Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is even.

$$T(0) = 5 \qquad T(n) = 3T(n-2) + n^2$$

(a) The height: $\frac{n}{2}$

(b) The number of leaves (please simplify): $3^{\frac{n}{2}} = (\sqrt{3})^n$

(c) Value in each node at level k: $(n-2k)^2$

Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

$n \qquad n\log(17n) \qquad \sqrt{n}+18 \qquad 8n^2 \qquad 2^n+n! \qquad 2^{\log_4 n}+5^n \qquad 0.001n^3+3^n$

Solution:

$\sqrt{n}+18 \ll n \ll n\log(17n) \ll 8n^2 \ll 0.001n^3+3^n \ll 2^{\log_4 n}+5^n \ll 2^n+n!$

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) Recall that f is $O(g)$ if and only if there are positive reals c and k such that $0 \leq f(x) \leq cg(x)$ for every $x \geq k$. Prof. Snape claims that there is a function f (from the reals to the reals) that can never be involved in a big-O relationship. Is he correct?

   Solution: He is right. Our definition will never be satisfied if one (or both) of the functions produces only negative outputs.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = c$
$T(n) = 2T(n/2) + n$

$\Theta(\log n)$ ☐	$\Theta(\sqrt{n})$ ☐	$\Theta(n)$ ☐	$\Theta(n \log n)$ ☑
$\Theta(n^2)$ ☐	$\Theta(n^3)$ ☐	$\Theta(2^n)$ ☐	$\Theta(3^n)$ ☐

$T(1) = d$
$T(n) = T(n/2) + c$

$\Theta(\log n)$ ☑	$\Theta(\sqrt{n})$ ☐	$\Theta(n)$ ☐	$\Theta(n \log n)$ ☐
$\Theta(n^2)$ ☐	$\Theta(n^3)$ ☐	$\Theta(2^n)$ ☐	$\Theta(3^n)$ ☐

$n^{1.5}$ is $\Theta(n^{1.614})$ ☐ $O(n^{1.614})$ ☑ neither of these ☐

$n^{log_3 5}$ grows faster than n^2 ☐ slower than n^2 ☑ at the same rate as n^2 ☐

CS 173, Spring 19 Examlet 10, white

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is even.

   $T(8) = 5 \qquad T(n) = 3T(n-2) + c$

   (a) The height: $\frac{n}{2} - 4$

   (b) The number of nodes at level k: 3^k

   (c) Value in each node at level k: c

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

   $3n^2 \qquad \frac{n \log n}{7} \qquad (10^{10^{10}})n \qquad 0.001n^3 \qquad 30\log(n^{17}) \qquad 8n! + 18 \qquad 3^n + 11^n$

   Solution:

   $30\log(n^{17}) \ll (10^{10^{10}})n \ll \frac{n \log n}{7} \ll 3n^2 \ll 0.001n^3 \ll 3^n + 11^n \ll 8n! + 18$

Name: ________________________________

NetID: ____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) Suppose that f and g are functions from the reals to the reals. Define precisely what it means for g to be $\Theta(f)$. Your definition can be in terms of other primitives such as $\ll$ and big-O.

   Solution: g is $\Theta(f)$ if and only if g is $O(f)$ and f is $O(g)$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = d$
$T(n) = T(n-1) + c$

$\Theta(\log n)$ ☐	$\Theta(\sqrt{n})$ ☐	$\Theta(n)$ ☑	$\Theta(n \log n)$ ☐
$\Theta(n^2)$ ☐	$\Theta(n^3)$ ☐	$\Theta(2^n)$ ☐	$\Theta(3^n)$ ☐

$T(1) = d$
$T(n) = 2T(n/2) + c$

$\Theta(\log n)$ ☐	$\Theta(\sqrt{n})$ ☐	$\Theta(n)$ ☑	$\Theta(n \log n)$ ☐
$\Theta(n^2)$ ☐	$\Theta(n^3)$ ☐	$\Theta(2^n)$ ☐	$\Theta(3^n)$ ☐

Suppose f and g produce only positive outputs and $f(n) \ll g(n)$. Will $g(n)$ be $O(f(n))$?

no ☑ sometimes ☐ yes ☐

$n^{log_2 4}$ grows

faster than n^2 ☐ slower than n^2 ☐
at the same rate as n^2 ☑

CS 173, Spring 19 Examlet 10, white

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is a multiple of 3.

   $T(3) = 7 \qquad T(n) = 2T(n-3) + c$

   (a) The height: $\frac{n}{3} - 1$

   (b) The number of leaves (please simplify): $2^{\frac{n}{3}-1}$

   (c) Total work (sum of the nodes) at level k (please simplify): There are 2^k nodes at level k, each containing value c. So the total work is $c2^k$.

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

   $n \qquad n\log(17n) \qquad \sqrt{n}+2^n+18 \qquad 8n^2 \qquad 2^n+n! \qquad 2^{\log_4 n} \qquad 0.001n^3+3^n$

   Solution:

   $2^{\log_4 n} \ll n \ll n\log(17n) \ll 8n^2 \ll \sqrt{n}+2^n+18 \ll 0.001n^3+3^n \ll 2^n+n!$

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (7 points) Suppose that f, g, and h are functions from the reals to the reals, such that $f(x)$ is $O(h(x))$ and $g(x)$ is $O(h(x))$. Must $f(x)g(x)$ be $O(h(x))$?

   Solution: This is false.

   Suppose that $f(x) = g(x) = h(x) = x$. Then $f(x)$ is $O(h(x))$ and $g(x)$ is $O(h(x))$, but $f(x)g(x) = x^2$ is not $O(h(x))$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = c$
$T(n) = 3T(n/3) + n$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	$\surd$
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

$T(1) = c$
$T(n) = 2T(n/2) + n^2$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
$\Theta(n^2)$	$\surd$	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

Dividing a problem of size n into k subproblems, each of size n/m, has the best big-Θ running time when

$k < m$	$\surd$	$k = m$	☐
$k > m$	☐	$km = 1$	☐

$n^{log_2 5}$ grows

faster than n^2	$\surd$	slower than n^2	☐
at the same rate as n^2	☐		

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is a power of 2.

   $T(8) = 7 \qquad T(n) = 4T\left(\frac{n}{2}\right) + n$

   (a) The height: $\log_2 n - 3$

   (b) Total work (sum of the nodes) at level k (please simplify): There are 4^k nodes at level k. Each one contains the value $\frac{n}{2^k}$. So the total for the level is $2^k n$.

   (c) The number of leaves (please simplify): $4^{\log_2 n - 3} = \frac{1}{4^3} 4^{\log_2 n}$
   $4^{\log_2 n} = 4^{\log_4 n \log_2 4} = (4^{\log_4 n})^{\log_2 4} == n^{\log_2 4} = n^2$
   So the number of leaves is $\frac{1}{4^3} n^2$.

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

   $3^n \qquad 4^{\log_2 n} \qquad 2^{3n} \qquad 3^{\log_2 4} \qquad 0.1n \qquad (5n)! \qquad \sqrt{n}$

   Solution:
   $3^{\log_2 4} \ll \sqrt{n} \ll 0.1n \ll 4^{\log_2 n} \ll 3^n \ll 2^{3n} \ll (5n)!$

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) Suppose that f and g are functions from the reals to the reals. Define precisely when $f \ll g$.

   Solution: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = d$	$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
$T(n) = 3T(n-1) + c$	$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☑ $\surd$

$T(1) = d$	$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☑ $\surd$	$\Theta(n \log n)$	☐
$T(n) = T(n/2) + n$	$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

3^n is	$\Theta(5^n)$	☐	$O(5^n)$	☑ $\surd$	neither of these	☐
3^n is	$\Theta(2^n)$	☐	$O(2^n)$	☐	neither of these	☑ $\surd$

CS 173, Fall 18 Examlet 10, Part B

Name:____

NetID:____________________ **Lecture:** **B**

Discussion: **Friday** **11** **12** **1** **2** **3** **4**

1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is a power of 4.

   $T(4) = 7 \qquad T(n) = 2T\left(\frac{n}{4}\right) + d$

   (a) The height: $\log_4 n - 1$

   (b) Number of nodes at level k: 2^k

   (c) Sum of the work in all the leaves (please simplify):
   Each leaf contains the value 7, and there are $2^{\log_4 n - 1} = \frac{1}{2}2^{\log 4} = \frac{1}{2}\sqrt{n}$ leaves. So the sum is $\frac{7}{2}\sqrt{n}$.

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

   $2^n + 3^n \qquad n^3 \qquad 100\log n \qquad 3^{31} \qquad 3n\log(n^3) \qquad 7n! + 2 \qquad 173n - 173$

   Solution:

   $3^{31} \ll 100\log n \ll 173n - 173 \ll 3n\log(n^3) \ll n^3 \ll 2^n + 3^n \ll 7n! + 2$

Name:____

NetID:____________________ **Lecture:** **B**

Discussion: **Friday** **11** **12** **1** **2** **3** **4**

1. (7 points) You found the following claim on a hallway whiteboard. Suppose that f and g are increasing functions from the reals to the reals, for which all output values are > 1. If $f(x)$ is $O(g(x))$, then $\log(f(x))$ is $O(\log(g(x)))$. Is this true? Briefly justify your answer.

   Solution:

   Yes, it is true. Suppose that $f(x)$ is $O(g(x))$. Then there are positive reals c and k such that $f(x) \leq cg(x)$ for all $x \geq k$. Then $\log(f(x)) \leq \log c + \log(g(x))$ for all $x \geq k$. Since $g(x)$ is an increasing function and c isn't, there is some $K \geq k$ such that $\log c \leq \log(g(x))$. So then $\log(f(x)) \leq 2\log(g(x))$ for all $x \geq K$. So $\log(f(x))$ is $O(\log(g(x)))$.

   [You don't need this much technical detail for full credit. We can ignore the other inequality from the definition of big-O because the two functions always output positive values.]

2. (8 points) Check the (single) box that best characterizes each item.

   $T(1) = d$
   $T(n) = T(n-1) + n$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
$\Theta(n^2)$	☑ $\surd$	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

   $T(1) = d$
   $T(n) = 3T(n/3) + c$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☑ $\surd$	$\Theta(n \log n)$	☐
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

   $\log_5 n$ is

$\Theta(\log_3 n)$	☑ $\surd$	$O(\log_3 n)$	☐	neither of these	☐

   Dividing a problem of size n into m subproblems, each of size n/k, has the best big-Θ running time when

$k < m$	☐	$k = m$	☐
$k > m$	☑ $\surd$	$km = 1$	☐

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is a power of 4.

   $T(1) = 7 \qquad T(n) = 2T\left(\frac{n}{4}\right) + n$

   (a) The height: $\log_4 n$

   (b) Number of leaves: $2^{\log_4 n} = n^{1/2} = \sqrt{n}$
   [Ok to stop simplifying at $n^{1/2}$.]

   (c) Total work (sum of the nodes) at level k (please simplify):
   There are 2^k nodes at level k. Each of these nodes contains the value $n/4^k$. So the total work is $2^k \cdot n/4^k = n/2^k$.

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

   $42n!$ $\quad 7^n$ $\quad 100\log n$ $\quad n\log(n^7)$ $\quad 2^{3n}$ $\quad \log(2^n)$ $\quad (n^3)^7$

   Solution:

   $100\log n \ll \log(2^n) \ll n\log(n^7) \ll (n^3)^7 \ll 7^n \ll 2^{3n} \ll 42n!$

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (7 points) In class, Prof. Snape made the following claim about all functions g and h from the reals to the reals whose output values are always > 1. If $g(x) \ll h(x)$, then $\log(g(x)) \ll \log(h(x))$. Is this true? Briefly justify your answer.

   Solution:

   This is not true. Consider $f(x) = x$ and $g(x) = x^2$. Then $\log(g(x)) = 2\log(f(x))$ So it can't be the case that $\log(f(x)) \ll \log(g(x))$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = d$ $T(n) = T(n/2) + c$	$\Theta(\log n)$	$\checkmark$	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
	$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐
$T(1) = d$ $T(n) = 2T(n-1) + c$	$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
	$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	$\checkmark$	$\Theta(3^n)$	☐

2^n is	$\Theta(3^n)$	☐	$O(3^n)$	$\checkmark$	neither of these	☐
Suppose f and g produce only positive outputs and $f(n) \ll g(n)$. Will $f(n)$ be $\Theta(g(n))$?	no	$\checkmark$	perhaps	☐	yes	☐

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) Suppose that f, g, and h are functions from the reals to the reals, such that f is $O(g)$ and g is $O(h)$. Must f be $O(h)$? Briefly justify your answer.

   Solution: This is true. Since f is $O(g)$ and g is $O(h)$, there are positive reals c, k, C and K such that $0 \leq f(x) \leq cg(x)$ and $0 \leq g(y) \leq Ch(y)$ for every $x \geq k$ and $y \geq K$.

   But then if we let $p = cC$, we have $O \leq f(x) \leq ph(x)$ for every $x \geq \max(k, K)$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = d$
$T(n) = T(n/2) + n$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☑ $\surd$	$\Theta(n \log n)$	☐
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

$T(1) = d$
$T(n) = 3T(n-1) + c$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☑ $\surd$

$n^{log_2 5}$ grows

faster than n^2	☑ $\surd$	slower than n^2	☐
at the same rate as n^2	☐		

Suppose $f(n)$ is $O(g(n))$.
Will $g(n)$ be $O(f(n))$?

no	☐	perhaps	☑ $\surd$	yes	☐

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) Suppose that f, g, and h are functions from the reals to the reals, such that $f(x)$ is $O(h(x))$ and $g(x)$ is $O(h(x))$. Must $f(x)g(x)$ be $O(h(x)h(x))$?

   Solution: This is true. Since $f(x)$ is $O(h(x))$ and $g(x)$ is $O(h(x))$, there are positive reals c, k, C and K such that $0 \leq f(x) \leq ch(x)$ and $0 \leq g(y) \leq Ch(y)$ for every $x \geq k$ and $y \geq K$.

   But then if we let $p = cC$, we have $O \leq f(x) \leq ph(x)h(x)$ for every $x \geq \max(k, K)$.

2. (8 points) Check the (single) box that best characterizes each item.

Suppose $f(n)$ is $O(g(n))$.
Will $f(n)$ be $\Theta(g(n))$?

no ☐	perhaps ☑	yes ☐

$17n^3$

$\Theta(n^3)$ ☑	$O(n^3)$ ☐	neither of these ☐

$T(1) = c$
$T(n) = 2T(n/2) + n^2$

$\Theta(\log n)$ ☐	$\Theta(\sqrt{n})$ ☐	$\Theta(n)$ ☐	$\Theta(n \log n)$ ☐
$\Theta(n^2)$ ☑	$\Theta(n^3)$ ☐	$\Theta(2^n)$ ☐	$\Theta(3^n)$ ☐

$T(1) = d$
$T(n) = T(n/3) + c$

$\Theta(\log n)$ ☑	$\Theta(\sqrt{n})$ ☐	$\Theta(n)$ ☐	$\Theta(n \log n)$ ☐
$\Theta(n^2)$ ☐	$\Theta(n^3)$ ☐	$\Theta(2^n)$ ☐	$\Theta(3^n)$ ☐

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is a power of 2.

   $T(1) = 1$ $\quad\quad$ $T(n) = 4T\left(\frac{n}{2}\right) + n^2$

   (a) Value in each node at level k: $\left(\frac{n}{2^k}\right)^2 = \frac{n^2}{4^k}$

   (b) Total work (sum of the nodes) at level k (please simplify): Level k has 4^k nodes, each containing the value $\frac{n^2}{4^k}$. So the total for the level is $4^k \frac{n^2}{4^k} = n^2$

   (c) Sum of the work in all internal (non-leaf) nodes (please simplify):

   The number of non-leaf levels is the height of the tree, which is $\log n$. The work at each level is n^2. So the total work in all the non-leaf nodes is $n^2 \log n$.

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

   $n \log n$ $\quad$ $\log(n^{17})$ $\quad$ $\sqrt{n} + n! + 18$ $\quad$ 2^n $\quad$ $8n^2$ $\quad$ $8^{\log_8 n}$ $\quad$ $0.001n^3$

   Solution:

   $\log(n^{17}) \ll 8^{\log_8 n} \ll n \log n \ll 8n^2 \ll 0.001n^3 \ll 2^n \ll \sqrt{n} + n! + 18$

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) Suppose that f and g are functions from the reals to the reals. Define precisely what it means for f to be $O(g)$.

   Solution: There are positive reals c and k such that $0 \leq f(x) \leq cg(x)$ for every $x \geq k$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = c$
$T(n) = 3T(n/3) + n$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☑ $\surd$
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

$T(1) = d$
$T(n) = 2T(n/2) + c$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☑ $\surd$	$\Theta(n \log n)$	☐
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

2^n $\Theta(n!)$ ☐ $O(n!)$ ☑ $\surd$ neither of these ☐

Suppose f and g produce only positive outputs and $f(n) \ll g(n)$. Will $f(n)$ be $O(g(n))$? no ☐ perhaps ☐ yes ☑ $\surd$

CS 173, Fall 17 Examlet 10, Part B

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is a power of 3.

   $T(3) = 7 \qquad T(n) = 4T\left(\frac{n}{3}\right) + 5n$

   (a) The height: $\log_3 n - 1$

   (b) Value in each node at level k: Each node at level k contains the value $\frac{5n}{3^k}$.

   (c) Sum of the work in all the leaves (please simplify): The number of leaves is $4^{\log_3 n - 1} = \frac{1}{4}4^{\log_3 n}$
   $4^{\log_3 n} = 4^{\log_4 n \log_3 4} = (4^{\log_4 n})^{\log_3 4} == n^{\log_3 4}$
   So the work at the leaves is $\frac{7}{4}n^{\log_3 4}$.

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

   $(3^n)^2 \qquad 10 \qquad 0.001n^3 \qquad 30\log n \qquad n\log(n^7) \qquad 8n! + 18 \qquad 3n^2$

   Solution:

   $10 \ll 30\log n \ll n\log(n^7) \ll 3n^2 \ll 0.001n^3 \ll (3^n)^2 \ll 8n! + 18$

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) Suppose that f, g, and h are functions from the reals to the reals, such that $f(x)$ is $O(h(x))$ and $g(x) \ll f(x))$. Must $f(x) + g(x)$ be $O(h(x))$?

   Solution: This is true. Since $g(x)$ is asymptotically smaller than $f(x)$, $f(x) + g(x)$ grows at the same rate as $f(x)$. We know this is $O(h(x))$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = d$
$T(n) = 3T(n/3) + c$

$\Theta(\log n)$ ☐	$\Theta(\sqrt{n})$ ☐	$\Theta(n)$ ☑	$\Theta(n \log n)$ ☐
$\Theta(n^2)$ ☐	$\Theta(n^3)$ ☐	$\Theta(2^n)$ ☐	$\Theta(3^n)$ ☐

$T(1) = c$
$T(n) = 3T(n/3) + n$

$\Theta(\log n)$ ☐	$\Theta(\sqrt{n})$ ☐	$\Theta(n)$ ☐	$\Theta(n \log n)$ ☑
$\Theta(n^2)$ ☐	$\Theta(n^3)$ ☐	$\Theta(2^n)$ ☐	$\Theta(3^n)$ ☐

$n!$ — $O(2^n)$ ☐ $\Theta(2^n)$ ☐ neither of these ☑

$n^{log_2 4}$ grows — faster than n^2 ☐ slower than n^2 ☐ at the same rate as n^2 ☑

Name:____

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) Suppose that f, g, and h are functions from the reals to the reals, such that $f(x)$ is $\Theta(h(x))$, $g(x)$ is $\Theta(h(x))$, and $f(x) > g(x)$ for any input x. Must $f(x) - g(x)$ be $\Theta(h(x))$?

   Solution: This is false.

   Suppose that $g(x) = h(x) = x^2$ and $f(x) = x^2 + x$. Then $f(x) - g(x) = x$, which is not $\Theta(x^2)$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = c$
$T(n) = 2T(n/2) + n$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	√
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

$T(1) = d$
$T(n) = 2T(n-1) + c$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	√	$\Theta(3^n)$	☐

$n^{log_2 3}$ grows

faster than n	√	slower than n	☐
at the same rate as n	☐		

Suppose $f(n)$ is $\Theta(g(n))$.
Will $g(n)$ be $\Theta(f(n))$?

no	☐	perhaps	☐	yes	√

CS 173, Fall 17 Examlet 10, Part B

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) You found the following claim on a hallway whiteboard. Suppose that f and g are increasing functions from the reals to the reals, for which all output values are > 1. If $f(x)$ is $O(g(x))$, then $\log(f(x))$ is $O(\log(g(x)))$. Is this true? Briefly justify your answer.

   Solution:

   Yes, it is true. Suppose that $f(x)$ is $O(g(x))$. Then there are positive reals c and k such that $f(x) \leq cg(x)$ for all $x \geq k$. Then $\log(f(x)) \leq \log c + \log(g(x))$ for all $x \geq k$. Since $g(x)$ is an increasing function and $\log c$ isn't, There is some $K \geq k$ such that $\log c \leq \log(g(x))$. So then $\log(f(x)) \leq 2\log(g(x))$ for all $x \geq K$. So $\log(f(x))$ is $O(\log(g(x)))$.

   [You don't need this much technical detail for full credit. We can ignore the other inequality from the definition of big-O because the two functions always output positive values.]

2. (8 points) Check the (single) box that best characterizes each item.

3^n is	$\Theta(2^n)$ ☐	$O(2^n)$ ☐	neither of these ☑

Dividing a problem of size n into m subproblems, each of size n/k, has the best big-Θ running time when	$k < m$ ☐	$k = m$ ☐
	$k > m$ ☑	$km = 1$ ☐

$T(1) = d$ $T(n) = T(n/2) + n$	$\Theta(\log n)$ ☐	$\Theta(\sqrt{n})$ ☐	$\Theta(n)$ ☑	$\Theta(n \log n)$ ☐
	$\Theta(n^2)$ ☐	$\Theta(n^3)$ ☐	$\Theta(2^n)$ ☐	$\Theta(3^n)$ ☐

$T(1) = d$ $T(n) = T(n-1) + n$	$\Theta(\log n)$ ☐	$\Theta(\sqrt{n})$ ☐	$\Theta(n)$ ☐	$\Theta(n \log n)$ ☐
	$\Theta(n^2)$ ☑	$\Theta(n^3)$ ☐	$\Theta(2^n)$ ☐	$\Theta(3^n)$ ☐

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is a power of 2.

   $T(4) = 7 \qquad T(n) = 5T\left(\frac{n}{2}\right) + n$

   (a) The height: $\log_2 n - 2$

   (b) The number of leaves (please simplify): $5^{\log_2 n - 2} = \dfrac{1}{25}5^{log_2 n} = \dfrac{1}{25}5^{log_5 n \log_2 5} = \dfrac{1}{25}n^{\log_2 5}$

   (c) Value in each node at level k: $\frac{n}{2^k}$

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

   $30\log(n^{17}) \qquad \sqrt{n} + n! + 18 \qquad \frac{n\log n}{7} \qquad (10^{10^{10}})n \qquad 0.001n^3 \qquad 2^n \qquad 8n^2$

   Solution:

   $30\log(n^{17}) \ll (10^{10^{10}})n \ll \frac{n\log n}{7} \ll 8n^2 \ll 0.001n^3 \ll 2^n \ll \sqrt{n} + n! + 18$

CS 173, Fall 17 Examlet 10, Part B

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) Suppose that f and g are functions from the reals to the reals, such that f is $\Theta(g)$. Must g be $O(f)$?

   Solution: This is true. The definition of Θ is that the big-O relationship holds in both directions.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = c$ $T(n) = 2T(n/2) + n^2$	$\Theta(\log n)$ ☐	$\Theta(\sqrt{n})$ ☐	$\Theta(n)$ ☐	$\Theta(n \log n)$ ☐
	$\Theta(n^2)$ ☑ $\surd$	$\Theta(n^3)$ ☐	$\Theta(2^n)$ ☐	$\Theta(3^n)$ ☐
$T(1) = d$ $T(n) = T(n-1) + c$	$\Theta(\log n)$ ☐	$\Theta(\sqrt{n})$ ☐	$\Theta(n)$ ☑ $\surd$	$\Theta(n \log n)$ ☐
	$\Theta(n^2)$ ☐	$\Theta(n^3)$ ☐	$\Theta(2^n)$ ☐	$\Theta(3^n)$ ☐

$n^{log_4 2}$ grows: faster than n^2 ☐ slower than n^2 ☑ $\surd$ at the same rate as n^2 ☐

$\log_5 n$ is: $\Theta(\log_3 n)$ ☑ $\surd$ $O(\log_3 n)$ ☐ neither of these ☐

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is a multiple of 3.

   $T(3) = 7 \qquad T(n) = 2T(n-3) + c$

   (a) The height: $\frac{n}{3} - 1$

   (b) The number of leaves (please simplify): $2^{\frac{n}{3}-1}$

   (c) Total work (sum of the nodes) at level k (please simplify): There are 2^k nodes at level k, each containing value c. So the total work is $c2^k$.

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

   $3^n \qquad 4^{\log_2 n} \qquad 2^{3n} \qquad 3^{\log_2 4} \qquad 0.1n \qquad (5n)! \qquad \sqrt{n}$

   Solution:

   $3^{\log_2 4} \ll \sqrt{n} \ll 0.1n \ll 4^{\log_2 n} \ll 3^n \ll 2^{3n} \ll (5n)!$

Name:____

NetID:________________________ Lecture: A B

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

1. (7 points) Prof. Flitwick claims that for any functions f and g from the reals to the reals whose output values are always > 1, if $f(x) \ll g(x)$ then $\log(f(x)) \ll \log(g(x))$ Is this true? Briefly justify your answer.

   Solution: This is not true. Consider $f(x) = x$ and $g(x) = x^2$. Then $\log(g(x)) = 2\log(f(x))$ So it can't be the case that $\log(f(x)) \ll \log(g(x))$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = c$
$T(n) = 4T(n/2) + n$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
$\Theta(n^2)$	☑ $\surd$	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

$T(1) = d$
$T(n) = T(n/2) + c$

$\Theta(\log n)$	☑ $\surd$	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

Suppose $f(n) \ll g(n)$.
Is $g(n) \ll f(n)$?

no ☑ $\surd$ perhaps ☐ yes ☐

Suppose f and g produce only positive outputs and $f(n) \ll g(n)$.
Will $g(n)$ be $O(f(n))$?

no ☑ $\surd$ perhaps ☐ yes ☐

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (7 points) Suppose that f, g, and h are functions from the reals to the reals, such that $f(x)$ is $\Theta(h(x))$ and $g(x)$ is $\Theta(h(x))$. Must $f(x) - g(x)$ be $\Theta(h(x))$?

   Solution: This is false.

   Suppose that $g(x) = h(x) = x^2$ and $f(x) = x^2 + x$. Then $f(x) - g(x) = x$, which is not $\Theta(x^2)$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = d$
$T(n) = 3T(n-1) + c$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	$\boxed{\surd}$

$T(1) = d$
$T(n) = 2T(n/2) + c$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	$\boxed{\surd}$	$\Theta(n \log n)$	☐
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

Dividing a problem of size n into k subproblems, each of size n/m, has the best big-Θ running time when

$k < m$	$\boxed{\surd}$	$k = m$	☐
$k > m$	☐	$km = 1$	☐

$n^{log_3 2}$ grows

faster than n	☐	slower than n	$\boxed{\surd}$
at the same rate as n	☐		

Name:____

NetID:_______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is a power of 3.

   $T(9) = 7$ $\qquad T(n) = T\left(\frac{n}{3}\right) + n^2$

   (a) The height: $\log_3 n - 3$

   (b) Number of nodes at level k: One. (This tree does not branch.)

   (c) Value in each node at level k: At level k, the problem size is $\frac{n}{3^k}$. So the value in each node is $(\frac{n}{3^k})^2$.

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

   $(\sqrt{n})^4 \qquad 200\log_5 n \qquad \log(2^n) \qquad 2^n + n! \qquad 7^n \qquad 3^{57} \qquad 55n\log n$

   Solution:

   $3^{57} \ll 200\log_5 n \ll \log(2^n) \ll 55n\log n \ll (\sqrt{n})^4 \ll 7^n \ll 2^n + n!$

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is a power of 2.

   $T(8) = 7$ $\quad\quad T(n) = 4T\left(\frac{n}{2}\right) + n$

   (a) The height: $\log_2 n - 3$

   (b) Total work (sum of the nodes) at level k (please simplify): There are 4^k nodes at level k. Each one contains the value $\frac{n}{2^k}$. So the total for the level is $2^k n$.

   (c) The number of leaves (please simplify): $4^{\log_2 n - 3} = \frac{1}{4^3} 4^{\log_2 n}$
   $4^{\log_2 n} = 4^{\log_4 n \log_2 4} = (4^{\log_4 n})^{\log_2 4} == n^{\log_2 4} = n^2$
   So the number of leaves is $\frac{1}{4^3} n^2$.

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

   $15n!$ $\quad\quad \log(n^5)$ $\quad\quad 127(2^n)$ $\quad\quad n \log_2 4$ $\quad\quad 7^n$ $\quad\quad 47n^3$ $\quad\quad 20n$

   Solution:

   $\log(n^5) \ll n \log_2 4 \ll 20n \ll 47n^3 \ll 127(2^n) \ll 7^n \ll 15n!$

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

1. (7 points) Suppose that f and g are functions from the reals to the reals. Define precisely what it means for g to be $O(f)$.

   Solution: There are positive reals c and k such that $0 \leq g(x) \leq cf(x)$ for every $x \geq k$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = d$
$T(n) = T(n-1) + c$

$\Theta(\log n)$		$\Theta(\sqrt{n})$		$\Theta(n)$	√	$\Theta(n \log n)$	
$\Theta(n^2)$		$\Theta(n^3)$		$\Theta(2^n)$		$\Theta(3^n)$	

$T(1) = d$
$T(n) = T(n/3) + c$

$\Theta(\log n)$	√	$\Theta(\sqrt{n})$		$\Theta(n)$		$\Theta(n \log n)$	
$\Theta(n^2)$		$\Theta(n^3)$		$\Theta(2^n)$		$\Theta(3^n)$	

Suppose $f(n) \ll g(n)$.
Is $g(n) \ll f(n)$?

no	√	perhaps		yes	

Suppose $f(n)$ is $\Theta(g(n))$.
Will $g(n)$ be $O(f(n))$?

no		perhaps		yes	√