CS 173, Spring 19 Examlet 2, colored

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NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Use proof by contrapositive to prove the following claim, using your best mathematical style and working directly from the definitions of "odd" and "even." (You may assume that odd and even are opposites.)

For all integers p and q, if $p^2(q^2 - 4)$ is odd, then p and q are odd.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for all integers p and q, if p is even or q is even, then $p^2(q^2 - 4)$ is even.

Let p and q be integers. Suppose that p is even or q is even.

There are two cases:

Case (1) p is even. Then there is an integer m such that $p = 2m$. So $p^2(q^2 - 4) = 4m(q^2 - 4) = 2(2mq^2 - 8m)$. $t = 2mq^2 - 8m$ is an integer, since m and q are integers. So $p^2(q^2 - 4) = 2t$ is even.

Case (2) q is even. Then there is an integer n such that $q = 2n$. So $p^2(q^2 - 4) = p^2(4n^2 - 4) = 2(2n^2p^2 - 2p^2)$. $r = 2n^2p^2 - 2p^2$ is an integer because n and p are integers. So $p^2(q^2 - 4) = 2r$ is even.

In both cases, $p^2(q^2 - 4)$ is even, which is what we needed to prove.

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(15 points) Prove the following claim, using your best mathematical style and the following definition of congruence mod k: $x \equiv y \pmod{k}$ if and only if $x = y + nk$ for some integer n.

> For all integers x, y, p, q and m, with $m > 0$, if $x \equiv p \pmod{m}$ and $y \equiv q \pmod{m}$, then $x^2 + xy \equiv p^2 + pq \pmod{m}$.

Solution: Let x, y, p, q and m be integers, with $m > 0$. Suppose that $x \equiv p \pmod{m}$ and $y \equiv q \pmod{m}$.

By the definition of congruence mod k, this means that $x = p + am$ and $y = q + bm$, for some integers a and b. Then we can calculate

$$\begin{aligned} x^2 + xy &= (p+am)^2 + (p+am)(q+bm) \\ &= (p+am)(p+am+q+bm) \\ &= (p+am)(p+q) + (p+am)(am+bm) \\ &= (p+am)(p+q) + m(p+am)(a+b) \end{aligned}$$

Let $t = (p+am)(a+b)$. Then we have

$$\begin{aligned} x^2 + xy &= (p+am)(p+q) + mt \\ &= p(p+q) + am(p+q) + mt = p^2 + pq + m(ap + aq + t) \end{aligned}$$

$(ap + aq + t)$ is an integer because a, b, m, p, q are all integers. So $x^2 + xy \equiv p^2 + pq \pmod{m}$.

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(15 points) For any two real numbers x and y, the harmonic mean is $H(x,y) = \frac{2xy}{x+y}$. Using this definition and your best mathematical style, prove the following claim:

For any real numbers x and y, if $0 < x < y$, then $H(x,y) < y$.

Solution: Let x and y be real numbers. Suppose that $0 < x < y$.

Since $x < y$ and y is positive, $xy < y^2$.

Adding xy to both sides gives us $2xy < xy + y^2 = y(x+y)$.

So $2xy < y(x+y)$. Since x and y are both positive, $x+y$ is positive. So, we can divide both sides by $(x+y)$ to get $\frac{2xy}{x+y} < y$.

So, $H(x,y) < y$, which is what we needed to prove.

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(15 points) Recall that a real number p is rational if there are integers m and n (n non-zero) such that $p = \frac{m}{n}$. Use this definition and your best mathematical style to prove the following claim by contrapositive.

> For all real numbers x and y, if x is not rational, then $2x + 3y$ is not rational or y is not rational.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for all real numbers x and y, if $2x + 3y$ is rational and y is rational, then x is rational.

Let x and y be real numbers. Suppose that $2x + 3y$ is rational and y is rational. Then $2x + 3y = \frac{a}{b}$ and $y = \frac{m}{n}$, where a, b, m, n are integers, b and n non-zero.

Then $2x + 3\frac{m}{n} = \frac{a}{b}$

So $2x = \frac{a}{b} - \frac{3m}{n} = \frac{an-3bm}{bn}$

So $x = \frac{an-3bm}{2bn}$

$an - 3bm$ and $2bn$ are both integers because a, b, m, n are integers. Also $2bn$ is non-zero because b and n are non-zero. So x is rational.

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(15 points) A triple (a, b, c) of positive integers is Pythagorean if $a^2 + b^2 = c^2$. Use proof by contrapositive to prove the following claim, using your best mathematical style and working directly from the definitions of "odd" and "even." (You may assume that odd and even are opposites.)

For any Pythagorean triple (a, b, c), if c^2 is odd, then a is even or b is even.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for any Pythagorean triple (a, b, c), if a and b are odd, then c^2 is even.

So suppose (a, b, c) is Pythagorean and a and b are odd. Then $a^2 + b^2 = c^2$ by the definition of Pythagorean. Also, by the definition of odd, $a = 2m + 1$ and $b = 2p + 1$, where m and n are integers.

Then $c^2 = a^2+b^2 = (2m+1)^2+(2p+1)^2 = (4m^2+4m+1)+(4p^2+4p+1) = 2(2m^2+2m+2p^2+2p+1)$

Let $t = 2m^2 + 2m + 2p^2 + 2p + 1$. t is an integer because m and p are integers. And $c^2 = 2t$. So c^2 is even.

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(15 points) For any two real numbers x and y, the harmonic mean is $H(x,y) = \frac{2xy}{x+y}$. Using this definition and your best mathematical style, prove the following claim:

For any real numbers x and y, if $0 < x < y$, then $x < H(x,y)$.

Solution: Let x and y be real numbers. Suppose that $0 < x < y$.

Since $x < y$ and x is positive, $x^2 < xy$.

Adding xy to both sides gives us $x^2 + xy < 2xy$.

So $x(x+y) < 2xy$. Since x and y are both positive, $x+y$ is positive. So, we can divide both sides by $(x+y)$ to get $x < \frac{2xy}{x+y}y$.

So, $x < H(x,y)$, which is what we needed to prove.

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(15 points) Prove the following claim, using your best mathematical style and the following definition of congruence mod k: $x \equiv y \pmod{k}$ if and only if $x = y + nk$ for some integer n.

For all integers a, b, c, p and k (c positive), if $ap \equiv b \pmod{c}$ and $k \mid a$ and $k \mid c$, then $k \mid b$.

Solution:

Let a, b, c, p and k be integers, with c positive. Suppose that $ap \equiv b \pmod{c}$ and $k \mid a$ and $k \mid c$.

By the definition of congruence mod k, $ap \equiv b \pmod{c}$ implies that $ap = b + nc$ for some integer n. By the definition of divides, $k \mid a$ and $k \mid c$ imply that $a = ks$ and $c = kt$ for some integers s and t.

Since $ap = b + nc$, $b = ap - nc$. So then we have

$b = ap - nc = ksp - nkt = k(sp - nt)$

$sp - nt$ is an integer since s, p, n, and t are integers. So this implies that $k \mid b$.

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(15 points) Prove the following claim, using your best mathematical style. Hint: look at remainders and use proof by cases.

For any integer n, $n^2 + 2$ is not divisible by 4.

Solution: Let n be an integer. From the Division Algorithm (aka definition of remainder), we know that there are integers q and r such that $n = 4q + r$.

There are four cases, depending on what the remainder r is:

Case 1: $n = 4q$. Then $n^2 + 2 = 16q^2 + 2 = 4(4q^2) + 2$.

Case 2: $n = 4q + 1$. Then $n^2 + 2 = 16q^2 + 8q + 3 = 4(4q^2 + 2q) + 3$.

Case 3: $n = 4q + 2$. Then $n^2 + 2 = 16q^2 + 16q + 6 = 4(4q^2 + 4q + 1) + 2$.

Case 4: $n = 4q + 3$. Then $n^2 + 2 = 16q^2 + 24q + 11 = 4(4q^2 + 6q + 2) + 3$.

In all four cases, the remainder of $n^2 + 2$ divided by 4 is not zero, so $n^2 + 2$ isn't divisible by 4.

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(15 points) Use proof by contrapositive to prove the following claim, using your best mathematical style and working directly from the definitions of "odd" and "even." (You may assume that odd and even are opposites.)

For all integers x and y, if $3x + y^2 + 2$ is odd, then x is even or y is even.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for all integers x and y, if x is odd and y is odd, then $3x + y^2 + 2$ is even.

Let x and y be integers. Suppose that x and y are both odd. Then there are integers p and q such that $x = 2p + 1$ and $y = 2q + 1$.

Then

$$\begin{aligned}
3x + y^2 + 2 &= 3(2p + 1) + (2q + 1)^2 + 2 \\
&= (6p + 3) + (4q^2 + 4q + 1) + 2 \\
&= 6p + 4q^2 + 4q + 6 \\
&= 2(3p + 2q^2 + 2q + 3)
\end{aligned}$$

Let $t = 3p + 2q^2 + 2q + 3$. The above shows that $3x + y^2 + 2 = 2t$. Furthermore t must be an integer because p and q are integers. So $3x + y^2 + 2$ must be even.

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(15 points) Recall that a real number p is rational if there are integers m and n (n non-zero) such that $p = \frac{m}{n}$. Use this definition and your best mathematical style to prove the following claim:

For all rational numbers x, y and z, if y is non-zero, then $5(\frac{x}{y}) - 2z$ is rational.

Solution: Let x, y and z be rational numbers and suppose that y is non-zero.

By the definition of rational, $x = \frac{a}{b}$, $y = \frac{c}{d}$ and $z = \frac{e}{f}$, where the numbers a to f are all integers and b, d, and f are non-zero. Since y is non-zero, c must also be non-zero.

We can then compute

$$\begin{aligned}
5(\frac{x}{y}) - 2z &= 5(\frac{\frac{a}{b}}{\frac{c}{d}}) - 2\frac{e}{f} \\
&= 5(\frac{ad}{bc}) - 2\frac{e}{f} \\
&= \frac{5adf - 2ebc}{bcf}
\end{aligned}$$

Let $p = 5adf - 2ebc$ and $q = bcf$. p and q are integers because a, b, c, d, e, and f are all integers. Futhermore, q is non-zero, because b, c, and f are all non-zero.

Therefore, $5(\frac{x}{y}) - 2z = \frac{p}{q}$, where p and q are integers and q is non-zero. So $5(\frac{x}{y}) - 2z$ is rational.

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(15 points) Prove the following claim, using direct proof and your best mathematical style.

For any integers m and k, if $k \leq 7$ and $0 < m - 3 \leq \frac{k}{7}$, then $m^2 - 9 \leq k$.

Solution: Let m and k be integers and suppose that $k \leq 7$ and $0 < m - 3 \leq \frac{k}{7}$.

Since $0 < m - 3 \leq \frac{k}{7}$, $0 < \frac{k}{7}$, so k must be positive.

Since $k \leq 7$ and $0 < m - 3 \leq \frac{k}{7}$, $0 < m - 3 \leq \frac{7}{7} = 1$.

We can deduce two things from $0 < m - 3 \leq 1$. First, since k is positive, $k(m - 3) \leq k$. Second, by adding 6 to both sides, we get $m+3 \leq 7$. Since $m-3$ is positive, this implies that $(m+3)(m-3) \leq 7(m-3)$.

Combining this with $k(m - 3) \leq k$, we get $(m + 3)(m - 3) \leq 7(m - 3) \leq k$.

But $k^2 - 9 = (m + 3)(m - 3)$. So $k^2 - 9 \leq k$, which is what we needed to show.

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(15 points) Prove the following claim, using your best mathematical style and the following definition of congruence mod k: $a \equiv b \pmod{k}$ if and only if $a - b = nk$ for some integer n.

Claim: For all integers a, b, c, d, j and k (j and k positive), if $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ and $j \mid k$, then $a + c \equiv b + d \pmod{j}$.

Solution:

Let a, b, c, d, j and k be integers, with j and k positive. Suppose that $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ and $j \mid k$.

By the definition of congruence mod k, $a \equiv b \pmod{k}$ implies that $a - b = nk$ for some integer n. Similarly $c \equiv d \pmod{k}$ implies that $c - d = mk$ for some integer m. By the definition of divides, $j \mid k$ implies that $k = pj$ for some integer p.

We can then calculate

$$(a + c) - (b + d) = (a - b) + (c - d) = nk + mk = (n + m)k = (n + m)pj$$

Notice that $(n + m)p$ is an integer, since n, m, and p are integers. So, by the definition of congruence mod k, $a + c \equiv b + d \pmod{j}$.

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(15 points) Use proof by contrapositive to prove the following claim, using your best mathematical style and working directly from the definition of "divides." ($p \nmid q$ is the negation of $p \mid q$.)

For all integers k, a, b, if $k \nmid ab$, then $k \nmid a$ and $k \nmid b$.

You must begin by explicitly stating the contrapositive of the claim.

Solution: Let's prove the contrapositive. That is, for all integers k, a, b, if $k \mid a$ or $k \mid b$, then $k \mid ab$.

So k, a, and b be integers and suppose that $k \mid a$ or k |b. There are two cases:

Case 1: $k \mid a$. Then $a = kp$ where p is an integer. Then $ab = kpb$. Let $s = pb$. Then $ab = ks$. s is an integer because p and b are integers. So $k \mid ab$.

Case 2: $k \mid b$. Then $b = kq$ where q is an integer. Then $ab = akq$. Let $t = aq$. Then $ab = kt$. t is an integer because q and a are integers. So this means that $k \mid ab$.

In both cases $k \mid ab$, which is what we needed to prove.

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(15 points) Recall that a real number p is rational if there are integers m and n (n non-zero) such that $p = \frac{m}{n}$. Use this definition and your best mathematical style to prove the following claim:

For all real numbers p and q ($p \neq -1$), if $\frac{2}{p+1}$ and $p+q$ are rational, then q is rational.

Solution: Let p and q be real numbers, where $p \neq -1$. Suppose that $\frac{2}{p+1}$ and $p+q$ are rational.

By the definition of rational, this means that $\frac{2}{p+1} = \frac{m}{n}$ and $p+q = \frac{a}{b}$, where m, n, a, and b are integers with n and b non-zero. Notice that $\frac{2}{p+1}$ is non-zero, and therefore m is non-zero.

Since $\frac{2}{p+1} = \frac{m}{n}$, $2n = m(p+1)$, so $p+1 = \frac{2n}{m}$. This means that $p = \frac{2n}{m} - 1 = \frac{2n-m}{m}$.

Since $p+q = \frac{a}{b}$, $q = \frac{a}{b} - p = \frac{a}{b} - \frac{2n-m}{m} = \frac{am-b(2n-m)}{bm}$.

Since a, b, n, and m are integers, $am - b(2n-m)$ and bm are both integers. Moreover, bm must be non-zero because m and b are both non-zero. So q is the ratio of two integers, with the denominator non-zero. Therefore q is rational.

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(15 points) A natural number n is "snarky" if and only if $n = 3m + 1$, where m is a natural number. Use this definition and your best mathematical style to prove the following claim:

For all natural numbers x and y, if x and y are snarky, then $(x + y)^2$ is snarky.

Solution: Let x and y be natural numbers, and suppose that x and y are both snarky. By the definition of "snarky," this means that $x = 3m + 1$ and $y = 3p + 1$, where m and p are natural numbers.

By substitution, we then have $(x + y)^2 = (3m + 1 + 3p + 1)^2 = (3m + 3p + 2)^2$.

Let $t = m + p$. Then $(x + y)^2 = (3t + 2)^2 = 9t^2 + 12t + 4 = 3(3t^2 + 4t + 1) + 1$.

Now let $z = 3t^2 + 4t + 1$. Then $(x + y)^2 = 3z + 1$.

Notice that t is a natural number because m and p are natural numbers. And therefore z must also be a natural number. So $(x + y)^2 = 3z + 1$ means that $(x + y)^2$ is snarky, which is what we needed to prove.

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(15 points) A pair of positive integers (a, b) is defined to be a *partition* of a positive integer n if and only if $ab = n$. Using this definition and your best mathematical style, prove the following claim:

> For all positive integers a, b, and n, if (a, b) is a partition of n and $1 < a < \sqrt{n}$, then $\sqrt{n} < b < n$.

Solution: Let a, b, and n be positive integers. Suppose that (a, b) is a partition of n and $1 < a < \sqrt{n}$.

By the definition of partition, since (a, b) is a partition of n, we know that $ab = n$.

We know that $1 < a$. b was given to be positive. So $b < ab$. But $ab = n$. So $b < n$.

We know that $a < \sqrt{n}$. So $ab < b\sqrt{n}$. Since $ab = n$, this means that $n < b\sqrt{n}$. Dividing both sides by $\sqrt{n}$ gives us $\sqrt{n} < b$.

Since $b < n$ and $\sqrt{n} < b$, $\sqrt{n} < b < n$, which is what we needed to prove.

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(15 points) Prove the following claim, using direct proof and your best mathematical style.

For any integers x, y, and z, if $100x + 10y + z$ is divisible by 9, then $x + y + z$ is divisible by 9.

Hint: analyze the difference between $100x + 10y + z$ and $x + y + z$.

Solution: Let a, b, and c be integers and suppose that $100x + 10y + z$ is divisible by 9.

By the definition of divides, $100x + 10y + z = 9k$, where k is an integer.

Notice that $100x + 10y + z = (99x + 9y) + (x + y + z)$. So $(x + y + z) = (100x + 10y + z) - (99x + 9y)$. Substituting $100x+10y+z = 9k$ into this, we get $(x+y+z) = 9k-(99x+9y)$. So $(x+y+z) = 9(k-11x-y)$.

Let $m = k - 11x - y$. m is an integer because k, x, and y are integers. So $(x + y + z) = 9m$, where m is an integer. So $(x + y + z)$ is divisible by 9.

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(15 points) Prove the following claim, working directly from the definitions of "remainder" and "divides", and using your best mathematical style.

For all real numbers k, m, n and r ($n \neq 0$), if $r = \text{remainder}(m, n)$, $k \mid m$, and $k \mid n$, then $k \mid r$.

Solution: Let k, m, n and r be real numbers ($n \neq 0$). Suppose that $r = \text{remainder}(m, n)$, $k \mid m$, and $k \mid n$.

By the definition of remainder, $m = nq + r$, where q is some integer. (Also r has to be between 0 and n, but that's not required here.) So $r = m - nq$.

By the definition of divides, $m = ks$ and $n = kt$, for some integers s and t. Substituting these into the previous equation, we get

$$r = m - nq = ks - ktq = k(s - tq)$$

$s - tq$ is an integer because s, t, and q are integers. So r is the product of k and an integer, which means that $k \mid r$.

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(15 points) Recall that a real number p is rational if there are integers m and n (n non-zero) such that $p = \frac{m}{n}$. Use this definition and your best mathematical style to prove the following claim:

For all real numbers x and y, $x \neq 0$, if x and $\frac{y+1}{2}$ are rational, then $\frac{5}{x} + y$ is rational.

Solution: Let x and y be real numbers, where $x \neq 0$. Suppose that x and $\frac{y+1}{2}$ are rational.

By the definition of rational, $x = \frac{m}{n}$ and $\frac{y+1}{2} = \frac{p}{q}$, where m, n, p, and q are rationals, n and q non-zero. Since x is non-zero, m is also non-zero.

Since $x = \frac{m}{n}$ and x is not zero, $\frac{5}{x} = \frac{5n}{m}$.

Since $\frac{y+1}{2} = \frac{p}{q}$, $y + 1 = \frac{2p}{q}$. So $y = \frac{2p}{q} - 1 = \frac{2p-q}{q}$.

Combining these, we get that $\frac{5}{x} + y = \frac{5n}{m} + \frac{2p-q}{q} = \frac{5nq+2pm-qm}{mq}$. $5nq + 2pm - qm$ and mq are integers, since n, m, p, and q are integers. mq can't be zero because m and q are both non-zero. So $\frac{5}{x} + y$ is the ratio of two integers and therefore rational.

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(15 points) Recall that $\gcd(m,n)$ is the largest integer that divides both m and n. Use this definition, the definition of divides, and your best mathematical style to prove the following claim by contrapositive.

For all integers p and q, if $p + 6q = 23$ then $\gcd(p,q) \neq 7$.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for all integers p and q, if $\gcd(p,q) = 7$, then $p + 6q \neq 23$.

Let p and q be integers and suppose that $\gcd(p,q) = 7$. Then $7 \mid p$ and $7 \mid q$ by the definition of gcd. By the definition of divides, this implies that $p = 7m$ and $q = 7n$, for some integers m and n.

So $p + 6q = 7m + 6(7n) = 7(m + 6n)$. This mean that $p + 6q$ is divisible by 7. Since we know that 23 isn't divisible by 7, $p + 6q$ can't be equal to 23.

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(15 points) An integer k is a perfect square if $k = n^2$ where n is a non-negative integer. Prove the following claim:

For any integer p, if $p \geq 8$ and $p+1$ is a perfect square, then p is composite (aka not prime).

Solution: Let p be an integer and suppose that $p \geq 8$ and $p+1$ is a perfect square.

By the definition of a perfect square, $p+1 = n^2$ where n is a non-negative integer. Then $p = n^2 - 1 = (n+1)(n-1)$.

Since $p \geq 8$, $n^2 = p+1 \geq 9 \geq 3^2$. Since n isn't negative, We must have $n \geq 3$. So then $n+1 \geq n-1 \geq 2$.

So we can factor p into two integers $n+1$ and $n-1$, neither of which can be one. So p is composite.

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(15 points) Prove the following claim, using your best mathematical style and the following definition of congruence mod k: $a \equiv b \pmod{k}$ if and only if $a = b + nk$ for some integer n.

Claim: For all integers a, b, c, d, and k (k positive), if $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ then $a^2 + c \equiv b^2 + d \pmod{k}$.

Solution:

Let a, b, c, d, and k be integers, with k positive. Suppose that $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$.

By the definition of congruence mod k, $a \equiv b \pmod{k}$ implies that $a = b + nk$ for some integer n. Similarly, $c \equiv d \pmod{k}$ implies that $c = d + mk$ for some integer m. Then we can calculate

$$a^2 + c = (b + nk)^2 + (d + mk) = b^2 + 2bnk + n^2k^2 + d + mk = b^2 + d + k(2bn + n^2k + m)$$

If we let $p = 2bn + n^2k + m$, then we have $a^2 + c = (b^2 + d) + kp$. Also, p must be an integer since b, n, k, and m are integers. So, by the definition of congruence mod k, $a^2 + c \equiv b^2 + d \pmod{k}$.

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(15 points) Notice that, for any integer p, $\lfloor p \rfloor = \lfloor p + \frac{1}{2} \rfloor = p$. Using this fact and your best mathematical style, prove the following claim:

For any integer n, if n is odd, then $\left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lfloor \frac{n}{2} \right\rfloor \geq \frac{1}{2} \left\lfloor \frac{n^2}{2} \right\rfloor$

Solution: Let n be an integer and suppose that n is odd. Since n is odd, we can write $n = 2k + 1$, where k is an integer.

Looking at the left side of our equation, we have $\left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k+1}{2} \right\rfloor^2 + \left\lfloor \frac{2k+1}{2} \right\rfloor = \left\lfloor k + \frac{1}{2} \right\rfloor^2 + \left\lfloor k + \frac{1}{2} \right\rfloor = k^2 + k$

On the right side, we have $\frac{1}{2} \left\lfloor \frac{n^2}{2} \right\rfloor = \frac{1}{2} \left\lfloor \frac{(2k+1)^2}{2} \right\rfloor = \frac{1}{2} \left\lfloor \frac{4k^2+4k+1}{2} \right\rfloor = \frac{1}{2} \left\lfloor 2k^2 + 2k + \frac{1}{2} \right\rfloor = \frac{1}{2}(2k^2 + 2k) = k^2 + k$. (Noting that $2k^2 + 2k$ must be an integer because k is an integer.)

So $\left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lfloor \frac{n}{2} \right\rfloor = \frac{1}{2} \left\lfloor \frac{n^2}{2} \right\rfloor$ and therefore $\left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lfloor \frac{n}{2} \right\rfloor \geq \frac{1}{2} \left\lfloor \frac{n^2}{2} \right\rfloor$.

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(15 points) Prove the following claim, using your best mathematical style and the following definition of congruence mod k: $a \equiv b \pmod{k}$ if and only if $a = b + nk$ for some integer n.

> Claim: for all integers a, b, c, d, j, and k (j and k positive), if $a \equiv b \pmod{j}$, $c \equiv d \pmod{k}$, and $j \mid k$, then $a + c \equiv b + d \pmod{j}$.

Solution: Let a, b, c, d, j, and k (j and k positive). Suppose that $a \equiv b \pmod{j}$, $c \equiv d \pmod{k}$, and $j \mid k$.

Using the given definition of congruence, since $a \equiv b \pmod{j}$, $a = b+nj$ for some integer n. Similarly, since $c \equiv d \pmod{k}$, $c = d + mk$ for some integer m.

Adding these two equations together, we get $a + c = (b + nj) + (d + mk)$.

Since $j \mid k$, $k = pj$ by the definition of divides. Substituting this into the above equation, we get:

$a + c = (b + nj) + (d + mpj) = b + d + (n + mp)j$

$n+mp$ is an integer because n, m, and p are all integers. So by the definition of congruence, $a+c \equiv b+d \pmod{j}$.

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(15 points) Recall that a real number p is rational if there are integers m and n (n non-zero) such that $p = \frac{m}{n}$. Use this definition and your best mathematical style to prove the following claim:

For all rational numbers x, y and z, if y is non-zero, then $5(\frac{x}{y}) - 2z$ is rational.

Solution: Let x, y and z be rational numbers and suppose that y is non-zero.

By the definition of rational, $x = \frac{a}{b}$, $y = \frac{c}{d}$ and $z = \frac{e}{f}$, where the numbers a to f are all integers and b, d, and f are non-zero. Since y is non-zero, c must also be non-zero.

We can then compute

$$\begin{aligned} 5(\frac{x}{y}) - 2z &= 5(\frac{\frac{a}{b}}{\frac{c}{d}}) - 2\frac{e}{f} \\ &= 5(\frac{ad}{bc}) - 2\frac{e}{f} \\ &= \frac{5adf - 2ebc}{bcf} \end{aligned}$$

Let $p = 5adf - 2ebc$ and $q = bcf$. p and q are integers because a, b, c, d, e, and f are all integers. Futhermore, q is non-zero, because b, c, and f are all non-zero.

Therefore, $5(\frac{x}{y}) - 2z = \frac{p}{q}$, where p and q are integers and q is non-zero. So $5(\frac{x}{y}) - 2z$ is rational.

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(15 points) Recall that a real number p is rational if there are integers m and n (n non-zero) such that $p = \frac{m}{n}$. Use this definition and your best mathematical style to prove the following claim by contrapositive.

> For all real numbers x and y, if x is not rational, then $2x + 3y$ is not rational or y is not rational.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for all real numbers x and y, if $2x+3y$ is rational and y is rational, then x is rational.

Let x and y be real numbers. Suppose that $2x + 3y$ is rational and y is rational. Then $2x + 3y = \frac{a}{b}$ and $y = \frac{m}{n}$, where a, b, m, n are integers, b and n non-zero.

Then $2x + 3\frac{m}{n} = \frac{a}{b}$

So $2x = \frac{a}{b} - \frac{3m}{n} = \frac{an-3bm}{bn}$

So $x = \frac{an-3bm}{2bn}$

$an - 3bm$ and $2bn$ are both integers because a, b, m, n are integers. Also $2bn$ is non-zero because b and n are non-zero. So x is rational.

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(15 points) Prove the following claim, using your best mathematical style. Hint: look at remainders and use proof by cases.

For any integer n, $n^2 + 2$ is not divisible by 4.

Solution: Let n be an integer. From the Division Algorithm (aka definition of remainder), we know that there are integers q and r such that $n = 4q + r$.

There are four cases, depending on what the remainder r is:

Case 1: $n = 4q$. Then $n^2 + 2 = 16q^2 + 2 = 4(4q^2) + 2$.

Case 2: $n = 4q + 1$. Then $n^2 + 2 = 16q^2 + 8q + 3 = 4(4q^2 + 2q) + 3$.

Case 3: $n = 4q + 2$. Then $n^2 + 2 = 16q^2 + 16q + 6 = 4(4q^2 + 4q + 1) + 2$.

Case 4: $n = 4q + 3$. Then $n^2 + 2 = 16q^2 + 24q + 11 = 4(4q^2 + 6q + 2) + 3$.

In all four cases, the remainder of n divided by 4 is not zero, so n isn't divisible by 4.

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(15 points) A triple (a, b, c) of positive integers is Pythagorean if $a^2 + b^2 = c^2$. Use proof by contrapositive to prove the following claim, using your best mathematical style and working directly from the definitions of "odd" and "even." (You may assume that odd and even are opposites.)

For any Pythagorean triple (a, b, c), if c is odd, then a is even or b is even.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for any Pythagorean triple (a, b, c), if a and b are odd, then c is even.

So suppose (a, b, c) is Pythagorean and a and b are odd. Then $a^2 + b^2 = c^2$ by the definition of Pythagorean. Also, by the definition of odd, $a = 2m + 1$ and $b = 2p + 1$, where m and n are integers.

Then $c^2 = a^2+b^2 = (2m+1)^2+(2p+1)^2 = (4m^2+4m+1)+(4p^2+4p+1) = 2(2m^2+2m+2p^2+2p+1)$

Let $t = 2m^2 + 2m + 2p^2 + 2p + 1$. t is an integer because m and p are integers. And $c^2 = 2t$. So c^2 is even.

[We hadn't actually intended you to have to prove the additional part shown below, so the TAs have instructions to be nice about the grading.]

If c was odd, then we'd have $c = 2k + 1$. This would mean that $c^2 = (2k + 1)^2 = 4k^2 + 4k + 1$ which is odd. So c has to be even.