CS 173, Spring 19 Examlet 8, colored

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(20 points) Suppose that $g : \mathbb{N} \to \mathbb{R}$ is defined by

$g(0) = 0 \qquad g(1) = \frac{4}{3}$

$g(n) = \frac{4}{3}g(n-1) - \frac{1}{3}g(n-2)$, for $n \geq 2$

Use (strong) induction to prove that $g(n) = 2 - \dfrac{2}{3^n}$

Solution: Proof by induction on n.

Base case(s): $n = 0$: $2 - \frac{2}{3^n} = 2 - \frac{2}{1} = 0 = g(0)$ So the claim holds.

$n = 1$: $2 - \frac{2}{3^n} = 2 - \frac{2}{3} = \frac{4}{3} = g(1)$ So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $g(n) = 2 - \frac{2}{3^n}$, for $n = 0, 1, \cdots, k-1$ for some integer $k \geq 2$.

Inductive Step:

We need to show that $g(k) = 2 - \dfrac{2}{3^k}$

$$\begin{aligned}
g(k) &= \frac{4}{3}g(k-1) - \frac{1}{3}g(k-2) && \text{[by the def, } k \geq 2\text{]} \\
&= \frac{4}{3}\left(2 - \frac{2}{3^{k-1}}\right) - \frac{1}{3}\left(2 - \frac{2}{3^{k-2}}\right) && \text{[Inductive Hypothesis]} \\
&= \frac{8}{3} - \frac{8}{3^k} - \frac{2}{3} + \frac{2}{3^{k-1}} \\
&= \frac{6}{3} - \frac{8}{3^k} + \frac{6}{3^k} \\
&= 2 - \frac{2}{3^k}.
\end{aligned}$$

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(20 points) Let function $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$f(0) = 2$

$f(1) = 7$

$f(n) = f(n-1) + 2f(n-2)$, for $n \geq 2$

Use (strong) induction to prove that $f(n) = 3 \cdot 2^n + (-1)^{n+1}$ for any natural number n.

Solution: Proof by induction on n.

Base case(s): For $n = 0$, we have $3 \cdot 2^0 + (-1)^1 = 3 - 1 = 2$ which is equal to $f(0)$. So the claim holds.

For $n = 1$, we have $3 \cdot 2^1 + (-1)^2 = 6 + 1 = 7$ which is equal to $f(1)$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $f(n) = 3 \cdot 2^n + (-1)^{n+1}$, for $n = 0, 1, \ldots, k-1$ where $k \geq 2$.

Rest of the inductive step:

$$\begin{aligned}
f(k) &= f(k-1) + 2f(k-2) && \text{by definition of } f \\
&= (3 \cdot 2^{k-1} + (-1)^k) \quad + \quad 2(3 \cdot 2^{k-2} + (-1)^{k-1}) && \text{by inductive hypothesis} \\
&= (3 \cdot 2^{k-1} + (-1)^k) \quad + \quad 3 \cdot 2^{k-1} + 2(-1)^{k-1} \\
&= 6 \cdot 2^{k-1} + (-1)^k - 2(-1)^k \\
&= 3 \cdot 2^k - (-1)^k \\
&= 3 \cdot 2^k (-1)^{k+1}
\end{aligned}$$

So $f(k) = 3 \cdot 2^k(-1)^{k+1}$, which is what we needed to show.

Name:____

NetID:______________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(20 points) Use (strong) induction to prove that the following claim holds:

Claim : For any integer $n \geq 2$, if $p_1, \ldots, p_n$ is a sequence of integers and $p_1 < p_n$, then there is an index j $(1 \leq j < n)$ such that $p_j < p_{j+1}$.

Solution:

Base case(s): Proof by induction on n. At $n = 2$: It's given that $p_1 < p_n$. But $p_n = p_2$. So $p_1 < p_2$ and so $j = 1$ is the required index.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that any sequence of integers $p_1, \ldots, p_n$ with $p_1 < p_n$ has an index j $(1 \leq j < n)$ such that $p_j < p_{j+1}$, for $n = 2, \ldots, k$.

Rest of the inductive step: Let $p_1, \ldots, p_{k+1}$ be a sequence of $k+1$ integers, with $p_1 < p_{k+1}$.

Consider p_k and p_{k+1}. There are two cases:

Case (1): $p_k < p_{k+1}$. Then the index $j = k$ works.

Case (2): $p_k \geq p_{k+1}$. Then we have $p_1 < p_{k+1}$ and $p_{k+1} \leq p_k$. So $p_1 < p_k$. So we can apply the inductive hypothesis to the shorter subsequence $p_1, \ldots, p_k$. That is, by the inductive hypothesis, there is an index j into the subsequence (i.e. $1 \leq j < k$) such that $p_j < p_{j+1}$. This (obviously) also works as an index into the longer sequence of $k+1$ integers.

In both cases, we have found an index j such that $p_j < p_{j+1}$, which is what we needed to find.

[Notes: it also works to remove the first element p_1 from the sequence, with small changes to the inductive step. Your inductive step doesn't need to be quite this detailed.]

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by is defined by

$f(1) = 5 \qquad f(2) = -5$

$f(n) = 4f(n-2) - 3f(n-1)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) = 2 \cdot (-4)^{n-1} + 3$

Solution: Proof by induction on n.

Base case(s): For $n = 1$, $2 \cdot (-4)^{n-1} + 3 = 2 \cdot (-4)^0 + 3 = 2 \cdot 1 + 3 = 5$, which is equal to $f(1)$.

For $n = 2$, $2 \cdot (-4)^{n-1} + 3 = 2 \cdot (-4)^1 + 3 = 2 \cdot (-4) + 3 = -5$, which is equal to $f(2)$.

So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) = 2 \cdot (-4)^{n-1} + 3$, for $n = 1, 2, \ldots, k-1$, for some integer $k \geq 3$

Rest of the inductive step:

Using the definition of f and the inductive hypothesis, we get

$f(k) = 4f(k-2) - 3f(k-1) = 4(2 \cdot (-4)^{k-3} + 3) - 3(2 \cdot (-4)^{k-2} + 3)$

Simplifying the algebra,

$$\begin{aligned}
4(2 \cdot (-4)^{k-3} + 3) - 3(2 \cdot (-4)^{k-2} + 3) &= 8 \cdot (-4)^{k-3} + 12 - 6 \cdot (-4)^{k-2} - 9 \\
&= -2 \cdot (-4)^{k-2} - 6 \cdot (-4)^{k-2} + 3 \\
&= -8 \cdot (-4)^{k-2} + 3 = 2 \cdot (-4)^{k-1} + 3
\end{aligned}$$

So $f(k) = 2 \cdot (-4)^{k-1} + 3$, which is what we needed to prove.

Name:____

NetID:______________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(20 points) Suppose that θ is a constant (but unknown) real number. For any real number p, the angle addition formulas imply the following two equations (which you can assume without proof):

$$\cos(\theta)\cos(p\theta) = \cos((p+1)\theta) + \sin(\theta)\sin(p\theta) \quad (1)$$

$$\cos(\theta)\cos(p\theta) = \cos((p-1)\theta) - \sin(\theta)\sin(p\theta) \quad (2)$$

Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$f(0) = 1$ $\quad f(1) = \cos(\theta)$

$f(n+1) = 2\cos(\theta)f(n) - f(n-1)$, for all $n \geq 2$.

Use (strong) induction to prove that $f(n) = \cos(n\theta)$ for any natural number n.

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $f(n) = f(0) = 1 = \cos(0) = \cos(0\theta) = \cos(n\theta)$.

At $n = 1$, $f(n) = f(1) = \cos\theta = \cos(1\theta) = \cos(n\theta)$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

$f(n) = \cos(n\theta)$ for $n = 0, \ldots, k$.

Rest of the inductive step: In particular, by the inductive hypothesis, $f(k) = \cos(k\theta)$ and $f(k-1) = \cos((k-1)\theta)$.

If we set $p = k$ in equations (1) and (2), and then add them together, we get

$$2\cos(\theta)\cos(k\theta) = \cos((k+1)\theta) + \cos((k-1)\theta)$$

So then we can compute

$$\begin{aligned} f(k+1) &= 2\cos(\theta)f(k) - f(k-1) \\ &= 2\cos(\theta)\cos(k\theta) - \cos((k-1)\theta) \qquad \text{(by the IH)} \\ &= \cos((k+1)\theta) + \cos((k-1)\theta) + \cos((k-1)\theta) \\ &= \cos((k+1)\theta) \end{aligned}$$

So $f(k+1) = \cos((k+1)\theta)$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) A Zellig graph consists of $2n$ ($n \geq 1$) nodes connected so as to form a circle. Half of the nodes have label 1 and the other half have label -1. As you move clockwise around the circle, you keep a running total of node labels. E.g. if you start at a 1 node and then pass through two -1 nodes, your running total is -1. Use (strong) induction to prove that there is a choice of starting node for which the running total stays ≥ 0.

Hint: remove an adjacent pair of nodes.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, there are only two nodes. If you start at the node with label 1, the running total stays ≥ 0.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that there is a choice of starting node for which the running total stays ≥ 0, for Zellig graphs with $2n$ nodes, where $n = 1, \ldots, k-1$.

Rest of the inductive step: Let G be a Zellig graph with $2k$ nodes. Find a 1 node that immediately precedes a -1 (going clockwise). Remove those two nodes m and s from G to create a smaller graph H.

By the inductive hypothesis, we can find a starting node p on H such that the running total stays ≥ 0. I claim that p also works as a starting node for G. Between p and m, we see the same sequence of nodes as in H, so the total stays ≥ 0. The total increases by 1 at m and the immediately decreases by 1 at s. So it can't dip below zero in that section of the circle. Between s and returning to p, we have the same running totals as in H.

So G has a starting point for which all the running totals stay ≥ 0, which is what we needed to prove.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) (20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$f(0) = 2 \qquad f(1) = 5 \qquad f(2) = 15$

$f(n) = 6f(n-1) - 11f(n-2) + 6f(n-3)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) = 1 - 2^n + 2 \cdot 3^n$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $f(0) = 2$ and $1 - 2^n + 2 \cdot 3^n = 1 - 1 + 2 = 2$

At $n = 1$, $f(1) = 5$ and $1 - 2^n + 2 \cdot 3^n = 1 - 2 + 6 = 5$

At $n = 2$, $f(2) = 15$ and $1 - 2^n + 2 \cdot 3^n = 1 - 4 + 18 = 15$

So the claim holds at all three values.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) = 1 - 2^n + 2 \cdot 3^n$ for $n = 0, 1, \ldots, k-1$.

Rest of the inductive step: By the definition of f and the inductive hypothesis, we get

$$\begin{aligned}
f(k) &= 6f(k-1) - 11f(k-2) + 6f(k-3) \\
&= 6(1 - 2^{k-1} + 2 \cdot 3^{k-1}) - 11(1 - 2^{k-2} + 2 \cdot 3^{k-2}) + 6(1 - 2^{k-3} + 2 \cdot 3^{k-3}) \\
&= (6 - 11 + 6) - (6 \cdot 2^{k-1} - 11 \cdot 2^{k-2} + 6 \cdot 2^{k-3}) + 2(6 \cdot 3^{k-1} - 11 \cdot 3^{k-2} + 6 \cdot 3^{k-3}) \\
&= 1 - (12 \cdot 2^{k-2} - 11 \cdot 2^{k-2} + 3 \cdot 2^{k-2}) + 2(18 \cdot 3^{k-2} - 11 \cdot 3^{k-2} + 2 \cdot 3^{k-2}) \\
&= 1 - 4 \cdot 2^{k-2} + 2 \cdot 9 \cdot 3^{k-2} = 1 - 2^k + 2 \cdot 2^k
\end{aligned}$$

So $f(k) = 1 - 2^k + 2 \cdot 2^k$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Use (strong) induction to prove that, for any integer $n \geq 8$, there are non-negative integers p and q such that $n = 3p + 5q$.

Solution: Proof by induction on n.

Base case(s): At $n = 8$, we can chose $p = 1$ and $q = 1$. At $n = 9$, we can chose $p = 3$ and $q = 0$. At $n = 10$, we can chose $p = 0$ and $q = 2$. In all three cases, $n = 3p + 5q$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that there are non-negative integers p and q such that $n = 3p + 5q$, for $n = 8, 9, \ldots, k - 1$, where $k \geq 11$.

Rest of the inductive step: Consider $n = k$.

Notice that $k \geq 11$, so $8 \leq k - 3 \leq k - 1$. So $k - 3$ is covered by the inductive hypothesiss. Therefore, there are non-negative integers r and q such that $k - 3 = 3r + 5q$.

Now, set $p = r + 1$. Then $k = (k - 3) + 3 = (3r + 5q) + 3 = 3(r + 1) + 5q = 3p + 5q$. p is non-negative since r is.

So there are non-negative integers p and q such that $k = 3p + 5q$, which is what we needed to prove.

CS 173, Fall 18 Examlet 8, Part A

Name: ____

NetID: ________________________ **Lecture:** **B**

Discussion: **Friday** **11** **12** **1** **2** **3** **4**

(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$f(1) = 3$ $\qquad f(2) = 5$

$f(n) = 3f(n-1) - 2f(n-2)$ for all $n \geq 3$.

Use (strong) induction to prove that $f(n) = 2^n + 1$

Solution: Proof by induction on n.

Base case(s):

$n = 1$: $f(1) = 3$. Also $2^1 + 1 = 3$.

$n = 2$: $f(2) = 5$. Also $2^2 + 1 = 5$.

So the claim holds for both $n = 1$ and $n = 2$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) = 2^n + 1$ for $n = 1, 2, \ldots, k-1$.

Rest of the inductive step:

By the definition of f and the inducive hypothesis, we get that

$$\begin{aligned} f(k) &= 3f(k-1) = 2f(k-2) \\ &= 3(2^{k-1} + 1) - 2(2^{k-2} + 1) \end{aligned}$$

Simplifying the algebra, we get:

$$\begin{aligned} f(k) &= 3 \cdot 2^{k-1} + 3 - 2^{k-1} - 2 \\ &= (3-1)2^{k-1} + (3-2) = 2^k + 1 \end{aligned}$$

Name:____

NetID:_____________________ **Lecture:** **B**

Discussion: **Friday** **11** **12** **1** **2** **3** **4**

(20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$f(0) = f(1) = f(2) = 1$

$f(n) = f(n-1) + f(n-3)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) \geq \frac{1}{2}(\sqrt{2})^n$. You may use the fact that $\sqrt{2}$ is smaller than 1.5.

Solution: Proof by induction on n.

Base case(s): For $n = 0$, $\frac{1}{2}(\sqrt{2})^n = \frac{1}{2}$. For $n = 1$, $\frac{1}{2}(\sqrt{2})^n = \frac{1}{2}(\sqrt{2}) = \frac{1}{\sqrt{2}}$. For $n = 2$, $\frac{1}{2}(\sqrt{2})^n = \frac{1}{2}(\sqrt{2})^2 = \frac{1}{2}(2) = 1$ In all three cases, $f(n)$ is 1 and the value of $\frac{1}{2}(\sqrt{2})^n$ is ≤ 1.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) \geq \frac{1}{2}(\sqrt{2})^n$ for $n = 0, 1, \ldots, k-1$.

Rest of the inductive step:

Using the definition of f and the inductive hypothesis, we get

$$f(k) = f(k-1) + f(k-3) \geq \frac{1}{2}(\sqrt{2})^{k-1} + \frac{1}{2}(\sqrt{2})^{k-3}$$

Simplifying this expression, we get

$$\begin{aligned} f(k) &\geq \frac{1}{2}(\sqrt{2})^{k-1} + \frac{1}{2}(\sqrt{2})^{k-3} = \frac{1}{2}(\sqrt{2})^{k-1} + \frac{1}{2}\frac{1}{2}(\sqrt{2})^{k-1} \\ &= \frac{1}{2}(\sqrt{2})^{k-1}(1 + \frac{1}{2}) = \frac{1}{2}(\sqrt{2})^{k-1}(1.5) \\ &\geq \frac{1}{2}(\sqrt{2})^{k-1}(\sqrt{2}) = \frac{1}{2}(\sqrt{2})^k \end{aligned}$$

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Let x be a non-zero real number such that $x + \frac{1}{x}$ is an integer. Use (strong) induction to prove that $x^n + \frac{1}{x^n}$ is an integer, for any natural number n.

Hint: $(a^n + b^n)(a + b) = (a^{n+1} + b^{n+1}) + ab(a^{n-1} + b^{n-1})$, for any real numbers a and b.

Solution: Let x be a non-zero real number such that $x + \frac{1}{x}$.

Proof by induction on n.

Base case(s): At $n = 0$, $x^n + \frac{1}{x^n} = 1 + 1 = 2$, which is an integer for any non-zero x.

At $n = 1$, $x^n + \frac{1}{x^n} = x + \frac{1}{x}$, so the claim is obviously true.

[Notice that we need two base cases because our inductive step will use the result at two previous values of n.]

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $x^n + \frac{1}{x^n}$ is an integer, for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

Using the hint, we get

$$x^{k+1} + \frac{1}{x^{k+1}} \quad = \quad (x^k + \frac{1}{x^k})(1 + \frac{1}{x}) - (x \cdot \frac{1}{x})(x^{k-1} + \frac{1}{x^{k-1}}) \quad = \quad (x^k + \frac{1}{x^k})(1 + \frac{1}{x}) - (x^{k-1} + \frac{1}{x^{k-1}})$$

By the inductive hypothesis, $x^k + \frac{1}{x^k}$ and $x^{k-1} + \frac{1}{x^{k-1}}$ are integers. We were also given that $(1 + \frac{1}{x})$ is an integer. The righthand side must be an integer since it's made by multiplying and subtracting integers. So the lefthand side $x^{k+1} + \frac{1}{x^{k+1}}$ must also be an integer. This is what we needed to show.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by

$h(1) = 1 \qquad h(2) = 7$

$h(n+1) = 7h(n) - 12h(n-1)$ for all $n \geq 2$

Use (strong) induction to prove that $h(n) = 4^n - 3^n$

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $h(1) = 1$ and $4^n - 3^n = 4 - 3 = 1$. So the claim holds.

At $n = 2$, $h(2) = 7$ and $4^n - 3^n = 16 - 9 = 7$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $h(n) = 4^n - 3^n$ for $n = 1, 2, \ldots, k$.

Rest of the inductive step:

Using the definition of h and the inductive step, we get

$$\begin{aligned} h(k+1) &= 7h(k) - 12h(k-1) \\ &= 7(4^k - 3^k) - 12(4k - 1 + 3^{k-1}) \\ &= 7(4^k - 3^k) - (3 \cdot 4k + 4 \cdot 3^k) \\ &= (7-3)4^k - (7-4)3^k = 4^{k+1} - 3^{k+1} \end{aligned}$$

So $h(k+1) = 4^{k+1} - 3^{k+1}$, which is what we needed to show.

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Recall that F_n is the nth Fibonacci number, and the positive Fibonacci numbers start with $F_1 = F_2 = 1$. Use (strong) induction to prove the following claim:

> Claim: Every positive integer can be written as the sum of (one or more) distinct Fibonacci numbers.

Hints: You can assume that the Fibonacci numbers are strictly increasing starting with F_1. To write x as the sum of Fibonacci numbers, start by including the largest Fibonacci number F_p such that $F_p \leq x$. (And therefore $x < F_{p+1}$.) How large is the remaining part of x?

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $n = 1 = F_1$. So n is the sum of a single Fibonacci number.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that n is the sum of (one or more) distinct Fibonacci numbers, for $n = 1, \ldots, k-1$.

Inductive Step: Consider k. Notice that we can assume that $k > 1$, since $n = 1$ was already covered in the base case. Let F_p be the largest Fibonacci number $\leq k$. There are two cases:

Case 1: $F_p = k$. Then k is the sum of a single Fibonacci number.

Case 2: $F_p < k$. Let $y = k - F_p$. Since F_p must be positive, y is less than k. So we can apply the inductive hypothesis to y. That is $y = F_{i_1} + \ldots + F_{i_j}$, where $F_{i_1} \ldots F_{i_j}$ are all distinct.

Notice that $x < F_{p+1} = P_p + P_{p-1}$ So $y = k - F_p < P_{p-1}$. This means that $F_{i_1} \ldots F_{i_j}$ are all smaller than F_p, so F_p can't be equal to any of them.

So then $k = y + F_p = (F_{i_1} + \ldots F_{i_j}) + F_p$ and the numbers in this sum are all distinct. So k is the sum of (one or more) distinct Fibonacci numbers, which is what we needed to prove.

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Use (strong) induction to prove the following claim:

> For any positive integer $n \geq 2$, if G is a graph with n nodes and more than $(n-1)(n-2)/2$ edges, then G is connected.

Hint: pick a node x. Perhaps x is connected to all the other nodes. If not, remove x to create a smaller graph H. What is the smallest number of edges that could remain in H? Notice that H has too few nodes to contain all the edges in G, so there is an edge from x to H.

Solution: Proof by induction on n.

Base case(s): $n = 2$ The graph has two nodes and one edge. There's only one such graph and it's connected.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: If G is a graph with n nodes and more than $(n-1)(n-2)/2$ edges, then G is connected, for $n = 2, \ldots, k-1$.

Rest of the inductive step: Let G be a graph with k nodes and more than $(k-1)(k-2)/2$ edges.

Pick a node x in G. Remove x (and its edges) to produce a smaller graph H.

Case 1: x is connected to all the other $k-1$ nodes in G. Then there is a path from any node to any other node, via x. So G is connected.

Case 2: x is connected to $k-2$ or fewer nodes. This means that H must have more than $(k-1)(k-2)/2 - (k-2)$ edges. $(k-1)(k-2)/2 - (k-2) = (k-1)(k-2)/2 - 2(k-2)/2 = (k-3)(k-2)/2$. So H has $k-1$ nodes and more than $(k-3)(k-2)/2$ edges. By the inductive hypothesis, H must be connected.

H has $k-1$ nodes. The maximum number of edges in H is $(k-1)(k-2)/2$, i.e. the number of edges in a complete graph. Since G has more edges than that, there must be at least one edge connecting x to a node of H.

Since H is connected, and x is connected to a node in H, the full graph G is connected, which is what we needed to prove.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Recall that F_n is the nth Fibonacci number, and these start with $F_0 = 0$, $F_1 = 1$. Use (strong) induction to prove the following claim:

Claim: $F_{n-1}F_{n+1} - (F_n)^2 = (-1)^n$ for any positive integer n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $F_{n-1}F_{n+1} - (F_n)^2 = F_0F_2 - (F_1)^2 = 0 - 1 = -1 = (-1)^n$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $F_{n-1}F_{n+1} - (F_n)^2 = (-1)^n$ for $n = 1, \dots, k$.

Inductive Step: By the definition of the Fibonacci numbers, $F_{k+2} = F_{k+1} + F_k$. So

$F_kF_{k+2} - (F_{k+1})^2 = F_k(F_{k+1} + F_k) - (F_{k+1})^2 = (F_k)^2 + F_kF_{k+1} - (F_{k+1})^2 = (F_k)^2 + F_{k+1}(F_k - F_{k+1})$

But since $F_{k+1} = F_k + F_{k-1}$, $F_k - F_{k+1} = -F_{k-1}$. So

$F_kF_{k+2} - (F_{k+1})^2 = (F_k)^2 - F_{k+1}F_{k-1} = (-1)(F_{k+1}F_{k-1} - (F_k)^2)$

By the inductive hypothesis, $F_{k-1}F_{k+1} - (F_k)^2 = (-1)^k$. Substituting this into the previous equation, we get

$F_kF_{k+2} - (F_{k+1})^2 = (-1)(-1)^k$

So $F_kF_{k+2} - (F_{k+1})^2 = (-1)^{k+1}$, which is what we needed to prove.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Let function $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$f(0) = 3$

$f(1) = 9$

$f(n) = f(n-1) + 2f(n-2)$, for $n \geq 2$

Use (strong) induction to prove that $f(n) = 4 \cdot 2^n + (-1)^{n-1}$ for any natural number n.

Solution: Proof by induction on n.

Base case(s): For $n = 0$, we have $4 \cdot 2^0 + (-1)^{-1} = 4 - 1 = 3$ which is equal to $f(0)$. So the claim holds.

For $n = 1$, we have $4 \cdot 2^1 + (-1)^0 = 8 + 1 = 9$ which is equal to $f(1)$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $f(n) = 4 \cdot 2^n + (-1)^{n-1}$ for $n = 0, 1, \ldots, k-1$ where $k \geq 2$.

Rest of the inductive step:

$$
\begin{aligned}
f(k) &= f(k-1) + 2f(k-2) && \text{by definition of } f \\
&= (4 \cdot 2^{k-1} + (-1)^{k-2}) \quad + \quad 2(4 \cdot 2^{k-2} + (-1)^{k-3}) && \text{by inductive hypothesis} \\
&= (4 \cdot 2^{k-1} + (-1)^{k-2}) \quad + \quad 4 \cdot 2^{k-1} + 2(-1)^{k-3} \\
&= 8 \cdot 2^{k-1} + (-1)^{k-2} - 2(-1)^{k-2} \\
&= 4 \cdot 2^k - (-1)^{k-2} \\
&= 4 \cdot 2^k + (-1)^{k-1}
\end{aligned}
$$

So $f(k) = 4 \cdot 2^k + (-1)^{k-1}$, which is what we needed to show.

CS 173, Fall 17 Examlet 8, Part A

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Use (strong) induction to prove that $a - b$ divides $a^n - b^n$, for any integers a and b and any natural number n.

Hint: $(a^n - b^n)(a + b) = (a^{n+1} - b^{n+1}) + ab(a^{n-1} - b^{n-1})$, for any real numbers a and b.

Solution: Let a and b be integers.

Proof by induction on n.

Base case(s):

At $n = 0$, $a^n - b^n = 1 - 1 = 0$, which is a multiple of any integer. So it's divisible by $a - b$.

At $n = 1$, $a^n - b^n = a - b$, so $a - b$ divides $a^n - b^n$.

[Notice that we need two base cases because our inductive step will use the result at two previous values of n.]

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $a - b$ divides $a^n - b^n$, for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

From the hint, we know that $a^{k+1} - b^{k+1} = (a^k - b^k)(a + b) - ab(a^{k-1} - b^{k-1})$

Notice that $(a^k - b^k)$ is divisible by $(a - b)$ by the inductive hypothesis. $(a + b)$ is an integer since a and b are integers. So $(a^k - b^k)(a + b)$ must be divisible by $(a - b)$.

Similarly, $(a^{k-1} - b^{k-1})$ is divisible by $(a - b)$ by the inductive hypothesis. Also ab is an integer because a and b are integers. So $ab(a^{k-1} - b^{k-1})$ is divisible by $(a - b)$.

So $(a^k - b^k)(a + b) - ab(a^{k-1} - b^{k-1})$ must be divisible by $(a - b)$, and therefore $a^{k+1} - b^{k+1}$ must be divisible by $(a - b)$, which is what we needed to show.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $f : \mathbb{N}^2 \to \mathbb{N}$ is defined by

$f(n, 0) = f(n, n) = 1$, for any natural number n

$f(n, a) = f(n-1, a-1) + f(n-1, a)$, for all n and a such that $1 \le a \le n-1$

Use (strong) induction to prove that $f(n, a) = \frac{n!}{a!(n-a)!}$ for any natural numbers a and n, where $n \ge a$. Hint: use n as your induction variable. At each step, make sure the equations work for an arbitrary natural number $a \le n$.

Solution: Proof by induction on n.

Base case(s): At $n = 0$, a must also be zero. So $f(n, a) = f(0, 0) = 1$. Also $\frac{n!}{a!(n-a)!} = \frac{0!}{0!0!} = 1$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $f(n, a) = \frac{n!}{a!(n-a)!}$ for $n = 0, \ldots, k-1$ and any natural number $a \le n$.

Rest of the inductive step: Let a be a natural number $\le k$. There are three cases:

Case 1: $a = k$. Then $f(k, a) = 1$ by the definition of f. Also $\frac{n!}{a!(n-a)!} = \frac{n!}{n!0!} = 1$. So the claim holds.

Case 2: $a = 0$. Then $f(k, a) = 1$ by the definition of f. Also $\frac{n!}{a!(n-a)!} = \frac{n!}{0!n!} = 1$. So the claim holds.

Case 3: $1 \le a \le k-1$. Then by the inductive hypothesis $f(k-1, a-1) = \frac{(k-1)!}{(a-1)!(k-a)!}$ and $f(k-1, a) = \frac{(k-1)!}{a!(k-1-a)!}$. Then

$$
\begin{aligned}
f(k, a) &= f(k-1, a-1) + f(k-1, a) &&= \frac{(k-1)!}{(a-1)!(k-a)!} + \frac{(k-1)!}{a!(k-1-a)!} \\
&= \frac{(k-1)!}{(a-1)!(k-a)!} + \frac{(k-1)!}{a!(k-1-a)!} &&= \frac{a(k-1)!}{a!(k-a)!} + \frac{(k-a)(k-1)!}{a!(k-a)!} \\
&= \frac{k(k-1)!}{a!(k-a)!} &&= \frac{k!}{a!(k-a)!}
\end{aligned}
$$

So $f(k, a) = \frac{k!}{a!(k-a)!}$, which is what we needed to show.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) (20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$f(0) = 2$ $\quad f(1) = 5$ $\quad f(2) = 15$

$f(n) = 6f(n-1) - 11f(n-2) + 6f(n-3)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) = 1 - 2^n + 2 \cdot 3^n$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $f(0) = 2$ and $1 - 2^n + 2 \cdot 3^n = 1 - 1 + 2 = 2$

At $n = 1$, $f(1) = 5$ and $1 - 2^n + 2 \cdot 3^n = 1 - 2 + 6 = 5$

At $n = 2$, $f(2) = 15$ and $1 - 2^n + 2 \cdot 3^n = 1 - 4 + 18 = 15$

So the claim holds at all three values.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) = 1 - 2^n + 2 \cdot 3^n$ for $n = 0, 1, \ldots, k-1$.

Rest of the inductive step: By the definition of f and the inductive hypothesis, we get

$$\begin{aligned}
f(k) &= 6f(k-1) - 11f(k-2) + 6f(k-3) \\
&= 6(1 - 2^{k-1} + 2 \cdot 3^{k-1}) - 11(1 - 2^{k-2} + 2 \cdot 3^{k-2}) + 6(1 - 2^{k-3} + 2 \cdot 3^{k-3}) \\
&= (6 - 11 + 6) - (6 \cdot 2^{k-1} - 11 \cdot 2^{k-2} + 6 \cdot 2^{k-3}) + 2(6 \cdot 3^{k-1} - 11 \cdot 3^{k-2} + 6 \cdot 3^{k-3}) \\
&= 1 - (12 \cdot 2^{k-2} - 11 \cdot 2^{k-2} + 3 \cdot 2^{k-2}) + 2(18 \cdot 3^{k-2} - 11 \cdot 3^{k-2} + 2 \cdot 3^{k-2}) \\
&= 1 - 4 \cdot 2^{k-2} + 2 \cdot 9 \cdot 3^{k-2} = 1 - 2^k + 2 \cdot 2^k
\end{aligned}$$

So $f(k) = 1 - 2^k + 2 \cdot 2^k$, which is what we needed to show.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is defined by

$f(0) = 0 \qquad f(1) = 1$

$f(n) = f(n-1) + f(n-2)$ for all $n \geq 2$.

Let $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. Use (strong) induction to prove that $f(n) = \dfrac{a^n - b^n}{a-b}$.

First show that $a^2 = a + 1$ **and** $b^2 = b + 1$**:** **Solution:** Notice that $a^2 = (\frac{1+\sqrt{5}}{2})^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = 1 + \frac{1+\sqrt{5}}{2} = 1 + a$.

Similarly $b^2 = (\frac{1-\sqrt{5}}{2})^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = 1 + \frac{1-\sqrt{5}}{2} = 1 + b$.

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $f(n) = 0$. Also $\frac{a^n - b^n}{a-b} = \frac{1-1}{a-b} = 0$.

At $n = 1$, $f(n) = 1$. Also $\frac{a^n - b^n}{a-b} = \frac{a-b}{a-b} = 1$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $f(n) = \frac{a^n - b^n}{a-b}$, for $n = 0, 1, \ldots, k-1$.

Rest of the inductive step: In particular, by the inductive hypothesis, $f(k-1) = \frac{a^{k-1} - b^{k-1}}{a-b}$ and $f(k-2) = \frac{a^{k-2} - b^{k-2}}{a-b}$. Then

$$\begin{aligned}
f(k) &= f(k-1) + f(k-2) = \frac{a^{k-1} - b^{k-1}}{a-b} + \frac{a^{k-2} - b^{k-2}}{a-b} \\
&= \frac{1}{a-b}(a^{k-1} - b^{k-1} + a^{k-2} - b^{k-2}) \\
&= \frac{1}{a-b}(a^{k-2}(a+1) - b^{k-2}(b+1)) \\
&= \frac{1}{a-b}(a^{k-2}(a^2) - b^{k-2}(b^2)) = \frac{1}{a-b}(a^k - b^k)
\end{aligned}$$

So $f(k) = \frac{a^k - b^k}{a-b}$, which is what we needed to show.

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $g : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ is defined by

$g(1) = 1$

$g(2) = 8$

$g(n) = g(n-1) + 2g(n-2)$

Use (strong) induction to prove that $g(n) = 3 \cdot 2^{n-1} + 2(-1)^n$.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $g(n) = 1$ and $3 \cdot 2^{n-1} + 2(-1)^n = 3 \cdot 2^0 + 2(-1) = 3 - 2 = 1$, so the claim holds.

At $n = 2$, $g(n) = 8$ and $3 \cdot 2^{n-1} + 2(-1)^n = 3 \cdot 2^1 + 2(-1)^2 = 6 + 2 = 8$, so the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $g(n) = 3 \cdot 2^{n-1} + 2(-1)^n$ for $n = 1, \ldots, k-1$.

Rest of the inductive step: In particular, by the inductive hypothesis, $g(k-1) = 3 \cdot 2^{k-2} + 2(-1)^{k-1}$ and $g(k-2) = 3 \cdot 2^{k-3} + 2(-1)^{k-2}$.

So

$$\begin{aligned}
g(k) &= g(k-1) - 2g(k-2) = (3 \cdot 2^{k-2} + 2(-1)^{k-1}) + 2(3 \cdot 2^{k-3} + 2(-1)^{k-2}) \\
&= 3 \cdot 2^{k-2} + 2(-1)^{k-1} + 6 \cdot 2^{k-3} + 4(-1)^{k-2} \\
&= 3 \cdot 2^{k-2} + 2(-1)^{k-1} + 3 \cdot 2^{k-2} - 4(-1)^{k-1} \\
&= 6 \cdot 2^{k-2} - 2(-1)^{k-1} = 3 \cdot 2^{k-1} + 2(-1)^k
\end{aligned}$$

So $g(k) = 3 \cdot 2^{k-1} + 2(-1)^k$, which is what we needed to show.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove that

$$\prod_{p=1}^{n} \frac{m+1-p}{p} = \frac{m!}{n!(m-n)!}$$

for any positive integers m and n where $m \geq n$. Hint: use n as your induction variable. At each step, make sure the the equations work for an arbitrary integer $m \geq n$.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $\prod_{p=1}^{n} \frac{m+1-p}{p} = \frac{m+1-1}{1} = m$. And $\frac{m!}{n!(m-n)!} = \frac{m!}{1!(m-1)!} = m$. So the claim holds for any $m \geq n$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\displaystyle\prod_{p=1}^{n} \frac{m+1-p}{p} = \frac{m!}{n!(m-n)!}$ for $n = 1, \ldots, k$ and any integer $m \geq n$.

Rest of the inductive step: Let m be any integer $\geq n+1$. Notice that $m \geq n$. So, by the inductive hypothesis, $\displaystyle\prod_{p=1}^{k} \frac{m+1-p}{p} = \frac{m!}{k!(m-k)!}$. Then

$$\begin{aligned}
\prod_{p=1}^{k+1} \frac{m+1-p}{p} &= \frac{(m+1)-(k+1)}{k+1} \cdot \prod_{p=1}^{k} \frac{m+1-p}{p} = \frac{(m+1)-(k+1)}{k+1} \cdot \frac{m!}{k!(m-k)!} \\
&= \frac{m-k}{k+1} \cdot \frac{m!}{k!(m-k)!} = \frac{m!}{(k+1)!(m-k-1)!} \\
&= \frac{m!}{(k+1)!(m-(k+1))!}
\end{aligned}$$

So $\displaystyle\prod_{p=1}^{k+1} \frac{m+1-p}{p} = \frac{m!}{(k+1)!(m-(k+1))!}$, i.e. our claim holds at $n = k+1$, which is what we needed to prove.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Also recall that $\sin 2y = 2\sin y \cos y$ for any real number y.

Suppose that x is a real number such that $\sin x$ is non-zero. Use (strong) induction to prove that $\displaystyle\prod_{p=0}^{n-1} \cos(2^p x) = \frac{\sin(2^n x)}{2^n \sin x}$, for any positive integer n.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $\prod_{p=0}^{n-1} \cos(2^p x) = \cos x$ and $\frac{\sin(2^n x)}{2^n \sin x} = \frac{\sin 2x}{2\sin x} = \frac{2\sin x \cos x}{2\sin x} = \cos x$. So the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\displaystyle\prod_{p=0}^{n-1} \cos(2^p x) = \frac{\sin(2^n x)}{2^n \sin x}$, for $n = 1, \ldots, k$.

Rest of the inductive step: In particular, by the inductive hypothesis, $\displaystyle\prod_{p=0}^{k-1} \cos(2^p x) = \frac{\sin(2^k x)}{2^k \sin x}$. Using this, we can compute:

$$\begin{aligned}
\prod_{p=0}^{k} \cos(2^p x) &= (\cos(2^k x)) \prod_{p=0}^{k-1} \cos(2^p x) = (\cos(2^k x)) \frac{\sin(2^k x)}{2^k \sin x} \\
&= \frac{\cos(2^k x)\sin(2^k x)}{2^k \sin x} = \frac{2\cos(2^k x)\sin(2^k x)}{2^{k+1} \sin x} \qquad (\text{since } \sin 2y = 2\sin y \cos y) \\
&= \frac{\sin(2 \cdot 2^k x)}{2^{k+1} \sin x} = \frac{\sin(2^{k+1} x)}{2^{k+1} \sin x}
\end{aligned}$$

So $\displaystyle\prod_{p=0}^{k} \cos(2^p x) = \frac{\sin(2^{k+1} x)}{2^{k+1} \sin x}$. This is what we needed show, since it is the claim at $n = k + 1$.

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by is defined by

$f(1) = 5$ $\quad$ $f(2) = -5$

$f(n) = 4f(n-2) - 3f(n-1)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) = 2 \cdot (-4)^{n-1} + 3$

Solution: Proof by induction on n.

Base case(s): For $n = 1$, $2 \cdot (-4)^{n-1} + 3 = 2 \cdot (-4)^0 + 3 = 2 \cdot 1 + 3 = 5$, which is equal to $f(1)$.

For $n = 2$, $2 \cdot (-4)^{n-1} + 3 = 2 \cdot (-4)^1 + 3 = 2 \cdot (-4) + 3 = -5$, which is equal to $f(2)$.

So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) = 2 \cdot (-4)^{n-1} + 3$, for $n = 1, 2, \ldots, k-1$, for some integer $k \geq 3$

Rest of the inductive step:

Using the definition of f and the inductive hypothesis, we get

$f(k) = 4f(k-2) - 3f(k-1) = 4(2 \cdot (-4)^{k-3} + 3) - 3(2 \cdot (-4)^{k-2} + 3)$

Simplifying the algebra,

$$\begin{aligned}
4(2 \cdot (-4)^{k-3} + 3) - 3(2 \cdot (-4)^{k-2} + 3) &= 8 \cdot (-4)^{k-3} + 12 - 6 \cdot (-4)^{k-2} - 9 \\
&= -2 \cdot (-4)^{k-2} - 6 \cdot (-4)^{k-2} + 3 \\
&= -8 \cdot (-4)^{k-2} + 3 = 2 \cdot (-4)^{k-1} + 3
\end{aligned}$$

So $f(k) = 2 \cdot (-4)^{k-1} + 3$, which is what we needed to prove.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) A "triangle-free" graph is a graph that doesn't contain any 3-cycles. Use (strong) induction to prove that a triangle-free graph with $2n$ nodes has $\leq n^2$ edges, for any positive integer n. Hint: in the inductive step, remove a pair of nodes joined by an edge. How many edges from those nodes to the rest of the graph?

Solution: Proof by induction on n.

Base case(s): At n=1. The graph has only two nodes, so it cannot have more than one edge. Since $n^2 = 1$, this means the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that any triangle-free graph with $2n$ nodes has $\leq n^2$ edges, for $n = 1, \ldots, k$.

Rest of the inductive step: Let G be a triangle-free graph with 2(k+1) nodes ($k \geq 1$). There are two cases:

Case 1: G has no edges. Since ($k \geq 1$), this means that G has $\leq (k+1)^2$ edges, which is what we needed to prove.

Case 2: G has at least one edge. Let's pick two nodes a and b that are joined by an edge. Let H be the graph we get by removing a, b, and all their edges from G. By the inductive hypothesis, H has $\leq k^2$ edges.

Consider a node v in H (i.e. a node that's not a or b). Because G is triangle-free, v cannot be joined to both of a and b. So v has either one edge to a or b, or no edges. Therefore, since H has 2k nodes, there can be no more than 2k edges from a/b to the nodes in H. We also have one extra edge: the edge joining a to b.

The total number of edges in G is the number in H, plus the nodes from a/b to a node in H, plus the edge joining a to b. That total is $\leq k^2 + 2k + 1 = (k+1)^2$. This is what we needed to prove.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$$f(0) = 1 \qquad f(1) = -5$$

$$f(n) = -7f(n-1) - 10f(n-2), \quad \text{for } n \geq 2$$

Use (strong) induction to prove that $f(n) = (-1)^n \cdot 5^n$

Solution: Proof by induction on n.

Base case(s): $f(0) = 1 = (-1)^0 \cdot 5^0$ and $f(1) = -5 = (-1)^1 \cdot 5^1$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) = (-1)^n * 5^n$ for $n = 0, 1, \ldots, k-1$, for some integer $k \geq 2$.

Rest of the inductive step:

From the inductive hypothesis, we know that $f(k-1) = (-1)^{k-1} * 5^{k-1}$ and $f(k-2) = (-1)^{k-2} * 5^{k-2}$

So then we have

$$\begin{aligned}
f(k) &= -7 \cdot f(k-1) - 10 \cdot f(k-2) \\
&= -7 \cdot (-1)^{k-1} * 5^{k-1} + -10 \cdot (-1)^{k-2} * 5^{k-2} \\
&= 7 \cdot (-1)^k * 5^{k-1} - 10 \cdot (-1)^k * 5^{k-2} \\
&= 7 \cdot (-1)^k * 5^{k-1} - 2 \cdot (-1)^k * 5^{k-1} \\
&= 5 \cdot (-1)^k * 5^{k-1} = 5 \cdot (-1)^k * 5^k
\end{aligned}$$

So $f(k) = 5 \cdot (-1)^k * 5^{k-1} = (-1)^k * 5^k$ which is what we needed to show.

Name:____

NetID:______________________ **Lecture: A B**

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

(20 points) Use (strong) induction to prove that, for all positive integers n, $x^2+y^2=z^n$ has a positive integer solution. (That is, a solution in which x, y, and z are all positive integers.) Hints: (1) notice that $3^2+4^2=5^2$ and (2) use the solution for $n=k-2$ (not $n=k-1$) to build a solution for $n=k$.

Solution: Proof by induction on n.

Base case(s): At $n=1$, one solution to $x^2+y^2=z$ is $x=1$, $y=2$, and $z=5$.

At $n=2$, one solution to $x^2+y^2=z^2$ is $x=3$, $y=4$, and $z=5$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that there is a positive integer solution to $x^2+y^2=z^n$ for $n=1,2,\ldots,k-1$.

Rest of the inductive step: From the inductive hypothesis, we know that there is a positive integer solution to $x^2+y^2=z^{k-2}$. That is, we have positive integers a, b, and c, such that $a^2+b^2=c^{k-2}$.

Consider $x=ac$, $y=bc$ and $z=c$. ac and bc are positive integers because a, b, and c are positive integers. Then

$$x^2+y^2=(ac)^2+(bc)^2=c^2(a^2+b^2)=c^2(c^{k-2})=c^k$$

So $x=ac$, $y=bc$ and $z=c$ is a positive integer solution to $x^2+y^2=z^k$, which is what we needed to show.

Name:____

NetID:_________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $g : \mathbb{N} \to \mathbb{R}$ is defined by

$$g(0) = 0 \qquad g(1) = \tfrac{4}{3}$$

$$g(n) = \tfrac{4}{3}g(n-1) - \tfrac{1}{3}g(n-2), \quad \text{for } n \geq 2$$

Use (strong) induction to prove that $g(n) = 2 - \dfrac{2}{3^n}$

Solution: Proof by induction on n.

Base case(s): $n = 0$: $2 - \frac{2}{3^n} = 2 - \frac{2}{1} = 0 = g(0)$ So the claim holds.

$n = 1$: $2 - \frac{2}{3^n} = 2 - \frac{2}{3} = \frac{4}{3} = g(1)$ So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $g(n) = 2 - \frac{2}{3^n}$, for $n = 0, 1, \cdots, k-1$ for some integer $k \geq 2$.

Inductive Step:

We need to show that $g(k) = 2 - \dfrac{2}{3^k}$

$$\begin{aligned}
g(k) &= \frac{4}{3}g(k-1) - \frac{1}{3}g(k-2) && \text{[by the def, } k \geq 2\text{]} \\
&= \frac{4}{3}\left(2 - \frac{2}{3^{k-1}}\right) - \frac{1}{3}\left(2 - \frac{2}{3^{k-2}}\right) && \text{[Inductive Hypothesis]} \\
&= \frac{8}{3} - \frac{8}{3^k} - \frac{2}{3} + \frac{2}{3^{k-1}} \\
&= \frac{6}{3} - \frac{8}{3^k} + \frac{6}{3^k} \\
&= 2 - \frac{2}{3^k}.
\end{aligned}$$