

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(10 points) Suppose we have a function f defined (for n a power of 4) by

$$\begin{aligned} f(1) &= 0 \\ f(n) &= 2f(n/4) + n \text{ for } n \geq 4 \end{aligned}$$

Your partner has already figured out that

$$f(n) = 2^k f(n/4^k) + n \sum_{p=0}^{k-1} 1/2^p$$

Finish finding the closed form for $f(n)$ assuming that n is a power of 4. Show your work and simplify your answer. Recall that $\log_b n = (\log_a n)(\log_b a)$.

Solution: To find the value of k at the base case, set $n/4^k = 1$. Then $n = 4^k$, so $k = \log_4 n$. Notice also that $2^{\log_4 n} = 2^{\log_2 n \log_4 2} = n^{1/2} = \sqrt{n}$

Substituting this into the above, we get

$$\begin{aligned} f(n) &= 2^{\log_4 n} f(1) + n \sum_{p=0}^{\log_4 n - 1} 1/2^p \\ &= 0 + n(2 - \frac{1}{2^{\log_4 n - 1}}) \\ &= n(2 - \frac{2}{2^{\log_4 n}}) = n(2 - \frac{2}{\sqrt{n}}) = 2(n - \sqrt{n}) \end{aligned}$$

Name: _____

NetID: _____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(10 points) Suppose we have a function F defined (for n a power of 2) by

$$\begin{aligned} F(2) &= c \\ F(n) &= F(n/2) + n \text{ for } n \geq 4 \end{aligned}$$

Your partner has already figured out that

$$F(n) = F(n/2^k) + \sum_{i=0}^{k-1} n \frac{1}{2^i}$$

Finish finding the closed form for F . Show your work and simplify your answer.**Solution:**

To find the value of k at the base case, we need to set $n/2^k = 2$. This means that $n = 2 \cdot 2^k$. So $n = 2^{k+1}$. So $k + 1 = \log n$. So $k = \log n - 1$. Substituting this value into the above equation, we get

$$\begin{aligned} F(n) &= F(n/2^k) + \sum_{i=0}^{k-1} n \frac{1}{2^i} = F(2) + \sum_{i=0}^{\log n - 2} n \frac{1}{2^i} \\ &= c + n \sum_{i=0}^{\log n - 2} \frac{1}{2^i} = c + n \left(2 - \frac{1}{2^{\log n - 2}} \right) \\ &= c + n \left(2 - \frac{1}{2^{\log n} \cdot 2^{-2}} \right) = c + n \left(2 - \frac{4}{2^{\log n}} \right) \\ &= c + n \left(2 - \frac{4}{n} \right) = c + 2n - 4 \end{aligned}$$

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (8 points) Suppose we have a function f defined by

$$\begin{aligned} f(0) &= f(1) = 3 \\ f(n) &= 5f(n-2) + d, \text{ for } n \geq 2 \end{aligned}$$

where d is a constant. Express $f(n)$ in terms of $f(n-6)$ (where $n \geq 6$). Show your work and simplify your answer. You do **not** need to find a closed form for $f(n)$.

Solution:

$$\begin{aligned} f(n) &= 5f(n-2) + d \\ &= 5(5f(n-4) + d) + d \\ &= 5(5(5f(n-6) + d) + d) + d \\ &= 5^3f(n-6) + (25 + 5 + 1)d \\ &= 5^3f(n-6) + 31d \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The chromatic number of the
4-dimensional hypercube Q_4

2 ☒3 ☐4 ☐5 ☐

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(10 points) Suppose we have a function f defined (for n a power of 4) by

$$\begin{aligned} f(1) &= 0 \\ f(n) &= 2f(n/4) + n \text{ for } n \geq 4 \end{aligned}$$

Express $f(n)$ in terms of $f(n/4^{13})$ (assuming n is large enough that this input hasn't reached the base case). Express your answer using a summation and show your work. Do **not** finish the process of finding the closed form for $f(n)$.

Solution:

$$\begin{aligned} f(n) &= 2f(n/4) + n \\ &= 2(2f(n/4^2) + n/4) + n \\ &= 2(2(2f(n/4^3) + n/4^2) + n/4) + n \\ &= 2^3 f(n/4^3) + n/2^2 + n/2 + n \\ &= 2^k f(n/4^k) + n \sum_{p=0}^{k-1} 1/2^p \\ &= 2^{13} f(n/4^{13}) + n \sum_{p=0}^{12} 1/2^p \end{aligned}$$

Name: _____

NetID: _____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (8 points) Suppose we have a function g defined (for n a power of 3) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 3g(n/3) + n \text{ for } n \geq 3 \end{aligned}$$

Express $g(n)$ in terms of $g(n/3^3)$ (where $n \geq 27$). Show your work and simplify your answer. You do **not** need to find a closed form for $g(n)$.

Solution:

$$\begin{aligned} g(n) &= 3g(n/3) + n \\ &= 3(3g(n/9) + n/3) + n \\ &= 3(3(3g(n/27) + n/9) + n/3) + n \\ &= 27g(n/27) + n + n + n \\ &= 27g(n/27) + 3n \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The number of nodes in the
4-dimensional hypercube Q_4

4 ☐16 ☒32 ☐64 ☐

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(10 points) Suppose we have a function g defined (for n a power of 4) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 2g(n/4) + n \text{ for } n \geq 4 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 2^k g(n/4^k) + n \sum_{p=0}^{k-1} \frac{1}{2^p}$$

Finish finding the closed form for $g(n)$ assuming that n is a power of 4. Show your work and simplify your answer. Recall that $\log_b n = (\log_a n)(\log_b a)$.

Solution: To find the value of k at the base case, set $n/4^k = 1$. Then $n = 4^k$, so $k = \log_4 n$. Notice also that $2^{\log_4 n} = 2^{\log_2 n \log_4 2} = n^{1/2} = \sqrt{n}$

Substituting this into the above, we get

$$\begin{aligned} g(n) &= 2^{\log_4 n} \cdot c + n \sum_{p=0}^{\log_4 n - 1} \frac{1}{2^p} \\ &= 2^{\log_4 n} \cdot c + n \left(2 - \frac{1}{2^{\log_4 n - 1}} \right) \\ &= c\sqrt{n} + n \left(2 - \frac{2}{\sqrt{n}} \right) \\ &= c\sqrt{n} + 2n - 2\sqrt{n} \\ &= 2n + (c - 2)\sqrt{n} \end{aligned}$$

Name: _____

NetID: _____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (8 points) Suppose we have a function g defined (for n a power of 2) by

$$\begin{aligned} g(1) &= 1 \\ g(n) &= 4g(n/2) + n^2 \text{ for } n \geq 2 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 4^k g(n/2^k) + kn^2$$

Finish finding the closed form for g . Show your work and simplify your answer.

Solution:

To find the value of k at the base case, set $n/2^k = 1$. Then $n = 2^k$, so $k = \log_2 n$. Substituting this into the above, we get:

$$\begin{aligned} g(n) &= 4^k g(n/2^k) + kn^2 \\ &= 4^{\log_2 n} g(1) + (\log_2 n)n^2 \\ &= 4^{\log_2 n} + n^2 \log_2 n \\ &= 4^{\log_4 n \log_2 4} + n^2 \log_2 n \\ &= (4^{\log_4 n})^{\log_2 4} + n^2 \log_2 n \\ &= n^{\log_2 4} + n^2 \log_2 n \\ &= n^2 + n^2 \log_2 n \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The Fibonacci numbers can be defined recursively by $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for all integers ...

$n \geq 0$ ☐

$n \geq 1$ ☐

$n \geq 2$ ☒

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(10 points) Suppose we have a function g defined (for n a power of 2) by

$$\begin{aligned} g(1) &= 3 \\ g(n) &= 4g(n/2) + n \text{ for } n \geq 2 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 4^k g(n/2^k) + \sum_{p=0}^{k-1} n2^p$$

Finish finding the closed form for $g(n)$ assuming that n is a power of 2. Show your work and simplify your answer. Recall that $\log_b n = (\log_a n)(\log_b a)$.

Solution: To find the value of k at the base case, set $n/2^k = 1$. Then $n = 2^k$, so $k = \log_2 n$. Notice also that $4^{\log_2 n} = 4^{\log_4 n \log_2 4} = n^2$.

Substituting this into the above, we get

$$\begin{aligned} g(n) &= 4^k g(n/2^k) + n \sum_{p=0}^{k-1} 2^p \\ &= 4^{\log_2 n} \cdot 3 + n \sum_{p=0}^{\log_2 n - 1} 2^p \\ &= 3n^2 + n(2^{\log_2 n} - 1) \\ &= 3n^2 + n(n - 1) = 4n^2 - n \end{aligned}$$

Name: _____

NetID: _____ Lecture: B

Discussion: Friday 11 12 1 2 3 4

1. (8 points) Suppose we have a function g defined by

$$\begin{aligned}g(0) &= g(1) = c \\g(n) &= kg(n-2) + n^2, \text{ for } n \geq 2\end{aligned}$$

where k and c are constants. Express $g(n)$ in terms of $g(n-6)$ (where $n \geq 6$). Show your work and simplify your answer. You do **not** need to find a closed form for $g(n)$.

Solution:

$$\begin{aligned}g(n) &= kg(n-2) + n^2 \\&= k(kg(n-4) + (n-2)^2) + n^2 \\&= k(k(kg(n-6) + (n-4)^2) + (n-2)^2) + n^2 \\&= k^3g(n-6) + k^2(n-4)^2 + k(n-2)^2 + n^2\end{aligned}$$

2. (2 points) Suppose that $f : \mathbb{N} \rightarrow \mathbb{N}$ is such that $f(n) = n!$. Give a recursive definition of f

Solution:

$f(0) = 1$, and $f(n) = nf(n-1)$ for $n \geq 1$.

You could also have used $f(n+1) = (n+1)f(n)$ for $n \geq 0$.

Name: _____

NetID: _____ Lecture: B

Discussion: Friday 11 12 1 2 3 4

1. (8 points) Suppose we have a function f defined by

$$\begin{aligned} f(1) &= 5 \\ f(n) &= 3f(n-1) + n^2 \text{ for } n \geq 2 \end{aligned}$$

Express $f(n)$ in terms of $f(n-3)$ (where $n \geq 4$). Show your work and simplify your answer. You do **not** need to find a closed form for $f(n)$.

Solution:

$$\begin{aligned} f(n) &= 3f(n-1) + n^2 \\ &= 3(3f(n-2) + (n-1)^2) + n^2 \\ &= 3(3(3f(n-3) + (n-2)^2) + (n-1)^2) + n^2 \\ &= 27f(n-3) + 9(n-2)^2 + 3(n-1)^2 + n^2 \end{aligned}$$

2. (2 points) Suppose that G_0 is the graph consisting of a single vertex. Also suppose that the graph G_n consists of a copy of G_{n-1} plus an extra vertex v and edges joining v to each vertex in G_{n-1} . Give a clear picture or precise description of G_4 .

Solution: This is a recursive construction of all the complete graphs, except for the indexing being off by one. So G_4 is just K_5 .

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(10 points) Suppose we have a function F defined (for n a power of 2) by

$$\begin{aligned} F(2) &= 17 \\ F(n) &= 3F(n/2), \text{ for } n \geq 4 \end{aligned}$$

Use unrolling to find the closed form for F . Show your work and simplify your answer.**Solution:**

$$\begin{aligned} F(n) &= 3F(n/2) = 3(3F(n/4)) = 3(3(3(F(n/2^3)))) \\ &= 3^3 F(n/2^3) \\ &= 3^k F(n/2^k) \end{aligned}$$

We'll hit the base case when $n/2^k = 2$, i.e. $n = 2^{k+1}$, i.e. $k + 1 = \log_2 n$, $k = \log_2 n - 1$. Substituting this value into the above equation, we get

$$\begin{aligned} F(n) &= 3^{\log_2 n - 1} \cdot 17 \\ &= 17/3 \cdot 3^{\log_2 n} = 17/3 \cdot 3^{\log_3 n \log_2 3} \\ &= 17/3 n^{\log_2 3} \end{aligned}$$

Name: _____

NetID: _____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (8 points) Suppose we have a function g defined (for n a power of 2) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 4g(n/2) + d \text{ for } n \geq 2 \end{aligned}$$

Express $g(n)$ in terms of $g(n/2^3)$ (where $n \geq 8$). Show your work and simplify your answer. You do **not** need to find a closed form for $g(n)$.

Solution:

$$\begin{aligned} g(n) &= 4g(n/2) + d \\ &= 4(4g(n/2^2) + d) + d \\ &= 4(4(4g(n/2^3) + d) + d) + d \\ &= 4^3g(n/2^3) + 21d \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

$f(n) = n!$ can be defined recursively by
 $f(0) = 1$, and $f(n+1) = (n+1)f(n)$
 for all integers ...

$n \geq 0$ ☒

$n \geq 1$ ☐

$n \geq 2$ ☐

Name: _____

NetID: _____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (8 points) Suppose we have a function g defined (for n a power of 3) by

$$\begin{aligned} g(9) &= 5 \\ g(n) &= 3g(n/3) + n \text{ for } n \geq 27 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 3^k g(n/3^k) + kn$$

Finish finding the closed form for g . Show your work and simplify your answer.

Solution:

To find the value of k at the base case, set $n/3^k = 9$. Then $n = 3^{k+2}$, so $k + 2 = \log_3 n$, so $k + 2 = \log_3 n - 2$. Substituting this into the above, we get:

$$\begin{aligned} g(n) &= 3^{\log_3 n - 2} g(9) + (\log_3 n - 2)n \\ &= 3^{\log_3 n} 3^{-2} 5 + n \log_3 n - 2n \\ &= \frac{5}{9}n + n \log_3 n - 2n = n \log_3 n - \frac{13}{9}n \end{aligned}$$

2. (2 points) Suppose that G_0 is the graph consisting of a single vertex. Also suppose that the graph G_n consists of a copy of G_{n-1} plus an extra vertex v and edges joining v to each vertex in G_{n-1} . Give a clear picture or precise description of G_4 .

Solution: This is a recursive construction of all the complete graphs, except for the indexing being off by one. So G_4 is just K_5 .

Name: _____

NetID: _____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(10 points) Suppose we have a function F defined (for n a power of 3) by

$$\begin{aligned} F(1) &= 5 \\ F(n) &= 3F(n/3) + 7 \text{ for } n \geq 3 \end{aligned}$$

Your partner has already figured out that

$$F(n) = 3^k F(n/3^k) + 7 \sum_{p=0}^{k-1} 3^p$$

Finish finding the closed form for F . Show your work and simplify your answer. Recall the following useful closed form (for $r \neq 1$): $\sum_{k=0}^n r^k = \frac{r^{n+1} - 1}{r - 1}$

Solution:

To find the value of k at the base case, set $n/3^k = 1$. Then $n = 3^k$, so $k = \log_3 n$. Substituting this into the above, we get

$$\begin{aligned} F(n) &= 3^{\log_3 n} \cdot 5 + 7 \sum_{p=0}^{\log_3 n - 1} 3^p \\ &= 5n + 7 \frac{3^{\log_3 n} - 1}{3 - 1} \\ &= 5n + 7 \frac{n - 1}{3 - 1} = 5n + \frac{7(n - 1)}{2} \end{aligned}$$

Name: _____

NetID: _____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (8 points) Suppose we have a function g defined (for n a power of 2) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 4g(n/2) + n \text{ for } n \geq 2 \end{aligned}$$

Express $g(n)$ in terms of $g(n/2^3)$ (where $n \geq 8$). Show your work and simplify your answer. You do **not** need to find a closed form for $g(n)$.

Solution:

$$\begin{aligned} g(n) &= 4g(n/2) + n \\ &= 4(4g(n/4) + n/2) + n \\ &= 4(4(4g(n/8) + n/4) + n/2) + n \\ &= 4^3g(n/8) + 4n + 2n + n \\ &= 4^3g(n/8) + 7n \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The diameter of the
4-dimensional hypercube Q_4

1

☐

2

☐

4

☒

16

☐

Name: _____

NetID: _____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (8 points) Suppose we have a function f defined (for n a power of 2) by

$$\begin{aligned} f(1) &= 5 \\ f(n) &= 3f(n/2) + n^2 \text{ for } n \geq 2 \end{aligned}$$

Express $f(n)$ in terms of $f(n/2^3)$ (where $n \geq 8$). Show your work and simplify your answer. You do **not** need to find a closed form for $f(n)$.

Solution:

$$\begin{aligned} f(n) &= 3f(n/2) + n^2 \\ &= 3(3f(n/4) + (n/2)^2) + n^2 \\ &= 3(3(3f(n/8) + (n/4)^2) + (n/2)^2) + n^2 \\ &= 27f(n/2^3) + (3/4)n^2 + (9/16)n^2 + n^2 \\ &= 27f(n/2^3) + (2 + 5/16)n^2 \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The n -dimensional

hypercube Q_n has an Euler circuit.

always

☐

sometimes

☒

never

☐

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(10 points) Suppose we have a function f defined by

$$\begin{aligned} f(0) &= f(1) = 3 \\ f(n) &= 5f(n-2) + d, \text{ for } n \geq 2 \end{aligned}$$

where d is a constant. Your partner has already figured out that

$$f(n) = 5^k f(n-2k) + \sum_{p=0}^{k-1} d5^p$$

Finish finding the closed form for $f(n)$ assuming that n is even. Show your work and simplify your answer. Recall the following useful closed form (for $r \neq 1$): $\sum_{k=0}^n r^k = \frac{r^{n+1} - 1}{r - 1}$

Solution:

To find the value of k at the base case, set $n - 2k = 0$. Then $n = 2k$, so $k = n/2$. Substituting this into the above, we get

$$\begin{aligned} f(n) &= 5^k f(n-2k) + \sum_{p=0}^{k-1} d5^p \\ &= 5^{n/2} \cdot 3 + d \sum_{p=0}^{n/2-1} 5^p \\ &= 3 \cdot 5^{n/2} + d \left(\frac{5^{n/2} - 1}{4} \right) \\ &= (3 + d/4)5^{n/2} - d/4 = (3 + d/4)(\sqrt{5})^n - d/4 \end{aligned}$$

Name: _____

NetID: _____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (8 points) Suppose we have a function f defined by

$$\begin{aligned} f(0) &= f(1) = 3 \\ f(n) &= 5f(n-2) + d, \text{ for } n \geq 2 \end{aligned}$$

where d is a constant. Express $f(n)$ in terms of $f(n-6)$ (where $n \geq 6$). Show your work and simplify your answer. You do **not** need to find a closed form for $f(n)$.

Solution:

$$\begin{aligned} f(n) &= 5f(n-2) + d \\ &= 5(5f(n-4) + d) + d \\ &= 5(5(5f(n-6) + d) + d) + d \\ &= 5^3 f(n-6) + (25 + 5 + 1)d \\ &= 5^3 f(n-6) + 31d \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

$f(n) = n!$ can be defined recursively
by $f(0) = 1$, and $f(n) = nf(n-1)$
for all integers ...

$n \geq 0$ ☐

$n \geq 1$ ☒

$n \geq 2$ ☐

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (8 points) Suppose we have a function g defined (for n a power of 3) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 3g(n/3) + n \text{ for } n \geq 3 \end{aligned}$$

Express $g(n)$ in terms of $g(n/3^3)$ (where $n \geq 27$). Show your work and simplify your answer. You do **not** need to find a closed form for $g(n)$.

Solution:

$$\begin{aligned} g(n) &= 3g(n/3) + n \\ &= 3(3g(n/9) + n/3) + n \\ &= 3(3(3g(n/27) + n/9) + n/3) + n \\ &= 27g(n/27) + n + n + n \\ &= 27g(n/27) + 3n \end{aligned}$$

2. (2 points) Suppose that $f : \mathbb{N} \rightarrow \mathbb{N}$ is such that $f(n) = n^2$. Give a recursive definition of f

Solution:

$$f(0) = 0, \text{ and } f(n+1) = f(n) + 2n + 1 \text{ for } n \geq 0.$$

You could also have used $f(n) = f(n-1) + 2n - 1$ for $n \geq 1$.

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(10 points) Suppose we have a function g defined (for n a power of 4) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 2g(n/4) + n \text{ for } n \geq 4 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 2^k g(n/4^k) + n \sum_{p=0}^{k-1} \frac{1}{2^p}$$

Finish finding the closed form for $f(n)$ assuming that n is a power of 4. Show your work and simplify your answer. Recall that $\log_b n = (\log_a n)(\log_b a)$.

Solution: To find the value of k at the base case, set $n/4^k = 1$. Then $n = 4^k$, so $k = \log_4 n$. Notice also that $2^{\log_4 n} = 2^{\log_2 n \log_4 2} = n^{1/2} = \sqrt{n}$

Substituting this into the above, we get

$$\begin{aligned} g(n) &= 2^{\log_4 n} \cdot c + n \sum_{p=0}^{\log_4 n - 1} \frac{1}{2^p} \\ &= 2^{\log_4 n} \cdot c + n \left(2 - \frac{1}{2^{\log_4 n - 1}} \right) \\ &= c\sqrt{n} + n \left(2 - \frac{2}{\sqrt{n}} \right) \\ &= c\sqrt{n} + 2n - 2\sqrt{n} \\ &= 2n + (c - 2)\sqrt{n} \end{aligned}$$

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(10 points) Suppose we have a function g defined (for n a power of 2) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 4g(n/2) + n \text{ for } n \geq 2 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 4^k g(n/2^k) + n \sum_{p=0}^{k-1} 2^p$$

Finish finding the closed form for $g(n)$ assuming that n is a power of 2. Show your work and simplify your answer. Recall that $\log_b n = (\log_a n)(\log_b a)$.

Solution: To find the value of k at the base case, set $n/2^k = 1$. Then $n = 2^k$, so $k = \log_2 n$. Notice also that $4^{\log_2 n} = 4^{\log_4 n \log_2 4} = n^2$

Substituting this into the above, we get

$$\begin{aligned} g(n) &= 4^{\log_2 n} \cdot c + n \sum_{p=0}^{\log_2 n - 1} 2^p \\ &= 4^{\log_2 n} \cdot c + n \sum_{p=0}^{\log_2 n - 1} 2^p \\ &= cn^2 + n(2^{\log_2 n} - 1) = cn^2 + n^2 - n = (c+1)n^2 - n \end{aligned}$$

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (8 points) Suppose we have a function g defined by

$$\begin{aligned} g(0) &= g(1) = c \\ g(n) &= kg(n-2) + n^2, \text{ for } n \geq 2 \end{aligned}$$

where k and c are constants. Express $g(n)$ in terms of $g(n-6)$ (where $n \geq 6$). Show your work and simplify your answer. You do **not** need to find a closed form for $g(n)$.

Solution:

$$\begin{aligned} g(n) &= kg(n-2) + n^2 \\ &= k(kg(n-4) + (n-2)^2) + n^2 \\ &= k(k(kg(n-6) + (n-4)^2) + (n-2)^2) + n^2 \\ &= k^3g(n-6) + k^2(n-4)^2 + k(n-2)^2 + n^2 \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The number of nodes in the
4-dimensional hypercube Q_4

4 ☐16 ☒32 ☐64 ☐

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

(10 points) Suppose we have a function f defined (for n a power of 4) by

$$\begin{aligned} f(1) &= 0 \\ f(n) &= 2f(n/4) + n \text{ for } n \geq 4 \end{aligned}$$

Express $f(n)$ in terms of $f(n/4^{13})$ (assuming n is large enough that this input hasn't reached the base case). Express your answer using a summation and show your work. Do **not** finish the process of finding the closed form for $f(n)$.

Solution:

$$\begin{aligned} f(n) &= 2f(n/4) + n \\ &= 2(2f(n/4^2) + n/4) + n \\ &= 2(2(2f(n/4^3) + n/4^2) + n/4) + n \\ &= 2^3 f(n/4^3) + n/2^2 + n/2 + n \\ &= 2^k f(n/4^k) + n \sum_{p=0}^{k-1} 1/2^p \\ &= 2^{13} f(n/4^{13}) + n \sum_{p=0}^{12} 1/2^p \end{aligned}$$

Name: _____

NetID: _____

Lecture: A B

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

1. (8 points) Suppose we have a function g defined (for n a power of 2) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 4g(n/2) + n^2 \text{ for } n \geq 2 \end{aligned}$$

Express $g(n)$ in terms of $g(n/2^3)$ (where $n \geq 8$). Show your work and simplify your answer. You do **not** need to find a closed form for $g(n)$.

Solution:

$$\begin{aligned} g(n) &= 4g(n/2) + n^2 \\ &= 4(4g(n/4) + (n/2)^2) + n^2 \\ &= 4(4(4g(n/8) + (n/4)^2) + (n/2)^2) + n^2 \\ &= 4^3g(n/8) + n^2 + n^2 + n^2 \\ &= 4^3g(n/8) + 3n^2 \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The Fibonacci numbers can be defined recursively by $F(0) = 0$, $F(1) = 1$, and

$F(n+1) = F(n) + F(n-1)$ for all integers ...

$n \geq 0$ ☐

$n \geq 1$ ☒

$n \geq 2$ ☐

Name: _____

NetID: _____ Lecture: A B

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

1. (8 points) Suppose we have a function f defined by

$$\begin{aligned} f(1) &= 5 \\ f(n) &= 3f(n-1) + n^2 \text{ for } n \geq 2 \end{aligned}$$

Express $f(n)$ in terms of $f(n-3)$ (where $n \geq 4$). Show your work and simplify your answer. You do **not** need to find a closed form for $f(n)$.

Solution:

$$\begin{aligned} f(n) &= 3f(n-1) + n^2 \\ &= 3(3f(n-2) + (n-1)^2) + n^2 \\ &= 3(3(3f(n-3) + (n-2)^2) + (n-1)^2) + n^2 \\ &= 27f(n-3) + 9(n-2)^2 + 3(n-1)^2 + n^2 \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The Fibonacci numbers can be defined

recursively by $F(0) = 0$, $F(1) = 1$, and

$F(n+2) = F(n) + F(n+1)$ for

all integers ...

$n \geq 0$ ☒

$n \geq 1$ ☐

$n \geq 2$ ☐

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

(10 points) Suppose we have a function g defined (for n a power of 2) by

$$\begin{aligned} g(1) &= 3 \\ g(n) &= 4g(n/2) + n \text{ for } n \geq 2 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 4^k g(n/2^k) + \sum_{p=0}^{k-1} n 2^p$$

Finish finding the closed form for $g(n)$ assuming that n is a power of 2. Show your work and simplify your answer. Recall that $\log_b n = (\log_a n)(\log_b a)$.

Solution: To find the value of k at the base case, set $n/2^k = 1$. Then $n = 2^k$, so $k = \log_2 n$. Notice also that $4^{\log_2 n} = 4^{\log_4 n \log_2 4} = n^2$.

Substituting this into the above, we get

$$\begin{aligned} g(n) &= 4^k g(n/2^k) + n \sum_{p=0}^{k-1} 2^p \\ &= 4^{\log_2 n} \cdot 3 + n \sum_{p=0}^{\log_2 n - 1} 2^p \\ &= 3n^2 + n(2^{\log_2 n} - 1) \\ &= 3n^2 + n(n - 1) = 4n^2 - n \end{aligned}$$

Name:_____

NetID:_____ Lecture: A B

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

(10 points) Suppose we have a function f defined (for n a power of 4) by

$$\begin{aligned} f(1) &= 0 \\ f(n) &= 2f(n/4) + n \text{ for } n \geq 4 \end{aligned}$$

Your partner has already figured out that

$$f(n) = 2^k f(n/4^k) + n \sum_{p=0}^{k-1} 1/2^p$$

Finish finding the closed form for $f(n)$ assuming that n is a power of 4. Show your work and simplify your answer. Recall that $\log_b n = (\log_a n)(\log_b a)$.

Solution: To find the value of k at the base case, set $n/4^k = 1$. Then $n = 4^k$, so $k = \log_4 n$. Notice also that $2^{\log_4 n} = 2^{\log_2 n \log_4 2} = n^{1/2} = \sqrt{n}$

Substituting this into the above, we get

$$\begin{aligned} f(n) &= 2^{\log_4 n} f(1) + n \sum_{p=0}^{\log_4 n - 1} 1/2^p \\ &= 0 + n(2 - \frac{1}{2^{\log_4 n - 1}}) \\ &= n(2 - \frac{2}{2^{\log_4 n}}) = n(2 - \frac{2}{\sqrt{n}}) = 2(n - \sqrt{n}) \end{aligned}$$

Name: _____

NetID: _____ Lecture: A B

Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6

1. (8 points) Suppose we have a function g defined (for n a power of 4) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 2g(n/4) + n \text{ for } n \geq 4 \end{aligned}$$

Express $g(n)$ in terms of $g(n/4^3)$ (where $n \geq 64$). Show your work and simplify your answer. You do **not** need to find a closed form for $g(n)$.

Solution:

$$\begin{aligned} g(n) &= 2g(n/4) + n \\ &= 2(2g(n/4^2) + n/4) + n \\ &= 2(2(2g(n/4^3) + n/4^2) + n/4) + n \\ &= 2^3g(n/4^3) + n(1/2^2 + 1/2 + 1) \\ &= 2^3g(n/4^3) + \frac{7}{4}n \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The chromatic number of the
4-dimensional hypercube Q_4

2 ☒3 ☐4 ☐5 ☐