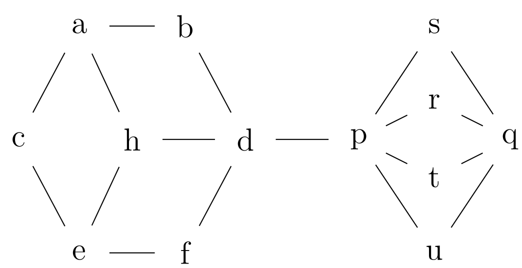
CS 173, Spring 19 Examlet 6, colored

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) How many isomorphisms are there from G (below) to itself? Justify your answer and/or show your work clearly .



Solution: Nodes d and p need to map to themselves. On the lefthand side, you can swap b and f (or not) and then the rest of the map is determined. On the righthand side, you can permute the nodes r, s, t, and u. So there are a total of $2 \cdot 4! = 48$ isomorphisms.

2. (5 points) The wheel graph W_{73} has 73 nodes on the rim. How many edges does it have?

Solution: It has 146 edges.

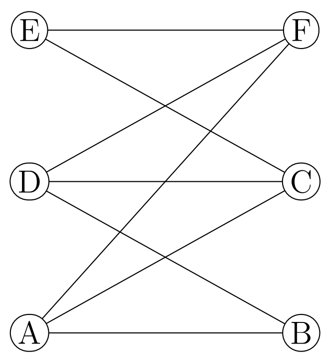
Name: ________________________________

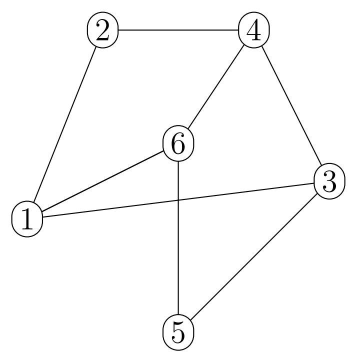
NetID: ____________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) Are graphs X and Y (below) isomorphic? Justify your answer.

Graph X Graph Y



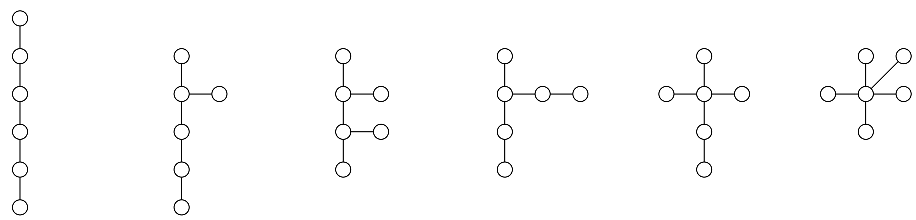


Solution: Yes, X and Y are isomorphic. We can map the nodes as follows

$f(A) = 6,\ f(B) = 5,\ f(C) = 1,\ f(D) = 3,\ f(E) = 2,\ f(F) = 4$

2. (5 points) Show four distinct (i.e. not isomorphic) graphs, each of which is connected and has six nodes and no cycles.

Solution: Any four out of the following set will work:



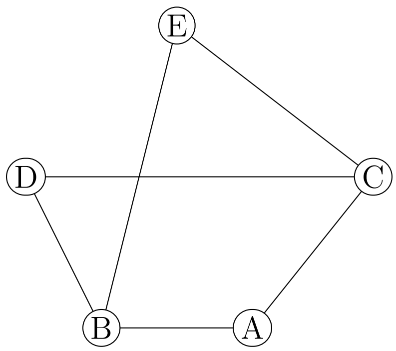
Name:____

NetID:____________________ **Lecture: A B**

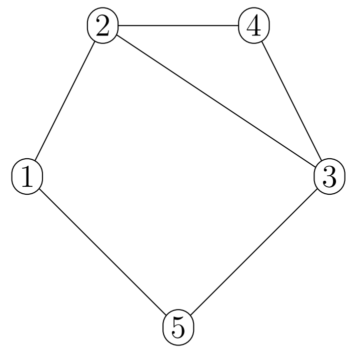
Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) Are graphs X and Y (below) isomorphic? Justify your answer.

Graph X



Graph Y



Solution: No, they are not isomorphic. Graph Y has a 5-cycle. Graph X is bipartite, so all of its cycles have even length.

2. (5 points) Is the cycle graph C_4 a subgraph of graph $K_{3,3}$? Briefly justify your answer.

Solution: Yes, it is. Pick two nodes on each side of $K_{3,3}$ and follow a path back-and-forth between the two sides.

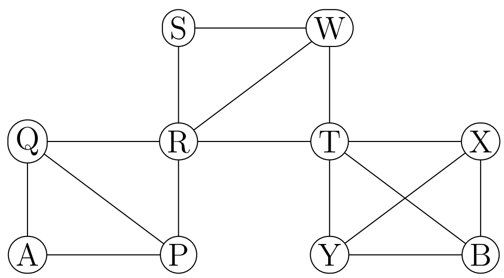
CS 173, Spring 19 **Examlet 6, colored**

Name:____

NetID:______________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) How many isomorphisms are there from G (below) to itself? Justify your answer and/or show your work clearly .



Solution: Nodes R and T must map to themselves. They are the only degree-5 nodes and only R is adjacent to a degree-2 node. So S, W, and A must also map to themselves. Q and P can be swapped. X, Y, and B can be permuted, creating 3! choices. So there are $2 \cdot 3!$ different isomorphisms of the whole graph.

2. (5 points) What is the difference between a path and an open walk?

Solution: A path uses each node only once. Open walks don't have to obey this constraint.

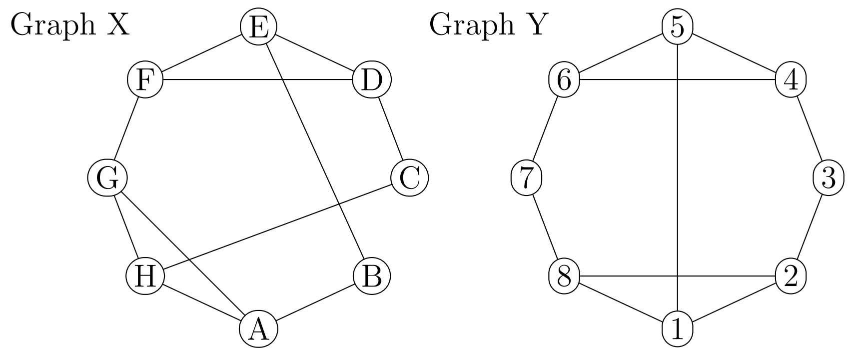
CS 173, Spring 19 Examlet 6, colored

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) Are graphs X and Y (below) isomorphic? Justify your answer.

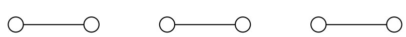


Solution: Yes, these graphs are isomorphic. Notice that the pairs of degree-2 nodes (B and C, 3 and 7) must be matched in one order or the other. Also notice that edge FG is distinctive because it connects the two triangles, so it must match 15 (in one direction or the other).

One possible map is f(C) = 7, f(B) = 3 for the degree-2 nodes. Then f(D) = 6, f(F) = 5, f(E) = 4 for one triangle. And f(H) = 8, f(A) = 2, and f(G) = 1 for the other triangle.

2. (5 points) The degree sequence of a graph is the list of the degrees of all the nodes in the graph, arranged in numerical order, largest to smallest. Suppose that graph G has degree sequence 1, 1, 1, 1, 1, 1. How many connected components does G have?

Solution: G must look as in the following picture. So G has three connected components.

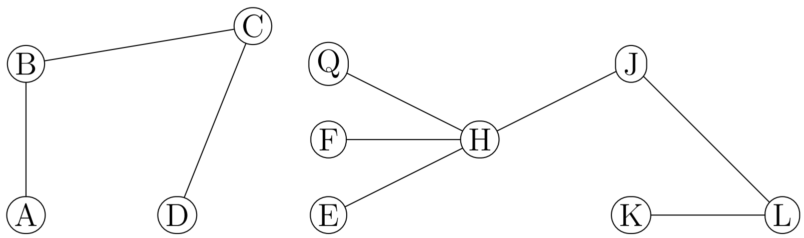


Name:____

NetID:______________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) How many isomorphisms are there from G (below) to itself? Justify your answer and/or show your work clearly .



Solution: B and C can be swapped or not (two choices). This determines the matches for A and D.

The three degree-one nodes directly connected to H (E, F, and Q) can be rotated, giving us 3! more choices.

J, K, and L must match themselves.

So the total number of choices is $2 \cdot 3! = 12$.

2. (5 points) Is the cycle graph C_{17} a subgraph of the wheel graph W_{23}? Briefly justify your answer.

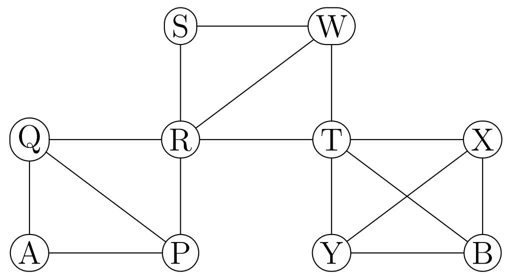
Solution: Yes, it is. Match 16 of the nodes in C_{17} with consecutive nodes on the rim of W_{23}. Then match the last node of C_{17} with the hub node of W_{23}.

Name:______________________________________

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) How many isomorphisms are there from G (below) to itself? Justify your answer and/or show your work clearly .



Solution: Nodes R and T must map to themselves. They are the only degree-5 nodes and only R is adjacent to a degree-2 node. So S, W, and A must also map to themselves. Q and P can be swapped. X, Y, and B can be permuted, creating 3! choices. So there are $2 \cdot 3!$ different isomorphisms of the whole graph.

2. (5 points) The degree sequence of a graph is the list of the degrees of all the nodes in the graph, arranged in numerical order, largest to smallest. Suppose graph G has degree sequence 1, 1, 1, 1, 2. Is G connected? Briefly justify your answer.

Solution: G isn't connected. The sum of the degrees is 6. So, by the Handshaking theorem, G has three edges. Suppose that we start with one base node. Connecting each of the other four nodes will require an edge, so at least four edges.

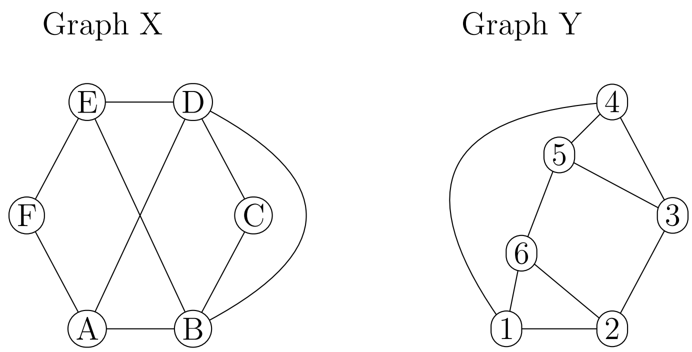
Or, if you prefer, G has to look as in the following picture. Two of the degree-1 nodes can be connected to the degree-2 node. But the only way to add two additional degree-1 nodes is to connect them to one another.

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) Are graphs X and Y (below) isomorphic? Justify your answer.



Solution: No, they are not isomorphic. Graph X has two nodes with degree 4. All the nodes in Graph Y have degree 3.

2. (5 points) Is the graph C_{10} bipartite? Briefly justify your answer.

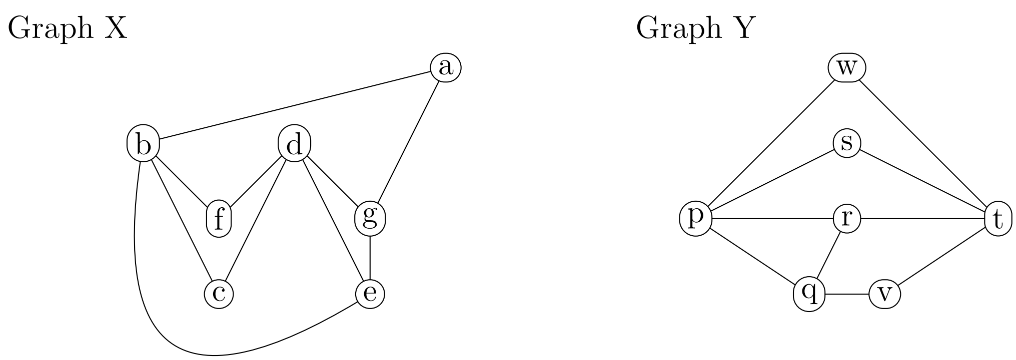
Solution: Yes, it is bipartite. As you walk around the cycle, assign nodes to the two subsets in an alternating manner.

Name:____

NetID:________________________ **Lecture: B**

Discussion: Friday 11 12 1 2 3 4

1. (10 points) Are graphs X and Y (below) isomorphic? Justify your answer.



Solution: Yes. We can map the nodes as follows:

f(b) = t, f(d) = p, f(f) = w, f(c) = s,, f(e) = r, f(g) = q), f(a) = v.

2. (5 points) The complete graph K_8 contains 8 nodes. How many edges does it have?

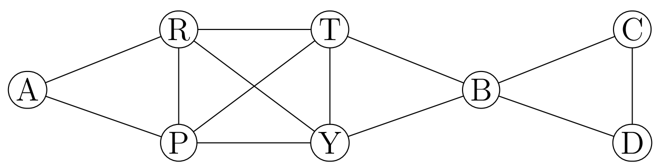
Solution: $\frac{8 \cdot 7}{2} = 28$

Name:____

NetID:______________________ **Lecture: B**

Discussion: Friday 11 12 1 2 3 4

1. (10 points) How many isomorphisms are there from G (below) to itself? Justify your answer and/or show your work clearly .



Solution: A and B must map onto themselves. R and P can swap (2 choices) T and Y can swap independently of R and P (2 choices). And C can also swap with D. So there are 8 choices total.

2. (5 points) Complete this statement of the Handshaking Theorem.

For any graph G with set of nodes V and set of edges E, ...

Solution: The sum of the degrees of all the nodes is equal to twice the number of edges.

Name: ____

NetID: ____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(10 points) Suppose that $g : \mathbb{N} \to \mathbb{N}$ is one-to-one. Let's define the function $f : \mathbb{N}^2 \to \mathbb{N}^2$ by the equation $f(x, y) = (x + g(y), g(x))$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

Solution: Let (x, y) and (a, b) be pairs of natural numbers and suppose that $f(x, y) = f(a, b)$.

By the definition of f, we know that $x + g(y) = a + g(b)$ and $g(x) = g(a)$.

Since g is one-to-one and $g(x) = g(a)$, $x = a$. Substituting this into $x + g(y) = a + g(b)$, we get $x + g(y) = x + g(b)$, so $g(y) = g(b)$.

Since g is one-to-one, $g(y) = g(b)$ implies that $y = b$.

Since $x = a$ and $y = b$, $(x, y) = (a, b)$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $g(x, y) = (f(x) + y, y + 3)$. Prove that g is onto.

Solution: Suppose that (a, b) is a pair of integers.

Consider $c = a - b + 3$. c is an integer, since a and b are integers. Since f is onto, this means there is an integer x such that $f(x) = c$.

Now, let $y = b - 3$. We can then calculate:

$$g(x, y) = (f(x) + y, y + 3) = (c + y, (b - 3) + 3) = ((a - b + 3) + (b - 3), b) = (a, b)$$

So we've found a point (x, y) such that $g(x, y) = (a, b)$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(10 points) Suppose that $f : (0, \infty) \to (\frac{5}{4}, \infty)$ is defined by $f(x) = \frac{5x^2+3}{4x^2}$. Proof that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

Solution: Let x and y be positive real numbers and suppose that $f(x) = f(y)$. By the definition of f, this translates into

$\frac{5x^2+3}{4x^2} = \frac{5y^2+3}{4y^2}$

So $\frac{5}{4} + \frac{3}{4}\frac{1}{x^2} = \frac{5}{4} + \frac{3}{4}\frac{1}{y^2}$

So $\frac{3}{4}\frac{1}{x^2} = \frac{3}{4}\frac{1}{y^2}$

So $\frac{1}{x^2} = \frac{1}{y^2}$

So $x^2 = y^2$.

Since x and y are known to be positive, this implies that $x = y$, which is what we needed to show.

CS 173, Spring 18 Examlet 6, Part A

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(10 points) Suppose that $f : \mathbb{Z}^2 \to \mathbb{Z}$ is defined by $f(x, y) = xy + yx^2 - x^2$. Prove that f is onto.

Solution:

Notice that $f(x, y) = xy + (y - 1)x^2$.

Let p be an integer. We need to find a pre-image for p.

Consider $m = (p, 1)$.

m is an element of $\mathbb{Z}^2$. We can compute

$$f(m) = p \cdot 1 + (1 - 1)p^2 = p + 0 \cdot p^2 = p$$

So m is a pre-image of p.

Since we can find a pre-image for an arbitrarily chosen integer, f is onto.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(10 points) Suppose that A and B are sets. Suppose that $f : B \to A$ and $g : A \to B$ are functions such that $f(g(x)) = x$ for every $x \in A$. Prove that g is one-to-one.

Solution: Let m and n be elements of A. Suppose that $g(m) = g(n)$.

Since $g(m) = g(n)$, $f(g(m)) = f(g(n))$ by substitution. Since $f(g(x)) = x$ for every $x \in A$, $f(g(m)) = m$ and $f(g(n)) = n$. So $f(g(m)) = f(g(n))$ implies that $m = n$.

Since $g(m) = g(n)$ implies that $m = n$ for any m and n in A, g is one-to-one, which is what we needed to prove.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(10 points) Suppose that $f : [0, \frac{1}{2}] \to [1, \frac{5}{2}]$ is defined by $f(x) = \frac{x^2+1}{1-2x^2}$ Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

Solution:

Let x and y be any numbers in $[0, \frac{1}{2}]$ and suppose $f(x) = f(y)$, that is

$$\begin{aligned} & \frac{x^2+1}{1-2x^2} = \frac{y^2+1}{1-2y^2} \\ \Rightarrow\ & (x^2+1)(1-2y^2) = (y^2+1)(1-2x^2) \\ \Rightarrow\ & x^2+1-2x^2y^2-2y^2 = y^2+1-2x^2y^2-2x^2 \\ \Rightarrow\ & 3x^2 = 3y^2 \\ \Rightarrow\ & x = y \end{aligned}$$

(The last step works because x and y are both positive.)

Therefore f is one-to-one.

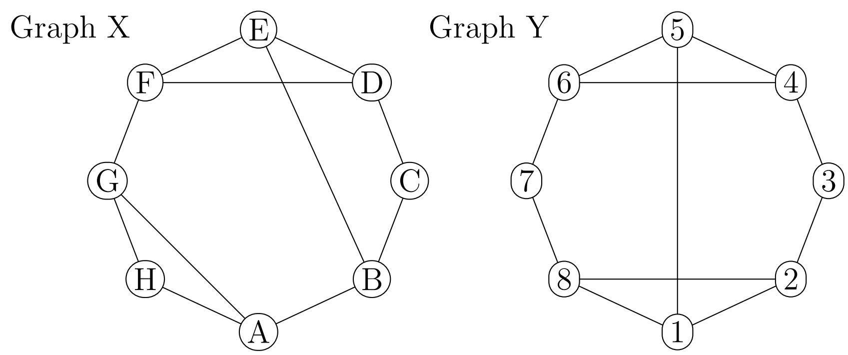
CS 173, Fall 17 Examlet 6, Part A

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

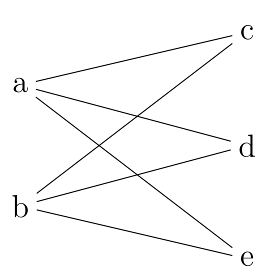
1. (10 points) Are graphs X and Y (below) isomorphic? Justify your answer.



Solution: No, they are not isomorphic. In graph X, one of the degree-2 nodes is part of a 3-cycle. In graph Y, neither degree-2 node is part of a 3-cycle.

2. (5 points) Draw a picture of the graph $K_{2,3}$.

Solution:

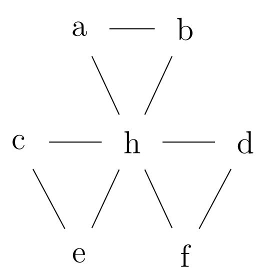


Name:____

NetID:______________________ **Lecture:** **A** **B**

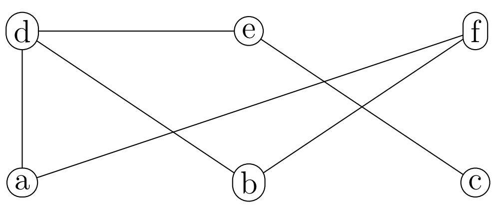
Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) How many isomorphisms are there from G (below) to itself? Justify your answer and/or show your work clearly .



Solution: Node h must map to itself. The three triangles can be permuted: 3! choices. Then each triangle can optionally be flipped over: 2 choices for each triangle. So we have a total of $3! \cdot 2^3$ isomorphisms.

2. (5 points) Is this graph bipartite? Briefly justify your answer.



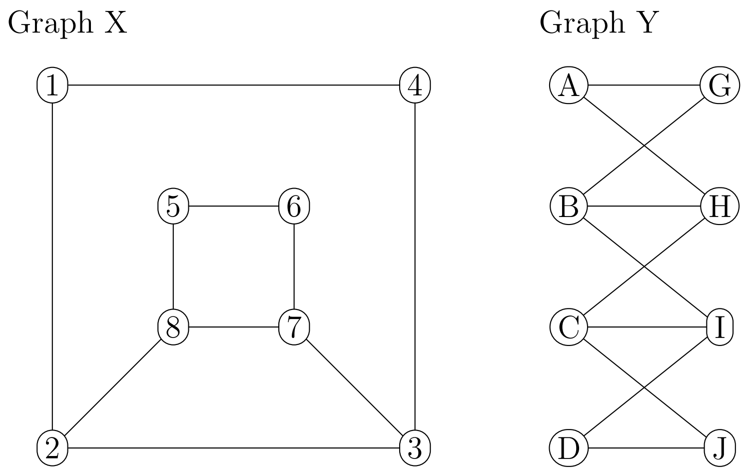
Solution: Yes, this is bipartite. Suppose we put nodes d, c, and f in the first set, and nodes a, b, and e in the second set. Then all edges connect a node from the first set to a node from the second set.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Are graphs X and Y (below) isomorphic? Justify your answer.



Solution: Yes, X and Y are isomorphic.

For example, use the node mapping $f(1) = A$, $f(4) = G$, $f(2) = H$, $f(3) = B$, $f(8) = C$, $f(7) = I$, $f(5) = J$, $f(6) = D$.

2. (5 points) The degree sequence of a graph is the list of the degrees of all the nodes in the graph, arranged in numerical order, largest to smallest. Is it possible to construct a (simple) graph with degree sequence: 4, 3, 3, 2, 0? Show how or briefly explain why this isn't possible.

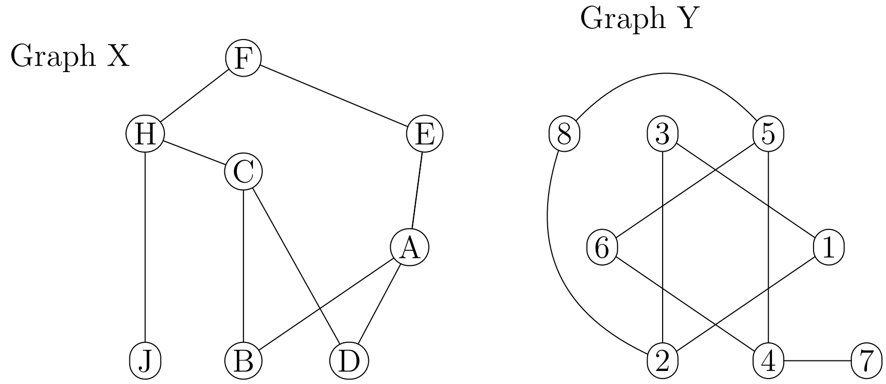
Solution: This isn't possible. Since one of the five nodes has degree 4, it's connected to all the other nodes. But then we can't have a node with degree 0.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Are graphs X and Y (below) isomorphic? Justify your answer.

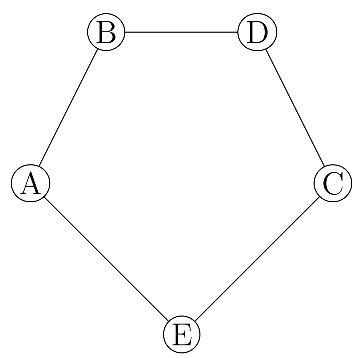


Solution: No, they are not isomorphic. Graph Y contains two 3-cycles but Graph X doesn't contain any 3-cycles.

2. (5 points) Suppose that $d(u, v)$ is the distance between nodes u and v (i.e. along the shortest path. Agent K claims that $d(u, v) + d(v, w) = d(u, w)$ for any nodes u, v, and w. Is he correct? Briefly explain why or give a counter-example.

Solution:

Agent K is wrong. Consider the graph to the right. $d(A, C) = 2$. $d(C, B)$ is also 2. So $d(A, C) + d(C, B) = 4$. But $d(A, B) = 1$. So $d(u, v) + d(v, w) \neq d(u, w)$ for these three nodes.



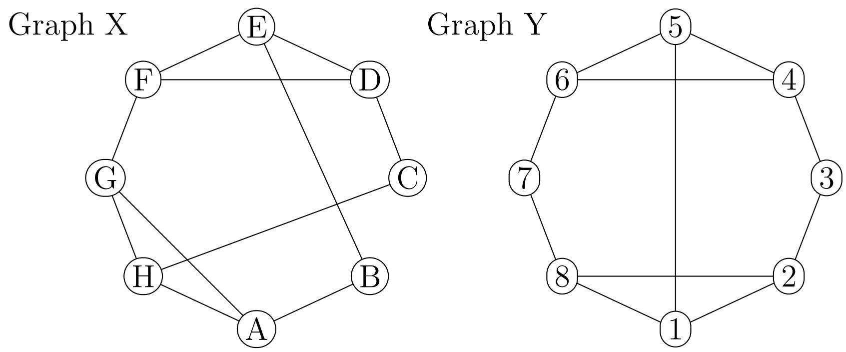
CS 173, Fall 17 Examlet 6, Part A

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Are graphs X and Y (below) isomorphic? Justify your answer.



Solution: Yes, these graphs are isomorphic. Notice that the pairs of degree-2 nodes (B and C, 3 and 7) must be matched in one order or the other. Also notice that edge FG is distinctive because it connects the two triangles, so it must match 15 (in one direction or the other).

One possible map is f(C) = 7, f(B) = 3 for the degree-2 nodes. Then f(D) = 6, f(F) = 5, f(E) = 4 for one triangle. And f(H) = 8, f(A) = 2, and f(G) = 1 for the other triangle.

2. (5 points) The degree sequence of a graph is the list of the degrees of all the nodes in the graph, arranged in numerical order, largest to smallest. Is it possible to construct a (simple) graph with degree sequence: 4, 3, 3, 2, 2, 1? Show how or briefly explain why this isn't possible.

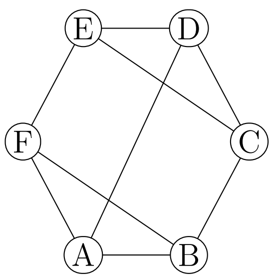
Solution: This isn't possible. By the Handshaking Theorem, the degrees have to add up to an even number. But these degrees add up to 15.

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) How many isomorphisms are there from G (below) to itself? Justify your answer and/or show your work clearly .



Solution: The triangle CDE can map onto itself or onto the triangle ABF (2 choices).

Having made that choice, we can permute the three nodes in CDE ($3! = 6$ choices). This determines the matches for the three nodes in the other triangle.

So there are $2 \cdot 3! = 12$ isomorphisms from the graph to itself.

2. (5 points) How many edges are in the complete bipartite graph $K_{10,5}$?

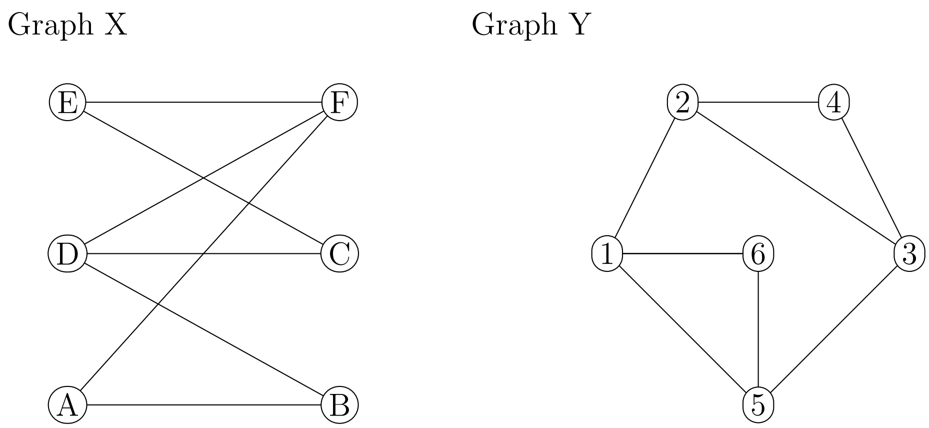
Solution: $10 \times 5 = 50$

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

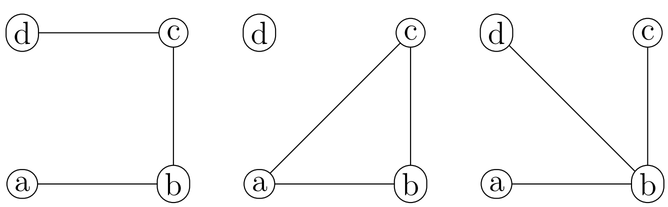
1. (10 points) Are graphs X and Y (below) isomorphic? Justify your answer.



Solution: No, they are not isomorphic. Graph X has 7 edges but graph Y has 8 edges. [And various other features fail to match as well.]

2. (5 points) Show three graphs, each with exactly four nodes and three edges, none of which are isomorphic.

Solution:

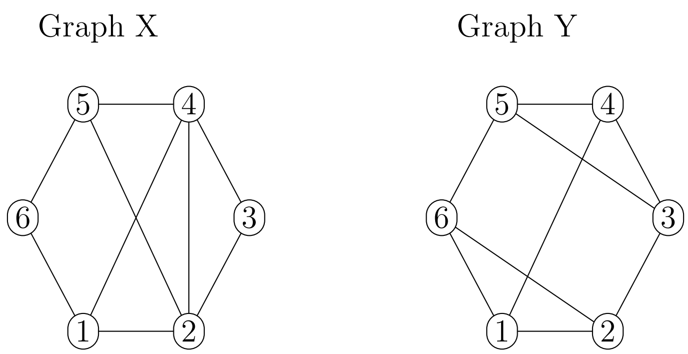


Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Are graphs X and Y (below) isomorphic? Justify your answer.



Solution: No, they are not isomorphic. Graph X has two nodes with degree 4. All the nodes in Graph Y have degree 3.

2. (5 points) The degree sequence of a graph is the list of the degrees of all the nodes in the graph, arranged in numerical order, largest to smallest. Is it possible to construct a graph with degree sequence: 5, 3, 2, 2, 2, 0? Show how or briefly explain why this isn't possible.

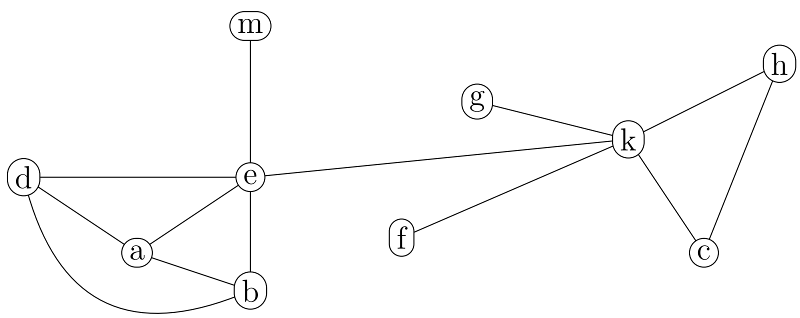
Solution: This isn't possible. Since one of the six nodes has degree 5, it's connected to all the other nodes. But then we can't have a node with degree 0.

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

1. (10 points) How many isomorphisms are there from G (below) to itself? Justify your answer and/or show your work clearly .

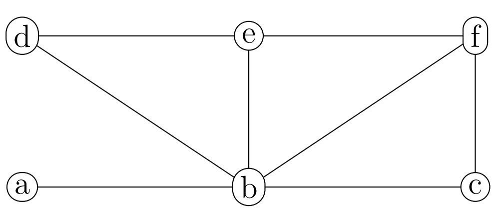


Solution: Nodes g and f can be swapped (2 choices). Nodes h and c can be swapped (2 choices). The nodes a, b, and d can be permuted (3! choices). So there are a total of $2 \cdot 2 \cdot 3! = 24$ isomorphisms.

2. (5 points) The degree sequence of a graph is the list of the degrees of all the nodes in the graph, arranged in numerical order, largest to smallest. Is it possible to construct a graph with degree sequence: 5, 3, 3, 2, 2, 1? Show how or briefly explain why this isn't possible.

Solution:

Yes. Here's a picture.



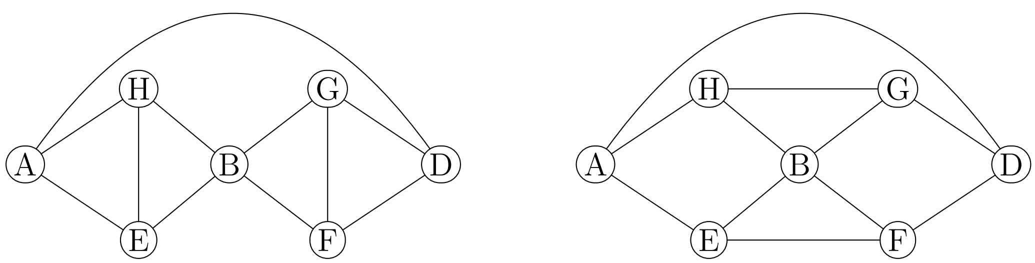
Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Are graphs X and Y (below) isomorphic? Justify your answer.

Graph X Graph Y



Solution: No, they are not isomorphic. Graph X contains four 3-cycles but Graph X contains only two 3-cycles.

2. (5 points) What is the difference between a cycle and a closed walk?

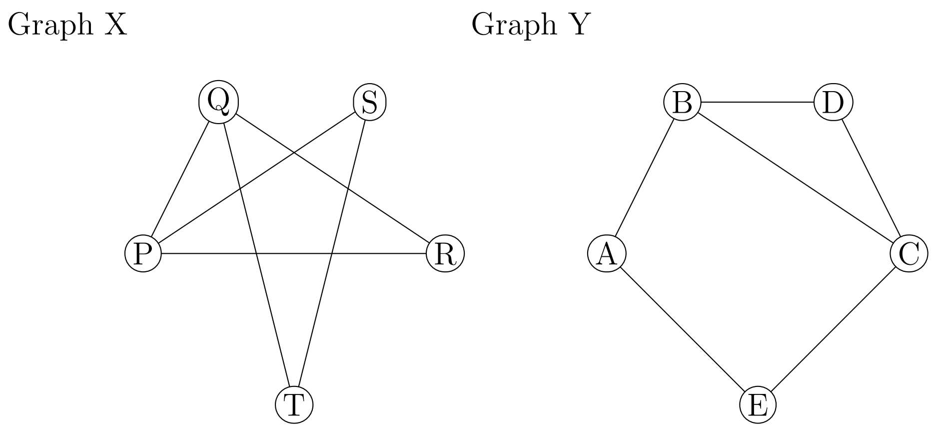
Solution: A cycle uses each node only once, except that the first and last nodes are the same. Also, a cycle must contain at least three nodes.

Name:____

NetID:____________________ Lecture: A B

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

1. (10 points) Are graphs X and Y (below) isomorphic? Justify your answer.



Solution: Yes, they are isomorphic. For example, use the map f, where f(R) = D, f(Q) = B, f(P) = C, f(S) = E, and f(T) = A.

2. (5 points) If G is a graph, its complement G' has the same nodes as G but G' has an edge between nodes x and y if and only G does not have an edge between x and y. Give a succinct high-level description of the complement of $K_{2,3}$. Briefly justify or show work.

Solution: Let's label the nodes a, b, c, d, and e, where a and b form one component of the bipartite graph and c, d, and e form the other. Then the complement contains all edges within a single component: ab, ad, de, and ec. So the complement consists of a 3-cycle, plus two nodes joined by an edge.

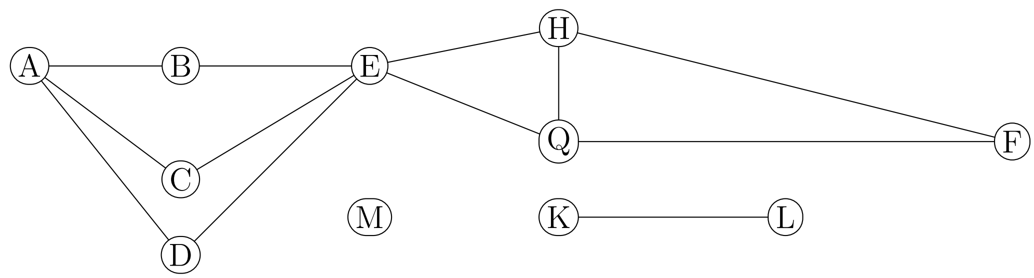
[A picture would also be a good way to justify your answer.]

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) How many isomorphisms are there from G (below) to itself? Justify your answer and/or show your work clearly .



Solution: Node E must map to itself. Nodes B, C, and D can be permuted (3! choices). Nodes H and Q can be swapped (2 choices), as can nodes K and L (2 choices). So there are $2 \cdot 2 \cdot 3! = 24$ isomorphisms from the graph to itself.

2. (5 points) What is the diameter of C_n?

Solution: The diameter of C_n is $n/2$ if n is even, and $(n-1)/2$ if n is odd.