1 Linear algebra proofs

1.1 $s(\vec{a} + \vec{b}) = s\vec{a} + s\vec{b}$

$$s(\vec{a} + \vec{b}) = s\left(\begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix}\right) = s\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ ... \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} s(a_1 + b_1) \\ s(a_2 + b_2) \\ ... \\ s(a_n + b_n) \end{bmatrix}$$

$$= \begin{bmatrix} sa_1 + sb_1 \\ sa_2 + sb_2 \\ ... \\ sa_n + sb_n \end{bmatrix} = \begin{bmatrix} sa_1 \\ sa_2 \\ ... \\ sa_n \end{bmatrix} + \begin{bmatrix} sb_1 \\ sb_2 \\ ... \\ sb_n \end{bmatrix} = s\vec{a} + s\vec{b}$$

1.2 $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ **if** $a, b, c \in R^3$

$$\vec{a} \times (\vec{b} + \vec{c})$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \left(\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} \right)$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x + c_x \\ b_y + c_y \\ b_z + c_z \end{bmatrix}$$

$$\begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x + c_x & b_y + c_y & b_z + c_z \end{vmatrix}$$

$$a_y(b_z+c_z)i + a_z(b_x+c_x)j + a_x(b_y+c_y)k - a_y(b_x+c_x)k - a_z(b_y+c_y)i - a_x(b_z+c_z)j$$

$$\begin{bmatrix} a_y(b_z + c_z) - a_z(b_y + c_y) \\ a_z(b_x + c_x) - a_x(b_z + c_z) \\ a_x(b_y + c_y) - a_y(b_x + c_x) \end{bmatrix}$$

$$\begin{bmatrix} a_y b_z + a_y c_z - a_z b_y - a_z c_y \\ a_z b_x + a_z c_x - a_x b_z - a_x c_z \\ a_x b_y + a_x c_y - a_y b_x - a_y c_x \end{bmatrix}$$

$$\begin{bmatrix} (a_y b_z - a_z b_y) + (a_y c_z - a_z c_y) \\ (a_z b_x - a_x b_z) + (a_z c_x - a_x c_z) \\ (a_x b_y - a_y b_x) + (a_x c_y - a_y c_x) \end{bmatrix}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} + \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}$$

$$\vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

1.3 $s(\vec{a} \cdot \vec{b}) = s\vec{a} \cdot \vec{b} = \vec{a} \cdot s\vec{b}$

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix}$$

$s(\vec{a} \cdot \vec{b})$

$s(a_1b_1 + a_2b_2 + ... + a_nb_n)$

$sa_1b_1 + sa_2b_2 + ... + sa_nb_n$

$(sa_1)b_1 + (sa_2)b_2 + ... + (sa_n)b_n$

$s\vec{a} \cdot \vec{b}$

$s(\vec{a} \cdot \vec{b})$

$s(a_1b_1 + a_2b_2 + ... + a_nb_n)$

$sa_1b_1 + sa_2b_2 + ... + sa_nb_n$

$a_1sb_1 + a_2sb_2 + ... + a_nsb_n$

$a_1(sb_1) + a_2(sb_2) + ... + a_n(sb_n)$

$\vec{a} \cdot s\vec{b}$

1.4 $|s| \cdot ||\vec{a}|| = ||s\vec{a}||$

$$\vec{a} \cdot \vec{a} = ||\vec{a}|| \cdot ||\vec{a}|| cos(0) = ||\vec{a}||^2$$

$$||\vec{a}|| = \sqrt{\vec{a} \cdot \vec{a}}$$

$||s\vec{a}||$

$\sqrt{s\vec{a} \cdot s\vec{a}}$

$\sqrt{s(s\vec{a} \cdot \vec{a})}$

$\sqrt{ss(\vec{a} \cdot \vec{a})}$

$\sqrt{s^2}\sqrt{\vec{a} \cdot \vec{a}}$

$|s|\sqrt{\vec{a} \cdot \vec{a}}$

$|s| \cdot ||\vec{a}||$

1.5 $\quad s(\vec{a} \times \vec{b}) = s\vec{a} \times \vec{b} = \vec{a} \times s\vec{b}$

$$s(\vec{a} \times \vec{b})$$

$$s(||\vec{a}|| \cdot ||\vec{b}||sin(\theta)\vec{n})$$

If $s \geq 0$,

$$||s\vec{a}|| \cdot ||\vec{b}||sin(\theta)\vec{n}$$

If $s < 0$,

$$-||s\vec{a}|| \cdot ||\vec{b}||sin(\theta + \frac{\pi}{2})\vec{n}$$

$$-- ||s\vec{a}|| \cdot ||\vec{b}||sin(\theta)\vec{n}$$

$$s\vec{a} \times \vec{b}$$

$$s(\vec{a} \times \vec{b})$$

$$s(||\vec{a}|| \cdot ||\vec{b}||sin(\theta)\vec{n})$$

If $s \geq 0$,

$$||\vec{a}|| \cdot ||s\vec{b}||sin(\theta)\vec{n}$$

If $s < 0$,

$$-||\vec{a}|| \cdot ||s\vec{b}||sin(\theta + \frac{\pi}{2})\vec{n}$$

$$-- ||\vec{a}|| \cdot ||s\vec{b}||sin(\theta)\vec{n}$$

$$\vec{a} \times s\vec{b}$$

1.6 Multiplying by a positive scalar does not affect a vector's orientation

$$cos(\alpha) = \frac{\vec{e_1} \cdot \vec{a}}{||\vec{a}||} \qquad\qquad \alpha = cos^{-1}\left(\frac{\vec{e_1} \cdot \vec{a}}{||\vec{a}||}\right)$$
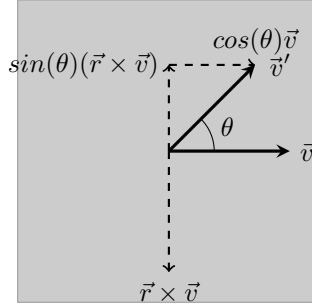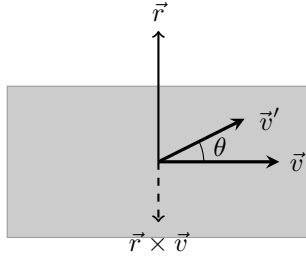
$$\alpha_{s\vec{a}}$$

$$cos^{-1}\left(\frac{\vec{e_1} \cdot s\vec{a}}{||s\vec{a}||}\right)$$

$$cos^{-1}\left(\frac{s(\vec{e_1} \cdot \vec{a})}{|s| \cdot ||\vec{a}||}\right)$$
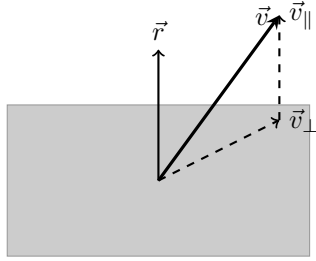
Keep in mind that $s \geq 0$.

$$cos^{-1}\left(\frac{\vec{e_1} \cdot \vec{a}}{||\vec{a}||}\right)$$

$$\alpha_{\vec{a}}$$

2 Linear albgebra derivations

2.0.1 3D rotation formula

If $\vec{v} \perp \vec{r}$, $\vec{v'} = cos(\theta)\vec{v} + sin(\theta)(\vec{r} \times \vec{v})$. We can therefore conclude that the general formula is $\vec{v'} = cos(\theta)\vec{v_\perp} + sin(\theta)(\vec{r} \times \vec{v_\perp}) + \vec{v_\parallel}$. Let's substitute $\vec{v_\perp}$ for $\vec{v} - \vec{v_\parallel}$

$$\vec{v_\parallel} = proj_{\vec{r}}\vec{v} = \frac{\vec{v} \cdot \vec{r}}{||\vec{r}||} \cdot \vec{r}$$

Because $||\vec{r}|| = 1$, $\vec{v_\parallel} = (\vec{v} \cdot \vec{r})\vec{r}$.

$$
\begin{aligned}
\vec{v'} &= cos(\theta)(\vec{v} - \vec{v_\parallel}) + sin(\theta)(\vec{r} \times (\vec{v} - \vec{v_\parallel})) + \vec{v_\parallel} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_\parallel} + (\vec{r} \times sin(\theta)(\vec{v} - \vec{v_\parallel})) + \vec{v_\parallel} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_\parallel} + (\vec{r} \times (sin(\theta)\vec{v} - sin(\theta)\vec{v_\parallel})) + \vec{v_\parallel} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_\parallel} + (\vec{r} \times sin(\theta)\vec{v}) - (\vec{r} \times sin(\theta)\vec{v_\parallel}) + \vec{v_\parallel} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_\parallel} + sin(\theta)(\vec{r} \times \vec{v}) - sin(\theta)(\vec{r} \times \vec{v_\parallel}) + \vec{v_\parallel} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_\parallel} + sin(\theta)(\vec{r} \times \vec{v}) + \vec{v_\parallel} \\
&= cos(\theta)\vec{v} + (1 - cos(\theta))\vec{v_\parallel} + sin(\theta)(\vec{r} \times \vec{v}) \\
&= cos(\theta)\vec{v} + (1 - cos(\theta))(\vec{v} \cdot \vec{r})\vec{r} + sin(\theta)(\vec{r} \times \vec{v})
\end{aligned}
\tag{1}
$$

3 Quaternion derivations

3.0.1 Multiplication

$$i^2 = j^2 = k^2 = ijk = -1$$

4

Substitution table:

	i	j	k
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

$$
\begin{aligned}
ab &= (a_s + a_i i + a_j j + a_k k)(b_s + b_i i + b_j j + b_k k) \\
&= a_s b_s + a_s b_i i + a_s b_j j + a_s b_k k \\
&\quad + a_i i b_s + a_i i b_i i + a_i i b_j j + a_i i b_k k \\
&\quad + a_j j b_s + a_j j b_i i + a_j j b_j j + a_j j b_k k \\
&\quad + a_k k b_s + a_k k b_i i + a_k k b_j j + a_k k b_k k \\
&= a_s b_s - a_i b_i - a_j b_j - a_k b_k \\
&\quad + a_s b_i i + a_j b_k i - a_k b_j i + a_i b_s i \\
&\quad + a_s b_j j + a_k b_i j - a_i b_k j + a_j b_s j \\
&\quad + a_s b_k k + a_i b_j k - a_j b_i k + a_k b_s k \\
&= (a_s b_s - a_i b_i - a_j b_j - a_k b_k) \\
&\quad + (a_i b_s + a_s b_i - a_k b_j + a_j b_k)i \\
&\quad + (a_j b_s + a_k b_i + a_s b_j - a_i b_k)j \\
&\quad + (a_k b_s - a_j b_i + a_i b_j + a_s b_k)k \\
&= (a_s b_s - a_i b_i - a_j b_j - a_k b_k \\
&\quad , a_i b_s + a_s b_i - a_k b_j + a_j b_k \\
&\quad , a_j b_s + a_k b_i + a_s b_j - a_i b_k \\
&\quad , a_k b_s - a_j b_i + a_i b_j + a_s b_k)
\end{aligned}
\tag{2}
$$

3.0.2 Multiplication in matrix form

$$
\begin{bmatrix}
a_s & -a_i & -a_j & -a_k \\
a_i & a_s & -a_k & a_j \\
a_j & a_k & a_s & -a_i \\
a_k & -a_j & a_i & a_s
\end{bmatrix}
\begin{bmatrix}
b_s \\ b_i \\ b_j \\ b_k
\end{bmatrix}
=
\begin{bmatrix}
(a_s b_s - a_i b_i - a_j b_j - a_k b_k \\
a_i b_s + a_s b_i - a_k b_j + a_j b_k \\
a_j b_s + a_k b_i + a_s b_j - a_i b_k \\
a_k b_s - a_j b_i + a_i b_j + a_s b_k
\end{bmatrix}
$$

3.0.3 Multiplication in vector form

$$ab = (a_s + a_i i + a_j j + a_k k)(b_s + b_i i + b_j j + b_k k)$$
$$= (a_s b_s - a_i b_i - a_j b_j - a_k b_k$$
$$, a_i b_s + a_s b_i - a_k b_j + a_j b_k$$
$$, a_j b_s + a_k b_i + a_s b_j - a_i b_k$$
$$, a_k b_s - a_j b_i + a_i b_j + a_s b_k)$$
$$= (a_s b_s - (a_i b_i + a_j b_j + a_k b_k) \tag{3}$$
$$, a_s b_i + a_j b_k - a_k b_j + a_i b_s$$
$$, a_s b_j + a_k b_i - a_i b_k + a_j b_s$$
$$, a_s b_k + a_i b_j - a_j b_i + a_k b_s)$$
$$= (a_s, \vec{a}_v)(b_s, \vec{b}_v)$$
$$= (a_s b_s - \vec{a}_v \cdot \vec{b}_v, \; a_s \vec{b}_v + b_s \vec{a}_v + \vec{a}_v \times \vec{b}_v)$$

3.0.4 Cross product in quaternionic form

$$xy = (0, \vec{x})(0, \vec{y}) = (-\vec{x} \cdot \vec{y}, \vec{x} \times \vec{y})$$
$$yx = (0, \vec{y})(0, \vec{x}) = (-\vec{y} \cdot \vec{x}, \vec{y} \times \vec{x}) = (-\vec{x} \cdot \vec{y}, -\vec{x} \times \vec{y})$$
$$xy - yx = (-\vec{x} \cdot \vec{y}, \vec{x} \times \vec{y}) - (-\vec{x} \cdot \vec{y}, -\vec{x} \times \vec{y}) = (0, 2(\vec{x} \times \vec{y}))$$
$$\vec{x} \times \vec{y} = \frac{1}{2}(xy - yx)$$

3.0.5 3D rotation formula

$$\text{If } \vec{v} \perp \vec{r}, \; \vec{v'} = cos(\theta)\vec{v} + sin(\theta)(\vec{r} \times \vec{v}).$$

$$v = (0, \vec{v})$$
$$v_\perp = (0, \vec{v}_\perp)$$
$$v_\parallel = (0, \vec{v}_\parallel)$$
$$v' = (0, \vec{v'})$$
$$r = (0, \vec{r})$$
$$rv = (\vec{r} \cdot \vec{v}, \vec{r} \times \vec{v})$$
$$(a, 0)(b, \vec{x}) = (ab, a\vec{x})$$

$$v' = cos(\theta)v + sin(\theta)rv$$
$$= (cos(\theta) + sin(\theta)r)v$$
$$= e^{\theta r} v$$
$$= (cos(\theta), sin(\theta)\vec{r})(0, \vec{v})$$
$$= (-sin(\theta)\vec{r} \cdot \vec{v}, cos(\theta)\vec{v} + sin(\theta)\vec{r} \times \vec{v}) \tag{4}$$
$$= (-\vec{v} \cdot sin(\theta)\vec{r}, cos(\theta)\vec{v} - \vec{v} \times sin(\theta)\vec{r})$$
$$= (0, \vec{v})(cos(\theta), -sin(\theta)\vec{r})$$
$$= ve^{-\theta r}$$

If $\vec{v}$ not $\perp \vec{r}$,

$$
\begin{aligned}
v' &= v_\parallel + v_\perp e^{-\theta r} \\
&= e^{\frac{\theta}{2}r} e^{-\frac{\theta}{2}r}(v_\parallel + v_\perp e^{-\theta r}) \\
&= e^{\frac{\theta}{2}r} e^{-\frac{\theta}{2}r} v_\parallel + e^{\frac{\theta}{2}r} e^{-\frac{\theta}{2}r} v_\perp e^{-\theta r} \\
&= e^{\frac{\theta}{2}r} e^{-\frac{\theta}{2}r} v_\parallel + e^{\frac{\theta}{2}r} e^{\frac{\theta}{2}r} v_\perp \\
&= e^{\frac{\theta}{2}r} e^{-\frac{\theta}{2}r} v_\parallel + e^{\frac{\theta}{2}r} v_\perp e^{-\frac{\theta}{2}r} \\
&= e^{\frac{\theta}{2}r}(cos(\frac{\theta}{2}), sin(\frac{\theta}{2})\vec{r})(0, \vec{v}_\parallel) + e^{\frac{\theta}{2}r} v_\perp e^{-\frac{\theta}{2}r} \\
&= e^{\frac{\theta}{2}r}(-(cos(\frac{\theta}{2})\vec{r})\cdot \vec{v}_\parallel, cos(\frac{\theta}{2})\vec{v}_\parallel + (sin(\frac{\theta}{2})\vec{r}) \times \vec{v}_\parallel) + e^{\frac{\theta}{2}r} v_\perp e^{-\frac{\theta}{2}r} \qquad (5) \\
&= e^{\frac{\theta}{2}r}(-\vec{v}_\parallel \cdot (cos(\frac{\theta}{2})\vec{r}), cos(\frac{\theta}{2})\vec{v}_\parallel - \vec{v}_\parallel \times (sin(\frac{\theta}{2})\vec{r})) + e^{\frac{\theta}{2}r} v_\perp e^{-\frac{\theta}{2}r} \\
&= e^{\frac{\theta}{2}r}(0, \vec{v}_\parallel)(cos(\frac{\theta}{2}), sin(\frac{\theta}{2})\vec{r}) + e^{\frac{\theta}{2}r} v_\perp e^{-\frac{\theta}{2}r} \\
&= e^{\frac{\theta}{2}r} v_\parallel e^{-\frac{\theta}{2}r} + e^{\frac{\theta}{2}r} v_\perp e^{-\frac{\theta}{2}r} \\
&= e^{\frac{\theta}{2}r}(v_\parallel + v_\perp) e^{-\frac{\theta}{2}r} \\
&= e^{\frac{\theta}{2}r} v e^{-\frac{\theta}{2}r}
\end{aligned}
$$