Chapter 7   <u>Taylor's formula</u>.

If f(x) has derivatives of orders 1,...,n at the point
x = a, then the polynomial function

$$T_n(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n,$$

where

$$a_m = \frac{f^{(m)}(a)}{m!}$$

for each m, is called the $n\underline{th}$ order <u>Taylor</u> <u>approximation</u> <u>to</u>
f <u>at</u> a. It has the crucial property that it agrees with f at
a, and that also its first n derivatives agree with those of
f at a. (See 7.1 and 7.2 of Apostol.)

In order to use this approximation for either practical
or theoretical purposes, we need to find a way to obtain an upper
bound on the difference between f and its Taylor approximation.
Let us define $E_n(x) = f(x) - T_n(x)$. Then we have the equation

$$f(x) = T_n(x) + E_n(x),$$

which is called <u>Taylor's</u> <u>formula</u>. Here $E_n(x)$ is called the
<u>remainder</u> <u>term</u>, or <u>error</u> <u>term</u>, in Taylor's formula.

There are a number of different formulas involving $E_n(x)$;
all involve (x-a) and $f^{(n+1)}(x)$ in some way. (See 7.7 of
Apostol, where four different ways of expressing $E_n(x)$ are

given.) Each of these expressions is useful in situations where the others are not much help; the study of such formulas leads to a general subject called Numerical Analysis. Here we are going to derive and use just one of these expressions, the one called the Lagrange form of the remainder.

First, we need a lemma.

Lemma 1. Suppose $h^{(n+1)}(x)$ exists on an open interval. I. (This implies that $h(x)$, $h'(x)$,..., $h^{(n)}(x)$ exist and are continuous on this interval.) Let a and b be points of this interval; suppose that
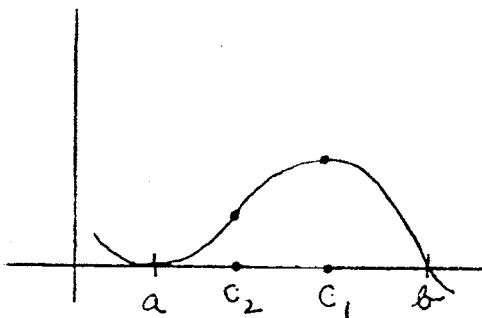
$$h(a) = h'(a) = \cdots = h^{(n)}(a) = 0,$$

and

$$h(b) = 0.$$

Then there is some point c between a and b for which

$$h^{(n+1)}(c) = 0.$$

Proof. We suppose for convenience that $a < b$. (The same proof works if $b < a$). Because $h(a) = h(b) = 0$, the mean value theorem tells us there is some point c_1 with $a < c_1 < b$ such that $h'(c_1) = 0$. Now because $h'(a) = h'(c_1) = 0$, the mean-value theorem, applied to $h'(x)$, tells us there is some point c_2 with $a < c_2 < c_1$ such that $h''(c_2) = 0$.

Similarly continue. At the nth stage, we have a point $c_n > a$ such that $h^{(n)}(c_n) = 0$. Since $h^{(n)}(a) = 0$, we can apply the mean-value theorem to $h^{(n)}(x)$ to find a point c with $a < c < c_n$ such that $h^{(n+1)}(c) = 0$. □

Theorem 2. Let $f^{(n+1)}(x)$ exist in an open interval I. Let a be a point of I. Let $T_n(x)$ be the nth order Taylor approximation to f at a; let $E_n(x) = f(x) - T_n(x)$. Then given x in I, there is a point c between a and x such that

$$E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

This expression is called the Lagrange form of the remainder.

Proof. Let b be a fixed point of the interval I different from a. We show there is a point c between a and b such that

$$E_n(b) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Since b is arbitrary, this will suffice.

Consider the function $E_n(x) = f(x) - T_n(x)$. Because the functions f and T_n and their first n derivatives agree at a, we have

$$(*) \qquad E_n(a) = E_n'(a) = \cdots = E_n^{(n)}(a) = 0.$$

Thus part of the hypothesis of the preceding lemma is satisfied. Of course, $E_n(b)$ is not zero, in general. We seek to modify E_n, by adding a suitably chosen function, so as to get a function $h(x)$ that does vanish at b, but which has the same derivatives as E_n does at a (so that condition $(*)$ still holds). A multiple of $(x-a)^{n+1}$ will do the job, for the first n derivatives of $(x-a)^{n+1}$ all vanish at a. The $(n+1)\underline{\mathrm{st}}$ derivative of $(x-a)^{n+1}$ does not vanish at a, of course. In fact, the $(n+1)\underline{\mathrm{st}}$ derivative is just the constant $(n+1)!$.

So let us define

$$h(x) = E_n(x) - A(x-a)^{n+1},$$

where we choose the constant A so that $h(b) = 0$. That is, we let $A = E_n(b)/(b-a)^{n+1}$. The hypotheses of the preceding lemma are then satisfied. We conclude there is some point c between a and b such that $h^{(n+1)}(c) = 0$.

Now we compute $h^{(n+1)}(x)$. Recall that

$$h(x) = E_n(x) - A(x-a)^{n+1} = f(x) - T_n(x) - A(x-a)^{n+1}.$$

Because $T_n(x)$ is a polynomial of degree n, its $(n+1)\underline{\mathrm{st}}$ derivative vanishes. Therefore we have

$$h^{(n+1)}(x) = f^{(n+1)}(x) - 0 - (n+1)!A$$

$$= f^{(n+1)}(x) - (n+1)!E_n(b)/(b-a)^{n+1}.$$

We substitute c for x, and recall that $h^{(n+1)}(c) = 0$. Solving for $E_n(b)$, we have

$$E_n(b) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

as desired. □

Theorem 3. If the $(n+1)\underline{st}$ derivative of f satisfies the inequalities

$$m \leqslant f^{(n+1)}(x) \leqslant M$$

for all x in some interval about a, then for all x in this interval, we have

$$\frac{m(x-a)^{n+1}}{(n+1)!} \leqslant E_n(x) \leqslant M\frac{(x-a)^{n+1}}{(n+1)!}$$

if $(x-a)^{n+1}$ is positive; otherwise, the reverse inequalities hold.

Proof. We apply the preceding theorem to calculate $E_n(x)$. Multiplying through

$$m \leqslant f^{(n+1)}(c) \leqslant M$$

by the number $(x-a)^{n+1}/(n+1)!$ gives us the desired inequalities.□

Application: Taylor's formula applied to indeterminate forms.

We can calculate limits for most familiar functions by using our basic theorems on limits, along with the continuity properties of the elementary functions. One situation where these theorems fail us is when we consider a limit of a quotient,

$$\lim_{x \to a} \frac{f(x)}{g(x)} \, ,$$

where the denominator approaches 0 as $x \longrightarrow a$. In this case, anything can happen. If $f(x)$ approaches a limit A different from zero and $g(x)$ approaches 0, then the limit of $f(x)/g(x)$ does not exist. If, however, $f(x)$ and $g(x)$ both approach zero, then the quotient may approach a finite limit. For example:

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} (x+2) = 4$$

$$\lim_{x \to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{x+2} = 0$$

$$\lim_{x \to 2} \frac{x^2-4}{(x-2)^2} = \lim_{x \to 2} \frac{x+2}{x-2} \, , \quad \text{which does not exist.}$$

Taylor's formula can sometimes be of help in computing limits, if one knows the Taylor polynomials of the functions involved. In general, if $f^{(n+1)}$ is continuous on an interval containing a, we have the formula

$$f(x) = f(a) + f'(a)(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + B(x)(x-a)^{n+1}$$

where $B(x) = f^{(n+1)}(c)/(n+1)!$. We do not know exactly what $B(x)$ is, but we do know it is <u>bounded</u> on the interval in question, because $f^{(n+1)}$ is continuous. We use the letter B to remind us it is bounded.

We have for example the formulas

$$e^x = 1 + x + x^2/2! + \ldots + x^n/n! + B(x)x^{n+1},$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + B(x)x^{n+1},$$

and so on for other elementary functions. Here is how these formulas can be used in calculating limits.

<u>Example 1</u>. Calculate the limit as $x \longrightarrow 0$ of $(\sin x)/x$. Now

$$\sin x = x - x^3/3! + Bx^5, \quad \text{so}$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + Bx^4,$$

which approaches 1 as x approaches 0. (Since B is a bounded function of x, we have $Bx^4 \longrightarrow 0$ as $x \longrightarrow 0$.)

Example 2. Calculate $\lim(\cos x-1)/x \sin x$ as $x \longrightarrow 0$.

$$\cos x = 1 - x^2/2! + Bx^4$$

$$\sin x = x - x^3/3! + Cx^5$$

$$\frac{\cos x - 1}{x \sin x} = \frac{- x^2/2! + Bx^4}{x^2 - x^4/3! + Cx^6} = \frac{- 1/2 + Bx^2}{1 - x^2/3! + Cx^4},$$

which approaches $- 1/2$ as $x \longrightarrow 0$.

Example 3. Calculate $\lim(x \cot x - 1)/x^2$ as $x \longrightarrow 0$.

$$\cot x = \frac{\cos x}{\sin x} = \frac{1 - x^2/2! + Bx^4}{x - x^3/3! + Cx^5}.$$

$$\frac{x \cot x - 1}{x^2} = \frac{(x-x^3/2!+Bx^5) - (x-x^3/3!+Cx^5)}{x^2(x-x^3/3!+Cx^5)}$$

$$= \frac{- x^3/3 + (B-C)x^5}{x^3 - x^5/3! + Cx^7}$$

$$= \frac{- 1/3 + (B-C)x^2}{1 - x^2/3! + Cx^4}$$

which approaches $- 1/3$ as $x \longrightarrow 0$.

Example 4. Calculate $\lim(\cos(ax) - \cos x)/x^2$ as $x \longrightarrow 0$.

$$\cos(ax) = 1 - (ax)^2/2 + B(ax)(ax)^4.$$

$$\cos(ax) - \cos x = -(ax)^2/2 + x^2/2 + B(ax)a^4x^4 - B(x)x^4.$$

Hence $(\cos(ax) - \cos x)/x^2 \longrightarrow (1-a^2)/2$ as $x \longrightarrow 0$, since both $B(ax)$ and $B(x)$ are bounded.

Example 5. Calculate the limit of $(\log(1+ax))/x$ as $x \longrightarrow 0$.

$$\log(1+ax) = (ax) - (ax)^2/2 + B(ax)(ax)^3;$$

so $(\log(1+ax))/x \longrightarrow a$ as $x \longrightarrow 0$.

Example 6. Show that $(1+ax)^{1/x} \longrightarrow e^a$ as $x \longrightarrow 0$. Now $(1+ax)^{1/x} = E((1/x)\log(1+ax))$. By continuity of the exponential function, it suffices to show that $(1/x)\log(1+ax) \longrightarrow a$. This is done in Example 5.

Exercises

Throughout, we use Taylor's formula with the <u>Lagrange</u> <u>form</u> of the remainder: $f(x) = T_n(x) + E_n(x)$, where

$$E_n(x) = \frac{f^{n+1}(c)}{(n+1)!} (x-a)^{n+1}.$$

1. Derive the following from Taylor's theorem:

   (a) Given x, there is a c between 0 and x such that

   $$e^x = \left[1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}\right] + \frac{e^c}{(n+1)!} x^{n+1}.$$

   (b) Given $x > -1$, there is a c between 0 and x such that

   $$\log(1+x) = \left[x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n-1} \frac{x^n}{n}\right] + \frac{(-1)^n}{(1+c)^{n+1}} \frac{x^{n+1}}{n+1}.$$

   (c) Given x, there is a c between 0 and x such that

   $$\sin x = \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right] + (-1)^{n+1} \frac{\cos c}{(2n+3)!} x^{2n+3},$$

   $$\cos x = \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!}\right] + (-1)^{n+1} \frac{\cos c}{(2n+2)!} x^{2n+2}.$$

<u>Learn</u> <u>the</u> <u>Taylor</u> <u>polynomials</u> <u>for</u> e^x, log $(1+x)$, sin x <u>and</u> cos x.

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2. (a) Use the third order Taylor approximation for e^x near

$x = 0$ and the fact that $e < 4$ to show that

$$\frac{8}{3} + \frac{1}{24} < e < \frac{8}{3} + \frac{1}{6},$$

whence it follows that $2.7 < e < 2.9$.

(b) Now use the same Taylor approximation and the fact that

$e < 3$ to show that

$$\frac{8}{3} + \frac{1}{24} < e < \frac{8}{3} + \frac{1}{8},$$

whence $2.7 < e < 2.8$.

3. Use the first and second order Taylor approximations to $\sqrt{x}$

near $a = 4$ to compute $\sqrt{3.8}$. (Actual value is 1.949359...)

4. Use the third order Taylor approximation to $\sin x$ near

$a = 0$ to compute $\sin(1/2)$. Obtain an upper bound on the

error.

5. What order Taylor polynomial should one use if one wishes to

compute e to two decimal places of accuracy (i.e.,

with an error less than .005)? Use the fact that $e < 3$.

Obtain an upper bound for the error. What about computing

$\log 2$?

6. (a) Show that the inequalities

$$x - x^3/3! \le \sin x \le x - x^3/3! + x^5/5!$$

hold for $0 \le x \le \pi/2$. [Hint: Consider the sign of the

error term.]

(b) Use these inequalities and your trusty pocket calculator to show that $\sin .523 < 1/2$ and $\sin .524 > 1/2$. (Be sure you allow for round-off error.)

(c) Since $\sin \pi/6 = 1/2$, conclude that

$$3.138 < \pi < 3.144.$$

These inequalities give the approximation $\pi \sim 3.14$ with two decimal places of accuracy.

7. Find the limit as $x \longrightarrow 0$ of the function

$$\frac{\sin x - xe^{(x^2)} + 7x^3/6}{(\sin^2 x)(\sin x^3)} \quad .$$

8. For what range of values of x can you replace $\cos x$ by $1 - x^2/2 + x^4/24$ with an error no greater than 5×10^{-4}?

9. The approximation $(1+x)^{1/3} \sim 1 + x/3$ is often used when $|x|$ is small. Find an upper bound for the error if $0 < x < .01$.

10. (a) Show that if $|x| < 1$, then

$$|e^x - (1+x+x^2/2)| < |x^3/2|.$$

(b) Show that if $|t| < 1$, then the approximation

$$\int_0^t e^{x^2}\, dx \sim t + t^3/3 + t^5/10$$

involves an error in absolute value no more than $|t^7/14|$.

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