CS 173, Spring 19 Examlet 5, colored

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}$ by $g(x, y) = f(x - 7)f(y)$. Prove that g is onto.

   Solution: Suppose that n is an integer.

   Since f is onto, there is an integer p such that $f(p) = 1$. Let $x = p + 7$. Then $f(x - 7) = f(p) = 1$.

   Also since f is onto, there is a natural number y such that $f(y) = n$.

   Now consider the pair (x, y). $g(x, y) = f(x - 7)f(y) = 1 \cdot n = n$. So (x, y) is a pre-image for n, which is what we needed to find.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : C \to M$ to be "onto." You must use explicit quantifiers. Do not assume the reader knows what the image of the function is.

   Solution: For every element y in M, there is an element x in C such that $g(x) = y$.

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $g(x, y) = (2f(x) + f(y), f(x) - f(y))$. Prove that g is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let (x, y) and (p, q) be elements of $\mathbb{Z}^2$ and suppose that $g(x, y) = g(p, q)$.

   By the definition of h, this means that $(2f(x) + f(y), f(x) - f(y)) = (2f(p) + f(q), f(p) - f(q))$. So $2f(x) + f(y) = 2f(p) + f(q)$ and $f(x) - f(y) = f(p) - f(q)$.

   Adding these two equations, we get $3f(x) = 3f(p)$. So $f(x) = f(p)$. Since f is one-to-one, this means that $x = p$.

   Subtracting twice the second equation from the first, we get $-3f(y) = -3f(q)$. So $f(y) = f(q)$. Since f is one-to-one, this means that $y = q$.

   Since $x = p$ and $y = q$, $(x, y) = (p, q)$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : C \to M$ to be "one-to-one." You must use explicit quantifiers; do not use words like "unique".

   Solution: For every elements x and y in C, if $g(x) = g(y)$, then $x = y$

Name:____

NetID:______________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) Suppose that A and B are sets. Suppose that $f : B \to A$ and $g : A \to B$ are functions such that $f(g(x)) = x$ for every $x \in A$. Prove that f is onto.

   Solution: Let m be an element of A. We need to find a pre-image for m.

   Consider $n = g(m)$. n is an element of B. Furthermore, since $f(g(x)) = x$ for every $x \in A$, we have $f(n) = f(g(m)) = m$.

   So n is a pre-image of m.

   Since we can find a pre-image for an arbitrarily chosen element of A, f is onto.

2. (5 points) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is one-to-one but not onto. Your answer must include a specific formula.

   Solution: Let $f(n) = n + 1$. Then f is one-to-one, but 0 isn't in the image of f.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}$ by $g(x, y) = f(x) + 2f(y) - 6$. Prove that g is onto.

   Solution: Let n be an arbitrary integer.

   Since f is onto, there is an integer input value y such that $f(y) = 3$. Similarly, there is an integer input value x such that $f(x) = n$.

   Now, consider (x, y). $g(x, y) = f(x) + 2f(y) - 6 = n + 2 \cdot 3 - 6 = n$. So (x, y) is a pre-image for n, which is what we needed to find.

2. (5 points) $A = \{0, 2, 4, 6, 8, 10, 12, \ldots\}$, i.e. the even integers starting with 0.

   $B = \{1, 4, 9, 16, 25, 36, 49, \ldots\}$, i.e. perfect squares starting with 1.

   Give a specific formula for a bijection $f : A \to B$. (You do not need to prove that it is a bijection.)

   Solution: $f(n) = (\frac{n}{2} + 1)^2$

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) Suppose that $f : A \to B$ and $g : B \to C$ are one-to-one. Prove that $g \circ f$ is one-to-one.

   Solution: Let x and y be elements of A and suppose that $g \circ f(x) = g \circ f(y)$. That is $g(f(x)) = g(f(y))$. Since g is one-to-one, this implies that $f(x) = f(y)$. Since f is one-to-one, this implies that $x = y$.

   We've shown that $g \circ f(x) = g \circ f(y)$ implies $x = y$ for any x and y in A. So $g \circ f$ is one-to-one.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : \mathbb{R} \to \mathbb{R}$ to be "increasing." You must use explicit quantifiers.

   Solution: For all x and y in $\mathbb{R}$, if $x \leq y$ then $g(x) \leq g(y)$.

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) Let P be the set of pairs of positive integers. Suppose that $f : P \to \mathbb{R}^2$ is defined by $f(x, y) = (\frac{x}{y}, x + y)$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let (x, y) and (p, q) be elements of P, i.e. pairs of positive integers. Suppose that $f(x, y) = f(p, q)$.

   By the definition of f, this means that $(\frac{x}{y}, x + y) = (\frac{p}{q}, p + q)$. So $\frac{x}{y} = \frac{p}{q}$ and $x + y = p + q$.

   Since $\frac{x}{y} = \frac{p}{q}$, $x = \frac{py}{q}$. Substituting this into $x + y = p + q$ gives us $\frac{py}{q} + y = p + q$. So $\frac{py+yq}{q} = p + q$. I.e. $\frac{y(p+q)}{q} = p + q$. So $\frac{y}{q} = 1$, and therefore $y = q$.

   Substituting $y = q$ into $x + y = p + q$ gives us $x + y = p + y$, so $x = p$.

   Therefore $(x, y) = (p, q)$, which is what we needed to prove.

2. (5 points) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is onto but not one-to-one. Your answer must include a specific formula.

   Solution: Let $f(n) = \lfloor n/2 \rfloor$ Then f is onto. But f isn't one-to-one because (for example) both 0 and 1 are mapped onto 0.

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) Suppose that $f : \mathbb{Z}^2 \to \mathbb{Z}$ is defined by $f(x, y) = 3x + 5y$. Prove that f is onto.

   Solution: Let p be an integer. We need to find a pre-image for p.

   Consider $m = (-3p, 2p)$.

   m is an element of $\mathbb{Z}^2$. We can compute

   $$f(m) = f(-3p, 2p) = 3(-3p) + 5(2p) = -9p + 10p = p$$

   So m is a pre-image of p.

   Since we can find a pre-image for an arbitrarily chosen integer, f is onto.

2. (5 points) $A = \{0, 1, 4, 9, 16, 25, 36, \ldots\}$, i.e. perfect squares starting with 0.

   $B = \{2, 4, 6, 8, 10, 12, 14, \ldots\}$, i.e. the even integers starting with 2.

   Give a specific formula for a bijection $f : A \to B$. (You do not need to prove that it is a bijection.)

   Solution: $f(n) = 2(\sqrt{n} + 1)$

Name:____

NetID:______________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) Let's define the function $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = \dfrac{4x-1}{2x+5}$. ($\mathbb{R}^+$ is the positive reals.) Prove that f is one to one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let x and y be positive reals. Suppose that $f(x) = f(y)$. By the definition of f, this means that $\frac{4x-1}{2x+5} = \frac{4y-1}{2y+5}$.

   Multiplying by the two denominators gives us $(4x-1)(2y+5) = (4y-1)2x+5$. That is $8xy - 2y + 20x - 5 = 8xy - 2x + 20y - 5$. So $-2y + 20x = -2x + 20y$. So $22x = 22y$. And therefore $x = y$, which is what we needed to prove.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : \mathbb{R} \to \mathbb{R}$ to be "strictly increasing." You must use explicit quantifiers.

   Solution: For all x and y in $\mathbb{R}$, if $x < y$ then $g(x) < g(y)$.

CS 173, Fall 18 Examlet 5, Part A

Name:____

NetID:________________________ **Lecture:** **B**

Discussion: **Friday** **11** **12** **1** **2** **3** **4**

1. (10 points) If a is any real number, (a, ∞) is the set of all real numbers greater than a. Let's define the function $f : (0, \infty) \to \left(\frac{1}{3}, \infty\right)$ by $f(x) = \dfrac{x^2 + 2}{3x^2}$. Prove that f is onto.

   Solution: Let $y \in \left(\frac{1}{3}, \infty\right)$. Then $y > \frac{1}{3}$, so $3y > 1$, and therefore $3y - 1 > 0$.

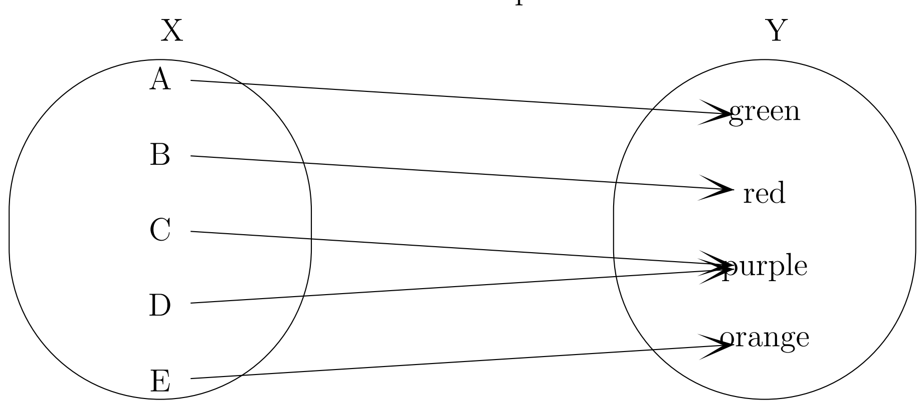
   So $\frac{2}{3y-1}$ is defined and positive. So consider $x = \sqrt{\frac{2}{3y-1}}$. x is defined and belongs to $(0, \infty)$.

   Then $x^2 = \frac{2}{3y-1}$. So $x^2 + 2 = \frac{2}{3y-1} + 2 = \frac{2+(6y-2)}{3y-1} = \frac{6y}{3y-1}$. And $3x^2 = \frac{6}{3y-1}$.

   Then $f(x) = \frac{x^2+2}{3x^2} = \frac{6y}{6} = y$.

   So we've found a pre-image for our original value y, which is what we needed to do.

2. (5 points) Complete this picture to make an example of a function that is onto but not one-to-one, by adding elements to the domain and arrows showing how input values map to output values. The elements of the domain must be letters of the alphabet.



Name:____

NetID:________________________ **Lecture:** **B**

Discussion: **Friday 11 12 1 2 3 4**

1. (10 points) Suppose that $h : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $f(x, y) = (h(x) - y, 3h(x) + 1)$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let (x, y) and (p, q) be elements of $\mathbb{Z}^2$ and suppose that $f(x, y) = f(p, q)$.

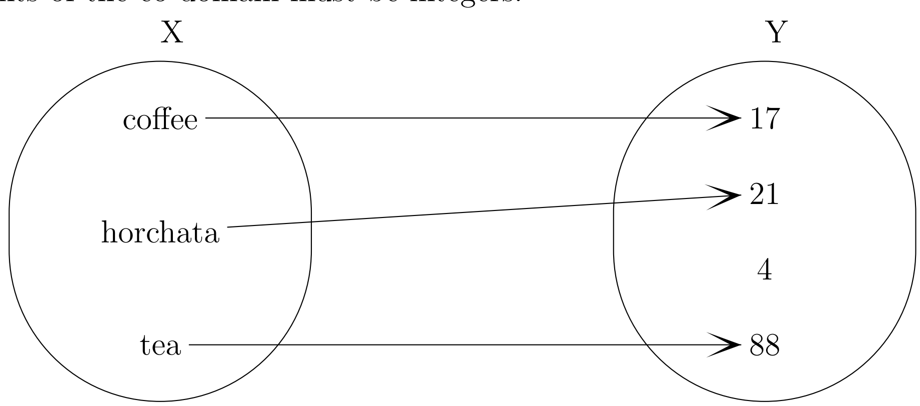
   By the definition of f, this means that $(h(x) - y, 3h(x) + 1) = (h(p) - q, 3h(p) + 1)$. So $h(x) - y = h(p) - q$ and $3h(x) + 1 = 3h(p) + 1$.

   Since $3h(x) + 1 = 3h(p) + 1$, $3h(x) = 3h(p)$. So $h(x) = h(p)$. Since h is one-to-one, this means that $x = p$.

   We now know that $h(x) = h(p)$ and $h(x) - y = h(p) - q$. Combining these equations, we get that $y = q$.

   Since $x = p$ and $y = q$, $(x, y) = (p, q)$, which is what we needed to prove.

2. (5 points) Complete this picture to make an example of a function that is one-to-one but not onto, by adding elements to the co-domain and arrows showing how input values map to output values. The elements of the co-domain must be integers.



Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is onto. Recall that the max function returns the larger of its two inputs. Let's define $g : \mathbb{N}^2 \to \mathbb{N}$ by $g(x, y) = f(\max(x - 7, 0)) + f(y)$. Prove that g is onto.

   Solution: Let n be an arbitrary natural number.

   Since f is onto, there is a natural number p such that $f(p) = 0$. Similarly, there is also a natural number r such that $f(r) = n$.

   Then $g(p + 7, r) = f(\max(p, 0)) + f(r) = f(p) + f(r) = f(p) + f(r) = 0 + n = n$. So $(p + 7, r)$ is a pre-image for n, which is what we needed to find.

2. (5 points) Recall that $\mathbb{Z}^+$ is the set of positive integers. Let's define the function $h : (\mathbb{Z}^+)^2 \to \mathbb{R}$ such that $h(d, e) = 2^d + \frac{1}{e}$. Is h one-to-one? Briefly justify your answer.

   Solution: The image of h consists of a little sequence of numbers near each power of two (starting with 2). The offset (from the power of 2) starts off with 1 and then continues with a set of increasingly small positive fractions.

   Notice that consecutive powers of 2, i.e. 2^d as we vary d, are separated by at least 2. For a fixed first input d, the output values (as we vary the second input e) are all distinct and lie between 2^d and $2^d + 1$. So the values produced for one first input d are not only distinct from each other but well-separated from the values produced by other values of d. So h is one-to-one.

CS 173, Fall 17 Examlet 5, Part A

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Let's define the function $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $f(x, y) = (x + y, 2x - 3y)$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let (x, y) and (a, b) be pairs of integers. Suppose that $f(x, y) = f(a, b)$.

   By the definition of f, $f(x, y) = f(a, b)$ implies that $(x + y, 2x - 3y) = (a + b, 2a - 3b)$. Therefore $x + y = a + b$ and $2x - 3y = 2a - 3b$.

   Subtracting twice $x + y = a + b$ from $2x - 3y = 2a - 3b$, we get $(2x - 3y) - (2x + 2y) = (2a - 3b) - (2a + 2b)$. Simplifying gives us $-5y = -5b$. So $y = b$.

   Substituting $y = b$ into $x + y = a + b$, we get $x + y = a + y$. So $x = a$.

   Since $x = a$ and $y = b$, $(x, y) = (a, b)$, which is what we needed to prove.

2. (5 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ are functions. Let's define the function $f + g$ by $(f + g)(x) = f(x) + g(x)$. Adele claims that if f and g are onto, then $f + g$ is onto. Is this correct? Briefly explain why it is or give a counter-example.

   Solution: This is not correct. Suppose that $g(x) = f(x)$ for every input x. Then $(f + g)(x)$ is even for any input x. So the odd numbers aren't in the image of $f + g$ and therefore it isn't onto.

Name:____

NetID:______________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $g : \mathbb{Z} \to \mathbb{Z}^2$ by $g(n) = (|n|, f(n)|n|)$. Prove that g is one-to-one.

   Solution:

   Let p and q be integers. Suppose that $g(p) = g(q)$.

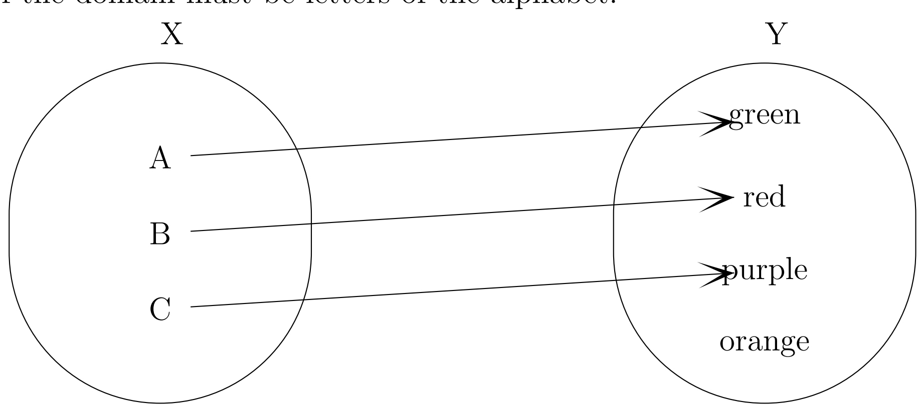
   By the definition of g, this means that $(|p|, f(p)|p|) = (|q|, f(q)|q|)$. So $|p| = |q|$ and $f(p)|p| = f(q)|q|$.

   Case 1: $|p| = 0$. Then $p = q = 0$. So $p = q$.

   Case 2: $|p|$ is non-zero. Substituting the first equation into the second, we get that $f(p)|p| = f(q)|p|$. So $f(p) = f(q)$. Since f is one-to-one, this means that $p = q$.

   So we've shown that $g(p) = g(q)$ implies that $p = q$, which means that g is one-to-one.

2. (5 points) Complete this picture to make an example of a function that is one-to-one but not onto, by adding elements to the domain and arrows showing how input values map to output values. The elements of the domain must be letters of the alphabet.



CS 173, Fall 17 Examlet 5, Part A

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) If a is any real number, (a, ∞) is the set of all real numbers greater than a. Let's define the function $f : (0, \infty) \to (\frac{5}{4}, \infty)$ by $f(x) = \frac{5x^2+3}{4x^2}$. Prove that f is onto.

   Solution: Let $y \in (\frac{5}{4}, \infty)$. Consider $x = \sqrt{\frac{3}{4y-5}}$. Since $y > \frac{5}{4}$, $4y - 5$ is always positive. So x is well defined (no dividing by zero). And, also x is positive, so x comes from the domain of f.

   Then $x^2 = \frac{3}{4y-5}$.

   So $f(x) = \frac{5x^2+3}{4x^2} = \frac{5\frac{3}{4y-5}+3}{4\frac{3}{4y-5}}$

   Multiplying by $(4y - 5)$, we get $f(x) = \frac{5 \cdot 3+3(4y-5)}{4 \cdot 3} = \frac{15+12y-15}{12} = \frac{12y}{12} = y$

   Since y was chosen arbitrarily from the co-domain of f, we've shown that f is onto.

2. (5 points) What's wrong with this attempt to define $f \circ g$?

   > If $f : A \to B$ and $g : B \to C$ are functions, then $f \circ g$ is the function from A to C defined by $(f \circ g)(x) = f(g(x))$.

   Solution: This is applying the two functions in the wrong order. The domain of $f \circ g$ is stated to be A, so x must be an element of A. But we're applying g first and the inputs to g must come from the set B. So this is secretly assuming some overlap among the three sets, which you can't do in a general definition of composition.

CS 173, Fall 17 Examlet 5, Part A

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}$ by $g(x, y) = f(x) + 2f(y) - 6$. Prove that g is onto.

   Solution: Let n be an arbitrary integer.

   Since f is onto, there is an integer input value y such that $f(y) = 3$. Similarly, there is an integer input value x such that $f(x) = n$.

   Now, consider (x, y). $g(x, y) = f(x) + 2f(y) - 6 = n + 2 \cdot 3 - 6 = n$. So (x, y) is a pre-image for n, which is what we needed to find.

2. (5 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is increasing (but perhaps not strictly increasing). Dumbledore claims that f must be one-to-one. Is he correct? Briefly explain why he is or give a concrete counter-example.

   Solution: He's wrong. Suppose that $f(n) = 0$ for every input value n. Then f is (non-strictly) increasing, but not one-to-one.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Suppose that $h : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $f(x, y) = (h(x) - y, 3h(x) + 1)$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let (x, y) and (p, q) be elements of $\mathbb{Z}^2$ and suppose that $f(x, y) = f(p, q)$.

   By the definition of f, this means that $(h(x) - y, 3h(x) + 1) = (h(p) - q, 3h(p) + 1)$. So $h(x) - y = h(p) - q$ and $3h(x) + 1 = 3h(p) + 1$.

   Since $3h(x) + 1 = 3h(p) + 1$, $3h(x) = 3h(p)$. So $h(x) = h(p)$. Since h is one-to-one, this means that $x = p$.

   We now know that $h(x) = h(p)$ and $h(x) - y = h(p) - q$. Combining these equations, we get that $y = q$.

   Since $x = p$ and $y = q$, $(x, y) = (p, q)$, which is what we needed to prove.

2. (5 points) Recall that $\mathbb{Z}^+$ is the set of positive integers. Let's define the function $h : (\mathbb{Z}^+)^2 \to \mathbb{R}$ such that $h(d, e) = 2^d + \frac{1}{e}$. Is h onto? Briefly justify your answer.

   Solution: The image of h consists of a little sequence of numbers near each power of two (starting wtih 2). The offset (from the power of 2) starts off with 1 and then continues with a set of increasingly small positive fractions. So every output value lies between a power of two and the next integer. This means that (for example) there are no output values between 17 (i.e. 16+1) and 32 (the next power of two). So h isn't onto.

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}$ by $g(x, y) = (1 - f(x))f(y)$. Prove that g is onto.

   Solution: Suppose that n is a natural number.

   Since f is onto, there is a natural number p such that $f(p) = 0$. Then $(1 - f(p)) = 1$

   Also since f is onto, there is a natural number q such that $f(q) = n$.

   Now consider the pair (p, q). $g(p, q) = (1 - f(p))f(q) = 1 \cdot n = n$. So (p, q) is a pre-image for n, which is what we needed to find.

2. (5 points) Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is one-to-one but not onto. Be specific.

   Solution: Let $f(n) = n + 1$ if $n \geq 0$, and $f(n) = n - 1$ if $n < 0$. Then f is one-to-one. However, it's not onto because 0 isn't in the image of f.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Let's define the function $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = \dfrac{x^2+2}{3x^2}$. ($\mathbb{R}^+$ is the positive reals.) Prove that f is one to one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let x and y be positive reals. Suppose that $f(x) = f(y)$. By the definition of f, this means that $\frac{x^2+2}{3x^2} = \frac{y^2+2}{3y^2}$.

   Multiplying both sides by the denominators, we get $(x^2+2)(3y^2) = (3x^2)(y^2+2)$. That is, $3x^2y^2 + 6y^2 = 3x^2y^2 + 6x^2$. Thus $6y^2 = 6x^2$. So $y^2 = x^2$. Because x and y are positive, this means that $x = y$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : M \to C$ to be "onto." You must use explicit quantifiers. Do not assume the reader knows what the image of the function is.

   Solution: For every element y in C, there is an element x in M such that $g(x) = y$.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Let P be the set of pairs of positive integers. Suppose that $f : P \to \mathbb{R}^2$ is defined by $f(x, y) = (\frac{x}{y}, x + y)$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) the behavior of increasing functions.

   Solution: Let (x, y) and (p, q) be elements of P, i.e. pairs of positive integers. Suppose that $f(x, y) = f(p, q)$.

   By the definition of f, this means that $(\frac{x}{y}, x + y) = (\frac{p}{q}, p + q)$. So $\frac{x}{y} = \frac{p}{q}$ and $x + y = p + q$.

   Since $\frac{x}{y} = \frac{p}{q}$, $x = \frac{py}{q}$. Substituting this into $x + y = p + q$ gives us $\frac{py}{q} + y = p + q$. So $\frac{py+yq}{q} = p + q$. I.e. $\frac{y(p+q)}{q} = p + q$. So $\frac{y}{q} = 1$, and therefore $y = q$.

   Substituting $y = q$ into $x + y = p + q$ gives us $x + y = p + y$, so $x = p$.

   Therefore $(x, y) = (p, q)$, which is what we needed to prove.

2. (5 points) $A = \{2, 3, 4, 5, 6, 7, 8 \ldots\}$, i.e. the integers ≥ 2

   $B = \{1, 2, 4, 8, 16, 32, 64, \ldots\}$, i.e. powers of 2 starting with 1

   Give a specific formula for a bijection $f : A \to B$. (You do not need to prove that it is a bijection.)

   Solution: $f(n) = 2^{n-2}$

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is onto. Let's define $g : \mathbb{N}^2 \to \mathbb{Z}$ by $g(m, n) = (2 - n)f(m)$. Prove that g is onto.

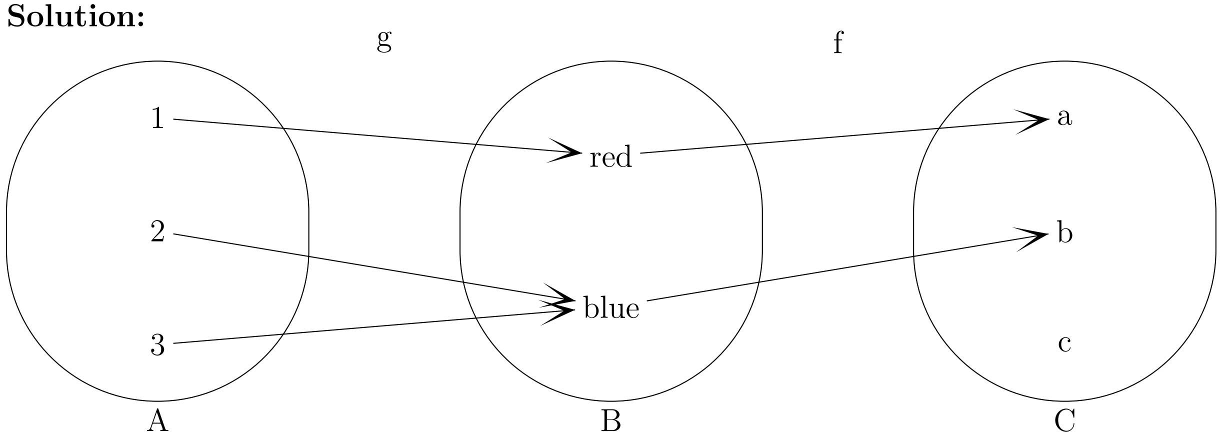
   Solution: Let y be an integer. Since f is onto, there exists $x \in \mathbb{N}$ such that $f(x) = |y|$. Then there are two cases:

   - Case 1: $y \geq 0$. Let $m = x$ and $n = 1$. Then $m, n \in \mathbb{N}$ and $g(m, n) = g(x, 1) = (2 - 1)f(x) = f(x) = |y| = y$.
   - Case 2: $y < 0$. Let $m = x$ and $n = 3$. Then $m, n \in \mathbb{N}$ and $g(m, n) = g(x, 3) = (2 - 3)f(x) = -f(x) = -|y| = -(-y) = y$.

   Since this argument works for any choice of y, we have shown that g is onto.

2. (5 points) Suppose that $g : A \to B$ and $f : B \to C$. Prof. Snape claims that if g is onto, then $f \circ g$ is onto. Disprove this claim using a concrete counter-example in which A, B, and C are all small finite sets.

   Solution:



Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Let $g : \mathbb{N} \to \mathbb{N}$ be onto, and let $f : \mathbb{N}^2 \to \mathbb{Z}$ be defined by $f(n, m) = (m-1)g(n)$. Prove that f is onto.

   Solution: Let a be an integer.

   Case 1) $a \geq 0$. Since g is onto, we can find a natural number n such that $g(n) = a$. Let $m = 2$. Then $f(n, m) = (2-1)g(n) = 1 \cdot a = a$.

   Case 2) $a \leq 0$. Then $(-a)$ is a natural number. Since g is onto, we can find a natural number n such that $g(n) = (-a)$. Let $m = 0$. Then $f(n, m) = (0-1)g(n) = (-1) \cdot (-a) = a$.

   So we've found a point (n, m) such that $g(n, m) = a$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : M \to C$ to be "one-to-one." You must use explicit quantifiers; do not use words like "unique".

   Solution: For every elements x and y in M, if $g(x) = g(y)$, then $x = y$

Name:____

NetID:_______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Let's define the function $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = \dfrac{4x-1}{2x+5}$. ($\mathbb{R}^+$ is the positive reals.) Prove that f is one to one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let x and y be positive reals. Suppose that $f(x) = f(y)$. By the definition of f, this means that $\frac{4x-1}{2x+5} = \frac{4y-1}{2y+5}$.

   Multiplying by the two denominators gives us $(4x-1)(2y+5) = (4y-1)2x+5$. That is $8xy - 2y + 20x - 5 = 8xy - 2x + 20y - 5$. So $-2y + 20x = -2x + 20y$. So $22x = 22y$. And therefore $x = y$, which is what we needed to prove.

2. (5 points) Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is onto but not one-to-one. Be specific.

   Solution: Let $f(n) = n$ if $n \geq 0$, and $f(n) = n + 1$ if $n < 0$. Then f is onto. However, it's not one-to-one because 0 has two pre-images.