Name:____

NetID:__________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(15 points) Use (strong) induction to prove the following claim:

Claim: $\sum_{p=1}^{n} \frac{p}{p+1} \leq \frac{n^2}{n+1}$ for all positive integers n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{p}{p+1} = \frac{1}{2}$ and $\frac{n^2}{n+1} = \frac{1}{2}$. So the claim holds at $n = 1$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} \frac{p}{p+1} \leq \frac{n^2}{n+1}$ for $n = 1, \ldots, k$.

Inductive Step:

First, let's prove the following lemma: $\frac{k^2}{k+1} \leq \frac{k(k+1)}{k+2}$.

Proof of lemma: Notice that $k(k+2) = k^2 + 2k \leq k^2 + 2k + 1 = (k+1)^2$. So $k(k+2) \leq (k+1)^2$. So (since k is positive) $\frac{k}{k+1} \leq \frac{k+1}{k+2}$. So $\frac{k^2}{k+1} \leq \frac{k(k+1)}{k+2}$.

Now by the inductive hypothesis $\sum_{p=1}^{k} \frac{p}{p+1} \leq \frac{k^2}{k+1}$ So

$$\begin{aligned}
\sum_{p=1}^{k+1} \frac{p}{p+1} &= \frac{k+1}{k+2} + \sum_{p=1}^{k} \frac{p}{p+1} \\
&\leq \frac{k+1}{k+2} + \frac{k^2}{k+1} \leq \frac{k+1}{k+2} + \frac{k(k+1)}{k+2} \\
&= \frac{k^2+2k+1}{k+2} = \frac{(k+1)^2}{k+2}
\end{aligned}$$

So $\sum_{p=1}^{k+1} \frac{p}{p+1} \leq \frac{(k+1)^2}{k+2}$ which is what we needed to show.

Name:____

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Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(15 points) Use (strong) induction to prove the following claim:

Claim: $(2n)!^2 < (4n)!$ for all positive integers.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $(2n)!^2 = (2!)^2 = 2^2 = 4$ And $(4n)! = 4! = 24$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(2n)!^2 < (4n)!$ for $n = 1, 2, \ldots, k$.

Inductive Step: At $n = k + 1$, we have

$(2(k+1))!^2 = (2k+2)!^2 = [(2k+2)(2k+1)(2k!)]^2 = (2k+2)(2k+2)(2k+1)(2k+1)(2k)!^2$

Also $(4(k+1))! = (4k+4)! = (4k+4)(4k+3)(4k+2)(4k+1)(4k)!$

Also notice that $(2k+2)(2k+2)(2k+1)(2k+1) < (4k+4)(4k+3)(4k+2)(4k+1)$ because each of the four terms on the left is smaller than the four terms on the right.

From the inductive hypothesis, we know that $(2k)!^2 < (4k)!$.

Putting this all together, we get

$$
\begin{aligned}
(2(k+1))!^2 &= (2k+2)(2k+2)(2k+1)(2k+1)(2k)!^2 \\
&< (2k+2)(2k+2)(2k+1)(2k+1)(4k)! \\
&< (4k+4)(4k+3)(4k+2)(4k+1)(4k)! \\
&= (4(k+1))!
\end{aligned}
$$

So $(2(k+1))!^2 < (4(k+1))!$, which is what we needed to prove.

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Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(15 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

Claim: For any positive integer n and any reals $a_1, \ldots, a_n$ between 0 and 1 (inclusive)

$$\prod_{p=1}^{n}(1-a_p) \geq 1 - \sum_{p=1}^{n} a_p$$

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\prod_{p=1}^{n}(1-a_p) = 1-a_1$ and $1-\sum_{p=1}^{n} a_p = 1-a_1$ so $\prod_{p=1}^{n}(1-a_p) \geq 1-\sum_{p=1}^{n} a_p$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=1}^{n}(1-a_p) \geq 1-\sum_{p=1}^{n} a_p$ for $n = 1, \ldots, k$ and any real numbers $a_1, \ldots, a_n$ between 0 and 1 (inclusive).

Inductive Step: Let $a_1, \ldots, a_{k+1}$ be real numbers between 0 and 1 (inclusive). By the inductive hypothesis, we know that $\prod_{p=1}^{k}(1-a_p) \geq 1-\sum_{p=1}^{k} a_p$. Since $(1-a_{k+1})$ is positive, this means that $(1-a_{k+1})\prod_{p=1}^{k}(1-a_p) \geq (1-a_{k+1})(1-\sum_{p=1}^{k} a_p)$. Then we have

$$\begin{aligned}
\prod_{p=1}^{k+1}(1-a_p) &= (1-a_{k+1})\prod_{p=1}^{k}(1-a_p) \\
&\geq (1-a_{k+1})(1-\sum_{p=1}^{k} a_p) \ = \ 1 - a_{k+1} + a_{k+1}\sum_{p=1}^{k} a_p - \sum_{p=1}^{k} a_p \\
&\geq 1 - a_{k+1} - \sum_{p=1}^{k} a_p \quad \text{because all values } a_p \text{ are positive} \\
&= 1 - \sum_{p=1}^{k+1} a_p
\end{aligned}$$

So $\prod_{p=1}^{k+1}(1-a_p) \geq 1-\sum_{p=1}^{k+1} a_p$, which is what we needed to show.

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Use (strong) induction to prove the following claim.

Claim: For any positive integer n, $\displaystyle\sum_{p=1}^{n} \frac{(-1)^{p-1}}{p} > 0$

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{(-1)^{p-1}}{p} = 1 > 0$. So the claim holds.
At $n = 2$, $\sum_{p=1}^{n} \frac{(-1)^{p-1}}{p} = 1 - 1/2 = 1/2 > 0$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{p=1}^{n} \frac{(-1)^{p-1}}{p} > 0$ for $n = 1, 2, \ldots, k$.

Inductive Step: There are two cases:

Case 1) k is even.

$\sum_{p=1}^{k+1} \frac{(-1)^{k-1}}{k} = \frac{(-1)^k}{k+1} + \sum_{p=1}^{k} \frac{(-1)^{p-1}}{p}$.

From the inductive hypothesis, we know that $\sum_{p=1}^{k} \frac{(-1)^{p-1}}{p}$ is positive. Since k is even, we know that $\frac{(-1)^k}{k+1}$ is positive. Since $\sum_{p=1}^{k+1} \frac{(-1)^{p-1}}{p}$ is the sum of two positive numbers, it must be positive.

Case 2) k is odd. Then remove two terms from the summation:

$\sum_{p=1}^{k+1} \frac{(-1)^{p-1}}{p} = \frac{(-1)^{k-1}}{k} + \frac{(-1)^k}{k+1} + \sum_{p=1}^{k-1} \frac{(-1)^{p-1}}{p}$.

From the inductive hypothesis, we know that $\sum_{p=1}^{k-1} \frac{(-1)^{p-1}}{p}$ is positive. Since k is odd, $\frac{(-1)^{k-1}}{k} + \frac{(-1)^k}{k+1} = \frac{1}{k} + \frac{-1}{k+1} = \frac{1}{k} - \frac{1}{k+1}$. Since $\frac{1}{k}$ is larger than $\frac{1}{k+1}$, $\frac{1}{k} - \frac{1}{k+1}$ is positive. Since $\sum_{p=1}^{k+1} \frac{(-1)^{p-1}}{p}$ is the sum of two positive numbers, it must be positive.

In both cases, we have show that $\sum_{p=1}^{k+1} \frac{(-1)^k}{k+1} > 0$, which is what we needed to show.

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Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(15 points) Use (strong) induction to prove the following claim:

Claim: $\frac{(2n)!}{n!n!} > 2^n$, for all integers $n \geq 2$

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, $\frac{(2n)!}{n!n!} = \frac{4!}{2!2!} = \frac{24}{4} = 6 > 4 = 2^n$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\frac{(2n)!}{n!n!} > 2^n$, for $n = 2, \ldots, k$.

Inductive Step: By the inductive hypothesis, $\frac{(2k)!}{k!k!} > 2^k$.

Also notice that $2k+1 > k+1$ because $k \geq 0$. So $\frac{2k+1}{k+1} > 1$.

Then we can compute

$$
\begin{aligned}
\frac{(2(k+1))!}{(k+1)!(k+1)!} &= \frac{(2k+2)(2k+1)(2k)!}{(k+1)k!(k+1)k!} = \frac{(2k+2)(2k+1)}{(k+1)^2}\frac{(2k)!}{k!k!} \\
&> \frac{(2k+2)(2k+1)}{(k+1)^2}2^k \\
&= \frac{(k+1)(2k+1)}{(k+1)^2}2^{k+1} = \frac{2k+1}{k+1}2^{k+1} > 2^{k+1}
\end{aligned}
$$

So $\frac{(2(k+1))!}{(k+1)!(k+1)!} > 2^{k+1}$, which is what we needed to show.

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Use (strong) induction to prove the following claim:

Claim: For any natural number n and any real number x, where $0 < x < 1$, $(1-x)^n \geq 1-nx$.

Let x be a real number, where $0 < x < 1$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 0$, $(1-x)^n = (1-x)^0 = 1$ and $1 - nx = 1 + 0 = 1$. So $(1-x)^n \geq 1 - nx$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(1-x)^n \geq 1 - nx$ for any natural number $n \leq k$, where k is a natural number.

Inductive Step: By the inductive hypothesis $(1-x)^k \geq 1 - kx$. Notice that $(1-x)$ is positive since $0 < x < 1$. So $(1-x)^{k+1} \geq (1-x)(1-kx)$.

But $(1-x)(1-kx) = 1 - x - kx + kx^2 = 1 - (1+k)x + kx^2$.

And $1 - (1+k)x + kx^2 \geq 1 - (1+k)x$ because kx^2 is non-negative.

So $(1-x)^{k+1} \geq (1-x)(1-kx) \geq 1-(1+k)x$, and therefore $(1-x)^{k+1} \geq 1-(1+k)x$, which is what we needed to show.

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Use (strong) induction to prove the following claim:

Claim: $\displaystyle\sum_{p=1}^{n} \frac{1}{p} \;\le\; \frac{n}{2} + 1$, for any positive integer n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\displaystyle\sum_{p=1}^{n} \frac{1}{p} = 1$. Also $\frac{n}{2} + 1 = 1.5$, which is larger. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\displaystyle\sum_{p=1}^{n} \frac{1}{p} \;\le\; \frac{n}{2} + 1$, for $n = 1, \ldots, k$.

Inductive Step: In particular, by the inductive hypothesis, $\sum_{p=1}^{k} \frac{1}{p} \;\le\; \frac{k}{2} + 1$. Also notice that k is positive, so $k + 1 \ge 2$, and therefore $\frac{1}{k+1} \le \frac{1}{2}$. Thus $\frac{1}{k+1} - \frac{1}{2} \le 0$. So

$$
\begin{aligned}
\sum_{p=1}^{k+1} \frac{1}{p} &= \frac{1}{k+1} + \sum_{p=1}^{k} \frac{1}{p} \;\le\; \frac{1}{k+1} + \frac{k}{2} + 1 \\
&= \left(\frac{k+1}{2} - \frac{k+1}{2}\right) + \left(\frac{1}{k+1} + \frac{k}{2} + 1\right) && \text{based on backwards scratch work} \\
&= \left(\frac{k+1}{2} + 1\right) + \frac{1}{k+1} + \left(\frac{k}{2} - \frac{k+1}{2}\right) && \text{rearrange terms} \\
&= \left(\frac{k+1}{2} + 1\right) + \frac{1}{k+1} - \frac{1}{2} \\
&\le \frac{k+1}{2} + 1 && \text{because } \frac{1}{k+1} - \frac{1}{2} \le 0
\end{aligned}
$$

So $\sum_{p=1}^{k+1} \frac{1}{p} \le \frac{k+1}{2} + 1$, which is what we needed to show.

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

Claim: For any positive integer n and any positive reals $a_1, \ldots, a_n$,

$$\prod_{p=1}^{n}(1+a_p) \geq 1 + \sum_{p=1}^{n} a_p$$

Solution:

Proof by induction on n.

Base Case(s): At $n=1$, $\prod_{p=1}^{n}(1+a_p) = 1+a_1$ and $1+\sum_{p=1}^{n} a_p = 1+a_1$ so $\prod_{p=1}^{n}(1+a_p) \geq 1+\sum_{p=1}^{n} a_p$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=1}^{n}(1+a_p) \geq 1+\sum_{p=1}^{n} a_p$ for $n = 1, \ldots, k$ and any positive real numbers $a_1, \ldots, a_n$.

Inductive Step: Let $a_1, \ldots, a_{k+1}$ be positive real numbers. By the inductive hypothesis, we know that $\prod_{p=1}^{k}(1+a_p) \geq 1+\sum_{p=1}^{k} a_p$. Then we have

$$\begin{aligned}
\prod_{p=1}^{k+1}(1+a_p) &= (1+a_{k+1})\prod_{p=1}^{k}(1+a_p) \\
&\geq (1+a_{k+1})(1+\sum_{p=1}^{k} a_p) = 1 + a_{k+1} + a_{k+1}\sum_{p=1}^{k} a_p + \sum_{p=1}^{k} a_p \\
&\geq 1 + a_{k+1} + \sum_{p=1}^{k} a_p \quad \text{because all values } a_p \text{ are positive} \\
&= 1 + \sum_{p=1}^{k+1} a_p
\end{aligned}$$

So $\prod_{p=1}^{k+1}(1+a_p) \geq 1+\sum_{p=1}^{k+1} a_p$, which is what we needed to show.

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Discussion: **Friday** **11** **12** **1** **2** **3** **4**

(15 points) Use (strong) induction to prove the following claim:

Claim: For all integers $n \geq 2$, $(2n)! > 2^n n!$

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, $(2n)! = 4! = 24$. $2^n n! = 4 \cdot 2 = 8$. So $(2n)! > 2^n n!$

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(2n)! > 2^n n!$ for all $n = 2, 3, \ldots, k$ for some integer $k \geq 2$.

Inductive Step: Notice that $2k + 1 \geq 1$ because k is positive. And $(2k)! > 2^k k!$ by the induction hypothesis.

So then

$(2(k+1))! = (2k+2)(2k+1)(2k)! \geq (2k+2)(2k)! > (2k+2)(2^k k!) = (k+1)2^{k+1}k! = 2^{k+1}(k+1)!$.

So $(2(k+1))! > 2^{k+1}(k+1)!$ which is what we needed to show.

Name:____

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Discussion: **Friday** **11** **12** **1** **2** **3** **4**

(15 points) Use (strong) induction to prove the following claim:

Claim: For any natural number n and any real number $x > -1$, $(1+x)^n \geq 1 + nx$.

Let x be a real number with $x > -1$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 0$, $(1+x)^n = (1+x)^0 = 1$ and $1 + nx = 1 + 0 = 1$. So $(1+x)^n \geq 1 + nx$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(1+x)^n \geq 1 + nx$ for any natural number $n \leq k$, where k is a natural number.

Inductive Step: By the inductive hypothesis $(1+x)^k \geq 1 + kx$. Notice that $(1+x)$ is positive since $x > -1$. So $(1+x)^{k+1} \geq (1+x)(1+kx)$.

But $(1+x)(1+kx) = 1 + x + kx + kx^2 = 1 + (1+k)x + kx^2$.

And $1 + (1+k)x + kx^2 \geq 1 + (1+k)x$ because kx^2 is non-negative.

So $(1+x)^{k+1} \geq (1+x)(1+kx) \geq 1 + (1+k)x$, and therefore $(1+x)^{k+1} \geq 1 + (1+k)x$, which is what we needed to show.

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NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Use (strong) induction to prove the following claim:

Claim: For any natural number n and any real number $x > -1$, $(1+x)^n \geq 1 + nx$.

Let x be a real number with $x > -1$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 0$, $(1+x)^n = (1+x)^0 = 1$ and $1 + nx = 1 + 0 = 1$. So $(1+x)^n \geq 1 + nx$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(1+x)^n \geq 1 + nx$ for any natural number $n \leq k$, where k is a natural number.

Inductive Step: By the inductive hypothesis $(1+x)^k \geq 1 + kx$. Notice that $(1+x)$ is positive since $x > -1$. So $(1+x)^{k+1} \geq (1+x)(1+kx)$.

But $(1+x)(1+kx) = 1 + x + kx + kx^2 = 1 + (1+k)x + kx^2$.

And $1 + (1+k)x + kx^2 \geq 1 + (1+k)x$ because kx^2 is non-negative.

So $(1+x)^{k+1} \geq (1+x)(1+kx) \geq 1 + (1+k)x$, and therefore $(1+x)^{k+1} \geq 1 + (1+k)x$, which is what we needed to show.

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NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Use (strong) induction to prove the following claim:

Claim: For any positive integer n, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq \sqrt{n}$

You may use the fact that $\sqrt{n+1} \geq \sqrt{n}$ for any natural number n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} = 1$ Also $\sqrt{n} = 1$. So $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq \sqrt{n}$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq \sqrt{n}$ for $n = 1, 2, \ldots, k$, for some integer $k \geq 1$.

Inductive Step: $\sum_{p=1}^{k} \frac{1}{\sqrt{p}} \geq \sqrt{k}$ by the inductive hypothesis.

So

$$\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} = \frac{1}{\sqrt{k+1}} + \sum_{p=1}^{k} \frac{1}{\sqrt{p}} \geq \frac{1}{\sqrt{k+1}} + \sqrt{k} = \frac{1+\sqrt{k}\sqrt{k+1}}{\sqrt{k+1}} \geq \frac{1+\sqrt{k}\sqrt{k}}{\sqrt{k+1}} = \frac{1+k}{\sqrt{k+1}} = \sqrt{k+1}$$

So $\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} \geq \sqrt{k+1}$, which is what we needed to show.

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Let function $f : \mathbb{Z}^+ \to \mathbb{R}$ be defined by

$f(1) = f(2) = 1$

$f(n) = \frac{1}{2}f(n-1) + \frac{1}{f(n-2)}$

Use (strong) induction to prove that $1 \leq f(n) \leq 2$ for all positive integers n.

Hint: prove both inequalities together using one induction.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$ and $n = 2$, $f(n) = 1$. So $1 \leq f(n) \leq 2$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $1 \leq f(n) \leq 2$ for $n = 1, 2, \ldots k - 1$.

Inductive Step: From the inductive hypothesis, we know that $1 \leq f(k-1) \leq 2$ and $1 \leq f(k-2) \leq 2$.

So $\frac{1}{2} \leq \frac{1}{2}f(k-1) \leq \frac{1}{2} \cdot 2 = 1$ and $\frac{1}{2} \leq \frac{1}{f(k-2)} \leq \frac{1}{1} = 1$.

Using the upper bounds from these equations: $f(k) = \frac{1}{2}f(k-1) + \frac{1}{f(k-2)} \leq 1 + 1 = 2$.

Using the lower bounds from these equations: $f(k) = \frac{1}{2}f(k-1) + \frac{1}{f(k-2)} \geq \frac{1}{2} + \frac{1}{2} = 1$.

So $1 \leq f(k) \leq 2$, which is what we needed to show.

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Recall the following fact about real numbers

Triangle Inequality: For any real numbers x and y, $|x+y| \leq |x| + |y|$.

Use this fact and (strong) induction to prove the following claim:

Claim: For any real numbers $x_1, x_2, \ldots, x_n$ ($n \geq 2$), $|x_1+x_2+\ldots+x_n| \leq |x_1|+|x_2|+\ldots+|x_n|$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, the claim is exactly the Triangle Inequality, which we're assuming to hold.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|$ for any list of n real numbers $x_1, x_2, \ldots, x_n$, where $2 \leq n \leq k$.

Inductive Step: Let $x_1, x_2, \ldots, x_{k+1}$ be a list of $k + 1$ real numbers.

Using the Triangle Inequality, we get

$|x_1 + x_2 + \ldots + x_k + x_{k+1}| = |(x_1 + x_2 + \ldots + x_k) + x_{k+1}| \leq |(x_1 + x_2 + \ldots + x_k)| + |x_{k+1}|$

But, by the inductive hypothesis $|x_1 + x_2 + \ldots + x_k| \leq |x_1| + |x_2| + \ldots + |x_k|$.

Putting these two equations together, we get

$|x_1 + x_2 + \ldots + x_k + x_{k+1}| = |(x_1 + x_2 + \ldots + x_k) + x_{k+1}| \leq (|x_1| + |x_2| + \ldots + |x_k|) + |x_{k+1}|$.

So $|x_1 + x_2 + \ldots + x_k + x_{k+1}| \leq |x_1| + |x_2| + \ldots + |x_k| + |x_{k+1}|$, which is what we needed to show.

Name:____

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

Claim: $\prod_{p=1}^{n} \frac{2p-1}{2p} < \frac{1}{\sqrt{2n+1}}$ for all integers $n \geq 1$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\prod_{p=1}^{n} \frac{2p-1}{2p} = \frac{1}{2} = \frac{1}{\sqrt{4}} < \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{2n+1}}$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=1}^{n} \frac{2p-1}{2p} < \frac{1}{\sqrt{2n+1}}$ for $n = 1, \ldots, k$.

Inductive Step:

Notice that $(2k+1)(2k+3) = 4k^2 + 8k + 3 < 4k^2 + 8k + 4 = (2k+2)^2$.

So $\frac{2k+1}{(2k+2)^2} < \frac{1}{2k+3}$. So $\frac{(2k+1)^2}{(2k+2)^2} < \frac{2k+1}{2k+3}$.

Taking the square root of both sides gives us $\frac{2k+1}{2k+2} < \frac{\sqrt{2k+1}}{\sqrt{2k+3}}$. And therefore $\frac{2k+1}{2k+2}\frac{1}{\sqrt{2k+1}} < \frac{1}{\sqrt{2k+3}}$.

Using this fact and the inductive hypothesis, we have

$$\prod_{p=1}^{k+1} \frac{2p-1}{2p} = \frac{2k+1}{2k+2}\left(\prod_{p=1}^{k} \frac{2p-1}{2p}\right) < \frac{2k+1}{2k+2}\frac{1}{\sqrt{2k+1}} < \frac{1}{\sqrt{2k+3}}$$

So $\prod_{p=1}^{k+1} \frac{2p-1}{2p} < \frac{1}{\sqrt{2k+3}}$, which is what we needed to show.

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(15 points) Suppose that $0 < q < \frac{1}{2}$. Use (strong) induction to prove the following claim:

Claim: $(1+q)^n \leq 1 + 2^n q$, for all positive integers n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $(1+q)^n = 1+q$ Also $1 + 2^n q = 1 + 2q$. So $(1+q)^n \leq 1 + 2^n q$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(1+q)^n \leq 1+2^n q$, for $n = 1, 2, \ldots, k$.

Inductive Step: From the inductive hypothesis, we know that $(1+q)^k \leq 1 + 2^k q$.

At $n = k+1$, we have

$$\begin{aligned}(1+q)^{k+1} &= (1+q)(1+q)^k \leq (1+q)(1+2^k q) \\ &= 1 + q + 2^k q + 2^k q^2 = 1 + q(1 + 2^k + 2^k q)\end{aligned}$$

Recall that $q < \frac{1}{2}$, so $2^k q < 2^{k-1}$. Also notice that $1 \leq 2^{k-1}$. Using these facts, we get

$$(1+q)^{k+1} \quad \leq \quad = 1 + q(1 + 2^k + 2^k q) \leq 1 + q(2^{k-1} + 2^k + 2^{k-1}) = 1 + 2^{k+1} q$$

So $(1+q)^{k+1} \leq 1 + 2^{k+1} q$, which is what we needed to show.

CS 173, Fall 17 Examlet 10, Part A

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

Claim: For any positive integer n and any positive reals $a_1, \ldots, a_n$,

$$\prod_{p=1}^{n}(1+a_p) \geq 1 + \sum_{p=1}^{n} a_p$$

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\prod_{p=1}^{n}(1+a_p) = 1+a_1$ and $1+\sum_{p=1}^{n} a_p = 1+a_1$ so $\prod_{p=1}^{n}(1+a_p) \geq 1+\sum_{p=1}^{n} a_p$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=1}^{n}(1+a_p) \geq 1+\sum_{p=1}^{n} a_p$ for $n = 1, \ldots, k$ and any positive real numbers $a_1, \ldots, a_n$.

Inductive Step: Let $a_1, \ldots, a_{k+1}$ be positive real numbers. By the inductive hypothesis, we know that $\prod_{p=1}^{k}(1+a_p) \geq 1+\sum_{p=1}^{k} a_p$. Then we have

$$\begin{aligned}
\prod_{p=1}^{k+1}(1+a_p) &= (1+a_{k+1})\prod_{p=1}^{k}(1+a_p) \\
&\geq (1+a_{k+1})(1+\sum_{p=1}^{k} a_p) \;=\; 1 + a_{k+1} + a_{k+1}\sum_{p=1}^{k} a_p + \sum_{p=1}^{k} a_p \\
&\geq 1 + a_{k+1} + \sum_{p=1}^{k} a_p \quad \text{because all values } a_p \text{ are positive} \\
&= 1 + \sum_{p=1}^{k+1} a_p
\end{aligned}$$

So $\prod_{p=1}^{k+1}(1+a_p) \geq 1+\sum_{p=1}^{k+1} a_p$, which is what we needed to show.

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(15 points) Use (strong) induction to prove the following claim:

Claim: For all integers $n \geq 2$, $(2n)! > 2^n n!$

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, $(2n)! = 4! = 24$. $2^n n! = 4 \cdot 2 = 8$. So $(2n)! > 2^n n!$

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(2n)! > 2^n n!$ for all $n = 2, 3, \ldots, k$ for some integer $k \geq 2$.

Inductive Step: Notice that $2k + 1 \geq 1$ because k is positive. And $(2k)! > 2^k k!$ by the induction hypothesis.

So then

$(2(k+1))! = (2k+2)(2k+1)(2k)! \geq (2k+2)(2k)! > (2k+2)(2^k k!) = (k+1)2^{k+1}k! = 2^{k+1}(k+1)!$.

So $(2(k+1))! > 2^{k+1}(k+1)!$ which is what we needed to show.

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(15 points) Use (strong) induction to prove the following claim:

Claim: For any sets A_1, A_2, ..., A_n, $|A_1 \cup A_2 \cup \ldots \cup A_n| \leq |A_1| + |A_2| + \ldots + |A_n|$

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$ the claim reduces to $|A_1| \leq |A_1|$, which is clearly true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $|A_1 \cup A_2 \cup \ldots \cup A_n| \leq |A_1| + |A_2| + \ldots + |A_n|$, for any sets A_1, A_2, ..., A_n, where $n = 1, 2, \ldots, k$.

Inductive Step: Let A_1, A_2, ..., A_{k+1} be sets. Let $S = A_1 \cup A_2 \cup \ldots \cup A_k$.

We know that $|S \cup A_{k+1}| = |S| + |A_{k+1}| - |S \cap A_{k+1}|$ by the Inclusion-Exclusion formula. So $|S \cup A_{k+1}| \leq |S| + |A_{k+1}|$ because $|S \cap A_{k+1}|$ cannot be negative.

By the inductive hypothesis $|S| = |A_1 \cup A_2 \cup \ldots \cup A_k| \leq |A_1| + |A_2| + \ldots + |A_k|$.

So $|A_1 \cup A_2 \cup \ldots \cup A_{k+1}| = |S \cup A_{k+1}| \leq |S| + |A_{k+1}| \leq (|A_1| + |A_2| + \ldots + |A_k|) + |A_{k+1}|$.

So $|A_1 \cup A_2 \cup \ldots \cup A_{k+1}| \leq |A_1| + |A_2| + \ldots + |A_{k+1}|$, which is what we needed to prove.

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NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Use (strong) induction to prove the following claim. You may use the fact that $\sqrt{2} \le 1.5$.

Claim: For any positive integer n, $\displaystyle\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \ge 2\sqrt{n+1} - 2$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\displaystyle\sum_{p=1}^{n} \frac{1}{\sqrt{p}} = 1$. Also $2\sqrt{n+1} - 2 = 2\sqrt{2} - 2 \le 2 \cdot 1.5 - 2 = 1$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\displaystyle\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \ge 2\sqrt{n+1} - 2$ for $n = 1, 2, \ldots, k$.

Inductive Step: First, notice that $(\sqrt{k+1} - \sqrt{k+2}\,)^2 \ge 0$. Multiplying this out gives us $(k+1) - 2\sqrt{k+1}\sqrt{k+2} + (k+2) \ge 0$. So $2k+3 \ge 2\sqrt{k+1}\sqrt{k+2}$.

From the inductive hypothesis, we know that $\displaystyle\sum_{p=1}^{k} \frac{1}{\sqrt{p}} \ge 2\sqrt{k+1} - 2$. So then

$$\begin{aligned}
\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} &= \frac{1}{\sqrt{k+1}} + \sum_{p=1}^{k} \frac{1}{\sqrt{p}} \ \ge\ \frac{1}{\sqrt{k+1}} + 2\sqrt{k+1} - 2 \\
&= \frac{1}{\sqrt{k+1}} + \frac{2(k+1)}{\sqrt{k+1}} - 2 = \frac{1 + 2(k+1)}{\sqrt{k+1}} - 2 = \frac{2k+3}{\sqrt{k+1}} - 2 \\
&\ge \frac{2\sqrt{k+1}\sqrt{k+2}}{\sqrt{k+1}} - 2 = 2\sqrt{k+2} - 2
\end{aligned}$$

So $\displaystyle\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} \ge 2\sqrt{k+2} - 2$, which is what we needed to show.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Let function $f : \mathbb{Z}^+ \to \mathbb{N}$ be defined by

$f(1) = 0$

$f(n) = 1 + f(\lfloor n/2 \rfloor)$, for $n \geq 2$,

Use (strong) induction on n to prove that $f(n) \leq \log_2 n$ for any positive integer n. You cannot assume that n is a power of 2. However, you can assume that the log function is increasing (if $x \leq y$ then $\log x \leq \log y$) and that $\lfloor x \rfloor \leq x$.

Solution:

Proof by induction on n.

Base Case(s):

$f(1) = 0$ and $\log_2 1 = 0$ So $f(1) \leq \log_2 1$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) \leq \log_2 n$ for $n = 1, \ldots, k-1$.

Inductive Step:

We can assume that $k \geq 2$ (since we did $n = 1$ for the base case). So $\lfloor k/2 \rfloor$ must be at least 1 and less than k. Therefore, by the inductive hypothesis, $f(\lfloor k/2 \rfloor) \leq \log_2(\lfloor k/2 \rfloor)$.

We know that $f(k) = 1 + f(\lfloor k/2 \rfloor)$, by the definition of f. Substituting the result of the previous paragraph, we get that $f(k) \leq 1 + \log_2(\lfloor k/2 \rfloor)$.

$\lfloor k/2 \rfloor \leq k/2$. So $\log_2(\lfloor k/2 \rfloor) \leq \log_2(k/2) = (\log_2 k) + (\log_2 1/2) = (\log_2 k) - 1$.

Since $f(k) \leq 1 + \log_2(\lfloor k/2 \rfloor)$ and $\log_2(\lfloor k/2 \rfloor) \leq (\log_2 k) - 1$, $f(k) \leq 1 + (\log_2 k) - 1 = (\log_2 k)$. This is what we needed to show.

Name:____

NetID:_________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(15 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

Claim: For any positive integer n, and any positive reals $a_1, \ldots, a_n$,

$$\prod_{p=1}^{n}(1-a_p) \geq 1 - \sum_{p=1}^{n} a_p$$

Solution:

This problem should have also required that $a_1, \ldots, a_n$ be ≤ 1. This shouldn't have a major impact on grading because it looks like many folks assumed the critical step would work.

Proof by induction on n.

Base Case(s): At $n = 1$, $\prod_{p=1}^{n}(1-a_p) = 1-a_1$ and $1-\sum_{p=1}^{n} a_p = 1-a_1$ so $\prod_{p=1}^{n}(1-a_p) \geq 1-\sum_{p=1}^{n} a_p$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=1}^{n}(1-a_p) \geq 1-\sum_{p=1}^{n} a_p$ for $n = 1, \ldots, k$ and any real numbers $a_1, \ldots, a_n$ between 0 and 1 (inclusive).

Inductive Step: Let $a_1, \ldots, a_{k+1}$ be real numbers between 0 and 1 (inclusive). By the inductive hypothesis, we know that $\prod_{p=1}^{k}(1-a_p) \geq 1-\sum_{p=1}^{k} a_p$. Since $(1-a_{k+1})$ is positive, this means that $(1-a_{k+1})\prod_{p=1}^{k}(1-a_p) \geq (1-a_{k+1})(1-\sum_{p=1}^{k} a_p)$. Then we have

$$\begin{aligned}
\prod_{p=1}^{k+1}(1-a_p) &= (1-a_{k+1})\prod_{p=1}^{k}(1-a_p) \\
&\geq (1-a_{k+1})(1-\sum_{p=1}^{k} a_p) \;=\; 1-a_{k+1}+a_{k+1}\sum_{p=1}^{k} a_p - \sum_{p=1}^{k} a_p \\
&\geq 1-a_{k+1}-\sum_{p=1}^{k} a_p \quad \text{because all values } a_p \text{ are positive} \\
&= 1-\sum_{p=1}^{k+1} a_p
\end{aligned}$$

So $\prod_{p=1}^{k+1}(1-a_p) \geq 1-\sum_{p=1}^{k+1} a_p$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Use (strong) induction to prove the following claim:

Claim: $\frac{(2n)!}{n!n!} > 2^n$, for all integers $n \geq 2$

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, $\frac{(2n)!}{n!n!} = \frac{4!}{2!2!} = \frac{24}{4} = 6 > 4 = 2^n$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\frac{(2n)!}{n!n!} > 2^n$, for $n = 2, \ldots, k$.

Inductive Step: By the inductive hypothesis, $\frac{(2k)!}{k!k!} > 2^k$.

Also notice that $2k + 1 > k + 1$ because $k \geq 0$. So $\frac{2k+1}{k+1} > 1$.

Then we can compute

$$\begin{aligned}
\frac{(2(k+1))!}{(k+1)!(k+1)!} &= \frac{(2k+2)(2k+1)(2k)!}{(k+1)k!(k+1)k!} = \frac{(2k+2)(2k+1)}{(k+1)^2}\frac{(2k)!}{k!k!} \\
&> \frac{(2k+2)(2k+1)}{(k+1)^2}2^k \\
&= \frac{(k+1)(2k+1)}{(k+1)^2}2^{k+1} = \frac{2k+1}{k+1}2^{k+1} > 2^{k+1}
\end{aligned}$$

So $\frac{(2(k+1))!}{(k+1)!(k+1)!} > 2^{k+1}$, which is what we needed to show.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Use (strong) induction to prove the following claim:

Claim: For any natural number n and any real number x, where $0 < x < 1$, $(1-x)^n \geq 1-nx$.

Let x be a real number, where $0 < x < 1$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 0$, $(1-x)^n = (1-x)^0 = 1$ and $1 - nx = 1 + 0 = 1$. So $(1-x)^n \geq 1 - nx$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(1-x)^n \geq 1 - nx$ for any natural number $n \leq k$, where k is a natural number.

Inductive Step: By the inductive hypothesis $(1-x)^k \geq 1 - kx$. Notice that $(1-x)$ is positive since $0 < x < 1$. So $(1-x)^{k+1} \geq (1-x)(1-kx)$.

But $(1-x)(1-kx) = 1 - x - kx + kx^2 = 1 - (1+k)x + kx^2$.

And $1 - (1+k)x + kx^2 \geq 1 - (1+k)x$ because kx^2 is non-negative.

So $(1-x)^{k+1} \geq (1-x)(1-kx) \geq 1-(1+k)x$, and therefore $(1-x)^{k+1} \geq 1-(1+k)x$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Use (strong) induction to prove the following claim:

Claim: $(2n)!^2 < (4n)!$ for all positive integers.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $(2n)!^2 = (2!)^2 = 2^2 = 4$ And $(4n)! = 4! = 24$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(2n)!^2 < (4n)!$ for $n = 1, 2, \ldots, k$.

Inductive Step: At $n = k + 1$, we have

$(2(k+1))!^2 = (2k+2)!^2 = [(2k+2)(2k+1)(2k!)]^2 = (2k+2)(2k+2)(2k+1)(2k+1)(2k)!^2$

Also $(4(k+1))! = (4k+4)! = (4k+4)(4k+3)(4k+2)(4k+1)(4k)!$

Also notice that $(2k+2)(2k+2)(2k+1)(2k+1) < (4k+4)(4k+3)(4k+2)(4k+1)$ because each of the four terms on the left is smaller than the four terms on the right.

From the inductive hypothesis, we know that $(2k)!^2 < (4k)!$.

Putting this all together, we get

$$\begin{aligned}
(2(k+1))!^2 &= (2k+2)(2k+2)(2k+1)(2k+1)(2k)!^2 \\
&< (2k+2)(2k+2)(2k+1)(2k+1)(4k)! \\
&< (4k+4)(4k+3)(4k+2)(4k+1)(4k)! \\
&= (4(k+1))!
\end{aligned}$$

So $(2(k+1))!^2 < (4(k+1))!$, which is what we needed to prove.

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

(15 points) Use (strong) induction to prove the following claim:

Claim: $\sum_{p=1}^{n} \frac{1}{p^2} > \frac{3n}{2n+1}$ for all integers $n \geq 2$

Solution:

Lemma: Suppose $k \geq 2$. Then $\frac{3k}{2k+1} - \frac{3k-3}{2k-1} = \frac{6k-3k}{4k^2-1} - \frac{6k^2-3k-3}{4k^2-1} = \frac{3}{4k^2-1}$

Notice that $3k^2 < 4k^2 - 1$, since $k \geq 2$. So $\frac{3}{4k^2-1} < \frac{1}{k^2}$.

Combining these two equations, we get $\frac{3k}{2k+1} - \frac{3k-3}{2k-1} < \frac{1}{k^2}$.

Proof by induction on n.

Base Case(s): At $n = 2$, $\sum_{p=1}^{n} \frac{1}{p^2} = 1 + \frac{1}{4} > 1 + \frac{1}{5} = \frac{6}{5} = \frac{3n}{2n+1}$ So the claim holds at $n = 2$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{p=1}^{n} \frac{1}{p^2} > \frac{3n}{2n+1}$ for $n = 2, \ldots, k-1$.

Inductive Step:

By the inductive hypothesis $\sum_{p=1}^{k-1} \frac{1}{p^2} > \frac{3k-3}{2k-1}$.

By the lemma above, $\frac{1}{k^2} + \frac{3k-3}{2k-1} > \frac{3k}{2k+1}$.

So $\sum_{p=1}^{k} \frac{1}{p^2} = \frac{1}{k^2} + \sum_{p=1}^{k-1} \frac{1}{p^2} > \frac{1}{k^2} + \frac{3k-3}{2k-1} > \frac{3k}{2k+1}$.

So $\sum_{p=1}^{k} \frac{1}{p^2} > \frac{3k}{2k+1}$, which is what we needed to show.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Let function $f : \mathbb{Z}^+ \to \mathbb{R}$ be defined by

$f(1) = f(2) = 1$

$f(n) = \frac{1}{2}f(n-1) + \frac{1}{f(n-2)}$

Use (strong) induction to prove that $1 \leq f(n) \leq 2$ for all positive integers n.

Hint: prove both inequalities together using one induction.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$ and $n = 2$, $f(n) = 1$. So $1 \leq f(n) \leq 2$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $1 \leq f(n) \leq 2$ for $n = 1, 2, \ldots k-1$.

Inductive Step: From the inductive hypothesis, we know that $1 \leq f(k-1) \leq 2$ and $1 \leq f(k-2) \leq 2$.

So $\frac{1}{2} \leq \frac{1}{2}f(k-1) \leq \frac{1}{2} \cdot 2 = 1$ and $\frac{1}{2} \leq \frac{1}{f(k-2)} \leq \frac{1}{1} = 1$.

Using the upper bounds from these equations: $f(k) = \frac{1}{2}f(k-1) + \frac{1}{f(k-2)} \leq 1 + 1 = 2$.

Using the lower bounds from these equations: $f(k) = \frac{1}{2}f(k-1) + \frac{1}{f(k-2)} \geq \frac{1}{2} + \frac{1}{2} = 1$.

So $1 \leq f(k) \leq 2$, which is what we needed to show.

Name:

NetID: **Lecture: A B**

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

(15 points) Use (strong) induction to prove the following claim:

Claim: $\frac{(2n)!}{n!n!} < 4^n$, for all integers $n \geq 2$

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, $\frac{(2n)!}{n!n!} = \frac{4!}{2!2!} = \frac{24}{4} = 6 < 16 = 4^n$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\frac{(2n)!}{n!n!} < 4^n$, for $n = 2, \ldots, k$.

Inductive Step: By the inductive hypothesis, $\frac{(2k)!}{k!k!} > 4^k$.

Then we can compute

$$\begin{aligned}
\frac{(2(k+1))!}{(k+1)!(k+1)!} &= \frac{(2k+2)(2k+1)(2k)!}{(k+1)k!(k+1)k!} = \frac{(2k+2)(2k+1)}{(k+1)^2}\frac{(2k)!}{k!k!} \\
&< \frac{(2k+2)(2k+1)}{(k+1)^2}4^k \\
&< \frac{(2k+2)(2k+2)}{(k+1)^2}4^k = \frac{4(k+1)(k+1)}{(k+1)^2}4^k \\
&= 4 \cdot 4^k = 4^{k+1}
\end{aligned}$$

So $\frac{(2(k+1))!}{(k+1)!(k+1)!} < 4^{k+1}$, which is what we needed to show.