1 Linear algebra proofs

1.1 $s(\vec{a}+\vec{b}) = s\vec{a}+s\vec{b}$

$$s(\vec{a}+\vec{b}) = s(\begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix}) = s\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ ... \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} s(a_1+b_1) \\ s(a_2+b_2) \\ ... \\ s(a_n+b_n) \end{bmatrix}$$

$$= \begin{bmatrix} sa_1+sb_1 \\ sa_2+sb_2 \\ ... \\ sa_n+sb_n \end{bmatrix} = \begin{bmatrix} sa_1 \\ sa_2 \\ ... \\ sa_n \end{bmatrix} + \begin{bmatrix} sb_1 \\ sb_2 \\ ... \\ sb_n \end{bmatrix} = s\vec{a}+s\vec{b}$$

1.2 $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ if $a, b, c \in R^3$

$$\vec{a}\times(\vec{b}+\vec{c})$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \left(\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} \right)$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x+c_x \\ b_y+c_y \\ b_z+c_z \end{bmatrix}$$

$$\begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x+c_x & b_y+c_y & b_z+c_z \end{vmatrix}$$

$$a_y(b_z+c_z)i+a_z(b_x+c_x)j+a_x(b_y+c_y)k-a_y(b_x+c_x)k-a_z(b_y+c_y)i-a_x(b_z+c_z)j$$

$$\begin{bmatrix} a_y(b_z+c_z)-a_z(b_y+c_y) \\ a_z(b_x+c_x)-a_x(b_z+c_z) \\ a_x(b_y+c_y)-a_y(b_x+c_x) \end{bmatrix}$$

$$\begin{bmatrix} a_yb_z+a_yc_z-a_zb_y-a_zc_y \\ a_zb_x+a_zc_x-a_xb_z-a_xc_z \\ a_xb_y+a_xc_y-a_yb_x-a_yc_x \end{bmatrix}$$

$$\begin{bmatrix} (a_yb_z-a_zb_y)+(a_yc_z-a_zc_y) \\ (a_zb_x-a_xb_z)+(a_zc_x-a_xc_z) \\ (a_xb_y-a_yb_x)+(a_xc_y-a_yc_x) \end{bmatrix}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} + \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}$$

$$\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

1.3 $s(\vec{a} \cdot \vec{b}) = s\vec{a} \cdot \vec{b} = \vec{a} \cdot s\vec{b}$

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix}$$

$$s(\vec{a} \cdot \vec{b})$$
$$s(a_1b_1 + a_2b_2 + ... + a_nb_n)$$
$$sa_1b_1 + sa_2b_2 + ... + sa_nb_n$$
$$(sa_1)b_1 + (sa_2)b_2 + ... + (sa_n)b_n$$
$$s\vec{a} \cdot \vec{b}$$

$$s(\vec{a} \cdot \vec{b})$$
$$s(a_1b_1 + a_2b_2 + ... + a_nb_n)$$
$$sa_1b_1 + sa_2b_2 + ... + sa_nb_n$$
$$a_1sb_1 + a_2sb_2 + ... + a_nsb_n$$
$$a_1(sb_1) + a_2(sb_2) + ... + a_n(sb_n)$$
$$\vec{a} \cdot s\vec{b}$$

1.4 $|s| \cdot ||\vec{a}|| = ||s\vec{a}||$

$$\vec{a} \cdot \vec{a} = ||\vec{a}|| \cdot ||\vec{a}||cos(0) = ||\vec{a}||^2$$
$$||\vec{a}|| = \sqrt{\vec{a} \cdot \vec{a}}$$

$$||s\vec{a}||$$
$$\sqrt{s\vec{a} \cdot s\vec{a}}$$
$$\sqrt{s(s\vec{a} \cdot \vec{a})}$$
$$\sqrt{ss(\vec{a} \cdot \vec{a})}$$
$$\sqrt{s^2}\sqrt{\vec{a} \cdot \vec{a}}$$
$$|s|\sqrt{\vec{a} \cdot \vec{a}}$$
$$|s| \cdot ||\vec{a}||$$

1.5 $s(\vec{a} \times \vec{b}) = s\vec{a} \times \vec{b} = \vec{a} \times s\vec{b}$

$$s(\vec{a} \times \vec{b})$$
$$s(||\vec{a}|| \cdot ||\vec{b}||sin(\theta)\vec{n})$$

If $s \geq 0$,

$$||s\vec{a}|| \cdot ||\vec{b}||sin(\theta)\vec{n}$$

If $s < 0$,

$$-||s\vec{a}|| \cdot ||\vec{b}||sin(\theta + \frac{\pi}{2})\vec{n}$$
$$- - ||s\vec{a}|| \cdot ||\vec{b}||sin(\theta)\vec{n}$$
$$s\vec{a} \times \vec{b}$$

$$s(\vec{a} \times \vec{b})$$
$$s(||\vec{a}|| \cdot ||\vec{b}||sin(\theta)\vec{n})$$

If $s \geq 0$,

$$||\vec{a}|| \cdot ||s\vec{b}||sin(\theta)\vec{n}$$

If $s < 0$,

$$-||\vec{a}|| \cdot ||s\vec{b}||sin(\theta + \frac{\pi}{2})\vec{n}$$
$$- - ||\vec{a}|| \cdot ||s\vec{b}||sin(\theta)\vec{n}$$
$$\vec{a} \times s\vec{b}$$

1.6 Multiplying by a positive scalar does not affect a vector's orientation

$$cos(\alpha) = \frac{\vec{e_1} \cdot \vec{a}}{||\vec{a}||} \qquad \alpha = cos^{-1}\left(\frac{\vec{e_1} \cdot \vec{a}}{||\vec{a}||}\right)$$

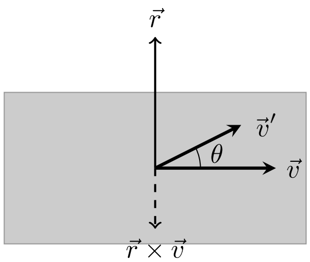
$$\alpha_{s\vec{a}}$$
$$cos^{-1}\left(\frac{\vec{e_1} \cdot s\vec{a}}{||s\vec{a}||}\right)$$
$$cos^{-1}\left(\frac{s(\vec{e_1} \cdot \vec{a})}{|s| \cdot ||\vec{a}||}\right)$$

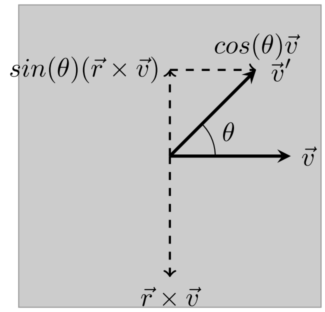
Keep in mind that $s \geq 0$.

$$cos^{-1}\left(\frac{\vec{e_1} \cdot \vec{a}}{||\vec{a}||}\right)$$
$$\alpha_{\vec{a}}$$

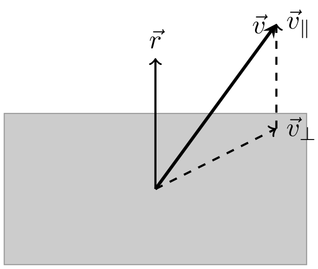
2 Linear albgebra derivations

2.0.1 3D rotation formula





If $\vec{v} \perp \vec{r}$, $\vec{v'} = cos(\theta)\vec{v} + sin(\theta)(\vec{r} \times \vec{v})$. We can therefore conclude that the general formula is $\vec{v'} = cos(\theta)\vec{v_\perp} + sin(\theta)(\vec{r} \times \vec{v_\perp}) + \vec{v_\|}$. Let's substitute $\vec{v_\perp}$ for $\vec{v} - \vec{v_\|}$



$$\vec{v_\|} = proj_{\vec{r}}\vec{v} = \frac{\vec{v} \cdot \vec{r}}{||\vec{r}||} \cdot \vec{r}$$

Because $||\vec{r}|| = 1$, $\vec{v_\|} = (\vec{v} \cdot \vec{r})\vec{r}$.

$$\begin{aligned}
\vec{v'} &= cos(\theta)(\vec{v} - \vec{v_\|}) + sin(\theta)(\vec{r} \times (\vec{v} - \vec{v_\|})) + \vec{v_\|} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_\|} + (\vec{r} \times sin(\theta)(\vec{v} - \vec{v_\|})) + \vec{v_\|} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_\|} + (\vec{r} \times (sin(\theta)\vec{v} - sin(\theta)\vec{v_\|})) + \vec{v_\|} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_\|} + (\vec{r} \times sin(\theta)\vec{v}) - (\vec{r} \times sin(\theta)\vec{v_\|}) + \vec{v_\|} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_\|} + sin(\theta)(\vec{r} \times \vec{v}) - sin(\theta)(\vec{r} \times \vec{v_\|}) + \vec{v_\|} \\
&= cos(\theta)\vec{v} - cos(\theta)\vec{v_\|} + sin(\theta)(\vec{r} \times \vec{v}) + \vec{v_\|} \\
&= cos(\theta)\vec{v} + (1 - cos(\theta))\vec{v_\|} + sin(\theta)(\vec{r} \times \vec{v}) \\
&= cos(\theta)\vec{v} + (1 - cos(\theta))(\vec{v} \cdot \vec{r})\vec{r} + sin(\theta)(\vec{r} \times \vec{v})
\end{aligned} \tag{1}$$

3 Quaternion derivations

3.0.1 Multiplication

$$i^2 = j^2 = k^2 = ijk = -1$$

Substitution table:

	i	j	k
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

$$\begin{aligned} ab &= (a_s + a_i i + a_j j + a_k k)(b_s + b_i i + b_j j + b_k k) \\ &= a_s b_s + a_s b_i i + a_s b_j j + a_s b_k k \\ &\quad + a_i i b_s + a_i i b_i i + a_i i b_j j + a_i i b_k k \\ &\quad + a_j j b_s + a_j j b_i i + a_j j b_j j + a_j j b_k k \\ &\quad + a_k k b_s + a_k k b_i i + a_k k b_j j + a_k k b_k k \\ &= a_s b_s - a_i b_i - a_j b_j - a_k b_k \\ &\quad + a_s b_i i + a_j b_k i - a_k b_j i + a_i b_s i \\ &\quad + a_s b_j j + a_k b_i j - a_i b_k j + a_j b_s j \\ &\quad + a_s b_k k + a_i b_j k - a_j b_i k + a_k b_s k \\ &= (a_s b_s - a_i b_i - a_j b_j - a_k b_k) \\ &\quad + (a_i b_s + a_s b_i - a_k b_j + a_j b_k) i \\ &\quad + (a_j b_s + a_k b_i + a_s b_j - a_i b_k) j \\ &\quad + (a_k b_s - a_j b_i + a_i b_j + a_s b_k) k \\ &= (a_s b_s - a_i b_i - a_j b_j - a_k b_k \\ &\quad , a_i b_s + a_s b_i - a_k b_j + a_j b_k \\ &\quad , a_j b_s + a_k b_i + a_s b_j - a_i b_k \\ &\quad , a_k b_s - a_j b_i + a_i b_j + a_s b_k) \end{aligned} \tag{2}$$

3.0.2 Multiplication in matrix form

$$\begin{bmatrix} a_s & -a_i & -a_j & -a_k \\ a_i & a_s & -a_k & a_j \\ a_j & a_k & a_s & -a_i \\ a_k & -a_j & a_i & a_s \end{bmatrix} \begin{bmatrix} b_s \\ b_i \\ b_j \\ b_k \end{bmatrix} = \begin{bmatrix} (a_s b_s - a_i b_i - a_j b_j - a_k b_k \\ a_i b_s + a_s b_i - a_k b_j + a_j b_k \\ a_j b_s + a_k b_i + a_s b_j - a_i b_k \\ a_k b_s - a_j b_i + a_i b_j + a_s b_k \end{bmatrix}$$

3.0.3 Multiplication in vector form

$$
\begin{aligned}
ab &= (a_s + a_i i + a_j j + a_k k)(b_s + b_i i + b_j j + b_k k) \\
&= (a_s b_s - a_i b_i - a_j b_j - a_k b_k \\
&\quad , a_i b_s + a_s b_i - a_k b_j + a_j b_k \\
&\quad , a_j b_s + a_k b_i + a_s b_j - a_i b_k \\
&\quad , a_k b_s - a_j b_i + a_i b_j + a_s b_k) \\
&= (a_s b_s - (a_i b_i + a_j b_j + a_k b_k) \\
&\quad , a_s b_i + a_j b_k - a_k b_j + a_i b_s \\
&\quad , a_s b_j + a_k b_i - a_i b_k + a_j b_s \\
&\quad , a_s b_k + a_i b_j - a_j b_i + a_k b_s) \\
&= (a_s, \vec{a}_v)(b_s, \vec{b}_v) \\
&= (a_s b_s - \vec{a}_v \cdot \vec{b}_v,\ a_s \vec{b}_v + b_s \vec{a}_v + \vec{a}_v \times \vec{b}_v)
\end{aligned} \tag{3}
$$

3.0.4 Cross product in quaternionic form

$$xy = (0, \vec{x})(0, \vec{y}) = (-\vec{x} \cdot \vec{y}, \vec{x} \times \vec{y})$$

$$yx = (0, \vec{y})(0, \vec{x}) = (-\vec{y} \cdot \vec{x}, \vec{y} \times \vec{x}) = (-\vec{x} \cdot \vec{y}, -\vec{x} \times \vec{y})$$

$$xy - yx = (-\vec{x} \cdot \vec{y}, \vec{x} \times \vec{y}) - (-\vec{x} \cdot \vec{y}, -\vec{x} \times \vec{y}) = (0, 2(\vec{x} \times \vec{y}))$$

$$\vec{x} \times \vec{y} = \frac{1}{2}(xy - yx)$$

3.0.5 3D rotation formula

If $\vec{v} \perp \vec{r}$, $\vec{v'} = cos(\theta)\vec{v} + sin(\theta)(\vec{r} \times \vec{v})$.

$$
\begin{aligned}
v &= (0, \vec{v}) \\
v_\perp &= (0, \vec{v}_\perp) \\
v_\parallel &= (0, \vec{v}_\parallel) \\
v' &= (0, \vec{v}\,') \\
r &= (0, \vec{r}) \\
rv &= (\vec{r} \cdot \vec{v}, \vec{r} \times \vec{v}) \\
(a, 0)(b, \vec{x}) &= (ab, a\vec{x})
\end{aligned}
$$

$$
\begin{aligned}
v' &= cos(\theta)v + sin(\theta)rv \\
&= (cos(\theta) + sin(\theta)r)v \\
&= e^{\theta r} v \\
&= (cos(\theta), sin(\theta)\vec{r})(0, \vec{v}) \\
&= (-sin(\theta)\vec{r} \cdot \vec{v}, cos(\theta)\vec{v} + sin(\theta)\vec{r} \times \vec{v}) \\
&= (-\vec{v} \cdot sin(\theta)\vec{r}, cos(\theta)\vec{v} - \vec{v} \times sin(\theta)\vec{r}) \\
&= (0, \vec{v})(cos(\theta), -sin(\theta)\vec{r}) \\
&= v e^{-\theta r}
\end{aligned} \tag{4}
$$

If $\vec{v}$ not $\perp$ $\vec{r}$,

$$\begin{aligned}
v' &= v_{\parallel} + v_{\perp} e^{-\theta r} \\
&= e^{\frac{\theta}{2} r} e^{-\frac{\theta}{2} r} (v_{\parallel} + v_{\perp} e^{-\theta r}) \\
&= e^{\frac{\theta}{2} r} e^{-\frac{\theta}{2} r} v_{\parallel} + e^{\frac{\theta}{2} r} e^{-\frac{\theta}{2} r} v_{\perp} e^{-\theta r} \\
&= e^{\frac{\theta}{2} r} e^{-\frac{\theta}{2} r} v_{\parallel} + e^{\frac{\theta}{2} r} e^{\frac{\theta}{2} r} v_{\perp} \\
&= e^{\frac{\theta}{2} r} e^{-\frac{\theta}{2} r} v_{\parallel} + e^{\frac{\theta}{2} r} v_{\perp} e^{-\frac{\theta}{2} r} \\
&= e^{\frac{\theta}{2} r} (cos(-\frac{\theta}{2}), sin(-\frac{\theta}{2})\vec{r})(0, \vec{v}_{\parallel}) + e^{\frac{\theta}{2} r} v_{\perp} e^{-\frac{\theta}{2} r} \\
&= e^{\frac{\theta}{2} r} (-(cos(-\frac{\theta}{2})\vec{r}) \cdot \vec{v}_{\parallel}, cos(-\frac{\theta}{2})\vec{v}_{\parallel} + (sin(-\frac{\theta}{2})\vec{r}) \times \vec{v}_{\parallel}) + e^{\frac{\theta}{2} r} v_{\perp} e^{-\frac{\theta}{2} r} \qquad (5) \\
&= e^{\frac{\theta}{2} r} (-\vec{v}_{\parallel} \cdot (cos(-\frac{\theta}{2})\vec{r}), cos(-\frac{\theta}{2})\vec{v}_{\parallel} - \vec{v}_{\parallel} \times (sin(-\frac{\theta}{2})\vec{r})) + e^{\frac{\theta}{2} r} v_{\perp} e^{-\frac{\theta}{2} r} \\
&= e^{\frac{\theta}{2} r} (0, \vec{v}_{\parallel})(cos(-\frac{\theta}{2}), sin(-\frac{\theta}{2})\vec{r}) + e^{-\frac{\theta}{2} r} v_{\perp} e^{-\frac{\theta}{2} r} \\
&= e^{\frac{\theta}{2} r} v_{\parallel} e^{-\frac{\theta}{2} r} + e^{\frac{\theta}{2} r} v_{\perp} e^{-\frac{\theta}{2} r} \\
&= e^{\frac{\theta}{2} r} (v_{\parallel} + v_{\perp}) e^{-\frac{\theta}{2} r} \\
&= e^{\frac{\theta}{2} r} v e^{-\frac{\theta}{2} r}
\end{aligned}$$