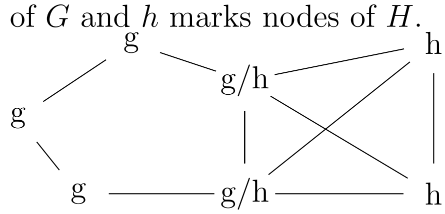
# CS 173, Spring 19 Examlet 7, white 

**Name:**________________________________________

**NetID:**________________________ **Lecture: A B**

**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

1. (11 points) If $G$ is a graph, recall that $\chi(G)$ is its chromatic number. Suppose that $G$ is a graph with at least one edge and $H$ is another graph with at least one edge, not connected to $G$. Now, pick a specific edge $e$ from $G$ and an edge $f$ from $H$ and merge the two edges, creating a combined graph $T$. For example, suppose that $G$ is $C_5$ and $H$ is $K_4$. Then $T$ might look as follows, where $g$ marks nodes of $G$ and $h$ marks nodes of $H$.




Describe how $\chi(T)$ is related to $\chi(G)$ and $\chi(H)$, justifying your answer. Your answer should handle any choice for $G$ and $H$.

**Solution:** $\chi(T) = \max(\chi(G), \chi(H))$

Lower bound: $G$ is a subgraph of $T$, so $\chi(T) \geq \chi(G)$. Similarly, $\chi(T) \geq \chi(H)$. So $\chi(T) \geq \max(\chi(G), \chi(H))$.

Upper bound: Suppose that $k = \max(\chi(G), \chi(H))$. We can color $G$ with $k$ colors, because $k \geq \chi(G)$. The merged edge in $H$ is already colored. Because $k \geq \chi(H)$, we can extend this to a coloring of $H$ with $k$ colors. So $\chi(T) \leq k = \max(\chi(G), \chi(H))$.

2. (4 points) Check the (single) box that best characterizes each item.

$\sum_{i=1}^{p-1} i$ $\frac{p(p-1)}{2}$ ☑ $\frac{(p-1)^2}{2}$ ☐ $\frac{p(p+1)}{2}$ ☐ $\frac{(p-1)(p+1)}{2}$ ☐

Leal team's bridge held 100 pounds without collapsing. 100 pounds is _____ on how much the bridge can hold.

an upper bound on ☐ exactly ☐
a lower bound on ☑ not a bound on ☐

**Name:**________________________________________

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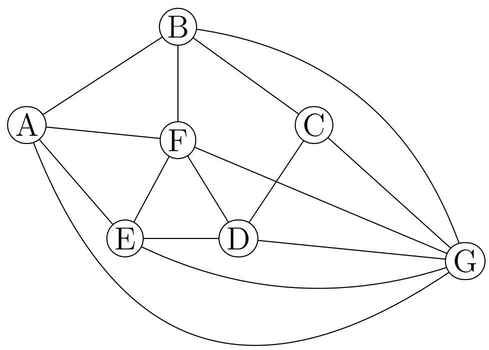
**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number is 4. We can color the graph with four colors as follows: A and D first color, B and E second color, F and C third color, G fourth color. The graph can't be colored with only 3 colors, because The graph contains an odd wheel (rim A,B, C, D, E and hub G).

2. (6 points) Check the (single) box that best characterizes each item.

| | | | | |
|---|---|---|---|---|
| Chromatic number of a graph containing a $C_7$. | $\geq 2$ ☐ | $\geq 3$ ☑ $\surd$ | $\leq 3$ ☐ | can't tell ☐ |
| $\sum_{i=1}^{p-1} i$ | $\frac{(p-1)^2}{2}$ ☐ | $\frac{(p-1)(p+1)}{2}$ ☐ | $\frac{p(p+1)}{2}$ ☐ | $\frac{p(p-1)}{2}$ ☑ $\surd$ |

| $\tau \leq 1.3$ | | |
|---|---|---|
| | an upper bound on $\tau$ ☑ $\surd$ | exactly $\tau$ ☐ |
| | a lower bound on $\tau$ ☐ | not a bound on $\tau$ ☐ |

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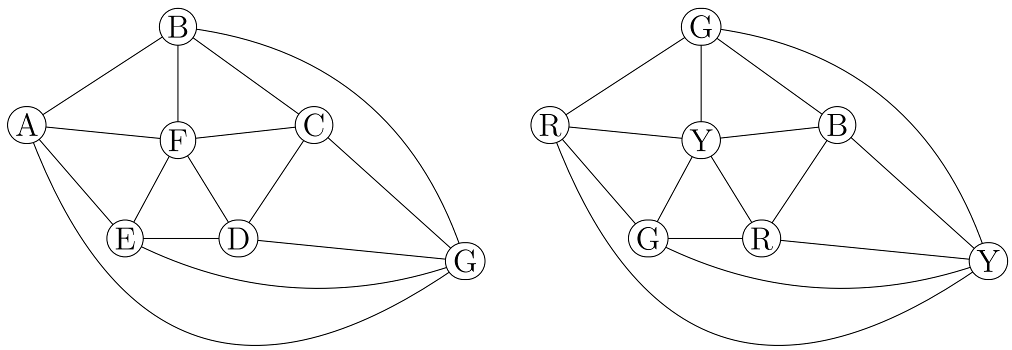
**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number is four. The picture above shows how to color it with four colors (upper bound).

For the lower bound, the graph contains a $W_5$ whose hub is F and whose rim contains nodes A, B, C, D, E. Coloring a $W_5$ requires four colors.

2. (6 points) Check the (single) box that best characterizes each item.

| | | | | |
|---|---|---|---|---|
| Chromatic number of a graph containing a $W_n$. | $\leq 3$ ☐ | $\geq 3$ ☑ | $\geq n$ ☐ | can't tell ☐ |
| $\sum_{i=1}^{p-1} \frac{i}{p}$ | $\frac{p(p-1)}{2}$ ☐ | $\frac{p(p+1)}{2}$ ☐ | $\frac{(p+1)}{2}$ ☐ | $\frac{(p-1)}{2}$ ☑ |

| | | |
|---|---|---|
| Putting 10 people in the canoe caused it to sink. 10 is _____ how many people the canoe can carry. | an upper bound on ☑ | exactly ☐ |
| | a lower bound on ☐ | not a bound on ☐ |

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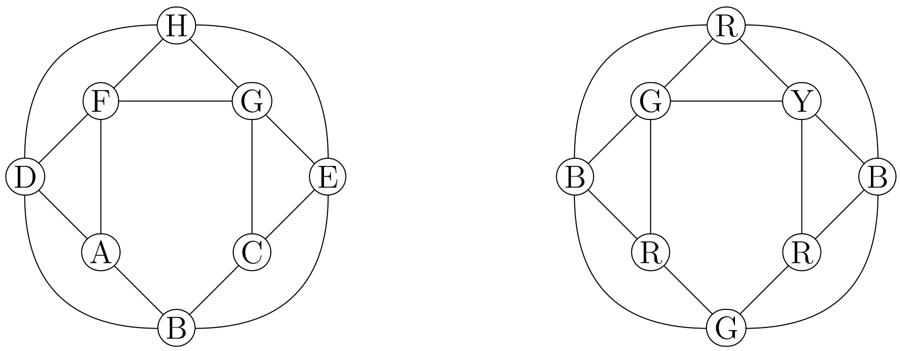
**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** This graph has chromatic number four. The picture above shows that four colors are enough. If you delete node H (and all its edges), you get the special graph presented towards the end of section 10.3 in the textbook, which we know to require four colors.

Alternatively, you can directly argue the lower bound as follows. Suppose we try to color this with three colors. Color the triangle A, B, D with R, G, B respectively. Then F must have color G. Nodes E and C must have colors R and B in some order. But then node G has neighbors with all three colors, so we're stuck. Therefore three colors is not enough.

2. (6 points) Check the (single) box that best characterizes each item.

Chromatic number of an connected acyclic graph with 5 nodes. $\leq 2$ ☐ $= 2$ ☑ $\leq 5$ ☐ can't tell ☐

$\sum_{i=0}^{k-1}(k \cdot i + 2)$

$\frac{k^2(k+1)}{2} + 2k$ ☐ $\frac{k(k+1)}{2} + 2(k-1)$ ☐

$\frac{k^2(k-1)}{2} + 2k$ ☑ $\frac{k(k-1)}{2} + 2(k-1)$ ☐

$\pi \geq 1.3$

an upper bound on $\pi$ ☐ exactly $\pi$ ☐

a lower bound on $\pi$ ☑ not a bound on $\pi$ ☐

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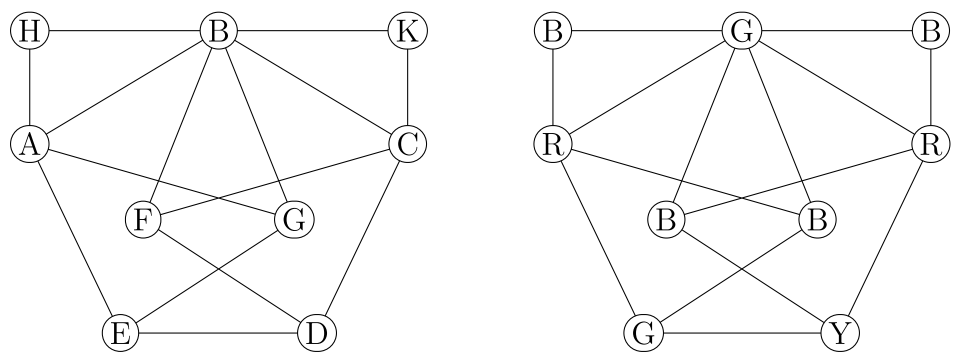
**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number of this graph is four.

The above picture shows that it can be colored with four colors (upper bound).

To show the lower bound, we could notice that if we delete the nodes H and K, we get a graph that we've shown to have chromatic number 4 (see section 10.3 of the textbook). If you want to argue this directly, color the triangle ABG with three colors as shown above. Then E must have the same color (G) as B. F and C must have the other two colors (in either order). This means that D has neighbors of all three colors. So three colors is not enough and therefore four is a lower bound.

2. (6 points) Check the (single) box that best characterizes each item.

| | | | | |
|---|---|---|---|---|
| Chromatic number of a connected graph with 10 nodes. | $\leq 2$ ☐ | $= 2$ ☐ | $\geq 2$ ☑ | can't tell ☐ |
| $\sum_{k=-2}^{n} k^2$ | $\sum_{p=0}^{n+2} (p+2)^2$ ☐ | $\sum_{p=0}^{n-2} (p-2)^2$ ☐ | $\sum_{p=0}^{n+2} (p-2)^2$ ☑ | $\sum_{p=0}^{n+2} p^2$ ☐ |

We have 30 tablespoons of filling. Each bun requires exactly one tablespoon of filling. 30 is _____ on how many buns we can make.

| | |
|---|---|
| an upper bound on ☑ | exactly ☐ |
| a lower bound on ☐ | not a bound on ☐ |

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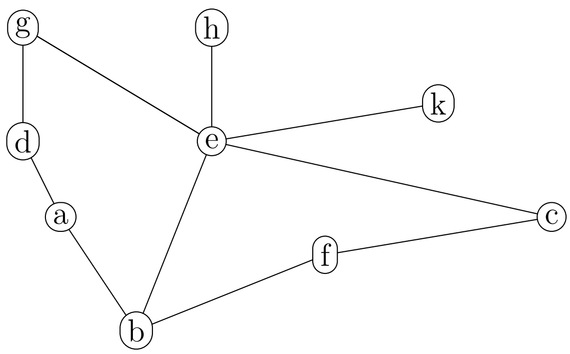
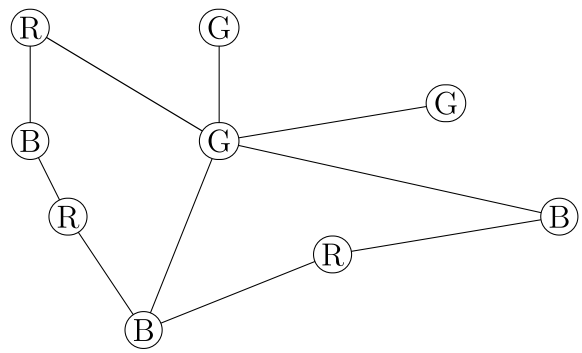
**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number is three. The picture above shows how to color it with three colors (upper bound). For the lower bound, the graph contains a $C_5$ made up of nodes a, b, e, g, and d.

2. (6 points) Check the (single) box that best characterizes each item.

Chromatic number of a graph containing a $W_7$. $\geq 3$ [ ] $\geq 4$ [$\surd$] $\geq 7$ [ ] can't tell [ ]

$\sum_{k=1}^{n} k!$ $\sum_{p=0}^{n+1}(p+1)!$ [ ] $\sum_{k=0}^{n+1}(k-1)!$ [ ] $\sum_{k=0}^{n-1}(k+1)!$ [$\surd$] $\sum_{p=0}^{n+1} k!$ [ ]

10 people rowed across Lake Tahoe in my canoe. 10 is _____ how many people the canoe can carry.

an upper bound on [ ] exactly [ ]
a lower bound on [$\surd$] not a bound on [ ]

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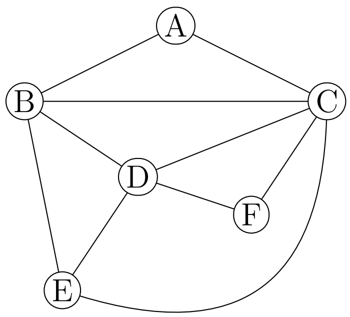
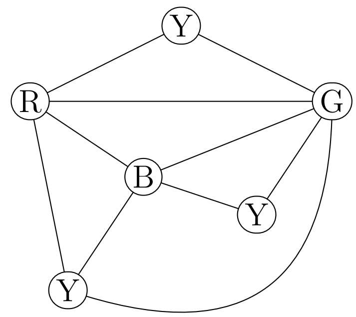
**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number is four. The picture above shows how to color it with four colors (upper bound). For the lower bound, the graph contains a $K_4$ made up of nodes B, C, D, E.

2. (6 points) Check the (single) box that best characterizes each item.

Chromatic number of a graph containing a $K_n$.

| | | | |
|---|---|---|---|
| $\leq n$ ☐ | $= n$ ☐ | $\geq n$ ☑ √ | can't tell ☐ |

$\sum_{k=0}^{n} k!$

| | | | |
|---|---|---|---|
| $\sum_{p=1}^{n+1} (p+1)!$ ☐ | $\sum_{k=1}^{n+1} (k-1)!$ ☑ √ | $\sum_{k=1}^{n-1} (k+1)!$ ☐ | $\sum_{p=1}^{n+1} k!$ ☐ |

I heated 2 liters of milk in my big pot. 2 liters is _____ how much the pot holds.

| | |
|---|---|
| an upper bound on ☐ | exactly ☐ |
| a lower bound on ☑ √ | not a bound on ☐ |

**Name:**__________________________________________

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**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (11 points) Let's define two sets as follows:

$$A = \{(x, y) \in \mathbb{R}^2 : y = x^2 - 4x + 3\}$$
$$B = \{(t + 2, t^2 - 1) : t \in \mathbb{R}\}$$

Prove that $A = B$ by proving two subset inclusions.

**Solution:**

$\mathbf{B} \subseteq \mathbf{A}$: Let $(x, y) \in B$. Then $(x, y) = (t + 2, t^2 - 1)$ for some real number $t$. So $x = t + 2$ and $y = t^2 - 1$.

Then $x^2 - 4x + 3 = (t + 2)^2 - 4(t + 2) + 3 = (t^2 + 4t + 4) - (4t + 8) + 3 = t^2 - 1 = y$. So $y = x^2 - 4x + 3$ and therefore $(x, y) \in A$.

$\mathbf{A} \subseteq \mathbf{B}$: Let $(x, y) \in A$. Then $y = x^2 - 4x + 3$.

Let $t = x - 2$. Then $x = t + 2$. Also $t^2 - 1 = (x - 2)^2 - 1 = x^2 - 4x + 4 - 1 = x^2 - 4x + 3 = y$. So $(x, y) = (t + 2, t^2 - 1)$ for this choice of $t$. Therefore $(x, y) \in B$.

Since $A \subseteq B$ and $B \subseteq A$, $A = B$, which is what we needed to show.

2. (4 points) Check the (single) box that best characterizes each item.

Chromatic number of $K_{m,n}$. (Assume $m \geq 1$, $n \geq 1$.) 2 ☑ 3 ☐ 4 ☐ can't tell ☐

$\pi \leq 10$ an upper bound on $\pi$ ☑ exactly $\pi$ ☐ a lower bound on $\pi$ ☐ not a bound on $\pi$ ☐

# CS 173, Fall 18 Examlet 7, Part B

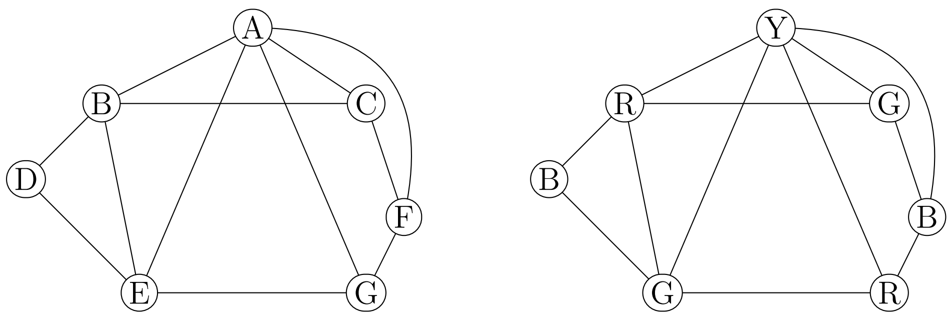


**Name:**________________________________________

**NetID:**________________________ **Lecture:** **B**

**Discussion:** **Friday** **11** **12** **1** **2** **3** **4**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number is four. The picture above shows how to color it with four colors (upper bound). For the lower bound, the graph contains a $W_5$: the hub is node A and the rim contains nodes B, C, F, G, and E.

2. (6 points) Check the (single) box that best characterizes each item.

Chromatic number of a bipartite graph with at least one edge — 1 ☐ 2 ☑ 3 ☐ can't tell ☐

Suppose I want to estimate $\frac{103}{20}$. 3 is _____ — an upper bound ☐ a lower bound ☑ an exact answer ☐ not a bound on ☐

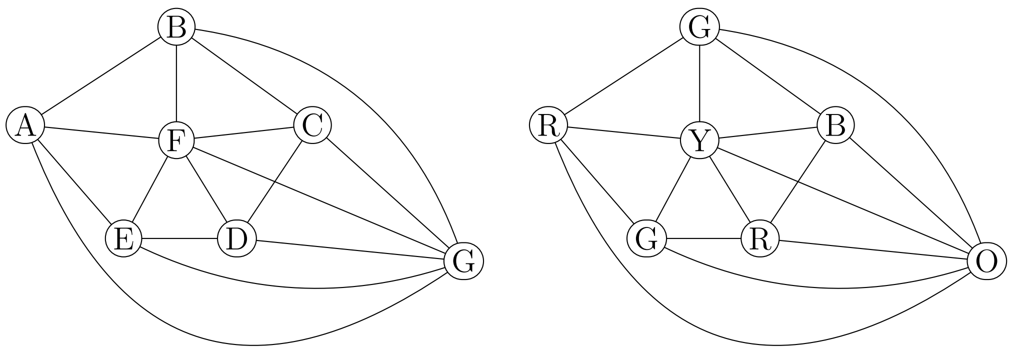
$\sum_{k=3}^{n} k^7$ — $\sum_{p=1}^{n-2} p^9$ ☐ $\sum_{p=1}^{n-2} k^7$ ☐ $\sum_{p=1}^{n-2} k^9$ ☐ $\sum_{p=1}^{n-2} (p+2)^7$ ☑

**Name:**________________________________________

**NetID:**________________________ **Lecture: B**

**Discussion: Friday 11 12 1 2 3 4**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number is five. The picture above shows how to color it with five colors (upper bound).

For the lower bound, the graph contains a $W_5$ whose hub is F and whose rim contains nodes A, B, C, D, E. Coloring a $W_5$ requires four colors. Then the node G is connected to all six nodes in the $W_5$, so it needs a different, fifth color.

2. (6 points) Check the (single) box that best characterizes each item.

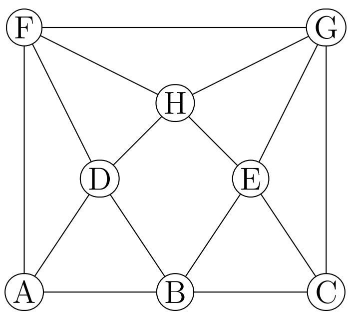
| | | | | |
|---|---|---|---|---|
| Chromatic number of $W_n$. | 2 ☐ | 3 ☐ | $\leq 3$ ☐ | $\leq 4$ ☑ $\surd$ |
| All elements of $M$ are also elements of $X$. | $M = X$ ☐ | $M \subseteq X$ ☑ $\surd$ | $X \subseteq M$ ☐ | |
| $\displaystyle\sum_{k=0}^{n} \frac{1}{2^k}$ | $1 - (\frac{1}{2})^{n-1}$ ☐ | $2 - (\frac{1}{2})^{n}$ ☑ $\surd$ | $1 - (\frac{1}{2})^{n}$ ☐ | $2 - (\frac{1}{2})^{n-1}$ ☐ |

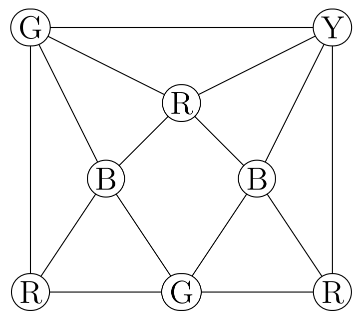
**Name:**________________________________________

**NetID:**________________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** This graph has chromatic number four. The picture above shows that four colors are enough. If you delete node H (and all its edges), you get the special graph presented towards the end of section 10.3 in the textbook, which we know to require four colors.

Alternatively, you can directly argue the lower bound as follows. Suppose we try to color this with three colors. Color the triangle A, B, D with R, G, B respectively. Then F must have color G. Nodes E and C must have colors R and B in some order. But then node G has neighbors with all three colors, so we're stuck. Therefore three colors is not enough.

2. (6 points) Check the (single) box that best characterizes each item.

Chromatic number of a graph with no cycles and at least one edge — 1 [ ] 2 [√] 3 [ ] can't tell [ ]

15 guests are invited to brunch. Each guest will eat at least two buns. 20 is _____ on how many buns we will need. — an upper bound on [ ] a lower bound on [√] exactly [ ] not a bound on [ ]

$\sum_{k=0}^{n-1} \frac{1}{2^k}$ — $1-(\frac{1}{2})^{n-1}$ [ ] $2-(\frac{1}{2})^{n}$ [ ] $1-(\frac{1}{2})^{n}$ [ ] $2-(\frac{1}{2})^{n-1}$ [√]

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**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (11 points) Let's define two sets as follows:

$$A = \{x \in \mathbb{R} \ : \ |x+1| \leq 2\}$$
$$B = \{w \in \mathbb{R} \ : \ w^2 + 2w - 3 \leq 0\}$$

Prove that $A = B$ by proving two subset inclusions.

**Solution:** $A \subseteq B$: Let $x$ be a real number and suppose $x \in A$. Then $|x+1| \leq 2$. Therefore, $-2 \leq x+1 \leq 2$ so $-3 \leq x \leq 1$. Therefore $x+3 \geq 0$ and $x-1 \leq 0$. So $x^2+2x-3 = (x+3)(x-1) \leq 0$. So $x \in B$.

$B \subseteq A$: Let $x$ be a real number and suppose $x \in B$. Then $x^2 + 2x - 3 \leq 0$. Factoring this polynomial, we get $(x+3)(x-1) \leq 0$. So $(x+3)$ and $(x-1)$ must have opposite signs. Since $x+3 > x-1$, it must be the case that $x+3 \geq 0$ and $x-1 \leq 0$. Therefore, $-3 \leq x \leq 1$. So $-2 \leq x+1 \leq 2$. So $|x+1| \leq 2$, and therefore $x \in A$.

Since $A \subseteq B$ and $B \subseteq A$, $A = B$.

2. (4 points) Check the (single) box that best characterizes each item.

| | | | | |
|---|---|---|---|---|
| $\sum_{i=1}^{p} i$ | $\frac{p(p-1)}{2}$ ☐ | $\frac{(p-1)^2}{2}$ ☐ | $\frac{p(p+1)}{2}$ ☑ $\surd$ | $\frac{(p-1)(p+1)}{2}$ ☐ |
| Chromatic number of $C_n$. | 2 ☐ | 3 ☐ | $\leq 3$ ☑ $\surd$ | $\leq 4$ ☐ |

# CS 173, Spring 18 Examlet 7, Part B

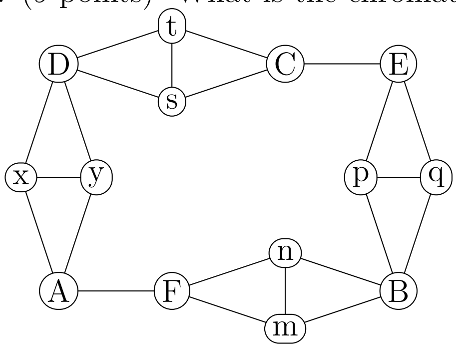


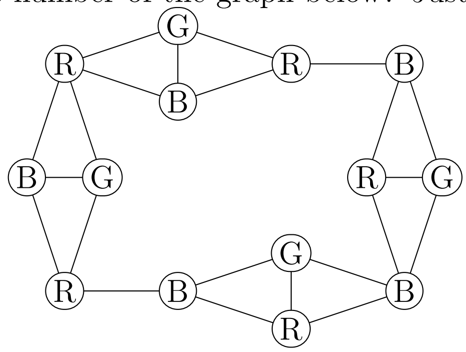
**Name:**________________________________________

**NetID:**____________________ **Lecture: A B**

**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number is three. The picture above shows that it can be colored with three colors (upper bound). Since it contains triangles, we also have a lower bound of three.

2. (6 points) Check the (single) box that best characterizes each item.

$\sum_{k=1}^{n-1} \frac{1}{2^k}$ $1-(\frac{1}{2})^n$ $\square$ $2-(\frac{1}{2})^n$ $\square$ $1-(\frac{1}{2})^{n-1}$ $\boxed{\surd}$ $2-(\frac{1}{2})^{n-1}$ $\square$

10 guests are invited to brunch. Each guest will eat at least two buns. 30 is ____ on how many buns we will need.

an upper bound on $\square$ exactly $\square$
a lower bound on $\square$ not a bound on $\boxed{\surd}$

Chromatic number of a graph with maximum vertex degree $D$

$= D$ $\square$ $= D+1$ $\square$
$\leq D+1$ $\boxed{\surd}$ $\geq D+1$ $\square$

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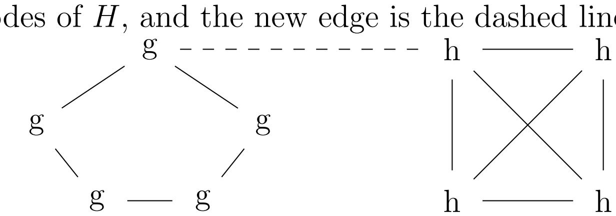
**NetID:**________________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (11 points) If $G$ is a graph, recall that $\chi(G)$ is its chromatic number. Suppose that $G$ is a graph and $H$ is another graph, not connected to $G$. Now, create a new graph $T$ which consists of a copy of $G$, a copy of $H$, and a new edge that connects some node of $G$ to some node of $H$. For example, suppose that $G$ is $C_5$ and $H$ is $K_4$. Then $T$ might look as follows, where $g$ marks nodes of $G$ and $h$ marks nodes of $H$, and the new edge is the dashed line.




Describe how $\chi(T)$ is related to $\chi(G)$ and $\chi(H)$, justifying your answer. Your answer should handle any choice for $G$ and $H$.

**Solution:** $\chi(T) = \max(\chi(G), \chi(H), 2)$

Lower bound: $G$ is a subgraph of $T$, so $\chi(T) \geq \chi(G)$. Similarly, $\chi(T) \geq \chi(H)$. We also know that $\chi(T) \geq 2$, because $T$ contains at least one edge: the one connecting $G$ to $H$. So $\chi(T) \geq \max(\chi(G), \chi(H), 2)$.

Upper bound: Suppose that $k = \max(\chi(G), \chi(H), 2)$. We can color $G$ with $k$ colors, because $k \geq \chi(G)$. Similarly, we can color $H$ with the same $k$ colors.

Now, suppose that $x$ and $y$ are the nodes connected by the new edge. If we have already assigned them different colors, then we are done.

If $x$ and $y$ have the same color, we must swap two of the color labels in $H$'s coloring. That is, if $x$ and $y$ have color $C$, we find some other color $D$ in our set. This second color $D$ always exists because we've forced $k$ to be at least 2. In the graph $H$, we then relabel each $C$ node with $D$ and each $D$ node with $C$. Now $x$ and $y$ have different colors.

Now we have a coloring of $T$ with $k$ colors. So So $\chi(T) \leq k = \max(\chi(G), \chi(H), 2)$.

2. (4 points) Check the (single) box that best characterizes each item.

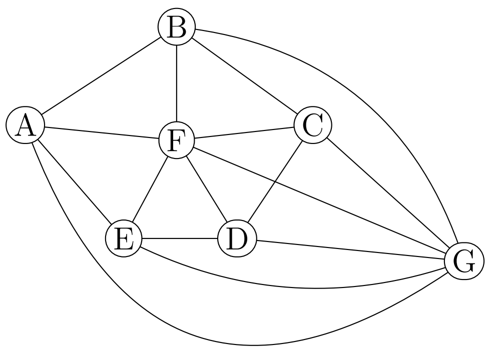
| | | | | |
|---|---|---|---|---|
| $\sum_{k=0}^{n-1} 2^k$ | $2^n - 2$ ☐ | $2^n - 1$ ☑ | $2^{n-1} - 1$ ☐ | $2^{n+1} - 1$ ☐ |
| All elements of $X$ are also elements of $M$. | $M = X$ ☐ | $M \subseteq X$ ☐ | $X \subseteq M$ ☑ | |

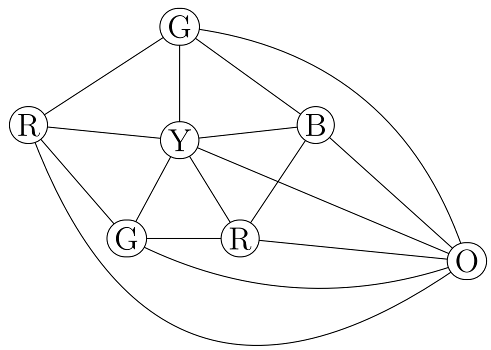
**Name:**________________________________________

**NetID:**______________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number is five. The picture above shows how to color it with five colors (upper bound).

For the lower bound, the graph contains a $W_5$ whose hub is F and whose rim contains nodes A, B, C, D, E. Coloring a $W_5$ requires four colors. Then the node G is connected to all six nodes in the $W_5$, so it needs a different, fifth color.

2. (6 points) Check the (single) box that best characterizes each item.

$\sum_{k=1}^{n} \frac{1}{2^k}$ $1-(\frac{1}{2})^{n-1}$ ☐ $2-(\frac{1}{2})^{n}$ ☐ $1-(\frac{1}{2})^{n}$ ☑ $2-(\frac{1}{2})^{n-1}$ ☐

Graph $H$ has 6 nodes. 7 is ____ the chromatic number of $H$.

an upper bound on ☑ exactly ☐
a lower bound on ☐ not a bound on ☐

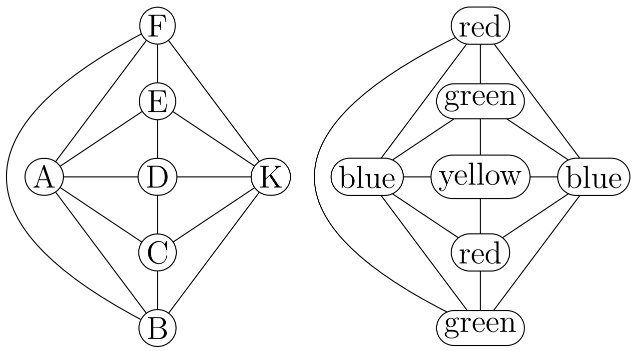
Chromatic number of $G$ $\mathcal{C}(G)$ ☐ $\phi(G)$ ☐ $\chi(G)$ ☑ $\| G \|$ ☐

**Name:**________________________________________

**NetID:**________________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number is four. The picture shows the graph colored with four colors (upper bound). For the lower bound, notice that it contains a $W_5$: the rim is BCDEF and the hub is A.

2. (6 points) Check the (single) box that best characterizes each item.

$\sum_{k=1}^{n} k$ | $\sum_{p=1}^{n}(n-p+1)$ ☑ | $\sum_{p=1}^{n}(n-p)$ ☐ | $\sum_{p=0}^{n}(n-p)$ ☐ | $\sum_{p=1}^{n+1}(n-p)$ ☐

10 students drove home in John's van. 10 is _____ how many students the van can carry.

an upper bound on ☐ | exactly ☐
a lower bound on ☑ | not a bound on ☐

Chromatic number of a graph (with at least one node) and no edges.

1 ☑ | 2 ☐ | 3 ☐ | can't tell ☐

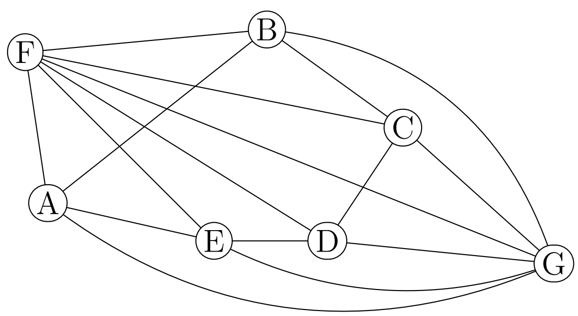
# CS 173, Fall 17 Examlet 7, Part B 

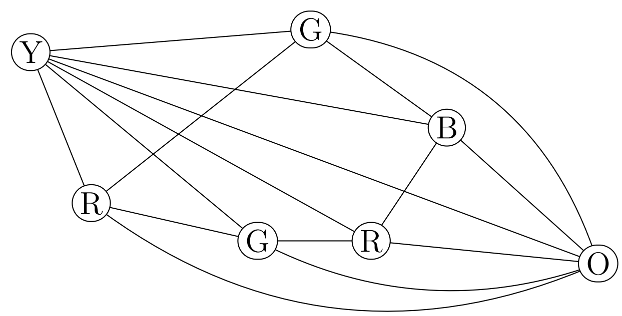
**Name:**________________________________________

**NetID:**______________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number is five. The picture above shows how to color it with five colors (upper bound).

For the lower bound, the graph contains a $W_5$ whose hub is F and whose rim contains nodes A, B, C, D, E. Coloring a $W_5$ requires four colors. Then the node G is connected to all six nodes in the $W_5$, so it needs a different, fifth color.

2. (6 points) Check the (single) box that best characterizes each item.

| | | | | |
|---|---|---|---|---|
| Chromatic number of a graph with maximum vertex degree $D$ | $= D$ | ☐ | $= D + 1$ | ☐ |
| | $\geq D + 1$ | ☐ | $\leq D + 1$ | √ |

| | | | | |
|---|---|---|---|---|
| We have 30 tablespoons of filling and each bun will require one tablespoon. 30 is _____ on how many buns we can make. | an upper bound on | √ | exactly | ☐ |
| | a lower bound on | ☐ | not a bound on | ☐ |

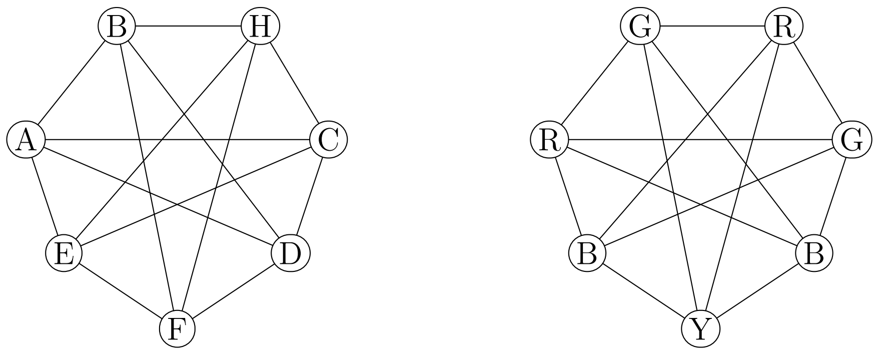
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\sum_{k=0}^{n+1} 2^k$ | $2^{n+1} + 1$ ☐ | | $2^{n+2} - 1$ √ | | $2^{n+2} - 2$ ☐ | | $2^{n+1} - 1$ ☐ | |

**Name:**________________________________________

**NetID:**______________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number is 4. The picture above shows that the graph can be colored with four colors (upper bound).

To show the lower bound, let's try to color the graph with three colors. First color the triangle ABD as shown in the above picture. Then C must be colored G and E must be colored B. The colorings on C and E imply that H must be colored R.

But none of the three colors is possible for F. So three colors isn't enough, i.e. we have a lower bound of 4.

2. (6 points) Check the (single) box that best characterizes each item.

$\sum_{i=1}^{p-1} \frac{i}{p}$ $\frac{p(p-1)}{2}$ ☐ $\frac{p(p+1)}{2}$ ☐ $\frac{(p+1)}{2}$ ☐ $\frac{(p-1)}{2}$ ☑ √

10 people rowed across Lake Tahoe in my canoe. 10 is _____ how many people the canoe can carry.

an upper bound on ☐ exactly ☐
a lower bound on √ not a bound on ☐

Chromatic number of a graph containing a $W_7$. $\geq 3$ ☐ $\geq 4$ √ $\geq 7$ ☐ can't tell ☐

# CS 173, Fall 17 Examlet 7, Part B

**Name:**________________________________________

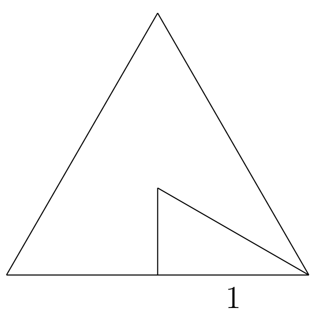
**NetID:**____________________ **Lecture:** **A** **B**

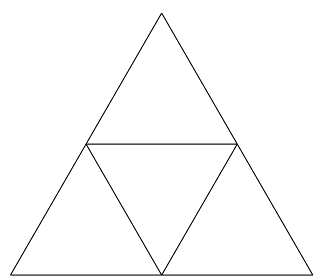
**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) Tomas wants to plant his tomatoes so that plants are more than 1 foot apart. His garden bed is an an equilateral triangle with each side 2 feet long. Prove that four is the maximum number of tomatoes he can plant.

   **Solution:** We need to show that four is possible (lower bound) and that five is not possible (upper bound).

   Lower bound: Put one tomato at each corner of the bed and one tomato in the exact center. Plants at two corners are 2 feet apart. You can see from the lefthand figure below that the plant in the center is more than a foot from each corner.



   Upper bound: Divide the bed into four small triangles with side length 1, as shown above right. Two points in the same small triangle are $\leq 1$ foot apart, so we can't put two tomatoes in the same small triangle. So we can't plant more than four tomatoes.

2. (4 points) Check the (single) box that best characterizes each item.

| | | | | |
|---|---|---|---|---|
| $\sum_{i=1}^{p-1} i$ | $\frac{(p-1)^2}{2}$ ☐ | $\frac{(p-1)(p+1)}{2}$ ☐ | $\frac{p(p+1)}{2}$ ☐ | $\frac{p(p-1)}{2}$ ☑ √ |

| | | |
|---|---|---|
| Putting 10 people in the canoe caused it to sink. 10 is _____ how many people the canoe can carry. | an upper bound on ☑ √ | exactly ☐ |
| | a lower bound on ☐ | not a bound on ☐ |

| | | | | |
|---|---|---|---|---|
| Chromatic number of a connected graph with 10 nodes. | $\leq 2$ ☐ | $= 2$ ☐ | $\geq 2$ ☑ √ | can't tell ☐ |

# CS 173, Fall 17 Examlet 7, Part B

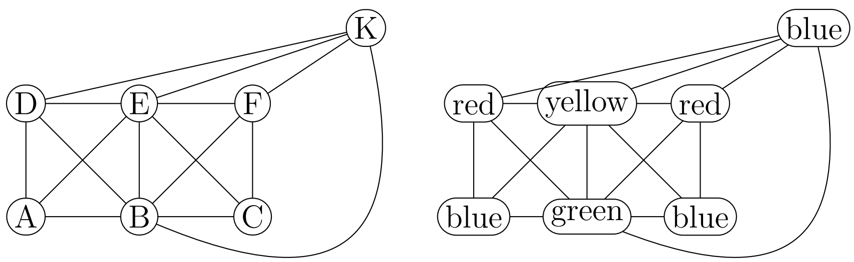


**Name:**________________________________________

**NetID:**____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number is 4. The righthand picture shows that four colors are sufficient (upper bound).

To show that four colors are required (lower bound), notice that A, B, C, and E form a $K_4$.

2. (6 points) Check the (single) box that best characterizes each item.

$\sum_{k=0}^{n-1} 2^k$ $2^n - 2$ ☐ $2^n - 1$ ☑ √ $2^{n-1} - 1$ ☐ $2^{n+1} - 1$ ☐

$C_5$ is a subgraph of graph $H$. 3 is _____ the chromatic number of $H$.

| | | | |
|---|---|---|---|
| an upper bound on | ☐ | exactly | ☐ |
| a lower bound on | √ | not a bound on | ☐ |

Exactly 40 books fit in my suitcase by volume, but I haven't checked their total weight. 40 is _____ how many books the suitcase can hold.

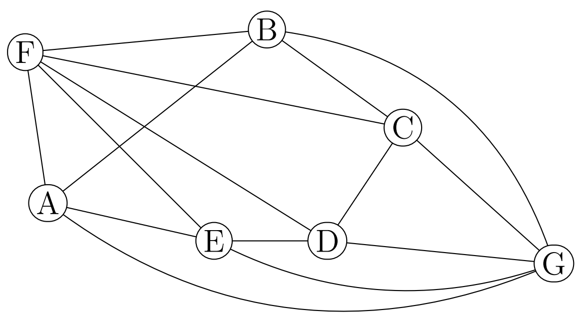
| | | | |
|---|---|---|---|
| an upper bound on | √ | exactly | ☐ |
| a lower bound on | ☐ | not a bound on | ☐ |

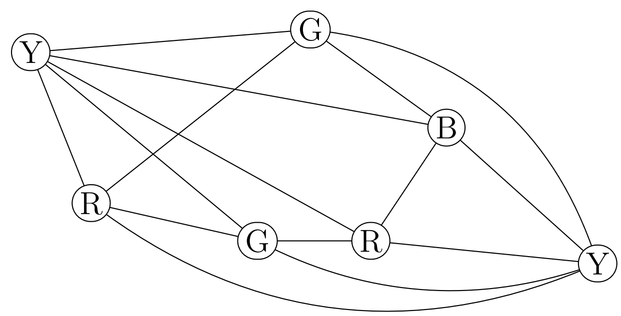
**Name:**________________________________________

**NetID:**____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of the graph below? Justify your answer.




**Solution:** The chromatic number is four. The picture above shows how to color it with four colors (upper bound).

For the lower bound, the graph contains a $W_5$ whose hub is F and whose rim contains nodes A, B, C, D, E. Coloring a $W_5$ requires four colors.

2. (6 points) Check the (single) box that best characterizes each item.

| | | | | |
|---|---|---|---|---|
| $\sum_{k=1}^{n} 2^k$ | $2^{n+1}-1$ ☐ | $2^{n+1}-2$ √ | $2^{n+1}-3$ ☐ | $2^n-1$ ☐ |

| | | |
|---|---|---|
| $C_5$ is a subgraph of graph $H$. 5 is _____ the chromatic number of $H$. | an upper bound on ☐ | exactly ☐ |
| | a lower bound on ☐ | not a bound on √ |

| | | | | |
|---|---|---|---|---|
| Chromatic number of a bipartite graph with at least one edge | 1 ☐ | 2 √ | 3 ☐ | can't tell ☐ |

**Name:**________________________________________

**NetID:**______________________ **Lecture: A B**

**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

1. (11 points) Let's define two sets as follows:

$$A = \{(4 - t^2, t + 1) : t \in \mathbb{R}\}$$

$$B = \{(x, y) \in \mathbb{R}^2 : x = 3 + 2y - y^2\}$$

Prove that $A = B$ by proving two subset inclusions.

**Solution:**

$\mathbf{A} \subseteq \mathbf{B}$: Let $(x, y) \in A$. Then $(x, y) = (4 - t^2, t + 1)$ for some real number $t$. So $x = 4 - t^2$ and $y = t + 1$. Then $t = y - 1$. So $x = 4 - t^2 = 4 - (y - 1)^2 = 4 - (y^2 - 2y + 1) = 3 + 2y - y^2$. So $(x, y) \in B$.

$\mathbf{B} \subseteq \mathbf{A}$: Let $(x, y) \in B$. Then $x = 3 + 2y - y^2$. Let $t = y - 1$. Then $y = t + 1$. Furthermore $x = 4 - (1 - 2y + y^2) = 4 - (y - 1)^2 = 4 - t^2$. So $(x, y) = (4 - t^2, t + 1)$, where $t$ is a real number. And therefore $(x, y) \in A$.

Since $A \subseteq B$ and $B \subseteq A$, $A = B$, which is what we needed to show.

2. (4 points) Check the (single) box that best characterizes each item.

| | | | | |
|---|---|---|---|---|
| $\sum_{k=3}^{n} k^7$ | $\sum_{p=1}^{n-2} p^9$ ☐ | $\sum_{p=1}^{n-2} k^7$ ☐ | $\sum_{p=1}^{n-2} k^9$ ☐ | $\sum_{p=1}^{n-2} (p+2)^7$ ☑ $\surd$ |
| Chromatic number of $K_{m,n}$. | 2 ☑ $\surd$ | 3 ☐ | 4 ☐ | can't tell ☐ |

**Name:**________________________________________

**NetID:**________________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (11 points) Let's define two sets as follows:

$$A = \{(x, y) \in \mathbb{R}^2 : y = 3x + 7\}$$

$$B = \{\lambda(-2, 1) + (1 - \lambda)(1, 10) : \lambda \in \mathbb{R}\}$$

Prove that $A = B$ by proving two subset inclusions.

**Solution:**

Lemma: $\lambda(-2, 1) + (1 - \lambda)(1, 10) = (-2\lambda + (1 - \lambda), \lambda + (10 - 10\lambda)) = (1 - 3\lambda, 10 - 9\lambda)$

**A** $\subseteq$ **B**: Let $(x, y)$ be an element of A. Then $y = 3x + 7$ by the definition of A.

Consider $\lambda = \frac{1-x}{3}$. Then, using our lemma above, we can calculate:

$$\begin{aligned} \lambda(-2, 1) + (1 - \lambda)(1, 10) &= (1 - 3\lambda, 10 - 9\lambda) = (1 - 3\frac{1-x}{3}, 10 - 9\frac{1-x}{3}) \\ &= (1 - (1 - x), 10 - 3(1 - x)) = (x, 3x + 7) = (x, y) \end{aligned}$$

So we've shown that (x,y) is an element of B. Since every element of A is also an element of B, $A \subseteq B$.

**B** $\subseteq$ **A**: Let (x,y) be an element of B. By the definition of B, we know that $(x, y) = \lambda(-2, 1) + (1 - \lambda)(1, 10)$ for some real number $\lambda$.

So, using our lemma above, $(x, y) = (1 - 3\lambda, 10 - 9\lambda)$. Then $3x + 7 = 3(1 - 3\lambda) + 7 = 3 - 9\lambda + 7 = 10 - 9\lambda = y$. So 3x+7 = y, which means that (x,y) is an element of A. Since every element of B is also an element of A, $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, $A = B$, which is what we needed to show.

2. (4 points) Check the (single) box that best characterizes each item.

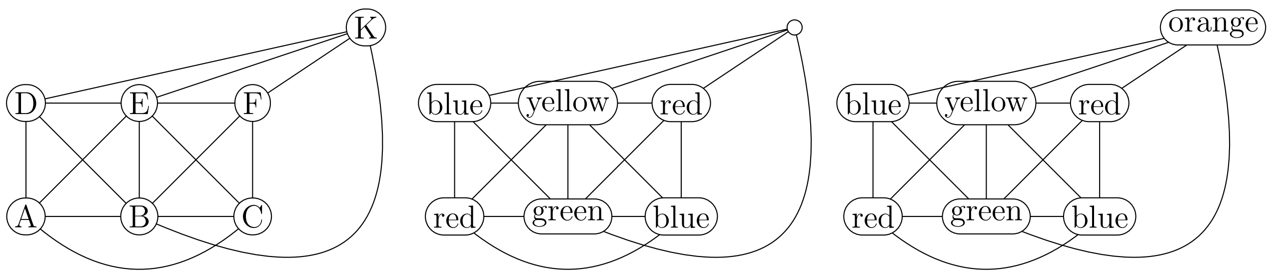
| | | | | |
|---|---|---|---|---|
| $\sum_{k=0}^{n} \frac{1}{2^k}$ | $1 - (\frac{1}{2})^{n-1}$ ☐ | $2 - (\frac{1}{2})^{n}$ ☑ | $1 - (\frac{1}{2})^{n}$ ☐ | $2 - (\frac{1}{2})^{n-1}$ ☐ |
| Chromatic number of $C_n$. | 2 ☐ | 3 ☐ | $\leq 3$ ☑ | $\leq 4$ ☐ |

**Name:**________________________________________

**NetID:**______________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of graph G (below)? Justify your answer.




**Solution:** The chromatic number is 5. The righthand picture shows that five colors are sufficient (upper bound).

To show that four colors isn't enough (lower bound), notice that A, B, C, and E form a $K_4$. So color these nodes with four colors as shown in the middle picture. Then F must be colored red and D must be colored blue. But then K cannot be any of the four colors.

2. (6 points) Check the (single) box that best characterizes each item.

$\sum_{k=1}^{n} k!$ $\sum_{p=0}^{n+1} (p+1)!$ ☐ $\sum_{k=0}^{n+1} (k-1)!$ ☐ $\sum_{k=0}^{n-1} (k+1)!$ [√] $\sum_{p=0}^{n+1} k!$ ☐

All elements of $M$ are also elements of $X$. $M = X$ ☐ $M \subseteq X$ [√] $X \subseteq M$ ☐

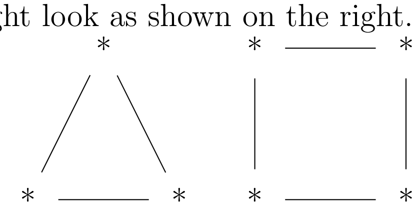
Chromatic number of $G$ $\mathcal{C}(G)$ ☐ $\phi(G)$ ☐ $\chi(G)$ [√] $\| G \|$ ☐

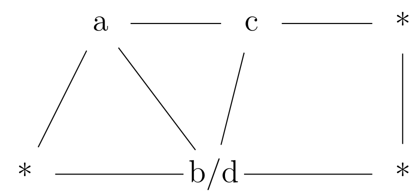
## Name:________________________________________

## NetID:____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (11 points) Recall that if $G$ is a graph, then $\chi(G)$ is its chromatic number. Suppose that $G$ is a graph and $H$ is another graph not connected to $G$. Suppose $G$ and $H$ each have at least two nodes and at least one edge. Dr. Evil picks two adjacent nodes $a$ and $b$ from $G$, and also two adjacent nodes $c$ and $d$ from $H$. He merges $G$ and $H$ into a single graph $T$ by merging $b$ and $d$ into a single node, and adding an edge connecting $a$ and $c$. So, if $G$ and $H$ are as shown on the left, then $T$ might look as shown on the right.






Describe how $\chi(T)$ is related to $\chi(G)$ and $\chi(H)$, justifying your answer.

**Solution:** $\chi(T) = \max(\chi(G), \chi(H), 3)$

The output graph contains a triangle, so it definitely requires at least three colors.

Without loss of generality, suppose that $k = \chi(G) \geq \chi(H)$. Then $\chi(T)$ must be at least $k$ because $G$ is a subgraph of $T$. Also notice that $k$ is at least 2 because the two input graphs each contain an edge.

First, suppose $k$ is at least 3. To color $T$ with $k$ colors, first color the part of $T$ corresponding to $G$. We have a coloring of $H$ that uses $\leq k$ colors, but the color choices might not be compatible with how we've started coloring $T$. If the two merged nodes $b$ and $d$ have different colors, swap the names of two colors to make them same. If $a$ and $c$ have the same color, swap the color of $c$ with some third color, remembering that $k$ is at least 3. Adjust the rest of the coloring for $H$ to use these same choices of color names.

Special case: if $k = 2$, then we carry out the same procedure. However, we won't have any third color available to fix the color of $c$, so we'll have to allocate an extra color.

2. (4 points) Check the (single) box that best characterizes each item.

$\sum_{k=-2}^{n} k^2$ $\sum_{p=0}^{n+2}(p+2)^2$ $\square$ $\sum_{p=0}^{n-2}(p-2)^2$ $\square$ $\sum_{p=0}^{n+2}(p-2)^2$ $\boxed{\checkmark}$ $\sum_{p=0}^{n+2}p^2$ $\square$

$W_7$ is a subgraph of graph $H$. 4 is _____ the chromatic number of $H$.

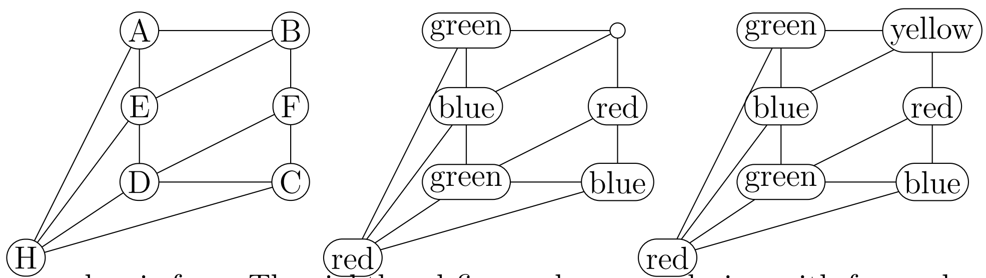
| | | | |
|---|---|---|---|
| an upper bound on | $\square$ | exactly | $\square$ |
| a lower bound on | $\boxed{\checkmark}$ | not a bound on | $\square$ |

**Name:**________________________________________

**NetID:**____________________ **Lecture: A B**

**Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6**

1. (9 points) What is the chromatic number of graph G (below)? Justify your answer.




**Solution:** The chromatic number is four. The righthand figure shows a coloring with four colors (an upper bound).

To show that three colors is not enough (lower bound), first pick three colors for the triangle CDH, as shown in the middle figure. With only three colors, F must then be colored red and E blue. Then A must be colored green. But then none of the three colors will work for node B.

2. (6 points) Check the (single) box that best characterizes each item.

| | | | | |
|---|---|---|---|---|
| $\displaystyle\sum_{i=0}^{k-1}(k \cdot i + 2)$ | $\frac{k^2(k+1)}{2} + 2k$ | ☐ | $\frac{k(k+1)}{2} + 2(k-1)$ | ☐ |
| | $\frac{k^2(k-1)}{2} + 2k$ | √ | $\frac{k(k-1)}{2} + 2(k-1)$ | ☐ |

| | | | | |
|---|---|---|---|---|
| When I poured 5 gallons of water into the bucket, some spilled over the top. 5 gallons is ____ how much the bucket holds. | an upper bound on | √ | exactly | ☐ |
| | a lower bound on | ☐ | not a bound on | ☐ |

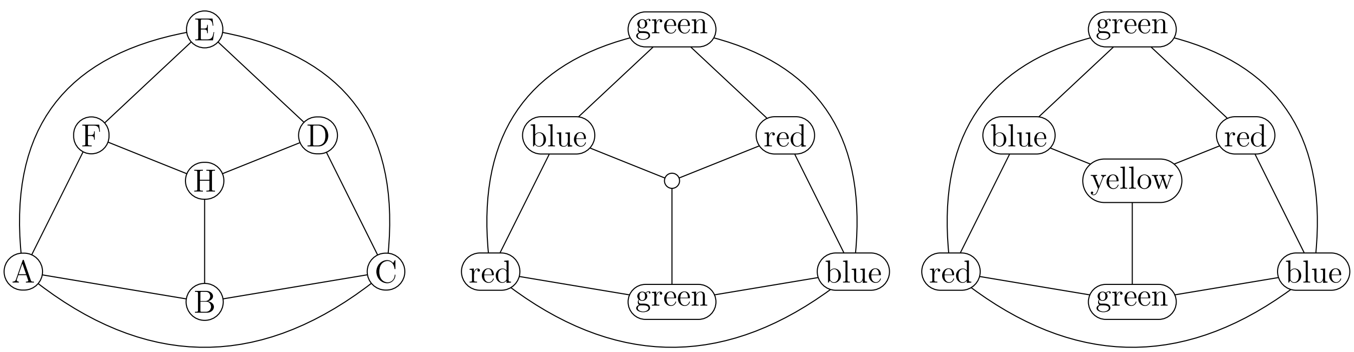
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Chromatic number of a bipartite graph with at least two vertices. | 1 | ☐ | 2 | ☐ | 3 | ☐ | can't tell | √ |

**Name:**______________________________________

**NetID:**____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (9 points) What is the chromatic number of graph G (below)? Justify your answer.




**Solution:** The chromatic number is four. The picture on the right shows the graph colored with four colors (upper bound).

To show that three colors isn't enough, suppose that we color the triangle ABC as shown in the middle figure. Then E must be green because it's connected to A and C. But then C must be red and F blue. Then none of the three colors will work for H.

2. (6 points) Check the (single) box that best characterizes each item.

$\sum_{i=1}^{p-1} i$ $\frac{p(p-1)}{2}$ [√] $\frac{(p-1)^2}{2}$ [ ] $\frac{p(p+1)}{2}$ [ ] $\frac{(p-1)(p+1)}{2}$ [ ]

I heated 2 liters of milk in my big pot. 2 liters is _____ how much the pot holds.

an upper bound on [ ] exactly [ ]
a lower bound on [√] not a bound on [ ]

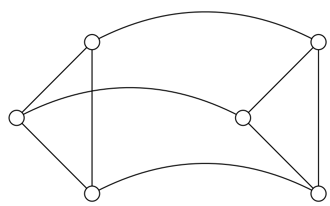
Chromatic number of a graph containing a $W_n$. $\geq 2$ [√] $\leq 3$ [ ] $\geq n$ [ ] can't tell [ ]

**Name:**________________________________________

**NetID:**____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (11 points) Recall that if $G$ is a graph, then $\chi(G)$ is its chromatic number. Let's define the "doubled" version of a graph $G$ as follows: make two copies of $G$ and add an edge joining each pair of corresponding nodes. For example, the doubled version of $C_3$ looks like:



Suppose that $T$ is the doubled version of a graph $G$. Describe how $\chi(T)$ is related to $\chi(G)$, justifying your answer. Your answer should handle any choice for $G$, not just $C_3$.

**Solution:**

$\chi(T) = \max(2, \chi(G))$.

First, let's suppose that $\chi(G) \geq 2$. If $\chi(G) = n$, then we can start coloring $T$ by coloring one copy of $G$ with $n$ colors. Let's call the colors $c_1, c_2, \ldots, c_n$. Now color the second copy of $G$ using the rule that if a node in the first copy has color $c_i$, then the corresponding node in the second copy has color $c_{i+1}$ if $i + 1 \leq n$ or $c_1$ if $i + 1 = n$. This shows that $\chi(T) = \chi(G)$.

This construction won't work if $\chi(G)$ is 1. In this case, there aren't any edges in $G$. So the only edges in $T$ connect pairs of corresponding nodes. This means that $T$ requires two colors.

2. (4 points) Check the (single) box that best characterizes each item.

Chromatic number of $W_n$. 2 ☐ 3 ☐ $\leq 3$ ☐ $\leq 4$ $\boxed{\surd}$

10 people can row the canoe but 11 people caused it to sink. 10 is _____ how many people the canoe can carry.

an upper bound on ☐ exactly $\boxed{\surd}$

a lower bound on ☐ not a bound on ☐