**LINAN CHEN:**   Hello. Welcome back to recitation. I'm sure you are becoming more and more familiar with the determinants of matrices. In the lecture, we also learned the geometric interpretation of the determinant. The absolute value of the determinant of a matrix is simply equal to the volume of the parallelepiped spanned by the row vectors of that matrix. So today, we're going to apply this fact to solve the following problem.

I have a tetrahedron, T, in this 3D space. And the vertices of T are given by O, which is the origin, A_1, A_2, and A_3. So I have highlighted this tetrahedron using the blue chalk. So this is T.

And our first goal is to compute the volume of T using the determinant. And the second part is: if I fix A_1 and A_2, but move A_3 to another point, A_3 prime, which is given by this coordinate, I ask you to compute the volume again. OK.

So since we want to use the fact that the determinant is related to the volume, we have to figure out which volume we should be looking at. We know that the determinant is related to the volume of a parallelepiped. But here, we only have a tetrahedron. So the first goal should be to find out which parallelepiped you should be working with.

OK, why don't you hit pause and try to work it out yourself. You can sketch the parallelepiped on this picture. And I will return in a while and continue working with you.

All right. How did your computation go? Now let's complete this picture together. As we were saying, we need a parallelepiped so that we can use the fact that the determinant is related to the volume.

Here, I have a tetrahedron. And let's look at these three edges, OA_1, OA_2, and OA_3. All of them meet at the origin. So why don't we just consider the parallelepiped spanned by those same three edges? It's a natural choice, because at least it shares three edges with T.

OK, now let's move on to this picture here. As you can see, I have drawn this parallelepiped in red chalk. So the blue part is my original T.

And the red part, let me call it P. It's the parallelepiped spanned by edges OA_1, OA_2, and OA_3. So that's the parallelepiped that I'm going to work with.

Now the next step is to relate the volume of T to the volume of P. OK, let's recall together what is the volume of a tetrahedron. We note that the volume of a tetrahedron is going to be equal to 1/3 of the area of the base times the height, right?

Of course, you can choose any side, any face to be the base. But for convenience, we're going to choose the triangle $OA_1A_2$ to be the base of T. So the volume of T is going to be equal to 1/3 times the area of triangle $OA_1A_2$.

So I use this A to indicate the area. Then times the height. Well, if I choose this to be my base, then this $A_3$ becomes the apex.

Then the height is simply equal to the distance from $A_3$ to the triangle $OA_1A_2$. Let me use letter h to denote this quantity. So that's the height.

And the volume of T is equal to 1/3 of the product of the area of the base times the height. OK, this is the volume of T. Now let's see what the volume of P is.

So P is a parallelepiped. The volume of a parallelepiped is simply equal to the area of the base times its height. This time, which face would you choose to be the base? Well, of course, you would like to choose this parallelogram to be the base because it contains the base of T.

If we do so, so we want to choose this parallelogram to be the base, then what is the area of this? Well, it clearly contains two copies of the triangle $OA_1A_2$. So the area of the parallelogram is simply equal to twice the area of the triangle $OA_1A_2$.

Then what is the height of P? Again, if you choose this face to be the base, then $A_3$ becomes the apex again. Then the height of the parallelepiped is equal to the distance from $A_3$ to the base, which is the same as the distance from $A_3$ to triangle $OA_1A_2$. So here, the height is also equal to h.

Now you can compare these two formulae. You see that volume of T is simply equal to 1/6 of volume of P. That's the connection between the volume of the tetrahedron with the volume of the parallelepiped. In order to compute the volume of T, we only need to compute the volume of P.

Now let's compute the volume of this parallelepiped. We know that it's related to the determinant of a matrix. And the row vectors of that matrix are given by these three edges. So because all of them start from 0, we only need the coordinate of $A_1$, $A_2$, and $A_3$.

So here, volume of P is equal to the absolute value. So don't forget the absolute value sign. Absolute value of the determinant of a three by three matrix.

So we just need to copy the coordinates of the vertices down here. The first one is [2, 2, -1]. These two are too close. -1. And $A_2$ is [1, 3, 0]. A3 is [-1, 1, 4]. The absolute value of this determinant.

And if you compute this, this is a three by three matrix. The determinant should be easy to compute. And the result should be 12. So that's the volume of P, which means the volume of the tetrahedron T is equal to 12 over 6, which is 2. Did you get the correct answer?

OK, so in order to compute the volume of T, we have related to a parallelepiped, P, which contains T. All right. Now let's look at the second part.

The second part says that if I keep $A_1$ and $A_2$ unchanged, but I move $A_3$ to a new point-- so A3 is going to be moved to a point given by $A_3$ prime, and the coordinate is -201, -199, and 104. And I'm asking you to compute the volume of the new tetrahedron.

Well as you can see, this point seems to be far away from the origin. I'm not even able to draw this point here. But you can imagine, as this point goes far away from the origin, this spike is going to become more and more pointy. In other words, the entire tetrahedron looks more and more like a needle. But nonetheless, we can use the same method to compute the volume.

So we follow the same idea, the volume of T is going to be equal to 1/6 of the volume of P. And in this case, that's going to become the absolute value of the determinant of a three by three matrix whose row vectors are given by these three edges. So in this case, we again copy down the coordinates of three vertices.

The first one is [2, 2, -1]. The second one is [1, 3, 0]. The third one should become this, so [-201, 199, 104].

OK. That should give me the volume of the new tetrahedron. Let me call it T prime just to differentiate it from the previous tetrahedron. And of course, you can compute this determinant explicitly. If you do so, you will see the answer should be 2 again.

But in fact, there is a way that you can just read out the answer directly without any real computation. Let's pay attention to the last row. In other words, let's pay attention to this new $A_3$, well, $A_3$ prime. What do you observe here?

A_3 prime, if you consider the difference between A_3 and A_3 prime, in other words, if you consider how much you have moved your apex, you will see that's equal to -100*A_1. Right? So A_3 is [-1, 1, 4]. A_3 prime is this. That's exactly equal to -100 times A_1.

What does that mean? Well from the point of view of the matrix, you are subtracting from the original third row 100 times the first row. But this row operation does not change the determinant. In other words, you know that this determinant should be the same as the previous one. So you can write out 2 directly.

From this picture, we can also see that fact. So from this picture, we know that this section is saying that I need to move A_3 in the inverse direction of A_1 by 100 times A_1. So you're moving A_3 parallel to A_1.

But it doesn't matter how far you've moved your apex. You're moving in a way that remains parallel to the base, which is saying that this movement does not change the height. Since A_1 and A_2 are fixed, your base is fixed, and you're not changing the height. So of course, the volume is not changing. That's also a reason, another way to see that the volume of T prime is simply equal to the volume of T.

OK, this completes this problem. I hope this example was helpful to you. And we should keep in mind that the fact that the determinant is related to volume sometimes can become very handy in computing the volume of certain geometric objects. Thank you for watching, and I'm looking forward to seeing you soon.