

Simple Linear Regression

William Brasic

The University of Arizona

Simple Linear Regression

Definition 1: Simple Linear Regression

The population **simple linear regression** model for $i = 1, \dots, n$ is given as

$$y_i = \beta_0 + \beta_1 x_i + u_i.$$

- y_i is often called a dependent variable, response variable, predicted variable, or outcome variable.
- x_i is often called an independent variable, explanatory variable, regressor, or covariate.
- u_i is often called the error term or idiosyncratic shock.
 - ▶ It represents all other factors than x_i that explain y_i .
- β_0 is the intercept/constant term.
- β_1 is the slope parameter.
 - ▶ It represents the effect of x on y .

How do we Estimate the Parameters?

Question 1: How do we Estimate the Parameters?

How would you estimate β_0 and β_1 ?

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How would you estimate β_0 and β_1 ?

Answer to Question 1

Ordinary Least Squares (OLS).

- We try to minimize the squared difference between our observed y_i and our predicted \hat{y}_i for each observation i .
 - ▶ \hat{y}_i is often called a fitted value for observation i .

Residuals

Definition 2: Residual

A **residual** \hat{u}_i is defined as

$$\hat{u}_i = y_i - \hat{y}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i.$$

- The **residual** is the estimated error for each i .
- OLS wants to minimize these residuals for each i .

Sum of Squared Residuals (SSR)

Definition 3: Sum of Squared Residuals

The **sum of squared residuals (SSR)** is defined as

$$\begin{aligned} SSR &= \sum_{i=1}^n \hat{u}_i^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2 \\ &= \sum_{i=1}^n \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)^2. \end{aligned}$$

- The squaring of the residuals makes everything positive.
- We use *SSR* instead of the summing the absolute values because the latter is not differentiable at zero.

Ordinary Least Squares (OLS)

Definition 4: Ordinary Least Squares (OLS)

The **ordinary least squares (OLS)** algorithm minimizes the *SSR* to obtain $(\hat{\beta}_0, \hat{\beta}_1)$:

$$\arg \min_{b_0, b_1} \sum_{i=1}^n \hat{u}_i^2 = (\hat{\beta}_0, \hat{\beta}_1).$$

How to Minimize the SSR?

Question 2: How to Minimize the SSR?

How do we minimize the SSR ?

How to Minimize the SSR?

Question 2: How to Minimize the SSR?

How do we minimize the *SSR*?

Answer to Question 2

Calculus! Take the derivative of the *SSR* with respect to each parameter. Then, set the derivatives equal to zero and solve the equations for $(\hat{\beta}_0, \hat{\beta}_1)$.

The OLS Solution

Theorem 1: The OLS Solution

The **OLS Solution** is given by

$$\begin{aligned}\hat{\beta}_0 &= \frac{1}{n} \sum_{i=1}^n y_i - \hat{\beta}_1 \frac{1}{n} \sum_{i=1}^n x_i \\ &= \bar{Y} - \hat{\beta}_1 \bar{X} \\ \hat{\beta}_1 &= \frac{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{X})(y_i - \bar{Y})}{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{X})^2} \\ &= \frac{\hat{\sigma}_{xy}}{\hat{\sigma}_x^2}.\end{aligned}$$

Proof of the OLS Solution

Proof 1: OLS Solution's Constant Term Part 1

The problem is to find the intercept term $\hat{\beta}_0$ by minimizing the *SSR*:

$$\min_{\hat{\beta}_0} \sum_{i=1}^n \hat{u}_i^2 = \min_{\hat{\beta}_0} \sum_{i=1}^n \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)^2.$$

To do so, we take the derivative of the *SSR* with respect to $\hat{\beta}_0$ and set it equal to zero (first order condition).

Proof of the OLS Solution

Proof 1: OLS Solution's Constant Term Part 2

$$\begin{aligned}\frac{\partial SSR}{\partial \hat{\beta}_0} &= 2 \sum_{i=1}^n \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right) (-1) \\ &= -2 \sum_{i=1}^n \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)\end{aligned}$$

Proof of the OLS Solution

Proof 1: OLS Solution's Constant Term Part 2

$$\begin{aligned}\frac{\partial SSR}{\partial \hat{\beta}_0} &= 2 \sum_{i=1}^n \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right) (-1) \\ &= -2 \sum_{i=1}^n \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)\end{aligned}$$

Setting this equal to zero we have

$$-2 \sum_{i=1}^n \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right) = 0 \iff \sum_{i=1}^n \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right) = 0.$$

Proof of the OLS Solution

Proof 1: OLS Solution's Constant Term Part 3

Solving the equation for $\hat{\beta}_0$ we get

$$\begin{aligned}\sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) &= 0 \iff \sum_{i=1}^n y_i - \sum_{i=1}^n \hat{\beta}_0 - \sum_{i=1}^n \hat{\beta}_1 x_i = 0 \\ &\iff \sum_{i=1}^n \hat{\beta}_0 = \sum_{i=1}^n y_i - \sum_{i=1}^n \hat{\beta}_1 x_i\end{aligned}$$

Proof of the OLS Solution

Proof 1: OLS Solution's Constant Term Part 4

Lastly,

$$\begin{aligned}\sum_{i=1}^n \hat{\beta}_0 &= \sum_{i=1}^n y_i - \sum_{i=1}^n \hat{\beta}_1 x_i \iff n\hat{\beta}_0 = \sum_{i=1}^n y_i - \hat{\beta}_1 \sum_{i=1}^n x_i \\ &\iff \hat{\beta}_0 = \frac{1}{n} \sum_{i=1}^n y_i - \hat{\beta}_1 \frac{1}{n} \sum_{i=1}^n x_i \\ &\iff \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}. \quad \square\end{aligned}$$

Hooray!

Proof of the OLS Solution

Proof 2: OLS Solution's Slope Term Part 1

The problem is to find the slope term $\hat{\beta}_1$ by minimizing the *SSR*:

$$\min_{\hat{\beta}_1} \sum_{i=1}^n \hat{u}_i^2 = \min_{\hat{\beta}_1} \sum_{i=1}^n \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)^2$$

Proof of the OLS Solution

Proof 2: OLS Solution's Slope Term Part 1

The problem is to find the slope term $\hat{\beta}_1$ by minimizing the *SSR*:

$$\begin{aligned}\min_{\hat{\beta}_1} \sum_{i=1}^n \hat{u}_i^2 &= \min_{\hat{\beta}_1} \sum_{i=1}^n \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)^2 \\ &= \min_{\hat{\beta}_1} \sum_{i=1}^n \left(y_i - \left(\bar{Y} - \hat{\beta}_1 \bar{X} \right) - \hat{\beta}_1 x_i \right)^2\end{aligned}$$

Proof of the OLS Solution

Proof 2: OLS Solution's Slope Term Part 1

The problem is to find the slope term $\hat{\beta}_1$ by minimizing the *SSR*:

$$\begin{aligned}\min_{\hat{\beta}_1} \sum_{i=1}^n \hat{u}_i^2 &= \min_{\hat{\beta}_1} \sum_{i=1}^n \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)^2 \\ &= \min_{\hat{\beta}_1} \sum_{i=1}^n \left(y_i - \left(\bar{Y} - \hat{\beta}_1 \bar{X} \right) - \hat{\beta}_1 x_i \right)^2 \\ &= \min_{\hat{\beta}_1} \sum_{i=1}^n \left(y_i - \bar{Y} + \hat{\beta}_1 \bar{X} - \hat{\beta}_1 x_i \right)^2\end{aligned}$$

Proof of the OLS Solution

Proof 2: OLS Solution's Slope Term Part 1

The problem is to find the slope term $\hat{\beta}_1$ by minimizing the *SSR*:

$$\begin{aligned}\min_{\hat{\beta}_1} \sum_{i=1}^n \hat{u}_i^2 &= \min_{\hat{\beta}_1} \sum_{i=1}^n \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)^2 \\ &= \min_{\hat{\beta}_1} \sum_{i=1}^n \left(y_i - \left(\bar{Y} - \hat{\beta}_1 \bar{X} \right) - \hat{\beta}_1 x_i \right)^2 \\ &= \min_{\hat{\beta}_1} \sum_{i=1}^n \left(y_i - \bar{Y} + \hat{\beta}_1 \bar{X} - \hat{\beta}_1 x_i \right)^2 \\ &= \min_{\hat{\beta}_1} \sum_{i=1}^n \left(y_i - \bar{Y} - \hat{\beta}_1 (x_i - \bar{X}) \right)^2.\end{aligned}$$

Proof of the OLS Solution

Proof 2: OLS Solution's Slope Term Part 2

Now we take the derivative of the *SSR* with respect to $\hat{\beta}_1$:

$$\frac{\partial SSR}{\partial \hat{\beta}_1} = -2 \sum_{i=1}^n \left(y_i - \bar{Y} - \hat{\beta}_1 (x_i - \bar{X}) \right) (x_i - \bar{X}) .$$

Proof of the OLS Solution

Proof 2: OLS Solution's Slope Term Part 3

Setting this derivative equal to zero gives

$$-2 \sum_{i=1}^n \left(y_i - \bar{Y} - \hat{\beta}_1 (x_i - \bar{X}) \right) (x_i - \bar{X}) = 0$$

Proof of the OLS Solution

Proof 2: OLS Solution's Slope Term Part 3

Setting this derivative equal to zero gives

$$\begin{aligned} -2 \sum_{i=1}^n \left(y_i - \bar{Y} - \hat{\beta}_1 (x_i - \bar{X}) \right) (x_i - \bar{X}) &= 0 \\ \Leftrightarrow \sum_{i=1}^n \left(y_i - \bar{Y} - \hat{\beta}_1 (x_i - \bar{X}) \right) (x_i - \bar{X}) &= 0 \end{aligned}$$

Proof of the OLS Solution

Proof 2: OLS Solution's Slope Term Part 3

Setting this derivative equal to zero gives

$$-2 \sum_{i=1}^n \left(y_i - \bar{Y} - \hat{\beta}_1 (x_i - \bar{X}) \right) (x_i - \bar{X}) = 0$$

$$\iff \sum_{i=1}^n \left(y_i - \bar{Y} - \hat{\beta}_1 (x_i - \bar{X}) \right) (x_i - \bar{X}) = 0$$

$$\iff \sum_{i=1}^n \left((y_i - \bar{Y}) (x_i - \bar{X}) - \hat{\beta}_1 (x_i - \bar{X}) (x_i - \bar{X}) \right) = 0.$$

Proof of the OLS Solution

Proof 2: OLS Solution's Slope Term Part 4

Some algebra gives

$$\sum_{i=1}^n \left((y_i - \bar{Y}) (x_i - \bar{X}) - \hat{\beta}_1 (x_i - \bar{X}) (x_i - \bar{X}) \right) = 0$$

Proof of the OLS Solution

Proof 2: OLS Solution's Slope Term Part 4

Some algebra gives

$$\sum_{i=1}^n \left((y_i - \bar{Y}) (x_i - \bar{X}) - \hat{\beta}_1 (x_i - \bar{X}) (x_i - \bar{X}) \right) = 0$$
$$\iff \sum_{i=1}^n (y_i - \bar{Y}) (x_i - \bar{X}) - \sum_{i=1}^n \hat{\beta}_1 (x_i - \bar{X})^2 = 0$$

Proof of the OLS Solution

Proof 2: OLS Solution's Slope Term Part 4

Some algebra gives

$$\sum_{i=1}^n \left((y_i - \bar{Y}) (x_i - \bar{X}) - \hat{\beta}_1 (x_i - \bar{X}) (x_i - \bar{X}) \right) = 0$$

$$\iff \sum_{i=1}^n (y_i - \bar{Y}) (x_i - \bar{X}) - \sum_{i=1}^n \hat{\beta}_1 (x_i - \bar{X})^2 = 0$$

$$\iff \hat{\beta}_1 \sum_{i=1}^n (x_i - \bar{X})^2 = \sum_{i=1}^n (y_i - \bar{Y}) (x_i - \bar{X}).$$

Proof of the OLS Solution

Proof 2: OLS Solution's Slope Term Part 5

Solving for $\hat{\beta}_1$ yields

$$\begin{aligned}\hat{\beta}_1 \sum_{i=1}^n (x_i - \bar{X})^2 &= \sum_{i=1}^n (y_i - \bar{Y}) (x_i - \bar{X}) \\ \iff \hat{\beta}_1 &= \frac{\sum_{i=1}^n (y_i - \bar{Y}) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2}.\end{aligned}$$

Proof of the OLS Solution

Proof 2: OLS Solution's Slope Term Part 6

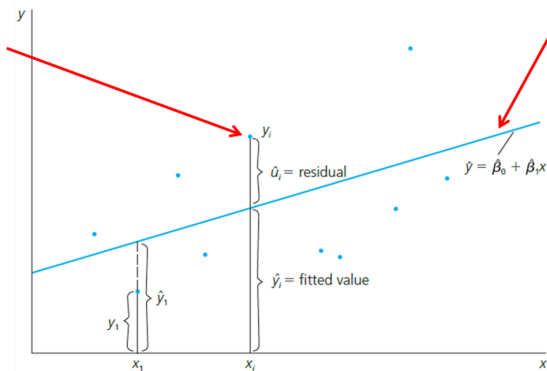
Lastly, multiplying and dividing by $\frac{1}{n-1}$ doesn't change anything. So,

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum_{i=1}^n (y_i - \bar{Y}) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \\&= \frac{\frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{Y}) (x_i - \bar{X})}{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{X})^2} \\&= \frac{\hat{\sigma}_{xy}}{\hat{\sigma}_x^2}. \quad \square\end{aligned}$$

Hurray!

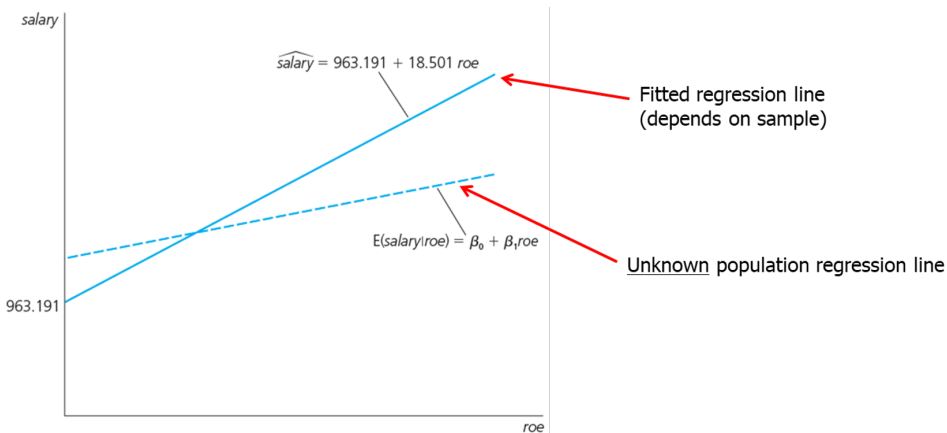
Visualization of What We Are Doing

For example, the i -th data point (x_i, y_i)



Fitted regression line

Visualization of What We Are Doing



Properties of OLS

Property 1: OLS Residuals Sum to Zero

Recall from the F.O.C. for $\hat{\beta}_0$ from Part 3 of the proof that

$$\sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = \sum_{i=1}^n \hat{u}_i = 0.$$

- This directly implies the mean of the residuals $\bar{U} = \frac{1}{n} \sum_{i=1}^n \hat{u}_i = 0$.
- If we don't included an intercept, then the residuals don't necessary sum to zero.

Properties of OLS

Property 2: Zero Sample Covariance Between Residuals and Regressor

Without first substituting in $\hat{\beta}_0$ for the F.O.C. for $\hat{\beta}_1$, we could've written it as

$$\sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) x_i = \sum_{i=1}^n \hat{u}_i x_i = 0.$$

Properties of OLS

Property 3: Zero Sample Covariance Between Residuals and Regressor

Without first substituting in $\hat{\beta}_0$ for the F.O.C. for $\hat{\beta}_1$, we could've written it as

$$\sum_{i=1}^n \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right) x_i = \sum_{i=1}^n \hat{u}_i x_i = 0.$$

Thus,

$$\begin{aligned} \hat{\sigma}_{x\hat{u}} &= \frac{1}{n-1} \sum_{i=1}^n (\hat{u}_i - \bar{U}) (x_i - \bar{X}) \\ &= \frac{1}{n-1} \sum_{i=1}^n \hat{u}_i x_i - \frac{\bar{X}}{n-1} \sum_{i=1}^n \hat{u}_i - \frac{\bar{U}}{n-1} \sum_{i=1}^n x_i + \frac{n\bar{X}\bar{U}}{n-1} \\ &= 0. \end{aligned}$$

Properties of OLS

Property 4: Sample Means of Dependent Variable and Regressor Lie on Regression Line

Recall $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$. So,

$$\bar{Y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{X}.$$

- If we try to predict y_i with \bar{X} , the prediction will be \bar{Y} .

Total Sum of Squares (SST)

Definition 5: Total Sum of Squares (SST)

The **total sum of squares** (SST) is defined as

$$SST = \sum_{i=1}^n (y_i - \bar{Y})^2.$$

- Measures how much the observed outcome varies with respect to its mean.

Explained Sum of Squares (SSE)

Definition 6: Explained Sum of Squares (SSE)

The **explained sum of squares** (SSE) is defined as

$$SSE = \sum_{i=1}^n (\hat{y}_i - \bar{Y})^2 .$$

- Measures how much the predicted outcome varies with respect to the mean of the observe outcome.

$$SST = SSE + SSR$$

Property 5: $SST = SSE + SSR$

We can write the SST in terms of the SSE and SSR as

$$SST = SSE + SSR.$$

R-squared

Definition 7: R-squared

The **R-squared** measures how well our regressor explains our outcome and is defined as

$$R^2 = \frac{SSE}{SST} = 1 - \frac{SSR}{SST}.$$

- R^2 lies between zero and one.
- Higher R^2 suggests better model fit.
- R^2 is overrated as it always increases as the number. covariates in model increases.

Adjusted R-squared

Definition 8: Adjusted R-squared

The **adjusted R-squared** is a modified version of the R-square defined as

$$\tilde{R}^2 = 1 - \frac{SSR/(n-k)}{SST/(n-1)} = 1 - \frac{(1-R^2)(n-1)}{n-k}$$

where k is the number of estimated parameters (including the intercept).

- Penalization for adding more regressors.
- Does not lie between zero and one.
 - ▶ Can be negative when a horizontal regression line (such as at \bar{Y}) predicts y better.

Regression with Logs

Property 6: Level-Level Regression Model

A **level-level** simple linear regression model is given by

$$y_i = \beta_0 + \beta_1 x_i + u_i.$$

It follows that

$$\frac{dy}{dx} = \beta_1.$$

- When x changes by 1 unit, y changes by β_1 units.
- When x changes by 3 units, y changes by $3 * \beta_1$ units

Regression with Logs

Property 7: Log-Level Regression Model

A **log-level** simple linear regression model is given by

$$\ln(y_i) = \beta_0 + \beta_1 x_i + u_i.$$

It follows that

$$\frac{d \ln(y)}{dx} = \beta_1.$$

- This represents a **semi-elasticity** of y with respect to x .
- When x changes by 1 unit, y changes by approximately $100 * \beta_1$ percent.
- When x changes by 5 units, y changes by approximately $100 * 5 * \beta_1$ percent.

Regression with Logs

Proof 3: Property 7 Part 1

If $\ln(y) = \beta_0 + \beta_1 x + u$, then

$$\frac{d \ln(y)}{dx} = \beta_1.$$

Note that $\ln(y)$ is a function of x . Let $u = \ln(y)$. By the chain rule

$$\frac{d \ln(y)}{dx} = \frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx}$$

Setting these equations equal to each other we see

$$\frac{1}{y} \frac{dy}{dx} = \beta_1 \iff \frac{dy}{y} = \beta_1 dx.$$

Regression with Logs

Proof 3: Property 7 Part 2

$\frac{dy}{y}$ represents the relative or proportional change in y telling us how much y changes relative to its current value. Think of dy as $y_1 - y$ then $\frac{dy}{y}$ is the exact formula for the proportional change in y . dx represents a small change in x . So, given a small change in x (dx), the proportional change in y ($\frac{dy}{y}$) is β_1 . To get the percentage change from the proportional change, we just multiply both sides of the equation by 100. This gives

$$100 \frac{dy}{y} = 100\beta_1 dx.$$

Thus, the percentage change in y ($100 \frac{dy}{y}$) given a small change in x (dx) is given by $100\beta_1$.

Regression with Logs

Property 8: Log-Log Regression Model

A log-log simple linear regression model is given by

$$\ln(y_i) = \beta_0 + \beta_1 \ln(x_i) + u_i.$$

It follows that

$$\frac{d \ln(y)}{d \ln(x)} = \beta_1.$$

- This represents an elasticity of y with respect to x .
- When x changes by 1 percent, y changes by approximately β_1 percent.
- When x changes by 5 percent, y changes by approximately $5\beta_1$ percent.

The Meaning of “Linear” in Linear Regression

Property 9: The Meaning of “Linear” in Linear Regression

When we estimate a linear regression model, we mean **linear in parameters**, not necessarily linear in covariates.

- The model $y_i = \beta_0 + \beta_1\sqrt{x_i} + u_i$ is linear in parameters so we are fine.
- The model $y_i = \beta_0 + \sqrt{\beta_1}x_i + u_i$ is not linear in parameters and we cannot cast it as a linear regression model.

The Meaning of “Linear” in Linear Regression

Question 3

Is the model $y_i = e^{\beta_0} x_i^{\beta_1} e^{u_i}$ **linear in parameters**? Could we cast it as a linear regression problem?

The Meaning of “Linear” in Linear Regression

Question 3

Is the model $y_i = e^{\beta_0} x_i^{\beta_1} e^{u_i}$ **linear in parameters**? Could we cast it as a linear regression problem?

Answer to Question 3

As it stands, it is not **linear in parameters** because y_i is a non-linear function of the parameters. However, we could transform it into a linear regression problem by taking the log of both sides of the equation giving

$$\ln(y_i) = \beta_0 + \beta_1 \ln(x_i) + u_i$$

which is a log-log model we can estimate via OLS.

OLS Does Not Necessarily Give us Causality

Property 10: OLS Does Not Necessarily Give us Causality

While OLS will generate the best possible line by minimizing the *SSR*, this does *not* mean $\hat{\beta}_1$ measures a **causal** relationship. To identify the **causal** relationship between y and x , we need

$$\mathbb{E} [\hat{\beta}_1] = \beta_1.$$

- This means $\hat{\beta}_1$ is unbiased for β_1 .
- We need the following four assumptions to be able to conclude $\hat{\beta}_1$ is unbiased for β_1 .

Linear in Parameters

SLR Assumption 1: Linear in Parameters

The population model is a **linear function of the parameters**.

- For instance, $y_i = \beta_0 + \beta_1 x_i + u_i$.

Random Sampling

SLR Assumption 2: Random Sampling

We have a **random (i.i.d.) sample** $\{(y_i, x_i)\}_{i=1}^n$ from the population of interest.

- This will ensure Assumption 4 holds for the entire sample and not just subsets.

Non-Zero Regressor Variance

SLR Assumption 3: Non-Zero Regressor Variance

The sample outcomes of our regressor, namely $\{x_i\}_{i=1}^n$, are **not all the same value**, i.e., $\hat{\sigma}_x^2 \neq 0$.

- Easy to verify by loading data into software and calculating the sample variance of $\{x_i\}_{i=1}^n$.

Zero Conditional Mean

SLR Assumption 4: Zero Conditional Mean

The expectation of the error term conditioned on the regressor is zero, i.e., $\mathbb{E}[u_i | x_i] = 0$ for each $i = 1, \dots, n$.

- Also called the **exogeneity assumption**
- Important implications include:
 1. $\mathbb{E}[u_i] = \mathbb{E}[\mathbb{E}[u_i | x_i]] = \mathbb{E}[0] = 0$.
 2. $\text{Cov}[u_i, x_i] = \mathbb{E}[u_i x_i] - \mathbb{E}[u_i]\mathbb{E}[x_i] = \mathbb{E}[u_i x_i] = \mathbb{E}[\mathbb{E}[u_i x_i | x_i]] = \mathbb{E}[x_i \mathbb{E}[u_i | x_i]] = 0$.
 3. $\mathbb{E}[y_i | x_i] = \mathbb{E}[\beta_0 + \beta_1 x_i + u_i | x_i] = \beta_0 + \beta_1 x_i$ so β_1 represents the average impact of x_i on y_i .
- ▶ Sample analogs of 1. and 2. are justified by Properties 1 and 2.

Unbiasedness of OLS

Theorem 2: Unbiasedness of OLS

Under SLR Assumptions 1-4, the OLS estimator is unbiased, i.e.,

$$\mathbb{E} \left[\hat{\beta}_0 \mid x_i \right] = \beta_0.$$

$$\mathbb{E} \left[\hat{\beta}_1 \mid x_i \right] = \beta_1.$$

- This means that on average if we take many samples and compute $\hat{\beta}_0$ and $\hat{\beta}_1$ we will get β_0 and β_1 .
 - ▶ The sampling distribution of our estimated parameters is centered around their true values.
- By the law of total expectation, $\mathbb{E} \left[\hat{\beta}_0 \right] = \beta_0$ and $\mathbb{E} \left[\hat{\beta}_1 \right] = \beta_1$.

Proof of the Unbiasedness of OLS

Proof 4: Unbiasedness of the OLS Intercept Term Part 1

$$\mathbb{E} \left[\hat{\beta}_0 \mid x_i \right] = \mathbb{E} \left[\bar{Y} - \hat{\beta}_1 \bar{X} \mid x_i \right]$$

.

Proof of the Unbiasedness of OLS

Proof 4: Unbiasedness of the OLS Intercept Term Part 1

$$\begin{aligned}\mathbb{E} \left[\hat{\beta}_0 \mid x_i \right] &= \mathbb{E} \left[\bar{Y} - \hat{\beta}_1 \bar{X} \mid x_i \right] \\ &= \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n y_i - \hat{\beta}_1 \bar{X} \mid x_i \right]\end{aligned}$$

.

Proof of the Unbiasedness of OLS

Proof 4: Unbiasedness of the OLS Intercept Term Part 1

$$\begin{aligned}\mathbb{E} \left[\hat{\beta}_0 \mid x_i \right] &= \mathbb{E} \left[\bar{Y} - \hat{\beta}_1 \bar{X} \mid x_i \right] \\ &= \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n y_i - \hat{\beta}_1 \bar{X} \mid x_i \right] \\ &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[y_i \mid x_i] - \mathbb{E} \left[\hat{\beta}_1 \bar{X} \mid x_i \right]\end{aligned}$$

.

Proof of the Unbiasedness of OLS

Proof 4: Unbiasedness of the OLS Intercept Term Part 1

$$\begin{aligned}\mathbb{E}[\hat{\beta}_0 | x_i] &= \mathbb{E}[\bar{Y} - \hat{\beta}_1 \bar{X} | x_i] \\&= \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n y_i - \hat{\beta}_1 \bar{X} | x_i\right] \\&= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[y_i | x_i] - \mathbb{E}[\hat{\beta}_1 \bar{X} | x_i] \\&= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\beta_0 + \beta_1 x_i | x_i] - \beta_1 \mathbb{E}[\bar{X} | x_i]\end{aligned}$$

.

Proof of the Unbiasedness of OLS

Proof 4: Unbiasedness of the OLS Intercept Term Part 1

$$\begin{aligned}\mathbb{E} \left[\hat{\beta}_0 \mid x_i \right] &= \mathbb{E} \left[\bar{Y} - \hat{\beta}_1 \bar{X} \mid x_i \right] \\&= \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n y_i - \hat{\beta}_1 \bar{X} \mid x_i \right] \\&= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[y_i \mid x_i] - \mathbb{E} \left[\hat{\beta}_1 \bar{X} \mid x_i \right] \\&= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\beta_0 + \beta_1 x_i \mid x_i] - \beta_1 \mathbb{E} \left[\bar{X} \mid x_i \right] \\&= \frac{1}{n} \sum_{i=1}^n (\beta_0 + \beta_1 x_i) - \beta_1 \bar{X}.\end{aligned}$$

Proof of the Unbiasedness of OLS

Proof 4: Unbiasedness of the OLS Intercept Term Part 2

$$\frac{1}{n} \sum_{i=1}^n (\beta_0 + \beta_1 x_i) - \beta_1 \bar{X} = \frac{1}{n} \sum_{i=1}^n \beta_0 + \beta_1 \frac{1}{n} \sum_{i=1}^n x_i - \beta_1 \bar{X}$$

Proof of the Unbiasedness of OLS

Proof 4: Unbiasedness of the OLS Intercept Term Part 2

$$\begin{aligned}\frac{1}{n} \sum_{i=1}^n (\beta_0 + \beta_1 x_i) - \beta_1 \bar{X} &= \frac{1}{n} \sum_{i=1}^n \beta_0 + \beta_1 \frac{1}{n} \sum_{i=1}^n x_i - \beta_1 \bar{X} \\ &= \beta_0 + \beta_1 \bar{X} - \beta_1 \bar{X}\end{aligned}$$

Proof of the Unbiasedness of OLS

Proof 4: Unbiasedness of the OLS Intercept Term Part 2

$$\begin{aligned}\frac{1}{n} \sum_{i=1}^n (\beta_0 + \beta_1 x_i) - \beta_1 \bar{X} &= \frac{1}{n} \sum_{i=1}^n \beta_0 + \beta_1 \frac{1}{n} \sum_{i=1}^n x_i - \beta_1 \bar{X} \\ &= \beta_0 + \beta_1 \bar{X} - \beta_1 \bar{X} \\ &= \beta_0. \quad \square\end{aligned}$$

Hooray!

Proof of the Unbiasedness of OLS

Proof 5: Unbiasedness of the OLS Slope Term Part 1

Using SLR Assumption 1 of $y_i = \beta_0 + \beta_1 x_i + u_i$ and Assumption

3 of $\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{X})^2 > 0$,

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (y_i - \bar{Y}) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2}$$

Proof of the Unbiasedness of OLS

Proof 5: Unbiasedness of the OLS Slope Term Part 1

Using SLR Assumption 1 of $y_i = \beta_0 + \beta_1 x_i + u_i$ and Assumption

3 of $\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{X})^2 > 0$,

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum_{i=1}^n (y_i - \bar{Y}) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \\ &= \frac{\sum_{i=1}^n (\beta_0 + \beta_1 x_i + u_i - \bar{Y}) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2}\end{aligned}$$

Proof of the Unbiasedness of OLS

Proof 5: Unbiasedness of the OLS Slope Term Part 1

Using SLR Assumption 1 of $y_i = \beta_0 + \beta_1 x_i + u_i$ and Assumption

3 of $\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{X})^2 > 0$,

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum_{i=1}^n (y_i - \bar{Y}) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \\&= \frac{\sum_{i=1}^n (\beta_0 + \beta_1 x_i + u_i - \bar{Y}) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \\&= \frac{\sum_{i=1}^n (\beta_0 + \beta_1 x_i + u_i - \hat{\beta}_0 - \hat{\beta}_1 \bar{X}) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2}.\end{aligned}$$

Proof of the Unbiasedness of OLS

Proof 5: Unbiasedness of the OLS Slope Term Part 2

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n \left(\beta_0 + \beta_1 x_i + u_i - \hat{\beta}_0 - \hat{\beta}_1 \bar{X} \right) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2}$$

Proof of the Unbiasedness of OLS

Proof 5: Unbiasedness of the OLS Slope Term Part 2

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum_{i=1}^n \left(\beta_0 + \beta_1 x_i + u_i - \hat{\beta}_0 - \hat{\beta}_1 \bar{X} \right) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \\ &= \frac{\sum_{i=1}^n \left((\beta_0 - \hat{\beta}_0) + \beta_1 (x_i - \bar{X}) + u_i \right) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2}\end{aligned}$$

Proof of the Unbiasedness of OLS

Proof 5: Unbiasedness of the OLS Slope Term Part 2

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum_{i=1}^n (\beta_0 + \beta_1 x_i + u_i - \hat{\beta}_0 - \hat{\beta}_1 \bar{X}) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \\&= \frac{\sum_{i=1}^n \left((\beta_0 - \hat{\beta}_0) + \beta_1 (x_i - \bar{X}) + u_i \right) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \\&= \frac{(\beta_0 - \hat{\beta}_0) \sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} + \frac{\beta_1 \sum_{i=1}^n (x_i - \bar{X})^2}{\sum_{i=1}^n (x_i - \bar{X})^2} + \frac{\sum_{i=1}^n (x_i - \bar{X}) u_i}{\sum_{i=1}^n (x_i - \bar{X})^2}\end{aligned}$$

Proof of the Unbiasedness of OLS

Proof 5: Unbiasedness of the OLS Slope Term Part 2

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum_{i=1}^n (\beta_0 + \beta_1 x_i + u_i - \hat{\beta}_0 - \hat{\beta}_1 \bar{X}) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \\&= \frac{\sum_{i=1}^n \left((\beta_0 - \hat{\beta}_0) + \beta_1 (x_i - \bar{X}) + u_i \right) (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \\&= \frac{(\beta_0 - \hat{\beta}_0) \sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} + \frac{\beta_1 \sum_{i=1}^n (x_i - \bar{X})^2}{\sum_{i=1}^n (x_i - \bar{X})^2} + \frac{\sum_{i=1}^n (x_i - \bar{X}) u_i}{\sum_{i=1}^n (x_i - \bar{X})^2} \\&= \frac{(\beta_0 - \hat{\beta}_0) \sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} + \beta_1 + \frac{\sum_{i=1}^n (x_i - \bar{X}) u_i}{\sum_{i=1}^n (x_i - \bar{X})^2}.\end{aligned}$$

Proof of the Unbiasedness of OLS

Proof 5: Unbiasedness of the OLS Slope Term Part 3

Now, using SLR Assumption 2 of random sampling, SLR Assumption 4 of exogeneity, and unbiasedness of $\hat{\beta}_0$,

$$\mathbb{E} [\hat{\beta}_1 | x_i] = \mathbb{E} \left[\frac{(\beta_0 - \hat{\beta}_0) \sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} | x_i \right] + \mathbb{E} [\beta_1 | x_i] + \mathbb{E} \left[\frac{\sum_{i=1}^n u_i (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} | x_i \right]$$

Proof of the Unbiasedness of OLS

Proof 5: Unbiasedness of the OLS Slope Term Part 3

Now, using SLR Assumption 2 of random sampling, SLR Assumption 4 of exogeneity, and unbiasedness of $\hat{\beta}_0$,

$$\begin{aligned}\mathbb{E}[\hat{\beta}_1 | x_i] &= \mathbb{E}\left[\frac{(\beta_0 - \hat{\beta}_0) \sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \mid x_i\right] + \mathbb{E}[\beta_1 | x_i] + \mathbb{E}\left[\frac{\sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2}\right] \\ &= \frac{(\beta_0 - \mathbb{E}[\hat{\beta}_0 | x_i]) \sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} + \beta_1 + \frac{\sum_{i=1}^n (x_i - \bar{X}) \mathbb{E}[\hat{\beta}_0 | x_i]}{\sum_{i=1}^n (x_i - \bar{X})^2}\end{aligned}$$

Proof of the Unbiasedness of OLS

Proof 5: Unbiasedness of the OLS Slope Term Part 3

Now, using SLR Assumption 2 of random sampling, SLR Assumption 4 of exogeneity, and unbiasedness of $\hat{\beta}_0$,

$$\begin{aligned}
 \mathbb{E} [\hat{\beta}_1 | x_i] &= \mathbb{E} \left[\frac{(\beta_0 - \hat{\beta}_0) \sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \mid x_i \right] + \mathbb{E} [\beta_1 | x_i] + \mathbb{E} \left[\frac{\sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \right] \\
 &= \frac{(\beta_0 - \mathbb{E} [\hat{\beta}_0 | x_i]) \sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} + \beta_1 + \frac{\sum_{i=1}^n (x_i - \bar{X}) \mathbb{E} [\hat{\beta}_0 | x_i]}{\sum_{i=1}^n (x_i - \bar{X})^2} \\
 &= \frac{(\beta_0 - \beta_0) \sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} + \beta_1 + \frac{\sum_{i=1}^n (x_i - \bar{X}) 0}{\sum_{i=1}^n (x_i - \bar{X})^2}
 \end{aligned}$$

Proof of the Unbiasedness of OLS

Proof 5: Unbiasedness of the OLS Slope Term Part 3

Now, using SLR Assumption 2 of random sampling, SLR Assumption 4 of exogeneity, and unbiasedness of $\hat{\beta}_0$,

$$\begin{aligned}
 \mathbb{E} [\hat{\beta}_1 | x_i] &= \mathbb{E} \left[\frac{(\beta_0 - \hat{\beta}_0) \sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \mid x_i \right] + \mathbb{E} [\beta_1 \mid x_i] + \mathbb{E} \left[\frac{\sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} \right] \\
 &= \frac{(\beta_0 - \mathbb{E} [\hat{\beta}_0 \mid x_i]) \sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} + \beta_1 + \frac{\sum_{i=1}^n (x_i - \bar{X}) \mathbb{E} [\frac{1}{\sum_{i=1}^n (x_i - \bar{X})^2}]}{\sum_{i=1}^n (x_i - \bar{X})^2} \\
 &= \frac{(\beta_0 - \beta_0) \sum_{i=1}^n (x_i - \bar{X})}{\sum_{i=1}^n (x_i - \bar{X})^2} + \beta_1 + \frac{\sum_{i=1}^n (x_i - \bar{X}) 0}{\sum_{i=1}^n (x_i - \bar{X})^2} \\
 &= \beta_1. \quad \square
 \end{aligned}$$

Hooray!

Variance of the Parameter Estimates

Question 4

Under SLR Assumptions 1-4 we will, on average, correctly estimate β_0 and β_1 . However, what is the variability of this estimate?

- By “on average”, we mean the sampling distribution of $\hat{\beta}_0$ and $\hat{\beta}_1$ will be centered around β_0 and β_1 , respectively.

Homoskedasticity

SLR Assumption 5: Homoskedastic Errors

Homoskedasticity states $\mathbb{V}[u_i | x_i] = \sigma^2$ for each $i = 1, \dots, n$.

- Under SLR Assumptions 1-5,
$$\mathbb{V}[u_i] = \mathbb{E}[\mathbb{V}[u_i | x_i]] + \mathbb{V}[\mathbb{E}[u_i | x_i]] = \mathbb{E}[\sigma^2] = \sigma^2.$$
- The variance of our error is constant across all observations when conditioning on our regressor.
- When this assumption fails, we say the errors are heteroskedastic.
- Really not that important since White's 1980 correction.

Variance of OLS Estimates

Theorem 3: Variance of OLS Estimates

Under SLR Assumptions 1-5, the **variance of the OLS estimators** $\hat{\beta}_0$ and $\hat{\beta}_1$ are

$$\mathbb{V}[\hat{\beta}_0 | x_i] = \frac{\sigma^2 n^{-1} \sum_{i=1}^n x_i^2}{\sum_{i=1}^n (x_i - \bar{X})^2}.$$

$$\mathbb{V}[\hat{\beta}_1 | x_i] = \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{X})^2}.$$

- As σ^2 increases, the variance of our estimators increase.
- As the *SST* of x (denominator) increases, the variance of our estimators decrease.
- As n increases the *SST* of x increases implying the variance of our estimates decreases.

Variance of OLS Estimates

Proof 6: Variance of OLS Slope Estimate Part 1

Recall from Proof 4, Part 2 that

$$\hat{\beta}_1 - \beta_1 = \frac{\sum_{i=1}^n (x_i - \bar{X}) u_i}{\sum_{i=1}^n (x_i - \bar{X})^2}.$$

Variance of OLS Estimates

Proof 6: Variance of OLS Slope Estimate Part 1

Recall from Proof 4, Part 2 that

$$\hat{\beta}_1 - \beta_1 = \frac{\sum_{i=1}^n (x_i - \bar{X}) u_i}{\sum_{i=1}^n (x_i - \bar{X})^2}.$$

Taking the variance of both sides of this equation yields

$$\mathbb{V} \left[\hat{\beta}_1 - \beta_1 \mid x_i \right] = \mathbb{V} \left[\frac{\sum_{i=1}^n (x_i - \bar{X}) u_i}{\sum_{i=1}^n (x_i - \bar{X})^2} \mid x_i \right].$$

Variance of OLS Estimates

Proof 6: Variance of OLS Slope Estimate Part 2

Inspecting the left hand side,

$$\mathbb{V} [\hat{\beta}_1 - \beta_1 | x_i] = \mathbb{V} [\hat{\beta}_1 | x_i] + \mathbb{V} [\beta_1 | x_i] - 2\text{Cov} [\hat{\beta}_1, \beta_1 | x_i].$$

Since we treat β_1 as a constant,

$$\mathbb{V} [\hat{\beta}_1 - \beta_1 | x_i] = \mathbb{V} [\hat{\beta}_1 | x_i].$$

Variance of OLS Estimates

Proof 6: Variance of OLS Slope Term Estimate Part 3

Inspecting the right hand side from Part 1,

$$\mathbb{V} \left[\frac{\sum_{i=1}^n (x_i - \bar{X}) u_i}{\sum_{i=1}^n (x_i - \bar{X})^2} \middle| x_i \right] = \frac{[\sum_{i=1}^n (x_i - \bar{X})]^2 \mathbb{V}[u_i | x_i]}{[\sum_{i=1}^n (x_i - \bar{X})^2]^2}$$

Variance of OLS Estimates

Proof 6: Variance of OLS Slope Term Estimate Part 3

Inspecting the right hand side from Part 1,

$$\begin{aligned}\mathbb{V} \left[\frac{\sum_{i=1}^n (x_i - \bar{X}) u_i}{\sum_{i=1}^n (x_i - \bar{X})^2} \middle| x_i \right] &= \frac{[\sum_{i=1}^n (x_i - \bar{X})]^2 \mathbb{V}[u_i | x_i]}{[\sum_{i=1}^n (x_i - \bar{X})^2]^2} \\ &= \frac{[\sum_{i=1}^n (x_i - \bar{X})]^2 \sigma^2}{[\sum_{i=1}^n (x_i - \bar{X})^2]^2}\end{aligned}$$

Variance of OLS Estimates

Proof 6: Variance of OLS Slope Term Estimate Part 3

Inspecting the right hand side from Part 1,

$$\begin{aligned}\mathbb{V} \left[\frac{\sum_{i=1}^n (x_i - \bar{X}) u_i}{\sum_{i=1}^n (x_i - \bar{X})^2} \middle| x_i \right] &= \frac{[\sum_{i=1}^n (x_i - \bar{X})]^2 \mathbb{V}[u_i | x_i]}{[\sum_{i=1}^n (x_i - \bar{X})^2]^2} \\ &= \frac{[\sum_{i=1}^n (x_i - \bar{X})]^2 \sigma^2}{[\sum_{i=1}^n (x_i - \bar{X})^2]^2} \\ &= \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{X})^2}.\end{aligned}$$

Variance of OLS Estimates

Proof 6: Variance of OLS Slope Term Estimate Part 4

Putting the right and left hand sides together, we have

$$\mathbb{V} \left[\hat{\beta}_1 \mid x_i \right] = \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{X})^2}.$$

Hooray!

- By the law of total variance, $\mathbb{V} \left[\hat{\beta}_1 \mid x_i \right] = \mathbb{V} \left[\hat{\beta}_1 \right]$.
- This is the true conditional variance of our estimator $\hat{\beta}_1$.
 - ▶ How do we obtain the estimate of this conditional variance?

Conditional Variance of Error Term

Property 11: Conditional Variance of Error Term

Under SLR Assumptions 1-5, the conditional variance of u_i for each $i = 1, \dots, n$ can be written as

$$\begin{aligned}\sigma^2 &= \mathbb{V}[u_i \mid x_i] \\ &= \mathbb{E}[u_i^2 \mid x_i] - \mathbb{E}[u_i \mid x_i]^2 \\ &= \mathbb{E}[u_i^2 \mid x_i].\end{aligned}$$

- If we had data on u_i , an unbiased estimate of σ^2 would then be

$$\frac{1}{n} \sum_{i=1}^n u_i^2.$$

- ▶ We only have data on \hat{u}_i though. What do we do?

Unbiased Estimator of the Variance of the Error Term

Theorem 4: Unbiased Estimator of the Variance of the Error Term

An unbiased estimator of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{n - k} \sum_{i=1}^n \hat{u}_i^2 = \frac{SSR}{n - k}.$$

where k is the number of estimated parameters (including intercept).

- In the simple linear regression model, $k = 2$.
- The $n - k$ term is called the *degree of freedom correction*.
- $\mathbb{E}[SSR] = (n - k)\sigma^2$.

Estimator of the Variance of the OLS Slope Term Estimator

Theorem 5: Estimator of the Variance of the OLS Slope Term Estimator

Under SLR Assumptions 1-5, an unbiased estimator of $\mathbb{V}[\hat{\beta}_1 | x_i]$ is

$$\hat{\mathbb{V}}[\hat{\beta}_1 | x_i] = \frac{\hat{\sigma}^2}{\sum_{i=1}^n (x_i - \bar{X})^2}.$$

Standard Errors

Definition 9: Standard Errors of OLS Estimates

The **standard errors** of our OLS estimators are

$$\begin{aligned}\text{se} \left[\hat{\beta}_0 \mid x_i \right] &= \sqrt{\hat{\mathbb{V}} \left[\hat{\beta}_0 \mid x_i \right]} \\ \text{se} \left[\hat{\beta}_1 \mid x_i \right] &= \sqrt{\hat{\mathbb{V}} \left[\hat{\beta}_1 \mid x_i \right]}.\end{aligned}$$

- The **standard error** of an estimator is simply its estimated standard deviation.

Thank You!