CS 173, Spring 19 Examlet 8, colored

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $g : \mathbb{N} \to \mathbb{R}$ is defined by

$$g(0) = 0 \qquad g(1) = \tfrac{4}{3}$$

$$g(n) = \tfrac{4}{3}g(n-1) - \tfrac{1}{3}g(n-2), \quad \text{for } n \geq 2$$

Use (strong) induction to prove that $g(n) = 2 - \dfrac{2}{3^n}$

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Let function $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$f(0) = 2$

$f(1) = 7$

$f(n) = f(n-1) + 2f(n-2)$, for $n \geq 2$

Use (strong) induction to prove that $f(n) = 3 \cdot 2^n + (-1)^{n+1}$ for any natural number n.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

CS 173, Spring 19 Examlet 8, colored

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(20 points) Use (strong) induction to prove that the following claim holds:

Claim : For any integer $n \geq 2$, if $p_1, \ldots, p_n$ is a sequence of integers and $p_1 < p_n$, then there is an index j $(1 \leq j < n)$ such that $p_j < p_{j+1}$.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

CS 173, Spring 19 Examlet 8, colored

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by is defined by

$f(1) = 5$ $\quad$ $f(2) = -5$

$f(n) = 4f(n-2) - 3f(n-1)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) = 2 \cdot (-4)^{n-1} + 3$

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that θ is a constant (but unknown) real number. For any real number p, the angle addition formulas imply the following two equations (which you can assume without proof):

$$\cos(\theta)\cos(p\theta) = \cos((p+1)\theta) + \sin(\theta)\sin(p\theta) \tag{1}$$
$$\cos(\theta)\cos(p\theta) = \cos((p-1)\theta) - \sin(\theta)\sin(p\theta) \tag{2}$$

Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$f(0) = 1 \qquad f(1) = \cos(\theta)$

$f(n+1) = 2\cos(\theta)f(n) - f(n-1)$, for all $n \geq 2$.

Use (strong) induction to prove that $f(n) = \cos(n\theta)$ for any natural number n.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) A Zellig graph consists of $2n$ ($n \geq 1$) nodes connected so as to form a circle. Half of the nodes have label 1 and the other half have label -1. As you move clockwise around the circle, you keep a running total of node labels. E.g. if you start at a 1 node and then pass through two -1 nodes, your running total is -1. Use (strong) induction to prove that there is a choice of starting node for which the running total stays ≥ 0.

Hint: remove an adjacent pair of nodes.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(20 points) (20 points) Suppose that $f : \mathbb{N} \rightarrow \mathbb{Z}$ is defined by

$f(0) = 2 \qquad f(1) = 5 \qquad f(2) = 15$

$f(n) = 6f(n-1) - 11f(n-2) + 6f(n-3)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) = 1 - 2^n + 2 \cdot 3^n$

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(20 points) Use (strong) induction to prove that, for any integer $n \geq 8$, there are non-negative integers p and q such that $n = 3p + 5q$.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:____________________ **Lecture:** **B**

Discussion: **Friday** **11** **12** **1** **2** **3** **4**

(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$f(1) = 3$ $\quad f(2) = 5$

$f(n) = 3f(n-1) - 2f(n-2)$ for all $n \geq 3$.

Use (strong) induction to prove that $f(n) = 2^n + 1$

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:____________________ **Lecture:** **B**

Discussion: **Friday** **11** **12** **1** **2** **3** **4**

(20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$f(0) = f(1) = f(2) = 1$

$f(n) = f(n-1) + f(n-3)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) \geq \frac{1}{2}(\sqrt{2})^n$. You may use the fact that $\sqrt{2}$ is smaller than 1.5.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:_________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Let x be a non-zero real number such that $x + \frac{1}{x}$ is an integer. Use (strong) induction to prove that $x^n + \frac{1}{x^n}$ is an integer, for any natural number n.

Hint: $(a^n + b^n)(a + b) = (a^{n+1} + b^{n+1}) + ab(a^{n-1} + b^{n-1})$, for any real numbers a and b.

Let x be a non-zero real number such that $x + \frac{1}{x}$.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:______________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(20 points) Suppose that $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by

$h(1) = 1 \qquad h(2) = 7$

$h(n+1) = 7h(n) - 12h(n-1)$ for all $n \geq 2$

Use (strong) induction to prove that $h(n) = 4^n - 3^n$

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Recall that F_n is the nth Fibonacci number, and the positive Fibonacci numbers start with $F_1 = F_2 = 1$. Use (strong) induction to prove the following claim:

> Claim: Every positive integer can be written as the sum of (one or more) distinct Fibonacci numbers.

Hints: You can assume that the Fibonacci numbers are strictly increasing starting with F_1. To write x as the sum of Fibonacci numbers, start by including the largest Fibonacci number F_p such that $F_p \leq x$. (And therefore $x < F_{p+1}$.) How large is the remaining part of x?

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

CS 173, Spring 18 Examlet 8, Part A

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(20 points) Use (strong) induction to prove the following claim:

> For any positive integer $n \geq 2$, if G is a graph with n nodes and more than $(n-1)(n-2)/2$ edges, then G is connected.

Hint: pick a node x. Perhaps x is connected to all the other nodes. If not, remove x to create a smaller graph H. What is the smallest number of edges that could remain in H? Notice that H has too few nodes to contain all the edges in G, so there is an edge from x to H.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Recall that F_n is the nth Fibonacci number, and these start with $F_0 = 0$, $F_1 = 1$. Use (strong) induction to prove the following claim:

Claim: $F_{n-1}F_{n+1} - (F_n)^2 = (-1)^n$ for any positive integer n.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Let function $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$f(0) = 3$

$f(1) = 9$

$f(n) = f(n-1) + 2f(n-2)$, for $n \geq 2$

Use (strong) induction to prove that $f(n) = 4 \cdot 2^n + (-1)^{n-1}$ for any natural number n.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Use (strong) induction to prove that $a - b$ divides $a^n - b^n$, for any integers a and b and any natural number n.

Hint: $(a^n - b^n)(a + b) = (a^{n+1} - b^{n+1}) + ab(a^{n-1} - b^{n-1})$, for any real numbers a and b.

Let a and b be integers.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(20 points) Suppose that $f : \mathbb{N}^2 \to \mathbb{N}$ is defined by

$f(n, 0) = f(n, n) = 1$, for any natural number n

$f(n, a) = f(n-1, a-1) + f(n-1, a)$, for all n and a such that $1 \leq a \leq n-1$

Use (strong) induction to prove that $f(n, a) = \frac{n!}{a!(n-a)!}$ for any natural numbers a and n, where $n \geq a$. Hint: use n as your induction variable. At each step, make sure the equations work for an arbitrary natural number $a \leq n$.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step: [First deal with two special cases: $f(k, 0)$ and $f(k, k)$.]

CS 173, Fall 17 Examlet 8, Part A

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) (20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$f(0) = 2$ $\quad f(1) = 5$ $\quad f(2) = 15$

$f(n) = 6f(n-1) - 11f(n-2) + 6f(n-3)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) = 1 - 2^n + 2 \cdot 3^n$

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

CS 173, Fall 17 Examlet 8, Part A

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is defined by

$f(0) = 0$ $\quad$ $f(1) = 1$

$f(n) = f(n-1) + f(n-2)$ for all $n \geq 2$.

Let $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. Use (strong) induction to prove that $f(n) = \dfrac{a^n - b^n}{a - b}$.

First show that $a^2 = a + 1$ and $b^2 = b + 1$:

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

CS 173, Fall 17 Examlet 8, Part A

Name: ____

NetID: ______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $g : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$g(1) = 1$

$g(2) = 8$

$g(n) = g(n-1) + 2g(n-2)$

Use (strong) induction to prove that $g(n) = 3 \cdot 2^{n-1} + 2(-1)^n$.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove that

$$\prod_{p=1}^{n} \frac{m+1-p}{p} = \frac{m!}{n!(m-n)!}$$

for any positive integers m and n where $m \geq n$. Hint: use n as your induction variable. At each step, make sure the the equations work for an arbitrary integer $m \geq n$.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:_________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Also recall that $\sin 2y = 2 \sin y \cos y$ for any real number y.

Suppose that x is a real number such that $\sin x$ is non-zero. Use (strong) induction to prove that $\displaystyle\prod_{p=0}^{n-1} \cos(2^p x) = \frac{\sin(2^n x)}{2^n \sin x}$, for any positive integer n.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ is defined by is defined by

$f(1) = 5$ $\quad$ $f(2) = -5$

$f(n) = 4f(n-2) - 3f(n-1)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) = 2 \cdot (-4)^{n-1} + 3$

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:_____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) A "triangle-free" graph is a graph that doesn't contain any 3-cycles. Use (strong) induction to prove that a triangle-free graph with $2n$ nodes has $\leq n^2$ edges, for any positive integer n. Hint: in the inductive step, remove a pair of nodes joined by an edge. How many edges from those nodes to the rest of the graph?

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$f(0) = 1 \qquad f(1) = -5$

$f(n) = -7f(n-1) - 10f(n-2), \quad \text{for } n \geq 2$

Use (strong) induction to prove that $f(n) = (-1)^n \cdot 5^n$

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Use (strong) induction to prove that, for all positive integers n, $x^2 + y^2 = z^n$ has a positive integer solution. (That is, a solution in which x, y, and z are all positive integers.) Hints: (1) notice that $3^2 + 4^2 = 5^2$ and (2) use the solution for $n = k - 2$ (not $n = k - 1$) to build a solution for $n = k$.

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step:

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(20 points) Suppose that $g : \mathbb{N} \to \mathbb{R}$ is defined by

$$g(0) = 0 \qquad g(1) = \tfrac{4}{3}$$

$$g(n) = \tfrac{4}{3}g(n-1) - \tfrac{1}{3}g(n-2), \quad \text{for } n \geq 2$$

Use (strong) induction to prove that $g(n) = 2 - \dfrac{2}{3^n}$

Proof by induction on n.

Base case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Rest of the inductive step: