

Cable Equation

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Definition

The cable equation is a mathematical equation derived from a circuit model of the membrane and its intracellular and extracellular space to provide a quantitative description of current flow and voltage change both within and between neurons, allowing a quantitative and qualitative understanding of how neurons function.

Detailed Description

Brief History of Cable Theory in Neuroscience

The origins of cable theory date back to 1855 to work by Lord (William Thomson) Kelvin, “On the Theory of the Electric Telegraph,” which provided the mathematical theory necessary for laying the transatlantic telegraph cable. The utility of this work for neuroscience, however, was not recognized until many years later, after nerve axons came to be regarded as “core conductors.” In the core conductor concept, nerve axons are considered to be long, thin, cylindrical tubes of membrane filled with an electrically conducting axoplasm and immersed in a highly conductive medium. Because the resistance across the membrane is much larger than the resistance through the axoplasm, over short distances along the cylinder, electric current will preferentially flow down the core. In fact current will flow through the core for a considerable distance before the total amount of current that has leaked out through the membrane becomes significant. This “core conductor” concept was given a mathematical framework in the 1870s when Weber derived and solved steady-state equations in cylindrical coordinates for three-dimensional current flow in and around an axon. Nevertheless, it was not until around 1900 that Hermann and others explicitly recognized that the reduction of these equations from three spatial dimensions to one spatial dimension (distance along the axon or cable) led to an equation equivalent to Lord Kelvin’s cable equation.

Quantitative comparison of cable equation calculations to experimental results requires the use of single fiber preparations, but such preparations were not available until the 1930s. Given the current emphasis on cable theory for dendritic neurons, it might seem surprising to many today to learn that the first applications of cable theory were to determine the passive electrotonic properties of axons. Axons are however the prototypical cable. Hodgkin and Rushton in their classic 1946 paper derived solutions of the cable equation for an infinite cable. They noted that weak subthreshold currents produce a voltage change that varies linearly with current, and they applied their cable equation solutions to recordings of such responses to estimate membrane capacity, membrane resistivity, and axoplasm and extracellular resistivity in lobster leg axons. Hodgkin and Rushton argued that an analysis of this “passive behavior is essential to any theory of excitation,” it “provides an insight into the structure of the surface membrane,” and “such an analysis must precede an understanding of the more complicated electrical changes which make up the nervous impulse itself.” Several

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investigators applied cable theory to estimate passive cable parameters in a number of giant invertebrate axons, and results have been summarized by Rall (1977; Table 1).

In the 1950s, the advent of glass micropipettes made it possible to stimulate and record intracellularly from neurons having diameters much smaller than those of the large invertebrate axons. Some of the early recordings were made from the soma of large motoneurons or pyramidal cells. It had been known from the time of Ramón y Cajal that these neurons possessed extensive dendritic trees, and in the late 1950s it was verified with the electron microscope that dendrites and dendritic spines were sites of synapses. What role did dendrites play in neuronal excitability, and how did dendritic properties affect integration of synaptic inputs? The early answers to these questions, based on interpretations of somatic recordings, turned out to be incorrect. It took the application of cable theory to these recordings by Rall (1957, 1959, 1960), along with subsequent work by Rall and others, to provide a correct understanding of the role dendrites play in signaling (for a detailed history, see commentaries and original papers in Segev et al. (1995)).

The application of cable theory to dendrites had to be different from the application to axons. First, while axons are long and can be effectively modeled as infinite cables, dendrites are much shorter, requiring solutions for a finite cable. Many such solutions are given by Rall (1959, 1960) and Jack et al. (1975). Second, axons tend to run for long distances without branching, but dendritic trees are highly branched. Solutions for branched dendrites are given in Rall (1959) and also Rall and Rinzel (1973) and Rinzel and Rall (1974). Furthermore, Rall showed that under certain conditions a branched structure could be reduced to an “equivalent cylinder” allowing solutions for a finite cable to be applied to many cell types. Third, axons and dendrites play different roles in signaling, so the questions for cable theory to address are much different. For example, how far synapses are from the soma electrotonically is a fundamental question for dendritic neurons; simple electrotonic length formulas were derived by Rall (1969) to address this question, and much work in this area has been done by others since (for a review and references, see Rall and Agmon-Snir (1998)).

As information accumulates about voltage-dependent conductances in dendrites, one might question the relevance of passive cable theory to the study of dendritic neurons today. On this point, Rall and Agmon-Snir (1998) echo and extend the arguments for the importance of cable theory for axons given by Hodgkin and Rushton, quoted above, stating “the intuition and methods given by passive cable theory enhance our understanding of the integrative mechanisms even in excitable dendrites. The passive case is an important reference for the excitable case and helps us better understand the role of excitability. Moreover, the passive case is a useful approximation that, analyzed with powerful methods, gives rise to general rules concerning the role of the geometry of the dendritic tree and of passive biophysical properties.” Some of these general rules will be shown below.

The Cable Equation

Given that neurons can be represented as core conductors with a patch of membrane adequately approximated by an electric circuit (Fig. 1), the derivation of the cable equation is a straightforward application of Ohm’s law, Kirchhoff’s law, and the relationship $q = CV$. The most difficult part of the derivation is defining resistances, capacitances, and currents in the proper units and knowing how to convert these terms to specific resistances and capacitance. Given the circuit in Fig. 1, Ohm’s law leads to $\partial V_i / \partial x = -i_i r_a$ and $\partial V_e / \partial x = -i_e r_e$, Kirchhoff’s law provides $\partial i_i / \partial x = -i_m$ and $\partial i_e / \partial x = +i_m$, and $q = CV$ and Ohm’s law together give membrane current $i_m = c_m \partial V_m / \partial t + (V_m - E_r) / r_m$. When these equations are combined, the result is the cable equation:

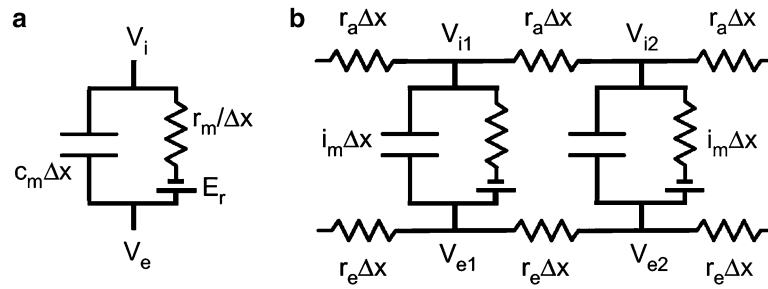


Fig. 1 Electric circuits for a membrane patch (a) and a neuron cable (b). r_m resistance of a unit length patch of membrane Ω cm, r_a axial resistance per unit length Ω /cm, c_m capacitance per unit length μ F/cm, i_m membrane current per unit length mA/cm. Note membrane resistivity $R_m = r_m \pi d$ in Ω cm², membrane capacity $C_m = c_m / \pi d$ in μ F/cm², axial resistivity $R_a = r_a \pi d^2 / 4$ in Ω cm (also called R_i), and d diameter

$$\frac{r_m}{r_a + r_e} \frac{\partial^2 V_m}{\partial x^2} = r_m c_m \frac{\partial V_m}{\partial t} + (V_m - E_r).$$

This equation can be rewritten as

$$\lambda^2 \frac{\partial^2 V}{\partial x^2} = \tau \frac{\partial V}{\partial t} + V \quad \text{where} \quad \lambda = \sqrt{\frac{r_m}{r_e + r_a}} \quad \text{and} \quad \tau = r_m c_m$$

with λ known as the space constant or length constant and τ known as the membrane time constant. In nondimensional form with $T = t/\tau$ and $X = x/\lambda$ this equation becomes

$$\frac{\partial V}{\partial T} = \frac{\partial^2 V}{\partial X^2} - V.$$

We usually neglect r_e (see Rall 1959) and express λ and τ in terms of specific resistances and capacitance as

$$\lambda = \sqrt{\frac{R_m d}{4 R_a}} \quad \text{and} \quad \tau = R_m C_m \quad \text{where } d \text{ is cable diameter.}$$

Note that the above cable equation assumes current flow in one dimension, down the core of the cable. This is a useful approximation because the length of axons and dendrites far exceeds the diameter making radial and angular flow of current negligible except in the immediate vicinity of a current source. A three-dimensional approach may be warranted when voltage changes in a large extracellular volume are of importance (Eisenberg and Johnson 1970), but in most cases, the one-dimensional cable equation is entirely appropriate.

Solutions for the Infinite Cable

For an axon, the assumption is that current is applied at a middle point $x = 0$, with the axon cable extending in both directions from this point to $\pm\infty$, with voltage being bounded. The solution obtained for these conditions by Hodgkin and Rushton (1946) and more directly by Jack et al. (1975) is

$$V(X, T) = \frac{r_a I_0 \lambda}{4} \left\{ \exp(-X) \operatorname{erfc}\left(\frac{X}{2\sqrt{T}} - \sqrt{T}\right) - \exp(X) \operatorname{erfc}\left(\frac{X}{2\sqrt{T}} + \sqrt{T}\right) \right\}$$

in nondimensional form, where I_0 is a constant current and erfc is the complementary error function. The solution above can be used to obtain two useful particular solutions – voltage at the stimulation site as a function of time, $V(0, T) = (r_a I_0 \lambda / 2) \operatorname{erfc}(\sqrt{T})$, and the steady-state voltage, $V(X, \infty) = (r_a I_0 \lambda / 2) \exp(-X)$. The former shows that voltage rises to 84 % of its final value at $T = t/\tau = 1$ (compared to 63 % for an isopotential sphere), and the latter illustrates the usefulness of λ as a measure of voltage decay with distance.

Steady-State Solutions for Semi-infinite and Finite Cables

In the steady-state $\partial V / \partial t = 0$, reducing the cable equation to an ordinary differential equation,

$$\lambda^2 \frac{d^2 V}{dx^2} - V = 0.$$

This equation is easily solved, and the general solution can be expressed in various equivalent forms:

$$\begin{aligned} V(x) &= A_1 \exp(-x/\lambda) + A_2 \exp(x/\lambda) \\ V(x) &= B_1 \cosh(x/\lambda) + B_2 \sinh(x/\lambda) \\ V(x) &= C_1 \cosh((\ell - x)/\lambda) + C_2 \sinh((\ell - x)/\lambda) \end{aligned}$$

where ℓ is the length of the cable. The arbitrary constants depend on the boundary conditions (BC) which specify either the voltage or the current (derivative of the voltage) at the ends of the cable:

Voltage BC : $V(0) = V_o$ and $V(\ell) = V_L$ or V is bounded (not infinite).

$$\text{Current BC : } -\frac{1}{r_a} \frac{dV}{dx} \Big|_{x=0} = i_i \quad \text{and} \quad \frac{1}{r_a} \frac{dV}{dx} \Big|_{x=\ell} = i_i.$$

Voltage boundary conditions are straightforward. The current boundary conditions come from Ohm's law and the convention that current is positive to the right, so one must be careful with the direction of the injected current, particularly at the end of the cable. This explains the lack of a minus sign for the current BC at $x = \ell$. Some commonly used current boundary conditions are:

$$\text{Sealed end: } -\frac{1}{r_a} \frac{dV}{dx} \Big|_{x=\ell} = 0.$$

$$\text{Leaky end: } -\frac{1}{r_a} \frac{dV}{dx} \Big|_{x=\ell} = V(\ell) G_\ell \quad \text{e.g., } -\frac{1}{r_a} \frac{dV}{dx} \Big|_{x=\ell} = V(\ell) \left(\frac{\pi d^2}{R_m 4} \right) \text{ for an end cap.}$$

$$\text{Current injection: } -\frac{1}{r_a} \frac{dV}{dx} \Big|_{x=0} = I_0.$$

Note that the leaky end condition would not have the minus sign if we considered leak at $x = 0$ because of the direction of the leak current. Simulation software packages typically assume a sealed

end condition at terminations because the leak conductance, G_ℓ , through an end cap of membrane with the same membrane resistivity, R_m , as the dendrite turns out to be negligible.

Application of boundary conditions to the general solution leads to special solutions that provide significant insight. First, consider the semi-infinite cable where boundary conditions are $V(0) = V_0$ and $V(\infty)$ is bounded. These conditions lead to

$$V(x) = V_0 \exp\left(\frac{-x}{\lambda}\right).$$

For current input, I_0 , at $x = 0$, V_0 would be replaced by $r_a I_0 \lambda$ which is a factor of 2 larger than noted above for the infinite cable. This solution shows the significance of λ as a measure of voltage decay with distance in that voltage drops 1/e or 37 % for each λ distance. It must be stressed that this result applies strictly only for semi-infinite and infinite cables – for finite cables with sealed ends, the decay is less.

As a second example, consider voltage clamp in a finite cable. Boundary conditions are $V(0) = V_0$ and sealed end at ℓ . The solution is

$$V(x) = V_0 \cosh\left(\frac{\ell - x}{\lambda}\right) / \cosh\left(\frac{\ell}{\lambda}\right).$$

One can use this expression to estimate how effective voltage clamp at the soma space clamps distal regions by considering the solution at $x = \ell$ or $V(\ell) = V_0 / \cosh(\ell/\lambda)$. For cells with an electrotonic length $L = \ell/\lambda = 1$, the solution shows that the voltage change at the end of the cable is only 65 % of the difference between V_0 and the resting potential. For transient inputs, the effective space clamp is considerably worse (Rall and Segev 1985).

A third useful case is the solution for current input in a finite cable. Boundary conditions are current input at $x = 0$ and sealed end at $x = \ell$. The solution is

$$V(x) = \frac{r_a \lambda I_0 \cosh((\ell - x)/\lambda)}{\sinh(\ell/\lambda)}.$$

This solution is useful for illustrating the dependence of input resistance, R_N , a measure of cell excitability, on various electrotonic and morphological factors:

$$V(0) = \frac{r_a \lambda I_0}{\tanh(\ell/\lambda)} \text{ so } R_N = \frac{r_a \lambda}{\tanh(\ell/\lambda)} = \left(\frac{2}{\pi}\right) \frac{\sqrt{R_m R_a} d^{-3/2}}{\tanh(\ell/\lambda)}.$$

The last expression shows the dependence of input resistance on $d^{3/2}$, a factor that proves to be important for branched case solutions and reduction of a branched tree to an equivalent cylinder. Note that as $\ell \rightarrow \infty$, $R_N \rightarrow r_a \lambda$ in agreement with the semi-infinite solution above. Alternatively, we say that the input conductance of a semi-infinite cylinder, $G_\infty = 1/(r_a \lambda)$. It is straightforward to convert the last expression above into the following useful equations:

$$R_N = \frac{R_m L}{A_D \tanh(L)} \quad \text{also} \quad R_m = \frac{R_N A_D \tanh(L)}{L}$$

where $L = \ell/\lambda$ is electrotonic length and A_D is the membrane area of the cable. The first of these expressions shows the dependence of input resistance on membrane area, electrotonic length, and

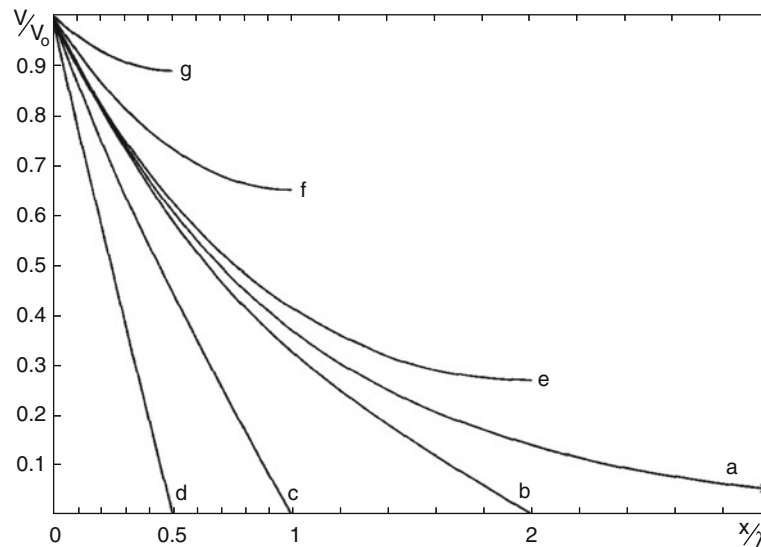


Fig. 2 Voltage attenuation with distance. *a*, Infinite cylinder; *b*, *c*, *d*, voltage clamp to rest in cylinders with $L = 2$, 1, and 0.5; *e*, *f*, *g*, sealed end in cylinders with $L = 2$, 1, and 0.5 (From Rall (1959), Fig. 3 with permission from Elsevier)

membrane resistivity, and the second is useful for estimating R_m when input resistance, membrane area, and electrotonic length are known or alternatively for estimating membrane area when input resistance, membrane resistivity, and electrotonic length are known. A means to estimate electrotonic length independently is described below.

Another useful case is current input at $x = 0$ and leaky end at $x = \ell$. In this case, the solution is

$$V(x) = \frac{r_a \lambda I_0 [\cosh((\ell - x)/\lambda) + (G_L/G_\infty) \sinh((\ell - x)/\lambda)]}{\sinh(\ell/\lambda) + (G_L/G_\infty) \cosh(\ell/\lambda)}$$

where G_L is the leaky end conductance and G_∞ is the conductance of an infinite extension of the cable (semi-infinite cylinder) as defined above. If $G_L = 0$ (sealed end), we get the solution shown earlier for current input into a finite cylinder. If $G_L = G_\infty$, then the leak is equivalent to an infinite extension of the cylinder and $V(x) = I_0/G_\infty \exp(-x/\lambda)$ as shown above for the semi-infinite cylinder. If $G_L = \infty$, then the solution represents the case where voltage at the end is clamped to 0. It is interesting to think about what other values of G_L/G_∞ might mean. A value of $G_L/G_\infty = 2$ might correspond to the attachment of an infinite extension of not one but two cylinders or perhaps flare in dendritic tree diameters; conversely, $G_L/G_\infty = 0.5$ might represent taper. Sample solutions for a number of different boundary conditions are shown in Fig. 2.

Steady-State Solutions for Branched Dendritic Trees

Steady-state solutions for branched dendritic trees can be obtained by letting each dendritic segment be represented by a finite cable and then connecting these cable segments at the branch points using continuity of voltage and conservation of current. It is useful to express the resulting equations in terms of the unknown voltages at the nodes – branch points, terminations, and the start. For a dendritic tree with n segments, the solution would require solving $n + 1$ simultaneous equations for voltages at the nodes. From these voltages, it is then possible to compute the voltage anywhere in the dendritic tree.

Another approach is to use the iterative algorithm proposed by Rall in 1959. In this algorithm, additional cylinders at branch points are considered as a leak conductance at the end of the parent

cylinder and use is made of the leaky end solution of the cable equation given just above. The leak conductance of the initial cylinder equals the sum of the input conductances of attached cylinders, which each have a leak conductance equal to the sum of the input conductances of their attached cylinders, and so forth iteratively, until terminations are reached. The leak conductance at terminations is assumed to be known, typically zero or sealed end. Given the leak conductance at terminations, the input conductance of the terminating branches can be computed, and the sum of input conductances of terminating branches having the same parent becomes the leak conductance of the parent branches, which allows the input conductance of the parent branches to be computed, iteratively, back to the initial segment. With input conductance of the initial cylinder known, along with the leak conductances G_L and G_∞ for all three segments, the voltage can then be computed iteratively from the starting segment out to terminating segments. Numerical examples of this procedure are given in Rall (1959) and Jack et al. (1975).

Although the above approaches, along with compartmental modeling approaches, can be used to obtain steady-state solutions for cells with arbitrarily branched geometry, these solutions do not provide the understanding and insight that can be obtained with closed form mathematical solutions. Rall and Rinzel (1973) constructed an idealized branched neuron model with suitable symmetry assumptions and equivalent cylinder constraints on branch diameters and were able to apply superposition to obtain mathematical solutions for the case when input was applied to only one distal branch of the dendritic tree. Explicit expressions were given for input resistance at the site of input and for voltage attenuation from the input location to the soma. Numerical illustrations showed how branch input resistance and voltage attenuation, factors important for the effectiveness of synaptic inputs, depend on certain morphological and electrotonic properties of the dendritic tree. A result of particular note is the asymmetric attenuation toward the soma compared with attenuation toward the periphery as shown in Fig. 3.

Transient Solutions to the Cable Equation

Transient solutions to the cable equation have been derived for various conditions. Mentioned above is the solution by Hodgkin and Rushton for the infinite cable, computed via Laplace transforms. A number of investigators have developed solutions with the use of Laplace transforms for various boundary conditions including Rall (1960), Jack and Redman (1971a, b), and others. The analysis from Rall (1960) was particularly important for showing that estimates of the membrane time constant from recordings of voltage transients were much too small when dendritic trees were not considered appropriately. Other methods of solution make use of Green's functions.

Transient solutions for branched dendrites can become very complicated and not very insightful without some simplifying assumptions, as evidenced by solutions with the Laplace transform approach by several investigators. However, the development by Rall (1962) of the equivalent cylinder model allowed the analysis to be simplified significantly (see entry “► [Equivalent Cylinder Model](#)”). Briefly, Rall showed that with a certain set of assumptions, a highly branched dendritic tree could be reduced mathematically to a single equivalent cylinder. Solutions that make use of the equivalent cylinder assumption to provide particularly insightful results have been obtained by Rall (1969) and Rinzel and Rall (1974) as we will now discuss.

Rall (1969) assumed that a neuron's dendritic tree could be approximated by an equivalent cylinder and used separation of variables to solve the cable equation, obtaining in nondimensional form:

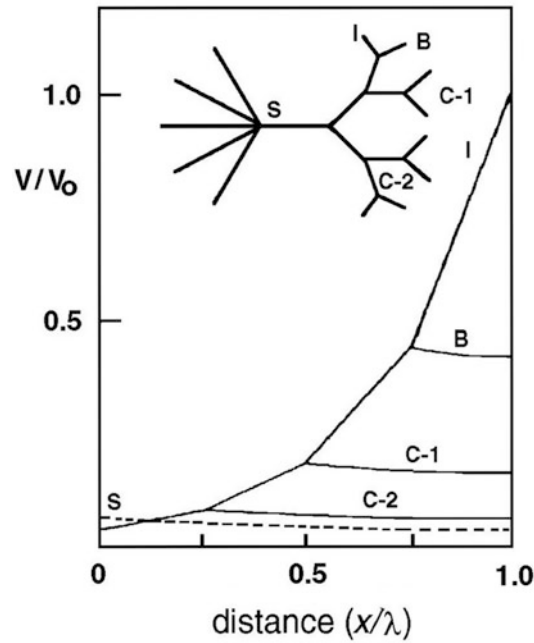


Fig. 3 Asymmetric voltage attenuation. Steady-state voltage attenuation with input at location I (*solid line*) or with the same input at the soma (*dotted line*). Note the steep attenuation from I to the soma and the lack of much attenuation from any branch point toward the periphery. Note also that the soma voltage for input at I equals the voltage at I for input at the soma. This illustrates the reciprocity property of linear systems (Figure reproduced from The Book of Genesis ©2003 James M Bower and David Beeman which was based on Fig. 4 from Rall and Rinzel (1973) reproduced with permission from Elsevier)

$$V(X,T) = (A \sin(\alpha X) + B \cos(\alpha X)) \exp[-(1 + \alpha^2)T].$$

Boundary conditions and the initial condition are then used to evaluate the constants A, B, and α . Applying sealed end boundary conditions at both ends gives

$$V(X,T) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi X}{L}\right) \exp\left[-\left(1 + \left(\frac{n\pi}{L}\right)^2\right)T\right]$$

where then B_n have to be obtained from the initial condition, $V(X,0)$. Regardless of the initial condition, the exponential term in the above equation shows that the voltage will decay back to rest with an infinite number of time constants defined by

$$\frac{1}{\tau_n} = \left[\frac{1 + (n\pi/L)^2}{\tau_0} \right] \quad \text{or} \quad \tau_n = \frac{\tau_0}{1 + (n\pi/L)^2}$$

where we have replaced T by t/τ and then include the factor $1 + (n\pi/L)^2$ to define the τ_n . Note that $\tau_0 = \tau_m$, the membrane time constant, when membrane properties are uniform. Then we can simplify our solution to:

$$V(X,t) = \sum_{n=0}^{\infty} C_n \exp(-t/\tau_n).$$

Rall called the τ_n “equalizing time constants.” In a cylinder, the initial voltage decay is fast, as it is governed by the fast time constants which describe how voltage “equalizes” among the dendrites, while the final slower decay is governed by the membrane time constant. A sphere or uniformly polarized cell has only one decay time constant, the slow membrane time constant. It was ignoring the fast time constants and trying to fit voltage transients with just one exponential term that led to the significant underestimation of the membrane time constant by some investigators in the 1950s and early 1960s. Note that voltage charging is also faster in a cylinder than in a sphere. This may seem counterintuitive if the fast time constants are “equalizing time constants.” To explain this paradox, it must be remembered that in a sphere, the capacitance of the whole cell has to be discharged before significant current flows through the membrane, whereas in a cylinder, only local capacitance has to be discharged before there is local membrane current flow; consequently, voltage charging begins earlier.

The usefulness of this transient solution was pointed out by Rall who showed that if τ_0 and τ_1 could be estimated from an experimental transient, then the electrotonic length of the dendritic tree $L = \ell/\lambda$ can be estimated with the following formula:

$$L = \frac{\pi}{\sqrt{\tau_0/\tau_1 - 1}}.$$

In early studies, the time constants were estimated by a procedure called “exponential peeling,” but with the advent of sophisticated computers and software, nonlinear curve fitting methods are used. Estimates of L tell us if the cell is electrotonically compact or not. This is important for estimating the effectiveness of synaptic inputs and in particular the strength of distal inputs compared to identical proximal inputs and also for estimating the extent of space clamp during voltage clamp. There are a number of other formulas that can be used to estimate L , although the one given above is considered to be the most robust. A caveat is that L estimates made with this formula assume that the cell can be approximated as an equivalent cylinder; deviations from this assumption will influence τ_1 and will cause errors in the L estimate (see entry “► [Electrotonic Length Formulas](#)”).

Rinzel and Rall (1974) developed a transient solution for a branched dendritic tree using the same idealized dendritic tree and similar symmetry assumptions and superposition methods as used in their steady-state solutions for a branched tree mentioned above. The mathematical expressions derived were used to illustrate the attenuation and delay characteristics of the voltage peak caused by a transient input in a distal branch as it proceeds to the soma. While voltage attenuation of transient inputs from the input site to the soma was considerably more severe than that observed previously for steady-state inputs, a significant fraction (about half) of the input charge (the integral of the voltage change) was nonetheless delivered to the soma. Expressed another way, the attenuation of the integral of the voltage between the input site and the soma equals the steady-state attenuation between these two locations. These calculations also showed that the time to peak and half-width of the voltage response at the soma depend on the distance of the input from the soma. Some of these results are illustrated in Fig. 4.

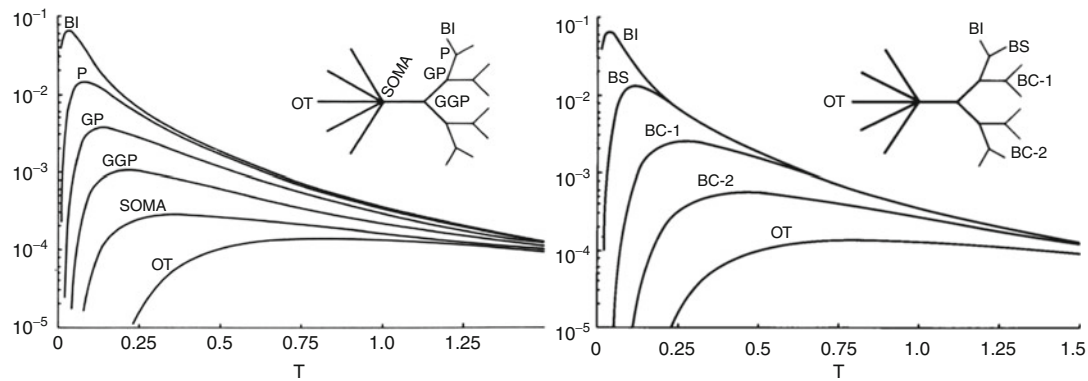


Fig. 4 Voltage attenuation for transient inputs. Input is at BI. *Left*: voltage attenuation to the branch points. *Right*: voltage attenuation to the terminals. Note how much less attenuation is from the branch points to the terminals than between branch points. Compare with Fig. 3 (Reproduced from Rinzel and Rall (1974) Figs. 4 and 5 with permission from Elsevier)

Insights from Cable Equation Solutions

Cable equation solutions have provided considerable insight leading to a quantitative and qualitative understanding of how neurons function. Insights may have come from solutions for simplified neuron structures with particular assumptions about branching and diameters at branch points and passive membrane with (usually) uniform properties, but their applicability is general. The list below is long but is not meant to be exhaustive:

1. *Space constant λ* . The space constant (length constant) λ is a useful qualitative measure of voltage decay with distance along a cable. However, as a quantitative measure, λ is the distance over which the steady-state voltage decays to $1/e$ or 37 % of its value at the origin *only* in infinite and semi-infinite cables. Electrotonic length and boundary conditions, as well as the presence of side branches, matter for steady-state voltage attenuation in finite cables.
2. *Membrane time constant τ* . The membrane time constant τ is a useful qualitative measure of how voltage decays in time. However, as a quantitative measure, τ is the time it takes for voltage to decay to $1/e$ or 37 % of its initial value *only* in an isopotential or uniformly polarized cell. Additional time constants affect the early voltage time course depending on morphology. Neglect of these additional time constants can lead to erroneous estimates of τ in dendritic neurons.
3. *Summation of inputs*. In a linear system, the solution for two current inputs is the sum of the solutions for each input. This is helpful for understanding spatial and temporal summation of synaptic inputs if they are small. However, it should be noted that synaptic inputs, being a conductance multiplied by a driving force, are not linear, so results should be interpreted with caution.
4. *Reciprocity*. In a linear system, the voltage response at the soma to current injection at a dendritic location will equal the voltage response at this dendritic location following the same current injection at the soma. This result applies generally for any two points in the cell and can be used to determine if a dendritic tree is passive or if blockers have made it so.
5. *Input resistance calculations*. Cable theory solutions can be used to calculate the input resistance in any passive branched dendritic tree. Input resistance can be much higher in distal dendrites than at the soma. The smaller diameter of the dendritic branch produces a larger voltage response to current injection. Boundary conditions also play a role.

6. *Steady-state voltage attenuation depends on cylinder electrotonic length and boundary conditions.* As mentioned above, steady-state voltage attenuation is described quantitatively by λ only for infinite and semi-infinite cables. For finite cylinders, both cylinder electrotonic length and boundary conditions affect attenuation. If the terminal boundary condition is a sealed end, attenuation with distance is less in electrotonically shorter cylinders than longer ones. If the terminal boundary condition is voltage clamp to rest, attenuation with distance is larger in electrotonically shorter cylinders than longer ones. The sealed end and voltage clamp to rest conditions represent two extremes that determine voltage attenuation (Fig. 2).
7. *Voltage attenuation is asymmetric.* A segment of dendrite within a dendritic tree will have a certain core resistance, but voltage attenuation across this core resistance will be asymmetric, depending on the boundary conditions at the ends of the segment. For an input in a distal branch, the boundary condition heading proximally toward the soma is a very leaky end, approaching a voltage clamp to rest condition, while the boundary condition heading distally is a not very leaky end, approaching a sealed end condition. Consequently, applying the previous point above, there will be severe attenuation heading proximally, but little attenuation heading distally. This explains the asymmetric attenuation in Fig. 3. The degree of voltage attenuation across a segment depends on how much the terminal boundary condition in the direction current is flowing that resembles a “sealed end” or “clamp to rest” condition.
8. *Transient voltage attenuation is more severe.* Voltage attenuation is more severe for transient inputs than for steady-state inputs. Nevertheless, the attenuation of the integral of the voltage change over time, with the integral computed at the input site and at the soma, equals the steady-state voltage attenuation between these two locations.
9. *Dendritic filtering.* In passive systems, dendrites provide low-pass filtering of the voltage response to synaptic inputs. As the signal travels to the soma, the voltage response is attenuated and becomes spread out. A shape index which plots EPSP half-width vs. time to peak can be used to predict the electrotonic location of the synapse. Note however that the shape index will depend not only on the synaptic location but also on the time course of the synaptic conductance and the electrotonic length of the dendritic tree as well.
10. *The $d^{3/2}$ and signal attenuation.* If dendrite diameters at branch points satisfy the “3/2 rule,” $d_p^{3/2} = d_{d1}^{3/2} + d_{d2}^{3/2}$, where d_p is the diameter of the parent branch and d_{d1} and d_{d2} are diameters of daughter branches, then there is impedance matching at branch points, and the attenuation with electrotonic distance remains unchanged. However, if diameters at branch points do not satisfy this condition, attenuation will be steeper or less steep after the branch point depending on whether the exponent is less than 3/2 or greater than 3/2, respectively. Changes in the summed $d^{3/2}$ along dendrites control signal attenuation.
11. *Consequences for voltage clamp. Attenuation out from soma is relatively small.* Recall that the analytic solution for voltage at the end of a cylinder given voltage clamp at the origin is $V(L) = V(0)/\cosh(L)$. This equation allows us to estimate the extent of space clamp in the cell when the soma is voltage clamped. Although much of the voltage change at the soma is felt at the distal dendrites, the space clamp is far from perfect at distal locations, and this can be very important in interpreting results.
12. *Implications for placement of excitatory and inhibitory inputs.* The differences in input resistance at the soma and distal dendrites, the asymmetry of voltage attenuation, and the solution for voltage clamp at the soma have implications for the placement of excitatory and inhibitory inputs. Regarding inhibition, inhibition is most effective when placed at the soma because the effect is felt throughout the cell. Regarding excitation, soma input will not produce the large changes in voltage seen with more distal input, so excitation at the soma is less likely to activate

voltage-gated conductances, suggesting that excitation might be better placed in the dendrites. If there are voltage-gated conductances in the dendrites, the large distal voltage changes could activate them and amplify the signal.

13. *Electrotonic length*. The electrotonic length, L , of the dendritic tree can be estimated from experimental measurements of τ_0 and τ_1 if the cell can be approximated as an equivalent cylinder. However, one must be careful in the interpretation of the time constants when the cell deviates from an equivalent cylinder. Most neurons have an electrotonic length of 1 or less.
14. *Dendritic spines*. Dendritic spines represent a special case of asymmetric voltage attenuation. Voltage attenuation will be steep from the spine head to the dendrite because the boundary condition at the spine-dendrite junction resembles a very leaky end. However, there will be negligible voltage attenuation from the dendrite to the spine head because the boundary condition at the tip of the spine head is practically a sealed end.
15. *Effect of reconstruction errors*. Cable theory can explain effects of reconstruction errors on parameter value estimates. Dendrite diameters are notoriously difficult to measure accurately because diameters are near the optical resolution of the microscope. Length values can be inaccurate because of tissue shrinkage, particularly in the z-axis. Cable theory says if the reconstruction diameters are uniformly off by a factor of x , then the R_m , C_m , and R_a values used in models should be equal to the actual R_m multiplied by x , the actual C_m divided by x , and the actual R_a multiplied by x^2 . If reconstruction lengths are uniformly off by a factor y , then the R_m , C_m , and R_a values used in models should be equal to the actual R_m multiplied by y , the actual C_m divided by y , and the actual R_a divided by y .
16. *Optimal diameters*. Cable theory calculations suggest that a neuron can optimize the effectiveness of distal inputs with its morphology or electrotonic parameters. For a cylinder of given length, if diameter is small, input resistance is large, and attenuation is steep, whereas if diameter is large, input resistance is small, and there is much less attenuation. Because of this trade-off between input resistance and attenuation, there will be a diameter that will maximize the effectiveness inputs delivered at the end of the cable.
17. *Modeling cells as an equivalent cylinder*. To a first approximation, dendrites can be modeled as an equivalent cylinder. This approximation is useful for providing insights into how voltage changes in dendritic trees.

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Further Reading

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