{\|\vec{OP}\|}\right), \tag{6.44}$$

(or another function having the same variations) where O is the center of the structuring element. This morphological definition is illustrated in Fig. 6.5.

In the second step, the evaluation of relative position of A with respect to R is given by a function of $\mu_\alpha(R)(x)$ and $\mu_A(x)$ for all $x \in \mathcal{S}$. The histogram of $\mu_\alpha(R)$ conditionally to μ_A is such a function. While this histogram gives the most complete information about the relative spatial position of two objects, it is difficult to reason in an efficient way with it. A summary of the information contained in the histogram could be more useful in practice. An appropriate tool for defining this summary is the fuzzy pattern matching approach [44] which provides an evaluation as two numbers: a necessity degree N (a pessimistic evaluation) defined as a degree of inclusion and a possibility degree Π (an optimistic evaluation) defined as a degree of intersection. They can also be interpreted in terms of fuzzy

Fig. 6.5 Left: a fuzzy reference object. Right: fuzzy landscape representing the relationship "to the left of" obtained by a fuzzy dilation by a directional structuring element

mathematical morphology, since the possibility is equal to the dilation of μ_A by $\mu_\alpha(R)$ at origin, while the necessity is equal to the erosion, as shown in [17]. These two interpretations, in terms of set theoretical operations and in terms of morphological ones, explain how the shape of the objects is taken into account. Several other functions combining $\mu_\alpha(R)$ and $\mu_A(x)$ can be constructed. The extreme values provided by the fuzzy pattern matching are interesting because of their morphological interpretation, and because they provide a pair of extreme values and not only a single value and may better capture the ambiguity of the relation if any. One drawback of these measures is that they are sensitive to noise, since they rely on infimum and supremum computation. An average measure can also be useful from a practical point of view (it is much less sensitive to noise).

6.4.8 Discussion and Examples

A formal comparison of these approaches is given in [18], based on their properties, the type of basic elements on which they rely, their behavior in extreme situations, in case of concavities, of distant objects, and on their computational cost. The main conclusions that can be drawn are as follows. While most approaches reduce the representation of objects to points, segments, or projections, only the morphological approach considers the objects as a whole and therefore better accounts for their shape. Approaches providing evaluation as intervals or fuzzy numbers are better suited for representing the ambiguity inherent to such relationships. If the distance between objects increases, an object is seen as a point from the reference object, which could be intuitively expected, for all methods but the projection approach. All approaches have a similar complexity, except the centroid method which is computationally simpler but which also reduces too much the amount of information.

The extension to 3D objects requires to represent a direction by two angles, which is generally straightforward but may increase the complexity. The extension of the angle histogram method to 3D objects amounts to computing a bi-dimensional histogram, i.e., as a function of these two angles, and then applying the same principle using the relations defined in 3D. The computation of the histogram is heavy in 2D and becomes even more so in 3D. Another problem when computing bi-histograms is that the domain of possible angle values may be under-represented, depending on the size and the sampling of the considered objects. This may result in a noisy and hole-containing histogram. This effect already appears in 2D. Extension of the force histogram method to 3D objects could be probably done, but with a high complexity. The morphological approach is directly applicable in 3D, without changing the complexity with respect to the number of points. All approaches can been extended to fuzzy objects using their α-cuts, at the price of a high computational cost. The morphological approach can be directly applied, without cost increase. The angle histogram approach can be extended easily too, by computing weighted histograms.

Another very important aspect is the type of questions each definition is able to answer, or dedicated to. This could be the main ingredient for a choice among all possible approaches. These questions can take different forms, e.g.:

- What are the spatial relationships between two given objects?
- To which degree a given spatial relation holds between two objects?
- What are the regions of the space where a spatial relationship is satisfied (to some degree) with respect to a reference object?

An important feature of angle histogram and force histogram is that they provide a general description of the directional relationships. From this general information, several relations can be deduced, as the degree of satisfaction of one specific relationship (for a particular direction), or the dominant relationship. This is not easy to obtain with the morphological approach that needs one computation for each direction of interest. Therefore, the angle histogram approach is more dedicated to cases where some specified relations should be evaluated.

For problems where the relative position of several objects with one reference object has to be evaluated, the morphological approach may be more appropriate if the computation time is a strong requirement.

Another advantage of the morphological approach is that the first step directly provides a spatial representation of a directional constraint with respect to a reference object (thus answering the third type of question), which can be used to guide the search for another object [11, 21] (see examples for image understanding in Chap. 9).

In order to illustrate the reviewed definitions, we choose two simple examples, shown in Fig. 6.6. Despite their apparent simplicity, they lead to eloquent results and allow us in particular to show how different parts of the objects can be taken into account.

Table 6.7 shows the results for object A with respect to object R, according to various methods. They all agree to say that A is mainly to the right of R. The degree of being to the right increases with the value of r, since the part of A which is to the right of R is the closest one to R. By contrast, the degree of being above decreases with r. The values are somewhat different for all approaches, but since the ranking

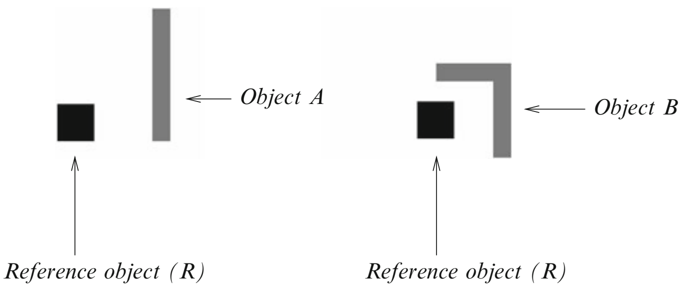


Fig. 6.6 Two examples where the relative position of objects with respect to the reference object is difficult to define in an "all-or-nothing" manner

Table 6.7 Relative position of object A (rectangle) with respect to object R (square) of Fig. 6.6, using centroid, aggregation compatibility methods, and using the morphological approach (the $[N, \Pi]$ intervals are given, as well as the average value). Angle or force histograms are computed using $r = 0$, $r = 2$, and $r = 5$ (the angle histogram method corresponds to $r = 0$)

Object A with respect to object R								
		Aggregation			Compatibility			Morpho. approach
Relation	Centroid	$r = 0$	$r = 2$	$r = 5$	$r = 0$	$r = 2$	$r = 5$	$[N, \Pi]$ average
Left	0.00	0.00	0.00	0.00	0.00	0.00	0.00	[0.00, 0.00] 0.00
Right	0.76	0.73	0.79	0.86	0.62	0.67	0.75	[0.50, 1.00] 0.81
Below	0.00	0.00	0.01	0.01	0.05	0.06	0.06	[0.00, 0.35] 0.05
Above	0.24	0.27	0.20	0.13	0.38	0.33	0.25	[0.00, 0.73] 0.44

and the general behavior is the same, no conclusion concerning a more favorable approach can be derived from this example.

Table 6.8 shows the results for object B with respect to object R. For these objects, two main relations are satisfied: right and above. The centroid method does not account well for the above relation, for which it gives a very low value. This shows one of the limitations of this approach which is too simple in that it reduces too much the data. Since the part of B which is above R is closer than the one to its right, the values of being right decrease with r while the values of being above increase. The morphological approach highlights the ambiguity of the relations for these objects. Parts of B satisfy completely the above relation, for instance, while other parts do not satisfy it at all. The non-zero degrees obtained for the relation below, for instance, are due to some points of B that are indeed partially below R.

More examples will be presented in Chap. 9, in particular for brain image understanding.

Table 6.8 Relative position of object B with respect to object R of Fig. 6.6, using centroid, aggregation, compatibility methods, and using the morphological approach (the $[N, \Pi]$ intervals are given, as well as the average value). Angle or force histograms are computed using $r = 0$, $r = 2$, and $r = 5$

Object B with respect to object R								
		Aggregation			Compatibility			Morpho. approach
Relation	Centroid	$r = 0$	$r = 2$	$r = 5$	$r = 0$	$r = 2$	$r = 5$	$[N, \Pi]$ average
Left	0.00	0.00	0.00	0.01	0.05	0.06	0.08	[0.00, 0.44] 0.02
Right	0.83	0.63	0.54	0.33	0.55	0.49	0.35	[0.29, 1.00] 0.81
Below	0.00	0.03	0.02	0.01	0.17	0.16	0.15	[0.00, 0.60] 0.11
Above	0.17	0.34	0.43	0.66	0.45	0.51	0.65	[0.00, 1.00] 0.52

6.5 Complex Relations: Surround, Between, Along, Across, Parallel, Aligned

While basic relations such as topological and metrical relations described in the previous sections are the most prominent ones, some more complex relations are also worth to be modeled, since they are used to describe complex arrangements of several objects in a scene. Again, their intrinsic imprecise nature gives naturally rise to fuzzy models. A few examples are presented in this section.

6.5.1 Surround

From directional relations, more complex relations such as "surround" can be defined. In [78], the tangent lines to an object A originating from a point of an object B are computed, as well as the angle θ between these lines. Then the degree of the relation surround is defined as:

$$\mu_{surround}(\theta) = \begin{cases} \cos^2 \frac{\theta}{2} & \text{if } 0 \leq \theta \leq \pi \\ 0 & \text{otherwise.} \end{cases} \tag{6.45}$$

If B is not reduced to a point, θ is defined for each point of B and the set of obtained values is handled as for the directional relations. A similar approach has been proposed in [64], but with a linear function of θ instead of a trigonometric one. The extension to fuzzy sets is done by integration over α-cuts, which can be computationally intensive. This type of definition is not necessarily adapted to any types of shapes and is difficult to extend to 3D objects.

Slightly different approaches for visual surround relation and topological surround relation have been proposed in [95].

Another approach consists in combining in a disjunctive way several directional relations such as left, right, above, below, etc. [8, 9, 78], which extends directly to

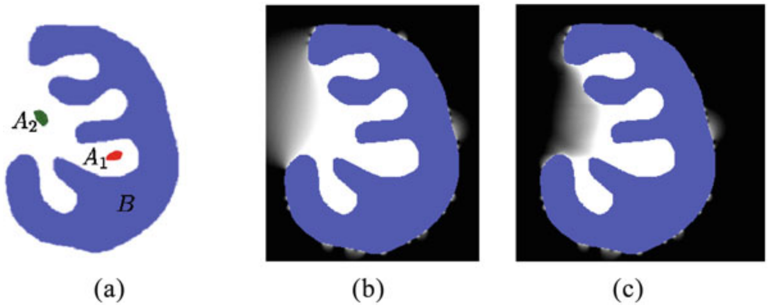


Fig. 6.7 Illustration of the fuzzy landscapes when not considering the distance of the target object and when considering it [107]. (**a**) Reference object B, and two target objects A_1 and A_2. (**b**) $\mu_{surround}(B)$ using a standard angular coverage measure. (**c**) $\mu_{surround}(B, \mu_n)$ taking into account also the distance between the target object A_2 and the reference object. Points in A_1 have higher degrees than points in A_2

3D and to fuzzy objects. But the difficulty is to distinguish between situations where A surrounds B and situations where B surrounds A.

Another definition, where the shape of the objects is taken into account, and that can hence deal with complex objects (crisp or fuzzy) [107], can be defined starting from ideas similar to the fuzzy landscape presented above. The membership of each point to this fuzzy landscape (representing to which degree this point is surrounded by the reference object) is defined as a decreasing function of an angular coverage measure (similar as in [95]), equal to the total angular length of the angular intervals for which this point is able to see the reference object, possibly tuned according to the distance to the reference object. This allows differentiating situations as in Fig. 6.7, where A_1 is more surrounded by B than A_2.

The surround relation was extended to more complex spatial scenarios, where the objects are nested, under the term of "enlacement or interlacement" in [33]. The construction is based on relations between cross-sections, as for the force histogram. This approach was used to analyze eye fundus (retinal images), showing a complex blood vessel network, and to differentiate between healthy and pathological eye fundus.

6.5.2 Between

The "between" relation is a typical example for which the definition should be contextual, i.e., it should depend on the relative spatial extension of the objects, on the type of problem at hand, on the application domain. For instance, the relation "between" cannot be defined in the same way whether the objects have similar spatial extensions or not. Its semantics changes depending on whether we consider a person between two buildings, a fountain between a house and a road, or a road passing between two houses. These differences have been discussed in cognitive and linguistic studies [75]. Similar examples can be found for other relations, such as "surround," "along," etc. It follows that it is impossible to propose a single definition that would apply to all possible contexts.

In the definition of [64], the degree to which an object B is between two objects A_1 and A_2 in the 2D space is computed based on a relation between points. For all $a_1 \in A_1, a_2 \in A_2, b \in B$, the angle θ at b between the segments $[b, a_1]$ and $[b, a_2]$ is computed. Then a function $\mu_{between}(\theta)$ is defined as a trapezoidal function with central modal value at π and is used to measure the degree to which b is between a_1 and a_2. For extended objects, the angle θ is averaged over all triplets of points (a_1, a_2, b) and the degree to which B is between A_1 and A_2 is defined as $\mu_{between}(\Theta)$, where Θ is this average angle. This approach is extended to fuzzy objects by integrating the results obtained on all α-cuts. Besides being computationally heavy in the case of fuzzy objects, this approach does not always match the intuition, as shown in [22].

The definition proposed in [76] also relies on computation of angles, but in a different spirit. In this work, what is actually computed is the degree to which a set B is between different connected components of a set A. This is based on the normalized angle or force histogram of A and B. Each α-cut H_α of this histogram is computed and some angles are defined to represent the intervals between connected components of H_α and the length of these connected components. The degree of the relation is then 0 if A is connected, and depends on the number of its connected components and on a function of these angles otherwise. Then the result is integrated over all α-cuts of the histogram. This approach has some major drawbacks: its computational complexity increases with the granularity of the quantification of α; parts of the objects which are opposite to the between area have an influence (while they should not be involved) and induce a bias in the results; the examples shown in [76] are sometime counter-intuitive when A has three connected components or more (a degree smaller than what would be expected is obtained), the problem being that the relation is then ambiguously defined and depends on which connected components build A_1 and which ones build A_2; finally, this approach does not deal appropriately with the non visible concavities of components of A.

The definition in [23] applies on one-dimensional fuzzy sets. It relies on the definition of fuzzy ordering based on a T-equivalence, from which unary ordering-based modifiers are defined. Without going into details, these modifiers correspond to left and right relations (or before and after if the space is the temporal axis) and are idempotent. The fuzzy region between two 1D fuzzy sets A_1 and A_2 is then defined as conjunctions and disjunctions of these modifiers applied to both fuzzy sets. The main difficulty in these definitions is that they rely on an ordering, which makes their extension to higher dimensions a more difficult task. However, the interpretation of these equations can provide some hints on possible extensions, exploited in definitions based on dilations [22] (described hereafter).

It is also interesting and useful to look at some linguistic and cognitive aspects to understand the meaning of the relation. As stated in [66], the area between two objects is cognitively understood as "the minimal space bounded by the pair of reference objects." In mathematical terms, this corresponds to the convex hull of the union of both objects. This is confirmed by the work of [75], where a simple definition of the common understanding is provided: the region between A_1 and A_2 is defined as the strict interior of the convex hull of $A_1 \cup A_2$ to which A_1 and A_2 are then suppressed. Obviously this applies only in simple situations, as explained below, but it justifies the choice of the notion of convex hull. An interesting idea is briefly mentioned in this work [75]: in cases where the objects have different spatial extensions, in particular when one object can be considered as infinite with respect to the other, then the proposed definition is not meaningful and more contextual definitions can be proposed, such as the area of the projection of the small object on the large one. However, as already mentioned, no general criteria covering all possible situations can be stated.

Let us now describe the approaches proposed in [22]. The intuitive answer given by most people when defining the region between two objects involves the convex hull of their union. This notion also appears in some work in the domain of linguistic and cognition [75]. Furthermore, links between logical operators for defining the between relation and convexity have been also exhibited [1]. These observations lead to the following simple definition. For any set X (closed and bounded), assumed first to be binary (crisp), X^C its complement, and $CH(X)$ its convex hull, we define the region of space between two objects A_1 and A_2, denoted by $\beta(A_1, A_2)$, as:

$$\beta_{CH}(A_1, A_2) = CH(A_1 \cup A_2) \cap A_1^C \cap A_2^C. \tag{6.46}$$

This definition is well adapted to convex objects or having concavities "facing each other." For more complex objects, the connected components of $CH(A_1 \cup A_2) \setminus (A_1 \cup A_2)$ which are not adjacent to both A_1 and A_2 should be suppressed. Unfortunately, a drawback of this approach is its lack of continuity with respect to small deformations of the objects. Although this approach can be easily extended to the case of fuzzy objects [22], it is often better to move to definitions that are more appropriate in a large class of situations.

One of these approaches uses the directional relative position between the two objects [22]. Directional dilation is then performed, using directional fuzzy structuring elements, with a similar approach as in [8]. For definition and properties of fuzzy dilation, please refer to Chap. 4. The main direction between two objects can be determined from the angle histogram [79]. In the case of simple objects, it may be sufficient to compute only the main direction α between the two objects from this histogram. Let D_α denote the dilation in direction α. The structuring element can be either a crisp segment in the direction α or a fuzzy structuring element where the membership function at a point (r, θ) (in polar coordinates) is a decreasing function of $|\theta - \alpha|$ [8]. From this dilation, we define:

$$\beta_\alpha(A_1, A_2) = D_\alpha(A_1) \cap D_{\pi+\alpha}(A_2) \cap A_1^C \cap A_2^2, \tag{6.47}$$

which is a fuzzy set when the structuring element is fuzzy (in the fuzzy case, complementation is replaced by a fuzzy complementation and intersection by a t-

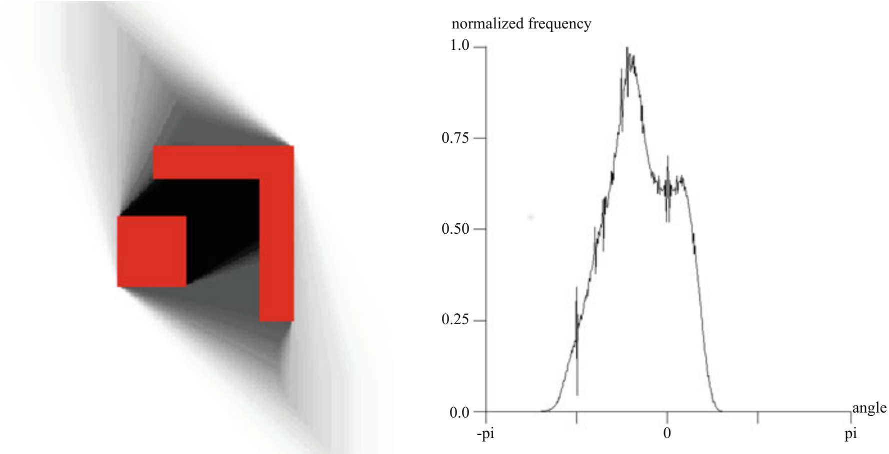


Fig. 6.8 Definition based on dilation by a structuring element derived from the angle histogram (Eq. 6.50). Objects A_1 and A_2 are displayed in red, and the membership values to $\beta(A_1, A_2)$ vary from 0 (white) to 1 (black). The angle histogram is shown on the right

norm). For more complex objects, it can be difficult or meaningless to find only one main direction. Then the histogram of angles can be used directly as a fuzzy structuring element. Let us define two fuzzy structuring elements ν_1 and ν_2 from the normalized angle histogram $H_{(A_1,A_2)}(\theta)$ as:

$$\nu_1(r, \theta) = H_{(A_1,A_2)}(\theta), \tag{6.48}$$

$$\nu_2(r, \theta) = H_{(A_1,A_2)}(\theta + \pi) = \nu_1(r, \theta + \pi). \tag{6.49}$$

Several definitions of the between region can be envisaged. The simplest one is:

$$\beta_{FDil1}(A_1, A_2) = D_{\nu_2}(A_1) \cap D_{\nu_1}(A_2) \cap A_1^C \cap A_2^C, \tag{6.50}$$

which is illustrated in Fig. 6.8.

Another definition is inspired by Bodenhofer [23]:

$$\beta_{FDil2}(A_1, A_2) = [D_{\nu_1}(A_1) \cup D_{\nu_1}(A_2)] \cap [D_{\nu_2}(A_1) \cup D_{\nu_2}(A_2)], \tag{6.51}$$

and, by removing the concavities which are not "facing each other," we obtain:

$$\beta_{FDil3}(A_1, A_2) = D_{\nu_2}(A_1) \cap D_{\nu_1}(A_2) \cap A_1^C \cap A_2^C \cap [D_{\nu_1}(A_1) \cap D_{\nu_1}(A_2)]^C \cap [D_{\nu_2}(A_1) \cap D_{\nu_2}(A_2)]^C, \tag{6.52}$$

which is illustrated in Fig. 6.9.

Fig. 6.9 Definition based on dilation by a structuring element derived from the angle histogram, with Eq. 6.52

A potential problem with these definitions is that the result may be spatially too large. Possible solutions are to threshold the angle histogram, or to take the intersection of the result with $CH(A_1 \cup A_2)$. Since these solutions require some parameters that may seem to be chosen in an ad hoc way, a formalization of these ideas will be achieved with the visibility approach, presented next.

The extension of this approach to the case of fuzzy objects A_1 and A_2 is straightforward, by computing a weighted histogram of angles. The extension to 3D objects (fuzzy or not) is also straightforward. Illustrations are provided here in 2D only for the sake of visualization simplicity.

Another powerful approach relies on the notion of visibility [22], based on the notion of admissible segments as introduced in [95]: a segment $]x_1, x_2[$, with x_1 in A_1 and x_2 in A_2, is said to be admissible if it is included in $A_1^C \cap A_2^C$. Note that x_1 and x_2 then necessarily belong to the boundary of A_1 and A_2, respectively (A_1 and A_2 are supposed to be compact sets). The visible points are those which belong to admissible segments. The region $\beta_{Adm}(A_1, A_2)$ between A_1 and A_2 can then be defined as the union of admissible segments. However, the definition of admissible segments may be too strict in some cases. In order to get more flexibility, the notion of approximate (or fuzzy) visibility extends both the crisp definition of visibility and the definition proposed in [64] in the sense that the information is not reduced to an average angle. This is achieved by relaxing the admissibility to semi-admissibility through the introduction of an intermediary point P on the segments. A segment $]a_1, P]$ with $a_1 \in A_1$ (respectively, $[P, a_2[$ with $a_2 \in A_2$) is said semi-admissible if it is included in $A_1^C \cap A_2^C$. At each point P of space, the angle the closest to π between two semi-admissible segments from P to A_1 and A_2, respectively, is computed. This is formally defined as:

$$\theta_{min}(P) = \min\{|\pi - \theta|, \theta = \angle([a_1, P], [P, a_2]), (a_1, P] \text{ and } [P, a_2) \text{ semi-admissible}\}. \tag{6.53}$$

The region between A_1 and A_2 is then defined as the fuzzy region of space with membership function:

$$\beta_{FVisib}(A_1, A_2)(P) = f(\theta_{min}(P)), \tag{6.54}$$

where f is a decreasing function from $[0, \pi]$ to $[0, 1]$ such that $f(0) = 1$, and f is set to 0 at the largest acceptable distance to π (this value can be tuned according to the context). If the objects are fuzzy, with membership functions μ_{A_1} and μ_{A_2}, the notion of admissible or semi-admissible segments has to be changed by replacing the inclusion by a fuzzy inclusion. This concerns both the membership of the extremities of the segments a_1 and a_2, and the inclusion of $]a_1, a_2[$ in $A_1^C \cap A_2^C$, which can be expressed as a usual degree of inclusion. The extension to 3D is again straightforward.

Finally, if the objects have different spatial extensions, a "myopic" vision can be adopted [22]. Assume that one of the objects, say A_2, can be considered to have infinite size with respect to the other. Intuitively, the between area should be considered between A_1 and the only part of A_2 which is the closest to A_1, instead of considering A_2 globally. Hence the idea of projecting A_1 onto A_2 in some sense, and to consider the "umbra" of A_1. Here an additional assumption, largely verified in most situations, can be made by approximating the part closest to A_1 by a segment. Let us denote the segment direction by $\vec{u}$. The between region can then be defined by dilating A_1 by a structuring element defined as a segment orthogonal to $\vec{u}$ and limiting this dilation to the half-plane defined by the segment of direction $\vec{u}$ and containing A_1. However, this may appear as too restrictive and a fuzzy dilation [17] by a structuring element having decreasing membership degrees when going away from the direction orthogonal to $\vec{u}$ is more flexible and matches better the intuitive idea. The projection segment can be defined by dilating the part of A_2 closest to A_1 (obtained by a distance map computation) conditionally to A_2 and computing the axis of inertia of the result. This approach is illustrated in Fig. 6.10. In terms of visibility, it corresponds to a "myopic" vision, in which the parts of A_2 which are too far from A_1 are not seen. A visibility constraint can be added as well, as in the previous approach.

Once the region of space $\beta(A_1, A_2)$ between two objects A_1 and A_2 is defined, the degree to which an object B is between A_1 and A_2 can be defined and computed, by comparing B and $\beta(A_1, A_2)$, using inclusion measures, for instance.

Properties and extensions of these definitions are detailed in [22].

Let us now illustrate some of the proposed definitions on anatomical objects (these examples are reproduced from [22]). The first example concerns brain structures. Figure 6.11 shows a few brain structures, on a 2D slice. The between relation is used in many anatomical descriptions, such as:

- The internal capsule (IC) is *between* the caudate nucleus (CN) and the lenticular nucleus (LN).
- The corpus callosum (CC) is *between* the lateral ventricles (LV) and the cingulate gyrus (CG).

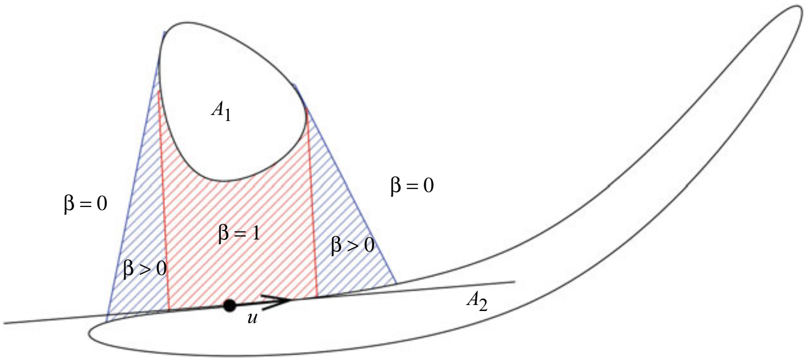


Fig. 6.10 Illustration of the definition of region β in case of an extended object (myopic vision). In the areas indicated by $\beta > 0$, the relation is satisfied to some degree between 0 and 1. They can be more or less spread depending on the structuring element, i.e., on the semantics of the relation

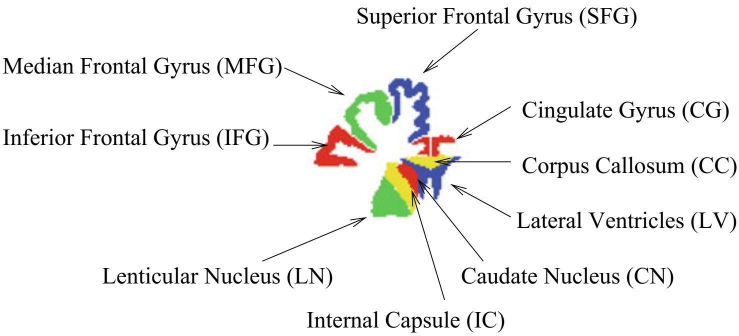


Fig. 6.11 A few brain structures (a 2D slice extracted from a 3D atlas)

- The medial frontal gyrus (MFG) is *between* the inferior frontal gyrus (IFG) and the superior frontal gyrus (SFG).

The above definitions were applied to define the region between the aforementioned brain structures. Figure 6.12 illustrates the third example and shows the region "between" IFG and SFG using the directional dilation (a-d), the convex hull approach (e), the admissible segments (f) and the fuzzy visibility (g). It is clear that the convex hull definition fails in the presence of concavities. This problem is solved by the directional dilation and visibility approaches. In (c) and (e), non-visible concavities are included in the result, while they have been adequately suppressed in (d), (f), and (g). Also, it should be noted that fuzzy methods (directional dilation and fuzzy visibility) are more appropriate than crisp ones (convex hull and admissible segments): the median frontal gyrus is partly outside the β region defined by crisp approaches while remaining in areas with high membership degrees when using the fuzzy ones. This last result better fits the intuition, and is confirmed by the high values obtained for the overlap measure. Note that the region where $\beta = 1$

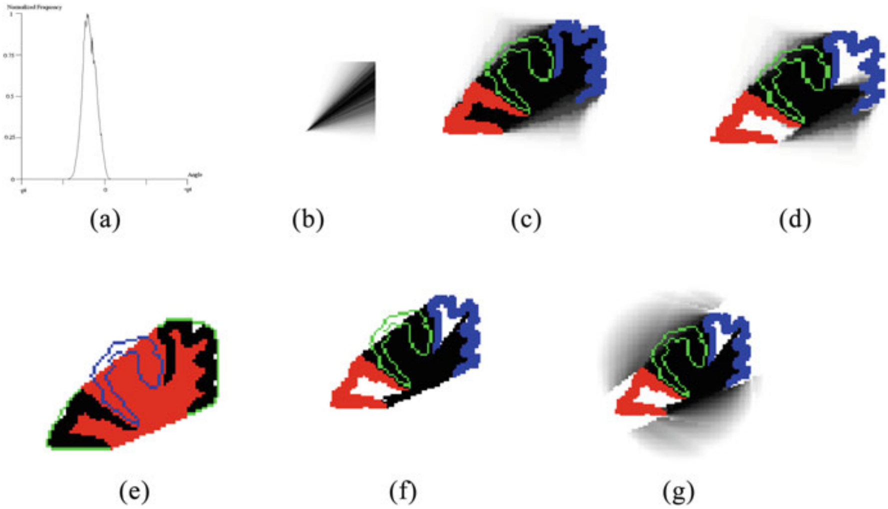


Fig. 6.12 (**a**) Angle histogram of objects A_1 and A_2 (superior and inferior frontal gyri, displayed in red and blue in (**c**)). (**b**) Corresponding structuring element ν_1 (ν_2 is its symmetric with respect to the origin). (**c**) Definition based on fuzzy dilation (Eq. 6.50). Membership values to $\beta(A_1, A_2)$ vary from 0 (white) to 1 (black). The contours of the median frontal gyrus are superimposed in green. (**d**) Definition based on fuzzy dilation, with Eq. 6.52. (**e**) Convex hull approach. (**f**) Definition using the admissible segments. (**g**) Fuzzy visibility approach

is the same in (f) and (g) and almost the same in (d), therefore these approaches are equivalent for objects completely included in this region. Visibility methods and fuzzy dilations are better in handling concavities than the convex hull approach since the only concavities which are visible from the other object are kept in β. The fact that parts of object B which are outside the convex hull of $A_1 \cup A_2$ have a non-zero membership degree to the area β is consistent with the way of speaking about the relation in many situations. For instance, it is often said that "the nose is between the eyes", which is understood by anybody, but does not mean that the nose is in the convex hull of the union of both eyes (generally it is not!).

To evaluate the relation "B is between A_1 and A_2," the normalized intersection of B and β is computed: $\frac{|\beta \cap B|}{|B|}$. A few results are shown in Table 6.9. They correspond to what is intuitively expected. Higher degrees are obtained with fuzzy methods, which again indicates that they are more appropriate when objects are not completely included in the crisp β region. The measures are, however, quite similar for all approaches, since none of the objects B is located in a concavity. The fifth row corresponds to a case where only a part of B is between A_1 and A_2, the relation being thus satisfied with a lower degree than in the previous cases. This case can be compared to the ones obtained in the fourth row. Indeed the corpus callosum (CC) is more between the caudate nucleus (CN) and the cingulate gyrus (CG) than the lateral ventricles (LV), which have their lower part outside the between area. All methods preserve the order intuitively expected between the different situations.

Table 6.9 A few results obtained with different methods: convex hull (1), fuzzy directional dilation (using Eq. 6.52) (2), admissible segments (3) and with the fuzzy visibility approach (4)

A_1	A_2	B	$\frac{\lvert\beta\cap B\rvert}{\lvert B\rvert}$ (1)	$\frac{\lvert\beta\cap B\rvert}{\lvert B\rvert}$ (2)	$\frac{\lvert\beta\cap B\rvert}{\lvert B\rvert}$ (3)	$\frac{\lvert\beta\cap B\rvert}{\lvert B\rvert}$ (4)	$[N, \Pi]$ (1)	$[N, \Pi]$ (4)
CN	LN	IC	0.85	0.84	0.84	0.94	[0, 1]	[0.2, 1]
LV	CG	CC	1.00	0.93	1.00	1.00	[1, 1]	[1, 1]
IFG	SFG	MFG	0.78	0.92	0.76	0.95	[0, 1]	[0.7, 1]
CG	CN	CC	0.88	0.90	0.88	0.97	[0, 1]	[0.6, 1]
CG	CN	LV	0.47	0.63	0.47	0.79	[0, 1]	[0, 1]
IFG	SFG	IC	0.00	0.02	0.00	0.16	[0, 0]	[0, 0.6]
IFG	SFG	LN	0.00	0.00	0.00	0.04	[0, 0]	[0, 0.3]

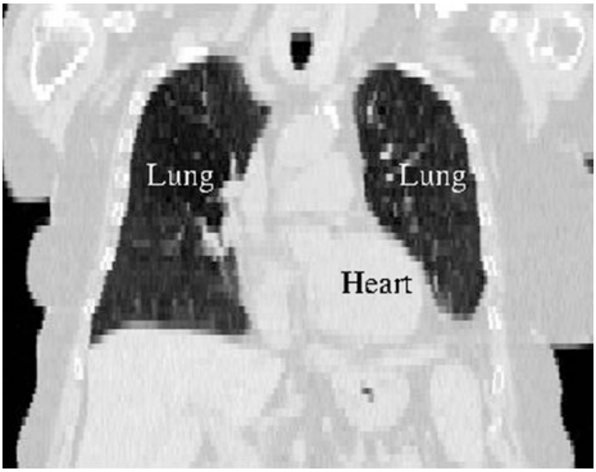
However, ratios cannot be directly compared. The last two rows correspond to cases where the relation is not satisfied. Low but non-zero values are obtained with the fuzzy approaches, because of the tolerance on angles. Fuzzy methods provide higher degrees than non-fuzzy ones (after suppressing the non-visible concavities), since they are more tolerant, as seen in Fig. 6.12, and do not restrict the between area to the convex hull of the union of both objects (or a subset of it). Results for the interval $[N, \Pi]$ are also provided. For the convex hull and admissible segments methods, this measure is not really significant (the results for both methods are exactly the same). By contrast, the overlap measure corresponds to what is intuitively expected. The intervals obtained for the fuzzy visibility method are more interesting since they better discriminate different types of situations. These intervals are similar for the fuzzy dilations and are therefore not displayed.

Let us consider another example to illustrate the usefulness of the fuzzy approaches, still in medical imaging, but now in the thoracic area (in 3D). A useful way to segment the heart in computerized tomography (CT) images consists in reducing the search area in order to avoid confusion with other soft tissues such as the liver. This search area can be intuitively defined as the area between the lungs, as can be seen in Fig. 6.13.

The method based on crisp admissible segments provides an interesting result, but somewhat restricted since a part of the heart is outside of the resulting region, as shown in Fig. 6.14.

Figure 6.15 illustrates the results obtained by fuzzy dilation based on angle histograms. Note that the results are displayed on 2D slices, but the whole computation is done in 3D. The fuzzy region includes, with decreasing degrees, parts outside the convex hull of the union of the lungs, providing a better region of interest for the heart, compared to the crisp version. The contours of the heart have been added to this figure in order to illustrate this feature of the fuzzy dilation approach. Similar results are obtained with the fuzzy visibility method (i.e., semi-admissible segments). The satisfaction degree of the relation "the heart is between the lungs" is equal to 0.87 for the crisp method (admissible segments), and to 0.99 for both fuzzy methods (fuzzy dilation and fuzzy visibility). The higher degree

Fig. 6.13 A coronal slice of a CT image in the thoracic area



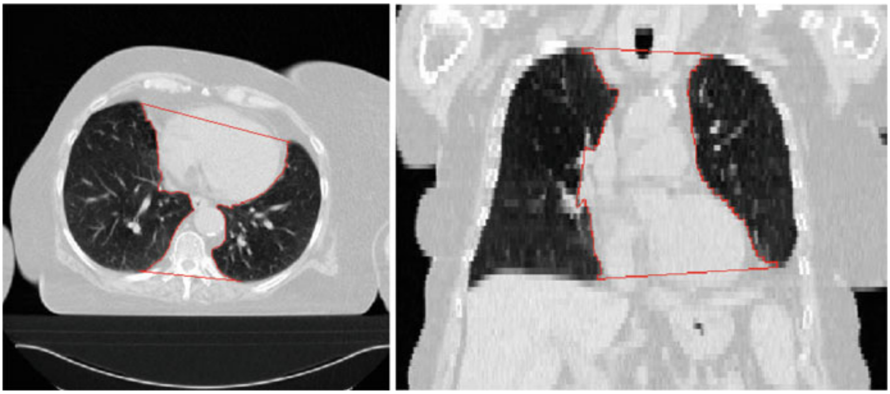


Fig. 6.14 Contours of β_{Adm} representing the region between the lungs, superimposed on an axial slice and on a coronal slice

obtained for the fuzzy methods confirms the usefulness of such approaches for modeling this type of anatomical information.

6.5.3 Across

The relation "to go across" can have several meanings, as analyzed in [109] for a line and a region. The first meaning is very permissive and only considers the fact that a line enters or starts at the border of the region and leaves or ends at the border of the region. This refers to topological relations between some parts of the objects and can also be named "go through." It was modeled in [109] by a conjunction of degrees of intersection / non-intersection between the line or its end points and the region.

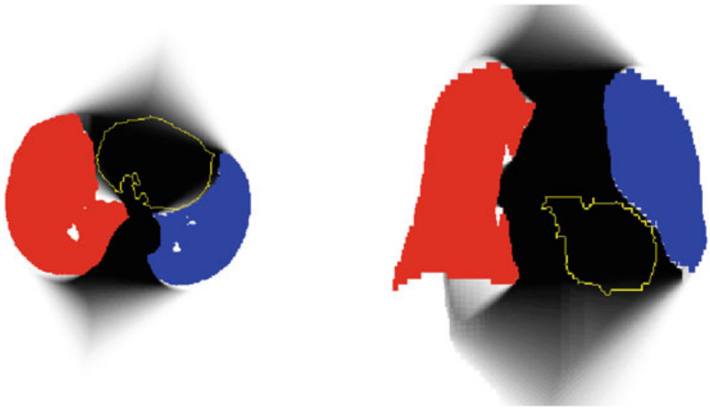


Fig. 6.15 Fuzzy region β_{FDil3} between the lungs, superimposed on an axial slice and on a coronal slice of the segmented lungs. The contours of the heart and aorta are superimposed too

Two more restrictive meanings also consider the geometry: (i) when the line goes from one side of the region to the opposite one; (ii) when the line goes deeply inside the region and passes close to its middle point. Case (i) has been modeled from the notion of approximate opposite sides, while Case (ii) also includes a degree of how deep inside the region the line goes (based on a distance map to the boundary of the region).

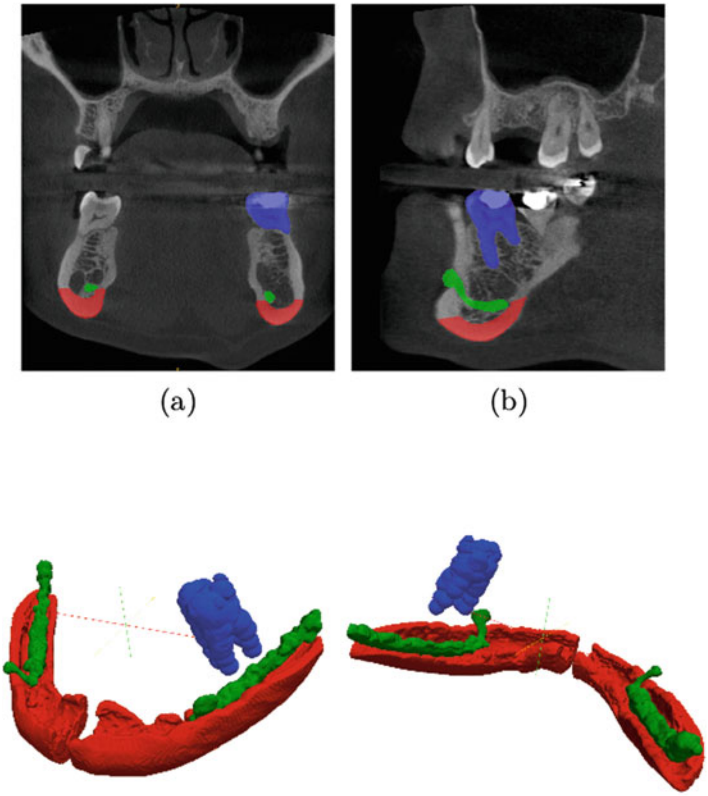
These three definitions were also successfully applied in satellite imaging (for instance, for characterizing paths and roads with respect to some regions).

6.5.4 Along

The relation "along" was modeled for crisp and fuzzy objects, in 2D and in 3D in [45, 103]. Two objects A and B are said to be along each other if they are side by side, with at least one of them being stretched in a direction almost perpendicular to the $A - B$ axis. Side by side implicitly means that the two objects are close to each other comparatively to their environment. Such conditions imply a specific shape for the space between A and B, which should be elongated. Looking at the inter-objects region allows computing the relation only between parts of objects, based on the distance between them. The global method to define the relation is then composed of three steps:

1. Compute the inter-objects region $\beta(A, B)$.
2. Define the elongation of β.
3. Compute the degree of satisfaction of the relation.

The first step uses one of the methods to compute the between relation described in Sect. 6.5.2. The elongation is then defined as a decreasing function of the



(a) (b)

(c) (d)

Fig. 6.16 Segmented objects of the maxillo-facial area in CBCT images: mandibular canals are in green, mandibular floors in red and one tooth in blue [45]. (**a**) Segmented objects overlayed on a coronal slice. (**b**) Segmented objects overlayed on a sagittal slice. (**c**) 3D facial view. (**d**) 3D dorsal view

compactness (ratio between volume and surface), which can include distance between the objects, in order to give more weights to the parts that are closer to each other.

As an example, let us illustrate these relations on dental imaging. Two orthogonal slices of a cone-beam CT 3D image are displayed in Fig. 6.16, along with three segmented structures. Computing the along relations between these objects leads to a high degree of satisfaction of the relation between the mandibular canals and the mandibular floors, and to a lower degree between the tooth and the other structures. Taking into account potential imprecision in the segmentation using fuzzy spatial objects leads to similar results.

6.5.5 Aligned

Both parallelism (described in the next section) and alignment between low-level features have been widely studied in computer vision. Some examples are parallelism between segments, between groups of points, or between linear segments, as collinearity in digital images [39]. Most of these works fall in the framework of perceptual organization (see Sect. 6.6), where the main objective is to find how to organize low-level features, such as edge segments, into groups, such as aligned segments or parallel segments. The groups are evaluated according to their perceptually significance based on the grouping laws of the Gestalt theory [63] (in particular proximity and similarity laws). Alignment between objects is often considered as alignment between their barycenters, which may lead to counter-intuitive results if the objects are spatially extended, not necessarily convex, and possibly of different sizes.

In [108] a method is proposed, which is robust to segmentation errors and can be applied to fuzzy objects, as well as alignments different from barycenter alignment. The method starts by computing the orientation histogram between two objects, which is simply an angle histogram where the angles are computed modulo π:

$$O(A, B)(\theta) = \frac{\displaystyle\sum_{p,q \in \mathcal{S} | mod(\angle(\vec{pq}, \vec{u}_x), \pi) = \theta} \mu_A(p) \wedge \mu_B(q)}{\displaystyle\max_{\phi \in [0,\pi[} \sum_{p,q \in \mathcal{S} | mod(\angle(\vec{pq}, \vec{u}_x), \pi) = \phi} \mu_A(p) \wedge \mu_B(p)}, \tag{6.55}$$

where μ_A and μ_B denote the membership functions of the two objects, and $\vec{u}_x$ denotes the x-axis in the spatial domain $\mathcal{S}$. The orientation histogram is a fuzzy subset of $[0, \pi[$ that represents the orientation between two objects. It has the same properties as the angle histogram and, in addition, is symmetrical in A and B.

To measure the degree to which two orientation histograms are similar, the orientation histograms are interpreted as fuzzy sets, and comparison measures such as similarity, distances, degree of intersection can be used. To account for potential imprecisions, a dilation can be applied to the histograms, to gain in robustness. Then a similarity is, for instance:

$$sim(O(A, B), O(C, D)) = \max_{\theta \in [0,\pi[} \left[D_{\nu_0}(O(A, B))(\theta) \wedge D_{\nu_0}(O(C, D))(\theta) \right], \tag{6.56}$$

where $D_{\nu_0}(O(X, Y))$ is the fuzzy morphological dilation of $O(X, Y)$ by a structuring element ν_0 chosen according to the imprecision (see Chap. 4). When the orientation histograms are not similar, a zero similarity value is obtained, as desired. This similarity measure can be extended to compare several orientation histograms. Let $\{O(A_i, A_j), i \in \{0, ...N\}, j \neq i\}$ be a set of orientation histograms, we define the similarity degree between them as:

$$sim(\{O(A_i, A_j), i \in \{0, ...N\}, j \neq i\}) = \max_{\theta \in [0,\pi[} \bigwedge_{i=0, j\neq i}^{N} D_{\nu_0}(O(A_i, A_j))(\theta). \tag{6.57}$$

Next, a neighborhood relation $Neigh(A, B)$ is considered, here a proximity relation, or approximated adjacency (see Sect. 6.1.1). A group $\mathcal{G}$ of objects is called connected by the $Neigh$ relation if for every $A, B \in \mathcal{G}$, there exist $C_0, \dots C_M$ objects in $\mathcal{G}$, such that $C_0 = A$, $C_M = B$ and for every $m = 0, \dots, M-1$, the relation $Neigh(C_m, C_{m+1})$ is satisfied.

According to the semantic meaning of the alignment relation, the group $\mathcal{G}$ is aligned if the following conditions are satisfied:

(i) $\mathcal{G}$ is connected by the $Neigh$ relation.
(ii) $|\mathcal{G}| \geq 3$.
(iii) There exists $\theta \in [0, \pi[$ such that for every $A \in \mathcal{G}$, A is able to see the rest of the group in direction θ or $\theta + \pi$ with respect to the horizontal axis.

The first condition ensures that the consecutive members of the group are neighbors, and thus that the group is not "divided." The second condition states that an aligned group should have at least three elements. To verify the third condition we measure that all the orientation histograms $O(A_i, \mathcal{G} \setminus \{A_i\})$ (orientation between A_i and the rest of the group) are similar for all $A_i \in \mathcal{G}$. This measure is robust with respect to small deviations in the group. For instance, if two pairs of objects of the group have dissimilar orientations, but there is still a general tendency of alignment within the group, then the orientation dissimilarity between these pairs will not affect the whole conjunction, because the comparison is done between the orientations of the whole group with respect to its members. The group in Fig. 6.17a has a pair of objects with an orientation different from the orientations between the other pairs. However, there is a tendency in the orientation of the whole group. This tendency is reflected in the dilated histograms $D_{\nu_0}(O(A_i, \mathcal{S} \setminus \{A_i\}))$ of Fig. 6.17b and in their aggregation in Fig. 6.17c, which gives a similarity measure of 0.81.

Finally, the *global* degree of alignment is defined as follows. Let $\mathcal{G} = \{A_0, \dots, A_N\}$, with $N \geq 3$, be a group of objects in $\mathcal{S}$, connected by the $Neigh$ relation. Then, the degree of alignment of $\mathcal{G}$ is given by:

$$\mu_{ALIG}(\mathcal{S}) = sim\left(O(A_0, \mathcal{S} \setminus \{A_0\}), \dots, O(A_N, \mathcal{S} \setminus \{A_N\})\right). \tag{6.58}$$

This definition is appropriate for measuring the degree of alignment of groups, as illustrated in Fig. 6.17. However, to find the aligned groups of objects within a set of objects then this measure is not sufficient, since it would be necessary to measure the degree of alignment for every group connected by the $Neigh$ relation with more than three members than can be formed from the set. Therefore a local measure is defined, to help determine the candidates for aligned groups of objects. This measure is called *local* alignment. Let $\mathcal{G} = \{A_0, \dots, A_N\}$, with $N \geq 3$, be a group of objects in $\mathcal{S}$, connected by the $Neigh$ relation. The degree of *local* alignment of $\mathcal{G}$ is defined as:

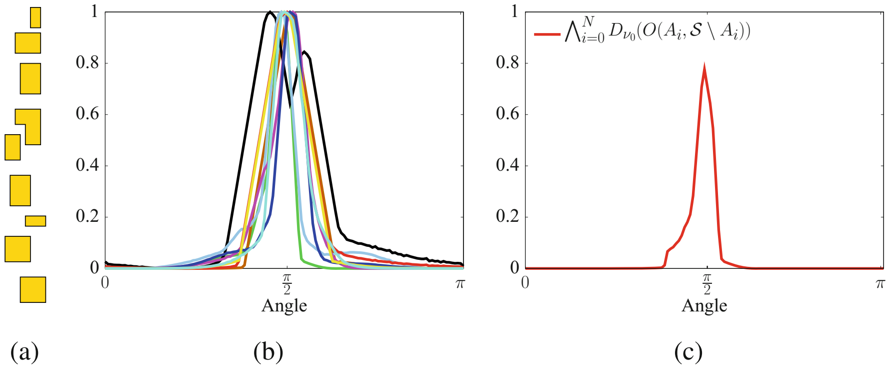


Fig. 6.17 (**a**) A group of objects for which the orientation between the 5th and 6th objects (counting from the bottom) is different from the orientation between the other pairs of objects of the group. (**b**) The dilated orientation histograms $D_{\nu_0}(O(A_i, \mathcal{S} \setminus \{A_i\}))$ for objects in (**a**), using the same structuring element to perform the dilation of each histogram. (**c**) Aggregation of the histograms in (**b**) using the t-norm of Lukasiewicz ($a \wedge b = \max(0, a + b - 1)$)

$$\mu_{LA}(\mathcal{S}) = \min_{\substack{X,Y,Z: \\ Neigh(X,Y) \wedge Neigh(Y,Z)}} sim(O(X, Y), O(Y, Z)). \tag{6.59}$$

The degree μ_{LA} measures that for every pair of elements $B, C \in \mathcal{G}$ connected to A by $Neigh$, the orientations $O(A, B)$ and $O(A, C)$ are similar. A group of objects $\mathcal{G}$ is then said *locally* aligned to a degree β if $\mu_{LA}(\mathcal{G}) \geq \beta$.

An efficient graph-based method to determine which are the locally and the globally aligned groups of objects from a set of segmented objects is described in [108]. An example of identification of alignment buildings in a satellite image is shown in Fig. 6.18.

6.5.6 Parallel

The first definition of parallelism between fuzzy lines (see Sect. 3.4.1) was proposed in [28]. It simply consists of a degree of non-intersection (see Sects. 3.2 and 3.4.4). However, approximating an object by a line, even fuzzy, is often too restrictive, and more general definitions of parallelism between objects are then better suited. In [108] such a definition was proposed, based on the notion of visibility, as used for the between relation.

Suppose A is a linear or elongated object and B is an object that it is not necessarily linear (both of which may be fuzzy). Let θ_A be the orientation of A and $\vec{u}_{\theta_A + \frac{\pi}{2}}$ be the normal unit vector to the principal axis of A. Then, the relation "A is parallel to B" depends on two conditions:

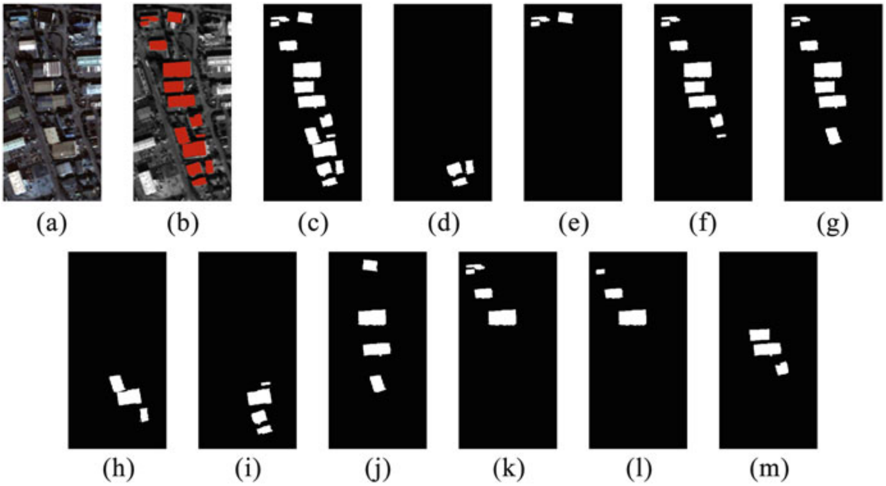


Fig. 6.18 (**a**) Original image. (**b**) Buildings for which we extracted the groups shown in Figs. (**c**)–(**m**). The degrees of *global* alignment of each group are (**c**) 0.75 (**d**) 0.3 (**e**) 0.97 (**f**) 0.98 (**g**) 0.96 (**h**) 0.98 (**i**) 0.93 (**j**) 1.0 (**k**) 0.97 (**l**) 1.0 (**m**) 0.98 [108]

Fig. 6.19 Illustration of the notion of visibility [108]. From left to right: objects A and B; directional dilation $D_{\nu\theta_A+\frac{\pi}{2}}(A)$, which represents the visual field of A in the direction of $\theta_A + \frac{\pi}{2}$, where θ_A is the orientation of A (the white pixels have a high membership value of being observed by A in the direction $\theta_A + \frac{\pi}{2}$); membership function of $B_{vis(A,\theta_A+\frac{\pi}{2})}$, the white pixels are the points with a high membership value

(i) There should be a large proportion of A that sees B in the direction $\vec{u}_{\theta_A+\frac{\pi}{2}}$.
(ii) The orientation of A and the orientation of the boundary of B that is facing A and that is seen by A in the direction $\vec{u}_{\theta_A+\frac{\pi}{2}}$ should be similar.

Both conditions rest on the notion of visibility, which can be modeled using directional dilations, as for the between relation (see Sect. 6.5.2). Let Y be a fuzzy object with membership function μ_Y not intersecting X. We denote by $X_{vis(Y,\theta)}$ the subset of X that is seen by the points on the boundary of Y, i.e., the subset of X that is seen by Y, in the direction $\vec{u}_\theta$, and it is defined by:

$$\mu_{X_{vis(Y,\theta)}}(x) = \mu_X(x) \wedge D_{\nu_\theta}(\mu_Y)(x), \tag{6.60}$$

where $D_{\nu_\theta}(\mu_Y)(x)$ is the directional dilation of μ_Y in the direction $\vec{u}_\theta$. This definition is illustrated in Fig. 6.19. When A and B are linear segments, $A_{vis(B,\theta_A+\frac{\pi}{2})}$ can be interpreted as the projection of B onto A.

For the first condition of parallelism, we are interested in the proportion of A that sees B in the direction $\vec{u}_{\theta_A+\frac{\pi}{2}}$. The subset of A that sees B in the direction $\vec{u}_{\theta_A+\frac{\pi}{2}}$ is $A_{vis(B,\theta_A-\frac{\pi}{2})}$. To measure this proportion, the ratio of the fuzzy volumes of this subset and of μ_A is computed. For the second condition we are interested only in the subset of the boundary of B that faces A and that is seen by the boundary of A in the direction θ_A. This is defined from the extremities b of admissibility segments:

$$\mu_{adm}(b) = \sup_{a \in Supp(A)} \inf_{p \in]a,b[} \min(1 - \mu_A(p), 1 - \mu_B(p))$$

for $b \in Supp(B)$, where $Supp(B)$ is the support of μ_B, and $\mu_{adm}(b) = 0$ for $b \notin Supp(B)$. The subset of the boundary of B that faces A and that is seen by the boundary of A in the direction $\theta_A + \frac{\pi}{2}$ is a fuzzy subset with the membership for a point x equal to the conjunction of its membership to B, the degree of being the extremity of an admissible segment and the degree of being seen by A:

$$\mu_{\delta B_{vis(A,\theta_A+\frac{\pi}{2})}}(x) = \mu_{adm}(x) \wedge \mu_{B_{vis(A,\theta_A+\frac{\pi}{2})}}(x), \tag{6.61}$$

where μ_{adm} represents the degree of being the extremity of an admissible segment.

Finally, the relation "A is parallel to B" is given by the following measure:

$$\mu_{\parallel}(A, B) = \frac{V_n(\mu_{A_{vis(B,\theta_A-\frac{\pi}{2})}})}{V_n(\mu_A)} \bigwedge \nu_0(\theta_{\delta B_{vis(A,\theta_A+\frac{\pi}{2})}} - \theta_A), \tag{6.62}$$

where $\nu_0(\theta)$ is a trapezoidal membership function which evaluates the degree to which $\theta_{\delta B_{vis(A,\theta_A+\frac{\pi}{2})}}$, the normal angle to $\delta B_{vis(A,\theta_A+\frac{\pi}{2})}$ (computed as the direction orthogonal to the main orientation of δB_{vis}), and θ_A are "approximately" equal.

This definition is invariant with respect to geometric transformations (translation, rotation, scaling). It is not transitive, which is required in case of fuzzy complex objects. However, a partial result holds in the crisp case. Suppose that A, B, C are linear crisp segments, if $\mu_{\parallel}(A, B) = 1$, $\mu_{\parallel}(B, C) = 1$ and $\theta_A = \theta_B = \theta_C$, then $\mu_{\parallel}(A, C) = 1$. This result shows that in the crisp case we have transitivity. It is clear that the relation is reflexive. However, depending on the context we may not want to consider intersecting objects as parallel. Thus, we can combine in a conjunctive way the previous degree with a degree of non-intersection between the two sets.

It is often useful to combine this relation with the alignment relation presented in the previous section. When considering parallelism with a globally aligned group of objects, the group has a similar role as the linear object. However, the visibility constraint should be computed in a modified way, as discussed in [108], where several examples are given, for example, to determine the residential areas composed of organized houses, or to identify roads that are parallel to aligned houses.

Another approach to model approximate parallelism was proposed in [96], based on the local derivative of the thickness of an object. This was particularly useful to detect approximately parallel lines in ophthalmologic imaging (layers of the retina in optical coherence tomography, or walls of the retina vessels in eye fundus images and in adaptive optics imaging).

6.6 Fuzzy Perceptual Organization for Image Understanding

One of the problems in image understanding is to elicit from the results of the image processing operators (e.g., edge detection) higher level features, which can be integrated into meaningful knowledge of the image contents. An interesting paper, written in 1983 [69], put forward the idea of perceptual organization as the basis for visual recognition. The basic concept of perceptual organization is that of grouping of data (image tokens) based on various grouping criteria. For instance, in inferring reliable line segments in an image the result of an edge detector can be input to a grouping procedure which evaluates various conditions under which these can be grouped. Various works have explored perceptual organization. They cover basic algorithms on the implementation of grouping conditions, as well as specific applications [39, 50, 68, 83, 84, 91, 92, 104, 108, 114].

This section is based on [91, 92] and the references therein, where perceptual organization is considered in the framework of fuzzy sets theory, which provides an ideal vehicle for quantification of the qualitative grouping properties. The focus is on the grouping of primitive components and object level structures which can be interpreted or used to explain the image. It is assumed that noise reduction and edge operators have been applied, and therefore, the input is a collection of (possibly short) line segments fitted from the results of edge detection. The goal is to obtain contours in the image which are reasonable approximations of the objects present in the image. For many reasons, physical and mathematical (lighting conditions, inter-object reflections, shadows, occlusion, general purpose edge detectors, etc.), many linear segments which should be present in the image get segmented and displaced, making it combinatorially explosive to reason with these detected line segments. Near junctions/corners or close presence of other strong features, these line segments get displaced from the straight lines that correspond to the real region boundaries, thus making simple collinearization impossible. In Gestalt terminology one can say that grouping is restoring the wholes (in this case longer line segments, junctions, strands, or closed regions). Primitive image elements (initially fitted line segments), which are typically generated by low-level segmentation processes, are the initial tokens input to the fuzzy perceptual organization operators, which are applied recursively to produce higher level tokens which describe the structural interrelationships. In turn, these can be used for high-level reasoning and for explanation. Grouping basic elements strongly relies on spatial relations and their combination. The relations are again best modeled in the framework of fuzzy sets, even for crisp elements (segments in the sequel).

6.6.1 Fuzzy Grouping Operator to Produce Straight Line Segments

The objective of the straight line segments grouping operator can be described as follows: given a collection S of straight line segments (fitted from the results of edge detection, after noise reduction) obtain a collection S_1 of straight segments that summarizes S in the following sense:

1. $|S_1| < |S|$, where $|\cdot|$ denotes set cardinality.
2. $\forall s \in S_1, \exists S_s \subseteq S$, such that s is obtained via a grouping from S_s. That is, each segment in S_1 is produced by grouping segments in a subset of S.
3. $\forall s \in S_1$, and S_s as above, $l(s) \geq \max\{l(t) \mid t \in S_s\}$, where $l(s)$ denotes the length of segment s. That is, the segment s produced by grouping of segments in S_s is at least as long as the longest segment in S_s.

The grouping properties for producing straight line segments are *similarity* (equal slope), shown in Fig. 6.20, and *proximity* (close distance), shown in Fig. 6.21. As it can be seen from Fig. 6.21, the proximity can be measured in two ways—using the *perpendicular distance* between segments, and what can be called the *parallel distance* between segments (equivalent to the endpoint distance of [58]).

The proximity properties are aggregated using a general t-norm operator for conjunction (e.g., min), to implement the intuition that two segments are close when the distances, perpendicular *and* parallel distances, between them are small. The parameters for the fuzzy sets describing the grouping properties are either provided by the user or they can be learned, based on knowledge of scene, camera position, etc.

Two or more line segments can be grouped to form a longer segment, which eventually corresponds to a meaningful image feature. The *collinearity* of two segments is defined as the aggregation of similarity and proximity, that is:

$$Collinearity = H(similarity, proximity), \tag{6.63}$$

where $H : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is an aggregation operator (e.g., mean) (see also Chap. 5 for other operators). Given a collection of line segments, S, for each $L_0 \in S$,

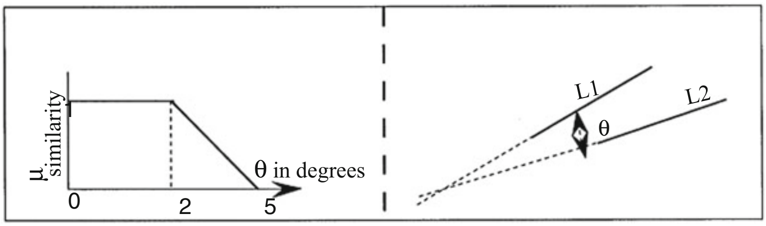


Fig. 6.20 Fuzzy set describing the similarity between two segments

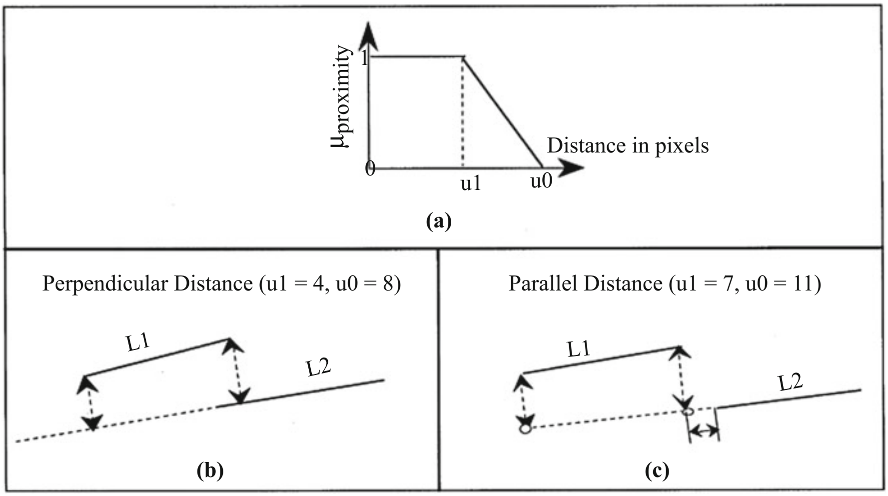


Fig. 6.21 Fuzzy set describing the similarity between two segments

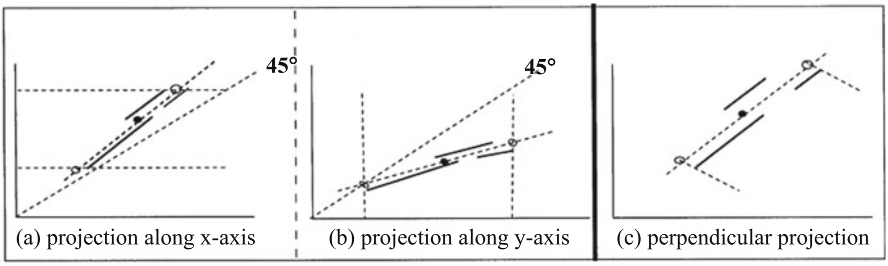


Fig. 6.22 Determining the extent of the inferred segments

with maximum length, *Collinearity* determines a fuzzy set of segments collinear with L_0, denoted by $coll_0$. Since L_0 is not necessarily unique, a segment $s \in S$ will be assigned to that $coll_0$ to which it has maximum degree of collinearity, and in the case of ties, a random assignment is made. Thus $coll_0$ can be thought of as a *fuzzy segment*, which is next defuzzified in order to obtain a crisp representative. To do this one needs to determine (i) the location of the segment, and (ii) the extent of the segment. For the first step, the slope θ and one point M on the line containing the representative segment are obtained by defuzzifying the fuzzy set of slopes Θ_0, and the fuzzy set of midpoints M_0, for the segments in $coll_0$. For the second step, the extent of the segment is determined from the max / min projections of the endpoints of the segments in $coll_0$ on the line determined in the first step. These projections can be true projections, that is, perpendicular to this line, as shown in Fig. 6.22c, or along the x- and y-axis, as shown in Fig. 6.22a and b.

Algorithm 1 summarizes the above discussion on grouping straight line segments.

Algorithm 1 Grouping line segments

procedure GROUPING(L, $coll_L$) ▷ L: initial collection of line segments; $coll_L$ collection of segment groups
    Sort L in non-increasing order of segment lengths
    while Grouping is possible in L **do**
        Select L_0 the *next* longest segment in L
        Calculate $coll_{L_0}$: the fuzzy set of segments collinear with L_0
        $L = L - coll_{L_0}$ ▷ Update L
        $coll_L = coll_L \cup coll_{L_0}$ ▷ Update $coll_L$
    end while
    Determine location & extent ▷ Summarize each cluster in $coll_L$
end procedure

6.6.2 Discrimination: Overlap of Two Segments

Experiments show that collinearity is usually overridden for segments which overlap (in a sense to be explained below). This suggests that the overlap is acting as a *discrimination operator* between the groupings to which the overlapping segments belong. In a strict sense the overlap is defined between segments which are strictly collinear, that is, they are on the same line: a and b overlap if $a \cap b \neq \emptyset$. Moreover, the *degree of overlap*, $Overlap$, of two strictly collinear segments is based on the length of their intersection relative to that of their union. That is,

$$Overlap = \frac{|a \cap b|}{|a \cup b|}. \tag{6.64}$$

However, since the segments produced by edge detection are seldom (if ever) strictly collinear, the concept of overlap must be extended to segments for which strict collinearity does not hold. To this end, for segments a and b with endpoints a_1, a_2 and b_1, b_2, respectively, and the usual (Euclidean) distance between two points, consider successively (with $\wedge = \min$ and $\vee = \max$):

1. $E = \{d_{ij} \mid d_{ij} = d(a_i, b_j), i, j = 1, 2\}$, the collection of endpoint distances.
2. $dps(p, b)$, the distance from a point, p, to segment, b: $dps(p, b) = \min\{d(p, x) \mid x \in b\}$. Letting Δ be the line on which segment b lies, and $q = Pr_\Delta(p)$, the projection of p on Δ, it can be seen that

$$dps(p, b) = \begin{cases} d(p, b_1) \wedge d(p, b_2) & \text{if } q \notin b \\ |pq| & \text{otherwise.} \end{cases} \tag{6.65}$$

3. Directed distance between two segments a and b:

$$dd(a, b) = \sup\{dps(x, b) \mid x \in a\}$$

   It can be seen that $dd(a, b) = dps(a_1, b) \vee dps(a_2, b)$.

4. Define next the quantity $N = \max\{d \mid d \in E\} - [dd(a, b) + dd(b, a)]$.
5. The overlap between segments a and b is finally defined as

$$Overlap(a, b) = \frac{N \vee 0}{\max\{d \mid d \in E\}}. \tag{6.66}$$

It is easy to see that the definition for overlap given in Eq. 6.66 has the following properties:

1. When a and b are strictly collinear, (6.66) reduces to (6.64).
2. When a and b are perpendicular, $Overlap(a, b) = 0$.
3. When a and b are parallel, $Overlap(a, b)$ decreases as $dd(a, b) + dd(b, a)$ increases.

Remark 1 In a similar way to Eq. 6.66, $nonOverlap(a, b)$ can be defined as

$$nonOverlap(a, b) = \frac{abs(N \wedge 0)}{\max\{d \,|\, d \in E\}}.$$

For strictly collinear segments, this means that the non-overlap is equal to the gap size between the nearest endpoints relative to the largest endpoint distance.

6.6.3 Obtaining Junctions

Junctions between line segments in an image are more complex structures, which are very important to detect [99, 100]. They are inferred from line segments output by the fuzzy perceptual grouping of line structures. They are of different types, of which the L-junctions and curvilinear junctions are basic. L-junctions are those where the inner angle formed by the segments at their co-termination endpoint is (approximately) 90°. Curvilinear junctions are special cases of L-junctions in which the inner angle is very obtuse or very acute. Other possible junctions are defined in terms of L-junctions which share a common segment and which satisfy the similarity and proximity criteria. Figure 6.23 illustrates the most common types of junctions.

Fig. 6.23 Examples of junctions: L-junctions, arrow, fork, trees

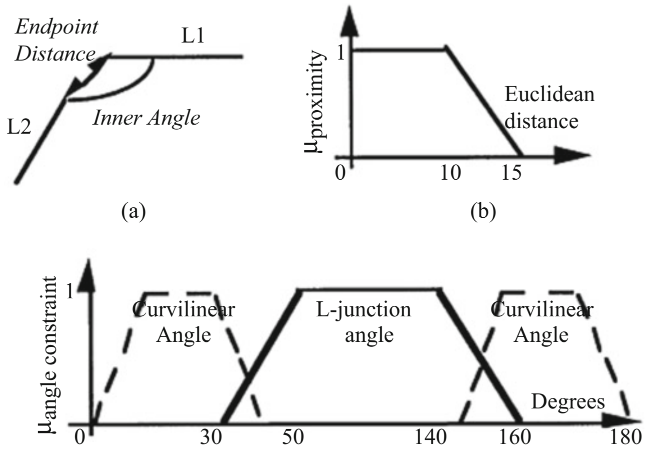


Fig. 6.24 Membership functions for junction angle constraint and junction proximity: (**a**) example of L-junction (endpoint distance and inner angle); (**b**) proximity membership function; (**c**) L-junction angle membership function (dashed membership functions are for proximity of collinear segments)

Since in practice, segments rarely terminate exactly at the same point (instead they terminate in a small common region), fuzzy sets can be employed to represent the end points co-terminacy condition. To construct and detect junctions, the grouping properties of *junction proximity* and *junction angle constraint* are defined, in terms of the minimum endpoint distance between line segments, and the inner angle, respectively. Highly collinear segments are not considered for junctions. In the simplest, yet possibly computationally expensive approach, the input set of line structures to be considered for junctions is taken in a pairwise fashion and evaluated with respect to proximity and angle constraints. Figure 6.24 illustrates membership functions for junction angle constraint and junction proximity, respectively. Different membership functions over the universe of inner angles are used to distinguish L-junction inner angles from curvilinear junction inner angles.

6.6.4 Obtaining Symmetric Line Structures

Symmetry between line (curve) structures can be defined as the one-to-one correspondence of the points of the two structures. Symmetry is a measure of the correspondence of the reflection of one line segment on to the other across a straight axis, the symmetry axis, which emerges as a result of segments being symmetric. Line segment lengths and angle are used to evaluate the symmetry of two line segments. Symmetry of line segments is useful at many levels in the detection and explanation of structures such as U-structures, quadrilaterals, or the region of a symmetric object. To compute degrees of symmetry one distinguishes between different cases as follows.

Symmetry of Non-parallel Line Segments

Let $s_i = a_i b_i, i = 1, 2$ denote two line segments, and let p be the intersection point of the lines on which the two segments lie. Define the nearest/farthest endpoint distance from p to s_i, $d_i = \min\{d(p, a_i), d(p, b_i)\}$, and $D_i = \max\{d(p, a_i), d(p, b_i)\}$, $i = 1, 2$. The degree symmetry of s_1 and s_2 is defined as

$$Sym(s_1, s_2; \text{non parallel}) = \frac{D_1 \wedge D_2 + \alpha(d_1 \wedge d_2)}{D_1 \vee D_2 + \alpha(d_1 \vee d_2)}, \tag{6.67}$$

where

$$\alpha = \begin{cases} +1 & \text{if} \quad p \in s_1 \cup s_2 \\ -1 & \text{if} \quad p \notin s_1 \text{ and } p \notin s_2. \end{cases}$$

It is easy to see that Sym defined by Eq. 6.67 is in [0, 1], and that it measures symmetry adequately. In particular, if Δ is the bisector of angle at p then $Sym(s_1, s_2; \text{non parallel}) = Overlap(Pr_\Delta s_1, Pr_\Delta s_2)$, where $Pr_\Delta s_i$ is the segment obtained by projecting s_i onto Δ (which is the axis of symmetry). However, using Eq. 6.67, there is no need to construct first the axis of symmetry, instead this emerges as a result of the symmetry. Moreover, since Sym is computed only in terms of the segment data (rather than projections of segments), it provides a more efficient way to evaluate symmetry.

Symmetry of Parallel Line Structures

The symmetry of two parallel segments is given by the degree of overlap between one segment and the projection of the other segment. More precisely, for segments s_1 and s_2, if Δ is the line on which s_1 lies, Sym can be defined as:

$$Sym(s_1, s_2; \text{parallel}) = Overlap(s_1, Pr_\Delta s_2) = \frac{|s_1 \cap Pr_\Delta s_2|}{|s_1 \cup Pr_\Delta s_2|}. \tag{6.68}$$

A special case is that of collinear segments, in which case Eq. 6.68 can be applied by taking p as the mid-point of the segment formed by the nearest endpoints of s_1 and s_2, and $\alpha = +1$. Thus, $d_1 = d_2 = d$, which yields

$$Sym(s_1, s_2; \text{collinear}) = \frac{D_1 \vee D_2 + d}{D_1 \wedge D_2 + d}. \tag{6.69}$$

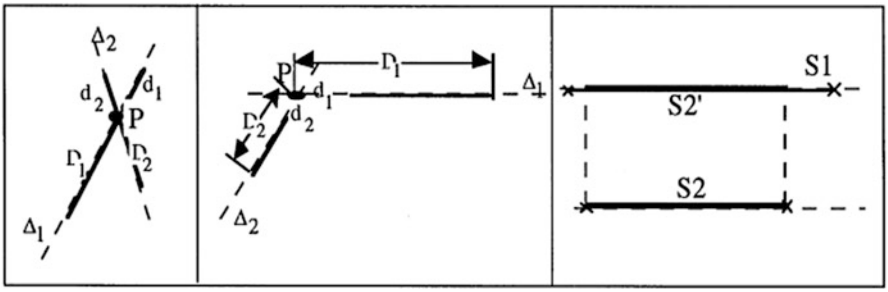


Fig. 6.25 Segment symmetry: parallel and non-parallel cases

Figure 6.25 illustrates segment symmetry for both the parallel and nonparallel cases. Other degrees of symmetry could be used as well, such as in [105].

6.6.5 Obtaining Curves and Closed Regions

The scenes, indoors or outdoors, which generate most images, as well as biomedical images, such as tissue or blood vessels images, usually contain a mixture of straight edged and curved surface components. Curves are very strong data features which are useful in detecting object level features including object regions, component regions, or objects themselves. The Gestalt concepts of *proximity*, *similarity*, and *continuity* play a key role in the construction of curves.

The perceptual organization process for constructing convex curves takes as input a set of L-junctions and curvilinear junctions, and links junctions which share a common line structure and which preserve convexity to form curve segments. The constructed curve segments then serve as input for detection of closed regions in the image. Closed regions may correspond to object surfaces, object components, or actual objects. The same Gestalt concepts of proximity and continuity with adequate new definitions corresponding to the new structures come into play. Strands can be thought of as curves which do not bound an image region and are detected in a similar fashion to convex regions; however, the convexity criterion is not utilized.

Summarizing this section, perceptual organization for image understanding echoes the Gestalt principle according to which the whole is recovered by grouping of parts and forms the basis of obtaining meaningful knowledge of the image contents. Fuzzy set theory provides the necessary concepts and techniques for describing and evaluating grouping criteria, as illustrated by the grouping examples in this section.

6.7 Comparison of Spatial Relations

Image interpretation based on structural information rests on modeling and computing spatial relations, as well as on an ability to compare them. For instance, a relation between two objects or image regions may have to be compared to some prior knowledge about this relation (e.g., from a map or an atlas), or to the same relation on an image acquired at a different date, etc. Obviously such comparisons, modeled as similarity measures or distances, depend to the representation of the relation. A few examples are given in this section. They are used in the reasoning methods presented in Chap. 9.

In the following discussion, a similarity will be denoted by s and a distance (or dissimilarity) by d or δ (we may have simply $d = 1 - s$). The domain on which the relation is defined is denoted by M.

6.7.1 *Relations Represented as Numbers or Intervals*

If relations are evaluated as numbers or intervals, usual comparison tools can be used. Two cases have to be distinguished. If M is the real line (or a part of it), then d is defined from an L_p norm, for instance, in 1D:

$$d(x, y) = |x - y|,$$

where x and y denote two relation evaluations to be compared. For periodic relations, such as directional relations, M is a circular domain (e.g., $[0, 2\pi]$ where 0 and 2π represent the same evaluation). Then a geodesic distance on M should be considered. If the period is ρ, we use:

$$d(x, y) = \min(|x - y|, \rho - |x - y|) = \frac{\rho}{2} - \left| |x - y| - \frac{\rho}{2} \right|.$$

In the case of angles, with $\rho = 2\pi$, this distance is expressed as:

$$d(\theta, \theta') = \min(|\theta - \theta'|, 2\pi - |\theta - \theta'|) = \pi - \big| |\theta - \theta'| - \pi \big|. \tag{6.70}$$

This formulation allows considering that values close to 0 and 2π, respectively, are at a short distance from each other.

These expressions can be extended to evaluations expressed as intervals, using interval arithmetic or distances between intervals considered as subsets of the real line. Intervals are also particular cases of representations as distributions.

For recognition purposes, situations where a relation is satisfied and situations where it is not may have to be handled differently. For instance, if a scene model provides information on the adjacency between objects, the fact that two objects are adjacent in an image may be more relevant to help recognizing these objects than the fact that they are not. Let x and y be the degrees of satisfaction of a relation to be compared, represented as numbers in [0, 1] (e.g., between two objects in a model for x and between two objects in the image for y). The similarity proposed in [20] (in the case of adjacency relation, but it can be applied to other relations as well) is:

$$s(x, y) = f(x, y),$$

where f is a t-norm or a symmetric sum with the appropriate behavior. Convenient symmetric sums are, for instance, associative ones (except median) that behave in a conjunctive way if both values are low (less than 0.5), in a disjunctive way if both values are high (greater than 0.5) or else as a compromise [43]. An example of such a symmetric sum is:

$$\forall (a, b) \in [0, 1]^2,\ f(a, b) = \frac{ab}{1 - a - b + 2ab}. \tag{6.71}$$

6.7.2 Relations Represented as Distributions

As shown in the previous sections, a spatial relation is often represented as a distribution, in particular as a fuzzy number. If one of the two evaluations is a distribution and the other one a number x, then a direct comparison is provided by the value of the distribution at x. More interesting is the case where two distributions have to be compared. One possible approach has been presented above for directional relations, based on the compatibility fuzzy set between the two distributions.

The Hausdorff distance d_H is also a good choice for comparing sets or functions, since it has all the properties of a metric on compact sets. In [16], methods based on morphological dilations and on optimal transport have been proposed both for distribution defined on the real line and for periodic distributions defined on a circle. Again, the link between Hausdorff distance and dilations is used. The distance between two distributions is defined as the distance between the corresponding cumulative distributions, denoted by F and G. In [16], it is shown that the Hausdorff distance between F and G can be computed efficiently using dilations. The fuzzy Hausdorff distance defined in Sect. 6.2 can also be used to obtain a fuzzy number instead as a crisp number. Interestingly, Lévy and Prokhorov distances, used in optimal transport, can be also expressed in a Hausdorff-like manner and benefit from the links with dilations. In particular, these links allow for adaptations and extensions to the case of periodic distributions, which would have been more difficult to address otherwise. Mathematical details can be found in [16].

As an illustration, these approaches have been used to compare the evolution of the distance between two brain structures between images acquired at two different dates [15]. The images and the two objects are illustrated in Fig. 6.26.

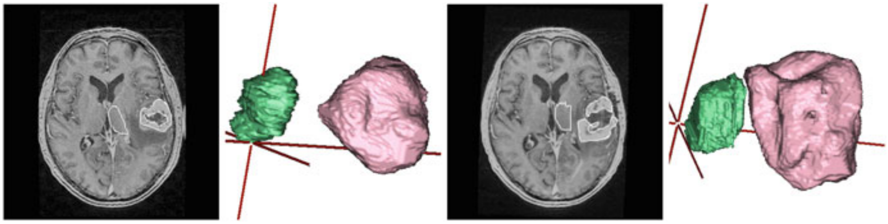


Fig. 6.26 A slice of a 3D MRI image, at two different dates, and 3D views of the two objects of interest (thalamus in green and tumor in red)

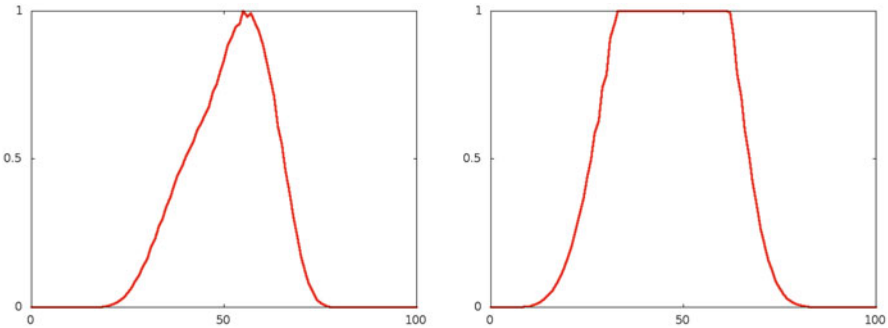


Fig. 6.27 Distance histograms between the two objects, at the two dates

The distance histograms between the two objects are displayed in Fig. 6.27. Computing the Hausdorff distance between these two distributions leads to the conclusion that the tumor, during its evolution, has become closer to the thalamus by about 20 mm.

6.7.3 *Relations Represented as Spatial Fuzzy Sets*

When the relations with respect to a reference object are represented as spatial fuzzy sets (or fuzzy landscape), then similarity and distances between fuzzy sets can be taken used to compare them, using one of the definitions presented in Sect. 6.2.

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Chapter 7
Fuzzy Sets and Machine Learning

The issue of learning is central to fuzzy set based methods. What we mean by this is that while the notion on a fuzzy set is natural and easy to grasp, for applications, the actual membership function must be determined and this issue appears regardless of whether the fuzzy sets are used in supervised/unsupervised learning, on their own, or in a hybrid type of approach in conjunction with other machine learning approaches. Thus, we have fuzzy if-then rules systems for diagnosis, fuzzy-neuro/neuro-fuzzy systems, fuzzy clustering, fuzzy self-organizing maps, fuzzy genetic algorithms, and fuzzy deep learning systems. The fuzzy systems span a very large variety of applications, including biomedical applications which are of interest to this volume. The membership functions can be selected from a family of membership functions (e.g., triangular) in which case learning them is done through an iterative *tuning procedure*, or by optimization of an objective function (as in fuzzy clustering).

This chapter discusses various machine learning approaches using fuzzy sets, each illustrated by biomedical applications. Due to the sheer volume of the relevant literature we will illustrate various approaches through a few selected works.

7.1 Fuzzy IF-THEN Rules

Fuzzy IF-THEN rules have a very special place in the literature of fuzzy systems as they were used in the first successful fuzzy (control) systems. As with many other concepts introduced in the fuzzy set theory and applications, the idea of such rules is due to Lotfi Zadeh (who invented fuzzy sets), in his seminal paper of 1973 [42]. Hence their use goes back to the mid 1970s when the AI concept known as *expert systems* was very popular. This section covers, in a necessarily limited way, the use of fuzzy IF-THEN rules in medical applications. To emphasize the continuous relevance of this approach to the design and implementation of fuzzy systems, it refers mainly to more recent works. These works also provide a reader with a rich

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bibliography on these systems. Further insight on fuzzy logic is given in Chap. 8 (Sect. 8.5).

In [27] the authors discuss a fuzzy rule-based system for assessing coronary heart disease. The approach described is a hybrid of expert-designed rule-based system, with rules whose certainty factors are subject to learning. Like the rules in other approaches, a typical rule in this system is:

$$\textbf{RULE R} \text{ :IF } x_1 \text{is } A_1 \text{ and } \ldots \text{ and } x_n \text{ is } A_n \text{ THEN class } C \text{ with } CF_R \tag{7.1}$$

In the antecedent of each rule, $[x_1, \ldots, x_n]$ is an n-dimensional vector of observations, A_i are (trapezoidal) fuzzy sets. In the consequent, C is the class to which the input pattern belongs. CF is the certainty factor which qualifies the rule. The systems comprises 114 of such rules to classify the input pattern into four classes 1 (normal, which has 39 subjects), s.v (stenosis in one single vessel, which has 35 subjects), 2.v (stenosis in two vessels, with 17 subjects), and 3.v (stenosis in three vessels, with 23 subjects). Since the rules are to a large extent manually constructed, the training of this system consists mainly in the adjustment of the certainty factors. Denoting by μ_i the membership function of the fuzzy sets A_i, the overall compatibility between the input and the antecedent is computed (the product is used as a conjunction operator). That is, for the pattern $\mathbf{x} = [x_1, \ldots, x_n]$,

$$\mu_R(\mathbf{x} \mid C, CF_R) = \Pi_{i=1}^{n} \mu_i(x_i).$$

Next, for each class, $C = 1, \ldots, 4$, an overall score $\beta_C(R)$ is computed as

$$\beta_C(R) = \sum_{\mathbf{x} \in C} \mu_R(\mathbf{x} \mid C, CF) \cdot w_{\mathbf{x}},$$

where $w_{\mathbf{x}}$ is a weight assigned to the training pattern $\mathbf{x} = [x_1, \ldots, x_n]$. The class for rule R is then decided based on $M = \{\arg \max_{C=\{1,\ldots,4\}} \beta_C(R)\}$, namely

$$\widehat{C} = \begin{cases} M & \text{if } |M| = 1 \\ \emptyset & \text{if } |M| > 1, \end{cases}$$

where $|M|$ denotes the cardinality of the set M. The certainty factor CF_R for rule R is calculated as

$$CF_R = \frac{\beta_{\widehat{C}}(R) - \overline{\beta}}{\sum_{C \neq \widehat{C}} \beta_C(R)},$$

where $\overline{\beta}$ is the average of the β scores for R over the remaining classes, i.e.,

$$\overline{\beta} = \frac{\sum_{C \neq \widehat{C}} \beta_C(R)}{3}.$$

The classification of a new pattern $\mathbf{x} = [x_1, \ldots, x_n]$ is according to the maximum of a new class score which introduces the certainty factors, that is

$$\mathbf{x} \in \arg \max_{C \in \{1,\ldots,4\}} \alpha_C(\mathbf{x}), \text{ where } \alpha_C(\mathbf{x}) = \max_{h \in \{1,\ldots 4\}} \mu_R(\mathbf{x} \mid h, CF_R).$$

The main learning aspect, the adjustment of the certainty factors CF_R, proceeds as follows: starting with a certainty factor CF_R for each rule R, upon unsuccessful classification of a training pattern $\mathbf{x}$, by R, CF_R is adjusted according to the equation

$$CF_R^{new} = (1 - \eta \cdot w_{\mathbf{x}}) CF_R^{old},$$

where $\eta \in (0, 1)$ is the learning rate, and $w_{\mathbf{x}}$ is the weight for $\mathbf{x}$. The final rule-based classification system used ten input variables (age, sex, diabetes, cholesterol level, triglyceride level, low density lipoprotein (LDL), systolic blood pressure, summed stress score (SSS), smoking, and genetic factor) whose values were represented by three fuzzy sets (low/normal, borderline/middle, high) for all, except the last two whose values were represented by four fuzzy sets (low/normal, middle/mild, high/moderate, and high/severe). Table 7.1 illustrates Rule 6 of the fuzzy system obtained.

Evaluating the risk of heart disease by a rule-based fuzzy system is also described in [5, 6]. Similarly to [27] these papers use some input from experts (medical doctors) in the form of guidelines for designing the rule-based system. Quoting [5] these guidelines include:

Table 7.1 Fuzzy IF-THEN rule 6 of [27]

IF	THEN	CF
Middle_Age and	CAD class is 3.v	1.00
Borderline Cholesterol and		
Borderline Triglycerides and		
LDL is *Middle* and		
Middle Systolic blood pressure and		
Moderate Summed Stress Score and		
Female and No Diabetes and		
Not smoking and Genetic factor present		

Table 7.2 Fuzzy IF-THEN rules 3 of [5]

IF	THEN
HeartRate is *Normal* and	RiskLevel is *High*
BreathingRate is *Tachypnea*and	
SpO_2 is *Critical* and	
LipsColor is *Regular*	

- When all the vital signs exhibit standard values, the risk level is low.
- When one vital sign exhibits a nonstandard value, the risk is medium.
- When two vital signs exhibit some nonstandard values, the risk is high.
- When three vital signs exhibit some nonstandard values, the risk is very high.

The vital signal data is extracted *contactless* (an important feature in the aftermath of the COVID-19 pandemic) from subjects. For each rule, the antecedent is a conjunction of conditions on the four vital signals extracted for each subject,

$$\{HeartRate, BreathingRate, SpO_2, LipsColor\}.$$

The consequent assesses the variable *RiskLevel*, which can take values in

$$\{Low, Medium, High, VeryHigh\},$$

and a typical rule is shown in Table 7.2.

The values of the vital sign variables are extracted contactless, using a visual system. The face is detected from video frames using a pre-trained frontal face detector. The main contribution of this paper with respect to data acquisition is inspired from photoplethysmography (PPG), an example of which is the pulse oximeter, a device which measures the oxygen and pulse. The *contactless* inexpensive, yet accurate device, a "smart mirror," which can detect changes in the blood flow through the human tissue was built. Once the frontal face image has been detected, two regions of interest (ROI) are considered: (1) the forehead, which according to the literature cited, is best to provide the changes in the blood flow, and hence the heart rate, the breathing rate and the Oxygen saturation, and (2) the mouth region, for the lips color, which as it is well known from medical practice, changes in the lips' color gives clues to the risk of heart problems.

With respect to the forehead region, the signals are separated into RGB channels and spatially averaged for each of the N frames collected, thus forming a $3 \times N$ data matrix V_{RGB}. Denoising and other preprocessing steps are applied before they are passed through a band-pass filter of fixed frequency in [0.6 Hz, 4 Hz], corresponding to heart rate [36 bpm, 240 bpm], for heart rate estimation. For estimation of the heart and breathing rate, blind source separation based on independent component analysis (ICA) with three sources is applied, followed by spectral analysis and identification of the component containing the strongest signal. The frequency f_h

Table 7.3 The input linguistic variables and their values from [5, 6]

Linguistic variable	Range	Fuzzy values
HR	[30, 52]	Bradycardia
	[48, 100]	Normal
	[95, 180]	Tachycardia
BR	[0, 8]	Bradypnea
	[7, 23]	Normal
	[20, 80]	Tachypnea
SpO_2	[75, 90]	Critical
	[7,23]	Normal
	[20, 80]	Tachypnea
LipsColor	[60, 90]	Regular
	[5, 10]	Altered
	[8, 16]	Purplish

is defined as the maximum peak intensity in the same band of the filter [0.85 Hz, 3.5 Hz]. This corresponds to the range [51 bpm, 210 bpm] of the heart rate. The frequency f_b is defined as the maximum peak intensity in the same band of the filter [0.15 Hz, 0.5 Hz]. This corresponds to the range [9 bpm, 30 bpm] of the breathing rate. The values of the heart rate (HR) and breath rate (BR) are obtained as average beats per minute, by multiplying the corresponding frequency values by 60. That is, $HR = 60 f_h$ and $BR = 60 f_b$. Oxygen blood saturation is defined as

$$SpO_2 = \frac{HbO_2}{HbO_2 + Hb} \times 100\%,$$

where HbO_2 and Hb are oxygenated hemoglobin and deoxygenated hemoglobin, respectively. SpO_2 is computed based on the standard deviations of the red and blue bands at each time point, which are used as the pulsatile alternating current (AC) component. The non-pulsatile direct current (DC) component is computed as the red and blue band mean intensities at each time point. Therefore,

$$SPO_2 = A - B\frac{AC_{red}/DC_{blue}}{AC_{blue}/DC_{blue}},$$

where $A = 125$ and $B = 26$ are determined empirically. The range of lip color is described by the fuzzy terms *normal*, *altered*, and *purplish*. Table 7.3 and Fig. 7.1 (which reproduce Table 1 and Fig 7 of [5], respectively) show the values for the input linguistic variables and their representations as fuzzy sets, respectively. Figure 7.2 is the diagram of the fuzzy inference system.

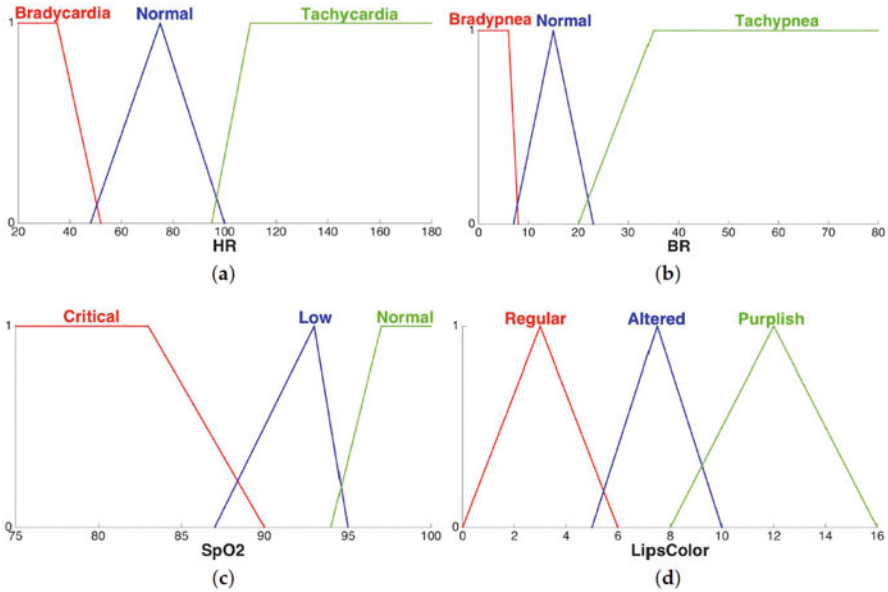


Fig. 7.1 Fuzzy sets of the linguistic variables related to the vital signs [5, 6]: (**a**) heart rate, (**b**) breath rate, (**c**) oxygen saturation in blood, (**d**) color of lips (obtained by courtesy)

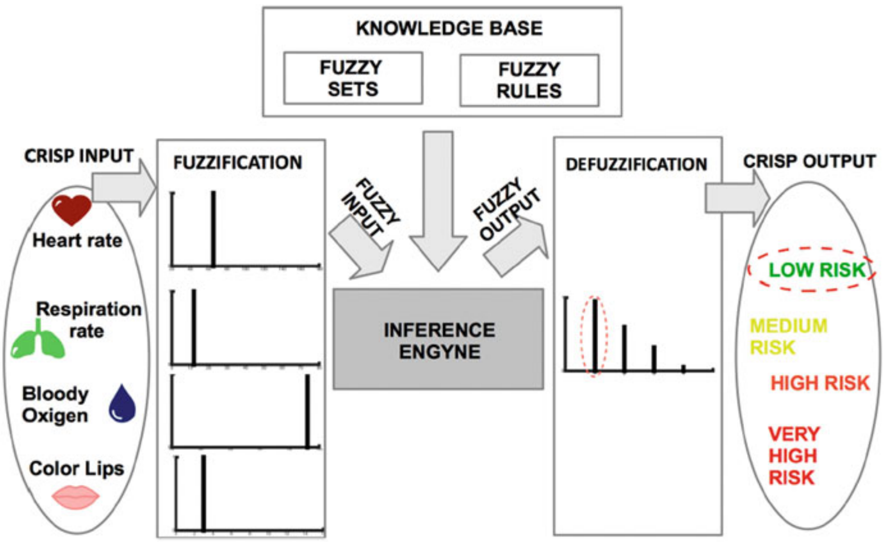


Fig. 7.2 Diagram of the overall system from [5, 6] (obtained by courtesy)

7.2 Unsupervised Learning

7.2.1 *Fuzzy Clustering*

Although this book focuses on higher level image description and understanding, this section includes a discussion of fuzzy clustering as a device for low-level image processing, namely for image segmentation. From the onset of computer-based image processing and understanding, clustering techniques, based on various image features, and using different approaches, have received a lot of attention. The implicit assumption is that clusters correspond to meaningful image content—objects. The use of fuzzy sets to represent clusters is intuitively very appealing, as gray levels, color intensity, texture complexity are easily seen to be a matter of degree. The difficulty arises when one has to actually find a way to assign individual data points to the fuzzy clusters. One of the first papers on fuzzy clusters published in 1969, [35] illustrated the point of usefulness of fuzzy sets on a simple toy data set, which came to be known as the "butterfly clustering". Starting with the early 1970s we witness increased activity in the development of fuzzy set based clustering approaches, notably fuzzy c-means type of algorithms (see, for example, [3, 10, 11, 19] to name only a few). Indeed, the volume of publications on this subject is quite staggering even when we restrict our attention to a special application field, as is the case in this book, that of biomedical images. For example, a cursory search on Google Scholar with the key words *fuzzy c-means clustering biomedical images* returns 11,800 titles for the period 2018 to mid-2022, more than half of the total of 22,100 entries returned for the key word *fuzzy c-means clustering* for the same period. What explains this volume of research on this algorithm, in particular in relation to its application to biomedical images? One possible explanation is that, unlike many images of various usual objects (e.g., objects of daily living), or even of natural scenes, biomedical images (of various organs, lesions, etc.) are very complex, with regions often ill defined, with blurred boundaries, irregular shapes. Another explanation, connected to the first, is that fuzzy clustering algorithms (fuzzy c-means in particular) can be quite easily modified when applied for different data sets. This section explores *fuzzy c-means algorithm*, the most popular of the fuzzy clustering approaches, some of the ways in which it has been modified, enhanced, and some of the biomedical imaging applications. For obvious reasons, in the light of the large number of works on this algorithm, this section is limited and many works are omitted. However, at the same time, it is hoped that the examples used to illustrate some of the approaches will enable the readers to judge for themselves the usefulness of the algorithm.

To the best of our knowledge, the trigger of the work on the fuzzy c-means algorithm was Dunn's paper [10] which presented an ISODATA-like clustering algorithm using fuzzy sets. Recall here that ISODATA refers to the famous (crisp/non-fuzzy) clustering algorithm developed by Duda and Hart [9]. Subsequently, Bezdek and his collaborators published the first work in which the clustering algorithm was named fuzzy c-means (FCM) [3]. Like any clustering algorithm, FCM seeks to obtain a (fuzzy) partition of the data set, that is, a collection of subsets, which completely cover the data set. The condition of non-overlap (from the definition of a crisp partition) is dropped, and elements within a cluster are assigned a membership degree, a value in [0, 1], to the cluster.

To set our discussion on a concrete basis, we review the basic FCM algorithm, which is set as an optimization problem. Given a data set $X = \{x_1, \ldots, x_n\}$, the number of clusters sought, $0 < c < n$ (usually, c is much smaller than n), v_k, $k = 1, \ldots, c$ the cluster centers, and $1 \leq m < \infty$, FCM seeks to minimize

$$J_{m,c}(X) = \sum_{i=1}^{n} \sum_{k=1}^{c} \mu_{ik}^{m} \|x_i - v_k\|^2 \tag{7.2}$$

subject to the conditions

$$\mu_{ik} \in [0, 1] \text{ for any } (i, k), \tag{7.3}$$

$$\sum_{k=1}^{c} \mu_{ik} = 1, \text{ for any } i = 1, \ldots, n \tag{7.4}$$

$$0 < \sum_{i=1}^{n} \mu_{ik} < n, \text{ for any } n, \tag{7.5}$$

where μ_{ik} is the membership degree of x_i to cluster k. These conditions correspond to the requirement that each data point belongs to at least one cluster, and that overall, the clusters account fully for each data point. At each step, the cluster centers v_k, and μ_{ik} are updated according to the following equations:

$$v_k = \frac{\sum_{i=1}^{n} x_i \mu_{ik}^{m}}{\sum_{i=1}^{n} \mu_{ik}^{m}} \qquad \mu_{ik} = \frac{1}{\sum_{l=1}^{c} \left(\frac{\|x_i - v_k\|}{\|x_i - v_l\|} \right)^{\frac{2}{m}}} \tag{7.6}$$

The algorithm starts by deciding the value of c, the number of clusters (a feature that it shares with its crisp version k-means), and by initializing the clusters centers, usually in a random manner. Then the operations described by Eqs. 7.6 are performed iteratively: matrix $U = (\mu_{ik})_{ik}$, $i = 1 \ldots n$, $k = 1, \ldots, c$ is computed, cluster centers are updated, until a stopping criterion is met. The stopping criterion is either a cutoff in the number of iterations or based on a threshold on the difference between two successive updates of J_m or v_k. The parameter m determines the fuzziness of the resulting clusters, and for this reason it is usually referred to as the *fuzzifier*. There are no exact rules for selecting its value. Most often this parameter is set to $m = 2$. Crisp clusters can be obtained from the matrix U by assigning a data point to that cluster to which it has the largest membership degree. That is, for

$i = 1, \ldots, n, x_i \in \arg\max\{\mu_{ik} \mid k = 1, \ldots, c\}$. The algorithm converges towards a local minimum of J_m, hence the importance of the initialization.

There are objections to FCM, both from theoretical and from application point of view, especially in its application to the image domain. The main theoretical objection concerns the inconsistent behavior of the cluster membership functions, whose values decrease as the distance to the cluster center increases up to a point, only to increase again, leading to a situation in which data points which are not (or should not be) in a cluster may be assigned membership degrees to that cluster greater than or equal to the membership degrees of some of the data points which are in the cluster. From the application point of view, various papers point the insensitivity to noise and to the nonhomogeneity in the image, failure to include spatial information and to capture cluster shapes. Several authors traced the theoretical issues to the condition expressed by Eq. 7.4, and therefore, versions of FCM have been developed where this condition is replaced by another (e.g., $\sum$ is replaced by max) or it is dropped altogether.

7.2.2 Spatial Information and Bias

In most of the work described below, the condition expressed in Eq. 7.4 is preserved, the modifications to the FCM algorithm consisting mainly in altering the objective function. Citing the need to deal with intensity non-uniformity (INU) and noise, FCM is modified by the introduction of spatial information and the b_k, for each cluster center, v_k, such that the objective function, becomes [30]:

$$J_m(X, c) = \sum_{k=1}^{c} \sum_{i=1}^{n} \mu_{ki}^m \|x_i - b_k - v_k, \|, \tag{7.7}$$

where $\sum_{k=1}^{c} \mu_{ik} = 1$, $\mu_{ki} \in [0, 1]$, and $0 \le \sum_{i=1}^{n} \mu_{ki} \le n$. The resulting update equations must then also involve b_k:

$$\mu_{ik} = \frac{1}{\sum_{l=1}^{c} \left(\frac{\|x_i - b_k - v_k\|}{\|x_i - b_l - v_l\|} \right)^{\frac{2}{m}}}, \qquad v_k = \frac{\sum_{i=1}^{n} \mu_{ik}^m (x_i - b_k)}{\sum_{i=1}^{n} \mu_{ik}^m}, \qquad b_k = \frac{\sum_{i=1}^{n} \mu_{ik}^m (x_i - v_k)}{\sum_{i=1}^{n} \mu_{ik}^m}. \tag{7.8}$$

The matrix $U = (\mu_{ik})_{ik}$ of membership degrees is convoluted with a mean 3×3 filter. This introduces spatial information from the 3×3 pixel neighborhood. According to the authors, when compared with the simple FCM algorithm, the

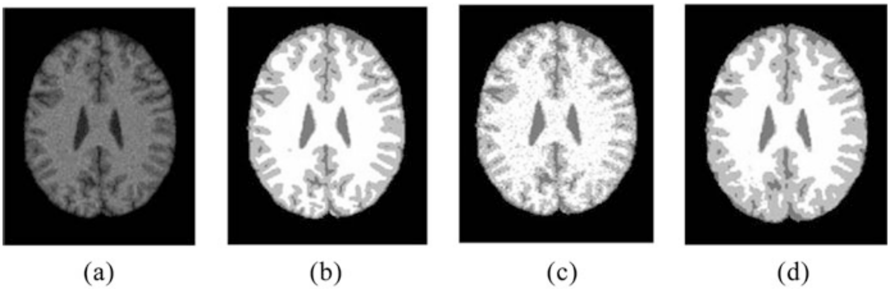


Fig. 7.3 Images from Figure 5 of [30] (obtained by courtesy): (**a**) is a T1 weighted original MR brain image from BrainWeb [7], with 0%INU and 9% Noise; (**b**) shows the ground truth for segmentation; (**c**) shows the segmentation obtained by the regular FCM; (**d**) shows the segmentation by Spatial FCM with bias correction

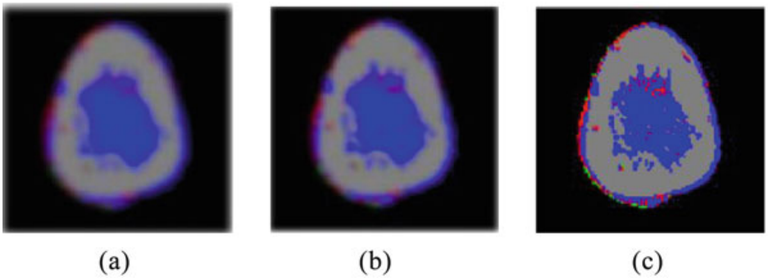


Fig. 7.4 Some of the images in Figure 19 of [17] (obtained by courtesy): (**a**) shows a right breast MRI corrupted by Gaussian noise; (**b**) the segmentation results obtained by FCM; (**c**) the result segmented image obtained by the algorithm of [17]

results show significant improvement in all, but three combinations (noise, INU) = $\{(5\%, INU) \mid INU = 0\%, 20\%, 40\%\}$. Figure 7.3 illustrates the effect of using the FCM enhanced with bias and spatial information proposed in [30]. According to the authors, the improvement in results is also conveyed by quantitative evaluation of the segmentation result.

Considering distances as spatial information [17] takes an approach in which the distance in the FCM objective function is replaced by the distance in the implicit feature space defined in terms of a kernel. This is used along with an algorithm for initialization of the cluster centers, which can also be viewed as an attempt to get a better hold on spatially meaningful information concerning the location of cluster centers. The results are evaluated with respect to the width of the cluster silhouettes (introduced in 1987 [34] for cluster validation), which also captures some spatial information in the form of distances between data points. The results are illustrated on benchmark data sets and also MRI images of the brain and breast. Figure 7.4 shows the results obtained on breast images.

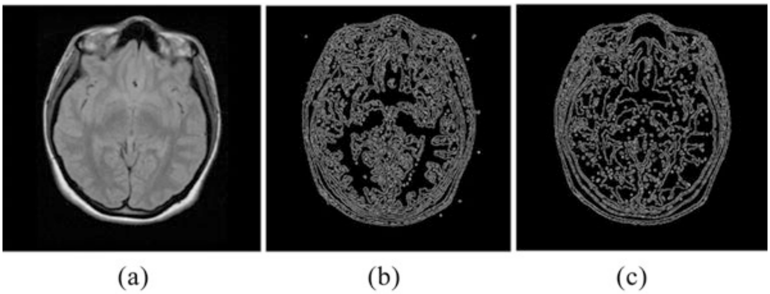


Fig. 7.5 Original image, image segmented using PSOFCM, image segmented using GAFCM (from Figure 2 of [22]). Both approaches used the Canny edge detection operator (obtained by courtesy)

An interesting approach in [22] uses FCM in conjunction with Particle Swarm Optimization (PSO) for edge detection in MRI brain images. Two approaches are compared: PSOFCM, in which FCM is used in conjunction with PSO, and GAFCM, in which FCM is combined with Genetic Algorithms, used in the previous work by the authors [21]. Figure 7.5 illustrates the type of results obtained with each of these approaches.

Citing the same drawbacks signaled by various authors (and already mentioned in this chapter), that is, inability to capture spatial information, and difficulties to cope with noise of the original FCM algorithm, a modified FCM algorithm is used in [41]. Claimed to be robust to noise, this algorithm aims to capture two types of spatial information - *local* and *non-local*. This is done by considering explicitly neighborhoods and distances within neighborhoods (local spatial information) and between neighborhoods (non-local spatial information). In a way, this is similar to the ideas underlying the concept of silhouette [34], with the difference that the two types of distance are aggregated by a convex combination and therefore, one can control their individual impact. Informally, the approach has the following steps:

1. Define an $w \times w$ neighborhood, N, around a pixel, x_i.
2. Define a *local distance* measure, d_l, between two pixels, as the Euclidean distance weighted by the slope of the Gaussian within N, and centered at one of the pixels.
3. Define a *non-local distance* measure, d_{nl} between two pixels, as the distance between their respective neighborhoods, expressed as vectors.
4. Finally, define the *overall distance* measure, D, between two pixels, as a convex combination of d_l and d_{nl}: $D^2(x_i, x) = \lambda_i d_l^2(x_i, x) + (1 - \lambda_i) d_{nl}^2(x_i, x)$, where $\lambda_i \in [0, 1]$ are also determined for each i.

Once D is defined, the usual steps of the FCM algorithm are applied. When evaluated on MRI images the authors claim much better results than using the

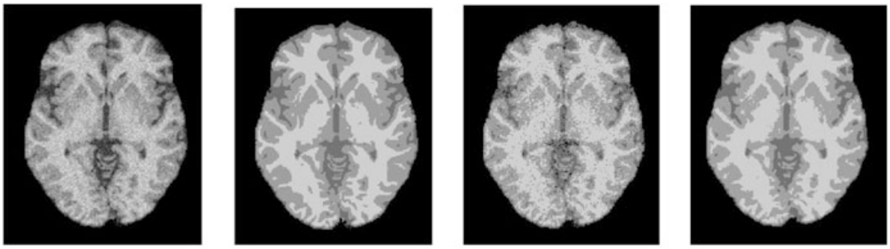


Fig. 7.6 Comparison of the segmentation results obtained in [41], on simulated MRI brain image (BrainWeb Simulated Brain Database, McConnell Brain Imaging Centre of the Montreal Neurological Institute (MNI), McGill University) [7]: (**a**) original image with 9% noise; (**b**) the ground truth segmentation; (**c**) results of the standard FCM algorithm; (**d**) the result of the modified FCM algorithm proposed in [41] (obtained by courtesy)

original FCM algorithm. Figure 7.6 illustrates the difference in the FCM and the proposed modified FCM algorithms.

As already mentioned, the parameter m from the FCM objective function determines the fuzziness of the clusters produced. This means that for different choices of m the membership degrees of the same points will be different. It is then quite natural to think that it may be more appropriate if m was allowed a range of values, rather than a fixed specific value (although, as it will be seen, this introduces more complexity in the overall approach). The starting point of the work in [31] is to let m belong to the interval $[m_1, m_2]$. The (infinitely many) membership functions obtained by clustering are bounded by those corresponding to m_1 and m_2 (in effect then, the interval for m is collapsed to its end points). An interval type-2 fuzzy set, defined by a pair of *upper* and *lower* membership functions $\overline{\mu}$ and $\underline{\mu}$, is obtained. The implicit hypothesis made in this approach is that the insensitivity to noise of the basic FCM is due to the requirement of specifying an *exact* value for m, and that, therefore, relaxing this requirement will produce a type 2 interval-valued FCM-like algorithm (IT2FCM), which is more sensitive to noise. The problem is therefore reduced to finding ways of defining $\overline{\mu}$ and $\underline{\mu}$. To do this, the authors use c, the number of clusters, and ratios of the pairwise distances between pixels. By analogy with the definition of μ for the basic FCM algorithm, the upper and lower membership functions are defined as follows (Equation (4) in [31]):

$$\overline{\mu}_{ik} = \begin{cases} 1/\sum\limits_{j=1}^{c}\left(\frac{d_{ik}}{d_{jk}}\right)^{2/(m_1-1)} & \text{if } \frac{d_{ik}}{\sum\limits_{j=1}^{c} d_{jk}} < c \\ 1/\sum\limits_{j=1}^{c}\left(\frac{d_{ik}}{d_{jk}}\right)^{2/(m_2-1)} & \text{otherwise} \end{cases} \tag{7.9}$$

and

$$\underline{\mu}_{ik} = \begin{cases} 1/\sum\limits_{j=1}^{c}\left(\frac{d_{ik}}{d_{jk}}\right)^{2/(m_2-1)} & \text{if } \frac{d_{ik}}{\sum\limits_{j=1}^{c} d_{jk}} < c \\ \\ 1/\sum\limits_{j=1}^{c}\left(\frac{d_{ik}}{d_{jk}}\right)^{2/(m_1-1)} & \text{otherwise} \end{cases} \tag{7.10}$$

Type reduction of the resulting type-2 fuzzy set is then performed based on established techniques, and finally, cluster assignment is done based on the type-1 fuzzy set obtained.

To further improve the behavior with respect to noise, spatial information, in the form of pixel neighborhoods, is used in the modified algorithm. More precisely, letting N_x a neighborhood of pixel x and $a \in N_x$, a spatial membership function μ_{ik_s} is defined as follows:

$$\mu_{ik\_s} = \sum_{a \in N} \mu_{ia} \cdot w_{ak},$$

where μ_{ia} is the membership degree of a to the cluster centered at x_i; w_{ak} is a weight conveying the connection between x_k and a and satisfies the condition that $\sum\limits_{a \in N} w_{ak} = 1$. In order to ensure desirable properties for μ_{ik_s}, w_{ak} is defined as

$$w_{ak} = \frac{1}{\sum\limits_{x \in N_k}\left(\frac{d(a,x_k)}{d(x,x_k)}\right)^{2/(m-1)}}.$$

To summarize, the algorithm has actually three main stages as follows: (i) determine the IT2MF membership functions using the usual FCM algorithm, (ii) determine spatial membership function and use it to modify the IT2 MFs, (iii) update centers and apply type reduction to obtain a type-1 membership function. Eventually, the authors report results of the type illustrated in Fig. 7.7. Extensive quantitative results are provided (Table 1 of [31]) for the comparison with three other competing algorithms on synthetic images and MRI images in each case, with and without noise.

A two-stage, 3D version of FCM is proposed in [16] and applied again to brain MRI images. In the first stage the approach outputs what the authors evaluate as better cluster centers, which eventually are used to initialize the second stage, the actual FCM portion of the approach. The input to the first stage is the number of clusters and an initial guess of the cluster centers obtained from the 3D histogram on voxels. A $3 \times 3 \times 3$ filter is used to capture spatial information in the form of the neighborhood of a voxel. Then probabilities of voxels in a neighborhood to belong to a cluster are obtained and used in the spatial component, denoted

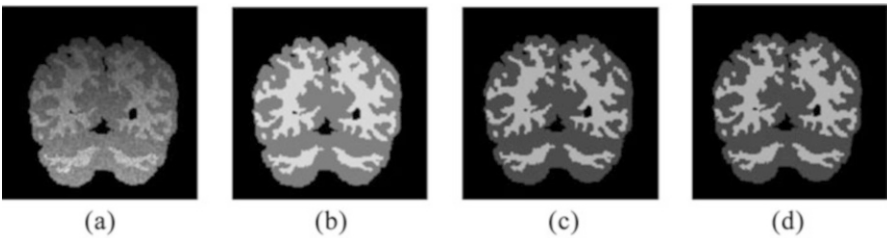


Fig. 7.7 Results of segmentation of MR brain images using the approach suggested in [31]: (**a**) original image; (**b**) ground truth segmentation; (**c**) the segmentation result using a competitor algorithm (RFCM); (**d**) segmentation result obtained using the proposed algorithm (obtained by courtesy)

as U_2, of the objective function. The component U_1 of the objective function is the usual FCM type of objective function. Therefore, the final objective function of the second stage of the algorithm is $J = U_1 + \alpha U_2$, where $\alpha \geq 0$ indicates the contribution of the spatial component. Obviously, for $\alpha = 0$ the usual FCM algorithm is obtained (presumably with random initialization of cluster centers). The algorithm then proceeds to optimize J, in each iteration updating the spatial information, i.e., the probabilities and spatial membership function defined using them, and the cluster centers. The results obtained are claimed to be superior to other competing approaches, including the usual FCM and other, modified versions of it. This is supported both visually (see Fig. 7.8) and by the statistical analysis provided in the paper on various measures of quantitative evaluation, including Tissue Segmentation Accuracy (TSA), and Cluster Validity Functions (Partition coefficient, and Partition Entropy).

An advantage of fuzzy clustering is that it can be easily integrated with other fuzzy set based descriptions of the image content, which can then be used to guide the recognition of objects in the image. For example, such an approach has been used in [12], which has shown that information about the possible location of objects in an image can assist in the selection of appropriate image processing operators (for example, color based segmentation in some areas of the image, texture based segmentation in other, or edge detection in yet other areas). For medical images, a similar approach was used quite early on in [39] for developing a clustering algorithm for vessel tracking in retinal images. Figure 7.9 shows a typical image of the eye fundus (right eye) used in [39]. Two linguistic variables (modeled as fuzzy sets), *vessel* and *non-vessel* are used. Thus, from classification/clustering point of view each pixel in the image is assigned with a certain degree to one of these two classes. An interesting aspect of the eye fundus image is the presence of useful spatial information. For example, one can describe the image by "the macula is in the center of the image (when the gaze is into the camera)"; "the optic nerve is a very *bright region* located either in the *left center* (for the right eye) or in the *right center* (for the left eye) of the image". These descriptions can be readily captured

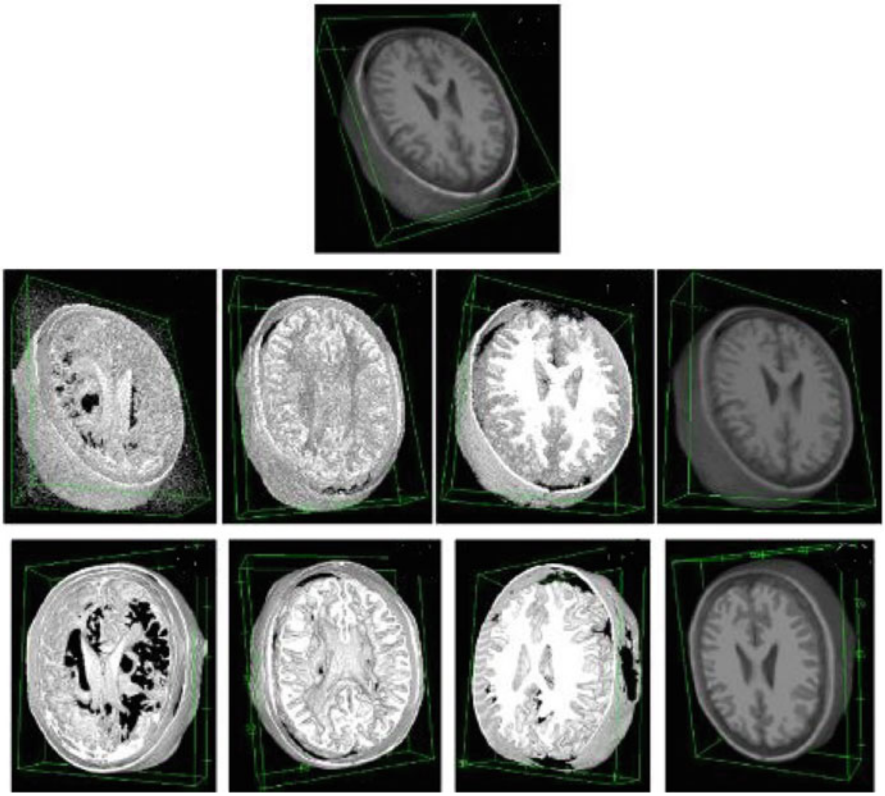


Fig. 7.8 Qualitative outputs of the different segmentation methods on a T1-weighted MRI brain image volume with 9% noise and 40% nonhomogeneity. top: Original MRI image volume; middle row: FCM; bottom row: Two-stage Multi-objective FCM [16] (obtained by courtesy)

quantitatively, using fuzzy sets and hedges (for modeling concepts such as *very*). Such approaches using spatial relations are further developed in Chaps. 8 and 9.

The approximate detection of the optic nerve is based on the descriptions mentioned above. A rough estimate of detecting candidate vessel starting points is carried out—vessels correspond to dark pixels. FCM with $c = 2$, is applied to distinguish two types of regions—*vessel* and *non-vessel*. The authors provide extensive descriptions of how they calculate the vessel segment diameter, and how vessels are tracked starting from the optic disk. Forks and junctions in the tracked vessels are also identified. Tracking a vessel ends when a measure of vessel contrast defined in terms of the prototypes of the *vessel* and *non-vessel* regions falls below the value 0.2, which corresponds to the minimum contrast that can be resolved by the human eye. Results are illustrated in Fig. 7.10.

A choice that each clustering algorithm must make concerns the distance to be used. Many FCM algorithms simply use the Euclidean distance. However, several studies, such as Gustafson et al. [14] and Krishnapuram et al. [20], have shown that

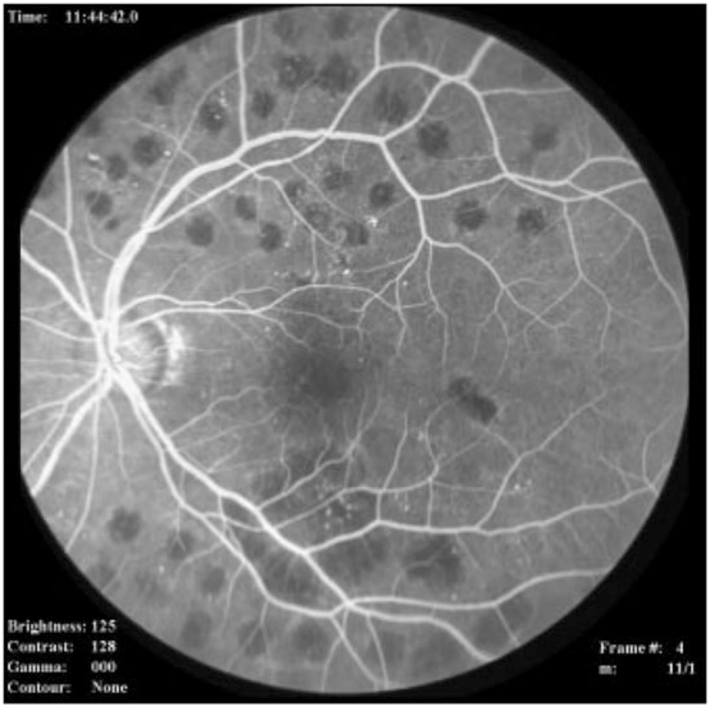


Fig. 7.9 Typical image of the fundus of the right eye. Since the gaze is into the camera, the macula is in the center; the optic disk is towards the nose. Image used in [39] (obtained by courtesy)

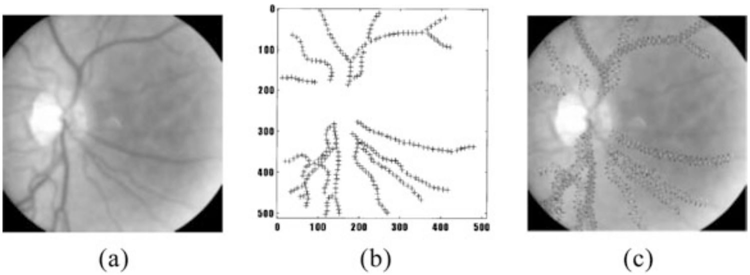


Fig. 7.10 Results of the eye vessel tracking algorithm: (**a**) test image; (**b**) center points that were detected; (**c**) vessel profiles that were examined and the vessel memberships. (Figure 10 of [39], obtained by courtesy)

other distances, such as the Mahalonobis distance, produce better results, especially when the cluster shapes are to be considered. Another study [40] pursued a variant of the FCM algorithm by modifying the feature weights used in the Euclidean distance. They proposed an approach to *feature weight learning* based on gradient descent and

Table 7.4 Correspondence between features of sleep EEG and fuzzy clustering [13]

	Sleep EEG signals environment	Clustering features
1.	Continuous transition between sleep stages	Groups are *not well separated*
2.	Intersubject variability of spectral features	Features have unequal variance: *variability in cluster shape*
3.	Variable number of segments and their features for different sleep stages	*variable cluster densities*
4.	The number of sleep stages may vary between subjects	The number of subgroups (clusters) *is not known a priori*

concluded that appropriate assignment of weights to feature vectors can improve the performance of fuzzy c-means.

A novel minimax approach to fuzzy c-means is discussed by Li et al. [24]. It is suggested an approach to minimize the maximum value of the set of weighted cluster variations in such a way that they satisfy a prior distribution. It is proved that the use of the prior distribution improves the quality of clustering results significantly.

Alternative approaches, departing from FCM, to fuzzy clustering with biomedical applications were also investigated. For example, an unsupervised fuzzy clustering algorithm is presented in [13]. The term “unsupervised” refers to the fact that the algorithm does not require as input the number of clusters. Instead, the algorithm infers the optimum number of clusters and the clusters in the training data. The approach relies on combining ideas from the fuzzy c-means algorithm and the *fuzzy maximum likelihood estimation* and it is supported by extensive experiments on synthetic and real data. The real data experiments use the scores of EEG data corresponding to various stages of sleep. The authors argue that the patterns of EEG segments during sleep naturally generate a fuzzy environment as shown in Table 7.4.

A modified FCM using multiple active contours has been applied for the segmentation of cytological images and of the tumoral lobules [36]. The modification of the FCM algorithm consists in modifying the objective function J by a probabilistic term. More precisely, the new objective function J_{prob} is

$$J_{prob} = J_{FCM} - \alpha \sum_{i=1}^{C} p_i \log p_i, \tag{7.11}$$

where J_{FCM} is the usual objective function from the original FCM algorithm, p_i the probability of cluster i is defined as the mean of the membership values to this cluster. Each pixel (i, j) is mapped into its membership degree to each of the C clusters. Therefore, one can consider that at this stage the image has been mapped into a multi-valued fuzzy image. From this point on, based on the set of initial cluster seeds, object boundaries are obtained by applying known image processing techniques, such as fast marching algorithm.

7.3 Fuzzy Sets and Connectionist Approaches

The use of fuzzy sets in conjunction with other approaches, notably with connectionist models has been an active and fruitful area of research. Such efforts go back to the 1990s. For example, in [25] a Fuzzy Hopfield Neural Network (FHNN) model was proposed and applied for segmentation of medical images. The problem of the image segmentation is regarded as an optimization based on a cost function defined as the Euclidean distance between the gray levels in a histogram to the cluster centers represented in the gray levels. The network is constructed as a $2D$ fully interconnected array with the columns representing number of classes and the rows representing the gray level of pixels taken as training samples. A training sample belongs to a cluster with a certain degree of membership. Thus these clusters are *fuzzy clusters* obtained using the fuzzy c-means clustering. The authors claim that the major strength of a FHNN is that it is computationally more efficient than the fuzzy c-means clustering alone due to the inherent parallel structures.

7.3.1 Conventional 2D Hopfield Neural Network

Following [25], a conventional 2-D parallel Hopfield network for the classification problem consists of $n \times c$ neurons which are fully interconnected neurons. With the notation $V_{x,i}$ for the binary state of neuron (x, i), $W_{x,i;y,j}$ the interconnection weight between the neuron (x, i) and the neuron (y, j), and a bias $I_{x,i}$ the net is described by

$$Net_{x,i} = \sum_{y=1}^{n} \sum_{j=1}^{c} W_{x,i;y,j} + I_{x,i}. \tag{7.12}$$

The Lyapunov energy function of the two-dimensional HNN is given by

$$E = -\frac{1}{2} \sum_{x=1}^{n} \sum_{y=1}^{n} \sum_{i=1}^{c} \sum_{j=1}^{c} V_{x,i} W_{x,i;y,j} V_{y,j} - \sum_{x=1}^{n} \sum_{i=1}^{c} I_{x,i} V_{x,i}. \tag{7.13}$$

Gradient descent of the energy function is used to train the network. Eventually, the following algorithm is applied:

Step 1 Input a set of training gray levels $Z = \{z_1, z_2, \ldots, z_n\}$ and their associated frequencies of occurrence $P = \{p_1, p_2, \ldots, p_n\}$, the fuzzification parameter m $(1 \leq m \leq \infty)$, the number of classes c, and randomly initialize the states for all neurons $U = (\mu_{x,i})_{x,i}$ (membership matrix);

Step 2 Compute the weight matrix $W = [W_{x,i;y,j}]$ using

$$W_{x,i;y,j} = \frac{p_y z_y}{\sum\limits_{h=1}^{n} p_h(\mu_{h,i})^m}.$$

Step 3 Calculate the input to each neuron (x, i) using

$$Net_{x,i} = \left[z_x - \sum_{y=1}^{n} \frac{p_y z_y (\mu_{y,i})^m}{\sum\limits_{h=1}^{n} p_h(\mu_{h,i})^m} \right]^2.$$

Step 4 Update the membership function for the xth row in a synchronized manner as

$$\mu_{x,i} = \left[\sum_{j=1}^{c} \left(\frac{Net_{x,i}}{Net_{x,j}} \right)^{2/m-1} \right]^{-1}, \text{ for all } i.$$

A synchronous iteration is defined as an updated fuzzy state for all neurons using a software simulation.

Step 5 Compute $\Delta = \max \|U^{(t+1)} - U^{(t)}\|$. If $\Delta > \epsilon$ then go to Step 2. Otherwise stop the process.

In addition to extensive experimental results on brain, liver, and chest images, theoretical results on the convergence of the FHNN are provided in [25], showing that the network will reach a stable state. Figure 7.11 is typical of the experimental results obtained. To evaluate the quality of segmentation, the measure of uniformity defined in [23] is used. Informally, this measure evaluates how homogeneous a segmented region is. It is based on the weighted average of the variance in pixel gray levels within the region of interest. Formally, for an image α, segmented into regions $R_1, \ldots, R_n$, the uniformity U_α is defined as follows:

$$U_\alpha = 1 - \frac{\sum_{R_i \in \alpha} w_i \sigma_i}{M},$$

where σ_i is the gray-level variance within region R_i, and

$$M = \sum_{R_i \in \alpha} w_i \frac{(f^i_{max} - f^i_{min})^2}{2},$$

with f^i_{max}, f^i_{min} the maximum and minimum gray levels in the region R_i, respectively. For example, for the liver images of Fig. 7.11, the uniformity measure for the hard c-means segmentation is 0.863, while for the FHNN segmentation it increases to 0.924.

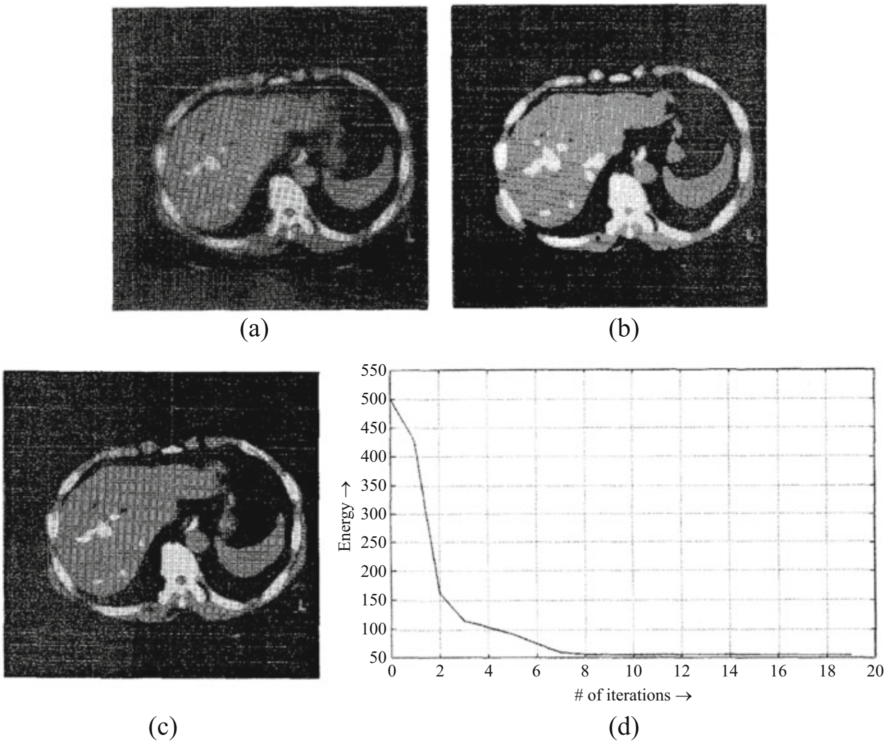


Fig. 7.11 (**a**) CT liver image for segmentation with $c = 3$. (**b**) Segmented result using the FHNN. (**c**) Segmented result using the c-means. (**d**) Energy curve in converging iterations in the FHNN (from [23], obtained by courtesy)

A hybrid approach combining two unsupervised learning paradigms—hierarchical self-organizing map (HSOM) and FCM—for image segmentation is presented in [28]. The approach is applied for the detection of various types of tissue, in particular tumors, from MRI images of the brain. Although based on already existing and well understood approaches, the authors show that the performance of the hybrid approach is superior. First, HSOM is applied. Its output provides the initial cluster centers assignment which is then further refined by FCM. An important aspect of this hybrid approach is that unlike when SOM is used in conjunction with clustering techniques, where the number of clusters must be decided in advance, HSOM does not require this. Instead this number is deduced from its output. Table 7.5 compares four different hybrid approaches, SOM with crisp and fuzzy clustering, and HSOM with crisp and fuzzy clustering. In addition to the number of weight vectors produced and size of the tumor region (number of pixels in the region), it details the execution time of the four possible combinations

Table 7.5 Weight vectors, detected tumor pixels and execution time for segmentation [28]

Types of segmentation	Number of weight vectors	Tumor size (number of pixels)	Execution time (sec)
SOM-kmeans	12	2772	24.98
SOM-fuzzy	8	3223	93.39
HSOM-kmeans	12	2772	45.63
HSOM-fuzzy	6	3223	100.03

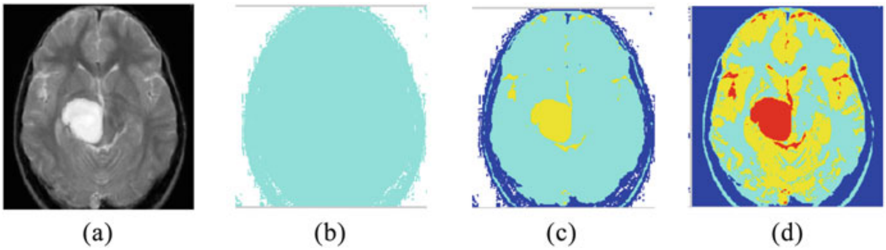


Fig. 7.12 Segmentation results from HSOM-FCM hybrid approach: (**a**) input image (256×256), (**b**) Gray matter (GM) segmented at level 1 (2×2), (**c**) GM,White Matter (WM) with tumor segmented at level 2 (4×4), (**d**) WM, GM, tumor and Cerebro-Spinal Fluid (CSF) segmented at level 3 (6×6) [28] (obtained by courtesy)

of SOM, HSOM, crisp k-means and fuzzy c-means clustering.[1] Figure 7.12 shows the segmentation results on a brain MRI image.

In [38] the authors are persuasive in explaining the advantages of adopting fuzzy set theory in conjunction with a connectionist model and introduce a *dynamic fuzzy neural network* (DFNN) model. This is applied to the detection of epilepsy seizures in the case of absence seizure (petit mal). The DFNN model is a network with unconstrained connectivity and with dynamic elements in its fuzzy processing units, called *feurons*. A feuron represents a single dynamic neuron with *a fuzzy activation function*. This architecture is illustrated in Fig. 7.13, which shows a DFNN with three feurons.

Basically the feuron model represents the biological neuron, which fires when its inputs are sufficiently excited. Firing is through a lag dynamics (i.e., Hopfield dynamics). The fuzzy activation function ϕ behaves as biological neurons. In this paper the authors used a standard fuzzy system as follows: (i) the input is fuzzified using a singleton fuzzifier, (ii) Gaussian membership functions are used, (iii) inference is based on the product of membership values, and (iv) the output is obtained by the center of gravity defuzzifier. The activation function of the ith feuron is then expressed as:

[1] The hybrid approach SOM/HSOM with the crisp k-means clustering is taken from [4].

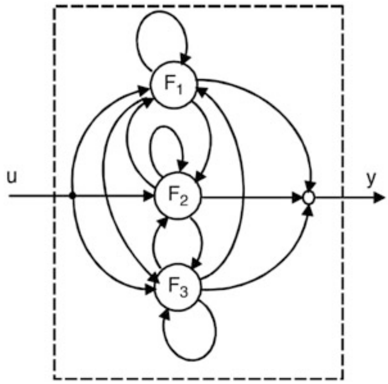


Fig. 7.13 The 3-feuron DFNN diagram [38] (obtained by courtesy)

$$\phi_i(x_i) = \frac{\sum\limits_{j=1}^{R_i} a_{ij}\mu_j(x_i)}{\sum_{j=1}^{R_i}\mu_j(x_i)}, \tag{7.14}$$

where $\mu_j(x_i) = \exp\left(-\frac{1}{2}\left(\frac{x_i - c_{ij}}{\sigma_{ij}}\right)^2\right)$, c_{ij} is center, and σ_{ij} is spread of the jth receptive field unit of the ith feuron. Other parameters of the model are described in detail in the paper. The inputs to the model are four wavelet coefficients extracted from the EEG signals. The DFNN classifier used as output two classes indicated by labels 0 and 1 for normal and epileptic cases, respectively. A dynamic learning rate was used which was adjusted in the opposite direction of the mean squared error (MSE). Evaluation of the model performance was done by reference to the board-certified neurologist to make independent judgment of presence or absence of seizures.

7.3.2 Fuzzy Sets and Deep Learning

Deep learning (DL) has fast become one of the most active areas of research in the field of Machine Learning (ML). Despite gaining popularity relative recently (approximately since 2012), its basic ideas go back to T. Senjowski and G. Hinton's research in the early 1980s. As computing resources have increased tremendously, the implementation of what has become to be known as deep learning has also become possible. One of the issues that drive ML algorithms can be summarized as linear vs non-linear, when seeking the boundary/decision surface between classes in a data set. Neural networks, support vector machine with non-linear kernels, fuzzy systems, they all attempt to address the case when the decision surface is non-linear. The extent to which any of these models is finally successful depends on many choices made, among which the most important are the features used to describe a data point, the type and number of features. DL is most successful for image processing or image understanding applications. Indeed for many other applications, researchers have designed ways in which the original data points have been rendered as images, prior to being used by a DL algorithm. Deep Learning is implemented as multilayer neural networks, also known under the name of convolutional neural networks (CNN), the term “deep” referring to the fact that there are several layers in the network. These layers perform various operations, e.g., convolution, max-pooling, etc. The goal is to mitigate the non-linearity in the data. As the network is trained, various features appear at different levels of the network. Such features could be of the type that a user might obtain (e.g., edges, by segmenting the image), or completely different (e.g., some complex texture features). Many different architectures for DL networks have been obtained to date. Designed specifically for the segmentation of biomedical data, the U-net architecture [33] does not require a massive number data points. Instead, by using data augmentation it generates additional training example, while also building robustness of the network to variations of the data. Various papers [8, 29, 37] have studied the theoretical aspects as well as the applications of the U-net architecture.

Fuzzy sets have been injected into this and other DL architectures. For example, to name a few, in [26] fuzzy sets are used in conjunction with U-net for developing an efficient approach for brain tumor detection, while in [18] a novel U-net neural network architecture obtained by the addition of fuzzy layers is explored and evaluated on the segmentation of cellular nuclei. In [2] concepts from fuzzy sets theory (L-numbers) and graph theory (minimal spanning tree) are used in conjunction with YOLO [32] deep learning algorithm for diagnosis of melanoma. In [15] a fuzzy enhanced deep learning architecture was used for early detection of Covid-19 from X-ray images of the chest.

An interesting study on the impact of using fuzzy sets in conjunction with several DL architectures is shown in [1] for semantic segmentation of breast cancer tumors in ultrasound images. The approach is along the following lines:

“Fuzzification” For each (i, j) pixel in the image I, define

$$\mu(i, j) = \frac{I(i, j) - Min}{Max - Min},$$

where $I(i, j)$ denotes the pixel value, $Min = \min_{(i,j)} I(i, j)$, and $Max = \max_{(i,j)} I(i, j)$. This step is actually a normalization of the gray levels.

Fuzzy Image Intensification Based on μ defined in the previous step, an intensification operator is applied, inspired from modifiers used in fuzzy set theory. This has the role of boosting large values of μ and decrease smaller values of μ. A concentration operator of the form $C : \mu \mapsto \mu^k$, with $k \geq 2$, has the property that $C(\mu) \leq \mu$. Then the image intensification is defined (for $k = 2$) as:

$$\mu'(i, j) = \begin{cases} 2\mu(i, j)^2 & \text{if } 0 \leq \mu(i, j) \leq 0.5 \\ 1 - 2\left(1 - \mu(i, j)^2\right) & \text{if } 0.5 < \mu(i, j) \leq 1 \end{cases}$$

Table 7.6 Semantic segmentation evaluation metrics for 200 *benign breast ultrasound images* before and after applying fuzzy enhancement, based on batch processing (Table 1 of [1], obtained by courtesy)

Semantic segmentation network	Global accuracy		Jaccard index		Mean border F1 score	
	%Before	%After	%Before	%After	%Before	%After
FCN-AlexNet	90.62	97.22	49.11	78.60	44.68	63.78
U-Net	91.08	97.40	49.09	78.65	44.93	77.22
SegNet-VGG16	90.99	98.14	49.13	83.77	45.31	82.31
SegNet-VGG19	90.88	98.16	49.13	84.00	45.26	82.66
DeepLabV3+ - ResNet18	90.54	98.01	48.87	83.84	45.14	81.91
DeepLabV3+ - ResNet50	90.47	98.38	48.80	85.70	45.10	83.49
DeepLabV3+ - MobileNet-V2	90.94	97.78	48.99	81.27	45.13	80.66
DeepLabV3+ - Xception	90.81	98.11	49.08	83.99	45.10	82.01

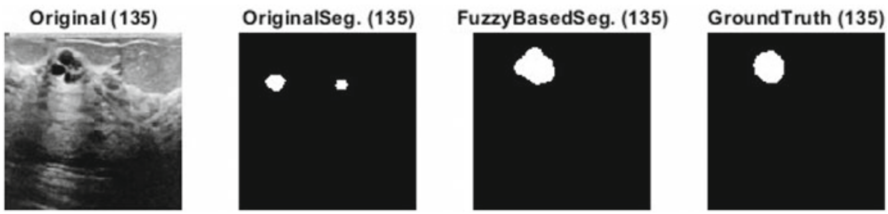
Table 7.7 Semantic segmentation evaluation metrics for 200 *malignant breast ultrasound images* before and after applying fuzzy enhancement, based on batch processing (Table 2 of [1], obtained by courtesy)

Semantic segmentation network	Global accuracy		Jaccard index		Mean border F1 score	
	%Before	%After	%Before	%After	%Before	%After
FCN-AlexNet	81.34	92.06	50.08	72.21	40.43	50.89
U-Net.	83.09	91.74	50.58	69.39	40.91	54.50
SegNet-VGG16	81.65	93.53	50.45	76.91	39.53	57.42
SegNet-VGG19	81.94	93.53	50.72	76.50	40.12	58.80
DeepLabV3+ - ResNet18	80.83	93.34	49.97	77.22	40.18	59.33
DeepLabV3+ - ResNet50	80.07	93.91	49.34	77.78	40.07	60.21
DeepLabV3+ - MobileNet-V2	81.71	92.80	50.31	73.81	40.47	55.93
DeepLabV3+ - Xception	80.34	93.02	50.03	75.39	39.63	58.06

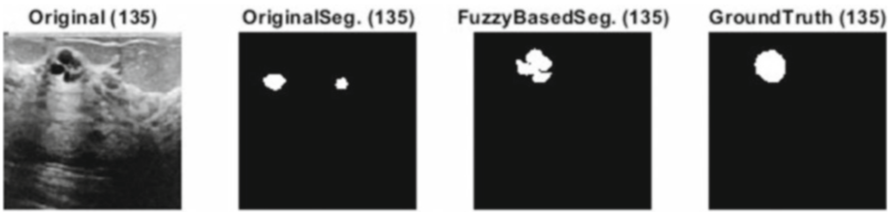
"Defuzzification" In this step, the enhanced fuzzy image is converted back into the initial range of values:

$$I_{out}(i, j) = Min + \mu'(i, j)(Max - Min).$$

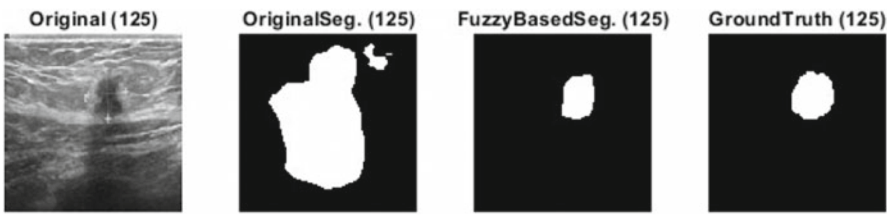
To evaluate the impact of the above image transformation on the performance of various semantic segmentation models and deep learning architectures, including FCN with AlexNet network, UNet network, SegNet using VGG16, SegNet using VGG19, DeepLabV3+ using ResNet18, DeepLabV3+ using ResNet50, DeepLabV3+ using MobileNet- V2, and DeepLabV3+ using Xception networks, the data were divided subsets of 200 images each, into benign and malignant before and after using the fuzzy image enhancement. Extensive experiments—batch and sequential processing were carried out and reported. Tables 7.6 and 7.7 illustrate the type of results obtained. Figure 7.14 also illustrates the effect of the fuzzy set based preprocessing of the image data.



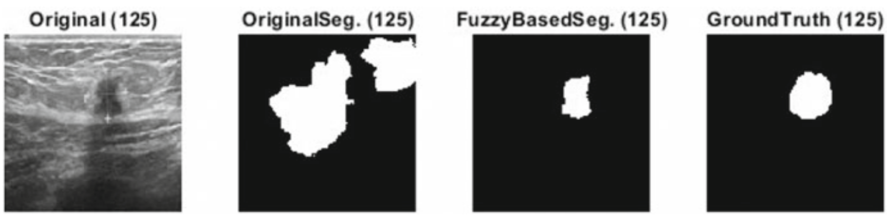
(a)



(b)



(c)



(d)

Fig. 7.14 Segmentation results before and after application of the fuzzy based image preprocessing (from [1], obtained by courtesy). (**a**) Benign tissue segmented by ResNet18. (**b**) Benign tissue segmented by U-Net. (**c**) Malignant tissue segmented by ResNet18. (**d**) Malignant tissue segmented by U-Net

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Chapter 8
Structural and Linguistic Representations

In this chapter, some formal models for knowledge and information representation are presented, focusing on their extensions to fuzzy sets. They will be used for image understanding in Chap. 9. The underlying idea, common to most models, is to represent spatial information and knowledge about it, with both spatial entities and spatial relations between these entities. See Chap. 6.4.1 for fuzzy models of relations. Now we explain how they are embedded in formal models, first of linguistic nature to establish links between the natural language and computational models (Sect. 8.2), and then of structural nature (Sects. 8.3–8.9). We start with a few general considerations in Sect. 8.1.

8.1 Fuzzy Representation of Image Information and of Related Knowledge

An overview on fuzzy sets in image processing and understanding can be found in [16]. In this section we focus on representation issues: fuzzy sets can be used to represent both image information, along with its imprecision (if any), and domain and expert knowledge.

8.1.1 Image Features

Fuzzy sets representing image information can be considered from two points of view. First, for spatial information representation, a membership function is defined from the image space into [0, 1], representing the membership degree of each point to a spatial fuzzy object. Such models may represent different types of imprecision, either on the boundary of the objects (due, for instance, to partial volume effect,

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or to the spatial resolution), on the variability of these objects, on the potential ambiguity between classes, etc. Secondly, a membership function can be defined from a space of attributes into [0, 1]. At numerical level, such attributes are typically the gray levels. The membership value then represents the degree to which a gray level supports the membership to an object or a class, described in vague terms, such as bright, dark, etc. At intermediate level, attributes can refer, for instance, to the shape of image regions. Membership functions then allow determining the degree to which an image region is elongated, regular, etc.

Many examples can be found in the literature. For instance, for applications in medical imaging, membership functions are defined in [28] for a few linguistic values of attributes such as size, gray level, shape, centrality and are then used to recognize brain structures in MRI images using similarities with a model. This type of modeling refers to the notion of linguistic variable (see Sects. 2.4 and 8.2.1 for other examples).

8.1.2 Knowledge and Semantics

With respect to knowledge representation, fuzzy sets are typically used to model, in a semi-quantitative way, symbolic or qualitative knowledge describing the expected content of the images (appearance and shape of the objects, spatial relations, type of objects, etc.). The concept of linguistic variable is then often used [86], as well as structural representations. These representations contribute to reduce the semantic gap (see next subsection), by associating a symbolic or qualitative value with a representation in a concrete domain (spatial domain or attribute domain). This applies to different types of knowledge useful in image understanding: generic knowledge on the type of observed scene and on the type of image, specific knowledge related to images, used for extracting meaningful information from these images, and knowledge linking image and model.

8.1.3 Semantic Gap

One important problem when reasoning at higher level, for instance, based on symbolic models, is the semantic gap [80], which addresses the following question: how to link visual percepts from the images to symbolic descriptions? To quote [80]:

The sensory gap is the gap between the object in the world and the information in a (computational) description derived from a recording of that scene. The semantic gap is the lack of coincidence between the information that one can extract from the visual data and the interpretation that the same data have for a user in a given situation.

In artificial intelligence, this is also known as the anchoring or symbol grounding problem [24, 44]:

We call anchoring the process of creating and maintaining the correspondence between symbols and sensor data that refer to the same physical objects. The anchoring problem is the problem of how to perform anchoring in an artificial system [24]. *How can the semantic interpretation of a formal symbol system be made intrinsic to the system, rather than just parasitic on the meanings in our heads? How can the meanings of the meaningless symbol tokens, manipulated solely on the basis of their (arbitrary) shapes, be grounded in anything but other meaningless symbols?* [44].

The relevance of fuzzy sets to answering this question relies in their capability to capture linguistic as well as quantitative knowledge and information. A typical example is the notion of linguistic variable [86] (see Sect. 2.4), where symbolic values (defined at an ontological level) have semantics defined by membership functions on a concrete domain (at the image or features level).

8.2 Linguistic Representations

It may happen that numerical representations are not adequate to describe a given situation. For instance, if a variable has a large range of variation, we can hardly attach one value to any specific situation and we may prefer to use a qualifier issued from the natural language to group coarsely some typical subsets of interest. For instance, to describe the size of an object, it may be more convenient to use only a few terms having rough frontiers, like large, medium, or small. This corresponds to a granulation of the information. According to [89], the concept of granule is the starting point in "computing with words," and it is defined as *a fuzzy set of points having the form of a clump of elements drawn together by similarity* (quoted from [89]). A word is then a label of a granule. When it comes to compute with these representations, specific tools are needed. Such representations are particularly useful in approximate reasoning.

Such representations are called linguistic variables (see the definition in Chap. 2, Sect. 2.4). They are variables whose values are words or sentences [86]. The advantage of such representations is that linguistic characterizations may be less specific than numerical ones (and therefore need less information) [86]. Their two levels (syntactic and semantic) allow, on the one hand, for approximate modeling of vague concepts and reasoning on them, and, on the other hand, solving the semantic gap issue by providing semantics in concrete domains, according to each specific context.

8.2.1 Description of Some Properties or Characteristics

It is natural to use linguistic variables to describe various image or object properties, such as shape, gray level, spatial relations…For each property, a set of linguistic values is defined. This level allows reasoning abstractly, in a symbolic way. For example, if an object has a "medium" gray value, it is easy to derive that it is darker than an object that has a "very bright" gray value. Now, according to the context, a semantic is defined for each of the linguistic values, as a fuzzy set in a concrete domain. This domain is an interval for gray values, the positive real half line for distances, angles for orientation and directional relations ($[0, \pi]$ or $[0, 2\pi]$ in 2D, $[0, 2\pi] \times [0, \pi]$ in 3D), etc. This second level allows reasoning in the concrete domain and with instances of the objects.

Let V be a variable, $\mathcal{D}$ its domain, and $\{V_1 \dots V_n\}$ the set of linguistic values. A membership function μ_i on $\mathcal{D}$ is associated with each value V_i, defined according to the context and the application. In a specific situation, let v be the value computed on the image objects (e.g., elongation of an object, gray value of an object, distance between two objects, etc.). Then the value $\mu_i(v)$ represents the degree to which the image objects are V_i. Note that if the situation is not described by a number, but rather by a fuzzy number or a distribution, then comparison tools should be used, as described in Chap. 6 for computing distances between fuzzy sets and for comparing spatial relations.

To illustrate this, let us consider the example of distances. The variable V is the distance, and $\mathcal{D} = \mathbb{R}^+$ (i.e., distance axis). Let us define the set of V_i as V_1 = close to, V_2 = at medium distance, and V_3 = far from (of course the granularity can be refined by choosing more linguistic values). The semantics are defined by membership functions depending on each concrete value of distance d: $\mu_i(d) = f(d)$. A simple example is illustrated in Fig. 8.1a, with trapezoidal shape membership functions, depending on four parameters $n_1 \dots n_4$. These parameters have to be specified for each V_i. For V_1 = close to, n_1 and n_2 are set to 0 (Fig. 8.1b). The parameters should also be adapted to the context. Obviously they will not be the same depending on whether we speak of anatomical structures that are close to each other in a MRI brain image, or we consider that a cell nucleus is close to another cell in a microscopy image.

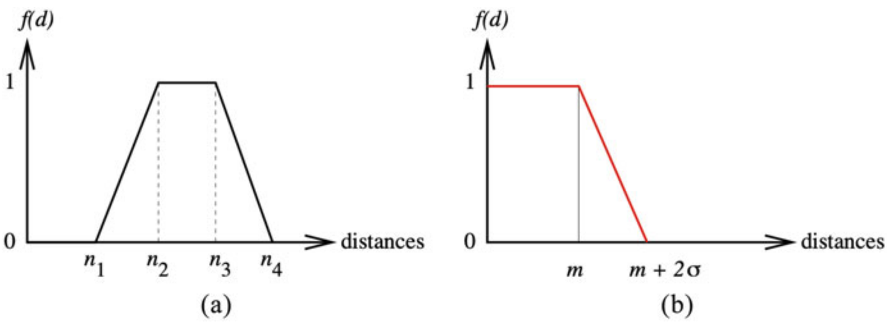

Fig. 8.1 Semantic of a distance linguistic value, defined as a fuzzy set on the real line. (**a**) Trapezoidal membership function, whose parameters are set for each linguistic value. (**b**) Membership function representing V_1 = close to (here the parameters $n_3 = m$ and $n_4 = m + 2\sigma$ can be learned from statistics on annotated data)

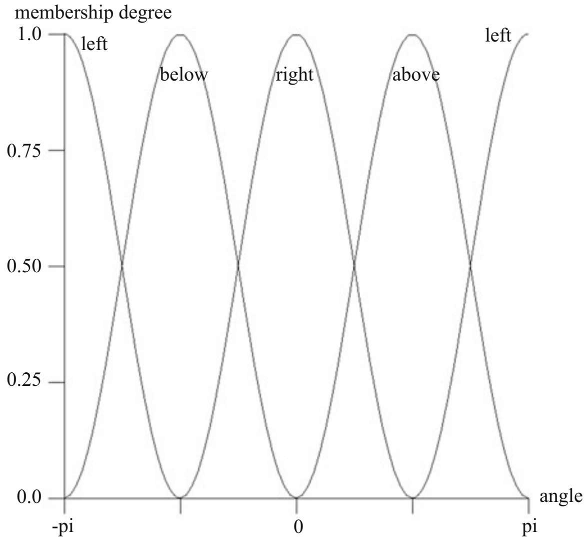

Fig. 8.2 Semantics of directions

Another example is illustrated in Fig. 8.2, where the membership functions provide the semantics of directions, with linguistic values "left," "below," "right," "above." Here $\mathcal{D} = [-\pi, \pi]$, i.e., angle interval in 2D. Again the shape and the parameters of these functions can be tuned according to the domain of application.

The membership functions and their parameters can be handcrafted, according to some expert knowledge on the application domain. They can also be learned. Let us provide an example of a learning procedure, reproduced from [4]. This method was experimented on MRI brain images, but can be adapted to any application domain. An interesting feature of this method is that it can deal with normal cases as well as with pathological cases. Furthermore, it allows learning the stability of the spatial relations in the presence of pathologies. The database is constituted of 18 healthy MRI and 16 pathological MRI where the main anatomical structures were manually segmented. The healthy cases are the images of the widely used IBSR database[1] (real clinical data). The pathological cases include brain tumors belonging to n different classes. The study focuses on the stability of spatial relations between internal brain structures, namely: ventricles, caudate nuclei, thalami, and putamen, which is a clinically relevant choice, according to medical experts' opinion. The first step concerns the structuration and clustering of the database according to a brain tumor ontology. A key point is the representativity of the database according to predominant spatial behaviors of brain tumors, i.e., their tendency to spread, to destroy (necrotic or not), to stem (cystic tumor, edema presence) and their location. From this database, the parameters involved in the construction of the

[1] available at http://neuro-www.mgh.harvard.edu/cma/ibsr/.

fuzzy representations of spatial relations (e.g., function f for distances) are learned. Let $K = (K^N, K^{P_1}, \dots K^{P_n})$ denote the learning database with K^N the set of healthy instances and $K^{P_i}, i \in 1 \dots n$, the set of pathological instances of class i. Let c be an instance of K (an image), O_c the set of segmented objects in c and R a spatial relation. We denote by μ_R^N the fuzzy subset in the image space corresponding to the relation R for a healthy case, and by $\mu_R^{P_i}$ the fuzzy subset in the image space corresponding to the relation R for a pathological case of class i (see Sect. 8.2.3 for such representations). A leave-one-out procedure is used to learn, for a given spatial relation R, the parameters of its fuzzy formulation μ_R. Since μ_R is defined in the spatial domain, it is possible to directly compare μ_R and the target objects. The parameters are optimized so as to maximize the inclusion of the target object in μ_R (i.e., the object has to fulfill the relation with the highest possible degree). For all $c \in K^k,\ k \in \{N, P_1, \dots P_n\},\ (A_c, B_c) \in O_c$, the fuzzy set μ_R^k with respect to A_c is computed, and for a given inclusion measure $\Delta_\subseteq$, the degree $\Delta_\subseteq(B_c, \mu_R^k)$ is derived. The fuzzy set optimizing this criterion is denoted by μ_R^c. In order to learn function f, the minimum or maximum of the distance values are computed for all instances, from which the function parameters are then determined. Let us detail the example of the relation "close to." The training consists in the computation of the maximum distance from a point x of the target object B_c to the reference object A_c:

$$d_{\max}^c = \max_{x \in B_c}(d_{A_c}(x)). \tag{8.1}$$

Then the mean m^k and standard deviation σ^k of the values $d_{\max}^c$ for all $c \in K^k$ are computed. The fuzzy interval f is then defined as a fuzzy subset of $\mathbb{R}^+$, with kernel $[0, m^k]$ and support $[0, m^k + 2\sigma^k]$ (see Fig. 8.1b). This allows taking into account the variability of the parameters in the training set. An example is illustrated in Fig. 8.3. A similar approach was applied for adjacency and directional relations. The stability of the spatial relations can now be assessed by comparing the learned parameters for specific cases and for healthy ones. Such cases are illustrated in Fig. 8.4. A suitable choice for such a comparison is a M-measure of resemblance, according to the classification proposed in [19].

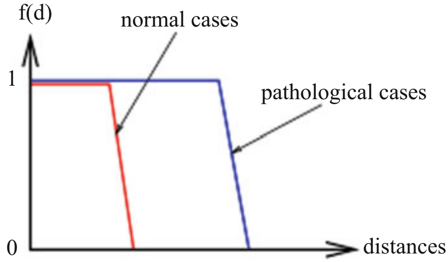

Fig. 8.3 Learning the relation "close to" between putamen and caudate nucleus μ_d on normal cases and on pathological cases for a class P_i (here high grade gliomas that shift the putamen away from the caudate nucleus)

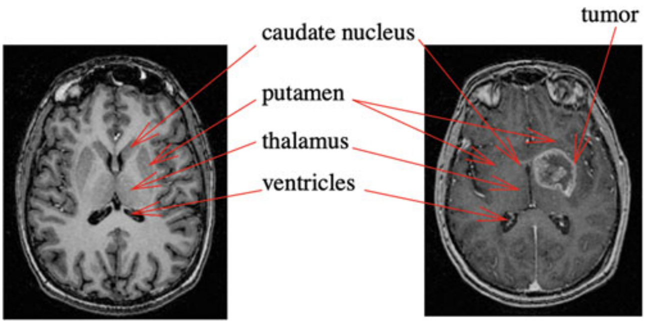

Fig. 8.4 A normal case and a pathological one. The caudate nuclei and the ventricles are adjacent in both cases (hence a high resemblance value for this relation). The putamen is deformed in the pathological case, thus modifying its distance to the caudate nuclei, which explains the low value of the resemblance found during the comparison

8.2.2 Quantifiers

In this section, the notion of quantifier is developed, because of its importance. As linguistic constructs, quantifiers are so common in daily verbal and linguistic interactions that to ask what they are seems almost a superfluous question. And yet, this question must be asked, because by asking, and more importantly, by trying to answer this question the complexity and importance of quantifiers will emerge. Quantifiers can be defined in a number of ways, each definition reflecting the point of view or the application for which they are intended. For example, one finds the following by way of definitions of a quantifier:

- A linguistic construct in which an assertion is made using a variable which ranges over some domain of discourse.
- A term expressive of quantity, especially one that binds variables in a logical formula.
- The act of discovering or expressing the quantity of something.
- A logical constant which indicates the quantity of a class which has a property: examples, such as "all," "no," and "some," are the most frequently studied (linguistic) quantifiers in English, as well as in logic to bind variables in a logical proposition.
- A specific type of determiner that gives an indication of quantity, answering the question how much/how many.
- In grammar a word that expresses a quantity (as "fifteen" or "many").

Naturally enough, the process of using quantifiers is referred to, as quantification, and it means *a way to talk about objects without being specific about the identity of the objects involved* or *the act of discovering or expressing the quantity* *of something*. Typically, there are two kinds of quantification—existential and universal—each quantification using one quantifier and one variable.

Generalized quantifiers are often used to great effect. Generalized quantifiers can be classified into:

Exact: e.g., *exactly five students*, *half of my money*, *John's books*.
Inexact: e.g., *most students*, *approximately half of the apples ...*, *a few dollars*, *the books on the upper shelves*, *the pixels in the middle of the image*.

In information processing, including image processing/understanding applications, quantifiers can be used in summarization procedures (e.g., *most young students are healthy*, or *a few of the tumor pixels are in the middle of the lung*). Thus, in general, if Q denotes a quantifier, and A and B are sets (or predicates), the resulting statements have the form $Q\ As$ are Bs. Besides being able to summarize *a thousand (pixel/data) values by a few words*, by the use of quantifiers, once such summarizations are obtained, an interesting issue is that of reasoning with them. More precisely, the interest is to reason with imprecise rules, typical of which is the following syllogistic rule where Q_1, Q_2, Q are quantifiers (e.g., *most, a few, exactly k*, etc.), H is a linguistic hedge (e.g., *very, a little more than, more or less*, etc.), and A, B, C, D, E, F are crisp or fuzzy sets (predicates):

$$\begin{array}{c} Q_1\ As\ are\ B \\ Q_2\ Cs\ are\ D \\ \hline ?Q\ Es\ are\ ?H\ F \end{array}$$

In logic the interest is to evaluate the truth of expressions quantified by imprecise quantifiers, with or without precise predicates. Thus, an expression such as $Qx \in X, p(x)$ (which states that for $Q\ \ x$ in the set X, the predicate $p(x)$ holds) is translated as $Q\ \ x \in X \longrightarrow p(x)$, (where $\longrightarrow$ denotes implication) when Q is the universal quantifier *for all* ($\forall$) or an extension of it, such as *most*.

There is a fascinating body of work on generalized quantifiers, spanning linguistics, philosophy, and more recently, computing, which is of interest to this chapter. Perhaps the first to provide a thorough formal treatment of generalized quantifiers was Richard Montague [64] in his paper, appropriately titled *The Proper Treatment of Quantification in Ordinary English* (PTQ), where he presents the view that *syntax, semantics, and pragmatism of language are part of mathematics*, and therefore, one might hasten to add, part of computing. Indeed, Montague states "*There is in my opinion no important theoretical difference between natural languages and the artificial languages of logicians; indeed, I consider it possible to comprehend the syntax and semantics of both kinds of languages within a single natural and mathematically precise theory.*" With Zadeh's *Fuzzy set based treatment of quantifiers: A computational approach to fuzzy quantifiers in natural language* [87], generalized quantifiers received the additional name, of *fuzzy quantifiers*. These so-called *fuzzy quantifiers* are, in fact, *inexact, generalized quantifiers* expressed, for computational purpose, as fuzzy sets. Thus, it needs to be stressed here the fact that what many call fuzzy quantifiers is just a treatment, using fuzzy sets, of generalized quantifiers.

Thus two ideas emerge in this discussion, namely that a quantifier is a set, and that a quantifier is a fuzzy set, a classical set being a particular case of a fuzzy set. The next idea that emerges as useful in discussing quantifiers, is that of *cardinality* (see Sect. 2.2.7 for the definition of the cardinality of a fuzzy set). According to Barwise and Cooper [8] the meaning of Q *As* is *the collection of subsets of* A *with cardinality (number of elements) equal to* Q. Similarly, an interesting discussion on the vagueness of generalized, imprecise quantifiers, such a *many*, *few* and the concept of cardinality, is also found in [66]. The cardinality-based approach is adopted here for the discussion of generalized/imprecise/fuzzy quantifiers. To begin with, an exact quantifier Q is identified by an indicator function $Q : \mathbb{N} \to \{0, 1\}$. For example, the quantifier $Q \equiv Exactly_{\mathbf{2}}$ is defined as

$$Exactly_{\mathbf{2}}(k) = \begin{cases} 1 \text{ if } k = 2 \\ 0 \text{ otherwise.} \end{cases} \tag{8.2}$$

A statement such as *Exactly 2 of the people in this room* means *the collection of all subsets of two people, from those in this room.* In general, given a universe of discourse $\mathcal{X}$ the meaning of $Exactly_2$ objects in $\mathcal{X}$ is

$$Exactly_{\mathbf{2}} \equiv \{\{x, y\} \mid x, y \in \mathcal{X}\} \equiv \{\mathcal{Y} \subseteq \mathcal{X}, |\mathcal{Y}| = 2\}$$

which is also captured by Eq. 8.2. Similarly, $AtLeast_{\mathbf{2}}$ objects in $\mathcal{X}$ is

$$AtLeast_{\mathbf{2}} \equiv \{\mathcal{Y} \subseteq \mathcal{X}, |\mathcal{Y}| \geq 2\}.$$

They are both exact quantifiers. An interesting feature of the exact quantifiers is that they are not context dependent, that is, one needs not know, or enumerate all the elements of the universe of discourse $\mathcal{X}$. This is also why mathematical statements involving the universal ($\forall$) and existential ($\exists$) quantifiers over an infinite universe of discourse (e.g., $\mathbb{R}$) are valid.

Transitioning to generalized quantifiers, one needs to see how the above cardinality-based representation of exact quantifiers translates to the generalized ones. For example, how should quantifiers such as *most*, *few*, *many* be represented? It is almost immediately obvious that these are now *context dependent*, that is, in the cardinality-based approach, their meaning depends on the size of the universe of discourse. As cited in [66], the difficulty of interpretation and understanding of generalized quantifiers with their inherent context dependency has been discussed as far back a Aristotle.[2]

[2] "Again, we say that there are many people in the village but few in Athens though there are many times more here than there; and that there are many in the house but few in the theater though there are many more there than here." (Aristotle, Categories).

Following [8], and the example of $Exactly_2$, the meaning of $most$ is the collection of all subsets of cardinality $most$. The first choice is an exact rule, according to which $most$ is defined as the collection of subsets that contain *more than half of the total number of elements*. For a universe of discourse $\mathcal{X}$ with $|\mathcal{X}| = N$, this collection would include all the subsets with $N/2 + j$, $j = 1, \ldots, N/2$ elements. However, while this rule eliminates all subsets of small cardinality (i.e., with fewer than $N/2$ elements) it does not distinguish between subsets which have more than $N/2$ elements (e.g., between subsets with $N/2 + 1$ elements and $N - 1$ elements, which, as N increases, are more and more different). The second choice, an inexact rule, Zadeh's idea put forward in [87], allows for graded distinction between subsets based on their cardinality and is adopted here. Thus, let a generalized quantifier be modeled as a fuzzy set in $\{0, 1, \ldots, N\}$, where N is a positive integer. More precisely, *Q of N* is represented as a fuzzy set with membership function:

$$\mu_Q : \{0, \ldots, n\} \to [0, 1], \tag{8.3}$$

where $\mu_Q(i)$ is the degree to which the quantifier "Q of n" is satisfied by the subset of cardinality i. Some concrete examples are useful here. The quantifier $MOST$ may be defined as:

$$\mu_{MOST_N}(x) = \begin{cases} 0 & \text{if } x \leq N/2 \\ \frac{x-N/2}{N/4} & \text{if } N/2 \leq x \leq 3N/4 \\ 1 & \text{if } 3N/4 \leq x \leq N \end{cases} \tag{8.4}$$

while $AboutHalf$ may be defined as:

$$\mu_{AboutHalf}(x) = \begin{cases} 0 & \text{if } x \leq N/4 \text{ or } x \geq 3N/4 \\ \frac{x-N/4}{N/4} & \text{if } N/4 \leq x \leq N/2 \\ \frac{3N/4-x}{N/4} & \text{if } N/2 \leq x \leq 3N/4 \end{cases} \tag{8.5}$$

Note that other definitions are possible.

Once a representation for quantifiers is selected, the first problem to pose concerns combination of quantifiers, which is illustrated by the following example: suppose *MOST* of N objects have property P_{MOST} and *a FEW* of these have property P_{FEW}. How many objects have property $P_{MOST} \wedge P_{FEW}$? In a more general setting let Q, R be determiners with membership functions, $\mu_{Q,N}, \mu_{R,N} : \{0, \ldots, N\} \to [0, 1]$, the composition $Q \circ R$ is defined by the following membership function:

$$\mu_{R \circ Q,N}(k) = \bigvee_{i=0}^{k} \mu_{R,k}(i) \wedge \mu_{Q,N}(k). \tag{8.6}$$

For more complex quantifiers, however, Eq. 8.6 may be computationally intensive, which raises the need to obtain the membership function for the composition in a more efficient way. This is possible by making use of the level sets, α-cuts, for the membership functions of the quantifiers involved. From Eq. 8.6 one obtains, as shown in [73]:

$$L_{R\circ Q,N}(\alpha) = \bigcup_{i \in L^{\alpha}_{Q,N}} L^{\alpha}_{R,i}. \tag{8.7}$$

Furthermore, when μ_Q, μ_R are quasi-convex (their level sets are intervals) explicit formulas for the level set of their combination can be obtained [73]: if the determiners Q, R are quasi-convex, with level sets $[q^1_{\alpha,N}, q^2_{\alpha,N}]$, $[r^1_{\alpha,N}, r^2_{\alpha,N}]$, then

$$L^{\alpha}_{R\circ Q,N} = \left[r^1_{\alpha,q^1_{\alpha,N}}, r^2_{\alpha,q^2_{\alpha,N}} \right]. \tag{8.8}$$

Eq. 8.8 is important as it provides a way to compute the membership function for the combination, as illustrated below for various combinations of $MOST$, $AboutHalf$.

The composition of quantifiers is illustrated below using the quantifiers $MOST$ and $AboutHalf$. First, using Eqs. 8.4 and 8.5, respectively, the level sets for these quantifiers are:

$$L^{\alpha}_{MOST \text{ of } N} = [(\alpha+2)N/4, N]. \tag{8.9}$$

$$L^{\alpha}_{AboutHalf \text{ of } N} = [(\alpha+1)N/4, (3-\alpha)N/4]. \tag{8.10}$$

Using Eq. 8.8 to compose $MOST$ with itself k times yields the α-level for the composition:

$$L^{\alpha}_{MOST^k \text{ of } N} = \left[N\left(\frac{\alpha+2}{4}\right)^k, N \right]. \tag{8.11}$$

For $\alpha = 0$ and $\alpha = 1$, Eq. 8.11 yields the support and core of the composition $MOST^k$:

$$\text{Support of } MOST^k \text{ of } N = \left[N\left(\frac{1}{2}\right)^k, N \right]. \tag{8.12}$$

$$\text{Core of } MOST^k \text{ of } N = \left[N\left(\frac{3}{4}\right)^k, N \right] \tag{8.13}$$

For $k = 1$ the support and core of the original definition for $MOST$ are recovered:

$$\begin{aligned}&\text{Support of } MOST \text{ of } N = \left[\tfrac{N}{2}, N\right]\\&\text{Core of } MOST \text{ of } N = \left[\tfrac{3N}{4}, N\right].\end{aligned} \tag{8.14}$$

Solving for α Eqs. 8.11, 8.12, and 8.13 we obtain the membership function $\mu_{MOST^k \text{ of } N}$:

$$\mu_{MOST^k \text{ of } N}(x) = \begin{cases} 0 & \text{if } x < \left(\frac{1}{2}\right)^k N \\ 4\left(\frac{x}{N}\right)^{1/k} - 2 & \text{if } \left(\frac{1}{2}\right)^k N \le x \le \left(\frac{3}{4}\right)^k N \\ 1 & \text{if } \left(\frac{3}{4}\right)^k N < x < N \end{cases} \tag{8.15}$$

Successive compositions of $MOST$ with itself lead to more and more vagueness, which at limit is captured by the quantifier $NoneToAll$. Indeed, it follows from Eq. 8.15 that as $k \to \infty$, $\mu_{MOST^k \text{ of } N}(x) \to \mu_{NoneToAll}(x) \equiv 1$.

In the cardinality-based treatment of quantifiers put forward by Barwise and Cooper [8], discussed in [66] and the citation therein, cardinality plays a central role. The reader is referred to Chap. 2 for definitions of the cardinality of a fuzzy set. It is obvious that the cardinality of a fuzzy set A can be viewed as the *fuzzy quantifier all*. Indeed, the fuzzy cardinality tells that, given the discrete fuzzy set A, with ordered membership degrees μ_i, there are k elements in it, that is, that the meaning of *all* the elements in A is k with degree $\mu_k \wedge (1 - \mu_{k+1})$.

Once evaluation of quantifiers is established, it becomes interesting to be able to evaluate statements such as Q *As are Bs*, where Q is a quantifier (possibly generalized, imprecise, fuzzy), A, B are predicates (also possibly generalized, imprecise, fuzzy) [73, 74, 76]. First, note that the meaning of Q *As* is the same as Q *of all As*, that is, the same as $Q_{of\ card A}$, and hence, the same as the composition $Q \circ R$ as defined in Eq. 8.6, which, replacing the quantifier R by All yields:

$$\mu_{Q \text{ of } A}(k) = \mu_{Q \text{ of } CardA}(k) = \bigvee_{i=0}^{k} \mu_{Q \text{ of } k}(i) \wedge \mu_{CardA}(k). \tag{8.16}$$

Alternatively, for a quicker solution $nCardA$ can be used instead of $CardA$. Similarly then, Q *As are Bs* means $Q_{(of\ CardA)}$ is the number of As which are Bs, that is $Card_{A \cap B}$. Thus, Q *As are Bs* is translated into the constraint:

$$Q_{(of\ CardA)} = Card_{A \cap B}. \tag{8.17}$$

For arbitrary fuzzy sets, Eq. 8.17 does not necessarily hold, and instead a measure how much it holds is computed, denoted $Q_{(of\ Card_A)} \circ Card_{A \cap B}$ and defined as:

$$\mu_{Q_{(of\ CardA)} \circ Card_{A\cap B}}(k) = \bigvee_{i=0}^{k} \mu_{Q\ of\ CardA}(i) \wedge \mu_{Card_{A\cap B}}(k), \tag{8.18}$$

for $k = 0, \ldots, N_{A\cap B}$, where $N_{A\cap B}$ is the size (cardinality) of the support of the fuzzy set $A \cap B$. This is, therefore, the (fuzzy) cardinality of the set of elements that satisfy *Q As are Bs* and provides the answer to the question *If Q As are B, How many As are B?*.

Given a *dictionary* of quantifier definitions and the associated linguistic labels, upon matching the result of Eq. 8.18 to these, statements such as *Many As are Bs*, *A few As are Bs*, *Approximately Half As are Bs* can be produced (eventually, each qualified by the degree of match).

A more interesting question would be *Which As are Bs?*. This is more interesting mainly because of its *predictive* flavor. For example, consider the statement *Good students will get good jobs*. Such a statement is a *disposition* in Zadeh's sense [88], that is, it contains implicitly one or more imprecise quantifiers. For this example, the implicit meaning is that *$MOST$ good students get good jobs*. In a group of n students, one can evaluate for each the degree of being a good student and for each job that these students got, the degree of being a good job. It is interesting to know (even roughly) not only how many students got good jobs (the answer will be a fuzzy cardinality), but, more interestingly who got good jobs. To answer such questions additional assumptions are needed. In general, knowing cardinality tells nothing (beyond size) about the set itself. However, knowing the fuzzy cardinality will tell the values of the membership function μ_k, although not their particular assignment to individual elements (for a set with n, there are $n!$ possible assignments):

$$\mu_k = \begin{cases} 1 - \mu_{CardA}(k-1) & \text{if } k \le k_0 \\ \mu_{CardA}(k) & \text{otherwise,} \end{cases} \tag{8.19}$$

where k_0 is obtained from $CardA$. In practice some assignments can be eliminated imposing constraints on the shape of the membership function, that is, various particular assignments can be hypothesized (e.g., obtain triangular membership functions, or enforce some consistency with membership functions in other sets).

A related question concerns the categorization of a data point: given *Q As are B*, and the data point $a \in A$ (with $\mu_A(a)$), is a a B? The answer, qualified by a degree, is related to the previous question, and additional assumptions are needed to obtain it.

The single most important aspect in the treatment of quantifiers presented in this section to is the view of the cardinality is a quantifier. In the crisp case, to say *All elements* ... of a set with N elements is the same as saying *N of N elements*. In the case of the fuzzy set to say *All elements* of a fuzzy set whose support has cardinality N is the same as saying *k out N* to the extent (degree) that the set has exactly k elements. It is the definition of cardinality [75, 77] used here which makes possible this approach. It is important to distinguish, once again, the mathematical definition of the quantifier from its linguistic label, and moreover, to realize that any fuzzy set on the set $\{0, 1, \ldots, N\}$, where N is the crisp cardinality of the support of a fuzzy set, is, in fact, a quantifier.

These quantifiers can be applied to image or object attributes whose linguistic values are modeled as fuzzy sets and could act as linguistic filters to select regions with specific characteristics, or to evaluate attributes of such regions.

8.2.3 Associating Linguistic Representations and the Spatial Domain

For spatial reasoning, it is often useful to represent spatial knowledge directly in the spatial domain. This contrasts with the representations in some feature space described in Sect. 8.2.1. This type of representation is particularly useful for spatial relations. Let R be a spatial relation to a reference object A (or a set of reference objects for n-ary relations). A spatial representation takes the form of fuzzy set $\mu_R(A)$, defined on the spatial domain $\mathcal{S}$, where for all x in $\mathcal{S}$, $\mu_R(A)(x)$ represents the degree to which x satisfies relation R with respect to A. Such an approach has been explained for several relations in Chap. 6, such as directional relations (with the morphological approach to compute fuzzy landscapes), surround, between. A common feature in all these definitions is the prominent role of morphological dilation (see Chap. 4).

Let us further illustrate this idea with distance relations [15]. Distance between objects is an important information for the assessment of spatial arrangement between objects in a scene. Therefore they are widely used in structural pattern recognition and spatial reasoning. Distances between objects A and B can be expressed in different forms, using linguistic expressions such as *the distance between A and B is equal to n*, *the distance between A and B is less (respectively, greater) than n*, *the distance between A and B is between n_1 and n_2*. These expressions are then translated in spatial regions or volumes of interest within $\mathcal{S}$, taking into account imprecision and uncertainty, since these statements are generally approximate (n, n_1, n_2 can be numbers but also intervals, fuzzy numbers, linguistic values, etc.). Distances between sets (average, Hausdorff, minimum distances) are usually defined by analytical expressions. But they also have equivalent expressions in set theoretical and algebraic terms by means of mathematical morphology. This allows us to easily include imprecision, and to deal with distances between fuzzy sets and with fuzzy distances, as detailed in Chap. 6. Let us assume that A is known as one already recognized object, or a known region of $\mathcal{S}$, and let us determine a distinct object B, subject to satisfy some distance relationship with A. According to the morphological expressions of distances, dilation of A is an adequate tool for this. Let us consider the following different cases, where $\delta^n(A)$ denotes the dilation of A by a structuring element of size n (in the discrete case):

- If knowledge expresses that $d(A, B) = n$, then the border of B should intersect the region defined by $\delta^n(A) \setminus \delta^{n-1}(A)$, which is made up of the points exactly at distance n from A, and B should be looked for in $\delta^{n-1}(A)^C$ (the complement of the dilation of size $n-1$).
- If knowledge expresses that $d(A, B) \leq n$, then B should be looked for in A^C (assuming that A and B are distinct, non-intersecting, objects), with the constraints that at least one point of B belongs to $\delta^n(A) \setminus A$.
- If knowledge expresses that $d(A, B) \geq n$, then B should be looked for in $\delta^{n-1}(A)^C$.
- If knowledge expresses that $n_1 \leq d(A, B) \leq n_2$, then B should be searched in $\delta^{n_1-1}(A)^C$ with the constraint that at least one point of B belongs to $\delta^{n_2}(A) \setminus \delta^{n_1-1}(A)$.

Generalizing to fuzzy sets and to fuzzy knowledge about distance, the constraints on the border lead to the definition of actually two fuzzy sets, one for constraining the object, and one constraining its border, as for adjacency. However, they can be avoided by considering both minimum and maximum (Hausdorff) distances, expressing, for instance, that B should lay between a distance n_1 and a distance n_2 of A. Therefore, the minimum distance should be greater than n_1 and the maximum distance should be less than n_2. In this case, the volume of interest for B is reduced to $\delta^{n_2}(A) \setminus \delta^{n_1-1}(A)$.

In cases where imprecision has to be taken into account, fuzzy dilations are used, with the corresponding equivalences with fuzzy distances [14, 17] (see Chap. 6). The extension to approximate distances calls for fuzzy structuring elements, defined through their membership function ν on $\mathcal{S}$. Structuring elements with a spherical symmetry can typically be used, where the membership degree only depends on the distance to the center of the structuring element. Let us consider the generalization to the fuzzy case of the last case (minimum distance of at least n_1 and maximum distance of at most n_2 to a fuzzy set μ). Instead of defining an interval $[n_1, n_2]$, we consider a fuzzy interval, defined as a fuzzy set on $\mathbb{R}^+$ having a core equal to the interval $[n_1, n_2]$. The membership function μ_n is increasing between 0 and n_1 and decreasing after n_2. Then two structuring elements are derived, as:

$$\nu_1(v) = \begin{cases} 1 - \mu_n(d_E(v, 0)) & \text{if } d_E(v, 0) \leq n_1 \\ 0 & \text{otherwise} \end{cases} \tag{8.20}$$

$$\nu_2(v) = \begin{cases} 1 & \text{if } d_E(v, 0) \leq n_2 \\ \mu_n(d_E(v, 0)) & \text{otherwise ,} \end{cases} \tag{8.21}$$

where d_E is the Euclidean distance in $\mathcal{S}$ and O the origin of space. The spatial fuzzy set expressing the approximate relationship about distance to μ is then defined as:

$$\mu_{dist} = \begin{cases} t[\delta_{\nu_2}(\mu), 1 - \delta_{\nu_1}(\mu)] & \text{if } n_1 \neq 0 \\ \delta_{\nu_2}(\mu) & \text{if } n_1 = 0, \end{cases} \tag{8.22}$$

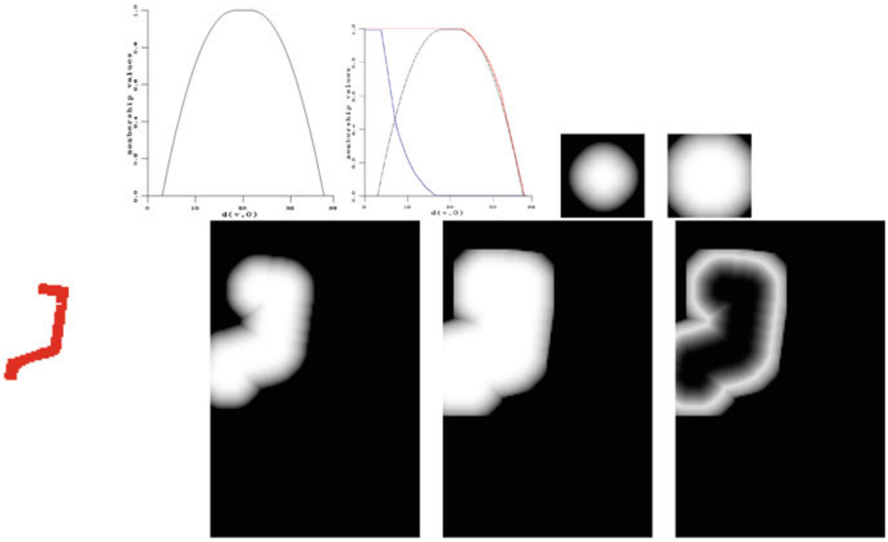

Fig. 8.5 Top: Membership function μ_n, structuring elements ν_1 and ν_2 represented on the real line and then in the spatial domain. Bottom: Reference object, dilations with these two structuring elements, and representation of μ_{dist}. See the text for the detailed equations

where t is a t-norm. The increasingness of fuzzy dilation with respect to both the set to be dilated and the structuring element [17] guarantees that these expressions do not lead to inconsistencies. Indeed, we have $\nu_1 \subseteq \nu_2$, $\nu_1(0) = \nu_2(0) = 1$, and therefore $\mu \subseteq \delta_{\nu_1}(\mu) \subseteq \delta_{\nu_2}(\mu)$. In the case where $n_1 = 0$, we do not have $\nu_1(0) = 1$ any longer, but in this case, only the dilation by ν_2 is considered. This case corresponds actually to a distance to μ less than "about n_2." These properties are indeed expected for representations of distance knowledge.

Figure 8.5 illustrates this approach. The two structuring elements ν_1 and ν_2 are derived from a fuzzy interval μ_n and used for dilation of an object, and μ_{dist} is computed to represent the approximate knowledge about the distance to this object.

From an algorithmic point of view, fuzzy dilations may be quite heavy if the structuring element has a large support. However, in the case of crisp objects and structuring elements with spherical symmetry, fast algorithms can be implemented. The distance to the object A is first computed using chamfer algorithms [18]. It defines a distance map in $\mathcal{S}$, which gives the distance of each voxel v to object A. This discrete distance can be made as precise as necessary [58]. Then the translation into a fuzzy volume of interest is made according to a simple look-up table derived from μ_n. This algorithm has a linear complexity in the cardinality of $\mathcal{S}$, whatever the dimension of space.

8.3 Knowledge-Based Systems

The important role played by knowledge in image interpretation explains the large development of knowledge-based systems (KBS) in this domain. A review of such systems can be found in [25, 53, 54, 81]. KBSs are inspired by human reasoning and consist in representing and modeling the knowledge relative to a domain. Their objective is to reason on this knowledge in order to solve a concrete problem, such as identification, recognition, classification, diagnosis, configuration and planning, among others [53]. These systems are usually composed of three parts: the knowledge base, the observation base and the reasoning components.

Knowledge can be represented as a graph, known as knowledge graph, which constitutes a fast developing field of research. A knowledge graph can be seen as a set of connections between entities and attributes and should represent a computational description of the world. This notion is closely related to the one of ontologies, described in Sect. 8.6.

For image interpretation, the knowledge base is typically composed of three types of knowledge:

- Image processing knowledge: it is used to extract the low-level features from the image and their numerical descriptions, so that they can help identifying the objects of interest in the image.
- Domain knowledge: it concerns knowledge about the semantics of the domain of the image.
- Knowledge about the mapping between image features and concepts: it establishes the link between the two previous types of knowledge. It concerns the knowledge used to map the low-level features and high-level concepts related to the domain of interest. This mapping problem is also known as the semantic gap.

The main idea in KBSs is to represent knowledge in a declarative form, to be able to separate knowledge into different categories, to use the same knowledge to achieve different objectives (related to the idea of shared knowledge and shared representations), and to create reusable inference mechanisms. Such systems can be seen as extensions and generalizations of the traditional expert systems. Typical examples are rule-based systems, frames [62] (declarative representation of attributes or properties, where classes with different granularities can be handled using hierarchy, inheritance, specialization, and instantiation links), semantic networks [72] representing in a graphical manner concepts and objects as vertices and relations as edges. As an example, a notion of fuzzy frames was developed, mainly through various examples, in [41]. It relies on the notions of linguistic variable and quantifiers (see Sect. 8.2). Note that all notions involved in the definition of a KBS can be extended to deal with fuzzy information: concepts or entities, attributes, relations, observations, reasoning methods.

Additionally, image interpretation requires reasoning strategies which can deal with the information imperfections, hence the usefulness of fuzzy sets. Moreover structural information and knowledge is of prime importance. The following sections describe a few models integrating fuzziness and structural information that are the bases of several KBSs, and on which reasoning strategies are built, as described in the next chapter.

8.4 Fuzzy Graphs and Hypergraphs

Graph representations are widely used for dealing with structural information, in different domains such as networks, psycho-sociology, image interpretation, pattern recognition, etc. Several researches use graphs to represent the knowledge and the information extracted, for instance, from images, where vertices represent the segments or entities of the image and edges show the relationships between them. Examples of areas in which this type of representation is used are cartography, robotics and autonomous agents, character recognition, and recognition of brain structures. One important problem to be solved when using such representations is graph matching. In order to achieve a good correspondence between two graphs, the most used concept is the one of graph isomorphism and a lot of work is dedicated to the search for the best isomorphism between two graphs or subgraphs. However, in a number of cases, the bijective condition is too strong, and the problem is expressed rather as an inexact graph matching problem, replacing isomorphisms by morphisms or homomorphisms. Relying on fuzzy sets is then useful to account for imprecision in the matching, in addition to the potential imprecision on the information carried by the graph.

In this section, definitions of fuzzy graphs and fuzzy morphisms are provided. Their use for structure recognition will be exemplified in Chap. 9, where the problem will then to find the best morphisms between two graphs. As an early work, we can mention the fuzzy morphisms defined between graph structures, applied to cognitive diagnosis [7, 49]. Here, the definitions of [68] are given, which are more general, since they include as particular cases the classical notions of graph or subgraph isomorphisms, and of set morphisms. It applies to most types of graphs (relational, attributed, fuzzy, fuzzy attributed, topological graphs...).

Let us first recall the definition of a fuzzy relation. Let S_1 and S_2 be two sets, and σ_1 and σ_2 the membership functions of two fuzzy subsets of S_1 and S_2, respectively. The function $\mu : S_1 \times S_2 \to [0, 1]$ is a fuzzy relation on $\sigma_1 \times \sigma_2$ if

$$\forall (x, y) \in S_1 \times S_2, \mu(x, y) \leq \sigma_1(x) \wedge \sigma_2(y), \tag{8.23}$$

where $\wedge$ denotes the minimum.

Two definitions of fuzzy graphs can be found in the literature.

1. The first one is simply a fuzzy subset of $S_1 \times S_2$ [50]. This corresponds to the definition of a fuzzy relation in the case where the membership functions σ_i are constant and equal to 1.
2. The second definition is restricted to the case where $S_1 = S_2 = S$ and applies on a fuzzy subset σ of S [78]: a fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : S \to [0, 1]$, $\mu : S \times S \to [0, 1]$ which satisfy Eq. 8.23 with $\sigma_1 = \sigma_2 = \sigma$.

This definition is probably the most used one and led to useful extensions of classical notions on graphs.

Given a fuzzy graph $G = (\sigma, \mu)$, for $\tau \subseteq \sigma$ (i.e., $\forall x, \tau(x) \leq \sigma(x)$), the fuzzy subgraph of G induced by τ is the maximal fuzzy subgraph of G that has τ as the fuzzy set of vertices, i.e., $G_\tau(\tau, \nu)$ such that $\forall x, y, \nu(x, y) = \tau(x) \wedge \tau(y) \wedge \sigma(x, y)$.

A path in G is a sequence of vertices $x_0 \dots x_n$ such that $\forall i, 1 \leq i \leq n, \mu(x_{i-1}, x_i) > 0$. The strength of the path is defined as $\wedge_{i=1}^{n} \mu(x_{i-1}, x_i)$. From this notion, several topological notions on graphs can be extended to fuzzy graphs, such as connected components, cliques and clusters, forests, etc. [78].

For image understanding, an important notion is the one of isomorphism, or more generally morphism (typically to handle inexact graph matching problems under uncertainty and imprecision). As will be seen in the next chapter, such morphisms can drive image understanding problems, where both knowledge and image information are represented as graphs or fuzzy graphs.

A fuzzy morphism (ρ_σ, ρ_μ) between graphs $G_1 = (N_1, E_1 \subseteq N_1 \times N1)$ and $G_2 = (N_2, E_2 \subseteq N_2 \times N_2)$ is a pair of mappings $\rho_\sigma : N_1 \times N_2 \to [0, 1]$ and $\rho_\mu : N_1 \times N_2 \times N_1 \times N_2 \to [0, 1]$ which satisfy the following inequality:

$$\forall (u_1, v_1) \in N_1 \times N_1, \forall (u_2, v_2) \in N_2 \times N_2, \rho_\mu(u_1, u_2, v_1, v_2) \leq \rho_\sigma(u_1, u_2) \wedge \rho_\sigma(v_1, v_2). \tag{8.24}$$

The mapping ρ_σ is called vertex morphism and the mapping ρ_μ is called edge morphism. They are linked by Eq. 8.24. The term morphism refers to algebra, where it denotes a mapping between two spaces endowed with an internal composition law, which is preserved by the mapping. Here the introduced morphism has a meaning which is similar to the one of algebra morphism, but not rigorously the same. The binary relation defining the edges between vertices of the graphs should be kept to some degree by the fuzzy morphism. It is a natural extension of the usual notion of graph isomorphism. It is easy to see that a fuzzy morphism is a fuzzy graph (second definition), by setting $S = N_1 \times N_2$. Interpretations as well as properties of fuzzy morphisms and their compositions can be found in [68]. In particular, the edge morphism can be interpreted in two complementary ways:

- $S \times S = (N_1 \times N_2) \times (N_1 \times N_2)$ which corresponds to the classical interpretation of the notion of association compatibility via the edges, called internal interpretation. The compatibility between object associations is often used in pattern recognition for quantifying the influence of the matching of two objects on the one of two other objects, by checking the consistency between both associations.

- $S \times S \rightsquigarrow (N_1 \times N_1) \times (N_2 \times N_2) \supseteq E_1 \times E_2$ which is the new notion of edge morphism introduced in [68]. The arrow $\rightsquigarrow$ indicates that we make $N_1 \times N_1$ and $N_2 \times N_2$ explicit, but it is not an equality. It is called external interpretation. The notion of edge morphism gives importance to the edges themselves by exchanging the order of the sets in the Cartesian product. This interpretation allows checking the correspondence between edges, and therefore to account for the structure of the graphs.

When more than two vertices are linked with an edge, graphs are extended to hypergraphs. A hypergraph is defined as $H = (V, E)$, where V is a vertex set, and E is a set of hyperedges, defined as subsets of V, with any cardinality [13, 20]. The extension to fuzzy hypergraphs dates back to the 1990s, with the work of Goetschel [38, 39]. In this work, the set V of vertices is defined as a crisp set, as in the classical definition, while the set E of hyperedges is a finite family of fuzzy subsets of V. Then classical notions of graph theory are extended to their fuzzy counter-part. A simple hypergraph is defined as a fuzzy hypergraph with no repeated hyperedge, and such that $\forall(\mu, \nu) \in E^2, \mu \leq \nu \Rightarrow \mu = \nu$ (i.e., an hyperedge cannot be a strict fuzzy subset of another hyperedge). Similarly a partial fuzzy hypergraph is a direct extension of a partial hypergraph, from which a partial ordering relation is derived. The extension of transversal to fuzzy hypergraphs is the main focus of [39]. Further extensions have been proposed, where fuzzy sets are replaced by "intuitionistic" or bipolar fuzzy sets and are not detailed here.

8.5 Fuzzy Logics and Fuzzy Rules

In this section, a rapid overview on fuzzy logics is provided, mentioning the main components that are useful for the rest of the book. The reader can refer to [29] for a very clear and complete account of fuzzy and possibilistic logics. As for any logic, language, syntax, and semantics have to be defined. Reasoning then relies on a set of axioms and rules of inference.

Recall that in propositional logic, the language is defined as a set of propositional symbols or variables (atomic formulas), and connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication). Formulas are defined syntactically as propositional variables and combination of formulas using connectives (and no others). A knowledge base is a set of formulas. The semantic is defined from an interpretation function $v : \mathcal{F} \rightarrow \{0, 1\}$, which provides for each formula in $\mathcal{F}$ a truth value (0 for false, 1 for true). A world is defined as an assignment to all variables. The satisfiability relation, denoted by $m \models A$, means that formula A is true in the world m (or m is a model for A, or m satisfies A). A knowledge base KB is satisfiable iff $\exists m, \forall \varphi \in KB, m \models \varphi$ (i.e., $Mod(KB) \neq \emptyset$). Usual axioms and inference rules are:

$$\mathcal{A}_1 : A \to (B \to A)$$

$$\mathcal{A}_2 : (A \to (B \to C)) \to ((A \to B) \to (A \to C))$$

$$\mathcal{A}_3 : (\neg A \to \neg B) \to (B \to A)$$

$$\text{Modus Ponens: } \frac{A, A \to B}{B}$$

These define a deductive system S for proving theorems. Let $\vdash$ denote the consequence relation ($H \vdash C$ iff C can be proved from H using a deduction system S). Then T is a theorem iff $\vdash T$.

Propositional logic can be enriched to be more expressive, leading to many other logics, such as predicate logic (built from constants, variables, functions, predicates expressing relations, the usual connectives, and quantifiers $\exists, \forall$), or modal logic (adding modalities $\Box, \Diamond$, and where a common semantics, called Kripke semantics is based on an accessibility relation between worlds). As an example, a fragment of first order (predicate) logic defines the description logic that is widely used to reason on ontologies (see Sect. 8.6).

All these logics have been extended to their fuzzy counter-part. The intuitive idea is that human reasoning is flexible, allows for imprecise statements, and is able to deal with gradual predicates, defined on continuous referential or representing typicality only. Therefore fuzzy logics, which formalize this intuition, bring an additional level of expressiveness. As highlighted in [29], a distinction has to be made between gradual information and formulas referring to imprecise properties. While the first type of imprecision leads to uncertainty degrees (e.g., in [0, 1]), the second type leads to truth-values that are not only true or false but can take intermediate values (e.g., in [0, 1]). These authors then distinguish between many-valued logics where truth is a matter of degree but consequencehood remains Boolean, and possibilistic logic for graded belief about Boolean statements. Both can be combined in a more general framework dealing with the two types of uncertainty or imprecision, and handling fuzzy or non-fuzzy Boolean statements, complete or incomplete information, fuzzy or non-fuzzy information, and all possible combinations of these. All these fuzzy logics make intensive use of fuzzy connectives (see Chap. 2 for the definitions), generalizing the classical ones: conjunction is modeled as a t-norm, disjunction as a t-conorm, negation as a complementation, and implication as a fuzzy implication, either derived from a negation and a disjunction or derived from a conjunction and the adjunction property $t(a, b) \leq c \Leftrightarrow a \leq I(b, c)$ (residual implication). Axioms and inference rules are then derived accordingly.

Let us give a simple example of fuzzy logic. Basic fuzzy propositions are expressed in the form $X \ is \ P$, where X is a variable taking values in some domain $\mathcal{U}$, P is a fuzzy subset of $\mathcal{U}$, defined by its membership function μ_P. Truth degrees in [0, 1] are defined from μ_P. A conjunction between two propositions $X \ is \ A \ \ and \ \ Y \ is \ B$ is then modeled as $\mu_{A \wedge B}(x, y) = t[\mu_A(x), \mu_B(y)]$, where t is a t-norm. A disjunction is expressed in a similar way, using a t-conorm. An implication such as $A \rightarrow B$ extends to $Imp(A, B) = \inf_{x \in \mathcal{U}} I(\mu_A(x)), \mu_B(x))$ where I is a fuzzy implication. Typical implications are $I(a, b) = \max(1 - a, b)$ (Kleene-Diene), $I(a, b) = \min(1, 1 - a + b)$ (Lukasiewicz), $I(a, b) = 1 - a + ab$ (Reichenbach).

Modus ponens can be extended as follows: let a rule be expressed as

$$if \ X \ is \ A \ then \ Y \ is \ B,$$

and assume a piece of knowledge or an observation is expressed as $X \ is \ A'$, where A' is an imprecise statement about A. Then the conclusion $Y \ is \ B'$ is modeled as $\mu_{B'}(y) = \sup_{x \in \mathcal{U}} t(\mu_{A \rightarrow B}(x, y), \mu_{A'}(x))$, where $\mu_{A \rightarrow B}$ represents the rule. Note that expressions such as $X \ is \ A$ are usually referring to linguistic variables [86].

Fuzzy rules may also involve connectives. Let us consider the following example:

$$\text{IF} \ \ (x \ is \ A \ \ \text{AND} \ \ y \ is \ B) \ \ \text{THEN} \ \ z \ is \ C$$

and denote by α the truth degree of $x \ is \ A$, β the truth degree of $y \ is \ B$ and γ the truth degree of $z \ is \ C$. The satisfaction degree of the rule is then computed as $I(t(\alpha, \beta), \gamma)$, where I is an implication and t a t-norm.

Let us briefly mention possibilistic logic [29, 32]. The main idea is to assign weights (values in [0, 1]) to formulas which are the lower bounds of necessity measures. Let (φ, α) be such a weighted formula, and N a necessity measure. We have $N(\varphi) \geq \alpha$. The modus ponens can be extended as follows: given a rule $N(A \rightarrow B) = \alpha$, a piece of knowledge or an observation $N(A) = \beta$, the conclusion is expressed as $\min(\alpha, \beta) \leq N(B) \leq \alpha$. Let us now consider a knowledge base KB, which can be stratified as $KB = \{(\varphi_i, \alpha_i), i = 1 \ldots n\}$, where α_i is the certainty degree or the priority of formula φ_i. This KB can be modeled as the following possibility distribution:

$$\pi_{(\varphi,\alpha)}(\omega) = \begin{cases} 1 & \text{if } \omega \models \varphi \\ 1 - \alpha & \text{otherwise} \end{cases}$$

for one formula (φ, α), and more generally as:

$$\pi_{KB}(\omega) = \min_{i=1\ldots n} \{1 - \alpha_i, \omega \models \neg\varphi_i\} = \min_{i=1\ldots n} \max(1 - \alpha_i, \varphi_i(\omega)\}.$$

More expressive logics have been extended as well to fuzzy logics (e.g., predicate logic, description logic, modal logics...). For example, modal logics with Kripke semantics can be extended by defining the accessibility relation as a fuzzy similarity relation (see, e.g., [37]). As another example, fuzzy modal operators can be defined, and $\Box_a A$ may mean that the plausibility measure of formula A is equal to a [63]. Fuzzy description logic will be mentioned in Sect. 8.6. Detailing all these extensions is outside of the scope of this book.

As an example that is useful for spatial reasoning and image understanding, logical expressions of Region Connection Calculus (RCC) have been described in Sect. 6.1.2, as well as their fuzzy extensions. For further logical models of space and spatial reasoning, the reader may want to refer to [2].

Representing knowledge as rules or fuzzy rules and reasoning about it to make inference on new examples refer to the domain of symbolic learning. Examples are described next, using fuzzy extensions of decision trees in Sect. 8.7, of association rules in Sect. 8.8, and of formal concept analysis in Sect. 8.9. Before moving to these formalisms, we briefly summarize in the next section how fuzzy knowledge can be formalized in an ontology, dealing specifically with spatial knowledge.

8.6 Ontologies

The semantic interpretation of images can benefit from representations of useful concepts and the links between them as ontologies. This is the aim of this section, which relies to a large part on excerpt of [46].

In knowledge engineering, an ontology is defined as *a formal, explicit specification of a shared conceptualization* [42]. An ontology encodes a partial view of the world, with respect to a given domain. It is composed of a set of concepts, their definitions and their relations which can be used to describe and reason about a domain. Ontological modeling of knowledge and information is crucial in many real world applications such as medicine, for instance [90]. However, most real world domains contain uncertain knowledge and imprecise and vague information. A major weakness of usual ontological technologies is their inability to represent and to reason with uncertainty and imprecision. As a consequence, extending ontologies in order to cope with these aspects is a major challenge. This problem has been stressed in the literature, and several approaches have been proposed to deal with uncertainty and imprecision in ontology engineering tasks [35, 79]. The first approach is based on probabilistic extensions of the standard OWL ontology language,[3] which will not be considered here. As the main ontology language OWL is based on description logics (DL) [6], another approach to deal with uncertainty and imprecision is to use fuzzy description logics [45, 56, 82, 83]. Fuzzy description logics can be classified according to the way fuzziness is introduced into the DL

[3] http://www.w3.org/TR/owl-features/.

Table 8.1 Description logics syntax and interpretation (examples inspired by Baader et al. [6])

Constructor	Syntax	Example	Semantics
Atomic concept	A	Human	$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
Individual	a	Lea	$a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$
Top	$\top$	Thing	$\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$
Bottom	$\perp$	Nothing	$\perp^{\mathcal{I}} = \emptyset^{\mathcal{I}}$
Atomic role	r	has-age	$R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
Conjunction	$C \sqcap D$	Human $\sqcap$ Male	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
Disjunction	$C \sqcup D$	Male $\sqcup$ Female	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
Negation	$\neg C$	$\neg$ Human	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
Existential restriction	$\exists r.C$	$\exists$has-child.Girl	$\{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}} : (x, y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
Universal restriction	$\forall r.C$	$\forall$has-child.Human	$\{x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}} : (x, y) \in R^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}}\}$
Value restriction	$\ni r.\{a\}$	$\ni$has-child.{Lea}	$\{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}} : (x, y) \in R^{\mathcal{I}} \Rightarrow y = a^{\mathcal{I}}\}$
Number restriction	$(\geq n R)$	($\geq$ 3 has-child)	$\{x \in \Delta^{\mathcal{I}} \mid \ \lvert\{y \mid (x, y) \in R^{\mathcal{I}}\}\rvert \geq n\}$
	$(\leq n R)$	($\leq$ 1 has-mother)	$\{x \in \Delta^{\mathcal{I}} \mid \ \lvert\{y \mid (x, y) \in R^{\mathcal{I}}\}\rvert \leq n\}$
Subsumption	$C \sqsubseteq D$	Man $\sqsubseteq$ Human	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
Concept definition	$C \equiv D$	Father $\equiv$ Man $\sqcap$	$C^{\mathcal{I}} = D^{\mathcal{I}}$
		$\exists$ has-child.Human	
Concept assertion	$a : C$	John:Man	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
Role assertion	$(a, b) : R$	(John,Helen):has-child	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$

formalism. A good review can be found in [26]. In particular, a common approach is to introduce fuzziness by using fuzzy predicates in concrete domains as described in [84].

Description logics [6] are a family of knowledge-based representation systems mainly characterized by **a set of constructors** that enable to build complex **concepts** and **roles** from atomic ones. Due to their well-defined semantics and to their powerful reasoning tools, description logics are perfect candidates for ontology languages as explained in [6]. In DLs, a semantics is associated with concepts, roles, and individuals using an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty set and $\cdot^{\mathcal{I}}$ is an interpretation function that maps a concept C to a subset $C^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$ or a role r to a subset $R^{\mathcal{I}}$ of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. **Concepts** correspond to classes. A concept C represents a set of individuals (a subset of the interpretation domain). **Roles** are binary relations between objects. Table 8.1 describes the main constructors and a syntax for description logics.

In a fuzzy setting, a fuzzy knowledge base or fuzzy ontology is defined as a finite set of axioms that comprises a fuzzy ABox A and a fuzzy TBox T. The fuzzy ABox is a finite set of fuzzy (concept or role) assertions, while the fuzzy TBox is a finite set of fuzzy General Concept Inclusions (GCIs), where the subsumption is defined as a fuzzy degree of subsumption. Logical operators of conjunction, disjunction, and complement are used as for other fuzzy logics (see Sect. 8.5).

Concrete domains are expressive means of DL that enable to describe concrete properties of real world objects such as their size, their spatial extension or their color. For instance, in the DL formalism given in Table 8.1, the concept Person $\sqcap \exists age. \leq_{20}$ denotes the set of persons whose age is lower than or equal to 20. In this example $\leq_{20}$ is a predicate over the concrete domain of natural numbers $\mathbb{N}$. As a consequence, a fuzzy extension can be obtained with fuzzy sets defined on the concrete domains. For instance, to denote the concept YoungPerson as YoungPerson $\equiv$ Person $\sqcap \exists$ age.Young, we can define the fuzzy concrete predicate over the natural numbers Young: $\mathbb{N} \rightarrow [0, 1]$ which represents the degree of youngness of a person according to usual modeling methods in fuzzy set theory [31]. Young can be represented by a trapezoidal membership function, for instance. In fuzzy description logics, **concepts** and **roles** are interpreted as fuzzy subsets of an interpretation domain, and axioms, rather than being satisfied (true) or unsatisfied (false) in an interpretation, are assigned a degree of truth in [0, 1]. More details about the semantics of fuzzy description logics can be found in [84].

As illustrated in the previous example, concrete domains can be natural, real, or rational numbers but they can also be more structured datatypes. For instance, in [43], a concrete domain *Polygon* is used to represent the spatial dimension and to combine spatial knowledge representation (restricted to topology in this work) and spatial reasoning in a unique paradigm. In image interpretation, we can consider the image domain as a concrete domain, or any feature space (color space, real line to represent distances, etc.). These predicates defined over concrete domains can be a means of reducing the semantic gap.

This approach was used in [46] to define an ontology of spatial relations, which is detailed next, as an example. This ontology can then be associated with other image information to design model-based image interpretation methods, as described in Chap. 9. An excerpt of the hierarchical organization of spatial relations in this ontology is displayed in Fig. 8.6. Each relation is modeled as a fuzzy set, and fuzzy representations are derived (see Chap. 6 and Sect. 8.2.3).

The spatial relations are then formalized in description logic. One important entity of the ontology is the concept ***SpatialObject*** (SpatialObject $\sqsubseteq \top$). Spatial relations are considered in two different ways: they are concepts with their own properties, and they are also links between concepts. For instance, the assertion "X **is to the right of** Y" can be interpreted and represented in two different ways:

1. As an "abstract" relation between X and Y that is either true or false.
2. As a physical spatial configuration between the two spatial objects X and Y.

As a consequence, a process of reification of spatial relations (illustrated in Fig. 8.7) is used as in [54]. A spatial relation is not considered as a role (property) between two spatial objects but as a concept on its own (***SpatialRelation***). Figure 8.8 represents the Venn diagram of the different concepts of the spatial relation ontology. The notations used in the following are those of Table 8.1.

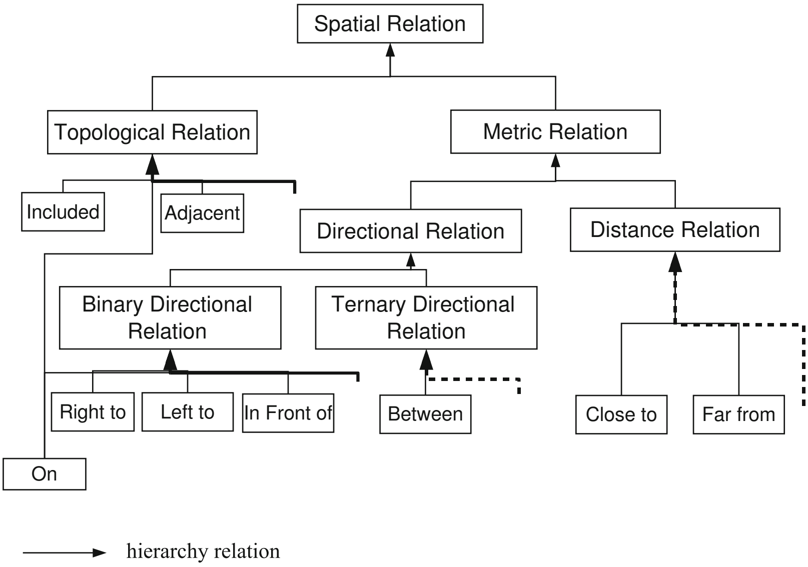

Fig. 8.6 Excerpt of the hierarchical organization of spatial relations in the ontology [46]

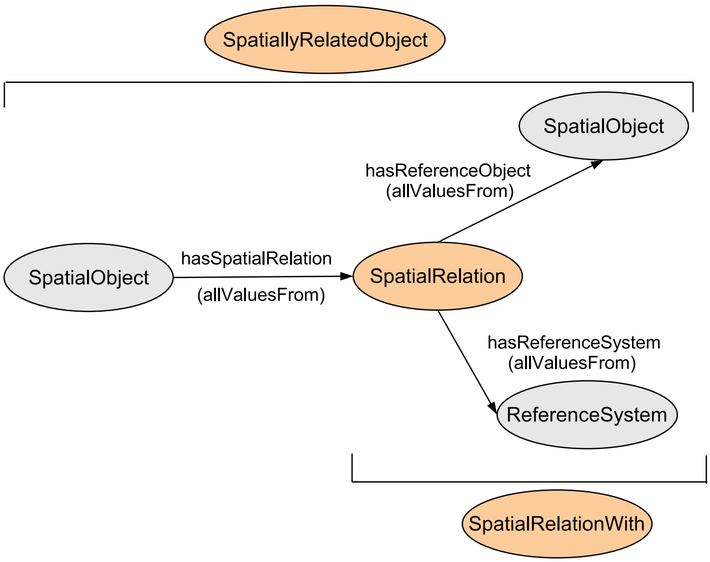

Fig. 8.7 Reification of spatial relations in the spatial relation ontology

- A ***SpatialRelation*** is subsumed by the general concept ***Relation***. It is defined according to a ***ReferenceSystem***.

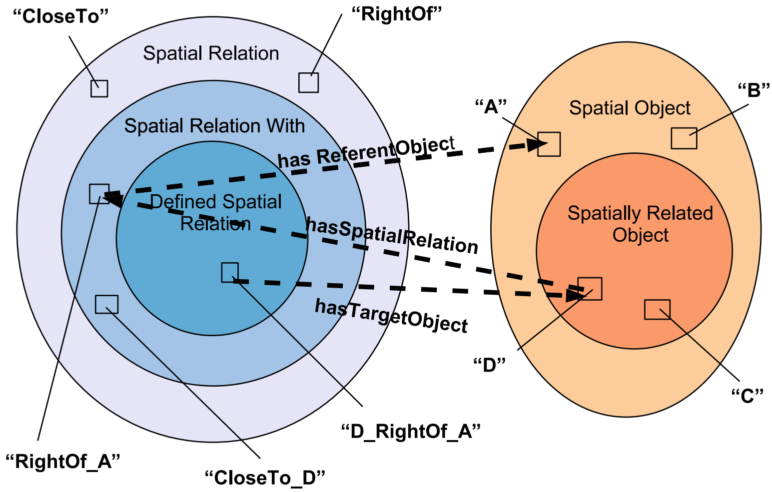

Fig. 8.8 Representation of the main concepts of the spatial relation ontology as a Venn diagram. This diagram illustrates the different concepts and how they are related. "A" is a ***SpatialObject***, it is the ***ReferentObject*** of the ***Spatial Relation With*** concept "RightOf_A". "D" is a ***SpatialObject*** which has the property of having as ***Spatial Relation*** the relation "RightOf_A"

$$\begin{aligned}\mathsf{SpatialRelation} \sqsubseteq\ & \mathsf{Relation} \sqcap \\ \ni\ & \mathsf{type}.\{\mathsf{Spatial}\} \sqcap \\ \exists\ & \mathsf{hasReferenceSystem.ReferenceSystem}\end{aligned}$$

SpatialRelation subsumes ***TopologicalRelation*** and ***MetricRelation*** which itself subsumes ***DirectionalRelation*** and ***DistanceRelation*** as shown in Fig. 8.6 (note that this is also in line with Kuipers' hierarchy of spatial relations [51]). For ***BinarySpatialRelation***, we can also specify *inverse spatial relations* and properties such as *reflexivity, irreflexivity, symmetry, antisymmetry, asymmetry* useful for qualitative spatial reasoning.

- The concept ***SpatialRelationWith*** refers to the set of spatial relations which are defined according to at least one or more reference spatial objects.

$$\begin{aligned}\mathsf{SpatialRelationWith} \equiv\ & \mathsf{SpatialRelation} \sqcap \\ \exists\ & \mathsf{hasReferentObject.SpatialObject} \sqcap \\ \geq\ & 1\mathsf{hasReferentObject}\end{aligned}$$

- The concept ***SpatiallyRelatedObject*** refers to the set of spatial objects which have at least one spatial relation with another spatial object. This concept is useful to describe spatial configurations.

$$\begin{aligned}\mathsf{SpatiallyRelatedObject} \equiv\ & \mathsf{SpatialObject} \sqcap \\ & \exists\ \mathsf{hasSpatialRelation.SpatialRelationWith} \sqcap \\ & \geq 1\mathsf{hasSpatialRelation}\end{aligned}$$

- At last, the concept ***DefinedSpatialRelation*** represents the set of spatial relations for which target and reference objects are defined.

$$\begin{aligned}\mathsf{DefinedSpatialRelation} \equiv\ & \mathsf{SpatialRelation} \sqcap \\ & \exists\ \mathsf{hasReferentObject.SpatialObject} \sqcap \\ & \geq 1\mathsf{hasReferentObject} \sqcap \\ & \exists\ \mathsf{hasTargetObject.SpatialObject} \sqcap \\ & = 1\mathsf{hasTargetObject}\end{aligned}$$

As in [65], an approach of modular semantics allows integrating the fuzzy model of spatial relations with the spatial relation ontology. The two various formalisms are combined in a modular way, taking the best of each of them. Moreover, the separation of the abstract domain (the spatial relation ontology) from its concrete domain on which fuzzy representations are defined contributes to reducing the semantic gap. This integration consists in linking concepts of the spatial relation ontology to their corresponding physical fuzzy representation in the image domain. Of course, the fuzzy representation depends on the type of question. For instance, for the relation "**Right of** R," we are interested in the region of the image space where the relation right of R can be satisfied. Therefore this concept is linked to a fuzzy landscape (i.e., spatial) representation, whereas the relation "**Right of**" is linked to a fuzzy subset of the set of angles representing the semantics of the relation. In the first case the concrete domain is the image support, while in the second one it is the real line. The fuzzy sets defined on these concrete domains provide the semantics of the relation.

Examples of reasoning based on these models will be given in Chap. 9.

8.7 Fuzzy Decision Trees

A convenient way to organize knowledge and then reason on concrete examples is provided by decision trees, in the domain of symbolic learning. One of the first extension as fuzzy decision trees was proposed in [22]. A fuzzy decision tree T is defined as a tree with root r, such that every vertex i in T which is not a leaf is associated with a decision function f_i (or test function at the node) and k ordered children $i_1 \dots i_k$. Each fuzzy decision function f_i is defined as $f_i : X \rightarrow [0, 1]^k$, where X is the input (e.g., the image to be interpreted, or the object to recognize), and the output k-uple defines the labels $v(i_1) \dots v(i_k)$ for all child vertices. The decision value along a path P of vertices in T is defined as $V(P) = \wedge_{y \in P} v(y)$, where $\wedge$ was initially defined as the product or the minimum in [22], and more generally can be any t-norm. The final decision of a new sample is obtained by computing the attributes of the sample and the decision functions at each node, descending the tree until the leaves by computing the value of the path that is followed. The final membership degrees to each class of interest are then obtained by aggregating the values of the leaves corresponding to this class. In this initial work, an example on hand-written number recognition was provided, where the decision functions at each node were defined as image characteristics, such as the presence of vertical lines, of broken loop, of concavities, etc.

There are two mains classes of approaches to define the membership functions for the test at the nodes: in the first class of approaches, a priori knowledge on the attributes or features is used (e.g., given by an expert); the second class comprises automatic methods that do not depend on such knowledge. Discrimination functions are needed in both cases to choose which attribute to consider so as to build a tree that is as small as possible.

The node test for a given attribute is usually defined from a separator function, based, for example, on histograms of attribute values computed for examples in the classes of interest, on which several criteria can be applied (entropy, ambiguity measure, contrast…). Different measures of discrimination have been described and included in a system for building fuzzy decision trees in [59], and an overview of different measures of discrimination that allow for the ranking of attributes can be found in [60].

Now when input data are fuzzy, and no more crisp, further extensions have to be designed in order to evaluate the degree to which the fuzzy value satisfies the node test, and to find the node tests from the fuzzy values of the examples of the training data. The first question can be solved by computing a degree of satisfiability. As for the second one, separator functions can be built based on fuzzy histograms of attribute values of the training samples. Such extensions of fuzzy decision trees have been addressed by many authors, e.g., [48] with details on the inference procedures, [85] where the inference is based on the reduction of classification ambiguity with fuzzy evidence. A review can be found in [47].

Figure 8.9 illustrates a fuzzy decision tree and how an example can be processed [69].

This method has been used in medical image understanding for classifying breast masses in digital breast tomosynthesis in [69]. A fuzzy segmentation of the masses is first performed, from which fuzzy attributes are computed, such as gradient, area, compacity, diameter, perimeter, statistics on the gradient direction and intensities. The method then allows distinguishing spiculated breast masses (as illustrated in Fig. 8.10) from circumscribed breast masses (as illustrated in Fig. 8.11).

An important feature of fuzzy decision trees is that they directly bring explainability to a decision system under imprecision, by highlighting which features led to a decision. This idea has also been exploited to provide explainability to other

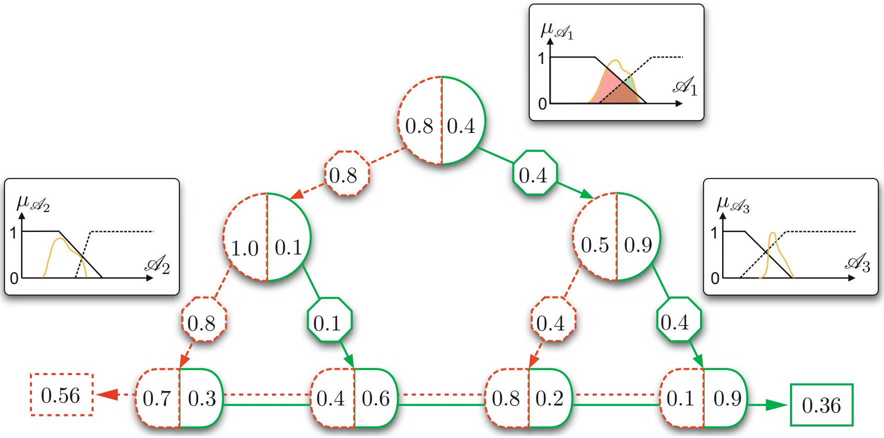

Fig. 8.9 Fuzzy decision tree and processing of a given fuzzy example of the database. For each node the separator (decision) functions are given along with the fuzzy attribute values. The intersection between the first fuzzy attribute value and the membership functions to the different classes is shaded to represent the satisfiability of the value to each class. The values in each leaf are the class densities of this leaf. The conjunction along a path is defined as the minimum, and the final aggregation of the leaf values is computed as a density-weighted maximum (source: [69])

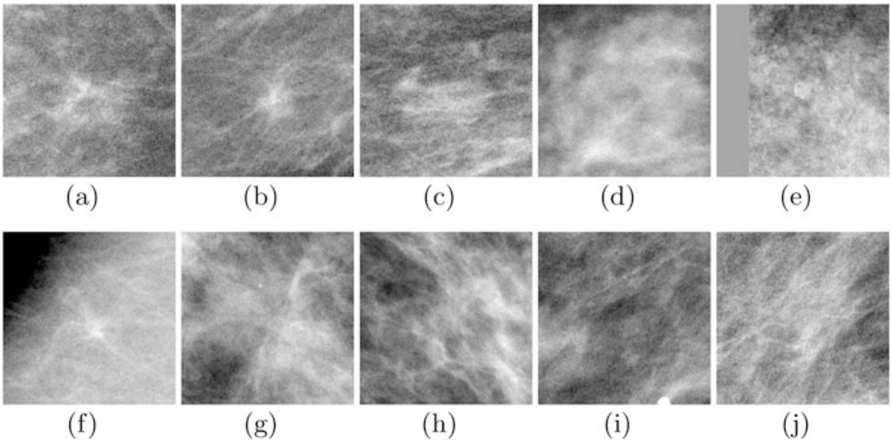

Fig. 8.10 Examples of spiculated breast masses [69]

artificial intelligence methods. Just to mention one example in the medical domain, a fuzzy decision tree was added on top of a convolutional network in [55] to design an explainable respiratory sound analysis system.

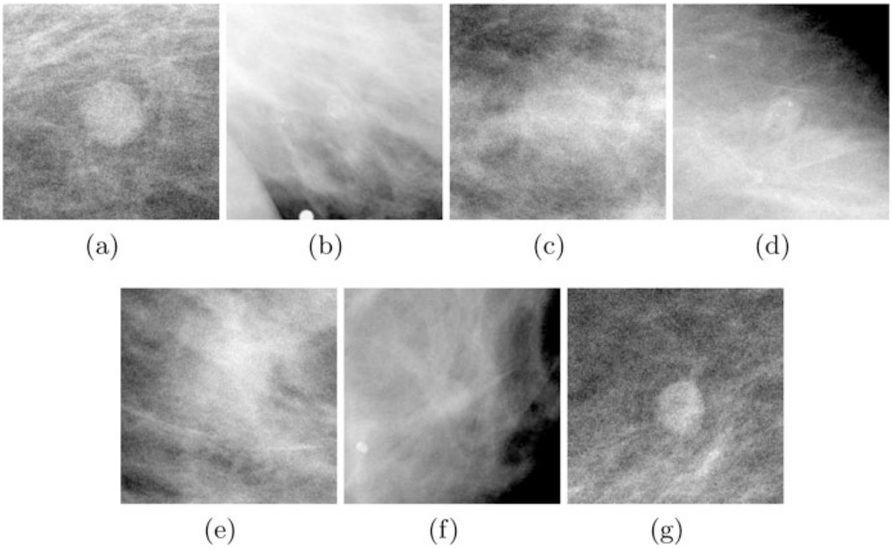

Fig. 8.11 Examples of circumscribed breast masses [69]

8.8 Fuzzy Association Rules

Another classical method in symbolic learning is based on mining frequent itemsets from a database, from which association rules can be extracted [1]. Let T be the database, also called set of transactions in this method. Let I be a set of items, and R a binary relation describing which items belongs to which transactions. Itemsets are characterized by their support: for $I_k \subseteq I$, $Supp(I_k) = \frac{|\{t \in T | \forall i \in I_k, R(t,i)\}|}{|T|}$. An itemset is said frequent if its support is larger than a threshold value. From a frequent itemset I_k, association rules can be inferred, by decomposing I_k as $I_k = A_k \cup B_k$, with $B_k = I_k \setminus A_k$. The rule then writes $A_k \Rightarrow B_k$ and can be quantified using its confidence value, defined as the ratio of the number of occurrences of I_k (i.e., A_k and B_k together in a transaction) to the number of occurrences of A_k.

Several extensions to fuzzy sets have been proposed. Let us briefly mention the main lines to incorporate fuzziness, at different levels. Attributes can be considered as fuzzy sets (as in [23], among others). Then an itemset is a (classical) set of fuzzy attributes. On the other hand, the relation can be considered as a fuzzy relation, which means that a transaction is then defined as a fuzzy subset of I (as, e.g., in [27]).

Extensions of rule mining methods rely on linguistic values in [5] or on the extension of the support to a degree of support in [52], and algorithms for deriving the association rules are direct extensions of Agrawal's A Priori algorithm (progressively building frequent itemsets of increasing cardinality, based on the monotony property of the support). This method was used, e.g., in [23], and rules are further made more expressive by using linguistic hedges or quantifiers (such as *very, more or less...*). Other methods first fuzzify the data, sort the transactions by membership values, and then use pattern growing algorithms where the database is first compacted in a tree structure. An example of this approach is the fuzzy frequent pattern tree of [57].

Recently, an extension of closed itemsets has been developed [70]. The idea (in the crisp case), exploits the potential correlations among the data, to first generate all the frequent closed itemsets from the database, and then derive all the frequent itemsets from this set [67]. The relation R is defined as a fuzzy relation, and $\forall i \in I, \forall t \in T, R(t, i) \in [0, 1]$.The closure operator is defined as two successive implications operators, as done for formal concept analysis (see Sect. 8.9), and its core is selected as a closed itemset. The closure of an itemset I_k is the set of items that belong to all the transactions that contain all the items in I_k. Note that in this approach, I_k is a crisp set. This approach has been used to infer interpretations of images, based on spatial relations between image structures. An example in medical imaging is illustrated in Fig. 8.12, where rules are generated using the frequent itemset approach and then used to explain the annotations of the organs [71].

Once the problem has been fuzzified, as well as the algorithms to obtain the association rules, the evaluation measures of the rules have to be adapted as well. This problem was addressed in [30], where the authors highlight the fact that this step is less straightforward. They propose to classify the data into three classes, i.e., examples of a rule, counter-examples, and data that are irrelevant to the rule. Note that if the rule writes $A \rightarrow B$, then examples of the rule are those that satisfy both A and B (and not data that satisfy the implication in a logical sense). In this framework, A and B are sets of fuzzy attributes over a domain D. Then, based on the semantics of fuzzy rules, three classes are defined as $S^+(A \rightarrow B) = \{x \in D \mid A(x) \wedge B(x)\}$, $S^-(A \rightarrow B) = \{x \in D \mid \neg(A(x) \rightarrow B(x))\}$, $S^\pm(A \rightarrow B) = \{x \in D \mid \neg A(x)\}$. These three classes become fuzzy when the attributes are fuzzy, i.e., $S^+(A \rightarrow B)(x) = t(A(x), B(x)) \in [0, 1]$, where t is a t-norm, etc. A constraint, related to the fact that reasoning on itemsets is done in terms of frequency, states that $S^+(A, B)(x) + S^-(A, B)(x) + S^\pm(A, B)(x) = 1$. The support of a rule is then defined as $Supp(A \rightarrow B) = |S^+|$ and the confidence as $Conf(A \rightarrow B) = \frac{|S^+|}{|S^+|+|S^-|}$. Connectives satisfying the constraint are then determined.

8.9 Fuzzy Formal Concept Analysis

Formal concept analysis is an algebraic theory of objects and attributes or properties of objects. Let us introduce the main definitions and notations in formal concept analysis (FCA) [36] that will be useful in this paper. A formal context is a triplet $\mathbb{K} = (G, M, I)$, where G is the set of objects, M the set of attributes, and $I \subseteq G \times M$ a relation between objects and attributes ($(g, m) \in I$ means that the object g has the attribute m). A formal concept of the context $\mathbb{K}$ is a pair (X, Y), with $X \subseteq G$ and $Y \subseteq M$, such that (X, Y) is maximal with the property $X \times Y \subseteq I$. The set X

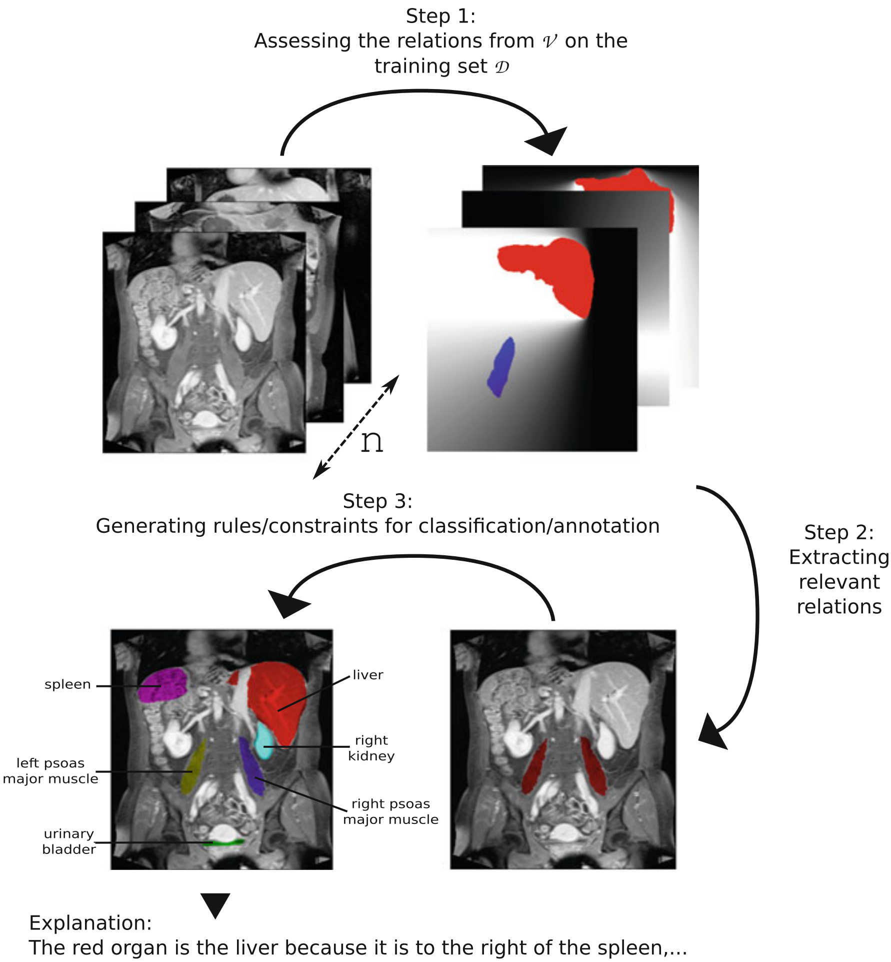

Fig. 8.12 Using fuzzy association rules based on spatial relations to explain annotations of medical images (source: [71])

is called the extent and the set Y is called the intent of the formal concept (X, Y). For any formal concept a, we denote its extent by $e(a)$ and its intent by $i(a)$, i.e., $a = (e(a), i(a))$.

The set of all formal concepts of a given context can be hierarchically ordered by inclusion of their extent (or equivalently by inclusion of their intent):

$$(X_1, Y_1) \preceq_C (X_2, Y_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow Y_2 \subseteq Y_1).$$

This order, which reflects the subconcept-superconcept relation, induces a complete lattice which is called the concept lattice of the context (G, M, I), denoted $\mathbb{C}(\mathbb{K})$, or simply $\mathbb{C}$.

For $X \subseteq G$ and $Y \subseteq M$, the derivation operators α and β are defined as:

$$\alpha(X) = \{m \in M \mid \forall g \in X, (g, m) \in I\},$$

and

$$\beta(Y) = \{g \in G \mid \forall m \in Y, (g, m) \in I\}.$$

The pair (α, β) is a Galois connection between the partially ordered power sets $(\mathcal{P}(G), \subseteq)$ and $(\mathcal{P}(M), \subseteq)$, i.e.,

$$\forall X \in \mathcal{P}(G), \forall Y \in \mathcal{P}(M), Y \subseteq \alpha(X) \Leftrightarrow X \subseteq \beta(Y).$$

Saying that (X, Y), with $X \subseteq G$ and $Y \subseteq M$, is a formal concept is equivalent to $\alpha(X) = Y$ and $\beta(Y) = X$.

Moving now to the fuzzy case, we will rely on a classical residuated lattice for fuzzy sets. Membership functions are taking values in L endowed with a lattice structure (typically $L = [0, 1]$ but all what follows extends directly for more general L-fuzzy sets [40]), and the corresponding residuated lattice is denoted by $(L, \leq, \wedge, \vee, *, \rightarrow)$, where $\wedge$ is the infimum, $\vee$ the supremum, and $*$ and $\rightarrow$ are adjoint conjunction and implication (see Chap. 2). Here, we use conjunctions defined as operators that are increasing in both arguments, commutative and associative, and admit 1 (or more generally the greatest element of the lattice L) as unit element, i.e., t-norms. Implications are defined as operators that are decreasing in the first argument, increasing in the second one, and satisfy $0 \rightarrow 0 = 0 \rightarrow 1 = 1 \rightarrow 1 = 1, 1 \rightarrow 0 = 0$ (or more general expressions by replacing 0 and 1 by the smallest and greatest elements of L, respectively). The adjunction property writes $c * a \leq b \Leftrightarrow c \leq a \rightarrow b$ and the implication defined by residuation from the conjunction is expressed as:

$$\forall (a, b) \in L^2, \ a \rightarrow b = \sup\{c \in L \mid c * a \leq b\}.$$

Non-commutative conjunctions can also be considered [61], with two associated implications, leading to adjoint triplets, and accordingly multi-adjoint concept lattices in the framework of formal concept analysis. They will not be detailed here.

The corresponding partial ordering on fuzzy sets is defined as:

$$\mu \preceq_F \nu \Leftrightarrow \forall x \in \mathcal{S}, \mu(x) \leq \nu(x),$$

where μ and ν are two fuzzy sets (or equivalently their membership functions), defined on an underlying space $\mathcal{S}$. The residuated lattice of fuzzy sets is denoted by $(\mathcal{F}, \preceq_F, \wedge^F, \vee^F, *, \rightarrow)$, with $\wedge^F = \min$ and $\vee^F = \max$ (or inf and sup more generally).

In particular we will use fuzzy sets defined on $\mathcal{S} = G$, i.e., $\mathcal{F} = L^G$, and on $\mathcal{S} = M$, i.e., $\mathcal{F} = L^M$.

Let us now move to fuzzy contexts, i.e., X and Y are fuzzy subsets of G and M, and I is a fuzzy relation ($I(g, m)$ now denotes the degree to which the object g has the property m). The residuated lattice introduced above is used, and degrees take values in any residuated lattice L. In the examples, we will use $L = [0, 1]$ for the sake of simplicity, but the theoretical results hold for any residuated lattice.

The derivation operators have been generalized to the fuzzy case in [9, 10] (see [11] for a discussion of various approaches for fuzzy concept analysis), leading to fuzzy sets $\alpha(X)$ and $\beta(Y)$ defined as:

$$\alpha(X)(m) = \wedge_{g \in G}(X(g) \rightarrow I(g, m)) \tag{8.25}$$

$$\beta(Y)(g) = \wedge_{m \in M}(Y(m) \rightarrow I(g, m)). \tag{8.26}$$

Note that in the early work [21], the implication was defined from a t-conorm and a complementation. We rely here on fuzzy implications related to a fuzzy conjunction by the adjunction property, i.e., residuated implications, such as in later works, which guarantees good properties, as detailed next.

As in the crisp case, a fuzzy formal concept is a pair of fuzzy sets (X, Y) such that $\alpha(X) = Y$ and $\beta(Y) = X$. From the classical partial ordering on fuzzy sets $\preceq_F$ (we use here the same notation for the ordering on L^G and on L^M), a partial ordering $\preceq_{FC}$ on fuzzy formal concepts is defined as:

$$(X_1, Y_1) \preceq_{FC} (X_2, Y_2) \Leftrightarrow X_1 \preceq_F X_2$$

and equivalently

$$(X_1, Y_1) \preceq_{FC} (X_2, Y_2) \Leftrightarrow Y_2 \preceq_F Y_1,$$

and this ordering induces a complete lattice structure on the fuzzy formal concepts, denoted $\mathbb{C}^F$. As shown in [10], the infimum and supremum of a family of fuzzy concepts $(X_t, Y_t)_{t \in T}$ are:

$$\wedge^{FC}{}_{t \in T}(X_t, Y_t) = \left(\wedge^F{}_{t \in T} X_t, \alpha\big(\beta(\vee^F{}_{t \in T} Y_t)\big)\right), \tag{8.27}$$

$$\vee^{FC}{}_{t \in T}(X_t, Y_t) = \left(\beta\big(\alpha(\vee^F{}_{t \in T} X_t)\big), \wedge^F{}_{t \in T} Y_t\right), \tag{8.28}$$

where $\wedge^F$ and $\vee^F$ are the classical intersection and union of fuzzy sets, defined as the pointwise infimum and supremum of the membership functions.

Interestingly enough, links can be established between derivation operators and morphological operators (in particular erosions and dilations), since they share the same properties, up to the reversal of the ordering relation either on L^M or L^G. Moreover, morphological operators can be defined on the concept lattice, leading to useful reasoning procedures. Examples are provided in Chap. 9. These links are detailed in [3]. Furthermore, derivation operators have also an interpretation in a possibilistic setting, as demonstrated in [33, 34], which brings an additional semantic layer. This interpretation extends to fuzzy derivation operators. See also [3] and the references therein.

Besides the notion of formal concepts and the lattice thereof, the theory of Formal Concept Analysis also deals with implications, in a logical sense, named attribute implications. In a fuzzy setting, an implication $A \Rightarrow B$, where A and B are fuzzy sets of attributes, expresses the degree to which such an implication holds and is defined from a fuzzy implication [12]. Here implications are really intended in a formal logic sense, in contrast to association rules, which convey a notion of frequency of occurrence instead. This means that, even in the crisp case, an association rule may not be always true, but only describes co-occurrences of frequent itemsets, while an attribute implication should always hold to be considered. Formal concept analysis, and its extension to fuzzy sets, is both a method to structure knowledge and data, and a method for symbolic learning.

Several methods and formalisms for knowledge representation, along with their extension to fuzzy sets, have been presented in this chapter. This constitutes the first main component of spatial reasoning. The second component is reasoning and is the focus of Chap. 9, specifically for image understanding.

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Chapter 9
Structural and Linguistic Reasoning for Image Understanding

A well-known problem in image and computer vision is the semantic gap (see Sect. 8.1.3) between the physical level of images, that is features extracted by image processing, and symbols expressed in a language. However, due to the nature of images and the difficulty of extracting meaningful features from them, language plays an important role. On the one hand, linguistic descriptions of prior knowledge about the images and the domains can be translated into formal models and algorithms to guide image retrieval, recognition, navigation, understanding. On the other hand, automatic annotation, that is generation of linguistic descriptions of image content, is an increasing, though recent, field of research that is likely to evolve rapidly. For instance, the question of providing, from image processing results, a high-level description in the language of the domain experts is yet to be addressed and can get inspiration from methods for linguistic summarization. We present in this chapter an overview of the state of the art in this domain, focusing on the use of fuzzy sets, with a few examples.

The meaning of image interpretation and understanding adopted here is as follows. The basic step is to recognize (often after or together with a segmentation step) individual objects or structures present in an image. However, image understanding goes some step further and aims at global scene recognition, to obtain high-level descriptions of the objects in their context, including their spatial arrangement. When images are not static but dynamic such as in video processing, then further interpretation steps may include recognition of movements or changes, and, for general image understanding not restricted to medical imaging, recognition of actions, gestures, emotions… Image understanding includes also semantic interpretation. Semantics is not present in the image itself, but requires some prior knowledge (for example, expressed as formal models) to extract it. All this should then lead to verbal or linguistic descriptions of the image content. This definition includes, but may cover a larger scope, the purely logical view in [83], where the interpretation of an image is defined as a logical model of three sets of axioms (image, scene, depiction). Surprisingly enough, such views relating linguistic models and image understanding

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were developed in the 1960s, then less addressed, and is now renewed. As early as 1968, a survey of linguistic methods for picture processing, defined as analysis and generation of pictures by computers, with or without human interaction, was proposed in [63]. In [26], a linguistic approach for picture interpretation was proposed, as a pattern description language. In [77], image understanding is defined as verbal descriptions of the image contents. The need for a semantic layer for spatial language was advocated again later in [10].

Importance of semantics related to images is acknowledged in several domains, including recognition, image understanding, cognitive vision, image retrieval, and annotations. Although this question has been recognized since the early works in image understanding and computer vision, it was renewed in recent work on semantic image annotation and retrieval, and in recent work linking vision and language, as evidenced, for instance, by recent workshops on vision and language (for instance, VL'15, and subsequent editions), as well as special issues on this topic in journals such as Computer Vision and Image Understanding. The need for semantics of object representation and their epistemic justification was also highlighted in the context of machine learning in [97]. Several aspects related more generally to the philosophy of pattern recognition can also be found in a dedicated issue of Pattern Recognition Letters in 2015 [76]. One main problem is the semantic gap, close to the symbol grounding problem (see Sect. 8.1.3), and fuzzy methods provide useful tools to deal with both ontological concepts (often provided as linguistic terms) and concrete domains, and to establish links between them.

Two main directions can be identified in connection with linguistic descriptions of images:

1. Using linguistic descriptions expressed in a model to guide image interpretation.
2. Deriving linguistic descriptions of images based on image features.

Our focus is on methods relying on fuzzy models (for representing vague knowledge, imprecision in images and in concepts, etc.), associated with symbolic and structural models.

The suitability of fuzzy sets for representing image information and knowledge has been summarized in Sect. 8.1. Based on these representations, the first direction, from linguistic descriptions to image understanding, is reviewed in Sect. 9.1, while the second one, from image analysis to image content descriptions, in summarized in Sect. 9.2. Examples in medical image understanding are given in Sect. 9.3.

9.1 From Linguistic Descriptions to Image Understanding

As mentioned in the introduction, to go beyond individual object recognition, image understanding requires descriptions of the spatial organization of objects in images. In knowledge-based approaches, the models should then include such structural knowledge. Models have then to be combined with image information, to finally lead to scene understanding. These steps are summarized in this section, which is to some extent taken from [15].

9.1.1 Representations of Structural Information

Let us first summarize the main structural representations on which the interpretation methods described next rely (see Chap. 8 for details). These representations aim at providing a computational framework to take into account both the properties of the objects and the relations between objects, in order to drive pattern recognition and scene interpretation processes, especially in model-based approaches. These models can be of iconic type, as an atlas, or of symbolic type, as linguistic descriptions, conceptual or semantic graphs, or ontologies.

While spatial relations constitute a very important information to guide the recognition of structures embedded in a complex environment, where object characteristics are prone to a higher variability, defining them in mathematical terms leading to efficient algorithms is a challenge. Mathematical models of several spatial relations have been proposed in the framework of fuzzy sets theory, strongly relying on mathematical morphology operators. For instance, as detailed previously, the semantic of a relation such as *close to, to the right of* can be modeled as a fuzzy structuring element, and the dilation of a reference object by this structuring element provides the fuzzy region of space where the corresponding relation is satisfied. More details on fuzzy mathematical morphology can be found, e.g., in [14] and in Chap. 4. See also Chap. 6 for a review of fuzzy models of spatial relations.

These fuzzy representations can enrich ontologies and contribute to reduce the semantic gap between symbolic concepts, as expressed in the ontology, and visual percepts, as extracted from the images [49]. Ontologies [45] have been extended to deal with uncertainty and imprecision, using probabilistic (e.g., [35, 107]) or fuzzy approaches, in particular using fuzzy description logics (e.g., [66, 92]). Several spatial ontologies have been proposed, in various domains, such as in [9, 23, 31] or [30, 36] in the medical domain. The ideas of linking ontologies expressing fuzzy spatial relations with images were used in particular in the segmentation and recognition methods described in [17, 18, 27, 67, 99]: a concept of the ontology is used for guiding the recognition by expressing its semantic as a fuzzy set, for instance in the image domain or in an attribute domain, which can therefore be directly linked to image information. While the concepts and their use can be defined in a general way, the fuzzy sets expressing the semantics may involve some parameters depending on the context (for instance, the notion of "close to" has a different meaning when speaking of brain structures in a medical image or of towns in a satellite image). These parameters can be learned, for instance, from annotated images [7].

Similarly, such spatial relations are useful attributes in graphs and fuzzy graphs and endow recognition and mining methods based on similarity between graphs with structural information [3, 25, 74], benefiting from the huge literature on fuzzy comparison tools (see, e.g., [19]). Spatial relations can also be embedded in conceptual graphs and their fuzzy extensions, as in [99]. Another example is the hierarchical model with fuzzy attributes proposed in [64] for modeling objects (recognition is then based on a fuzzy measure between the model and image processing results). The extension of structured representations to cope with fuzziness is detailed in Chap. 8, for several types of such representations.

9.1.2 Fusion

A lot of approaches for image understanding involve fusion steps. The information to be combined can be obtained from one or several images, can involve several relations between objects and several features of the objects, and can also come models, such as an anatomical atlas or a conceptual graph, or knowledge expressed in linguistic form or as ontologies. The advantages of fuzzy sets rely in the variety of combination operators, offering a lot of flexibility in their choice, that can be adapted to any situation at hand, and which may deal with heterogeneous information [38, 104]. These aspects have been detailed in Chap. 5. Examples can be found in various domains (e.g., [18, 27, 62, 67, 71, 84, 99]).

9.1.3 Scene Understanding

A survey of knowledge-based systems for image interpretation until 1997 can be found in [29]. Here we focus on more recent approaches, and using fuzzy formalisms.

Scene understanding using fuzzy approaches mostly belongs to the domain of spatial reasoning, which can be defined as the domain of spatial knowledge representation, in particular spatial relations between spatial entities, and of reasoning on these entities and relations. This field has been largely developed in artificial intelligence, in particular using qualitative representations based on logical formalisms [2]. In image interpretation and computer vision it is much less developed and is mainly based on quantitative representations. Using fuzzy approaches can then be seen as halfway between purely quantitative and purely qualitative reasoning, as well as a mean to federate different types of knowledge and data.

A typical example in this domain concerns model-based structure recognition in images, where the model represents spatial entities and relations between them. Two main components of this domain are spatial knowledge representation and reasoning. In particular spatial relations constitute an important part of the knowledge we have to handle. Imprecision is often attached to spatial reasoning in images and can occur at different levels, from knowledge to the type of question we want to answer. The reasoning component includes fusion of heterogeneous spatial knowledge, decision making, inference, recognition. Two types of questions arise when reasoning with spatial relations:

1. Given two objects (possibly fuzzy), assess the degree to which a relation is satisfied.
2. Given one reference object, define the region of space in which a relation to this reference is satisfied (to some degree).

It has been shown in [13] that the association of three frameworks in a unified way, namely mathematical morphology, fuzzy sets and logics, allows, on the one hand, matching two important requirements: expressiveness and completeness with respect to the types of spatial information to be represented [1], and, on the other hand, performing successful reasoning tasks for image understanding.

A common representation of structural information to guide image interpretation consists of a graph, where vertices represent objects or image regions (possibly with attributes such as shape, size, color, or gray level), and edges carry the structural information (spatial relations between objects, radiometric contrast between regions…). Although this type of representation has become popular in the last 30 years [28], there are still a number of open problems regarding their efficient use for interpretation. One type of approach consists in deriving a graph from the image itself, based on a preliminary segmentation of the image into homogeneous regions, and to express the recognition as a graph matching problem between the image graph and the model graph, which, however, raises combinatorial problems [21, 28]. In [46, 58, 72, 88, 108] an initial labeling of the image regions is performed, and spatial relations are used to refine this labeling or to extract the objects of interest.

All these approaches assume a correct initial segmentation of the images. However, this is known to be a very difficult problem in image processing, for which no universally acceptable solution exists: the segmentation is usually imperfect and no isomorphism exists between the graphs to be matched. This leads naturally to the need to find an inexact matching, for instance, by allowing several image regions to be assigned to one model vertex, or by relaxing the notion of morphism to the one of fuzzy morphism [25, 74] (see also Chap. 8). As an example, in [33, 34], an oversegmentation of the image is used, which is easier to obtain. A model structure is then explicitly associated with a set of regions and the recognition is expressed as a constraint satisfaction problem. Fuzzy relations can also be used once an initial matching is done, to get the final labeling and interpretation [40].

When fuzzy graphs are derived from the image (via an oversegmentation, typically) and from a model, and when it comes to find a fuzzy morphism between these graphs, another issue is that several morphisms can be found. Then additional information should be added, in the form of an objective function, involving comparison between vertex attributes of the two graphs and comparison between edge attributes, using measures of similarity or of resemblance, for instance [19]. Several methods for fuzzy graph comparison and matching have been proposed in that direction. The method in [73] relies on fuzzy relaxation [82] adapted to the generic definition of fuzzy morphisms in [74], described in Chap. 8. An illustration is provided in Fig. 9.1.

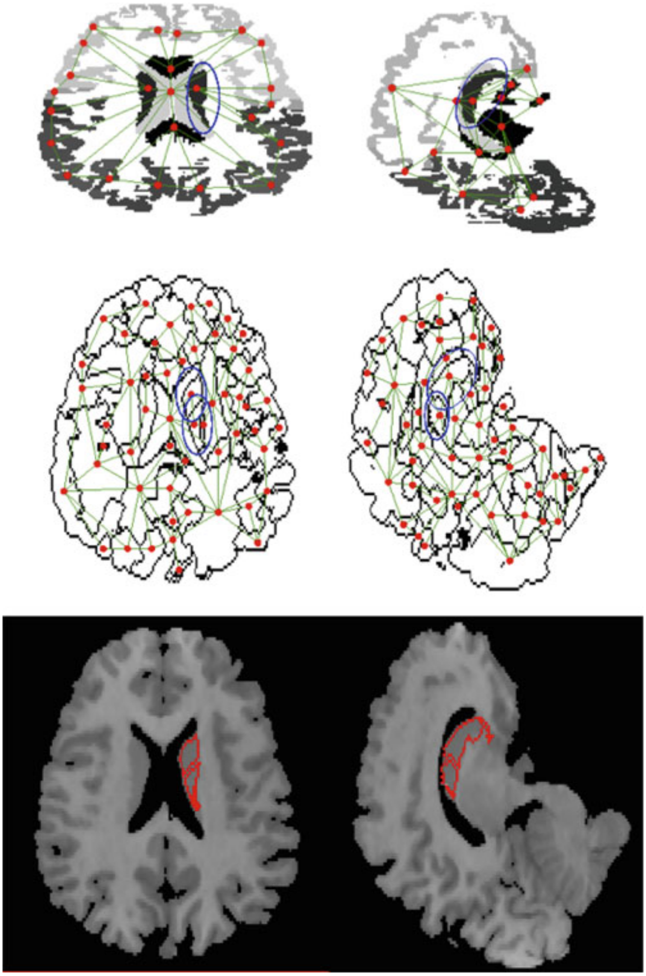

Fig. 9.1 Two slices (axial and sagittal) of an anatomical atlas of the brain, with an excerpt of the derived graph superimposed, two slices of an oversegmentation of a MRI brain image, with an excerpt of the derived graph superimposed, and results of the matching process for the caudate nucleus (red contours), where two regions of the oversegmentation have been assigned to one vertex (i.e., one anatomical structure) of the model [73]

Other optimization methods are based on genetic algorithms [75], estimation of distributions algorithms [12], or graph kernels involving spatial relations [3]. In all these approaches, the importance of edge information, in particular spatial relations, has been demonstrated. However, one of the main issues in these methods is the design of an appropriate objective function, guaranteeing that it is optimal for the right solution, which is a difficult task.

Still relying on a preliminary segmentation, recent approaches have been proposed, for instance using ontologies [49, 68], with fuzzy extensions, besides other types of methods (grammatical or probabilistic ones). Other approaches combining segmentation results and fuzzy models of shapes and spatial relations were proposed, e.g., in [48] for medical images, or in [98] for seismic images, using fuzzy rules. Fuzzy RCC (see Chap. 6) can also be used to identify objects based on their mereotopological relations, as done in the crisp case (e.g., [52] or [53], where a prior model of a given spatial organization, for instance around a village, is compared, via the satisfied RCC relations, to regions extracted from satellite images).

To overcome the difficulty of obtaining a relevant segmentation, the segmentation and the recognition can also be performed simultaneously. For instance, the method proposed in [17, 27] consists in sequentially segmenting and recognizing each object of interest, in a pre-calculated order [3, 25, 74]. The objects that are easier to segment are considered first and taken as reference. Spatial relations to these reference objects encoded in the structural model are used as constraints to guide the segmentation and recognition of other objects. However, the extraction of the first objects can be difficult if it is not sufficiently constrained, and due to the sequential nature of the process, the errors are potentially propagated. Backtracking may then be needed, as proposed in [41].

Similar approaches have been used for mobile robot navigation in [42], where linguistic descriptions of a scene, given by a human observer, are translated into fuzzy spatial regions. Another sequential approach was proposed in [94] for vessel tracking in MRI. Starting from an initial fuzzy classification, the authors apply fuzzy rules, involving both image information and geometrical characteristics, to track the vessels and handle the bifurcations. Sequential approaches based on fuzzy rules have been developed for the recognition of abdominal organs in CT images [55, 56], and for the recognition of brain structures [32]. The rules describe anatomical knowledge and involve spatial relations and object properties modeled as linguistic variables (e.g., the kidneys are close to the spine and on each side of it, the cerebrum is a large region, the gray-matter is a part of the cerebrum).

To overcome the problems raised by sequential approaches, while avoiding the need of an initial segmentation, another method, still relying on a structural model, but solving the problem in a global way, was proposed in [67]. A solution is the assignment of a region of space to each model object that satisfies the constraints expressed in the model. A solution is obtained by reducing progressively the solution domain for all objects by excluding assignments that are inconsistent with the structural model. Constraint networks [86] constitute an appropriate framework both for the formalization of the problem and for the optimization. This approach was extended in [99] to fuzzy constraint satisfaction problems (extending [39]) to deal with more complex relations, or involving an undetermined number of objects, and applied to the interpretation of high resolution remote sensing images.

Besides recognition and segmentation, fuzzy spatial relations and more generally fuzzy spatial information have proved useful for other interpretation tasks, such as multiple object tracking [102, 103], graph kernels for machine learning [3], facial expression understanding [78–81], navigation in unknown environments in robotics [22, 37, 43], among others.

9.2 From Image Analysis to Image Content Descriptions

Let us now consider the other way around, where the objective is to start from image features to derive descriptions of the image content in a way as close as possible to natural language. Usually referred to as image annotation, this task aims at identifying "tags" that are associated with images to describe their content. These tags are most often related to the recognition of one main object in an image, or a few objects given as an unstructured list. However, in more recent work, new approaches emerged to provide descriptions as whole sentences. This refers to the typical "show and tell" approaches that benefit from recent advances in machine learning (convolutional networks and deep learning), or use mostly clustering and probabilistic approaches [54, 87, 100]. Such approaches are also used to model queries in image retrieval (see, e.g., the reviews in [44, 91]). As an example using fuzzy models, let us cite [8] where fuzzy multimedia ontologies were developed for semantic image annotation. Tags were identified based on consistency of candidate concepts, obtained from SVM classification, and tested using fuzzy description logic reasoning.

Interesting methods using structural representations such as graphs or grammars are worth to mention [69, 93, 96]. For instance, in medical imaging, attributed grammars are applied to results of image processing (e.g., detection, skeletonization) to provide a syntactic description of results.

Although a large majority of approaches rely on probabilistic models or learning methods, some of them, in particular structural approaches using graphs or grammars, could be enhanced by fuzzy components to deal with imprecision, vagueness, variability. Still a few fuzzy approaches have been proposed, as described next.

One problem with neural networks is that it may be difficult to understand which rules or reasoning processes they have learned. This question was answered in [60] where satellite image classification was performed using fuzzy neural networks, also producing the fuzzy rules that are actually used by the system, and that are understandable by domain experts, thus providing a description of the image and of how this description was obtained.

Fuzzy sets learned from neural networks were used in [57] in the domain of art image retrieval. The linguistic variables describe "fuzzy aesthetic semantics," in terms of action, relaxation, joy, fear, etc., associated with degrees of satisfaction. Also using a neural network, associated with a fuzzy classifier and an expert system reasoning on low-level features, the work in [70] leads to descriptions of facial expressions.

Fuzzy rules were also exploited to generate simple linguistic descriptions of image content [4, 5, 50, 95]. This approach was used for various applications, such as circular structures on Mars, traffic, human gait, medical images… In [11], a clustering and compression method was proposed to provide a small number of fuzzy rules having a linguistic meaning, which constitute fuzzy models that provide linguistic descriptions of low-level features in images. Applications in matching were developed.

At a more structural level, a few authors included spatial relations to provide image descriptions. For instance, in [24], linguistic features describing regions were obtained by fuzzy segmentation, fuzzy spatial relations and locations. From a set of predefined linguistic terms, a brief and accurate description of the whole image is then generated. In [61, 90], previous work by the authors on computation of relative direction was used to derive linguistic descriptions of relative positions in images, associated with a qualitative validity of the description. A typical example of result is "the building is perfectly to the right of the reference object; the description is satisfactory." The methods for comparing spatial relations described in Sect. 6.7 also provide useful tools to describe linguistic descriptions, for instance, based on a comparison with values of linguistic variables (see, e.g., the example in Sect. 6.7 and [16]).

Note that, conversely, conceptual descriptions, in natural language, of visual scenes can be used to create an image matching these descriptions, and an example can be found in [65].

Another source of inspiration may come from work on summarization. For instance, in [89] the summarization of image databases was based on low-level features and fuzzy labels. While summarization was not much addressed until now for images, several works have emerged for time series and signals, see, for instance, the special issues on linguistic description of time series in [59], with several papers on automatic generation of linguistic descriptions of data, mapping from non-linguistic to linguistic expressions, linguistic summarization. Although some ideas and methods could probably be exploited, the problem when dealing with images is quite different, and there is some work to do to really account for the spatial nature of images and for structural information and knowledge.

One issue in all these approaches is the validation of the obtained linguistic descriptions of the images. Most of the time, a simple comparison with the description provided by a human is performed. This assumes defining a common vocabulary and language, which raises the issue of the level of the description. For instance, describing a brain image may take different forms depending on whether it is intended for a wide public audience, for a patient or for an neurology expert, ranging thus from "an abnormal structure is present in the brain," to "a peripheral non-enhanced tumor is present in the right hemisphere."

9.3 A Few Examples in Medical Image Understanding

In this section, we provide several examples of methods for brain MRI image interpretation driven by symbolic and structural models. All these methods make extensive use of fuzzy spatial relations and were developed with several PhD candidates at LTCI/Télécom Paris over several years.

9.3.1 Interpretation as Graph Reasoning

The first example deals with sequential joint segmentation and recognition of brain structures, based on a model [17, 27, 41]. The model is a graph, where each vertex represents an anatomical structure, and edges carry spatial relations between structures, namely distance, adjacency, directional relations, according to anatomical knowledge. The modeled spatial relations are based on fuzzy intervals in each appropriate parametric space that are chosen of trapezoidal shape. The parameters of each relation are learned according to the procedure in [7], which basically consists in enlarging the kernel and the support of the spatial relation in a way that all the targeted structures are included in this support (see also Sect. 8.2.1).

The proposed framework has two levels, as depicted in Fig. 9.2. The first level is a generic bottom-up module which allows selecting the next structure to segment. This level does not rely on an initial segmentation or classification, but instead on a focus of attention and a map of generic features. The sequential approach allows this level to use two types of knowledge: generic and domain independent features in unexplored area of the image to segment (based on visual saliency maps), and high-level knowledge such as spatial relations linked in the graph to

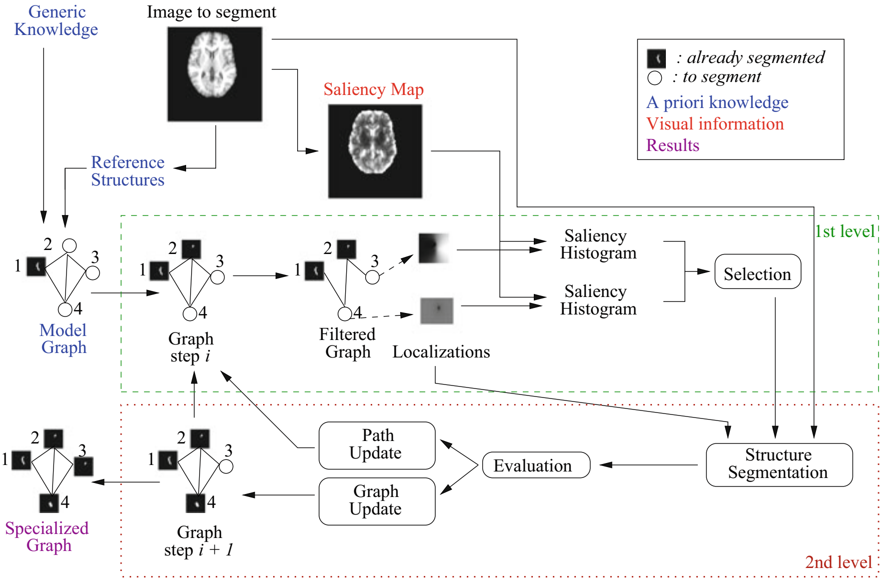

Fig. 9.2 General scheme of the sequential segmentation framework [41]

the already recognized structures. The fusion of the two kinds of information provides a localization map for each potential structure to be segmented, and criteria are defined to choose among these structures. The second level achieves recognition and segmentation of the selected structure, as well as the evaluation of the segmentation. The recognition of the structure is thus achieved at the same time as the segmentation. The segmentation is performed by a deformable model, minimizing an energy functional which includes classical terms (regularization and data fidelity) and an additional term derived from the spatial relations that the targeted structure has to satisfy with respect to previously segmented ones [27].

Due to the sequential nature of the process, during the segmentation of a particular structure, errors may occur and propagate. Therefore, the process must be able to detect errors immediately or a posteriori, and then to update its strategy, i.e., backtrack and change the sequence of segmentation even if this implies to discard previous structures segmentations. To this end, two criteria were proposed as well as a structure of control, which consists of a tree of all current and past segmentations, used to update the strategy during the process [41]. The evaluation of a segmentation is based on the spatial consistency of the result. As mentioned above, the parameters of each spatial relation are learned in a way that the targeted structure is included in the kernel of the relation. Accordingly, the spatial consistency criterion evaluates whether this assertion is still true once a new structure segmentation has been added. A fuzzy measure of satisfiability is used to this end [19]. To keep track of the history of the previous steps of the process a tree structure is maintained, which contains information about all the segmentations done by the process. In case of an error occurring during the segmentation of a structure and detected thanks to the previous criterion, the strategy of control of the process is simple: it consists in preventing the system of trying the same sequence, which is immediate thanks to the segmentation tree. This results in a higher number of recognized structures, as well as a better accuracy of the segmentation.

A few results are illustrated in Fig. 9.3. Noticeably, the recognition and segmentation of internal brain structures (gray nuclei) are correct, even in the presence of large deforming tumors.

In such cases, a preliminary detection of the tumor is performed using approximate symmetry analysis, using the method in Sect. 6.2.2. Once the symmetry plane is detected, the difference between the histograms in each hemisphere provides useful information about new intensity classes induced by the tumor [51] and allows for an automatic detection of the tumor. This is illustrated in Fig. 9.4. Then the other anatomical structures are segmented and recognized as described above. These examples illustrate the robustness of methods based on fuzzy models with respect to variations of tumor appearance and deformation of the normal structures.

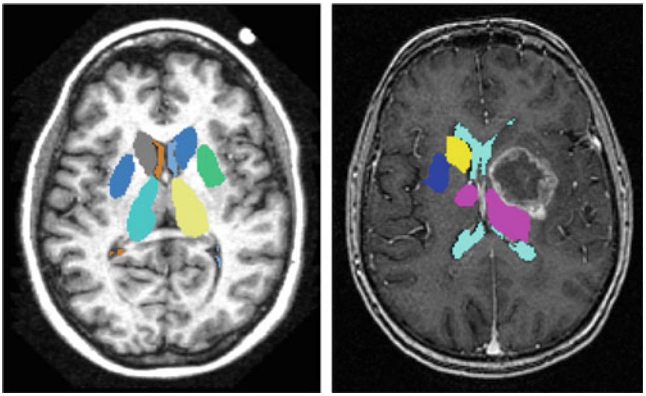

Fig. 9.3 Final segmentation of gray nuclei and ventricles [41]. Left: normal case. Right: pathological case

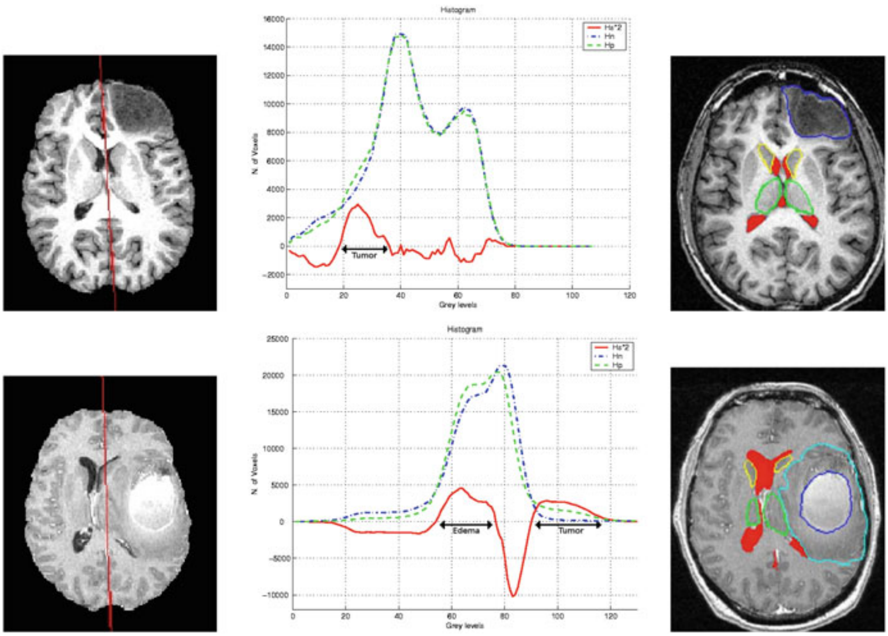

Fig. 9.4 Examples of tumor and normal structure segmentation for a non-enhanced tumor (top) and a full-enhanced tumor with edema (bottom). From left to right: symmetry plane, histograms in the two hemispheres and difference, exhibiting the tumor class, and segmentation results [51]. The whole process is performed in 3D, and only one slice is shown for visualization purpose

9.3.2 *Interpretation as Constraint Satisfaction Problem*

In this example, detailed in [67], the image interpretation problem is framed as a constraint satisfaction problem (CSP). Again, the structural model is a graph, as in the previous example, with similar spatial relations, but also intensity relations between anatomical structures (related to the type of image acquisition), and vertex attributes (such as prior information on the volume of each anatomical structure). The graph is then interpreted as a network of constraints that the results of the segmentation and recognition process should satisfy. The whole process is illustrated in Fig. 9.5.

Let $\mathcal{I} : \mathcal{S} \to \mathbb{N}^*$ be an image whose spatial domain $\mathcal{S}$ is a subset of $\mathbb{Z}^d$, where d is typically equal to 2 or 3. We wish to obtain regions in $\mathcal{S}$ for a set of n objects $\chi = \{O_i \mid i \in [1..n]\}$ that are visible in the image; these objects are the variables of the problem. As the image $\mathcal{I}$ provides a discrete view of the continuous world, the regions cannot be represented accurately as subsets of $\mathcal{S}$. The digital sampling and artifacts induced by the acquisition cause imprecision in $\mathcal{I}$ on the object boundaries. Regions are therefore represented as fuzzy subsets of $\mathcal{S}$ (see Chap. 3). The variables O_i are then represented by fuzzy subsets μ_i of $\mathcal{S}$ (i.e., $\mu_i : \mathcal{S} \to [0, 1]$). The set of all fuzzy subsets of $\mathcal{S}$ is denoted by $\mathcal{F}$.

The domains $\mathcal{D} = \{\mathcal{D}(A) \mid A \in \chi\}$ associated with the variables are then subsets of $\mathcal{F}$ ($\mathcal{D}(A) \subseteq \mathcal{F}$). If the problem is satisfiable, then the solutions are among these subsets. In fact, the cardinality of $\mathcal{F}$ depends exponentially on $|\mathcal{S}|$, and its size is $k^{|\mathcal{S}|}$ where k is the number of discrete levels used to represent the membership degrees, and the domains can be any subset of $\mathcal{F}$. To make the approach tractable, only two bounds are used to approximate each domain. With the usual partial ordering on fuzzy sets,[1] denoted by $\leq$, $(\mathcal{F}, \leq)$ is a complete lattice (see Sect. 4.1). Therefore, every subset of $\mathcal{F}$ has an upper bound and a lower bound that belong to $\mathcal{F}$. The upper bound $\overline{A}$ of the domain $\mathcal{D}(A)$ is therefore defined as follows: $\overline{A} = \bigvee\{\nu \in \mathcal{D}(A)\}$, where $\forall x \in \mathcal{S}$, $\overline{A}(x) = \sup_{\nu \in \mathcal{D}(A)} \nu(x)$. This bound is an over-estimation of the target fuzzy set μ_A. Similarly, the lower bound $\underline{A}$ is defined as follows: $\underline{A} = \bigwedge\{\nu \in \mathcal{D}(A)\}$, where $\forall x \in \mathcal{S}$, $\underline{A}(x) = \inf_{\nu \in \mathcal{D}(A)} \nu(x)$. This bound provides an under-estimation of μ_A.

Constraints are obtained from the structural model (i.e., the relational attributed graph). For instance, if the model contains the relation "A is to the right of B," then the recognition process must obtain an instantiation $\{(A, \mu_1), (B, \mu_2)\}$, where A and B represent structures of the model, satisfying the constraint $C^{dir}_{A,B}$, i.e., $(\mu_1, \mu_2) \in C^{dir}_{A,B}$. We denote these constraints by $\mathcal{C}$.

The segmentation and recognition problem is represented by a constraint network $N = \langle \chi, \mathcal{D}, \mathcal{C} \rangle$ and a solution of N should be a consistent instantiation of all variables in χ that satisfies all of the constraints. We assume that the problem is

[1] Let $\mu, \nu \in \mathcal{F}$, $\mu \leq \nu$ if $\forall x \in \mathcal{S},\ \mu(x) \leq \nu(x)$.

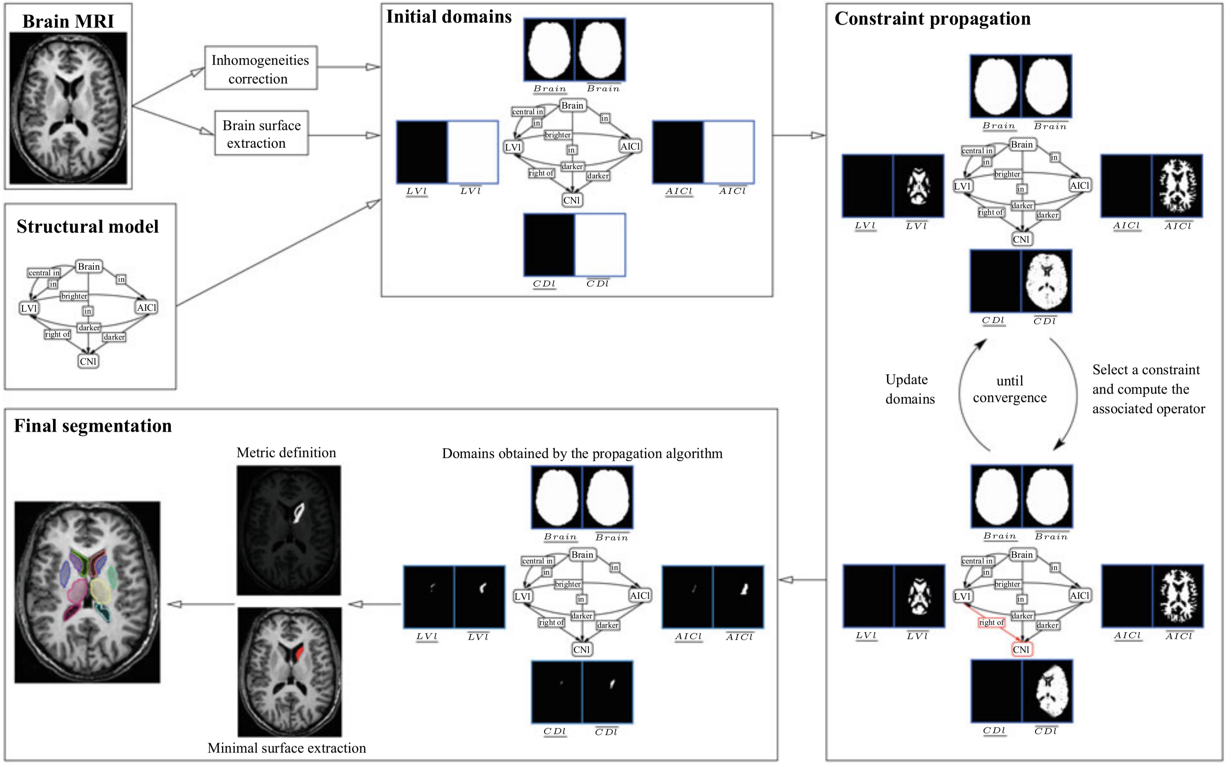

Fig. 9.5 Overview of the proposed approach for the brain structures example [67]. For instance, the solution space of the left caudate nucleus (CNl) is reduced based on the constraint that "the left caudate nucleus (CNl) is exterior (i.e., to the right in the image) to the left lateral ventricle (LVl)"

satisfiable, which means that such a solution exists, which is reasonable since the model is generic enough.

The cardinality of the search space is $k^{|\mathcal{S}|\times|\chi|}$, where $|\mathcal{S}|$ is approximately 10^7 for a typical MRI volume and $|\chi|$ is the number of structures in the model. Clearly, a backtracking algorithm cannot be applied. To obtain a solution, the constraint network is first simplified using a constraint propagation algorithm that removes as many inconsistent values as possible from the domains, according to the constraints. The propagation algorithm obtains the smallest possible element of $\mathcal{P}^{sol}_{N\,D}$ in polynomial time. For this purpose propagators were defined that are related to each constraint, and the constraint propagation algorithm sequentially applies these propagators.

Let us detail the directional constraint and the associated propagator (details for all other constraints can be found in [67]). The directional relation is modeled using a fuzzy dilation with a structuring element ν, as described in Chap. 6. Let A and B be two objects with a stable directional relative position characterized by a structuring element ν. The constraint $C^{dir\ \nu}_{A,B}$ is defined as follows:

$$C^{dir\ \nu}_{A,B} : \mathcal{D}(A) \times \mathcal{D}(B) \rightarrow \{0, 1\}$$
$$(\mu_1, \mu_2) \quad \mapsto \begin{cases} 1 \text{ if } \mu_2 \leq \delta_\nu(\mu_1), \\ 0 \text{ otherwise.} \end{cases}$$

The propagator $f_{C^{dir\ \nu}_{A,B}}$ that is associated with the directional relative position constraint between two structures A and B is defined as follows:

$$\frac{\langle A, B; (\underline{A}, \overline{A}), (\underline{B}, \overline{B}); C^{dir\ \nu}_{A,B} \rangle}{\langle A, B; (\underline{A}, \overline{A}), (\underline{B}, \overline{B} \wedge \delta_\nu(\overline{A})); C^{dir\ \nu}_{A,B} . \rangle}$$

Figure 9.6 illustrates the propagator associated with the relation "CNl is to the right of LVl." The domain of the caudate nucleus, $(\underline{CNl}, \overline{CNl})$, is reduced by removing elements that do not satisfy the directional relation.

Each application of a constraint, through the associated propagator, reduces the interval in which the solution for each structure can be found. Successive applications of the propagators then allow focusing on restricted areas, in which the segmentation becomes easy (again a deformable model is applied for this final step). As shown in [67], the propagators enjoy good properties, which allow applying them in any order. Because the result does not depend on the ordering, several criteria were applied for choosing the next propagator to be computed, including the magnitude of the changes in the domains since the last application of the propagator, the computational cost of the propagator, and a fine estimation of the maximal possible domain reduction. The constraint ordering algorithm enables a significantly more rapid convergence

Figure 9.7 shows a 3D reconstruction of the results obtained for the internal brain structures.

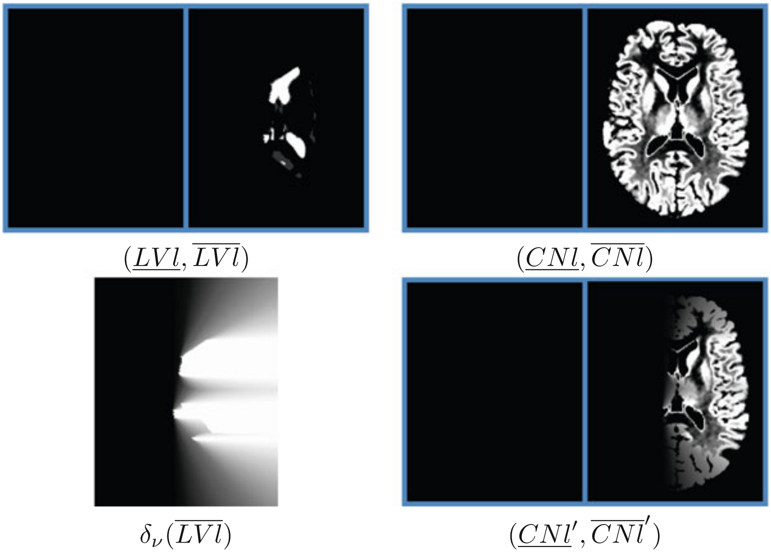

Fig. 9.6 Illustration of the propagator $f_{C^{dir\ \nu}_{LVl,CNl}}$ [67]. The upper bound of the caudate nucleus domain $\overline{CNl}$ is restricted to the subset of space to the right of the elements of $\mathcal{D}^I(LVl)$ obtained from the dilation $\delta_\nu(\overline{LVl})$

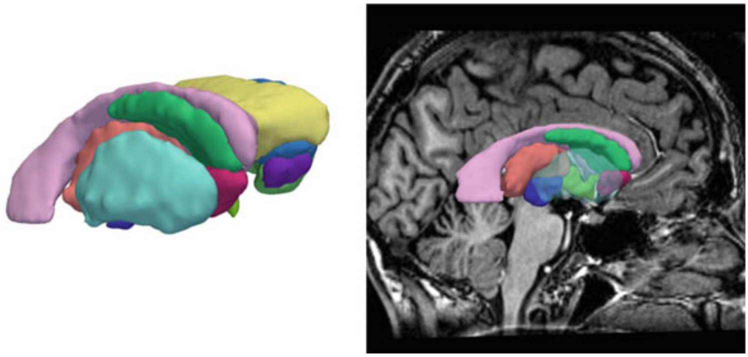

Fig. 9.7 3D reconstruction of the recognition results for the caudate nuclei, putamens, lateral ventricles, thalami, third ventricle, accumbens nuclei, and sub-thalami [67]

9.3.3 Recognition Based on Ontological Reasoning

In this section, partly reproduced from [49], we rely on the ontology described in Chap. 8 to recognize internal brain structures in magnetic resonance volumes. The elaboration of the domain ontology can benefit from the large amount of existing knowledge formalization models, some of them based on ontological engineering tools (such as the FMA [85]), that emerged from the medical informatics research field. While neuro-anatomy has not been much developed in these models, it is largely described in textbooks [101] and dedicated sites,[2] in linguistic form. These models involve concepts that correspond to anatomical objects, their characteristics, or the spatial relations between them. Human experts use intensively such concepts and knowledge to recognize visually anatomical structures in images. This motivates the use of the ontology of spatial relations for enriching the ontology of the domain (cerebral anatomy in this context). In the approach developed in [49], the ontology is imported in an ontology of the brain anatomy (excerpt of the Foundational Model of Anatomy (FMA) [85]) and is used to describe the spatial organization of brain anatomical components. We consider that each physical anatomical component is a spatial object. Then, spatial relations between these different spatial objects are described by using the spatial relation ontology. For instance, as illustrated in Fig. 9.8, the right caudate nucleus is to the right and close to the right ventricle and above the right thalamus.

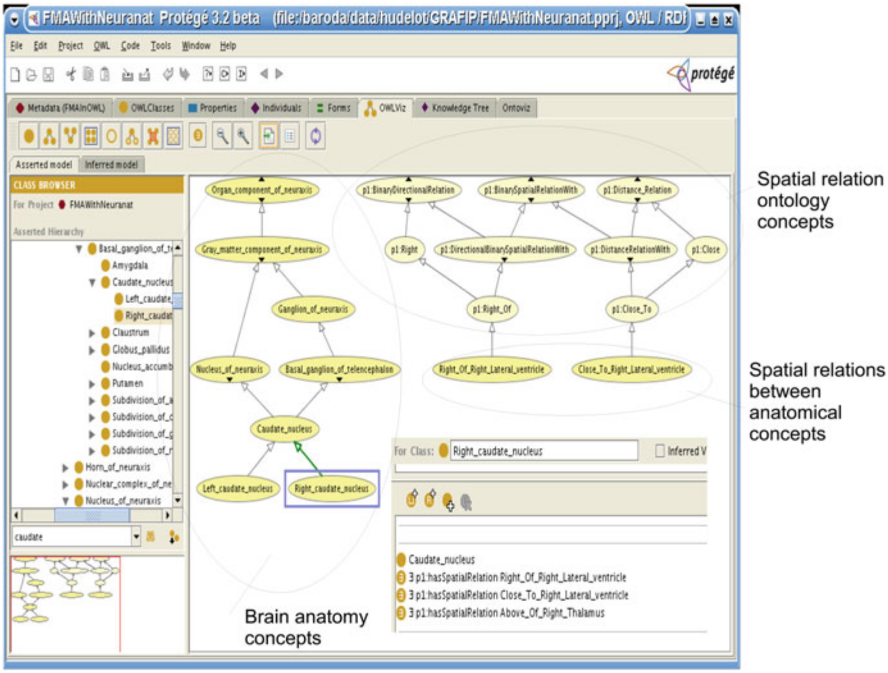

Fig. 9.8 Part of an ontology of the brain anatomy (excerpt of the FMA [85]). The concepts of the spatial relation ontology are prefixed by **p1** [49]

[2] http://www.chups.jussieu.fr/ext/neuranat/index.htmlforinstance.

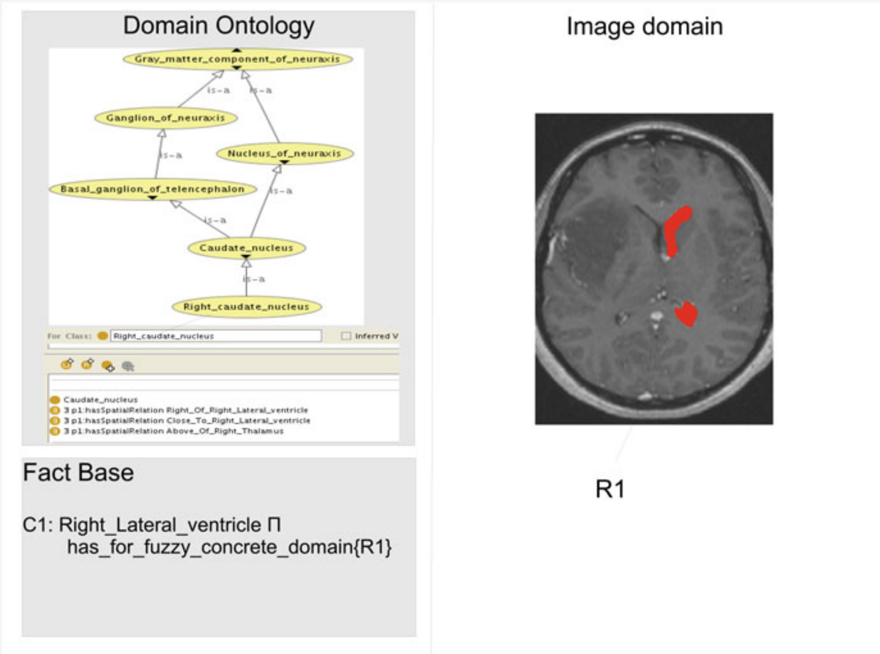

Fig. 9.9 The right lateral ventricle corresponds to the spatial region R1 in the image. The domain ontology describes spatial relations between the right caudate nucleus and the right lateral ventricle [49]. These relations are exploited to segment the right caudate nucleus

Moreover, the semantic enrichment by the fuzzy representations of spatial relations, learned on a database of examples, makes it possible to formalize the ontology concepts in an operational way, that facilitates pattern recognition and image interpretation. Here crisp spatial objects and fuzzy spatial relations are considered. Let us detail the recognition process of the right caudate nucleus, assuming that the right lateral ventricle has already been extracted. The situation is represented in Fig. 9.9.[3]

- A first step consists in extracting information from the domain ontology by querying it. The goal of the query is to find the spatial relations involving the right lateral ventricle and the right caudate nucleus. As the first one is already extracted and recognized, it is taken as a reference object. As a querying language, we use the nRQL language provided by RACER [47]. The nRQL request is expressed as:

[3] Here we do not use the "left is right" convention usually adopted in the medical imaging community, but for the sake of simplicity we denote by "right" structures that are on the right side in the figures (i.e., on the left side of the body).

```
(tbox-retrieve (?x)(and
(?y Right_Caudate_nucleus)
(?y ?x hasSpatialRelation)
(?z Right_Lateral_ventricle)
(?x ?z hasReferenceObject)))
```

An answer to such a query using our enriched domain ontology is: ***Right_Of_Right_Lateral_ventricle*** and ***Close_To_Right_Lateral_ventricle***. Indeed, according to the domain ontology "the right caudate nucleus is **to the right** and **close to** the right lateral ventricle and **above** the right thalamus" (see Fig. 9.9). Note that the last part of this knowledge is not used here since the thalamus is not recognized yet.

- Then, according to the ontology of spatial relations, concepts such as:

  Right_Of_Right_Lateral_ventricle or ***Close_To_Right_Lateral_ventricle***

  are derived from the concept ***SpatialRelationWith*** and their integration (i.e., fuzzy semantics in the image domain) corresponds here to a spatial fuzzy set. The fuzzy semantics is used to guide the operating mode (in this case, a fuzzy dilation with a structuring element defining the right direction). A similar reasoning is used for the relation **close to**, leading to another morphological operation.
- In the image domain, the search space of the "right caudate nucleus" corresponds to the area to the right and close to the right lateral ventricle, derived from the conjunctive fusion of the results of the two morphological operations, still performed in the spatial domain (Fig. 9.10).

The next step consists in segmenting the caudate nucleus. The fuzzy region of interest derived from the previous steps is used to constrain the search space and to drive the evolution of a deformable model. An initial surface is deformed towards the solution under a set of forces, including forces derived from spatial relations [6, 27, 41]. Once all pieces of knowledge are represented in the spatial domain, the process is the same as in the sequential approach described in Sect. 9.3.1.

While in the sequential approach, segmentation and recognition are performed simultaneously, in a global approach [12], several objects are first extracted from the image using a segmentation method and then recognized. The recognition can be achieved by assessing whether the spatial relations between two objects x and y are those existing in the domain ontology. As for the sequential approach, let us detail the process for a few internal structures.

- From the segmentation process (not described here), three structures that belong to the grey nuclei are extracted. The first step consists in assessing spatial relations between these structures. For the sake of simplicity we focus on relative directions. This situation is represented in Fig. 9.11.
- We are interested in finding all the directional spatial relations between R1, R2, R3, R4, where R1 represents the lateral ventricles and R2–R4 the three regions to be labeled. The ontology of spatial relations is used to select an adequate fuzzy representation of the concept "***X in directional relation with Y***." The derived

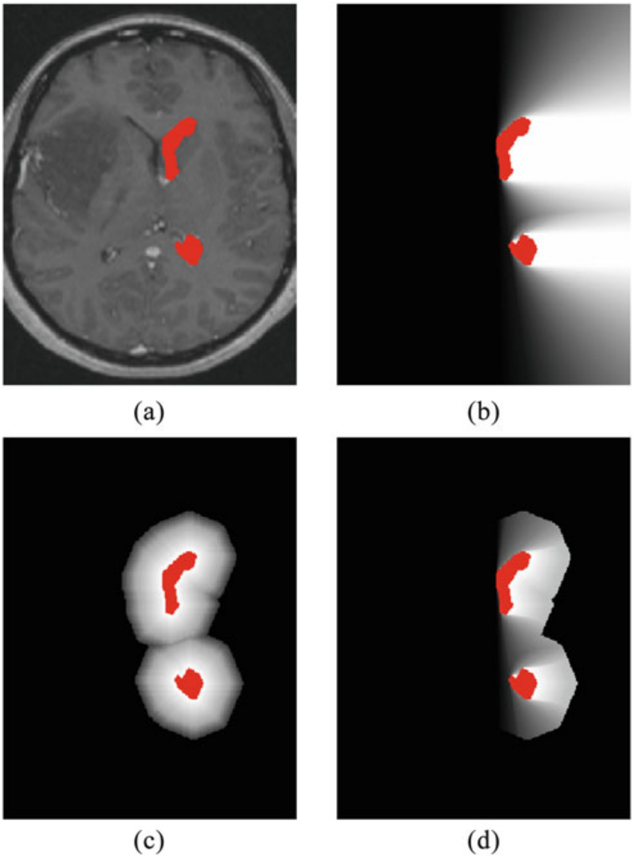

Fig. 9.10 (**a**) The right ventricle is superimposed on one slide of the original image. The search space of the object "caudate nucleus" corresponds to the conjunctive fusion of the spatial relations "**to the right of the right ventricle**" (**b**) and "**close to the right ventricle**" (**c**). The fusion result is shown in (**d**)

integration corresponds here to a histogram of angles (see Chap. 6). By using a *fuzzy interval* operating mode, the degrees of satisfaction of several directional relations between the segmented regions are computed. In this example, the following assertions yield high degrees of satisfaction: "***R2 is to the right of R1***," "***R2 is below R4***," "***R3 is to the right of R1***," "***R3 is to the right of R4***," "***R4 is to the right of R1***."

- The description of the concepts C1, C2, C3, C4 (Fig. 9.11) is completed with the predominant directional relations between R1, R2, R3, R4 and then are classified in the hierarchy using reasoners. This allows us to label, i.e., to recognize, each individual structure. In the example, structures R2, R3, and R4 are recognized as thalamus, putamen, and caudate nucleus, respectively.

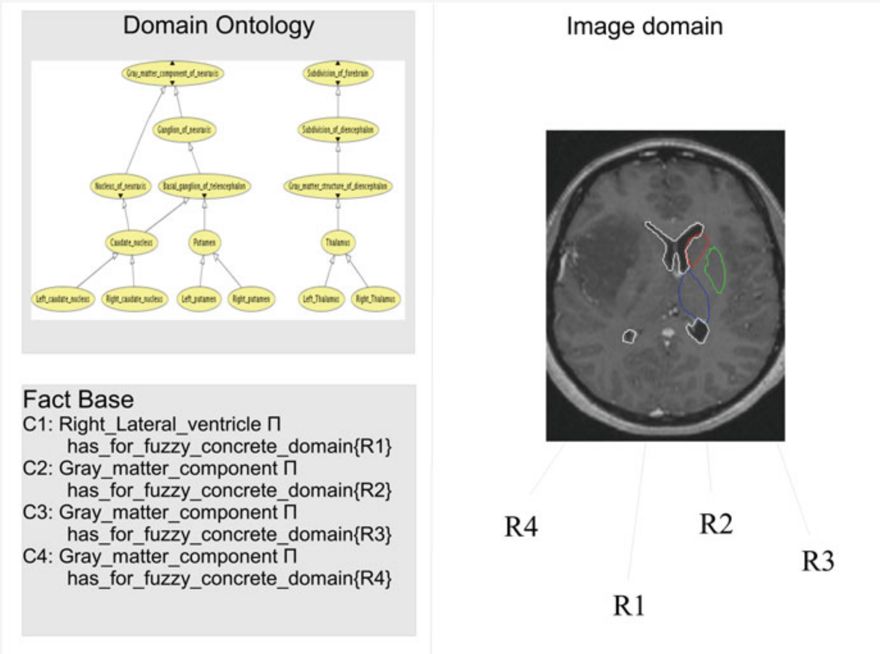

Fig. 9.11 The right lateral ventricle corresponds to the spatial region R1 on image. The domain ontology describes spatial relations between several grey nuclei and the lateral ventricles. These relations are exploited to identify each individual structure [49]

9.3.4 Interpretation as Abductive Reasoning

In this example, going one step further, the interpretation task is expressed as an abduction process, where the interpretation is formalized as the best explanation of the observation (i.e., the image, as in Fig. 9.12), according to the available knowledge, expressed in description logics. Here the segmentation is performed beforehand, and only the recognition and interpretation are addressed [105].

This example exploits the two directions described in this chapter. Starting from a linguistic description of the expert knowledge (here neuro-anatomy), a formal model is built (an ontology) and a knowledge base is derived in description logics, which will guide the image interpretation. At the end of the interpretation process, the result is expressed in the same logics, close to the expert natural language. The whole scheme is illustrated in Fig. 9.13.

The knowledge base used in this framework is composed of three blocks: terminologies (TBox), role axioms (RBox), and assertions (ABox) ($\mathcal{K} = \{\mathcal{T}, \mathcal{R}, \mathcal{A}\}$). An example of a knowledge base referring to brain anatomy is as follows, where LVl and LVr denote left and right lateral ventricles, and left and right caudate nuclei are denoted by CNl and CNr. The general knowledge is represented in the TBox,

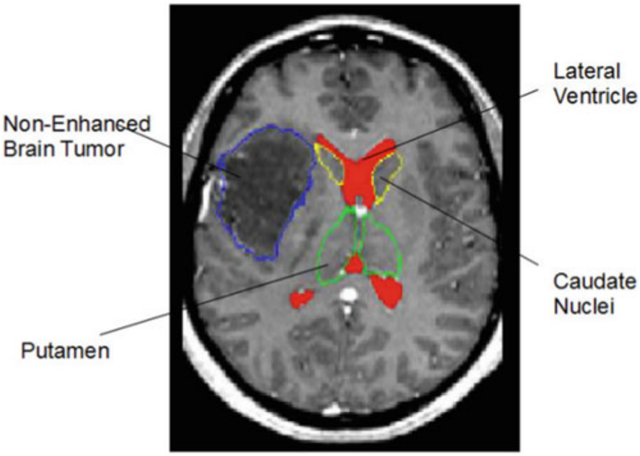

Fig. 9.12 A slice of a pathological brain volume (MRI acquisition), where some structures are annotated

which describes basic axioms of the background knowledge. The ABox represents the assertions, involving the facts in the observation (such as information extracted from an image). The complete knowledge base is given as follows:

$$
\begin{aligned}
TBox = \{Hemisphere &\sqsubseteq \exists isPartOf.Brain \\
BrainStructure &\sqsubseteq \exists isPartOf.Brain \\
BrainDisease &\sqsubseteq \exists isPartOf.Brain \sqcap \neg BrainStructure \\
Tumor &\sqsubseteq BrainDisease \\
LVl &\sqsubseteq BrainStructure \sqcap \exists(rightOf \sqcap closeTo).CNl \\
LVr &\sqsubseteq BrainStructure \sqcap \exists(leftOf \sqcap closeTo).CNr \\
CNl &\sqsubseteq BrainStructure \\
CNr &\sqsubseteq BrainStructure \\
PeripheralHemisphere &\sqsubseteq Hemisphere \\
CentralHemisphere &\sqsubseteq Hemisphere \sqcap \neg PeripheralHemisphere \\
PeripheralTumor &\sqsubseteq Tumor \sqcap \exists isPartOf.PeripheralHemisphere \sqcap \exists farFrom. \\
&\quad (LVl \sqcup LVr) \\
SmallDeformingTumor &\sqsubseteq Tumor \sqcap \exists closeTo.(CNl \sqcup CNr)\}
\end{aligned}
$$

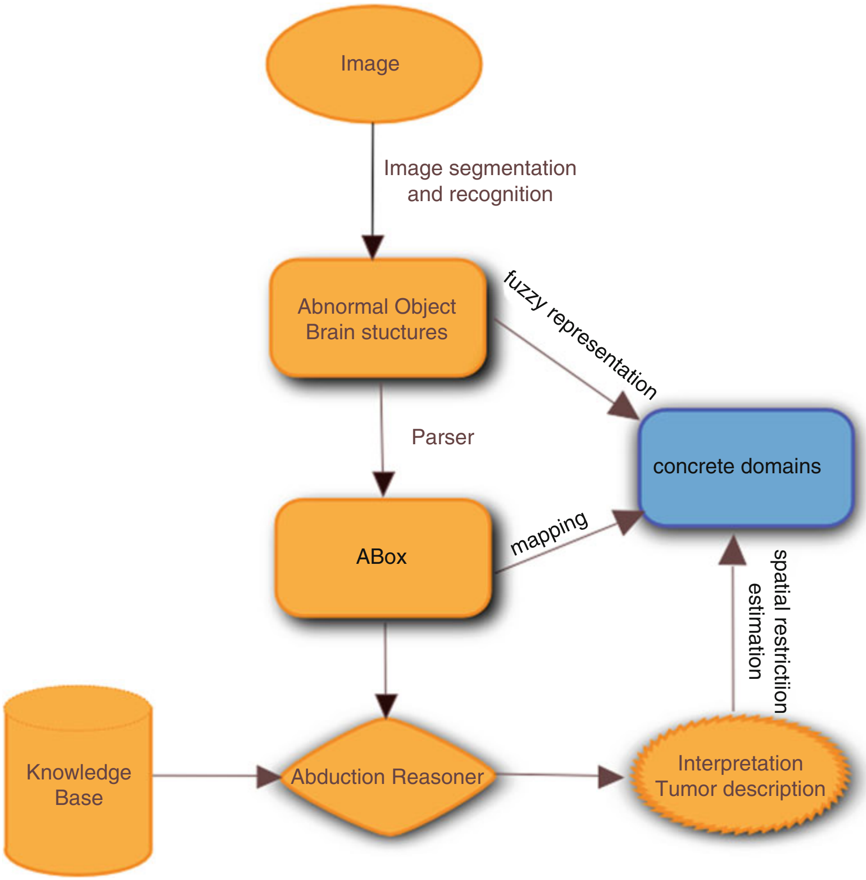

Fig. 9.13 Interpretation as abductive reasoning: overview [106]

$$
\begin{aligned}
RBox = \{rightOf &\equiv leftOf^- \\
above &\equiv below^- \\
closeTo &\equiv closeTo^- \\
farFrom &\equiv farFrom^- \\
isPartOf \circ isPartOf &\sqsubseteq isPartOf \\
hasPart \circ hasPart &\sqsubseteq hasPart \\
isPartOf &\equiv hasPart^- \}
\end{aligned}
\qquad
\begin{aligned}
ABox = \{a &: CNl \\
b &: unknown \\
c &: Brain \\
\langle a, b\rangle &: leftOf, closeTo \\
\langle b, c\rangle &: isPartOf\}
\end{aligned}
$$

This knowledge base example demonstrates a practical way to represent brain anatomy. For instance, $LVl \sqsubseteq BrainStructure \sqcap \exists(rightOf \sqcap closeTo).CNl$ expresses that the left lateral ventricle belongs to the brain structure which is on the right of and close to the left caudate nucleus. In the RBox, inverse relations ($rightOf \equiv leftOf^-$) and transitive relations ($hasPart \circ hasPart \sqsubseteq hasPart$) are used to represent spatial relation properties. In the ABox, a, b, c are individuals corresponding to observed objects in the image. $a : CNl$ is a concept assertion and $\langle b, c\rangle : isPartOf$ is a role assertion, expressing that b is a part of c. The semantic gap between these conceptual descriptions and the image information is again filled by modeling spatial relations as fuzzy sets in concrete domains.

The final interpretation is then obtained from an abductive reasoning based on a tableau method. The logical formalism of abduction in DLs is represented as follows: given an observation concept $\mathcal{O}$, a hypothesis is a concept $\mathcal{H}$ such that $\mathcal{K} \models \mathcal{H} \sqsubseteq \mathcal{O}$. In this example, the observation $\mathcal{O}$ is a concept for an unknown object (b) in the ABox) expressed, considering the background knowledge, as: $\mathcal{O} \equiv \exists(leftOf^- \sqcap closeTo^-).CNl \sqcap \exists isPartOf.Brain$. The subsumption problem can be converted into a test of satisfiability which requires to prove that $\mathcal{H} \sqcap \neg\mathcal{O}$ is unsatisfiable, and a potential hypothesis $\mathcal{H}$ is then as concept which makes the tableau of $\mathcal{H} \sqcap \neg\mathcal{O}$ closed, after applying expansion rules.

Among the obtained hypotheses, the one satisfying additional minimality criteria is chosen. Finally, the preferred hypothesis in this example is

$$\mathcal{H} = SmallDeformingTumor \sqcap \forall closeTo.\neg CNr$$

which is consistent by the medical expert interpretation. See [105, 106] for more details about this approach and the tableau computation.

9.3.5 *Deriving Linguistic Descriptions*

In the previous example, the image understanding process leads to a high-level interpretation expressed in the language of the considered logic. Another approach relies on the idea of comparing distributions, detailed in Chap. 6. In Sect. 6.7, an example illustrated how to compare spatial relations represented as distributions. A similar approach can be used to compare spatial relations between segmented and recognized objects in an image, and prior models of these relations. The numerical result α allows drawing conclusions such as “object A is close to object B” to degree α, hence a description of the image content in natural language.

9.4 Interpretability and Explainability

As a conclusion, let us briefly provide some hints on interpretability and explainability. While this topic is gaining a renewed interest in artificial intelligence with the large adoption of neural networks, in particular in deep learning, this is an old topic that was addressed, for instance, in the late 1800s by Charles Sanders Peirce with formal abductive reasoning. Interestingly enough, fuzzy models, establishing links between knowledge, linguistic descriptions, cognition, on the one hand, and concrete domains and data, on the other hand, provide naturally interpretable and explainable systems (see, e.g., [20]). All the representations described in this book, in particular in Chap. 8, and their use for spatial reasoning and image understanding as shown in this chapter, are evidences of this powerful feature. For instance, the rules that are fired to lead to a decision or that are obtained by a decision tree, the spatial relations actually involved in the recognition of a structure, the concepts of an ontology used to interpret an image, etc. all directly provide reasons why a specific decision was made.

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