**Name:**________________________________________

**NetID:**________________________ **Lecture: A B**

**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

(18 points) A Mouse tree is a binary tree containing 2D points such that:

- Each leaf node contains $(3, 1)$, $(-2, -5)$, or $(2, 2)$.
- An internal node with one child labelled $(a, b)$ has label $(a + 1, b - 1)$.
- An internal node with two childen labelled $(x, y)$ and $(a, b)$ has label $(x + a, y + b)$.

Use (strong) induction to prove that the point in the root node of any Mouse tree is on or below the line $x = y$.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**______________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(18 points) A palindrome is a string that is the same if you reverse it. For example, `abbabba` and `abaaba` are palindromes. The empty (zero-length) string $\epsilon$ counts as a palindrome.

Here is a grammar $G$, with start symbol $S$ and terminal symbols $a$ and $b$.

$$S \rightarrow a\,S\,a \mid b\,S\,b \mid a \mid b \mid \epsilon$$

Use (strong) induction to prove that any palindrome made out of characters a and b can be generated by grammar $G$. That is, show how to build parse trees for these strings. Hint: remove the first and last character from the string.

The induction variable is named ______ and it is the ________________ of/in the string.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**________________________ **Lecture: A B**

**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

(18 points) Recall that a node in a full binary tree must have either 0 or 2 children. A Shark tree is a full binary tree in which each node is colored orange or blue, such that:

- If $v$ is a leaf node, then $v$ is colored orange.
- If $v$ has two children of the same color, then $v$ is colored blue.
- If $v$ has two children of different colors, then $v$ is colored orange.

Use (strong) induction to show that the root of a Shark tree is blue if and only if the tree has an even number of leaves. You may assume basic divisibility facts e.g. the sum of two odd numbers is even.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**________________________ **Lecture: A B**

**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

(18 points) A Horse tree is a binary tree whose nodes contain integers such that

- Every leaf node contains 5, 17, or 23.
- A node with one child contains the same number as its child.
- A node with two children contains the value $x(y+1)$, where $x$ and $y$ are the values in its children.

Use strong induction to prove that the value in the root of a Horse tree is always positive.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**________________________ **Lecture: A B**

**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

(18 points) Here is a grammar $G$, with start symbols $N$ and $P$, and terminal symbols $a$ and $b$.

$$\begin{aligned} N &\rightarrow P\ a \mid b\ b \\ P &\rightarrow P\ N \mid a \end{aligned}$$

Use (strong) induction to prove that any tree matching (aka generated by) grammar $G$ has an even number of leaves if and only if its root has label $N$.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**________________________ **Lecture: A B**

**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

(18 points) Recall that a node in a full binary tree is either a leaf or has exactly two children. A Possum tree is a full binary tree whose leaves are all orange and whose root is blue.

Use (strong) induction to prove that a Possum tree contains a blue node with (at least) one orange child.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**________________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(18 points) Here is a grammar $G$, with start symbol $S$ and terminal symbols $a$ and $p$.

$$S \rightarrow S\,S \mid p\,S\,p \mid p\,p \mid a\,a$$

Use (strong) induction to prove that any tree matching (aka generated by) grammar $G$ has an even number of nodes with label p. Use $P(T)$ as shorthand for the number of p's in a tree $T$.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(18 points) Recall that a node in a full binary tree is either a leaf or has exactly two children. A Snake tree is a full binary tree whose leaves are all blue and whose root is orange.

Use (strong) induction to prove that a Snake tree contains an orange node with two blue children.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**________________________ **Lecture: B**

**Discussion: Friday 11 12 1 2 3 4**

(18 points) Here is a grammar $G$, with start symbol $S$ and terminal symbols $a$ and $b$.

$$\begin{aligned} S &\rightarrow a\ S\ a \mid S\ a\ S \mid a\ N\ a \\ N &\rightarrow a \mid b\ b \end{aligned}$$

Use (strong) induction to prove that any tree of height $h$ matching (aka generated by) grammar $G$ has at least $h$ nodes with label $a$. Use $A(T)$ as shorthand for the number of a's in a tree $T$.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**____________________ **Lecture:** **B**

**Discussion:** **Friday** **11** **12** **1** **2** **3** **4**

(18 points) Recall that a node in a full binary tree must have either 0 or 2 children. A Sleepy tree is a full binary tree in which each node is colored orange or blue, such that:

- If $v$ is a leaf node, then $v$ may be colored orange or blue.
- If $v$ has two children of the same color, then $v$ is colored blue.
- If $v$ has two children of different colors, then $v$ is colored orange.

Use (strong) induction to show that the root of a Sleepy tree is blue if and only if the tree has an even number of orange leaves. You may assume basic divisibility facts e.g. the sum of two odd numbers is even.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

# CS 173, Spring 18 Examlet 9, Part A

**Name:**________________________________________

**NetID:**______________________ **Lecture: A B**

**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

(18 points) Recall that $F_n$ is the nth Fibonacci number, and these start with $F_0 = 0$, $F_1 = 1$.

Let $T_n$ be the number of bit strings of length $n$ that don't contain any consecutive zeros. E.g. when counting strings of length 6, we include 010110, but not 101001. Prove that $T_n = F_{n+2}$ for any natural number $n$. Hint: if $w$ is a string with no consecutive zeros, either $w = 1x$, where x is a shorter string, or $w = 01y$, where y is a shorter string.

The induction variable is named ______ and it is the ________________ of/in the string.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

# CS 173, Spring 18 Examlet 9, Part B

**Name:**____________________________________________

**NetID:**________________________ **Lecture: A B**

**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

1. (8 points) Here is a grammar with start symbol $S$ and terminal symbol $a$. Draw three parse trees for the string `aaaaaa` that match this grammar.

$$S \rightarrow SS \mid aSa \mid aa$$

2. (4 points) Check the (single) box that best characterizes each item.

| | | | | | |
|---|---|---|---|---|---|
| Total number of leaves in a 3-ary tree of height $h$ | $3^h$ ☐ | $\leq 3^h$ ☐ | $\frac{1}{2}(3^{h+1}-1)$ ☐ | $3^{h+1}-1$ ☐ | |
| The level of a leaf node in a tree of height $h$. | 0 ☐ | 1 ☐ | $h-1$ ☐ | $\leq h$ ☐ | $h$ ☐ |

**Name:**________________________________________

**NetID:**____________________ **Lecture: A B**

**Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6**

(18 points) Here is a grammar $G$, with start symbol $S$ and terminal symbols $a$ and $b$.

$$\begin{aligned} S &\rightarrow a\ S\ a \mid S\ a\ S \mid a\ N\ a \\ N &\rightarrow a \mid b\ b \end{aligned}$$

Use (strong) induction to prove that any tree of height $h$ matching (aka generated by) grammar $G$ has at least $h$ nodes with label $a$. Use $A(T)$ as shorthand for the number of a's in a tree $T$.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(18 points) Recall that a node in a full binary tree is either a leaf or has exactly two children. A Lemon tree is a full binary tree such the two child subtrees of each internal node have heights that differ by at most one. Prove that every Lemon tree of height $h$ contains at least $F_{h+1}$ nodes, where $F_k$ is the $k$th Fibonacci number. (Recall: $F_0 = 0$, $F_1 = F_2 = 1$)

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

# CS 173, Fall 17 Examlet 9, Part A

**Name:**________________________________________

**NetID:**____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(18 points) Suppose that $G$ is a connected graph. A Friendly coloring of $G$ labels each node orange or blue, such that

- If $G$ contains only one node, it is colored orange, and
- Otherwise, every node of $G$ is adjacent to at least one node of the opposite color.

Use (strong) induction to prove that any connected graph can be given a Friendly coloring. Hint: remove any node $x$ (no special properties required) and color the rest of the graph. What color pattern is required if $x$ is the only neighbor of another node?

The induction variable is named ______ and it is the ________________ of/in the graph.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

# CS 173, Fall 17 Examlet 9, Part A

**Name:**________________________________________

**NetID:**____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(18 points) Recall that a node in a full binary tree must have either 0 or 2 children. A Happy tree is a full binary tree in which each node is colored orange or blue, such that:

- If $v$ is a leaf node, then $v$ may be colored orange or blue.
- If $v$ has two children of the same color, then $v$ is colored blue.
- If $v$ has two children of different colors, then $v$ is colored orange.

Use (strong) induction to show that the root of a Happy tree is blue if and only if the tree has an even number of orange leaves. You may assume basic divisibility facts e.g. the sum of two odd numbers is even.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(18 points) Recall that a node in a full binary tree is either a leaf or has exactly two children. A Tidy tree is a full binary tree such that leaf nodes have label 0 and an internal node with label $x$ matches one of these patterns:

- The left subtree is a leaf and the right subtree has root label $y$, where $y \equiv x - 1 \pmod 3$, or
- The roots of both subtrees have label $y$, where $y \equiv x - 1 \pmod 3$.

Use (strong) induction to prove that the root label of every Tidy tree is congruent to $h \pmod 3$, where $h$ is the height of the tree. You may assume basic facts about congruence and modular arithmetic (e.g. congruence is transitive).

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:** ________________________________________

**NetID:** _____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(18 points) Recall that a node in a full binary tree is either a leaf or has exactly two children. A Trim tree is a full binary tree such the two child subtrees of each internal node have heights that differ by at most one. Prove that every Trim tree of height $h$ contains at least $2^{h/2}$ leaves. You may use the fact that $\sqrt{2} > 1.4$.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

## Name:________________________________________

## NetID:________________________ **Lecture: A B**

**Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6**

(18 points) Sarah needs to saw a $m$ by $n$ by $p$ inch block of wood into one-inch cubes. ($m$, $n$, and $p$ are integers.) The saw can slice a block of wood at any integer position parallel to one of its sides. However, a safety feature prevents her from slicing more than one piece of wood at a time. Use (strong) induction to prove that it takes $mnp - 1$ cuts to divide the block of wood into one-inch cubes, for any sequence of cuts.

The induction variable is named ______ and it is the ________________ of/in the block.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

## Name:________________________________________

## NetID:____________________ **Lecture: A B**

**Discussion: Thursday Friday 10 11 12 1 2 3 4 5 6**

(18 points) Recall that a node in a full binary tree is either a leaf or has exactly two children. A Monkey tree is a full binary tree such the two child subtrees of each internal node have heights that differ by at most one. Prove that every Monkey tree of height $h$ contains at least $F_{h+1}$ leaves, where $F_k$ is the $k$th Fibonacci number. (Recall: $F_0 = 0$, $F_1 = F_2 = 1$)

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**________________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(18 points) Here is a grammar $G$, with start symbols $N$ and $P$, and terminal symbols $a$ and $b$.

$$
\begin{aligned}
N &\rightarrow P\ P \mid a\ b \\
P &\rightarrow N\ P \mid b
\end{aligned}
$$

Use (strong) induction to prove that any tree matching (aka generated by) grammar $G$ has an even number of leaves if and only if its root has label $N$.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**____________________________________________

**NetID:**______________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(18 points) A Camel tree is a binary tree whose nodes contain integers such that

- Every leaf node contains 7, 9, or 12.
- A node with one child contains the same number as its child.
- A node with two children contains the value $xy - y$, where $x$ and $y$ are the values in its children.

Use (strong) induction to prove that the value in the root of a Camel tree is always $\geq 7$

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**________________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(18 points) UIUC is considering hosting massive hackathons in rooms like the first floor of the Armory. Facilities will need to divide this $100h$ square foot room into $h$ workspaces, each 100 square feet, using expanding partitions. Each end of each partition must be attached to a wall of the room or to another partition. A partition can expand to any length but cannot cross another partition. The partitions are low enough that doors are not required. Use (strong) induction to prove that they will need $h-1$ partitions, no matter how they arrange the partitions.

The induction variable is named ______ and it is the ________________ of/in the room.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**

**Name:**________________________________________

**NetID:**____________________ **Lecture:** **A** **B**

**Discussion:** **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(18 points) Octopus trees are binary trees whose nodes are labelled with strings, such that

- Each leaf node has label `left`, `right`, or `back`
- If a node has one child, it has label $\alpha$`x` where where $\alpha$ is the child's label. E.g. if the child has label `left` then the parent has `leftx`.
- If a node has two children, it contains $\alpha\beta$ where $\alpha$ and $\beta$ are the child labels. E.g. if the children have labels `right` and `back`, then the parent has label `rightback`.

Let S(n) be the length of the label on node n. Let L(n) be the number of leaves in the subtree rooted at n. Use (strong) induction to prove that $S(n) \geq 4L(n)$ if $n$ is the root node of any Octopus tree.

The induction variable is named ______ and it is the ________________ of/in the tree.

**Base Case(s):**

**Inductive Hypothesis [Be specific, don't just refer to "the claim"]:**

**Inductive Step:**