1 Linear algebra proofs

1.1 $s(\vec{a}+\vec{b}) = s\vec{a}+s\vec{b}$

$$s(\vec{a}+\vec{b}) = s(\begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix}) = s\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ ... \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} s(a_1+b_1) \\ s(a_2+b_2) \\ ... \\ s(a_n+b_n) \end{bmatrix}$$

$$= \begin{bmatrix} sa_1+sb_1 \\ sa_2+sb_2 \\ ... \\ sa_n+sb_n \end{bmatrix} = \begin{bmatrix} sa_1 \\ sa_2 \\ ... \\ sa_n \end{bmatrix} + \begin{bmatrix} sb_1 \\ sb_2 \\ ... \\ sb_n \end{bmatrix} = s\vec{a}+s\vec{b}$$

1.2 $s(\vec{a}\cdot\vec{b}) = s\vec{a}\cdot\vec{b} = \vec{a}\cdot s\vec{b}$

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix}$$

$$s(\vec{a}\cdot\vec{b})$$
$$s(a_1b_1+a_2b_2+...+a_nb_n)$$
$$sa_1b_1+sa_2b_2+...+sa_nb_n$$
$$(sa_1)b_1+(sa_2)b_2+...+(sa_n)b_n$$
$$s\vec{a}\cdot\vec{b}$$

$$s(\vec{a}\cdot\vec{b})$$
$$s(a_1b_1+a_2b_2+...+a_nb_n)$$
$$sa_1b_1+sa_2b_2+...+sa_nb_n$$
$$a_1sb_1+a_2sb_2+...+a_nsb_n$$
$$a_1(sb_1)+a_2(sb_2)+...+a_n(sb_n)$$
$$\vec{a}\cdot s\vec{b}$$

1.3 $|s|\cdot||\vec{a}|| = ||s\vec{a}||$

$$\vec{a}\cdot\vec{a} = ||\vec{a}||\cdot||\vec{a}||cos(0) = ||\vec{a}||^2$$
$$||\vec{a}|| = \sqrt{\vec{a}\cdot\vec{a}}$$

$$||s\vec{a}||$$
$$\sqrt{s\vec{a}\cdot s\vec{a}}$$
$$\sqrt{s(s\vec{a}\cdot\vec{a})}$$
$$\sqrt{ss(\vec{a}\cdot\vec{a})}$$
$$\sqrt{s^2}\sqrt{\vec{a}\cdot\vec{a}}$$
$$|s|\sqrt{\vec{a}\cdot\vec{a}}$$
$$|s|\cdot||\vec{a}||$$

1.4 $s(\vec{a} \times \vec{b}) = s\vec{a} \times \vec{b} = \vec{a} \times s\vec{b}$

$$s(\vec{a} \times \vec{b})$$

$$s(||\vec{a}|| \cdot ||\vec{b}||sin(\theta)\vec{n})$$

If $s \geq 0$,

$$||s\vec{a}|| \cdot ||\vec{b}||sin(\theta)\vec{n}$$

If $s < 0$,

$$-||s\vec{a}|| \cdot ||\vec{b}||sin(\theta + \frac{\pi}{2})\vec{n}$$

$$- - ||s\vec{a}|| \cdot ||\vec{b}||sin(\theta)\vec{n}$$

$$s\vec{a} \times \vec{b}$$

$$s(\vec{a} \times \vec{b})$$

$$s(||\vec{a}|| \cdot ||\vec{b}||sin(\theta)\vec{n})$$

If $s \geq 0$,

$$||\vec{a}|| \cdot ||s\vec{b}||sin(\theta)\vec{n}$$

If $s < 0$,

$$-||\vec{a}|| \cdot ||s\vec{b}||sin(\theta + \frac{\pi}{2})\vec{n}$$

$$- - ||\vec{a}|| \cdot ||s\vec{b}||sin(\theta)\vec{n}$$

$$\vec{a} \times s\vec{b}$$

1.5 Multiplying by a positive scalar does not affect a vector's orientation

$$cos(\alpha) = \frac{\vec{e_1} \cdot \vec{a}}{||\vec{a}||} \qquad \alpha = cos^{-1}\left(\frac{\vec{e_1} \cdot \vec{a}}{||\vec{a}||}\right)$$

$$\alpha_{s\vec{a}}$$

$$cos^{-1}\left(\frac{\vec{e_1} \cdot s\vec{a}}{||s\vec{a}||}\right)$$

$$cos^{-1}\left(\frac{s(\vec{e_1} \cdot \vec{a})}{|s| \cdot ||\vec{a}||}\right)$$

Keep in mind that $s \geq 0$.

$$cos^{-1}\left(\frac{\vec{e_1} \cdot \vec{a}}{||\vec{a}||}\right)$$

$$\alpha_{\vec{a}}$$

2 Linear albgebra derivations

2.0.1 3D rotation formula

If $\vec{v} \perp \vec{r}$, $\vec{v'} = cos(\theta)\vec{v} + sin(\theta)(\vec{r} \times \vec{v})$. We can therefore conclude that the general formula is $\vec{v'} = cos(\theta)\vec{v_\perp} + sin(\theta)(\vec{r} \times \vec{v_\perp}) + \vec{v_\parallel}$. Let's substitute $\vec{v_\perp}$ for $\vec{v} - \vec{v_\parallel}$

$$\vec{v_\parallel} = proj_{\vec{r}}\vec{v} = \frac{\vec{v} \cdot \vec{r}}{||\vec{r}||} \cdot \vec{r}$$

Because $||\vec{r}|| = 1$, $\vec{v_{\parallel}} = (\vec{v} \cdot \vec{r})\vec{r}$.

$$\begin{aligned}
\vec{v'} &= cos(\theta)(\vec{v} - \vec{v_{\parallel}}) + sin(\theta)(\vec{r} \times (\vec{v} - \vec{v_{\parallel}})) + \vec{v_{\parallel}} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_{\parallel}} + (\vec{r} \times sin(\theta)(\vec{v} - \vec{v_{\parallel}})) + \vec{v_{\parallel}} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_{\parallel}} + (\vec{r} \times (sin(\theta)\vec{v} - sin(\theta)\vec{v_{\parallel}})) + \vec{v_{\parallel}} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_{\parallel}} + (\vec{r} \times sin(\theta)\vec{v}) - (\vec{r} \times sin(\theta)\vec{v_{\parallel}}) + \vec{v_{\parallel}} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_{\parallel}} + sin(\theta)(\vec{r} \times \vec{v}) - sin(\theta)(\vec{r} \times \vec{v_{\parallel}}) + \vec{v_{\parallel}} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_{\parallel}} + sin(\theta)(\vec{r} \times \vec{v}) + \vec{v_{\parallel}} \\
&= cos(\theta)\vec{v} + (1 - cos(\theta))\vec{v_{\parallel}} + sin(\theta)(\vec{r} \times \vec{v}) \\
&= cos(\theta)\vec{v} + (1 - cos(\theta))(\vec{v} \cdot \vec{r})\vec{r} + sin(\theta)(\vec{r} \times \vec{v})
\end{aligned} \tag{1}$$

3 Quaternion derivations

3.0.1 Multiplication

$$i^2 = j^2 = k^2 = ijk = -1$$

Substitution table:

	i	j	k
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

$$\begin{aligned}
ab &= (a_s + a_i i + a_j j + a_k k)(b_s + b_i i + b_j j + b_k k) \\
&= a_s b_s + a_s b_i i + a_s b_j j + a_s b_k k \\
&\quad + a_i i b_s + a_i i b_i i + a_i i b_j j + a_i i b_k k \\
&\quad + a_j j b_s + a_j j b_i i + a_j j b_j j + a_j j b_k k \\
&\quad + a_k k b_s + a_k k b_i i + a_k k b_j j + a_k k b_k k \\
&= a_s b_s - a_i b_i - a_j b_j - a_k b_k \\
&\quad + a_s b_i i + a_j b_k i - a_k b_j i + a_i b_s i \\
&\quad + a_s b_j j + a_k b_i j - a_i b_k j + a_j b_s j \\
&\quad + a_s b_k k + a_i b_j k - a_j b_i k + a_k b_s k
\end{aligned} \tag{2}$$