CS 173, Spring 19 Examlet 7, colored

Name:____

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Working directly from the definition of divides, use (strong) induction to prove the following claim:

Claim: $n^3 + 5n$ is divisible by 6, for all positive integers n.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $n^3 + 5n = 6$, which is clearly divisible by 6.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $n^3 + 5n$ is divisible by 6, for $n = 1, 2, \ldots, k$.

Rest of the inductive step: Notice that

$$(k+1)^3 + 5(k+1) = (k^3 + 3k^2 + 3k + 1) + (5k + 5) = (k^3 + 5k) + (3k^2 + 3k) + 6 = (k^3 + 5k) + 3k(k+1) + 6$$

$(k^3 + 5k)$ is divisible by 6 by the inductive hypothesis. $3k(k + 1)$ is divisible by 6 because one of k and (k+1) must be even. 6 is obviously divisible by 6. Since $(k + 1)^3 + 5(k + 1)$ is the sum of these three terms, it must also be divisible by 6, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim.

Claim: For any positive integer n, $\sum_{p=1}^{n} \log(p^2) = 2\log(n!)$

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $\sum_{p=1}^{n} \log(p^2) = \log(1^2) = \log 1 = 0$. Also $2\log(n!) = 2\log 1 = 2 \cdot 0 = 0$. So the two are equal.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} \log(p^2) = 2\log(n!)$. for $n = 1, \ldots, k$.

Rest of the inductive step: In particular, $\sum_{p=1}^{k} \log(p^2) = 2\log(k!)$. Then

$$\begin{aligned}
\sum_{p=1}^{k+1} \log(p^2) &= \log((k+1)^2) + \sum_{p=1}^{k} \log(p^2) = 2\log(k+1) + \sum_{p=1}^{k} \log(p^2) \\
&= 2\log(k+1) + 2\log(k!) = 2(\log(k+1) + \log(k!)) \\
&= 2\log((k+1)k!) = 2\log((k+1)!)
\end{aligned}$$

So $\sum_{p=1}^{k+1} \log(p^2) = 2\log((k+1)!)$, which is what we needed to show.

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(15 points) Let A be a constant integer. Use (strong) induction to prove the following claim. Remember that $0! = 1$.

Claim: For any integer $n \geq A$, $\sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{(n+1)!}{(A+1)!(n-A)!}$

Solution: Proof by induction on n.

Base case(s): At $n = A$, $\sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{A!}{A!0!} = 1 = \frac{(A+1)!}{(A+1)!0!} = \frac{(n+1)!}{(A+1)!(n-A)!}$

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{(n+1)!}{(A+1)!(n-A)!}$ is true for all $n = A, \ldots, k$.

Rest of the inductive step:

In particular, $\sum_{p=A}^{k} \frac{p!}{A!(p-A)!} = \frac{(k+1)!}{(A+1)!(k-A)!}$. So then

$$\begin{aligned}
\sum_{p=A}^{k+1} \frac{p!}{A!(p-A)!} &= \sum_{p=A}^{k} \frac{p!}{A!(p-A)!} + \frac{(k+1)!}{A!(k+1-A)!} \\
&= \frac{(k+1)!}{(A+1)!(k-A)!} + \frac{(k+1)!}{A!(k+1-A)!} \\
&= \frac{(k+1-A)(k+1)!}{(A+1)!(k+1-A)!} + \frac{(A+1)(k+1)!}{(A+1)!(k+1-A)!} \\
&= \frac{(k+2)(k+1)!}{(A+1)!(k+1-A)!} = \frac{(k+2)!}{(A+1)!(k+1-A)!}
\end{aligned}$$

So $\sum_{p=A}^{k+1} \frac{p!}{A!(p-A)!} = \frac{(k+2)!}{(A+1)!(k+1-A)!}$, which is what we needed to show.

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(15 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

$\prod_{p=2}^{n}(1-\frac{1}{p^2}) = \frac{n+1}{2n}$ for any integer $n \geq 2$.

Solution: Proof by induction on n.

Base case(s): At $n=2$, $\prod_{p=2}^{n}(1-\frac{1}{p^2}) = (1-\frac{1}{4}) = \frac{3}{4}$ and $\frac{n+1}{2n} = \frac{3}{4}$, so the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=2}^{n}(1-\frac{1}{p^2}) = \frac{n+1}{2n}$ for $n = 2, \ldots, k$

Rest of the inductive step: In particular, from the inductive hypothesis $\prod_{p=2}^{k}(1-\frac{1}{p^2}) = \frac{k+1}{2k}$.

So

$$\begin{aligned}
\prod_{p=2}^{k+1}(1-\frac{1}{p^2}) &= (\prod_{p=2}^{k}(1-\frac{1}{p^2}))(1-\frac{1}{(k+1)^2}) \\
&= (\frac{k+1}{2k})(1-\frac{1}{(k+1)^2}) = \frac{k+1}{2k} - \frac{k+1}{2k(k+1)^2} \\
&= \frac{k+1}{2k} - \frac{1}{2k(k+1)} = \frac{(k+1)^2}{2k(k+1)} - \frac{1}{2k(k+1)} \\
&= \frac{(k+1)^2-1)}{2k(k+1)} = \frac{k^2+2k)}{2k(k+1)} = \frac{k(k+2)}{2k(k+1)} = \frac{k+2}{2(k+1)}
\end{aligned}$$

So $\prod_{p=2}^{k+1}(1-\frac{1}{p^2}) = \frac{k+2}{2(k+1)}$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: for all natural numbers n, $\sum_{j=0}^{n} 2(-7)^j = \frac{1-(-7)^{n+1}}{4}$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $\sum_{j=0}^{n} 2(-7)^j = 2$ and $\frac{1-(-7)^{n+1}}{4} = \frac{1-(-7)}{4} = 2$. So the claim holds at $n = 0$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{j=0}^{n} 2(-7)^j = \frac{1-(-7)^{n+1}}{4}$ for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

In particular $\sum_{j=0}^{k} 2(-7)^j = \frac{1-(-7)^{k+1}}{4}$. So then

$$\begin{aligned}
\sum_{j=0}^{k+1} 2(-7)^j &= \left(\sum_{j=0}^{n} 2(-7)^j\right) + 2(-7)^{k+1} \\
&= \frac{1-(-7)^{k+1}}{4} + 2(-7)^{k+1} = \frac{1-(-7)^{k+1} + 8(-7)^{k+1}}{4} = \frac{1+7(-7)^{k+1}}{4} \\
&= \frac{1-(-7)^{k+2}}{4}
\end{aligned}$$

So $\sum_{j=0}^{k+1} 2(-7)^j = \frac{1-(-7)^{k+2}}{4}$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim.

Claim: For any positive integer n, $\displaystyle\sum_{p=1}^{n} \frac{1}{\sqrt{p-1}+\sqrt{p}} = \sqrt{n}$

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $\displaystyle\sum_{p=1}^{n} \frac{1}{\sqrt{p-1}+\sqrt{p}} = \frac{1}{\sqrt{1-1}+\sqrt{1}} = \frac{1}{\sqrt{0}+\sqrt{1}} = 1$ and $\sqrt{n} = \sqrt{1} = 1$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\displaystyle\sum_{p=1}^{n} \frac{1}{\sqrt{p-1}+\sqrt{p}} = \sqrt{n}$ for $n = 1, \ldots, k$.

Rest of the inductive step: In particular, $\displaystyle\sum_{p=1}^{k} \frac{1}{\sqrt{p-1}+\sqrt{p}} = \sqrt{k}$. So then

$$\begin{aligned}
\sum_{p=1}^{k+1} \frac{1}{\sqrt{p-1}+\sqrt{p}} &= \frac{1}{\sqrt{k}+\sqrt{k+1}} + \sum_{p=1}^{k} \frac{1}{\sqrt{p-1}+\sqrt{p}} = \frac{1}{\sqrt{k}+\sqrt{k+1}} + \sqrt{k} \\
&= \frac{\sqrt{k}-\sqrt{k+1}}{k-(k+1)} + \sqrt{k} = \frac{\sqrt{k}-\sqrt{k+1}}{-1} + \sqrt{k} \\
&= (\sqrt{k+1}-\sqrt{k}) + \sqrt{k} = \sqrt{k+1}
\end{aligned}$$

So $\displaystyle\sum_{p=1}^{k+1} \frac{1}{\sqrt{p-1}+\sqrt{p}} = \sqrt{k+1}$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$ for all positive integers n.

Solution:

Proof by induction on n.

Base case(s): $n = 1$. At $n = 1$, $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$. Also $\frac{n}{n+1} = \frac{1}{2}$. So the two sides of the equation are equal.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$ for $n = 1, \ldots, k$ for some integer $k \geq 1$.

Rest of the inductive step:

Consider $\sum_{j=1}^{k+1} \frac{1}{j(j+1)}$. By removing the top term of the summation and then applying the inductive hypothesis, we get

$$\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \frac{1}{(k+1)(k+2)} + \sum_{j=1}^{k} \frac{1}{j(j+1)} = \frac{1}{(k+1)(k+2)} + \frac{k}{k+1}.$$

Adding the two fractions together:

$$\frac{1}{(k+1)(k+2)} + \frac{k}{k+1} = \frac{1}{(k+1)(k+2)} + \frac{k(k+2)}{(k+1)(k+2)} = \frac{1+k(k+2)}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

So $\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \frac{k+1}{k+2}$, which is what we needed to show.

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(15 points) Use (strong) induction and the fact that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ to prove the following claim:

For all natural numbers n, $(\sum_{i=0}^{n} i)^2 = \sum_{i=0}^{n} i^3$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $(\sum_{i=0}^{n} i)^2 = 0^2 = 0 = \sum_{i=0}^{n} i^3$. So the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:
Suppose that $(\sum_{i=0}^{n} i)^2 = \sum_{i=0}^{n} i^3$ for $n = 0, 1, \ldots, k$.

Rest of the inductive step:
Starting with the lefthand side of the equation for $n = k + 1$, we get

$$\left(\sum_{i=0}^{k+1} i\right)^2 = \left((k+1) + \sum_{i=0}^{k} i\right)^2 = (k+1)^2 + 2(k+1)\sum_{i=0}^{k} i + \left(\sum_{i=0}^{k} i\right)^2$$

By the inductive hypothesis $\left(\sum_{i=0}^{k} i\right)^2 = \sum_{i=0}^{k} i^3$. Substituting this and the fact we were told to assume, we get

$$\left(\sum_{i=0}^{k+1} i\right)^2 = (k+1)^2 + 2(k+1)\frac{k(k+1)}{2} + \sum_{i=0}^{k} i^3 = (k+1)^2 + k(k+1)^2 + \sum_{i=0}^{k} i^3 = (k+1)^3 + \sum_{i=0}^{k} i^3 = \sum_{i=0}^{k+1} i^3$$

So $\left(\sum_{i=0}^{k+1} i\right)^2 = \sum_{i=0}^{k+1} i^3$ which is what we needed to show.

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Use (strong) induction to prove the following claim:

Claim: For all integers a, b, n, $n \geq 1$, if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$.

Use this definition in your proof: $x \equiv y \pmod p$ if and only if $x = y + kp$ for some integer k.

Solution:

Proof by induction on n.

Base case(s): At $n = 1$, our claim becomes "if $a \equiv b \pmod 7$ then $a \equiv b \pmod 7$" which is clearly true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$, for all integers a, b, n, where $n = 1, \ldots, k$,

a and b need to be introduced at some point in this proof, but there's several places you might do this. For example, you could say "let a and b be integers" right at the start. Then your inductive hypothesis would just be "if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$, for $n = 1, \ldots, k$." We won't get picky about this when grading.

Rest of the inductive step:

Let a and b be integers.

Suppose that $a \equiv b \pmod 7$. then $a = b + 7p$ for some integer p.

From the inductive hypothesis, we know that $a^k \equiv b^k \pmod 7$, So $a^k = b^k + 7q$ for some integer q.

Combining these two equations, we get that

$$a^{k+1} = (b + 7p)(b^k + 7q) = b^{k+1} + 7(pb^k + bq + 7pq)$$

$pb^k + bq + 7pq$ is an integer since p, q, and b are integers. So we know that $a^{k+1} \equiv b^{k+1} \pmod 7$, which is what we needed to prove.

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Use (strong) induction to prove the following claim

Claim: $\sum_{k=0}^{n} p^k = \frac{p^{n+1}-1}{p-1}$, for all natural numbers n and all real numbers $p \neq 1$.

Solution: Proof by induction on n.

Base case(s): at $n = 0$, $\sum_{k=0}^{n} p^k = p^0 = 1$. And $\frac{p^{n+1}-1}{p-1} = \frac{p-1}{p-1} = 1$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{k=0}^{n} p^k = \frac{p^{n+1}-1}{p-1}$, all real numbers $p \neq 1$. and all natural numbers $n = 0, \ldots, j$.

p needs to be introduced somewhere, but there are several options. For example, you could say "let p be a real number $\neq 1$" before you give the inductive hypothesis. We won't be picky about this when grading.

Notice that the moving variable for the summation is k, so you can't also use k for the bound on the induction variable. You need to use a fresh variable name for one of the two.

Rest of the inductive step: Let p be a real number $\neq 1$.

Then $\sum_{k=0}^{j+1} p^k = p^{j+1} + \sum_{k=0}^{j} p^k$

By the inductive hypothesis, we know that $\sum_{k=0}^{j} p^k == \frac{p^{j+1}-1}{p-1}$. Substituting this into the previous equation, we get

$$\sum_{k=0}^{j+1} p^k = p^{j+1} + \frac{p^{j+1}-1}{p-1} = \frac{p^{j+1}(p-1) + p^{j+1} - 1}{p-1} = \frac{p^{j+2} - p^{j+1} + p^{j+1} - 1}{p-1} = \frac{p^{j+2}-1}{p-1}$$

So $\sum_{k=0}^{j+1} p^k = \frac{p^{j+2}-1}{p-1}$ which is what we needed to show.

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Use (strong) induction to prove the following claim:

Claim: $\sum_{p=0}^{n}(p \cdot p!) = (n+1)! - 1$, for all natural numbers n.

Recall that 0! is defined to be 1.

Solution: Proof by induction on n.

Base case(s):

At $n = 0$, $\sum_{p=0}^{n}(p \cdot p!) = 0 \cdot 0! = 0$ Also $(n+1)! - 1 = 1! - 1 = 1 - 1 = 0$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=0}^{n}(p \cdot p!) = (n+1)! - 1$, for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

By the inductive hypothesis $\sum_{p=0}^{k}(p \cdot p!) = (k+1)! - 1$. So

$$\begin{aligned}
\sum_{p=0}^{k+1}(p \cdot p!) &= ((n+1) \cdot (n+1)!) + \sum_{p=0}^{k}(p \cdot p!) \\
&= ((k+1) \cdot (k+1)!) + \sum_{p=0}^{k}(p \cdot p!) \\
&= (n+1) \cdot (k+1)! + (k+1)! - 1 \\
&= (k+1) \cdot (k+1)! + (k+1)! - 1 \\
&= [(k+1) + 1] \cdot (k+1)! - 1 \\
&= (k+2) \cdot (k+1)! - 1 = (k+2)! - 1
\end{aligned}$$

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Use (strong) induction to prove the following claim:

Claim: $\displaystyle\sum_{j=2}^{n} \frac{1}{j(j-1)} = \frac{n-1}{n}$ for all integers $n \geq 2$.

Solution:

Proof by induction on n.

Base case(s): $n = 2$. At $n = 2$, $\sum_{j=2}^{n} \frac{1}{j(j-1)} = \frac{1}{2}$. Also $\frac{n-1}{n} = \frac{1}{2}$. So the two sides of the equation are equal.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{j=2}^{n} \frac{1}{j(j-1)} = \frac{n-1}{n}$. for $n = 2, \ldots, k$ for some integer $k \geq 2$.

Rest of the inductive step:

Consider $\sum_{j=2}^{k+1} \frac{1}{j(j-1)}$.

By removing the top term of the summation and then applying the inductive hypothesis, we get $\sum_{j=2}^{k+1} \frac{1}{j(j-1)} = \frac{1}{(k+1)k} + \sum_{j=2}^{k} \frac{1}{j(j-1)} = \frac{1}{(k+1)k} + \frac{k-1}{k}$.

Adding the two fractions together:

$\frac{1}{(k+1)k} + \frac{k-1}{k} = \frac{1}{(k+1)k} + \frac{(k+1)(k-1)}{(k+1)k} = \frac{1}{(k+1)k} + \frac{k^2-1}{(k+1)k} = \frac{k^2}{(k+1)k} = \frac{k}{(k+1)}$

So $\sum_{j=2}^{k+1} \frac{1}{j(j-1)} = \frac{k}{(k+1)}$ which is what we needed to show.

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Working directly from the definition of divides, use (strong) induction to prove the following claim:

Claim: $(4n)!$ is divisible by 8^n, for all positive integers n.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, the claim amounts to "4! is divisible by 8." $4! = 24$ which is clearly divisible by 8.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(4n)!$ is divisible by 8^n, for $n = 1, 2, \ldots, k$.

Rest of the inductive step: At $n = k + 1$, $(4n)! = (4(k + 1))! = (4k + 4)! = (4k + 4)(4k + 3)(4k + 2)(4k + 1)(4k)!$

Now, $(4k + 4)$ is divisible by 4, and $(4k + 2)$ is divisible by 2. So $(4k + 4)(4k + 3)(4k + 2)(4k + 1)$ is divisible by 8. By the inductive hypothesis, we know that $(4k)!$ is divisible by 8^k. Combining these two facts, $(4(k + 1))!$ is divisible by 8^{k+1}, which is what we needed to show.

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Use (strong) induction to prove the following claim:

Claim: $\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$, for all positive integers n.

Solution: Proof by induction on n.

Base case(s): $n = 1$. At $n = 1$, $\sum_{j=1}^{n} j(j+1) = 1(1+1) = 2$ Also, $\frac{n(n+1)(n+2)}{3} = \frac{1\cdot 2\cdot 3}{3} = 2$. So the two sides of the equation are equal at $n = 1$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$, for $n = 1, \ldots, k$, for some integer $k \geq 1$.

Rest of the inductive step:

Consider $\sum_{j=1}^{k+1} j(j+1)$. By removing the top term of the summation and applying the inductive hypothesis, we get

$$\sum_{j=1}^{k+1} j(j+1) = (k+1)(k+2) + \sum_{j=1}^{k} j(j+1) = (k+1)(k+2) + \frac{k(k+1)(k+2)}{3}$$

Simplifying the algebra:

$(k+1)(k+2) + \frac{k(k+1)(k+2)}{3} = \frac{3(k+1)(k+2)}{3} + \frac{k(k+1)(k+2)}{3} = \frac{3(k+1)(k+2)\ +\ k(k+1)(k+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3}$

So $\sum_{j=1}^{k+1} j(j+1) = \frac{(k+1)(k+2)(k+3)}{3}$, which is what we needed to show.

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Use (strong) induction to prove the following claim:

For all positive integers n, $\sum_{p=1}^{n} p2^p = (n-1)2^{n+1} + 2$.

Solution: Proof by induction on n.

Base case(s): $n = 1$. Then $\sum_{p=1}^{n} p2^p = 1 \cdot 2^1 = 2$ and $(n-1)2^{n+1} + 2 = 0 \cdot 2^2 + 2 = 2$. So the equation holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} p2^p = (n-1)2^{n+1} + 2$ for $n = 1, \ldots, k$.

Rest of the inductive step:

From the inductive hypothesis $\sum_{p=1}^{k} p2^p = (k-1)2^{k+1} + 2$.

Then

$$\begin{aligned}
\sum_{p=1}^{k+1} p2^p &= \left(\sum_{p=1}^{k} p2^p\right) + (k+1)2^{k+1} \\
&= ((k-1)2^{k+1} + 2) + (k+1)2^{k+1} \\
&= ((k-1) + (k+1))2^{k+1} + 2 = 2k2^{k+1} + 2 = k2^{k+2} + 2
\end{aligned}$$

So $\sum_{p=1}^{k+1} p2^p = k2^{k+2} + 2$, which is what we needed to show.

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Use (strong) induction to prove the following claim.

Claim: For any positive integer n, 2^{4n-1} ends in the digit 8. (I.e. when written out in base-10, the one's digit is 8.)

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $2^{4n-1} = 2^3 = 8$, which ends in the digit 8.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that 2^{4n-1} ends in the digit 8, for $n = 1, \ldots, k$.

Rest of the inductive step:

In particular, 2^{4k-1} ends in the digit 8. That is $2^{4k-1} = 10p + 8$, where p is an integer. Then

$$
\begin{aligned}
2^{4(k+1)-1} &= 2^{4k+4-1} = 2^{(4k-1)+4} = 2^{4k-1} \cdot 2^4 \\
&= (10p + 8) \cdot 2^4 = (10p + 8) \cdot 16 \\
&= 10(16p) + 8 \cdot 16 = 10(16p) + 128 \\
&= 10(16p) + 120 + 8 = 10(16p + 12) + 8
\end{aligned}
$$

$16p + 12$ is an integer, since p is an integer. So $2^{4(k+1)-1} = 10(16p + 12) + 8$ has a remainder of 8 when divided by 10. That is, its one's digit is 8, which is what we needed to prove.

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Working directly from the definition of divides, use (strong) induction to prove the following claim:

Claim: $2^{n+2} + 3^{2n+1}$ is divisible by 7, for all natural numbers n.

Solution:

Proof by induction on n.

Base case(s): At $n = 0$, $2^{n+2} + 3^{2n+1} = 2^2 + 3 = 7$ which is clearly divisible by 7.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $2^{n+2} + 3^{2n+1}$ is divisible by 7, for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

At $n = k + 1$, $2^{n+2} + 3^{2n+1}$ is equal to $2^{k+3} + 3^{2k+3}$.

$2^{k+3} + 3^{2k+3} = 2 \cdot 2^{k+2} + 9 \cdot 3^{2k+1} = 2(2^{k+2} + 3^{2k+1}) + 7(3^{2k+1})$

By the inductive hypothesis, $2^{k+2}+3^{2k+1}$ is divisible by 7. So $2(2^{k+2}+3^{2k+1})$ is divisible by 7. $7(3^{3k+1})$ is divisible by 7 because it contains a literal factor of 7 and the rest of the expression (3^{3k+1}) is an integer. So the sum of these two terms must be divisible by 7.

Thus, $2^{k+3} + 3^{2k+3}$ is divisible by 7, which is what we needed to show.

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If f is a function, recall that f' is its derivative. Recall the product rule: if $f(x) = g(x)h(x)$, then $f'(x) = g'(x)h(x) + g(x)h'(x)$. Assume we know that the derivative of $f(x) = x$ is $f'(x) = 1$.

Use (strong) induction to prove the following claim:

For any positive integer n, if $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

Solution: Proof by induction on n.

Base case(s): $n = 1$. Then $f(x) = x$. So $f'(x) = 1$. But also $nx^{n-1} = 1 \cdot n^0 = 1$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$, for $n = 1, \ldots, k$.

Rest of the inductive step: Suppose that $f(x) = x^{k+1}$. Let $g(x) = x$ and $h(x) = x^k$. By the product rule $f'(x) = g'(x)h(x) + g(x)h'(x)$.

Since $g(x) = x$, we know that $g'(x) = 1$. By the inductive hypothesis we know that $h'(x) = kx^{k-1}$.

So $f'(x) = g'(x)h(x) + g(x)h'(x) = 1 \cdot x^k + x \cdot kx^{k-1}$. Simplifying, we get $f'(x) = x^k + kx^k = (1+k)x^k$. So $f'(x) = (1+k)x^k$, which is what we needed to show.

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Use (strong) induction to prove the following claim:

For any natural number n, $\sum_{p=0}^{n} 3(-1/2)^p = 2 + (-1/2)^n$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $\sum_{p=0}^{n} 3(-1/2)^p = 3 \cdot (-1/2)^0 = 3$ and $2 + (-1/2)^n = 2 + (-1/2)^0 = 2 + 1 = 3$. So the equation holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=0}^{n} 3(-1/2)^p = 2 + (-1/2)^n$ for $n = 0, \ldots, k$.

Rest of the inductive step: From the inductive hypothesis, $\sum_{p=0}^{k} 3(-1/2)^p = 2 + (-1/2)^k$.

Then

$$\begin{aligned}
\sum_{p=0}^{k+1} 3(-1/2)^p &= \left(\sum_{p=0}^{k} 3(-1/2)^p\right) + 3(-1/2)^{k+1} \\
&= (2 + (-1/2)^k) + 3(-1/2)^{k+1} = 2 - 2(-1/2)^{k+1} + 3(-1/2)^{k+1} \\
&= 2 + (-1/2)^{k+1}
\end{aligned}$$

So $\sum_{p=0}^{k+1} 3(-1/2)^p = 2 + (-1/2)^{k+1}$, which is what we needed to show.

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Use (strong) induction to prove the following claim:

For all natural numbers n, $\displaystyle\sum_{p=0}^{n}(2p+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$

Solution: Proof by induction on n.

Base case(s): At $n=0$, $\displaystyle\sum_{p=1}^{n}(2p+1)^2 = 1^2 = 1$ and $\frac{(n+1)(2n+1)(2n+3)}{3} = \frac{1\cdot 1\cdot 3}{3} = 1$. So the equation holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:
$\displaystyle\sum_{p=0}^{n}(2p+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$ for $n = 0, \ldots, k$.

Rest of the inductive step: From the inductive hypothesis, we know that
$\displaystyle\sum_{p=0}^{k}(2p+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$.

Then

$$\begin{aligned}
\sum_{p=0}^{k+1}(2p+1)^2 &= \left(\sum_{p=0}^{k}(2p+1)^2\right) + (2(k+1)+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3} + (2(k+1)+1)^2 \\
&= \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2 = (2k+3)\frac{(k+1)(2k+1)+3(2k+3)}{3} \\
&= (2k+3)\frac{(2k^2+3k+1)+(6k+9)}{3} = (2k+3)\frac{2k^2+9k+10}{3} = \frac{(k+2)(2k+3)(2k+5)}{3}
\end{aligned}$$

So $\displaystyle\sum_{p=0}^{k+1}(2p+1)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$, which is what we needed to show

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Let's say that a set of polygonal regions in the plane is "properly colored" if regions sharing an edge never have the same color.

Suppose that we draw n lines in the plane, in general position (no lines are parallel, no point belongs to more than two lines). The lines divide up the plane into a set of regions. Use (strong) induction to prove that, for any positive integer n, this set of regions can be properly colored with two colors.

Solution: Proof by induction on n.

Base case(s): For $n = 1$, there is exactly one line dividing the plane. We can color one side of it red and the other side green.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that the set of regions formed by n lines can be properly colored with two colors, for $n = 1, \ldots, k$.

Rest of the inductive step:

Suppose that we are given k+1 lines in general position. Pick an arbitrary line L and remove it. By the inductive hypothesis, we can find a coloring for the regions formed by the remaining lines in which adjacent regions always have different colors.

Now, add L back (keeping the regions colored). Swap the two colors on the regions to one side of L. Then

- Regions on the un-altered side of L have the colors they had before, so adjacent regions on this side have different colors.
- Regions on the altered side of L have exactly the opposite colors they had before, so adjacent regions on this side have different colors.
- Adjacent regions with L as their common boundary now have different colors, because one has its original color and the other has had its color swapped.

So we have a proper coloring of the regions formed by our set of $k + 1$ lines.

[OK if you used pictures to help explain the construction in the inductive step.]

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Use (strong) induction to prove the following claim:

Claim: For all integers a, b, n, $n \geq 1$, if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$.

Use this definition in your proof: $x \equiv y \pmod p$ if and only if $x = y + kp$ for some integer k.

Solution:

Proof by induction on n.

Base case(s): At $n = 1$, our claim becomes "if $a \equiv b \pmod 7$ then $a \equiv b \pmod 7$" which is clearly true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$, for all integers a, b, n, where $n = 1, \ldots, k$,

a and b need to be introduced at some point in this proof, but there's several places you might do this. For example, you could say "let a and b be integers" right at the start. Then your inductive hypothesis would just be "if $a \equiv b \pmod 7$ then $a^n \equiv b^n \pmod 7$, for $n = 1, \ldots, k$." We won't get picky about this when grading.

Rest of the inductive step:

Let a and b be integers.

Suppose that $a \equiv b \pmod 7$. then $a = b + 7p$ for some integer p.

From the inductive hypothesis, we know that $a^k \equiv b^k \pmod 7$, So $a^k = b^k + 7q$ for some integer q.

Combining these two equations, we get that

$$a^{k+1} = (b + 7p)(b^k + 7q) = b^{k+1} + 7(pb^k + bq + 7pq)$$

$pb^k + bq + 7pq$ is an integer since p, q, and b are integers. So we know that $a^{k+1} \equiv b^{k+1} \pmod 7$, which is what we needed to prove.

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Use (strong) induction to prove the following claim:

Claim: $\sum_{p=1}^{n} \frac{2}{p^2+2p} = \frac{3}{2} - \frac{1}{(n+1)} - \frac{1}{(n+2)}$

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $\sum_{p=1}^{1} \frac{2}{p(p+2)} = \frac{2}{3} = \frac{3}{2} - \frac{1}{2} - \frac{1}{3}$

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} \frac{2}{p(p+2)} = \frac{3}{2} - \frac{1}{(n+1)} - \frac{1}{(n+2)}$ is true for all $n = 1, 2, \ldots, k$.

Rest of the inductive step:

Notice that $\sum_{p=1}^{k+1} \frac{2}{p(p+2)} = \sum_{p=1}^{k} \frac{2}{p(p+2)} + \frac{2}{(k+1)(k+3)}$. By the inductive hypothesis, we know that $\sum_{p=1}^{k} \frac{2}{p(p+2)} = \frac{3}{2} - \frac{1}{(k+1)} - \frac{1}{(k+2)}$. So we have

$$\begin{aligned}
\sum_{p=1}^{k+1} \frac{2}{p(p+2)} &= \frac{3}{2} - \frac{1}{(k+1)} - \frac{1}{(k+2)} + \frac{2}{(k+1)(k+3)} \\
&= \frac{3}{2} - \frac{1}{(k+2)} - \frac{1}{(k+1)} + \frac{2}{(k+1)(k+3)} \\
&= \frac{3}{2} - \frac{1}{(k+2)} + \frac{2-(k+3)}{(k+1)(k+3)} \\
&= \frac{3}{2} - \frac{1}{(k+2)} + \frac{-(k+1)}{(k+1)(k+3)} = \frac{3}{2} - \frac{1}{(k+2)} - \frac{1}{(k+3)},
\end{aligned}$$

So $\sum_{p=1}^{k+1} \frac{2}{p(p+2)} = \frac{3}{2} - \frac{1}{(k+2)} - \frac{1}{(k+3)}$, which is what we needed to prove.

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If f is a function, recall that f' is its derivative. Recall the product rule: if $f(x) = g(x)h(x)$, then $f'(x) = g'(x)h(x) + g(x)h'(x)$. Assume we know that the derivative of $f(x) = x$ is $f'(x) = 1$.

Use (strong) induction to prove the following claim:

For any positive integer n, if $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

Solution: Proof by induction on n.

Base case(s): $n = 1$. Then $f(x) = x$. So $f'(x) = 1$. But also $nx^{n-1} = 1 \cdot n^0 = 1$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$, for $n = 1, \ldots, k$.

Rest of the inductive step: Suppose that $f(x) = x^{k+1}$. Let $g(x) = x$ and $h(x) = x^k$. By the product rule $f'(x) = g'(x)h(x) + g(x)h'(x)$.

Since $g(x) = x$, we know that $g'(x) = 1$. By the inductive hypothesis we know that $h'(x) = kx^{k-1}$.

So $f'(x) = g'(x)h(x) + g(x)h'(x) = 1 \cdot x^k + x \cdot kx^{k-1}$. Simplifying, we get $f'(x) = x^k + kx^k = (1+k)x^k$. So $f'(x) = (1 + k)x^k$, which is what we needed to show.

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Let A be a constant integer. Use (strong) induction to prove the following claim. Remember that $0! = 1$.

Claim: For any integer $n \geq A$, $\sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{(n+1)!}{(A+1)!(n-A)!}$

Solution: Proof by induction on n.

Base case(s): At $n = A$, $\sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{A!}{A!0!} = 1 = \frac{(A+1)!}{(A+1)!0!} = \frac{(n+1)!}{(A+1)!(n-A)!}$

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{(n+1)!}{(A+1)!(n-A)!}$ is true for all $n = A, \ldots, k$.

Rest of the inductive step:

In particular, $\sum_{p=A}^{k} \frac{p!}{A!(p-A)!} = \frac{(k+1)!}{(A+1)!(k-A)!}$ So then

$$\begin{aligned}
\sum_{p=A}^{k+1} \frac{p!}{A!(p-A)!} &= \sum_{p=A}^{k} \frac{p!}{A!(p-A)!} + \frac{(k+1)!}{A!(k+1-A)!} \\
&= \frac{(k+1)!}{(A+1)!(k-A)!} + \frac{(k+1)!}{A!(k+1-A)!} \\
&= \frac{(k+1-A)(k+1)!}{(A+1)!(k+1-A)!} + \frac{(A+1)(k+1)!}{(A+1)!(k+1-A)!} \\
&= \frac{(k+2)(k+1)!}{(A+1)!(k+1-A)!} = \frac{(k+2)!}{(A+1)!(k+1-A)!}
\end{aligned}$$

So $\sum_{p=A}^{k+1} \frac{p!}{A!(p-A)!} = \frac{(k+2)!}{(A+1)!(k+1-A)!}$, which is what we needed to show.

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Use (strong) induction to prove the following claim:

Claim: $\sum_{p=1}^{n} 2(-1)^p p^2 = (-1)^n n(n+1)$, for all positive integers n

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $\sum_{p=1}^{n} 2(-1)^p p^2 = 2(-1)^1 1^2 = -2$. And $(-1)^n n(n+1) = (-1)^1 1 \cdot 2 = -2$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} 2(-1)^p p^2 = (-1)^n n(n+1)$, for $n = 1, 2, \ldots k$.

Rest of the inductive step:

$$\sum_{p=1}^{k+1} 2(-1)^p p^2 = 2(-1)^{k+1}(k+1)^2 + \sum_{p=1}^{k} 2(-1)^p p^2$$

By the inductive hypothesis, we know that $\sum_{p=1}^{k} 2(-1)^p p^2 = (-1)^k k(k+1)$. Substituting this into the previous equation, we get

$$\begin{aligned}
\sum_{p=1}^{k+1} 2(-1)^p p^2 &= 2(-1)^{k+1}(k+1)^2 + (-1)^k k(k+1) \\
&= (k+1)(-1)^{k+1}(2(k+1) - k) \\
&= (k+1)(-1)^{k+1}(k+2) = (-1)^{k+1}(k+1)(k+2)
\end{aligned}$$

So $\sum_{p=1}^{k+1} 2(-1)^p p^2 = (-1)^{k+1}(k+1)(k+2)$ which is what we needed to show.

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Use (strong) induction and the fact that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ to prove the following claim:

For all natural numbers n, $\left(\sum_{i=0}^{n} i\right)^2 = \sum_{i=0}^{n} i^3$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $\left(\sum_{i=0}^{n} i\right)^2 = 0^2 = 0 = \sum_{i=0}^{n} i^3$. So the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:
Suppose that $\left(\sum_{i=0}^{n} i\right)^2 = \sum_{i=0}^{n} i^3$ for $n = 0, 1, \ldots, k$.

Rest of the inductive step:
Starting with the lefthand side of the equation for $n = k + 1$, we get

$$\left(\sum_{i=0}^{k+1} i\right)^2 = \left((k+1) + \sum_{i=0}^{k} i\right)^2 = (k+1)^2 + 2(k+1)\sum_{i=0}^{k} i + \left(\sum_{i=0}^{k} i\right)^2$$

By the inductive hypothesis $\left(\sum_{i=0}^{k} i\right)^2 = \sum_{i=0}^{k} i^3$. Substituting this and the fact we were told to assume, we get

$$\left(\sum_{i=0}^{k+1} i\right)^2 = (k+1)^2 + 2(k+1)\frac{k(k+1)}{2} + \sum_{i=0}^{k} i^3 = (k+1)^2 + k(k+1)^2 + \sum_{i=0}^{k} i^3 = (k+1)^3 + \sum_{i=0}^{k} i^3 = \sum_{i=0}^{k+1} i^3$$

So $\left(\sum_{i=0}^{k+1} i\right)^2 = \sum_{i=0}^{k+1} i^3$ which is what we needed to show.

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The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

$\prod_{p=2}^{n}(1 - \frac{1}{p^2}) = \frac{n+1}{2n}$ for any integer $n \geq 2$.

Solution: Proof by induction on n.

Base case(s): At $n = 2$, $\prod_{p=2}^{n}(1 - \frac{1}{p^2}) = (1 - \frac{1}{4}) = \frac{3}{4}$ and $\frac{n+1}{2n} = \frac{3}{4}$, so the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=2}^{n}(1-\frac{1}{p^2}) = \frac{n+1}{2n}$ for $n = 2, \ldots, k$

Rest of the inductive step: In particular, from the inductive hypothesis $\prod_{p=2}^{k}(1 - \frac{1}{p^2}) = \frac{k+1}{2k}$.

So

$$\begin{aligned}
\prod_{p=2}^{k+1}(1 - \frac{1}{p^2}) &= (\prod_{p=2}^{k}(1 - \frac{1}{p^2}))(1 - \frac{1}{(k+1)^2}) \\
&= (\frac{k+1}{2k})(1 - \frac{1}{(k+1)^2}) = \frac{k+1}{2k} - \frac{k+1}{2k(k+1)^2} \\
&= \frac{k+1}{2k} - \frac{1}{2k(k+1)} = \frac{(k+1)^2}{2k(k+1)} - \frac{1}{2k(k+1)} \\
&= \frac{(k+1)^2 - 1)}{2k(k+1)} = \frac{k^2 + 2k)}{2k(k+1)} = \frac{k(k+2)}{2k(k+1)} = \frac{k+2}{2(k+1)}
\end{aligned}$$

So $\prod_{p=2}^{k+1}(1 - \frac{1}{p^2}) = \frac{k+2}{2(k+1)}$, which is what we needed to show.