CS 173, Spring 19 Examlet 12, colored

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(7 points) A triomino is a triangular tile with a number on each edge, visible on both front and back. In our set of triominos, the numbers range from 0 to 5. So possible tiles include 5-3-4, 0-4-4, and 3-3-3. A tile is the same if you turn it over or rotate it. So 5-3-4 is the same tile as 3-4-5, 4-5-3, and also 5-4-3. How many distinct tiles are in our set? Briefly justify your answer and/or show work.

Solution: All three sides have the same number: 6 tiles.

Two sides have the same number: 6 choices for the duplicated number, five choices for the single number. So 30 different tiles.

All three sides have different numbers: There are $\binom{6}{3} = 20$ ways to pick the three numbers. The order does not matter because we can get from any order to any other order by rotating and/or turning the tile over.

Total number of tiles is 56.

(6 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

For every dinosaur k, if k is blue, then k is not vegetarian or k is friendly.

Solution: There is a dinosaur k such that k is blue but k is vegetarian and k is not friendly.

(2 points) Check the (single) box that best characterizes each item.

The number of bit strings of length 20 with exactly 8 1's.

$\binom{26}{7}$ ☐	$\binom{27}{7}$ ☐	$\binom{20}{8}$ ☑ $\surd$
$\binom{20}{13}$ ☐	$\binom{20}{14}$ ☐	2^{20} ☐

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(7 points) This evening, Ollie the Owl wants to say `hoot` 8 times, `krick` 7 times, and `yeet` 3 times. How many distinct sequences of the 18 noises could he produce? Briefly justify your answer and/or show work.

Solution: In the sequence, he has $\binom{18}{8}$ choices for when to say `hoot`. After he has made that decision, he has $\binom{10}{7}$ options for when to say `krick`. And then `yeet` goes in the other three positions. So the total number of options is $\binom{18}{8}\binom{10}{7}$

(8 points) Use proof by contradiction to show that, in a party of n people ($n \geq 2$), there are (at least) two people who danced with the same number of different partners. Assume people always dance in pairs. Don't assume everyone danced.

Solution: Suppose not. That is, suppose that each of the n people danced with a different number of partners. Notice that the minimum number of partners is 0 and the maximum number is $n-1$. Since there are exactly n numbers between 0 and $n-1$, there's some person P who danced 0 people and another person Q who danced with $n-1$ people. But this is a contradiction. If Q danced with $n-1$ people, then Q must have danced with P, contradicting the fact that P danced with no one.

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(7 points) Each time Nancy presses her 2FA login token, it generate a sequence of 6 decimal digits. What is the chance that this sequence will contain at least one duplicate digit? Give an exact formula; don't try to figure out the decimal equivalent. Briefly justify your answer and/or show work.

Solution: There are 10^6 sequences of 6 decimal digits. There are $\frac{10!}{4!}$ sequences of 6 distinct decimal digits. So the chance of getting at least one duplicate digit is

$$1 - \frac{10!}{4!10^6}$$

.

(6 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

For every relish r, if r is orange and r is not spicy, then r is pungent.

Solution: There is a relish r, such that r is orange and r is not spicy but r is not pungent.

(2 points) Check the (single) box that best characterizes each item.

If you want to take 4 classes next semester, out of 35 classes being offered, how many different choices do you have?	$\frac{38!}{35!3!}$	☐	$\frac{35!}{31!4!}$	☑	35^4	☐
	4^{35}	☐	$\frac{35!}{31!}$	☐	4^4	☐

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(7 points) Suppose a car dealer is planning to buy a collection of Civics, Accords, and Fits (three kinds of cars). The dealer will buy ten cars in total and can buy any number of each type. How many different choices does he have? The collection is unordered, so three Civics and seven Fits is the same as seven Fits and three Civics. Briefly justify your answer and/or show work.

Solution: Using the formula for combinations with repetition, there are

$$\binom{10+2}{2}$$

choices.

(8 points) Use proof by contradiction to show that if x and y are positive integers, $x^2 - y^2 \neq 6$.

Solution: Suppose not. That is, suppose that there are positive integers x and y such that $x^2 - y^2 = 6$. Factoring the lefthand side, we get $(x-y)(x+y) = 10$. $(x-y)$ and $(x+y)$ must be integers since x and y are integers.

Because x and y are positive, $x+y$ is positive Since $(x-y)(x+y)$ is positive, $x-y$ must also be positive.

There are only two ways to factor 10 into two positive numbers: $2 \cdot 3$ or $1 \cdot 6$. In both cases, exactly one of the factors is odd, so the sum of the two factors is odd. But the sum of $(x-y)$ and $(x+y)$ is $2x$, which is even.

We have found a contradiction, so the original claim must have been correct.

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(7 points) Pearl's start-up sells sets of circular bands for labelling wine glasses. Each band has four beads, each of which can be orange, blue, or silver. Two bands are the same if they can be made to look the same by rotation and/or turning over. E.g. all bands with three blue beads and one silver bead are the same. How many distinct bands can she put in each set? Briefly justify your answer and/or show work.

Solution: All beads are the same color: 3 choices.

Three beads are the same color, one color is different: three choices for the first color, two for the second, so 6 choices total.

Two beads are the same color, two beads are a second color: 3 choices for the color that's left out, two choices for the pattern (are same-colored beads adjacent?), so 6 choices total.

Beads of all three colors: 3 choices for which color is represented twice, two choices for the pattern, so 6 choices total.

So there are $6 + 6 + 6 + 3 = 21$ distinct bands.

(6 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

For every mountain m, if m is tall or m is not in the north, then m has a snow cap.

Solution: There is a mountain m, such that m is tall or m is not in the north, but m does not have a snow cap.

(2 points) Check the (single) box that best characterizes each item.

Suppose you want to take 5 classes next semester, out of 25 classes being offered. You must take STATS 100. How many different choices do you have?

5^{24} ☐	$\frac{(24+3)!}{24!3!}$ ☐	$\frac{24!}{20!}$ ☐
24^5 ☐	$\frac{24!}{20!4!}$ ☑	$\frac{25!}{20!5!}$ ☐

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(7 points) Suppose that A is a set containing p elements and B is a set containing n elements. How many functions are there from A to $\mathbb{P}(B)$? How many of these functions are one-to-one? Briefly justify your answer and/or show work.

Solution: $\mathbb{P}(B)$ contains 2^n elements. So the total number of functions from A to $\mathbb{P}(B)$ is $(2^n)^p$. The number of one-to-one functions is $\frac{2^n!}{(2^n-p)!}$.

(8 points) Researchers recorded phone calls between pairs of two (different) people, never repeating the same pair of people. The experiment used n people ($n \geq 2$), but it's possible some of these n people were not in any conversation. Use proof by contradiction to show that two people were in the same number of conversations.

Solution: Suppose not. That is, suppose that each of the n people was in a different number of conversations. For each person, the minimum number of conversations is zero and the maximum is $n-1$. Since there are exactly n numbers between 0 and $n-1$, there's some person P who wasn't in any conversation and another person Q who was in $n-1$ conversations. But this is a contradiction. If Q talked to $n-1$ people, then Q must have talked to P, contradicting the fact that P didn't talk to anyone.

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(7 points) CMU's new robotic hummingbird Merrill travels in 3D. Each command makes it move one foot in a specified cardinal direction e.g. up/down, north/south, or east/west, but not diagonally. How many different sequences of 30 commands will get Merrill from position $(1, 10, 3)$ to position $(10, 6, 20)$? Briefly justify your answer and/or show work.

Solution: We'll need 9 commands that increase the first coordinate, 4 that decrease the second coordinate, and 17 that increase the third coordinate. There are $\binom{30}{9}$ ways to pick which of the 30 commands changes the first coordinate, and then $\binom{21}{4}$ choices for when to change the second coordinate. So our total choices are

$$\binom{30}{9}\binom{21}{4}$$

.

(6 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

There is a mushroom f such that f is not poisonous or f is blue.

Solution: For every mushroom f, f is poisonous and f is not blue.

(2 points) Check the (single) box that best characterizes each item.

E is the edge set of a tree with n nodes. $\lvert\mathbb{P}(E)\rvert =$	2^{n-1} ☑	2^n ☐	can't tell ☐
	2^{n+1} ☐	n ☐	$n-1$ ☐

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(7 points) Prof. Howard has 17 tuba players, ordered by ability, who must be distributed among three bands. Stellar will get the best players, Dismal the worst ones, and Normal the group in the middle of the range. Each band must be given at least one tuba player. How many options does he have? Briefly justify your answer and/or show work.

Solution: There are 16 positions between adjacent tuba players in the ability ordering. We need to pick two of these to be the division points. So there are $\binom{16}{2}$ possibilities

(8 points) Suppose we know that $\sqrt{6}$ is irrational. Use proof by contradiction to show that $\sqrt{2}+\sqrt{3}$ is irrational. (You must use the definition of "rational." You may not use facts about adding/subtracting rational numbers.)

Solution: Suppose not. That is, suppose that $\sqrt{2}+\sqrt{3}$ is rational. Then there are integers p and q (q non-zero) such that $\sqrt{2}+\sqrt{3}=\frac{p}{q}$.

Squaring both sides of this equation gives $2+2\sqrt{6}+3=\frac{p^2}{q^2}$. So $2\sqrt{6}=\frac{p^2}{q^2}-5=\frac{p^2-5q^2}{q^2}$. So $\sqrt{6}=\frac{p^2}{q^2}-5=\frac{p^2-5q^2}{2q^2}$. But notice that p^2-5q^2 and $2q^2$ are both integers since p and q are integers. So this means that $\sqrt{6}$ is the ratio of two integers and therefore rational. But we know that $\sqrt{6}$ is not rational.

Since our original assumption led to a contradiction, $\sqrt{2}+\sqrt{3}$ must be irrational.

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(8 points) Suppose that I have a set of p nodes, labelled 1 through p. How many different graphs can I make with this fixed set of nodes? (Isomorphic graphs with differently labelled nodes count as different for this problem.) Briefly justify your answer.

Solution: There are $\frac{p(p-1)}{2}$ possible edges for the graph. For each one, we can choose to include it or not. So there are $2^{\frac{p(p-1)}{2}}$ different possible graphs.

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

> For every computer game g, if g has trendy music or g has an interesting plotline, then g is not cheap.

Solution: There is a computer game g such that g has trendy music or an interesting plotline but g is cheap.

(2 points) Check the (single) box that best characterizes each item.

The number of ways to select a set of 17 flowers chosen from 4 possible varieties (zero or more of each variety).	$\binom{17}{5}$ ☐	$\binom{20}{4}$ ☐	$\binom{20}{3}$ $\boxed{\surd}$
	$\binom{17}{4}$ ☐	$\binom{21}{4}$ ☐	$\frac{17!}{4!}$ ☐

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(9 points) Use proof by contradiction to show that $\log_5 2$ is irrational.

Solution: Suppose not. That is, suppose that $\log_5 2$ is rational. Then $\log_5 2 = \frac{a}{b}$, where a and b are integers, b non-zero.

Raising 5 to the power of both sides, we get $2 = 5^{\frac{a}{b}}$. Raising both sides to the bth power, we get $2^b = 5^a$. Since 2 and 5 are both prime, this equation can hold only if $a = b = 0$. But we know that b is non-zero. So we have a contradiction.

Since its negation led to a contradiction, our original claim must have been true.

(6 points) Margaret's home is defended from zombies by wallnuts, peashooters, and starfruit. At each timestep, she can make one move, which adds or deletes one plant from her arsenal. If she starts with 3 wallnuts, 2 peashooters, and 19 starfruit, how many different sequences of 25 moves will get her to a configuration with 7 wallnuts, 13 peashooters, and 9 starfruit?

Solution: The sequence of 25 moves needs to add 4 wallnuts, add 11 peashooters, and delete 10 starfruit. So we need to pick 4 moves in the sequence to be the ones that add wallnuts, and then 11 of the remaining 21 moves to be ones that add peashooters. So our number of choices is

$$\binom{25}{4}\binom{21}{11}$$

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(8 points) The Emerald City bakery allows customers to special-order fruit pies. Each pie can contain one fruit or a mixture of 2 or 3 (distinct) fruits. The available fruits are raspberry, blueberry, pear, apple, cherry, apricot, and peach. How many choices do you have?

Solution:

There are 7 choices for the 1-fruit pies, $\binom{7}{2}$ for the 2-fruit pies, and $\binom{7}{3}$ for the 3-fruit pies. So the total number of choices is $7 + \binom{7}{2} + \binom{7}{3}$. If you want to turn this into a single number (not required), it is $7 + 21 + 35 = 63$.

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

There is a dorm room d, such that d has green walls and d has no window.

Solution: For every dorm room d, d has walls that aren't green or d has a window.

(2 points) Check the (single) box that best characterizes each item.

The number of ways to select an ordered sequence of 17 flowers chosen from 4 possible varieties.	$\binom{16}{3}$ ☐	$\binom{16}{4}$ ☐	$\binom{20}{3}$ ☐
	$\binom{20}{4}$ ☐	$\binom{21}{3}$ ☐	4^{17} ☑

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(9 points) Use proof by contradiction to show that there are no positive integer solutions to the equation $4x^2 - y^2 = 1$.

Solution: Suppose not. That is, suppose that there are positive integers x and y such that $4x^2 - y^2 = 1$. Factoring the lefthand side, we get $(2x - y)(2x + y) = 1$. $(2x - y)$ and $(2x + y)$ must be integers since x and y are integers. So $(2x - y)$ and $(2x + y)$ are either both 1 or both -1.

Case 1: $(2x - y) = 1$ and $(2x + y) = 1$. Adding the two equations gives us $4x = 2$, so $x = 1/2$.

Case 2: $(2x - y) = -1$ and $(2x + y) = -1$. Adding the two equations gives us $4x = -2$, so $x = -1/2$.

In both cases, x must have a non-integer value, contradicting our assumption that x is an integer.

(6 points) In the game Tic-tac-toe is played on a 3x3 grid and a move consists of the first player putting an X into one of the squares, or the second player putting an O into one of the squares. The board cannot be rotated, e.g. an X in the upper right corner is different from an X in the lower left corner. How many different board configurations are possible after four moves (i.e. two moves by each player)?

Solution: You need to pick 2 of the 9 squares to contain the X's, and then 2 of the remaining 7 squares to contain the O's. So the total number of choices is $\binom{9}{2}\binom{7}{2}$.

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(8 points) Anna has to climb 6 stairs to get onto the podium. She can climb a single step or two steps at a time. E.g. one possible (ordered) sequence of actions is: one step, a double step, three single steps. How many ways could she climb the stairs?

Solution: There is one way to do it with only single steps. Likewise, there is only one way to do it with three double-steps.

There are 5 ways to order one double step plus four single steps. And there are $\binom{4}{2} = 6$ ways to order two double steps plus two single steps.

So the total number of options is $1 + 1 + 5 + 6 = 13$.

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

> There is a bug b, such that for every plant p, if b pollinates p and p is showy, then p is poisonous.

Solution: For every bug b, there is a plant p, such that b pollinates p and p is showy, but p is not poisonous.

(2 points) Check the (single) box that best characterizes each item.

The number of ways to select a set of 17 flowers chosen from 4 possible varieties (zero or more of each variety).	$\binom{17}{5}$ ☐	$\binom{20}{4}$ ☐	$\binom{20}{3}$ ☑ $\surd$
	$\binom{17}{4}$ ☐	$\binom{21}{4}$ ☐	$\frac{17!}{4!}$ ☐

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(8 points) Ignatius Eggbert flips a coin 10 times. The coin is fair, i.e. equal chance of getting a head vs. a tail. What is the chance that he gets exactly 7 heads? Give an exact formula; don't try to figure out the decimal equivalent. Briefly explain your answer and/or show work.

Solution: There are 2^{10} sequences of 10 coin flips. There are $\binom{10}{7}$ ways to pick 7 of these flips to contain a head. So the chance of getting 7 heads is

$$\frac{\binom{10}{7}}{2^{10}}$$

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

For every Meerkat m, if m is in New York, then m is not in the wild or m is lost.

Solution: There is a Meerkat m, such that m is in New York, but m is in the wild and m is not lost.

(2 points) Check the (single) box that best characterizes each item.

The number of bit strings of length 20 with exactly 7 1's.	$\binom{26}{7}$ ☐	$\binom{27}{7}$ ☐	$\binom{26}{6}$ ☐
	$\binom{20}{13}$ ☑	$\binom{20}{14}$ ☐	2^{20} ☐

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(9 points) Use proof by contradiction to show that $\sqrt{2}+\sqrt{3} \leq 4$.

Solution: Suppose not. That is, suppose that $\sqrt{2}+\sqrt{3} > 4$.

Then $(\sqrt{2}+\sqrt{3})^2 > 16$. (All the numbers involved are positive.) So $2+2\sqrt{2}\sqrt{3}+3 > 16$. So $2\sqrt{2}\sqrt{3} > 11$.

Squaring both sides again, we get $4 \cdot 2 \cdot 3 > 121$. That is $24 > 121$. But this last equation is obviously false. So our original assumption must have been wrong and therefore $\sqrt{2}+\sqrt{3} \leq 4$.

(6 points) Use the binomial theorem to find a closed form for the summation $\sum_{k=0}^{n}(-1)^k\binom{n}{k}$. Make sure it's clear how you used the theorem.

Solution: The binomial theorem states that $(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k}\binom{n}{k}$.

Setting $x=-1$ and $y=1$ gives us $(-1+1)^n = \sum_{k=0}^{n}(-1)^k 1^{n-k}\binom{n}{k}$

That is $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

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(8 points) If w is a string of characters, a *substring* of w is a contiguous string that forms part of w. For example, "rtho" is a substring of "warthog" but "ahog" is not. Suppose that w is a string without any repeated characters, of length k. How many non-empty substrings does it have?

Solution: A substring is determined by its starting and ending positions. This is an unordered pair, because we only get a well-defined substring if the starting position precedes the ending one. (Or, if you like, the backwards pair defines the same substring.)

Method (1): Suppose a "position" is a character in the string. There are k substrings of length 1. For substrings of length $k > 1$, we need to pick a set of two distinct positions (starting and ending). We have $\binom{k}{2}$ ways to do this. Adding these two numbers gives us $\frac{k(k-1)}{2} + k = \frac{k(k+1)}{2}$.

Method (2): Suppose a "position" lies between two characters. Then we have $k+1$ positions, from which we must choose an unordered pair. We have $\binom{k+1}{2} = \frac{(k+1)k}{2}$ ways to make this choice.

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

There is a soup s such that s is tasty and s does not contain meat.

Solution: For every soup s, s is not tasty or s contains meat.

(2 points) Check the (single) box that best characterizes each item.

How many ways can I choose 6 bagels from among 8 varieties, if I can have any number of bagels from any type?	$\frac{8!}{6!2!}$ ☐	$\frac{13!}{6!7!}$ ☑	$\frac{14!}{9!5!}$ ☐
	$\frac{14!}{6!7!}$ ☐	8^6 ☐	6^8 ☐

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(8 points) Marsha has 25 cups, identical except that 10 are red, 8 are blue, and 7 are green. How many ways can she make an (ordered) sequence of 25 cups?

Solution: In the sequence, she has $\binom{25}{10}$ choices for where to put the red cups. Then she has $\binom{15}{8}$ choices for where to put the blue cups. So the total number of options is $\binom{25}{10}\binom{15}{8}$

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

There is a dorm room d, such that d has green walls and d has no window.

Solution: For every dorm room d, d has walls that aren't green or d has a window.

(2 points) Check the (single) box that best characterizes each item.

Suppose you want to take 5 classes next semester, out of 25 classes being offered. You must take STATS 100. How many different choices do you have?

$\frac{24!}{20!}$	☐	$\frac{24!}{20!4!}$	☑ √
$\frac{(24+3)!}{24!3!}$	☐	$\frac{25!}{20!5!}$	☐

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(9 points) Use proof by contradiction to show that, for any graph G and any two nodes a and b, the shortest walk from a to b does not contain any repeated nodes.

Solution: Suppose not. That is, suppose we have a graph G, two nodes a and b, and the shortest walk W from a to b contains at least one repeated node.

Since W contains a repeated pair of nodes, it must look like $a = w_1, w_2, \ldots, w_{n-1}, w_n = b$. Suppose that $w_i = w_k$, where $i < k$. We can then make a new walk W' from a to b by removing the nodes between w_i and w_k, merging w_i with w_k. W' is shorter than W, contradicting our assumption that W was the shortest walk between these two nodes.

(6 points) Suppose a set S has 11 elements. How many subsets of S have an even number of elements? Express your answer as a summation. Briefly justify or show work.

Solution: Subsets of S having an even number of elements would have $0, 2, 4, 6, 8$ or 10 elements. There are $\binom{11}{0}$ subset of S with no elements, $\binom{11}{2}$ subsets of S with 2 elements, and so on. So the number of subsets of S having an even number of elements is

$$\binom{11}{0} + \binom{11}{2} + \binom{11}{4} + \binom{11}{6} + \binom{11}{8} + \binom{11}{10} = \sum_{i=0}^{5} \binom{11}{2i}.$$

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(8 points) Suppose we know (e.g. from the binomial theorem) that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Use this to show that $\sum_{k=1}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$. Show your work.

Solution:

$k \cdot \binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$

So $\sum_{k=1}^{n} k \cdot \binom{n}{k} = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} = n \cdot \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} = n \cdot \sum_{k=1}^{n} \binom{n-1}{k-1}$

Changing the index of the summation and using the given identity gives us

$n \cdot \sum_{k=1}^{n} \binom{n-1}{k-1} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} = n \cdot 2^{n-1}$

Combining this with the previous equation, we get $\sum_{k=1}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

For every tree t, if t grows in Canada, then t is not tall or t is a conifer.

Solution: There is a tree t, such that g is tall and t is not a conifer, bug t grows in Canada.

(2 points) Check the (single) box that best characterizes each item.

How many ways can I choose a set of 5 bagels, if there are 10 types of bagels and all 5 bagels must be different types?

$\frac{10!}{5!5!}$	$\surd$	$\frac{14!}{10!4!}$	☐	$\frac{14!}{9!5!}$	☐
$\frac{15!}{10!5!}$	☐	10^5	☐	5^{10}	☐

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Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(9 points) Researchers recorded phone calls between pairs of two (different) people, never repeating the same pair of people. The calls used n people ($n \geq 2$), but it's possible some people were not in any conversation. Use proof by contradiction to show that two people were in the same number of conversations.

Solution: Suppose not. That is, suppose that each of the n people was in a different number of conversations. For each person, the minimum number of conversations is zero and the maximum is $n-1$. Since there are exactly n numbers between 0 and $n-1$, there's some person P who wasn't in any conversation and another person Q who was in $n-1$ conversations. But this is a contradiction. If Q talked to $n-1$ people, then Q must have talked to P, contradicting the fact that P didn't talk to anyone.

(6 points) Use the binomial theorem to find a closed form for the summation $\sum_{k=0}^{n}\binom{n}{k}$. Make sure it's clear how you used the theorem.

Solution: The binomial theorem states that $(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k}\binom{n}{k}$.

Setting $x = y = 1$ gives us $2^n = (1+1)^n = \sum_{k=0}^{n}(1)^k 1^{n-k}\binom{n}{k} = \sum_{k=0}^{n}\binom{n}{k}$

That is $\sum_{k=0}^{n}\binom{n}{k} = 2^n$.

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(9 points) Use proof by contradiction to show that, for any integer n, at least one of the three integers n, $2n+1$, $4n+3$ is not divisible by 7.

Solution: Suppose not. That is, suppose that n, $2n+1$, $4n+3$ are all divisible by 7. Then their sum $n+(2n+1)+(4n+3)$ must be divisible by 7. So $7n+4$ must be divisible by 7. But then 4 would need to be divisible by 7, which isn't true.

Since its negation led to a contradiction, our original claim must have been true.

(6 points) Suppose a set S has 11 elements. How many subsets of S have five or fewer elements? Express your answer as a summation. Briefly justify or show work.

Solution: There are $\binom{11}{0}$ subset of S with no elements, $\binom{11}{2}$ subsets of S with 2 elements, and so on. So the number of subsets of S having five or fewer elements is

$$\binom{11}{0} + \binom{11}{1} + \binom{11}{2} + \binom{11}{3} + \binom{11}{4} + \binom{11}{5} = \sum_{i=0}^{5} \binom{11}{i}.$$

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(8 points) Recall that nodes in a full binary tree have either zero or two children. Suppose we are building a full binary tree with unlabelled nodes whose leaves are all at levels k or $k+1$, with p leaves at level $k+1$. How many different ways can we construct such a tree? Briefly justify your answer.

Solution: Notice that nodes in each level come in pairs, because the tree is full. So p must be even. Let $p = 2n$.

The only thing we get to choose when constructing these trees is which nodes at level k have two children (vs. zero children). There are 2^k nodes at level k, of which n will have children. So we have $\binom{2^k}{n}$ options for constructing the tree.

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

If it is raining, then there is a cyclist c such that c is getting wet.

Solution: It is raining and for every cyclist c, c is not getting wet.

(2 points) Check the (single) box that best characterizes each item.

The number of ways to select a set of 17 flowers chosen from 4 possible varieties (one or more of each variety).	$\binom{16}{3}$ ☑	$\binom{16}{4}$ ☐	$\binom{20}{3}$ ☐
	$\binom{20}{4}$ ☐	$\binom{21}{3}$ ☐	4^{17} ☐

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(8 points) The Google interviewer suggests that $\binom{n}{k}$ can be computed very efficiently using the equation $\binom{n}{k} = \frac{n+1-k}{k}\binom{n}{k-1}$. Is this formula correct? Assume $k > 0$. Briefly justify your answer.

Solution: This formula is correct.

$$\frac{n+1-k}{k}\binom{n}{k-1} = \frac{n+1-k}{k}\frac{n!}{(k-1)!(n-(k-1))!} = \frac{(n+1-k)n!}{k(k-1)!(n-(k-1))!} = \frac{(n+1-k)n!}{k!(n-k+1)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

For every dog d, if d is a terrier, then d is not large and d is noisy.

Solution: There is a dog d, such that d is a terrier, but d is large or d is not noisy.

(2 points) Check the (single) box that best characterizes each item.

V is the vertex set of a tree with n edges. $\lvert\mathbb{P}(V)\rvert =$	2^{n-1}	☐	2^n	☐	not determined	☐
	2^{n+1}	$\surd$	n	☐		

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Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(9 points) Every hacker is a black hat or a white hat (and not both). White hats always tell the truth. Black hats always lie. Alfred says to you "I am a black hat." Use proof by contradiction to show that Alfred is not a hacker.

Solution: Suppose not. That is, suppose that Alfred is a hacker. Since Alfred is a hacker, there are two possibilities.

Case 1: Alfred is a white hat. This means his statement should be true. But he said he was a black hat.

Case 2: Alfred is a black hat. This means his statement should be fals. That is, since he said he was a black hat, he should be a white hat.

In both cases, we have a contradiction: Alfred is supposedly both a white hat and a black hat.

(6 points) If $x, y, z \in \mathbb{N}$, how many solutions are there to the equation $x + y + z = 25$?

Solution: Imagine that the number 25 represents 25 objects that can be chosen in three different types: type x, type y, and type z. To indicate type we divide the objects into three bins, i.e., place 3 dividers into the list of objects. The number of ways to place the objects into the bins is the number of different solutions and is given by the expression for combinations with repetition

$$\binom{25+3-1}{25} = \binom{27}{25} = \binom{27}{2} = 351$$

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Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(8 points) A triomino is a triangular tile with a number on each edge. In our set of triominos, the numbers range from 0 to 5. So possible tiles include 5-3-4, 0-4-4, and 3-3-3. Tiles can be turned over: Also notice that a tile is the same if you rotate it. So 5-3-4 is the same tile as 3-4-5, 4-5-3, and also 5-4-3. How many distinct tiles are in our set?

Solution: All three sides have the same number: 6 tiles.

Two sides have the same number: 6 choices for the duplicated number, five choices for the single number. So 30 different tiles.

All three sides have different numbers: There are $\binom{6}{3} = 20$ ways to pick the three numbers. The order does not matter because we can get from any order to any other order by rotating and/or turning the tile over.

Total number of tiles is 56.

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

For every tiger k, if k is orange, then k is large and k is not friendly.

Solution: There is a tiger k such that k is not large or k is friendly, but k is orange.

(2 points) Check the (single) box that best characterizes each item.

The number of ways to select a set of 4 flowers chosen from 17 possible varieties (zero or more of each variety).	$\binom{17}{5}$ ☐	$\binom{20}{4}$ ☑	$\binom{20}{3}$ ☐
	$\binom{17}{4}$ ☐	$\binom{21}{4}$ ☐	$\frac{17!}{4!}$ ☐

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Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(9 points) Use proof by contradiction to show that, in any group of 7 people, there is at least one person who knows an even number of people. (Assume that "knowing someone" is symmetric.)

Solution: Suppose not. That is, suppose that we have a group of 7 people, in which each person knows an odd number of the other people.

Since knowing someone is symmetric, the total number of "knows" relationships must be even. (Each relationship goes both ways.) However, if we add up the number of supposed relationships, we have the sum of 7 odd numbers, which must be odd.

We have found a contradiction, so the original claim must have been correct.

(6 points) Margaret's home is defended from zombies by wallnuts, peashooters, and starfruit. She has a row of 20 pedestals on which they can stand, and she needs to use at least one starfruit. How many options does she have for the placing defenders on the pedestals?

Solution: We're making an ordered arrangement, and we have three choices for each pedestal. If we ignore the constraint about one starfruit, there are 3^{20} possible arrangements.

There are 2^{20} arrangements that feature walnuts and peashooters, but no starfruits.

So we have $3^{20} - 2^{20}$ arrangements that meet all the requirements.

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(9 points) Use proof by contradiction to show that $\sqrt{\sqrt{2}}$ is not rational. (You may use the fact that $\sqrt{2}$ is not rational.)

Solution: Suppose not. That is, suppose that $\sqrt{\sqrt{2}}$ were rational.

Since $\sqrt{\sqrt{2}}$ is rational, then $\sqrt{\sqrt{2}} = \frac{m}{n}$, where m and n are integers, $n \neq 0$. Squaring both sides, we get $\sqrt{2} = \frac{m^2}{n^2}$. Since m and n are both integers, m^2 and n^2 are both integers. So $\sqrt{2} = \frac{m^2}{n^2}$ implies that $\sqrt{2}$ is rational. But we know that $\sqrt{2}$ is not rational.

Since the negated claim led to a contradiction, the original claim must be true.

(6 points) In the town of West Fork, the streets are laid out in a uniform square grid. Alvin's school lies 6 blocks east and 9 blocks north of his house. So (since there are no diagonal roads) he travels 15 blocks to school. How many different 15-block paths can he choose from? Show your work or justify your answer.

Solution: We need to pick 6 of the 15 moves to be the ones where he walks east. That is, we have $\binom{15}{6}$ ways to form a path.

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(9 points) Use proof by contradiction to show that there are no positive integer solutions to the equation $x^2 - y^2 = 10$.

Solution: Suppose not. That is, suppose that there are positive integers x and y such that $x^2 - y^2 = 10$. Factoring the lefthand side, we get $(x-y)(x+y) = 10$. $(x-y)$ and $(x+y)$ must be integers since x and y are integers.

Ignoring sign, there are only two ways to factor 10: $2 \cdot 5$ or $1 \cdot 10$. In both cases, exactly one of the factors is odd, so the sum of the two factors is odd. But the sum of $(x-y)$ and $(x+y)$ is $2x$, which is even.

We have found a contradiction, so the original claim must have been correct.

(6 points) Margaret's home is defended from zombies by wallnuts, peashooters, and starfruit. She has room to place 20 of these on her lawn. How many options does she have for her set of defenders?

Solution: This is a combinations with repetition problem. We have 20 objects and three types, thus two dividers. So the number of options is $\binom{20+2}{2} = \binom{22}{2}$.