CS 173, Spring 19 Examlet 4, colored

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let's define a relation T between natural numbers follows:

aTb if and only if $a = b + 2k$, where k is a natural number

Working directly from this definition, prove that T is antisymmetric.

Solution: Let a and b be natural numbers and suppose that aTb and bTa.

By the definition of T, this means that $a = b+2k$ and $b = a+2j$, where k and j are natural numbers.

Substituting one equation into the other, we get $a = (a + 2j) + 2k = a + 2(j + k)$. So $2(j + k) = 0$. So $j + k = 0$.

Notice that j and k are both non-negative. So $j + k = 0$ implies that $j = k = 0$.

So $a = b$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

A closed interval of the real line can be represented as a pair (c, r), where c is the center of the interval and r is its radius. Let $X = \{(c, r) \mid c, r \in \mathbb{R}, r \geq 0\}$ be the set of closed intervals represented this way.

Now, let's define the interval containment $\preceq$ on X as follows

$(c, r) \preceq (d, q)$ if and only if $r \leq q$ and $|c - d| + r \leq q$.

Prove that $\preceq$ is transitive.

Solution: Let (c, r), (d, q), and (f, s) be elements of X. Suppose that $(c, r) \preceq (d, q)$ and $(d, q) \preceq (f, s)$. By the definition of $\preceq$, this means that $r \leq q$ and $|c - d| + r \leq q$ and $q \leq s$ and $|d - f| + q \leq s$. So $r \leq s$. Also, $|c - d| + r + |d - f| + q \leq q + s$, which implies that $|c - f| + r \leq |c - d| + |d - f| + r \leq s$. So $(c, r) \prec (f, s)$, which is what we needed to show.

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Suppose that T is a relation on the integers which is transitive. Let's define a relation R on the integers as follows:

xRy if and only if there is an integer k such that xTk and kTy.

Prove that R is transitive.

Solution: Let a, b and c be integers. Suppose that aRb and bRc.

By the definition of R, aRb means that there is an integer k such that aTk and kTb. Since T is known to be transitive, this implies that aTb.

Similarly bRc means that there is an integer j such that bTj and jTc. And (because T is transitive), therefore bTc.

We now know that aTb and bTc, where b is an integer. So, by the definition of R, aRc, which is what we needed to show.

Name:____________________________________

NetID:____________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

Let T be the relation defined on $\mathbb{Z}^2$ by

$(x, y)T(p, q)$ if and only if $x < p$ or ($x = p$ and $y \leq q$)

Prove that T is antisymmetric.

Solution:

Let (x, y) and (p, q) be pairs of integers. Suppose that $(x, y)T(p, q)$ and $(p, q)T(x, y)$. By the definition of T $(x, y)T(p, q)$ means that $x < p$ or ($x = p$ and $y \leq q$). Similarly, $(p, q)T(x, y)$ means that $p < x$ or ($p = x$ and $q \leq y$).

There are four cases:

Case 1: $x < p$ and $p < x$. This is impossible.

Case 2: $x < p$ and $p = x$ and $q \leq y$. Also impossible.

Case 3: $p < x$ and $x = p$ and $y \leq q$. Impossible as well.

Case 4: $x = p$ and $y \leq q$ and $p = x$ and $q \leq y$. Since $y \leq q$ and $q \leq y$, $x = y$. So we have $(x, y) = (p, q)$.

$(x, y) = (p, q)$ is true, which is what we needed to show.

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

Define the relation $\sim$ on $\mathbb{Z}$ by

$x \sim y$ if and only if $5 \mid (3x + 7y)$

Working directly from the definition of divides, prove that $\sim$ is transitive.

Solution: Let x, y, and z be integers. Suppose that $x \sim y$ and $y \sim z$.

By the definition of $\sim$, $5 \mid (3x + 7y)$ and $5 \mid (3y + 7z)$. So $3x + 7y = 5m$ and $3y + 7z = 5n$, for some integers m and n.

Adding these two equations together, we get $3x+7y+3y+7z = 5m+5n$. So $3x+10y+7z = 5(m+n)$. So $3x + 7z = 5(m + n - 2y)$.

$m + n - 2y$ is an integer, since m, n and y are integers. So this means that $5 \mid 3x + 7z$ and therefore $x \sim z$, which is what we needed to show.

Name:____

NetID:______________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

Let $A = \{(x, y, z) \in \mathbb{Z}^3 : x \leq y \leq z\}$. Let's define a relation R on A as follows:

$(a, b, c)R(x, y, z)$ if and only if $a \leq x$ and $z \leq b$.

Working directly from this definition, prove that R is antisymmetric.

Solution: Let (x, y, z) and (a, b, c) be elements of A. Suppose that $(x, y, z)R(a, b, c)$ and $(a, b, c)R(x, y, z)$.

By the definition of R, $(a, b, c)R(x, y, z)$ implies that $a \leq x$ and $z \leq b$. Similarly, $(x, y, z)R(a, b, c)$ implies that $x \leq a$ and $c \leq y$.

We have $a \leq x$ and $x \leq a$, so $x = a$.

We also have $z \leq b$ and $c \leq y$. But notice that we also know that $x \leq y \leq z$ and $a \leq b \leq c$ from the definition of A. Combining these inequlities, we have

$$b \leq c \leq y \leq z \leq b$$

So $b = c = y = z$.

So $(x, y, z) = (a, b, c)$, which is what we needed to prove.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$, i.e. pairs of positive integers.

Define a relation $\gg$ on A as follows:

$(x, y) \gg (p, q)$ if and only if there exists an integer $n \geq 1$ such that $(x, y) = (np, nq)$.

Prove that $\gg$ is antisymmetric.

Solution: Let (x, y) and (p, q) be pairs of natural numbers and suppose that $(x, y) \gg (p, q)$ and $(p, q) \gg (x, y)$.

By the definition of $\gg$, $(x, y) = (np, nq)$ and $(p, q) = m(x, y)$, for some positive integers m and n. So $x = np$, $y = nq$, $p = mx$ and $q = my$.

Combining these equations, we get $x = n(mx) = (nm)x$ and $y = n(my) = (nm)y$. So $nm = 1$. But this means that $n = m = 1$ since n and m are positive integers. So $x = p$ and $y = q$. So $(x, y) = (p, q)$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let $A = \{(x, y, z) \in \mathbb{Z}^3 : x \leq y \leq z\}$. Let's define a relation R on A as follows:

$(a, b, c)R(x, y, z)$ if and only if $a \leq x$ and $z \leq b$.

Working directly from this definition, prove that R is transitive.

Solution: Let (x, y, z), (a, b, c), and (p, q, r) be elements of A. Suppose that $(x, y, z)R(a, b, c)$ and $(a, b, c)R(p, q, r)$.

By the definition of R, $(x, y, z)R(a, b, c)$ implies that $x \leq a$ and $c \leq y$. Similarly $(a, b, c)R(p, q, r)$ implies that $a \leq p$ and $r \leq b$.

So have $x \leq a$ and $a \leq p$, so $x \leq p$.

We also have $c \leq y$ and $r \leq b$. Notice that $a \leq b \leq c$ by the definitino of the set A. So we have $r \leq b \leq c \leq y$, and therefore $r \leq y$

Since $x \leq p$ and $r \leq y$, $(x, y, z)R(p, q, r)$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **B**

Discussion: **Friday** **11** **12** **1** **2** **3** **4**

Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$, i.e. pairs of positive integers. Consider the relation T on A defined by

$(x, y)T(p, q)$ if and only if $(xy)(p + q) = (pq)(x + y)$

Prove that T is transitive.

Solution: Let (a, b), (p, q), and (m, n) be elements of A. Suppose that $(a, b)T(p, q)$ and $(p, q)T(m, n)$. By the definition of T, this means that $(xy)(p + q) = (pq)(x + y)$ and $(pq)(m + n) = (mn)(p + q)$

Since $m+n$ is positive, we can divide both sides by it, to get $(pq) = (mn)(p+q)/(m+n)$. Substituting this into the first equation, we get

$$(xy)(p + q) = (mn)(p + q)/(m + n) \times (x + y)$$

Multiplying both sides by $(m + n)$, we get

$$(xy)(p + q)(m + n) = (mn)(p + q)(x + y)$$

Since $(p + q)$ is positive, we can cancel it from both sides to get

$$(xy)(m + n) = (mn)(x + y)$$

By the definition of T, this means that $(a, b)T(m, n)$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **B**

Discussion: **Friday** **11** **12** **1** **2** **3** **4**

Suppose that T is a relation on the integers which is antisymmetric. Let's define a relation R on pairs of integers such that $(p, q)R(a, b)$ if and only if $(a + b)T(p + q)$ and bTq. Prove that R is antisymmetric.

Solution: Let (a, b) and (p, q) be pairs of integers. Suppose that $(a, b)R(p, q)$ and $(p, q)R(a, b)$.

By the definition of R, this means that $(a, b)R(p, q)$ means that $(p+q)T(a+b)$ and qTb. Similarly,$(p, q)R(a, b)$ means that $(a + b)T(p + q)$ and bTq.

Because T is antisymmetric, qTb and bTq implies that $q = b$. Similarly, $(p + q)T(a + b)$ and $(a + b)T(p + q)$ implies that $p + q = a + b$.

Since $q = b$ and $p + q = a + b$, $p = a$. So $(p, q) = (a, b)$, which is what we needed to prove.

CS 173, Spring 18 Examlet 4, Part A

Name:____

NetID:_________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$, i.e. pairs of positive integers. Consider the relation T on A defined by

$(x, y)T(p, q)$ if and only if $(xy)(p + q) < (pq)(x + y)$

Prove that T is transitive.

Solution: Let (x, y), (p, q), and (m, n) be elements of A. Suppose that $(x, y)T(p, q)$ and $(p, q)T(m, n)$. By the definition of T, this means that $(xy)(p + q) < (pq)(x + y)$ and $(pq)(m + n) < (mn)(p + q)$

Since $m + n$ and $x + y$ are both positive, we can multiply the above equations by them to get: $(xy)(p+q)(m+n) < (pq)(x+y)(m+n)$ and $(pq)(m+n)(x+y) < (mn)(p+q)(x+y)$. Combining these two equations, we get $(xy)(p + q)(m + n) < (mn)(p + q)(x + y)$.

Since $(p + q)$ is positive, we can cancel it from both sides to get

$$(xy)(m + n) < (mn)(x + y)$$

By the definition of T, this means that $(x, y)T(m, n)$, which is what we needed to show.

Name:____

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Suppose that n is some integer ≥ 2. Let's define the relation R_n on the integers such that aR_nb if and only if $a \equiv b + 1 \pmod{n}$. Prove the following claim

Claim: If R_n is symmetric, then $n = 2$.

You must work directly from the definition of congruence mod k, using the following version of the definition: $x \equiv y \pmod{k}$ iff $x - y = mk$ for some integer m. You may use the following fact about divisibility: for any non-zero integers p and q, if $p \mid q$, then $|p| \leq |q|$.

Solution: Suppose n is an integer, with $n \geq 2$. Also, suppose R_n is symmetric, where aR_nb for integers a, b iff $a \equiv b + 1 \pmod{n}$.

Suppose, then, that aR_nb for some integers a, b. Using the above definition of congruence mod k, $a - b - 1 = mn$ for some integer m. Because R_n is symmetric, bR_na, so $b - a - 1 = jn$ for some integer j. So $b = jn + a + 1$. Substituting this into $a - b - 1 = mn$, we get $a - a - jn - 2 = mn$. So $-2 = jn + mn$, so $2 = (-j - m)n$. Therefore, $n|2$ by definition of divides, since j and m are integers. Using the above divisibility fact, $|n| \leq |2|$. But we know that $n \geq 2$. So $n = 2$, which is what we needed to prove.

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let $A = \{(x, y) \in \mathbb{R}^2 \mid x + y = 10\}$. Consider the relation T on A defined by

$(a, b)T(p, q)$ if and only if $aq \geq bp$

Prove that T is antisymmetric.

Solution: Let (a, b) and (p, q) be points in A. Suppose that $(a, b)T(p, q)$ and $(p, q)T(a, b)$.

By the definition of T, $(a, b)T(p, q)$ and $(p, q)T(a, b)$ imply that $aq \geq bp$ and $bp \geq aq$. So $aq = bp$.

Since (a, b) and (p, q) are in A, we know that $a + b = 10$ and $p + q = 10$. So $b = 10 - a$ and $q = 10 - p$. Substituting these equations into $aq = bp$, we get $a(10 - p) = (10 - a)p$. So $10a - ap = 10p - ap$. So $10a = 10p$. So $a = p$. But then $b = 10 - a = 10 - p = q$.

Since $a = p$ and $b = q$, $(a, b) = (p, q)$, which is what we needed to prove.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let T be the relation defined on $\mathbb{N}^2$ by

$(x, y)T(p, q)$ if and only if $x < p$ or $(x = p$ and $y \leq q)$

Prove that T is transitive.

Solution:

Let (x, y), (p, q) and (m, n) be pairs of natural numbers. Suppose that $(x, y)T(p, q)$ and $(p, q)T(m, n)$. By the definition of T, $(x, y)T(p, q)$ means that $x < p$ or $(x = p$ and $y \leq q)$. Similarly $(p, q)T(m, n)$ implies that $p < m$ or $(p = m$ and $q \leq n)$.

There are four cases:

Case 1: $x < p$ and $p < m$. Then $x < m$.

Case 2: $x < p$ and $p = m$. Then $x < m$.

Case 3: $x = p$ and $p < m$. Then $x < m$.

Case 4: $x = p$ and $p = m$. In this case, we must also have $y \leq q$ and $q \leq n$. So $x = m$ and $y \leq n$.

In all four cases, $(x, y)T(m, n)$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let's define a relation T between natural numbers follows:

aTb if and only if $a = b + 2k$, where k is a natural number

Working directly from this definition, prove that T is antisymmetric.

Solution: Let a and b be natural numbers and suppose that aTb and bTa.

By the definition of T, this means that $a = b + 2k$ and $b = a + 2j$, where k and j are natural numbers.

Substituting one equation into the other, we get $a = (a + 2j) + 2k = a + 2(j + k)$. So $2(j + k) = 0$. So $j + k = 0$.

Notice that j and k are both non-negative. So $j + k = 0$ implies that $j = k = 0$.

So $a = b$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Suppose that T is a relation on the integers which is antisymmetric. Let's define a relation R on pairs of integers such that $(p,q)R(a,b)$ if and only if $(a+b)T(p+q)$ and bTq. Prove that R is antisymmetric.

Solution: Let (a,b) and (p,q) be pairs of integers. Suppose that $(a,b)R(p,q)$ and $(p,q)R(a,b)$.

By the definition of R, this means that $(a,b)R(p,q)$ means that $(p+q)T(a+b)$ and qTb. Similarly,$(p,q)R(a,b)$ means that $(a+b)T(p+q)$ and bTq.

Because T is antisymmetric, qTb and bTq implies that $q=b$. Similarly, $(p+q)T(a+b)$ and $(a+b)T(p+q)$ implies that $p+q=a+b$.

Since $q=b$ and $p+q=a+b$, $p=a$. So $(p,q)=(a,b)$, which is what we needed to prove.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$, i.e. pairs of positive integers. Consider the relation T on A defined by

$(a, b)T(p, q)$ if and only if $ab \mid p$

Working directly from the definition of divides, prove that T is transitive.

Solution: Let (a, b), (p, q), and (m, n) be elements of A. Suppose that $(a, b)T(p, q)$ and $(p, q)T(m, n)$. By the definition of T, this means that $ab \mid p$ and $pq \mid m$.

By the definition of divides, we then have $abx = p$ and $pqy = m$, for some integers x and y. Substituting the first equation into the second, we get $(abx)qy = m$. That is $(ab)(xqy) = m$. Since x, y, and q are all integers, so is xqy. So this implies that $ab \mid m$. So $(a, b)T(m, n)$, which is what we needed to show.

Name:____

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let's define a relation R on $\mathbb{Z}^3$ as follows:

$(a, b, c)R(x, y, z)$ if and only if $c = x$, $a = y$, and $b = z$.

Working directly from this definition, prove that R is antisymmetric.

Solution: Let (a, b, c) and (x, y, z) be triples of integers. Suppose that $(a, b, c)R(x, y, z)$ and $(x, y, z)R(a, b, c)$.

By the definition of R, $(a, b, c)R(x, y, z)$ implies that $c = x$, $a = y$, and $b = z$.

Alos by the definition of R, $(x, y, z)R(a, b, c)$ implies $z = a$, $x = b$, and $y = c$.

Chaining these equalities together, we get

$a = y = c = x = b = z = a$

So all six integers must be equal. In particular, $a = x$, $b = y$, and $c = z$. So $(a, b, c) = (x, y, z)$.

CS 173, Fall 17 Examlet 4, Part A

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Define the relation $\sim$ on $\mathbb{Z}$ by

$x \sim y$ if and only if $5 \mid (3x + 7y)$

Working directly from the definition of divides, prove that $\sim$ is transitive.

Solution: Let x, y, and z be integers. Suppose that $x \sim y$ and $y \sim z$.

By the definition of $\sim$, $5 \mid (3x + 7y)$ and $5 \mid (3y + 7z)$. So $3x + 7y = 5m$ and $3y + 7z = 5n$, for some integers m and n.

Adding these two equations together, we get $3x+7y+3y+7z = 5m+5n$. So $3x+10y+7z = 5(m+n)$. So $3x + 7z = 5(m + n - 2y)$.

$m + n - 2y$ is an integer, since m, n and y are integers. So this means that $5 \mid 3x + 7z$ and therefore $x \sim z$, which is what we needed to show.

Name:____

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let's define the relation $\succeq$ on $\mathbb{N}^2$ by

$(x, y) \succeq (a, b)$ if and only if $x - a \geq 2$ and $y \geq b$.

Prove that $\succeq$ is transitive.

Solution: Let (a, b), (x, y), and (c, d) be elements of X. Suppose that $(x, y) \succeq (a, b)$ and $(a, b) \succeq (c, d)$.

By the definition of $\succeq$, $(x, y) \succeq (a, b)$ implies that $x - a \geq 2$ and $y \geq b$. Similarly, $(a, b) \succeq (c, d)$ implies that $a - c \geq 2$ and $b \geq d$.

Since $y \geq b$ and $b \geq d$, $y \geq d$.

We know that $x - a \geq 2$ and $a - c \geq 2$, Adding these two equations, we get $(x - a) + (a - c) \geq 4$. So $x - c \geq 4$. So $x - c \geq 2$.

Therefore $x - c \geq 2$ and $y \geq d$. This implies that $(x, y) \succeq (c, d)$, which is what we needed to show.

Name:____

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

A closed interval of the real line can be represented as a pair (c, r), where c is the center of the interval and r is its radius. Let $X = \{(c, r) \mid c, r \in \mathbb{R}, r \geq 0\}$ be the set of closed intervals represented this way.

Now, let's define the interval containment $\preceq$ on X as follows

$(c, r) \preceq (d, q)$ if and only if $r \leq q$ and $|c - d| + r \leq q$.

Prove that $\preceq$ is antisymmetric.

Solution: Let (c, r) and (d, q) be elements of X. Suppose that $(c, r) \preceq (d, q)$ and $(d, q) \preceq (c, r)$.

By the definition of $\preceq$, $(c, r) \preceq (d, q)$ means that $r \leq q$ and $|c - d| + r \leq q$. Similarly, $(d, q) \preceq (c, r)$ means that $q \leq r$ and $|d - c| + q \leq r$.

Since $r \leq q$ and $q \leq r$, $q = r$. Substituting this into $|c - d| + r \leq q$, we get $|c - d| + r \leq r$. So $|c - d| \leq 0$. Since the absolute value of a real number cannot be negative, this means that $|c - d| = 0$, so $c = d$.

Since $q = r$ and $c = d$, $(c, r) = (d, q)$, which is what we needed to prove.

CS 173, Fall 17 Examlet 4, Part A

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let $A = \mathbb{N} \times \mathbb{N}$, i.e. pairs of natural numbers.

Define a relation $\gg$ on A as follows:

$(x, y) \gg (p, q)$ if and only if there exists an integer $n \geq 1$ such that $(x, y) = (np, nq)$.

Prove that $\gg$ is transitive.

Solution: Let (x, y), (p, q) and (a, b) be pairs of natural numbers and suppose that $(x, y) \gg (p, q)$ and $(p, q) \gg (a, b)$.

By the definition of $\gg$, $(x, y) = (np, nq)$ and $(p, q) = m(a, b)$, for some positive integers m and n. So $x = np$, $y = nq$, $p = ma$ and $q = mb$.

Combining these equations, we get $x = np = n(ma) = (nm)a$ and $y = nq = n(mb) = (nm)b$. Let $s = nm$. Since m and n are positive integers, so is s. But $(x, y) = (sa, sb)$. So $(x, y) \gg (a, b)$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Suppose that T is a relation on the integers which is transitive. Let's define a relation R on pairs of integers such that $(p, q)R(a, b)$ if and only if $(a + b)T(p + q)$. Prove that R is transitive.

Solution: Let (a, b), (p, q), and (m, n) be pairs of integers. Suppose that $(a, b)R(p, q)$ and $(p, q)R(m, n)$.

By the definition of R, this means that $(a, b)R(p, q)$ means that $(p+q)T(a+b)$ Similarly,$(p, q)R(m, n)$ means that $(m + n)T(p + q)$.

Because T is transitive, $(m + n)T(p + q)$ and $(p + q)T(a + b)$ implies that $(m + n)T(a + b)$.

By the definition of R, $(m + n)T(a + b)$ implies that $(a, b)R(m, n)$, which is what we needed to show.

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Let T be the relation defined on $\mathbb{Z}^2$ by

$(x, y)T(p, q)$ if and only if $x < p$ or ($x = p$ and $y \leq q$)

Prove that T is antisymmetric.

Solution:

Let (x, y) and (p, q) be pairs of integers. Suppose that $(x, y)T(p, q)$ and $(p, q)T(x, y)$. By the definition of T $(x, y)T(p, q)$ means that $x < p$ or ($x = p$ and $y \leq q$). Similarly, $(p, q)T(x, y)$ means that $p < x$ or ($p = x$ and $q \leq y$).

There are four cases:

Case 1: $x < p$ and $p < x$. This is impossible.

Case 2: $x < p$ and $p = x$ and $q \leq y$. Also impossible.

Case 3: $p < x$ and $x = p$ and $y \leq q$. Impossible as well.

Case 4: $x = p$ and $y \leq q$ and $p = x$ and $q \leq y$. Since $y \leq q$ and $q \leq y$, $x = y$. So we have $(x, y) = (p, q)$.

$(x, y) = (p, q)$ is true, which is what we needed to show.

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Let $A = \{(a,b) \in \mathbb{R}^2 \ : \ a > 0, b > 0, \text{ and } a + b = 90\}$. That is, an element of A is a pair of positive reals that sum to 90 (e.g. the non-right angles in a right triangle). Now, let's define a relation $\sim$ on A as follows:

$(a,b) \sim (p,q)$ if and only if $a = p$ or $a = q$.

Prove that $\sim$ is transitive.

Solution: Let (a,b), (p,q), and (m,n) be elements of A. Suppose that $(a,b) \sim (p,q)$ and $(p,q) \sim (m,n)$.

Since $(a,b) \sim (p,q)$, $a = p$ or $a = q$. Since $(p,q) \sim (m,n)$, $p = m$ or $p = n$.

First, notice that $q = 90 - p$, $n = 90 - m$, and $m = 90 - n$.

There are four cases.

Case 1: $a = p$ and $p = m$. Then $a = m$

Case 2: $a = p$ and $p = n$. Then $a = n$.

Case 3: $a = q$ and $p = m$. Then $a = 90 - p = 90 - m = n$. So $a = n$.

Case 4: $a = q$ and $p = n$. Then $a = 90 - p = 90 - n = m$. So $a = m$.

In all four cases, $a = m$ or $a = n$. So, by the definition of $\sim$, we have $(a,b) \sim (m,n)$, which is what we needed to prove.

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For any two real numbers with $a \leq b$, the closed interval $[a, b]$ is defined by
$[a, b] = \{x \in \mathbb{R} \ : \ a \leq x \leq b\}$. Let J be the set containing all closed intervals $[a, b]$. Let's define the relation F on J as follows:

$[s, t]F[p, q]$ if and only if $q \leq s$

Prove that F is antisymmetric.

Solution: Let $[s, t]$ and $[p, q]$ be two closed intervals. Suppose that $[s, t]F[p, q]$ and $[p, q]F[s, t]$.

By the definition of F, this means that $q \leq s$ and $t \leq p$. By the definition of closed interval, $s \leq t$ and $p \leq q$. So we have

$$p \leq q \leq s \leq t \leq p$$

So $p = q = s = t$ and therefore $[s, t] = [p, q]$, which is what we needed to show.

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Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$, i.e. pairs of positive integers. Consider the relation T on A defined by

$(x, y)T(p, q)$ if and only if $x \leq p$ and $xy \leq pq$

Prove that T is antisymmetric.

Solution: Let (x, y) and (p, q) be elements of A. Suppose that $(x, y)T(p, q)$ and $(p, q)T(x, y)$.

By the definition of T, $(x, y)T(p, q)$ implies that $x \leq p$ and $xy \leq pq$.

Similarly $(p, q)T(x, y)$ implies that that $p \leq x$ and $pq \leq xy$.

Since $x \leq p$ and $p \leq x$, $x = p$. Since $xy \leq pq$ and $pq \leq xy$, $xy = pq$.

Notice that x and o are positive, by the definition of A. So $x = p$ and $xy = pq$ implies that $y = q$.

We now know that $x = p$ and $y = q$. So therefore $(x, y) = (p, q)$, which is what we needed to show.

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A closed interval of the real line can be represented as a pair (c, r), where c is the center of the interval and r is its radius. Let $X = \{(c, r) \mid c, r \in \mathbb{R}, r \geq 0\}$ be the set of closed intervals represented this way.

Now, let's define the interval "starts" relation $\preceq$ on X as follows

$(c, r) \preceq (d, q)$ if and only if $c + q = d + r$ and $c + r \leq d + q$

Prove that $\preceq$ is transitive.

Solution: Let (c, r), (d, q), and (f, s) be elements of X. Suppose that $(c, r) \preceq (d, q)$ and $(d, q) \preceq (f, s)$.

By the definition of $\preceq$, $(c, r) \preceq (d, q)$ means that $c+q = d+r$ and $c+r \leq d+q$ Similarly, $(d, q) \preceq (f, s)$ means that $d + s = f + q$ and $d + q \leq f + s$.

Since $c + r \leq d + q$ and $d + q \leq f + s$, $c + r \leq f + s$.

We also know that $c+q = d+r$ and $d+s = f+q$. We can rewrite the second equation as $d = f+q-s$. Substituting this into the first equation, we get $c+q = (f+q-s)+r$. So $c = f-s+r$. So $c+s = f+r$.

Since $c + s = f + r$ and $c + r \leq f + s$, $(c, r) \preceq (f, s)$, which is what we needed to show.