CS 173, Spring 19 Examlet 3, colored sheet

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

$A = \{(x, y) \in \mathbb{R}^2 \ : \ y = x^2 - 4\}$

$B = \{(p, q) \in \mathbb{Z}^2 \ : q < 0\}$

$C = \{(a, b) \in \mathbb{R}^2 \ : \ |a| \leq 1\}$

Prove that $A \cap B \subseteq C$.

Solution: Proof: Let $(x, y) \in A \cap B$. Then, $(x, y) \in A$ and $(x, y) \in B$. So, from the definition of A, we know that $y = x^2 - 4$. From the definition of B, we know that $y < 0$ and that x and y are both integers.

$y = x^2 - 4 = (x - 2)(x + 2)$. So since $y < 0$, $-2 < x < 2$. But x is an integer. So the only possible values in this range are -1, 0, and 1. Therefore $|x| \leq 1$. So $(x, y) \in C$, which is what we needed to prove.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < x < y - 1\}$

$B = \{(a, b, c) \in \mathbb{R}^3 \mid b^2 + 2 < c^2\}$

$C = \{(p, q, r) \in \mathbb{R}^3 \mid p^2 < r^2\}$

Prove that $A \cap B \subseteq C$.

Solution: Let $(x, y, z) \in A \cap B$. Then $(x, y, z) \in A$, so (x, y, z) is a triple of real numbers with $0 < x < y - 1$. Also $(x, y, z) \in B$, so $y^2 + 2 < z^2$.

We know that $0 < x < y - 1$. Since $y - 1 > 0$, $y > 0$, so $-2y < 0$. Squaring both sides of $x < y - 1$ and using the fact that both sides of the equation are positive, we get $x^2 < y^2 - 2y + 1$. So $x^2 < y^2 + 1 < y^2 + 2$. But we know that $y^2 + 2 < z^2$. So we have $x^2 < z^2$, and therefore $(x, y, z) \in C$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

$A = \{\alpha(2, -4) + (1 - \alpha)(-3, 6)) \mid \alpha \in \mathbb{R}\}$

$B = \{(a, b) \in \mathbb{R}^2 \mid a \geq 1\}$

$C = \{(p, q) \in \mathbb{R}^2 \mid q \leq 0\}$

Prove that $A \cap B \subseteq C$.

Solution: Let (x, y) be a 2D point and suppose that $(x, y) \in A \cap B$. Then $(x, y) \in A$ and $(x, y) \in B$.

Since $(x, y) \in A$, $(x, y) = \alpha(2, -4) + (1 - \alpha)(-3, 6))$ where α is a real number. So $x = 2\alpha - 3(1 - \alpha) = 5\alpha - 3$. And $y = -4\alpha + 6(1 - \alpha) = 6 - 10\alpha$.

Since $(x, y) \in B$, we know that $x \geq 1$. So $5\alpha - 3 \geq 1$. Therefore $\alpha \geq \frac{4}{5}$.

Substituting this into the equation for y, we get $y = 6 - 10\alpha \leq 6 - 10\frac{4}{5} = 6 - 8 = -2 \leq 0$. Since $y \leq 0$, $(x, y) \in C$, which is what we needed to show.

Name:____

NetID:______________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

$A = \{(x, y) \in \mathbb{R}^2 \;:\; y = x^2 - 2x - 1\}$

$B = \{(p, q) \in \mathbb{R}^2 \;:\; |p| \geq 3\}$

$C = \{(m, n) \in \mathbb{R}^2 \;:\; n \geq 0\}$

Prove that $A \cap B \subseteq C$.

Solution: Let $(x, y) \in \mathbb{R}^2$ and suppose that $(x, y) \in A \cap B$. Then $(x, y) \in A$ and $(x, y) \in B$.

Since $(x, y) \in A$, $y = x^2 - 2x - 1$. So $y = x(x - 2) - 1$.

Since $(x, y) \in B$, $|x| \geq 3$. There are two cases:

Case 1: $x \geq 3$. Then $x - 2 \geq 1$. So $y \geq 3 \cdot 1 - 1 = 2$.

Case 2: $x \leq -3$. Then $x - 2 \leq -5$. So $x(x - 2) \geq (-3)(-5) = 15$. Therefore $y = x(x - 2) - 1 \geq 14$.

In both cases, $y \geq 0$. So $(x, y) \in C$, which is what we needed to prove.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y, z) \in \mathbb{R}^3 \ : y = x^2 - 2x + 11\}$

$B = \{(a, b, c) \in \mathbb{R}^3 \ : \ b \leq c\}$

$C = \{(p, q, r) \in \mathbb{R}^3 \ : \ r \geq 5\}$

Prove that $A \cap B \subseteq C$.

Solution: Let $(x, y, z) \in A \cap B$. Then $(x, y, z) \in A$, so $y = x^2 - 2x + 11$. Also $(x, y, z) \in B$, so $y \leq z$.

We can rewrite the first equation as $y = (x-1)^2 + 10$. $(x-1)^2 \geq 0$ because it's the square of a real number. So $y \geq 10$.

We now have $y \geq 10$ and $y \leq z$. Combining these gives us $z \geq 10$. So $z \geq 5$. Therefore $(x, y, z) \in C$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y, z) \in \mathbb{R}^3 \;:\; |x + y + z| = 20\}$

$B = \{(a, b, c) \in \mathbb{N}^3 \;:\; a + b < 5\}$

$C = \{(p, q, r) \in \mathbb{R}^3 \;:\; r > 10\}$

Prove that $A \cap B \subseteq C$.

Solution: Let $(x, y, z) \in A \cap B$. Then $(x, y, z) \in A$, so $|x + y + z| = 20$. Also $(x, y, z) \in B$, so $x + y < 5$ and x, y, and z are all natural numbers.

Since x, y, and z are natural numbers, they can't be negative. So $x + y + z$ isn't negative. Therefore $x + y + z = |x + y + z| = 20$. So $z = 20 - (x + y)$.

Since $z = 20 - (x + y)$ and $x + y < 5$, $z > 15$. So $z > 10$, which means that $(x, y, z) \in C$. This is what we needed to show.

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

$A = \{(x, y) \in \mathbb{R}^2 \ : \ xy \leq -7\}$

$B = \{(p^3, p^2) \ : \ p \in \mathbb{R}\}$

$C = \{(a, b) \in \mathbb{R}^2 \ : \ a < 0\}$

Prove that $A \cap B \subseteq C$.

Solution: Proof: Let $(x, y) \in A \cap B$. Then, $(x, y) \in A$ and $(x, y) \in B$. So, from the definition of A, we know that $xy \leq -7$. From the definition of B, we know that $x = p^3$ and $y = p^2$, for some real number p.

Since $xy \leq 7 < 0$, we know x and y have opposite signs and neither is zero. Since $y = p^2$, we know that y is positive. So x must be negative.

Since x is negative, $(x, y) \in C$, which is what we needed to prove.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{a(1,0) + b(3,1) + c(2,4) \ : \ a, b, c \text{ are positive reals and } a + b + c = 1\}$

$B = \{(x, y) \in \mathbb{R}^2 \ : \ x \leq 3 \text{ and } y \geq 0\}$

Prove that $A \subseteq B$.

Solution: Let $(x, y) \in A$. By the definition of A, $(x, y) = a(1,0) + b(3,1) + c(2,4)$, where a, b, and c are positive reals and $a + b + c = 1$.

Then $(x, y) = (a + 3b + 2c, b + 4c)$. So $x = a + 3b + 2c$ and $y = b + 4c$.

We know that a, b, and c are positive, so $b + 4c$ must be positive. So $y \geq 0$.

Since a and c are positive and $a + b + c = 1$, we have

$$x = a + 3b + 2c \leq 3a + 3b + 3c = 3(a + b + c) = 3$$

So $y \geq 0$ and $x \leq 3$. Therefore (x, y) is in B, by the definition of the set B.

Name:____

NetID:________________________ **Lecture:** **B**

Discussion: **Thursday** **Friday** **11** **12** **1** **2** **3** **4**

$A = \{(a, b) \in \mathbb{R}^2 \ : \ a = 3 - b^2\}$

$B = \{(x, y) \in \mathbb{R}^2 \ : \ |x| \geq 1 \text{ or } |y| \geq 1\}$

Prove that A $\subseteq$ B. Hint: you may find proof by cases helpful.

Solution: Suppose that (a, b) is an element of A. Then, by the definition of A, $(a, b) \in \mathbb{R}^2$ and $a = 3 - b^2$.

Consider two cases, based on the magnitude of b:

Case 1: $|b| \geq 1$. Then (a, b) is an element of B. (Because it satisfies one of the two conditions in the OR.)

Case 2: $|b| < 1$. Then $b^2 < 1$. Then $a = 3 - b^2 > 3 - 1 = 2$. So $|a| \geq 1$, which means that (a, b) is an element of B.

So (a, b) is an element of B in both cases, which is what we needed to show.

Name:____

NetID:________________________ **Lecture: B**

Discussion: Thursday Friday 11 12 1 2 3 4

$A = \{(x, y) \in \mathbb{Z}^2 \mid 2xy + 6y - 5x - 15 \geq 0\}$

$B = \{(a, b) \in \mathbb{Z}^2 \mid a \geq 0\}$

$C = \{(p, q) \in \mathbb{Z}^2 \mid q \geq 0\}$

Prove that $(A \cap B) \subseteq C$.

Solution: Suppose that (x, y) is an element of $(A \cap B)$. This means that (x, y) is an element of A and (x, y) is an element of B. So $2xy + 6y - 5x - 15 \geq 0$ and $x \geq 0$, by the definitions of A and B.

Notice that $2xy + 6y - 5x - 15 = (x + 3)(2y - 5)$. So $(x + 3)(2y - 5) \geq 0$. We know that $x + 3$ is positive because $x \geq 0$. So we must have $(2y - 5) \geq 0$.

Now, if $(2y - 5) \geq 0$, then $2y \geq 5$. So $y \geq \frac{5}{2}$. So $y \geq 0$. This means that (x, y) is an element of C which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

For any integers s and t define $L(s,t)$ as follows:

$$L(s,t) = \{sx + ty \mid x, y \in \mathbb{Z}\}$$

Thus, $L(s,t)$ consists of all integers that can be expressed as the sum of multiples of s and t. Prove the following claim using your best mathematical style and the following definition of congruence mod k: $p \equiv q \pmod{k}$ if and only if $p = q + kn$ for some integer n.

Claim: For any integers a, b, r, where r is positive, if $a \equiv b \pmod{r}$, then $L(a,b) \subseteq L(r,b)$.

Solution: Let a, b and r be integers, where r is positive. And suppose that $a \equiv b \pmod{r}$. Then $a = b + rn$ for some integer n.

Let q be an element of $L(a,b)$. Then $q = ax + by$, where x and y are integers.

Substituting $a = b + rn$ into $q = ax + by$, we get $q = x(b + rn) + by$. So $q = (xn)r + (x + y)b$.

xn and $x + y$ are integers, because x, y, and n are integers. So $q = (xn)r + (x + y)b$ implies that $q \in L(r,b)$.

Since q was an arbitrarily chosen element of $L(a,b)$, we've shown that $L(a,b) \subseteq L(r,b)$.

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NetID:________________________ **Lecture:** **A** **B**

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$A = \{(p, q) \in \mathbb{R}^2 \ | \ p = 0\}$

$B = \{(x, y) \in \mathbb{R}^2 \ | \ (x-1)^2 + y^2 = 4\}$

$C = \{(s, t) \in \mathbb{R}^2 \ | \ (s+1)^2 + t^2 = 4\}$

Prove that $B \cap C \subseteq$ A.

Solution: Let $(x, y) \in \mathbb{R}$ and suppose that $(x, y) \in B \cap C$. Then $(x, y) \in B$ and $(x, y) \in C$.

By the definitions of B and C, this means that $(x-1)^2 + y^2 = 4$ and $(x+1)^2 + y^2 = 4$. So $(x-1)^2 + y^2 = (x+1)^2 + y^2$, which means that $(x-1)^2 = (x+1)^2$. Multiplying out the two sides, we get $x^2 - 2x + 2 = x^2 + 2x + 2$ So $-2x = 2x$, so $4x = 0$, so $x = 0$.

Since $x = 0$, $(x, y) \in A$, which is what we needed to show.

[This shows more algebra steps than I'd expect for full credit.]

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

$A = \{(x, y, z) \in \mathbb{Z}^3 \ : \ 2x + y = z - 1\}$

$B = \{(a, b, c) \in \mathbb{Z}^3 \ : \ 2a - b = c - 3\}$

$C = \{(p, q, r) \in \mathbb{Z}^3 \ : \ r \text{ is even}\}$

Prove that $A \cap B \subseteq C$. (Work directly from the definition of "even.")

Solution: Let $(x, y, z) \in A \cap B$. Then $(x, y, z) \in A$. So x, y, and z are integers and $2x + y = z - 1$. Also $(x, y, z) \in B$. So $2x - y = z - 3$.

Adding together $2x+y = z-1$ and $2x-y = z-3$, we get $4x = 2z-4$. So $2x = z-2$. So $z = 2(x+1)$. Since x is an integer, $x+1$ is an integer. So $z = 2(x+1)$ implies that z is even, which is what we needed to prove.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(a, b) \in \mathbb{R}^2 \ : \ a = 3 - b^2\}$

$B = \{(x, y) \in \mathbb{R}^2 \ : \ |x| \geq 1 \text{ or } |y| \geq 1\}$

Prove that A $\subseteq$ B. Hint: you may find proof by cases helpful.

Solution: Suppose that (a, b) is an element of A. Then, by the definition of A, $(a, b) \in \mathbb{R}^2$ and $a = 3 - b^2$.

Consider two cases, based on the magnitude of b:

Case 1: $|b| \geq 1$. Then (a, b) is an element of B. (Because it satisfies one of the two conditions in the OR.)

Case 2: $|b| < 1$. Then $b^2 < 1$. Then $a = 3 - b^2 > 3 - 1 = 2$. So $|a| \geq 1$, which means that (a, b) is an element of B.

So (a, b) is an element of B in both cases, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{\lambda(3,2) + (1-\lambda)(5,0) \ \mid \ \lambda \in [0,1]\}$

$B = \{(x,y) \in \mathbb{R}^2 \ \mid \ x \geq y\}$

Prove that $A \subseteq B$.

Solution:

Let $(x,y) \in A$. Then $(x,y) = \lambda(3,2) + (1-\lambda)(5,0)$ for some $\lambda \in [0,1]$. So $x = 3\lambda + 5(1-\lambda) = 5 - 2\lambda$ and $y = 2\lambda$.

Since $\lambda \in [0.1]$, $\lambda \leq 1$. So $4\lambda \leq 5\lambda \leq 5$. So $5 - 4\lambda \geq 0$. And therefore $x = 5 - 2\lambda \geq 2\lambda = y$.

Since $x \geq y$, $(x,y) \in B$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y, z) \in \mathbb{Z}^3 \ : \ 0 < x \le y \le z\}$

$B = \{(a, b, c) \in \mathbb{R}^3 \ : \ c^2 \le a\}$

$C = \{(p, q, r) \in \mathbb{R}^3 \ : \ p \ge 1\}$

Prove that $A \cap B \subseteq C$.

Solution: Let $(x, y, z) \in A \cap B$. Then $(x, y, z) \in A$, so x, y, and z are integers and $0 < x \le y \le z$. Also $(x, y, z) \in B$, so $z^2 \le x$.

From the first equation, we know that z is positive. Since $x \le y \le z$, we know that $x \le z$. Combining this with $z^2 \le x$, we have $z^2 \le z$. Since z is positive, this implies that $z \le 1$.

Notice that we now have $0 < x \le z \le 1$. So $0 < x \le 1$. Since x is an integer, this means that $x = 1$. So $x \ge 1$. So $(x, y, z) \in C$, which is what we needed to show.

CS 173, Fall 17 Examlet 3, Part A

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < x < y - 1\}$

$B = \{(a, b, c) \in \mathbb{R}^3 \mid b^2 + 2 < c^2\}$

$C = \{(p, q, r) \in \mathbb{R}^3 \mid p^2 < r^2\}$

Prove that $A \cap B \subseteq C$.

Solution: Let $(x, y, z) \in A \cap B$. Then $(x, y, z) \in A$, so (x, y, z) is a triple of real numbers with $0 < x < y - 1$. Also $(x, y, z) \in B$, so $y^2 + 2 < z^2$.

We know that $0 < x < y - 1$. Since $y - 1 > 0$, $y > 0$, so $-2y < 0$. Squaring both sides of $x < y - 1$ and using the fact that both sides of the equation are positive, we get $x^2 < y^2 - 2y + 1$. So $x^2 < y^2 + 1 < y^2 + 2$. But we know that $y^2 + 2 < z^2$. So we have $x^2 < z^2$, and therefore $(x, y, z) \in C$, which is what we needed to show.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y) \in \mathbb{R}^2 \mid x = \lfloor 3y + 5 \rfloor \}$

$B = \{(p, q) \in \mathbb{Z}^2 \mid 2p + q \equiv 3 \pmod 7 \}$

Prove that $A \cap \mathbb{Z}^2 \subseteq B$.

Use the following definition of congruence mod k: if s, t, k are integers, k positive, then $s \equiv t \pmod k$ if and only if $s = t + nk$ for some integer n.

Solution: Let (x, y) be an element of $A \cap \mathbb{Z}^2$. Then (x, y) is an element of A and, also, both x and y are integers.

By the definition of Z, $x = \lfloor 3y + 5 \rfloor$. Since y is an integer, $3y + 5$ must also be an integer. So $\lfloor 3y + 5 \rfloor = 3y + 5$. Therefore, $x = 3y + 5$.

Now, consider $2x + y$.

$$2x + y = 2(3y + 5) + y = 7y + 10 = 7(y + 1) + 3$$

$y + 1$ is an integer, since y is an integer. So this means that $2x + y \equiv 3 \pmod 7$. Therefore, (x, y) is an element of B, which is what we needed to show.

CS 173, Fall 17 **Examlet 3, Part A**

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y) \in \mathbb{R}^2 \mid y = x^2 - 3x + 2\}$

$B = \{(p, q) \in \mathbb{R}^2 \mid p \geq 0\}$

$C = \{(m, n) \in \mathbb{R}^2 \mid n \geq 1\}$

Prove that $A \subseteq B \cup C$.

Solution:

Suppose that (x, y) is an element of A. Then $y = x^2 - 3x + 2 = (x - 1)(x - 2)$. There are two cases:

Case 1: $x \geq 0$. Then $(x, y) \in B$ so $(x, y) \in B \cup C$.

Case 2: $x \leq 0$. Then $x - 1 \leq -1$ and $x - 2 \leq -2$. So $y = (x - 1)(x - 2) \geq 2 \geq 1$. So $(x, y) \in C$. And therefore $(x, y) \in B \cup C$.

In both cases $(x, y) \in B \cup C$, which is what we needed to show.

CS 173, Fall 17 Examlet 3, Part A

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y) \in \mathbb{R}^2 \ : \ |x + y + 2| < 5\}$

$B = \{(p, q) \in \mathbb{R}^2 \ : \ |p - q| < 3\}$

$C = \{(a, b) \in \mathbb{R}^2 \ : \ |ab| < 10\}$

Prove that $A \cap B \subseteq C$.

Solution: This claim isn't true as stated. It's ok if you did any of the following:

- An almost-right proof, with a gap or a dodgy step (e.g. involving the inequalities).
- A proof of a slightly modified version of the claim, correct or almost-right.
- A well-explained counter-example.

For the proofs, we'll be looking at whether your basic style and outline are sensible, even if the algebra didn't or couldn't work.

Name:____

NetID:__________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y) \in \mathbb{Z}^2 \mid 2xy + 6y - 5x - 15 \geq 0\}$

$B = \{(a, b) \in \mathbb{Z}^2 \mid a \geq 0\}$

$C = \{(p, q) \in \mathbb{Z}^2 \mid q \geq 0\}$

Prove that $(A \cap B) \subseteq C$.

Solution: Suppose that (x, y) is an element of $(A \cap B)$. This means that (x, y) is an element of A and (x, y) is an element of B. So $2xy + 6y - 5x - 15 \geq 0$ and $x \geq 0$, by the definitions of A and B.

Notice that $2xy + 6y - 5x - 15 = (x + 3)(2y - 5)$. So $(x + 3)(2y - 5) \geq 0$. We know that $x + 3$ is positive because $x \geq 0$. So we must have $(2y - 5) \geq 0$.

Now, if $(2y - 5) \geq 0$, then $2y \geq 5$. So $y \geq \frac{5}{2}$. So $y \geq 0$. This means that (x, y) is an element of C which is what we needed to show.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y, z) \in \mathbb{R}^3 \ : \ |x + y + z| = 20\}$

$B = \{(a, b, c) \in \mathbb{N}^3 \ : \ a + b < 5\}$

$C = \{(p, q, r) \in \mathbb{R}^3 \ : \ r > 10\}$

Prove that $A \cap B \subseteq C$.

Solution: Let $(x, y, z) \in A \cap B$. Then $(x, y, z) \in A$, so $|x + y + z| = 20$. Also $(x, y, z) \in B$, so $x + y < 5$ and x, y, and z are all natural numbers.

Since x, y, and z are natural numbers, they can't be negative. So $x + y + z$ isn't negative. Therefore $x + y + z = |x + y + z| = 20$. So $z = 20 - (x + y)$.

Since $z = 20 - (x + y)$ and $x + y < 5$, $z > 15$. So $z > 10$, which means that $(x, y, z) \in C$. This is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y) \in \mathbb{R}^2 \ : \ y = x^2 - 4\}$

$B = \{(p, q) \in \mathbb{Z}^2 \ : q < 0\}$

$C = \{(a, b) \in \mathbb{R}^2 \ : \ |a| \leq 1\}$

Prove that $A \cap B \subseteq C$.

Solution: Proof: Let $(x, y) \in A \cap B$. Then, $(x, y) \in A$ and $(x, y) \in B$. So, from the definition of A, we know that $y = x^2 - 4$. From the definition of B, we know that $y < 0$ and that x and y are both integers.

$y = x^2 - 4 = (x - 2)(x + 2)$. So since $y < 0$, $-2 < x < 2$. But x is an integer. So the only possible values in this range are -1, 0, and 1. Therefore $|x| \leq 1$. So $(x, y) \in C$, which is what we needed to prove.

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Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y) \in \mathbb{Z}^2 \mid 2xy + 6y - 5x - 15 \geq 0\}$

$B = \{(a, b) \in \mathbb{Z}^2 \mid a \geq 0\}$

$C = \{(p, q) \in \mathbb{Z}^2 \mid q \geq 0\}$

Prove that $(A \cap B) \subseteq C$.

Solution: Suppose that (x, y) is an element of $(A \cap B)$. This means that (x, y) is an element of A and (x, y) is an element of B. So $2xy + 6y - 5x - 15 \geq 0$ and $x \geq 0$, by the definitions of A and B.

Notice that $2xy + 6y - 5x - 15 = (x + 3)(2y - 5)$. So $(x + 3)(2y - 5) \geq 0$. We know that $x + 3$ is positive because $x \geq 0$. So we must have $(2y - 5) \geq 0$.

Now, if $(2y - 5) \geq 0$, then $2y \geq 5$. So $y \geq \frac{5}{2}$. So $y \geq 0$. This means that (x, y) is an element of C which is what we needed to show.

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$A = \{\alpha(2, -4) + (1 - \alpha)(-2, 5)) \mid \alpha \in \mathbb{R}\}$

$B = \{(a, b) \in \mathbb{R}^2 \mid b \leq -1\}$

$C = \{(p, q) \in \mathbb{R}^2 \mid p \geq 0\}$

Prove that $A \cap B \subseteq C$.

Solution: Let (x, y) be a 2D point and suppose that $(x, y) \in A \cap B$. Then $(x, y) \in A$ and $(x, y) \in B$.

Since $(x, y) \in A$, $(x, y) = \alpha(2, -4) + (1 - \alpha)(-2, 5))$ where α is a real number. So $x = 2\alpha - 2(1 - \alpha) = 4\alpha - 2$ And $y = -4\alpha + 5(1 - \alpha) = 5 - 9\alpha$

Since $(x, y) \in B$, $y \leq -1$. So we have $y = 5 - 9\alpha \leq -1$. So $6 \leq 9\alpha$. So $\alpha \geq \frac{2}{3}$.

So then $x = 4\alpha - 2 \geq 4\frac{2}{3} - 2 = \frac{8}{3} - 2 = \frac{2}{3}$.

So $x \geq 0$ and therefore $(x, y) \in C$, which is what we needed to show.

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$A = \{a(1,0) + b(3,1) + c(2,4) \ : \ a, b, c \text{ are positive reals and } a + b + c = 1\}$

$B = \{(x, y) \in \mathbb{R}^2 \ : \ x \leq 3 \text{ and } y \geq 0\}$

Prove that $A \subseteq B$.

Solution: Let $(x, y) \in A$. By the definition of A, $(x, y) = a(1,0) + b(3,1) + c(2,4)$, where a, b, and c are positive reals and $a + b + c = 1$.

Then $(x, y) = (a + 3b + 2c, b + 4c)$. So $x = a + 3b + 2c$ and $y = b + 4c$.

We know that a, b, and c are positive, so $b + 4c$ must be positive. So $y \geq 0$.

Since a and c are positive and $a + b + c = 1$, we have

$$x = a + 3b + 2c \leq 3a + 3b + 3c = 3(a + b + c) = 3$$

So $y \geq 0$ and $x \leq 3$. Therefore (x, y) is in B, by the definition of the set B.

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Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y) \in \mathbb{R}^2 \ : \ xy \leq -7\}$

$B = \{(p^3, p^2) : p \in \mathbb{R}\}$

$C = \{(a, b) \in \mathbb{R}^2 \ : \ a < 0\}$

Prove that $A \cap B \subseteq C$.

Solution: Proof: Let $(x, y) \in A \cap B$. Then, $(x, y) \in A$ and $(x, y) \in B$. So, from the definition of A, we know that $xy \leq -7$. From the definition of B, we know that $x = p^3$ and $y = p^2$, for some real number p.

Since $xy \leq 7 < 0$, we know x and y have opposite signs and neither is zero. Since $y = p^2$, we know that y is positive. So x must be negative.

Since x is negative, $(x, y) \in C$, which is what we needed to prove.

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Discussion: **Thursday** **Friday** **10** **11** **12** **1** **2** **3** **4** **5** **6**

$A = \{(x, y) \in \mathbb{R}^2 \ : \ y = x^2 - 4\}$

$B = \{(p, q) \in \mathbb{Z}^2 \ : q < 0\}$

$C = \{(a, b) \in \mathbb{R}^2 \ : \ |a| \leq 1\}$

Prove that $A \cap B \subseteq C$.

Solution: Proof: Let $(x, y) \in A \cap B$. Then, $(x, y) \in A$ and $(x, y) \in B$. So, from the definition of A, we know that $y = x^2 - 4$. From the definition of B, we know that $y < 0$ and that x and y are both integers.

$y = x^2 - 4 = (x - 2)(x + 2)$. So since $y < 0$, $-2 < x < 2$. But x is an integer. So the only possible values in this range are -1, 0, and 1. Therefore $|x| \leq 1$. So $(x, y) \in C$, which is what we needed to prove.