# Solution Guide for Chapter 4: Exponential Functions

## 4.1 EXPONENTIAL GROWTH AND DECAY

E-1. **Practice with exponents**:

(a) Separating the terms with $a$ and those with $b$, and using the quotient law, we have

$$\begin{aligned}\frac{a^3b^2}{a^2b^3} &= \frac{a^3}{a^2}\times\frac{b^2}{b^3}\\ &= a^{3-2}\times b^{2-3}\\ &= a^1\times b^{-1} = \frac{a}{b}.\end{aligned}$$

This expression can also be written as $ab^{-1}$.

(b) Using the power law, we have

$$\begin{aligned}\left(\left(a^2\right)^3\right)^4 &= \left(a^{2\times 3}\right)^4\\ &= a^{(2\times 3)\times 4} = a^{24}.\end{aligned}$$

(c) Separating the terms with $a$ and those with $b$, and using the product law, we have

$$\begin{aligned}a^3b^2a^4b^{-1} &= (a^3a^4)\times(b^2b^{-1})\\ &= a^{3+4}b^{2+(-1)} = a^7b^1 = a^7b.\end{aligned}$$

E-2. **Solving exponential equations**: To solve these equations for the variable $a$, we first divide to put terms with $a$ on one side and the remaining terms on the other, after which we raise each side of the equation to $1/(\text{power of } a)$.

(a)

$$\begin{aligned}5a^3 &= 7\\ a^3 &= \frac{7}{5}\\ a &= \left(\frac{7}{5}\right)^{1/3} = 1.12.\end{aligned}$$

(b)

$$\begin{aligned}
6a^4 &= 2a^2 \\
\frac{a^4}{a^2} &= \frac{2}{6} \\
a^2 &= \frac{2}{6} \quad \textbf{using the quotient law} \\
a &= \left(\frac{2}{6}\right)^{1/2} = 0.58.
\end{aligned}$$

(c) Since $b \neq 0$, we may divide by it.

$$\begin{aligned}
a^5b^2 &= ab \\
\frac{a^5}{a} &= \frac{b}{b^2} \\
a^4 &= \frac{1}{b} \quad \textbf{using the quotient law} \\
a &= \left(\frac{1}{b}\right)^{1/4}.
\end{aligned}$$

This can also be written as a $a = b^{-1/4}$.

(d) Assuming $t \neq 1$, we may divide by $t-1$, since $t-1 \neq 0$.

$$\begin{aligned}
a^t &= ab \\
\frac{a^t}{a} &= b \\
a^{t-1} &= b \quad \textbf{using the quotient law} \\
a &= b^{1/(t-1)}.
\end{aligned}$$

E-3. **Finding exponential functions**:

(a) Since $N$ is an exponential function with growth factor 6, $N = P \times 6^t$, and we want to find $P$. Now $N(2) = 7$, so $P \times 6^2 = 7$ and therefore $P = 7/(6^2) = 0.19$. Thus $N = 0.19 \times 6^t$.

(b) Since $N$ is an exponential function, $N = Pa^t$. The two equations $N(3) = 4$ and $N(7) = 8$ tell us that

$$\begin{aligned}
4 &= Pa^3 \\
8 &= Pa^7.
\end{aligned}$$

Dividing the bottom equation by the top gives

$$\frac{8}{4} = \frac{Pa^7}{Pa^3} = \frac{a^7}{a^3} = a^4.$$

Thus $a^4 = 2$ and so $a = 2^{1/4} = 1.19$.

Returning to the equation $N(3) = 4$, we now have $P \times 1.19^3 = Pa^3 = 4$ and so $P = 4/(1.19^3) = 2.37$. Thus $N = 2.37 \times 1.19^t$.

(c) Since $N$ is an exponential function with growth factor $a$ (we are assuming that $a$ is some known number) and $N(2) = 3$, we have $Pa^2 = 3$. Solving for $P$, we have $P = 3/(a^2) = 3a^{-2}$. Thus $N = 3a^{-2}a^t$, which can also be written as $N = 3a^{t-2}$.

(d) Since $N$ is an exponential function, $N = Pa^t$. The two equations $N(2) = m$ and $N(4) = n$ tell us that

$$\begin{aligned} m &= Pa^2 \\ n &= Pa^4. \end{aligned}$$

Dividing the bottom equation by the top gives

$$\frac{n}{m} = \frac{Pa^4}{Pa^2} = \frac{a^4}{a^2} = a^2.$$

Thus $a^2 = \dfrac{n}{m}$ and so

$$a = \left(\frac{n}{m}\right)^{1/2} = \sqrt{\frac{n}{m}}.$$

Returning to the equation $N(2) = m$, we now have

$$P \times \left(\sqrt{\frac{n}{m}}\right)^2 = Pa^2 = m$$

Using the power law and dividing we get

$$P = \frac{m}{\left(\frac{n}{m}\right)} = \frac{m^2}{n}.$$

Thus

$$N = \frac{m^2}{n}\left(\sqrt{\frac{n}{m}}\right)^t.$$

E-4. **Substitution**:

(a) Substituting $y = x^2$ into $x^4 - 3x^2 + 2 = 0$ yields $y^2 - 3y + 2 = 0$. This can be factored as $(y-1)(y-2) = 0$, so the solutions are $y = 1$ and $y = 2$. Since $x^2 = y$, then $x^2 = 1$ and $x^2 = 2$. Thus the solutions are $x = \pm 1$ and $x = \pm\sqrt{2}$, which can also be written as $x = \pm 1$ and $x = \pm 1.41$.

(b) Substituting $y = x^3$ into $x^6 - 5x^3 + 6 = 0$ yields $y^2 - 5y + 6 = 0$. This can be factored as $(y-2)(y-3) = 0$, so the solutions are $y = 2$ and $y = 3$. Since $x^3 = y$, then $x^3 = 2$ and $x^3 = 3$. Thus the solutions are $x = 2^{1/3}$ and $x = 3^{1/3}$, which can also be written as $x = 1.26$ and $x = 1.44$.

(c) Substituting $y = x^5$ into $x^{10} - 7x^5 + 12 = 0$ yields $y^2 - 7y + 12 = 0$. This can be factored as $(y-3)(y-4) = 0$, so the solutions are $y = 3$ and $y = 4$. Since $x^5 = y$, then $x^5 = 3$ and $x^5 = 4$. Thus the solutions are $x = 3^{1/5}$ and $x = 4^{1/5}$, which can also be written as $x = 1.25$ and $x = 1.32$.

E-5. **Finding initial value**: Since $N$ is an exponential function with growth factor 1.94, $N = P \times 1.94^t$, and we want to find the initial value $P$. Now $N(5) = 6$, so $P \times 1.94^5 = 6$ and therefore $P = 6/(1.94^5) = 0.22$. Thus $N = 0.22 \times 1.94^t$.

E-6. **Exponential function from two points**: Since $N$ is an exponential function, $N = Pa^t$. The two equations $N(4) = 9$ and $N(7) = 22$ tell us that

$$\begin{aligned} 9 &= Pa^4 \\ 22 &= Pa^7. \end{aligned}$$

Dividing the bottom equation by the top gives

$$\frac{22}{9} = \frac{Pa^7}{Pa^4} = \frac{a^7}{a^4} = a^3.$$

Thus $a^3 = 22/9$ and so $a = (22/9)^{1/3} = 1.347$.

Returning to the equation $N(4) = 9$, we now have $P \times 1.347^4 = Pa^4 = 9$ and so $P = 9/(1.347^4) = 2.733$. Thus $N = 2.733 \times 1.347^t$.

S-1. **Function value from initial value and growth factor**: Since the growth factor is 2.4 and the initial value is $f(0) = 3$, $f(2) = 3 \times 2.4 \times 2.4 = 17.28$. Since $f$ is an exponential function, we can get a formula for $f(x)$ in the form $Pa^x$, where $a$ is the growth factor 2.4 and $P$ is the initial value 3. Thus $f(x) = 3 \times 2.4^x$.

S-2. **Function value from initial value and decay factor**: Since the decay factor is 0.094 and the initial value is $f(0) = 400$, $f(2) = 400 \times 0.094 \times 0.094 = 3.53$. Since $f$ is an exponential function, we can get a formula for $f(x)$ in the form $Pa^x$, where $a$ is the decay factor 0.094 and $P$ is the initial value 400. Thus $f(x) = 400 \times 0.094^x$.

S-3. **Finding the growth factor**: The growth factor is the factor by which $f$ changes for a change of 1 in the variable. If $a$ is the growth factor, then

$$a = \frac{f(5)}{f(4)} = \frac{10}{8} = 1.25.$$

S-4. **Exponential decay**: The graph of exponential decay versus time is decreasing — see, for example, Figure 4.2 of the text.

S-5. **Rate of change**: The rate of change of an exponential function is proportional to the function value.

S-6. **Percentage growth**: If a function has an initial value of 10 and grows at a rate of 7% per year, then it is an exponential function with $P = 10$ and $a = 1 + r = 1 + 0.07 = 1.07$. Thus an exponential function which describes this is $10 \times 1.07^t$, with $t$ in years.

S-7. **Percentage decay**:If a function has an initial value of 10 and decays by 4% per year, then it is an exponential function with $P = 10$ and $a = 1 - r = 1 - 0.04 = 0.96$. Thus an exponential function which describes this is $10 \times 0.96^t$, with $t$ in years.

S-8. **Changing units**: If the yearly growth factor is 1.17, then, since a month is $1/12$th of a year, the monthly growth factor is $1.17^{1/12} = 1.013$. Since a decade is 10 years, the decade growth factor is $1.17^{10} = 4.81$.

S-9. **Percentage change**: If a bank account grows by 9% each year, then the yearly growth factor is $a = 1 + r = 1 + 0.09 = 1.09$. The monthly growth factor, since a month is $1/12$th of a year, is $1.09^{1/12} = 1.0072$, which is a growth of $0.72\%$ per month.

S-10. **Percentage change**: If the radioactive substance decays by 17% each year, then the yearly decay factor is $a = 1 - r = 1 - 0.17 = 0.83$. The monthly decay factor, since a month is $1/12$th of a year, is $0.83^{1/12} = 0.9846$. Now $r = 1 - a = 1 - 0.9846 = 0.0154$, which is a decay rate of $1.54\%$ per month.

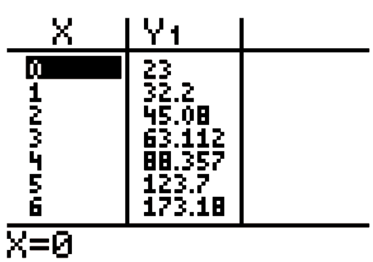
S-11. **Percentage growth**: If the population grows by 3.5% each year, then the yearly growth factor is $a = 1 + r = 1 + 0.035 = 1.035$. The decade growth factor is $1.035^{10} = 1.4106$ since a decade is 10 years. This represents a growth of 41.06% per decade.

S-12. **Percentage decline**: If the population declines by 1.5% each year, then the yearly decay factor is $a = 1 - r = 1 - 0.015 = 0.985$. The decade decay factor is $0.985^{10} = 0.8597$ since a decade is 10 years. Since $1 - 0.8597 = 0.1403$, this represents a decline of 14.03% per decade.

1. **Exponential growth with given initial value and growth factor**: The initial value is $P = 23$, and the growth factor is $a = 1.4$, so the formula for the function is

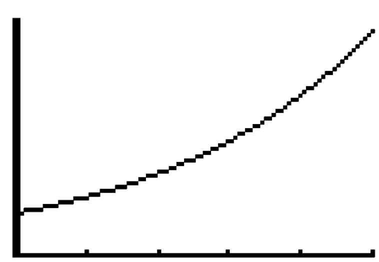
$$N(t) = 23 \times 1.4^t.$$

The table of values below leads us to choose a horizontal span of 0 to 5 and a vertical span of 0 to 130. The graph is shown below.



| X | Y1 |
|---|---|
| 0 | 23 |
| 1 | 32.2 |
| 2 | 45.08 |
| 3 | 63.112 |
| 4 | 88.357 |
| 5 | 123.7 |
| 6 | 173.18 |

X=0



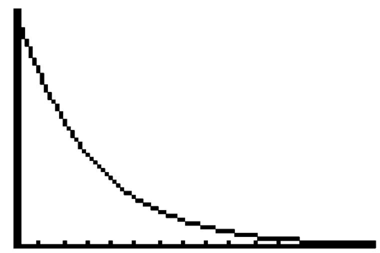
2. **Exponential decay with given initial value and decay factor**: The initial value is $P = 200$, and the decay factor is $a = 0.73$, so the formula for the function is

$$N(t) = 200 \times 0.73^t.$$

The table of values below led us to choose a horizontal span of 0 to 15 and a vertical span of 0 to 200. The graph is shown below.

| X | Y1 |
|---|---|
| 0 | 200 |
| 3 | 77.803 |
| 6 | 30.267 |
| 9 | 11.774 |
| 12 | 4.5804 |
| 15 | 1.7819 |
| 18 | .69317 |

X=0



3. **A population with given per capita growth rate**:

   (a) Since the population grows at a rate of 2.3% per year, the yearly growth factor for the function is $a = 1 + .023 = 1.023$. Since initially the population is $P = 3$ million, the exponential function is

   $$N(t) = 3 \times 1.023^t.$$

   Here $N$ is the population in millions and $t$ is time in years.

   (b) The population after 4 years is expressed by $N(4)$ using functional notation. Its value, using the formula from Part (a), is $N(4) = 3 \times 1.023^4 = 3.29$ million.

4. **Unit conversion with exponential growth**:

   (a) Now $N(1.5) = 3500 \times 1.77^{1.5} = 8241.91$, so the population has about 8242 individuals after 1.5 decades, that is, after 15 years.

   (b) The decade growth factor is $a = 1.77 = 1 + 0.77$, so $r = 0.77$. The population increases by 77% every decade.

   (c) Since $d$ is measured in decades then one year corresponds to a $d$ value of $\frac{1}{10}$. Since the decade growth factor is 1.77, the population will grow by a factor of

   $$1.77^{1/10} = 1.059$$

   in one year. The percentage growth rate for one year is found using the same method as in Part (a): $1.059 = 1 + 0.059$, so $r = 0.059$. Thus the percentage growth rate for one year is 5.9%.

(d) Since one century is 10 decades, one century corresponds to a $d$ value of $10$. The growth factor for one century is given by

$$(1.77)^{10} = 301.81.$$

Since the growth factor for one century is $301.81 = 1 + 300.81$, we find that $r = 300.81$. So the percentage growth rate for one century is $30{,}081\%$.

5. **Unit conversion with exponential decay**:

(a) Now $N(2) = 500 \times 0.68^2 = 231.2$, which means that after 2 years there are 231.2 grams of the radioactive substance present.

(b) The yearly decay factor is $0.68 = 1 - 0.32$, so $r = 0.32$. Thus the yearly percentage decay rate is 32%.

(c) Now $t = \frac{1}{12}$ corresponds to one month, so the monthly decay factor is

$$0.68^{1/12} = 0.968.$$

The monthly percentage decay rate is found using the same method as in Part (b): $0.968 = 1 - 0.032$, so $r = 0.032$. Thus the monthly percentage decay rate is 3.2%.

(d) There are $31{,}536{,}000$ seconds in a year, so one second corresponds to $\frac{1}{31{,}536{,}000}$ year. Thus the decay factor per second is

$$0.68^{1/31{,}536{,}000} = 0.9999999878 = 1 - 0.0000000122.$$

Thus $r = 0.0000000122$, and so the percentage decay rate per second is only 0.00000122%.

6. **A savings account**:

(a) The initial investment is $P = 500$, and since the decimal interest rate is 0.04 per year, the yearly growth factor is $a = 1.04$. The savings account balance is given by

$$B = 500 \times 1.04^t \text{ dollars.}$$

Here $t$ is the number of years since the initial investment.

(b) A month is $\frac{1}{12}$ of a year, so the monthly growth factor is

$$a^{1/12} = 1.04^{1/12} = 1.00327.$$

This gives a monthly interest rate of 0.327%.

(c) Since there are 10 years in a decade, the decade growth factor is $a^{10} = 1.04^{10} = 1.48$.

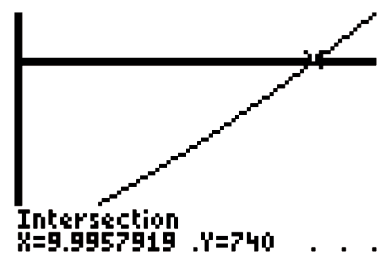
(d) We want to know when the balance is \$740. That is, we need to solve the equation $B = 740$, or

$$500 \times 1.04^t = 740.$$

We solve this using the crossing graphs method. The table below shows that the account is well over \$740 by the 12th year, and so we use 0 to 12 for a horizontal span. Since the account begins at \$500, and we are only looking up to \$740, we use 600 to 800 for a vertical span. In the right-hand figure below we have graphed both $B$ and the constant function 740 (thick line). The horizontal axis is years since initial investment, and the vertical axis is account balance. The graphs intersect at $t = 10$ years since the initial investment.

| X | Y1 |
|---|---|
| 0 | 500 |
| 3 | 562.43 |
| 6 | 632.66 |
| 9 | 711.66 |
| 12 | 800.52 |
| 15 | 900.47 |
| 18 | 1012.9 |

X=0




We found that the account reached \$740 in ten years, or 1 decade. We proceed to check this with the work in Part (c). We found the decade growth factor to be 1.48. Thus at the end of one decade, there will be $500 \times 1.48 = 740$ dollars, and this agrees with the answer found above that the account reaches \$740 at the end of 10 years.

7. **Half-life of heavy hydrogen**: If we start with 100 grams of $H_3$ then, according to Example 4.1, the decay is modeled by

$$A = 100 \times 0.783^t.$$

As in Example 4.1, the half-life of $H_3$ is obtained by solving $100 \times 0.783^t = 50$ to get $t = 2.83$ years. So, after 2.83 years we will have $50$ grams of $H_3$ left. This is the same length of time as found in Example 4.1, Part 4, for $50$ grams of $H_3$ to decay to 25 grams.

Since the amount we start with decays by half every 2.83 years, we see that after starting with 100 grams of $H_3$ we will have 50 grams after 2.83 years, 25 grams after $2 \times 2.83 = 5.66$ years, 12.5 grams after $3 \times 2.83 = 8.49$ years, and 6.25 grams after $4 \times 2.83 = 11.32$ years. This is four half-lives.

8. **How fast do exponential functions grow?** If we work from age 25 to age 65, we have worked a total of 40 years.

   **Retirement option 1**: Upon retirement, we would receive

   $$\$25{,}000 \times \text{ Years of service } = 25{,}000 \times 40 = \$1{,}000{,}000.$$

   **Retirement option 2**: The monthly growth factor for the account is $1 + 0.01 = 1.01$. Since the initial amount is $\$10{,}000$, the amount in the account after $t$ months is $10,000 \times 1.01^t$. Since 40 years is 480 months, we will receive

   $$10{,}000 \times 1.01^{480} = \$1{,}186{,}477.25.$$

   Retirement option 2 is the more favorable retirement package if you retire at age 65.

   If we work from age 25 to age 55, we have worked a total of 30 years.

   **Retirement option 1**: Upon retirement, we would receive

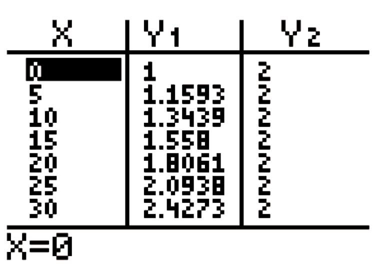
   $$\$25{,}000 \times \text{ Years of service } = 25{,}000 \times 30 = \$750{,}000.$$

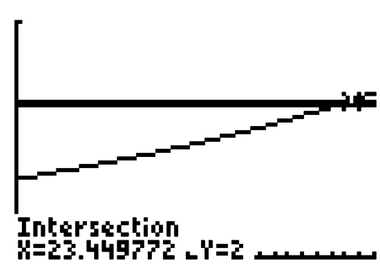
   **Retirement option 2**: The monthly growth factor for the account is $1 + 0.01 = 1.01$. Since the initial amount is $\$10{,}000$, the amount in the account after $t$ months is $10,000 \times 1.01^t$. Since 30 years is 360 months, we will receive

   $$10{,}000 \times 1.01^{360} = \$359{,}496.41.$$

   Retirement option 1 is the more favorable retirement package if you retire at age 55.

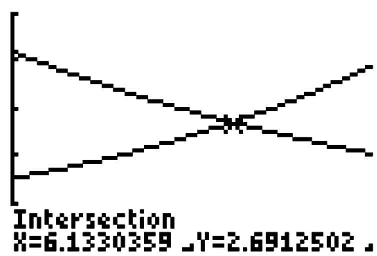
9. **Inflation**: The yearly growth factor for prices is given by $1 + 0.03 = 1.03$. We want to find the value of $t$ for which $1.03^t = 2$, since that is when the prices will have doubled. We solve this using the crossing graphs method. The table of values below leads us to choose a horizontal span of 0 to 25 years and a vertical span of 0 to 3. In the graph below, the horizontal axis is years, and the vertical axis is the factor by which prices increase. We see that the two graphs cross at $t = 23.45$ years. Thus, prices will double in 23.45 years. Note that we did not use the amount of the fixed income. The answer, 23.45 years, is the same regardless of the income.



10. **Rice production in Asia**: The annual rice yield in Asia was increasing by about 0.05 ton per hectare each year, so the yield has a constant rate of change. Thus the rice yield is a linear function of time. On the other hand, the predicted demand for Asian rice increases at about 2.1% per year, so the demand has a constant percentage rate of change. Thus the predicted demand is an exponential function of time with a growth factor of 1.021; in particular, it shows exponential growth. Thus future demand will far outstrip production.

11. **The MacArthur-Wilson Theory of biogeography**: According to the theory, stabilization occurs when the rate of immigration equals the rate of extinction, that is, $I = E$. This is equivalent to solving the equation $4.2 \times 0.93^t = 1.5 \times 1.1^t$. This can be solved using a table or a graph. The graph below, which uses a horizontal span of 0 to 10 and a vertical span of 0 to 5, shows that the solution occurs when $t = 6.13$ years, at which time the immigration and extinction rates are each 2.69 species per year.




12. **Long-term population growth**: The yearly growth factor for the population is $1+0.03 = 1.03$. Since the initial population was 3.93 million, the population $N$ (in millions) at the time $t$ years after 1790 is given by

$$N = 3.93 \times 1.03^t.$$

Now 210 years elapsed from 1790 to 2000, so we would predict the population in 2000 to be

$$N(210) = 3.93 \times 1.03^{210} = 1950.79 \text{ million people.}$$

This about 1.95 billion people, which is almost seven times as large as the actual figure of 281.42 million.

13. **The population of Mexico**:

(a) The yearly growth factor for the population is $1 + 0.026 = 1.026$, and the initial population is 67.38 million. The population $N$ (in millions) at the time $t$ years after 1980 is

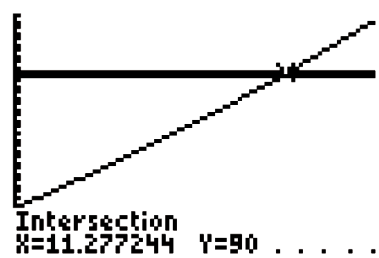
$$N = 67.38 \times 1.026^t.$$

(b) The population of Mexico is 1983 is expressed as $N(3)$ in functional notation, since 1983 is $t = 3$ years after 1980. The value of $N(3)$ is $67.38 \times 1.026^3 = 72.77$ million.

(c) The population of Mexico is 90 million when $67.38 \times 1.026^t = 90$. We solve this equation using the crossing graphs method. The table below leads us to choose a horizontal span of 0 to 15 and a vertical span of 60 to 100. The curve is the graph of $67.38 \times 1.026^t$, and the thick line is the graph of 90. We see that the two cross at 11.28 years after 1980. That is just over a quarter of the way through 1991.

| X | Y1 | Y2 |
|---|---|---|
| 0 | 67.38 | 90 |
| 3 | 72.773 | 90 |
| 6 | 78.599 | 90 |
| 9 | 84.89 | 90 |
| 12 | 91.685 | 90 |
| 15 | 99.024 | 90 |
| 18 | 106.95 | 90 |

X=0




14. **Cleaning contaminated water**:

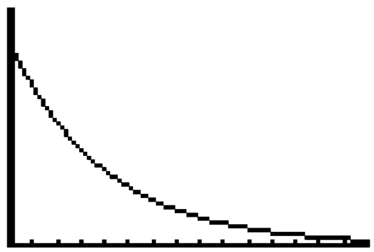
(a) The amount of salt is being decreased by 22% each hour, so the decay is at a constant proportional rate. This means that $S(t)$ is an exponential function. The hourly decay factor is $a = 1 - 0.22 = 0.78$.

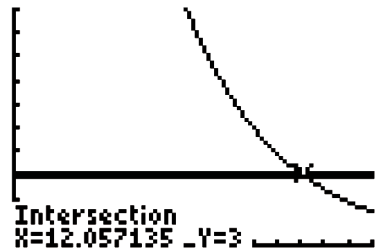
(b) The initial amount of salt was 60 pounds, so $P = 60$, and we already know that the decay factor is $a = 0.78$. Thus $t$ hours after the cleaning process begins the amount of salt is $S(t) = 60 \times 0.78^t$ pounds.

(c) We use a horizontal span of 0 to 15 as suggested. Since there are initially 60 pounds of salt in the tank, and this amount will decay toward 0, we use a vertical span of 0 to 70. The graph is the left-hand picture below. The horizontal axis is hours since the cleaning process began, and the vertical axis is pounds of salt left.

From the graph we can see that large amounts of salt are removed early in the process, but, as time passes, the rate of removal decreases.

(d) There are 3 pounds of salt left when $60 \times 0.78^t = 3$. We use the crossing graphs method to solve this. In the right-hand figure below, we have added the graph of 3 and calculated the intersection. (To better show the intersection point, we changed the vertical span to 0 to 10.) We see that the salt will decrease to 3 pounds in 12.06 hours.






(e) Since the cost is \$8000 per hour, the cost to reduce the salt to 3 pounds is $8000 \times 12.06 = 96{,}480$ dollars.

There is 0.1 pound left when $60 \times 0.78^t = 0.1$. Solving this using the crossing graphs method gives $t = 25.75$. So reducing the salt from 3 pounds to 0.1 pound will take an additional $25.75 - 12.06 = 13.69$ hours and cost an additional $8000 \times 13.69 = 109{,}520$ dollars.

It is worth noting that this is typical of functions showing exponential decay. Reducing from 60 to 3 takes just over 12 hours. But once the function is small, it decreases at a very slow rate, and it takes an additional 13.69 hours to reduce to the level of 0.1.

15. **Tsunami waves in Crescent City**:

(a) The probability $P$ is an exponential function of $Y$ because for each change in $Y$ of 1 year, the probability $P$ decreases by 2%, so $P$ exhibits constant percentage change.

(b) The decay factor for $P$ is 1 minus the decay percentage, so $a = 1 - r = 1 - 0.02 = 0.98$ per year.

(c) The initial value of $P$, $P(0)$, represents the probability of no tsunami wave of height 15 feet or more striking over a period of 0 years. Since this is a certainty (as a probability), the initial value of $P$ is 1.

(d) Since $P$ is an exponential function of $Y$ (by Part (a)) with a decay factor of 0.98 (by Part (b)) and an initial value of 1 (by Part (c)) the formula is $P = 1 \times 0.98^Y$, or simply $P = 0.98^Y$.

(e) The probability of no tsunami waves 15 feet or higher striking Crescent City over a 10-year period is $P(10) = 0.98^{10} = 0.82$. Thus the probability of no such tsunami wave striking over a ten-year period is 0.82 or 82%. Over a 100-year period, the probability is $P(100) = 0.98^{100} = 0.13$. Thus the probability of no such tsunami wave striking over a 100-year period is 0.13 or only 13%.

(f) To find the probability $Q$ that at least one wave 15 feet or higher will strike Crescent City over a period of $Y$ years, we first see that $P+Q = 1$ since $P+Q$ represents all possibilities. Thus $Q = 1 - P = 1 - 0.98^Y$.

16. **Growth of bacteria**:

(a) The number of *E. coli* present is an exponential function of time because the number of cells is doubling each 20 minutes, representing constant proportional change.

(b) Since the number of cells is doubling every 20 minutes, the 20-minute growth factor is 2. Since an hour is 3 periods of 20 minutes, the hourly growth factor is $2^3 = 8$.

(c) The population $N$ is an exponential function of time $t$ measured in hours, so $N$ can be written as $Pa^t$. Here $P$ is the initial value, which we denote by $N_0$, and $a = 8$ as seen in Part (b). Thus $N = N_0 \times 8^t$.

(d) The population will triple when $N = 3N_0$. Since $N = N_0 \times 8^t$, this occurs when $3 = 8^t$. Solving yields that the population triples when $t = 0.53$ hour, or about 32 minutes.

17. **Grains of wheat on a chess board**: The growth factor here is 2, and the initial amount is 1. Thus the function we need is $1 \times 2^t$, or simply $2^t$, where $t$ is measured in days since the first day. Since we count the first square as being on the initial day ($t = 0$), the second day corresponds to $t = 1$, and so on; thus the 64th day corresponds to $t = 63$. Hence the king will have to put $2^{63} = 9.22 \times 10^{18}$ grains of wheat on the 64th square. If we divide $9.22 \times 10^{18}$ by the number of grains in a bushel, namely $1{,}100{,}000$, we get $8.38 \times 10^{12}$ bushels. If each bushel is worth \$4.25, then the value of the wheat on the 64th square was

$$4.25 \times 8.38 \times 10^{12} = 3.56 \times 10^{13} = 35{,}600{,}000{,}000{,}000 \text{ dollars.}$$

This is over 35.6 trillion dollars, a sum that may trouble even a king.

18. **The Beer-Lambert-Bouguer law**:

    (a) Intensity $I$ is an exponential function of depth $d$ since $I$ decreases by a constant percentage as a function of depth.

    (b) Intensity $I$ is an exponential function of $d$, so $I$ can be written as $Pa^t$. Here $P$ is the initial value of the intensity, which we denote by $I_0$. The decay factor is $a = 1 - r = 1 - 0.75 = 0.25$, so $I = I_0 \times 0.25^d$.

    (c) The initial value of $I$, $I_0$, represents the intensity when $d = 0$, that is, at the surface of the lake.

    (d) The intensity of light will be one tenth of the intensity of light striking the surface when $I = 0.1I_0$. This occurs at a depth $d$ where $I_0 \times 0.25^d = 0.1I_0$, so when $0.25^d = 0.1$. Solving yields that the depth $d$ is 1.66 meters.

19. **The photic zone**: Near Cape Cod, Massachusetts, the depth of the photic zone is about 16 meters. We know that the photic zone extends to a depth where the light intensity is about 1% of surface light, that is, $I = 0.01I_0$. Since the Beer-Lambert-Bouguer law implies that $I$ is an exponential function of depth, $I = I_0 \times a^d$. Since $I(16) = 0.01I_0$, we have $I_0 \times a^{16} = 0.01I_0$, so $a^{16} = 0.01$. Thus the decay factor is $a = 0.01^{1/16} = 0.750$, representing a 25.0% decrease in light intensity for each additional meter of depth.

20. **Decibels**:

    (a) Since each increase of one decibel $D$ causes a 12.2% increase in pressure $P$, $P$ has constant percentage change. Therefore $P$ is an exponential function of $D$ with growth factor $a = 1 + r = 1 + 0.122 = 1.122$.

    (b) Since $P$ is an exponential function with growth factor $1.122$, $P = P(0) \times 1.122^D$. Now we know that $P(97) = 15$ and so $P(0) \times 1.122^{97} = 15$. Thus $P(0) = 15/(1.122^{97}) = 0.0002$. In practical terms, this means that a loudness of 0 decimals produces a pressure of 0.0002 dyne per square centimeter.

    (c) From Parts (a) and (b), we see that $P = P(0) \times 1.122^D = 0.0002 \times 1.122^D$.

    (d) The pressure on the ear reaches a level of about 200 dynes per square centimeter when $P = 200$ and therefore $0.0002 \times 1.122^D = 200$. Solving for $D$ using a table or graph shows that $D = 120.02$, or about 120, decibels is the level which should be considered dangerous.

21. **Headway on four-lane highways**:

    (a) A flow of 500 vehicles per hour is $\frac{500}{3600} = 0.14$ vehicle per second, so $q = 0.14$. Thus the probability that the headway exceeds 15 seconds is given by $P = e^{-0.14\times 15} = 0.122$, or about 12%.

    (b) On a four-lane highway carrying an average of 500 vehicles per hour, $q = 0.14$, and so $P = e^{-0.14t}$. Using the law of exponents, this can be written as $P = e^{-0.14t} = (e^{-0.14})^t$. In this form it is clear that the decay factor is $a = e^{-0.14} = 0.87$.

22. **APR and APY**:

    (a) If interest is compounded four times a year, then the $\frac{1}{4}$-year percentage growth rate is $\frac{0.1}{4}$, and so the $\frac{1}{4}$-year growth factor is $1 + \frac{0.1}{4}$. Since a year is 4 quarter-years, the yearly growth factor is

    $$\left(1 + \frac{0.1}{4}\right)^4 = 1.1038.$$

    (b) If interest is compounded $n$ times a year, then the $\frac{1}{n}$-year percentage growth rate is $\frac{0.1}{n}$, and so the $\frac{1}{n}$-year growth factor is $1 + \frac{0.1}{n}$. Since a year is $n$ $\frac{1}{n}$-years, the yearly growth factor is

    $$\left(1 + \frac{0.1}{n}\right)^n.$$

    (c) If interest is compounded daily, then $n = 365$, and so the yearly growth factor is

    $$\left(1 + \frac{0.1}{365}\right)^{365} = 1.1052.$$

23. **Continuous compounding**:

    (a) The limiting value of the function given by the formula

    $$\left(1 + \frac{0.1}{n}\right)^n$$

    can be found using a table or a graph. Scanning down a table, for example, shows that the limiting value is 1.1052 to four decimal places.

    (b) Since the APR is 0.1, $e^{\text{APR}} = e^{0.1} = 1.1052$ to four decimal places..

    (c) Since the answers to Parts (a) and (b) are the same, this shows that $e^{\text{APR}}$ is the yearly growth factor for continuous compounding in the case when the APR is 10%.

## 4.2 MODELING EXPONENTIAL DATA

E-1. **Testing for exponential data**: To test unevenly spaced data, we calculate the new $y$/old $y$ ratio to see if it equals $a$ raised to the difference in the $x$ values. For those that are exponential, this allows us to find $a$, then we use a value from the data table to calculate the initial value $P$.

(a) Calculating the ratios of the $y$ values with the appropriate powers of $a$, we get

$$\begin{aligned} a^3 = a^{5-2} &= \frac{3072}{48} = 64 \\ a^2 = a^{7-5} &= \frac{49{,}152}{3072} = 16 \\ a^4 = a^{11-7} &= \frac{12{,}582{,}912}{49{,}152} = 256. \end{aligned}$$

The three equations each give values for $a$, namely $64^{1/3} = 4$, $16^{1/2} = 4$, and $256^{1/4} = 4$. Since all three give the same value, the data are exponential with growth factor $a = 4$. The first data point allows us to find the initial value:

$$\begin{aligned} P \times a^2 &= 48 \\ P \times 4^2 &= 48 \\ P &= 48/(4^2) = 3. \end{aligned}$$

Thus the exponential model for the data is $y = 3 \times 4^x$.

(b) Calculating the ratios of the $y$ values with the appropriate powers of $a$, we get

$$\begin{aligned} a^2 = a^{4-2} &= \frac{1}{4} = 0.25 \\ a^4 = a^{8-4} &= \frac{0.0625}{1} = 0.0625 \\ a^2 = a^{10-8} &= \frac{0.015625}{0.0625} = 0.25. \end{aligned}$$

The three equations each give values for $a$, namely $0.25^{1/2} = 0.5$, $0.0625^{1/4} = 0.5$, and $0.25^{1/2} = 0.5$. Since all three give the same value, the data are exponential with growth factor $a = 0.5$. The first data point allows us to find the initial value:

$$\begin{aligned} P \times a^2 &= 4 \\ P \times 0.5^2 &= 4 \\ P &= 4/(0.5^2) = 16. \end{aligned}$$

Thus the exponential model for the data is $y = 16 \times 0.5^x$.

(c) Calculating the ratios of the $y$ values with the appropriate powers of $a$, we get

$$\begin{aligned} a^3 = a^{5-2} &= \frac{972}{36} = 27 \\ a^2 = a^{7-5} &= \frac{8748}{972} = 9 \\ a^2 = a^{9-7} &= \frac{88{,}740}{8748} = 10.14. \end{aligned}$$

The three equations each give values for $a$, namely $27^{1/3} = 3$, $9^{1/2} = 3$, and $10.14^{1/2} = 3.18$. Since the three do not give the same value, the data are not exponential.

E-2. **Finding missing points**: Since we know that data set is exponential, then $y = P \times a^x$. Since $y = 54$ when $x = 3$ and $y = 1458$ when $x = 6$, then $a^{6-3} = a^3$ must equal $\frac{1458}{54} = 27$ and so $a = 27^{1/3} = 3$. Thus the $y$ value associated with $x = 1$ has the property that it times 3 times 3 equals 54, so the $y$ value is $54/(3 \times 3) = 6$. Similarly $\frac{162}{54} = 3$, so the associated $x$ value must be 1 more than that for 3, that is $x = 4$. The missing data are displayed in the table below.

| | | | | |
|---|---|---|---|---|
| $x$ | 1 | 3 | **4** | 6 |
| $y$ | **6** | 54 | 162 | 1458 |

E-3. **Testing exponential data**: Calculating the ratios of the $y$ values with the appropriate powers of the growth factor (which we did not denote by $a$ since that letter is already used in the data), we get

$$\begin{aligned} \text{Growth factor}^3 = \text{Growth factor}^{(a+3)-a} &= \frac{b^3}{1} = b^3 \\ \text{Growth factor}^2 = \text{Growth factor}^{(a+5)-(a+3)} &= \frac{b^5}{b^3} = b^2 \\ \text{Growth factor}^1 = \text{Growth factor}^{(a+6)-(a+5)} &= \frac{b^6}{b^5} = b. \end{aligned}$$

Since the three equations each give the same value for the growth factor, the data are exponential with growth factor $b$, and thus the exponential model is $y = P \times b^x$. The first data point allows us to find the initial value $P$:

$$\begin{aligned} P \times b^a &= 1 \\ P &= 1/(b^a) = b^{-a}. \end{aligned}$$

Thus the exponential model for the data is $y = b^{-a} \times b^x$, which can also be written as $y = b^{x-a}$.

S-1. **Finding the growth factor**: Since $N$ is multiplied by 8 if $t$ is increased by 1, the growth factor $a$ is 8. Since the initial value $P$ is 7, a formula for $N$ is $N = P \times a^t = 7 \times 8^t$.

S-2. **Finding the growth factor**: Since $N$ is divided by 13 if $t$ is increased by 1, the decay factor $a$ is $\frac{1}{13}$. Since the initial value $P$ is 6, a formula for $N$ is $N = P \times a^t = 6 \times \left(\frac{1}{13}\right)^t$, which can also be written as $N = 6 \times 0.077^t$.

S-3. **Finding the growth factor**: Since $N$ is multiplied by 62 if $t$ is increased by 7, the growth factor $a$ satisfies the equation $a^7 = 62$. Thus $a = 62^{1/7} = 1.803$. Since the initial value $P$ is 12, a formula for $N$ is $N = P \times a^t = 12 \times 1.803^t$.

S-4. **Testing exponential data**: Calculating the ratios of successive terms of $y$, we get $\frac{7.8}{2.6} = 3$, $\frac{23.4}{7.8} = 3$, and $\frac{70.2}{23.4} = 3$. Since the $x$ values are evenly spaced and these ratios show a constant value of 3, the table does show exponential data.

S-5. **Testing exponential data**: Calculating the ratios of successive terms of $y$, we get $\frac{10}{5} = 2$, $\frac{20}{10} = 2$, and $\frac{40}{20} = 2$. Since the $x$ values are evenly spaced and these ratios show a constant value of 2, the table does show exponential data.

S-6. **Testing exponential data**: Calculating the ratios of successive terms of $y$, we get $\frac{9}{5} = 1.8$, $\frac{21}{9} = 2.33$, and $\frac{43}{21} = 2.05$. Since the $x$ values are evenly spaced and these ratios do not show a constant value, the table does not show exponential data.

S-7. **Modeling exponential data**: The data from Exercise S-4 show a constant ratio of 3 for $x$ values increasing by 1's and so the growth factor is $a = 3$. The initial value of $P = 2.6$ comes from the first entry in the table. Thus an exponential model for the data is $y = 2.6 \times 3^x$.

S-8. **Modeling exponential data**: The data from Exercise S-5 show a constant ratio of 2 for $x$ values increasing by 2's and so the growth factor $a$ satisfies $a^2 = 2$. Thus $a = 2^{1/2} = 1.41$. The initial value of $P = 5$ comes from the first entry in the table. Thus an exponential model for the data is $y = 5 \times 1.41^x$.

S-9. **Testing exponential data**: Calculating the ratios of successive terms of $y$, we get $\frac{18}{6} = 3$, $\frac{54}{18} = 3$, and $\frac{162}{54} = 3$. Since the $x$ values are evenly spaced and these ratios show a constant value of 3, the table does show exponential data.

S-10. **Modeling exponential data**: The data from Exercise S-9 show a constant ratio of 3 for $x$ values increasing by 2's and so the growth factor $a$ satisfies $a^2 = 3$. Thus $a = 3^{1/2} = 1.73$. The initial value of $P = 6$ comes from the first entry in the table. Thus an exponential model for the data is $y = 6 \times 1.73^x$.

S-11. **Testing exponential data**: Calculating the ratios of successive terms of $y$, we get $\frac{240}{1000} = 0.24$, $\frac{57.6}{240} = 0.24$, and $\frac{13.8}{57.6} = 0.24$. Since the $x$ values are evenly spaced and these ratios show a constant value of 0.24, the table does show exponential data.

S-12. **Modeling exponential data**: The data from Exercise S-11 show a constant ratio of 0.24 for $x$ values increasing by 4's and so the decay factor $a$ satisfies $a^4 = 0.24$. Thus $a = 0.24^{1/4} = 0.7$. The initial value of $P = 1000$ comes from the first entry in the table. Thus an exponential model for the data is $y = 1000 \times 0.7^x$.

1. **Making an exponential model**: To show that this is an exponential function we must show that the successive ratios are the same.

| $t$ increment | 0 to 1 | 1 to 2 | 2 to 3 | 3 to 4 | 4 to 5 |
|---|---|---|---|---|---|
| $\frac{\text{new}}{\text{old}}$ ratio of $f(t)$ | $\frac{3.95}{3.80} = 1.04$ | $\frac{4.11}{3.95} = 1.04$ | $\frac{4.27}{4.11} = 1.04$ | $\frac{4.45}{4.27} = 1.04$ | $\frac{4.62}{4.45} = 1.04$ |

Note that the ratios are not precisely equal, but they are equal when rounded to 2 decimal places. The table in the exercise tells us that the initial value is $P = 3.80$ (this corresponds to $t = 0$). Since $t$ increases in units of 1, we don't need to adjust the units, and so the growth factor is $a = 1.04$. The formula for the function is

$$f(t) = 3.80 \times 1.04^t.$$

2. **An exponential model with unit adjustment**: To show that this is an exponential function we must show that the successive ratios (rounded to three decimal places) are the same.

| $t$ increment | $\frac{\text{New}}{\text{Old}}$ ratio |
|---|---|
| 0 to 4 | $28.65/38.30 = 0.748$ |
| 4 to 8 | $21.43/28.65 = 0.748$ |
| 8 to 12 | $16.04/21.43 = 0.748$ |
| 12 to 16 | $11.99/16.04 = 0.748$ |
| 16 to 20 | $8.97/11.99 = 0.748$ |

Since all of the ratios are equal, the function is exponential. Since $t$ increases by 4 each time, the common ratio, $0.748$, is the growth factor for 4 units. To get the one-unit growth factor, we calculate

$$a = 0.748^{1/4} = 0.930.$$

The initial value is 38.30 and so the formula for the function is

$$g(t) = 38.30 \times 0.930^t.$$

3. **Data that are not exponential**: To show that these data are not exponential, we show that some of the successive ratios are not equal. It suffices to compute the first two ratios:

$$\frac{26.6}{4.9} = 5.43 \text{ and } \frac{91.7}{26.6} = 3.45.$$

Since these ratios are not equal and yet the changes in $t$ are the same, $h$ is not an exponential function.

4. **Linear and exponential data**: Successive ratios for Table A are always 1.16, so Table A represents an exponential function with $a = 1.16$, since $t$ increases by single units. The initial value is 6.70, and so the data is modeled by

$$f = 6.70 \times 1.16^t.$$

Successive differences for Table B are always 1.73, so Table B represents a linear function with slope $m = 1.73$, since $t$ increases by single units. The initial value is $b = 5.80$, so the formula for the function is

$$g = 1.73t + 5.80.$$

5. **Magazine sales**: The calculations of differences and ratios for the years in the data set are in the table below.

| Interval | Differences | Ratios |
|---|---|---|
| 1998–1999 | $8.82 - 7.76 = 1.06$ | $8.82/7.76 = 1.137$ |
| 1999–2000 | $9.88 - 8.82 = 1.06$ | $9.88/8.82 = 1.120$ |
| 2000–2001 | $10.94 - 9.88 = 1.06$ | $10.94/9.88 = 1.107$ |
| 2001–2002 | $12.00 - 10.94 = 1.06$ | $12.00/10.94 = 1.097$ |
| 2002–2003 | $13.08 - 12.00 = 1.08$ | $13.08/12.00 = 1.090$ |
| 2003–2004 | $14.26 - 13.08 = 1.18$ | $14.26/13.08 = 1.090$ |
| 2004–2005 | $15.54 - 14.26 = 1.28$ | $15.54/14.26 = 1.090$ |

From 1998 to 2002, the magazine sales exhibit a constant growth rate of $1.06$ thousand dollars per year. From 2002 to 2005, sales grew at a constant proportional rate, that is, each year's sales are 1.09 times those of the previous year—this is a growth rate of 9% per year.

6. **Population growth**: A table of ratios should reveal which data point fails to exhibit constant proportional growth.

| Interval | Ratio |
|---|---|
| 2000–2001 | $5.51/5.25 = 1.05$ |
| 2001–2002 | $5.79/5.51 = 1.05$ |
| 2002–2003 | $6.04/5.79 = 1.04$ |
| 2003–2004 | $6.38/6.04 = 1.06$ |
| 2004–2005 | $6.70/6.38 = 1.05$ |

The population is growing exponentially so we expect a common ratio, and, given the ratios calculated, we expect that ratio to be 1.05. The lower ratio of 1.04, followed by the higher ratio of 1.06, indicates that the entry of 6.04 for 2003 is in error. The correct entry should be $5.79 \times 1.05 = 6.08$ thousand.

7. **An investment**:

(a) The amount of money originally invested is the balance when $t = 0$. Hence the original investment was \$1750.00.

(b) Each successive ratio of new/old is $1.012$, which shows the data is exponential. Furthermore, since $t$ increases by single units, the common ratio 1.012 is the monthly growth factor. The formula for an exponential model is

$$B = 1750.00 \times 1.012^t.$$

Here $t$ is time in months, and $B$ is the savings balance in dollars.

(c) The growth factor is $1.012 = 1 + 0.012$, so $r = 0.012$. The monthly interest rate is 1.2%.

(d) To find the yearly interest rate we first have to find the yearly growth factor. There are 12 months in a year, so the yearly growth factor is $1.012^{12} = 1.154$. Now $1.154 = 1 + 0.154$, so $r = 0.154$, and thus the yearly interest rate is 15.4%.

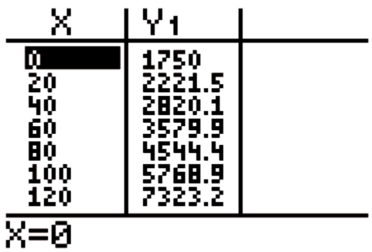
(e) The account will grow for 18 years or 216 months, so when she is 18 years old, her college fund account balance will be

$$B(216) = 1750 \times 1.012^{216} = \$23{,}015.94.$$

(f) The account will double the first time when it reaches \$3500 and double the second time when it reaches \$7000. That is, we need to solve the equations

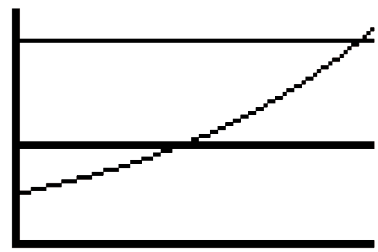
$$1750 \times 1.012^t = 3500 \text{ and } 1750 \times 1.012^t = 7000.$$

We do this using the crossing graphs method. The vertical span we want is 0 to 8000, and the table of values below for $B$ leads us to choose 0 to 120 months for a horizontal span. In the right-hand figure below we have graphed $B$ along with 3500 and 7000 (thick lines). The horizontal axis is months since the initial investment, and the vertical axis is account balance.

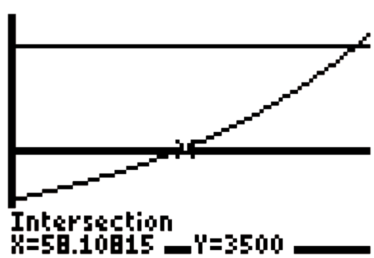


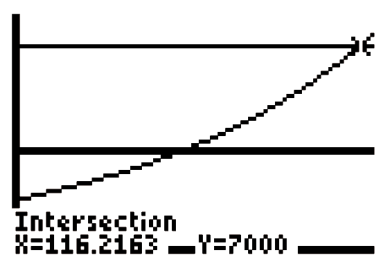
| X | Y1 |
|---|---|
| 0 | 1750 |
| 20 | 2221.5 |
| 40 | 2820.1 |
| 60 | 3579.9 |
| 80 | 4544.4 |
| 100 | 5768.9 |
| 120 | 7323.2 |

X=0



In the left-hand figure below, we have calculated the intersection of $B$ with 3500, and we find that the balance doubles in 58.11, or about 58.1, months. In the right-hand figure below, we have calculated the intersection with 7000, and we see the account reaches \$7000 after 116.22 months. Thus the second doubling occurs 58.11, or about 58.1, months after the first one. (In fact, the account doubles every 58.11 months.)




8. **A bald eagle murder mystery**: The key to this exercise is to find at what time the temperature of the bald eagle was 105 degrees, because that is when the eagle was still alive. Since the table gives us the difference between the air temperature, which is 62 degrees, and the eagle's temperature, we want to know when that difference was $105 - 62 = 43$ degrees. To do this we must first find the function that models the tabulated data. Notice that the successive ratios in the $D$ values are always 0.91, so the function is exponential. Since the $t$ values are measured in hours, the hourly decay factor is $a = 0.91$. The initial temperature difference is 26.83, so the function that models the tabulated data is

$$D = 26.83 \times 0.91^t.$$

We want to know when the temperature difference was 43. That is, we want to solve the equation $D(t) = 43$, or

$$26.83 \times 0.91^t = 43.$$

If we look at the table of values below in preparation for solving by the crossing graphs method, we can spot the answer without further work. The temperature difference is 43 degrees when $t = -5$. This means that the eagle's temperature was 105 degrees at the time 5 hours before it was discovered. Thus, because it was discovered at 3:00 p.m., the bird was killed at around 10:00 a.m. Since the second archer was in camp between 9:00 a.m. and 11:00 a.m., he is innocent. The other archer remains a suspect.

| X | Y1 | |
|---|---|---|
| -6 | 47.247 | |
| -5 | 42.995 | |
| -4 | 39.125 | |
| -3 | 35.604 | |
| -2 | 32.399 | |
| -1 | 29.484 | |
| 0 | 26.83 | |

X= -5

9. **A skydiver**:

(a) The successive ratios in $D$ values are always 0.42, so the data can be modeled with an exponential function. Since the $t$ values are measured every 5 seconds, 0.42 is the decay factor every 5 seconds. To find the decay factor every second we compute

$$0.42^{1/5} = 0.84.$$

The initial value is 176.00, and so the exponential model is

$$D = 176.00 \times 0.84^t.$$

(b) The decay factor per second is $0.84 = 1 - 0.16$, so $r = 0.16$, and the percentage decay rate is 16% per second. This means that the difference between the terminal velocity and the skydiver's velocity decreased by 16% each second.

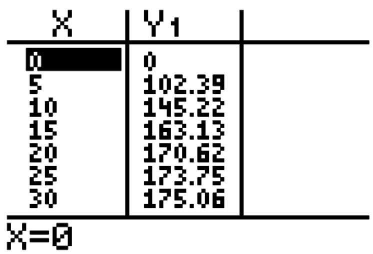
(c) Now $D$ represents the difference between terminal velocity of 176 feet per second and the skydiver's velocity, $V$. Thus $D = 176 - V$, and so $V = 176 - D$. Using the formula from Part (a) for $D$, we find that

$$V = 176.00 - 176.00 \times 0.84^t.$$

(d) Now 99% of terminal velocity is $0.99 \times 176 = 174.24$, and we want to know when the velocity reaches this value. That is, we want to solve the equation

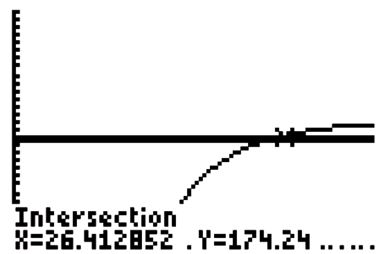
$$176 - 176 \times 0.84^t = 174.24.$$

We do this using the crossing graphs method. We know that the velocity starts at 0 and increases toward 176, and so we use a vertical span of 160 to 190. The table below leads us to choose a horizontal span of 0 to 35. In the right-hand figure below, we have graphed velocity and the target velocity of 174.24 (thick line). We see that the intersection occurs at $t = 26.41$. The skydiver reaches 99% of terminal velocity after 26.41 seconds.



| X | Y1 |
|---|---|
| 0 | 0 |
| 5 | 102.39 |
| 10 | 145.22 |
| 15 | 163.13 |
| 20 | 170.62 |
| 25 | 173.75 |
| 30 | 175.06 |

X=0




10. **The half-life of U$^{239}$**: Let $t$ be the time in minutes and $U$ the number of grams remaining.

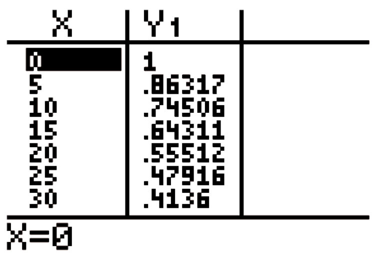
(a) The ratios of successive terms are always $0.971$, and so the data is exponential. The $t$ values are measured in units of one, so the decay factor per minute is 0.971. The initial value is 1 so the equation of the exponential function is

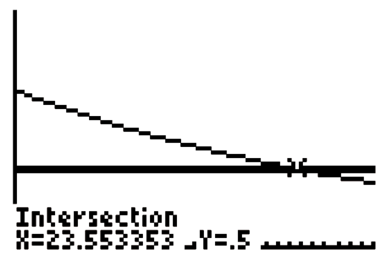
$$U(t) = 1 \times 0.971^t = 0.971^t.$$

(b) Since the decay factor per minute is $0.971 = 1 - 0.029$, $r = 0.029$. Thus the percentage decay per minute is 2.9%. This means that 2.9% of the remaining uranium decays each minute.

(c) In functional notation, the amount remaining after 10 minutes is $U(10)$. The value of $U(10)$ is $0.971^{10} = 0.745$ gram.

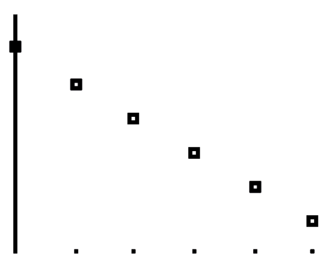
(d) To find the half-life we need to know when there is $\frac{1}{2} = 0.5$ gram remaining. That is, we want to solve the equation $0.971^t = 0.5$. We do this with the crossing graphs method. We know the uranium decays from 1 gram toward 0, and so we use a vertical span of 0 to 1.5. The table below leads us to choose a horizontal span of 0 to 30 minutes. In the right-hand figure below, we have graphed $U$ and the constant function 0.5 (thick line). The horizontal axis is minutes, and the vertical axis is the amount of uranium remaining. We see that the intersection occurs at $t = 23.553$. The half-life of $U^{239}$ is about 23.553 minutes.



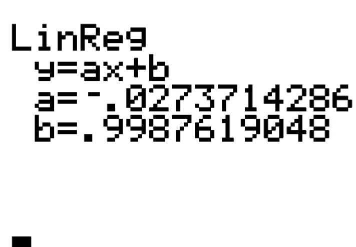



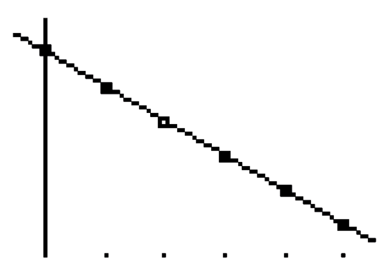

11. **An inappropriate linear model for radioactive decay**:

(a) As seen in the figure below, the data points do seem to fall on a straight line.



(b) In the left-hand figure below, we have calculated the regression line parameters, and the equation of the regression line is $U = -0.027t + 0.999$. In the right-hand figure, we have added the graph of the regression line to the data plot.



(c) Using the regression line equation for the decay, we see that the time required for decay to $0.5$ gram is given by solving $-0.027t + 0.999 = 0.5$. This shows that the time required would be 18.48, or about 18.5, minutes.

(d) The regression line predicts that after 60 minutes there will be $-0.621$ gram of the uranium 239 left, which is impossible.

Two items are worth noting here. The first is that the data has a linear appearance because we have sampled over too short a time period. Sampling over a longer time period would have clearly shown that the data is not linear. It is crucial in any experiment to gather enough data to allow for sound conclusions. Second, it is always dangerous to propose mathematical models based solely on the appearance of data. In the case of radioactive decay, physicists have good scientific reasons for believing that radioactive decay is an exponential phenomenon. Collected data serves to verify this, and to allow for the calculation of a constant such as the half-life for specific substances.

12. **Account growth**:

(a) To determine if the interest rate is constant, we calculate the ratios of new balances to old balances, as shown in the table below.

| Time interval | Ratios of New/Old Balances |
|---|---|
| $t = 0$ to $t = 1$ | $131.25/125.00 = 1.05$ |
| $t = 1$ to $t = 2$ | $137.81/131.25 = 1.05$ |
| $t = 2$ to $t = 3$ | $144.70/137.81 = 1.05$ |
| $t = 3$ to $t = 4$ | $151.94/144.70 = 1.05$ |

The data table indicates a constant growth factor of 1.05, or an interest rate of 5% per year.

(b) Since the initial value of $B$ is 125.00 and the growth factor is 1.05, a formula for $B$ is $125.00 \times 1.05^t$. Therefore $B(2.75) = 125.00 \times 1.05^{2.75} = 142.95$. In practical terms this means that the account balance after 2 years and 9 months is approximately \$142.95.

13. **Rates vary**: To determine the interest rates, we calculate the ratios of new balances to old balances, as shown in the table below.

| Time interval | Ratios of New/Old Balances |
|---|---|
| $t = 0$ to $t = 1$ | $262.50/250.00 = 1.050$ |
| $t = 1$ to $t = 2$ | $275.63/262.50 = 1.050$ |
| $t = 2$ to $t = 3$ | $289.41/275.63 = 1.050$ |
| $t = 3$ to $t = 4$ | $302.43/289.41 = 1.045$ |
| $t = 4$ to $t = 5$ | $316.04/302.43 = 1.045$ |
| $t = 5$ to $t = 6$ | $330.26/316.04 = 1.045$ |

Thus the yearly interest rate was 5.0% for the first 3 years, then 4.5% after that.

14. **Wages**:

(a) The ratios of successive terms are always 1.02 to two decimal places, which shows that the data is exponential.

(b) The yearly growth factor is the common ratio, so $a = 1.02$.

(c) The hourly wage was \$15.30 in 1991. Since $W$ is exponential with yearly growth factor 1.02, we have

$$\begin{aligned}\text{Hourly wage in 1991} &= \text{Hourly wage in 1990} \times \text{Growth factor} \\ 15.30 &= \text{Hourly wage in 1990} \times 1.02.\end{aligned}$$

We solve this by dividing both sides by 1.02:

$$\text{Hourly wage in 1990} = \frac{15.30}{1.02} = 15.00.$$

Thus the hourly wage at the beginning of 1990 was \$15.00.

(d) Now 1990 corresponds to $t = 0$, so the initial value of $W$ is 15.00. Since the growth factor is $a = 1.02$, we have

$$W = 15.00 \times 1.02^t.$$

(e) Since the growth factor is $a = 1.02$, we have $r = 0.02$, so the yearly percentage growth rate is 2%. Thus the worker received 2% raises each year.

(f) Over the decade of the 1990's, wages increased by 2% each year, so the decade growth factor is

$$1.02^{10} = 1.22.$$

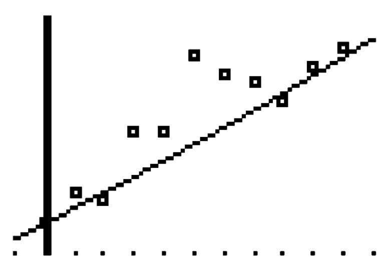
This means that wages increased by 22% over the decade, which is far below the decade inflation rate of 34%.

15. **Stochastic population growth**:

(a) The table below shows the Monte Carlo method applied by rolling a die ten times. The faces showing for these ten rolls are: 5, 2, 6, 3, 6, 1, 2, 1, 5, and 4. Your answer will presumably be different since it is highly unlikely that your die will roll the same values as these(!). Since the numbers are the size of a population, we will only use whole numbers.

| $t$ | Die face | Population % Change | Population change | Population |
|---|---|---|---|---|
| 0 | | | | 500 |
| 1 | 5 | Up 4% | +20 | 520 |
| 2 | 2 | Down 1% | $-5$ | 515 |
| 3 | 6 | Up 9% | +46 | 561 |
| 4 | 3 | No change | 0 | 561 |
| 5 | 6 | Up 9% | +50 | 611 |
| 6 | 1 | Down 2% | $-12$ | 599 |
| 7 | 2 | Down 1% | $-6$ | 593 |
| 8 | 1 | Down 2% | $-12$ | 581 |
| 9 | 5 | Up 4% | +23 | 604 |
| 10 | 4 | Up 2% | +12 | 616 |

(b) The exponential model is $500 \times 1.02^t$ since the initial value is 500 and the growth rate is 2%. A graph of the data points from Part (a) together with this exponential model is shown below. Although the points are scattered about and do not lie on the exponential curve, they do follow, very roughly, the general upward trend of the curve.



16. **Growth rate of a tubeworm**:

(a) We calculate the differences and the ratios of the growth rates in the table below.

| Lengths | Differences | Ratios |
|---|---|---|
| 0 – 0.5 | $0.0255 - 0.0510 = -0.026$ | $0.0255/0.0510 = 0.500$ |
| 0.5 – 1.0 | $0.0128 - 0.0255 = -0.013$ | $0.0128/0.0255 = 0.502$ |
| 1.0 – 1.5 | $0.0064 - 0.0128 = -0.006$ | $0.0064/0.0128 = 0.500$ |
| 1.5 – 2.0 | $0.0032 - 0.0064 = -0.003$ | $0.0032/0.0064 = 0.500$ |

Clearly the differences are not constant and so a linear function would not be good choice. On the other hand, the growth rate decreases at a constant proportion of almost exactly 0.50, so an exponential function with decay factor 0.5 per 0.5-meter length would be an appropriate model. The decay factor for 1-meter lengths is $0.50^2 = 0.25$. Since the initial value is 0.0510 meters per year, an exponential formula for $G$ is $0.0510 \times 0.25^L$. Here $G$ is the growth rate in meters per year and $L$ the length in meters. The decay factor of 0.5 indicates that the growth rate slows by 50% for each increase in length of 0.5 meter.

(b) The growth rate at a length of 0.64 meter is expressed in functional notation as $G(0.64)$. Its value is $G(0.64) = 0.0510 \times 0.25^{0.64} = 0.021$ meter per year.

## 4.3 MODELING NEARLY EXPONENTIAL DATA

E-1. **Population growth**: Because the population has abundant resources and there are few predators, an exponential model is appropriate.

E-2. **More population growth**: Because the population has limited resources but there are few predators, a logistic model is appropriate.

E-3. **Constant speed**: In this context the rate of change of the distance traveled is the speed, which is constant. Since the rate of change for the distance traveled is constant, a linear model is appropriate.

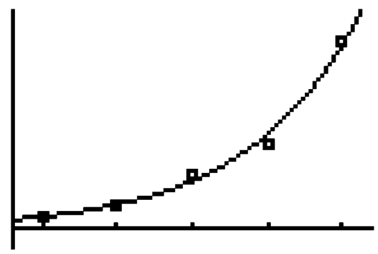
S-1. **Population growth**: Because the population has abundant resources, an exponential model should be appropriate.

S-2. **Inflation**: Because the price of a bag of groceries grows at a relatively constant percentage rate, an exponential model is appropriate.

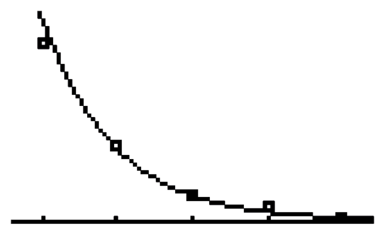
S-3. **Exponential regression**: The exponential model is $y = 51.01 \times 1.04^x$.

S-4. **Exponential regression**: The exponential model is $y = 42.98 \times 1.07^x$.

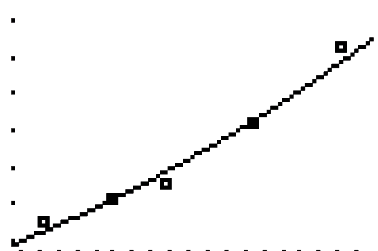
S-5. **Exponential regression**: The exponential model is $y = 2.22 \times 1.96^x$. A plot of the exponential model, together with the data, is shown below.



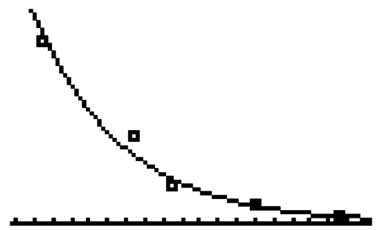
S-6. **Exponential regression**: The exponential model is $y = 2.21 \times 0.36^x$. A plot of the exponential model, together with the data, is shown below.



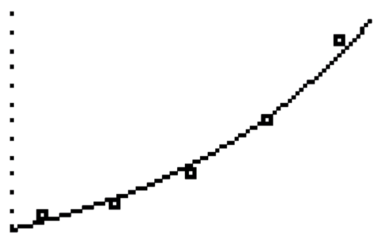
S-7. **Exponential regression**: The exponential model is $y = 6.35 \times 1.03^x$. A plot of the exponential model, together with the data, is shown below.



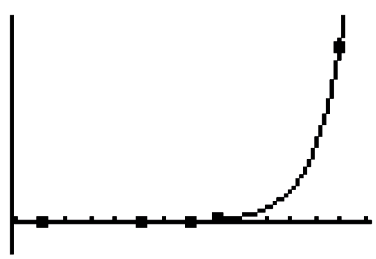
S-8. **Exponential regression**: The exponential model is $y = 5.59 \times 0.81^x$. A plot of the exponential model, together with the data, is shown below.



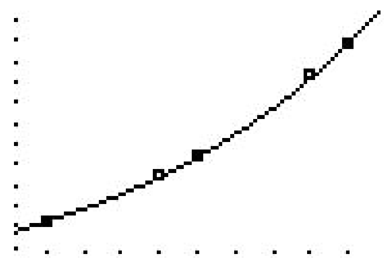
S-9. **Exponential regression**: The exponential model is $y = 2.39 \times 1.40^x$. A plot of the exponential model, together with the data, is shown below.



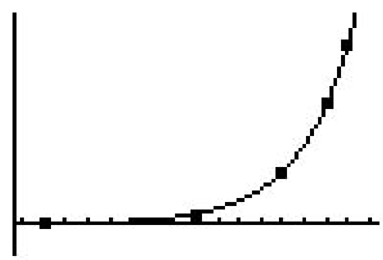
S-10. **Exponential regression**: The exponential model is $y = 2.61 \times 2.34^x$. A plot of the exponential model, together with the data, is shown below.



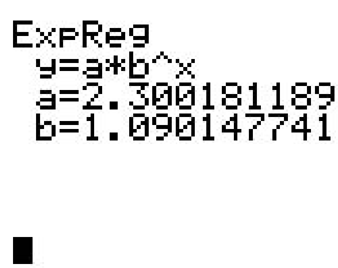
S-11. **Exponential regression**: The exponential model is $y = 3.17 \times 1.15^x$. A plot of the exponential model, together with the data, is shown below.



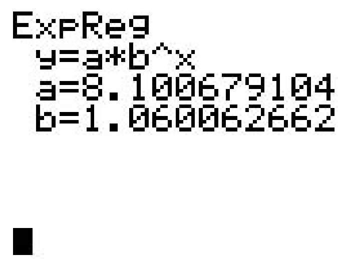
S-12. **Exponential regression**: The exponential model is $y = 0.20 \times 1.50^x$. A plot of the exponential model, together with the data, is shown below.



1. **Population growth**: Let $t$ be years since 2001 and $N$ the population, in thousands. The regression parameters for the exponential model are calculated in the figure below. We find the exponential model $N = 2.300 \times 1.090^t$.

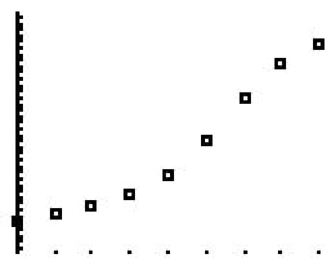


2. **Magazine sales**: Let $t$ be years since 2001 and $S$ the income from sales, in thousands of dollars. The regression parameters for the exponential model are calculated in the figure below. We find the exponential model $S = 8.101 \times 1.060^t$.




3. **Cable TV**: Let $t$ be years since 1976 and $C$ the percent with cable.

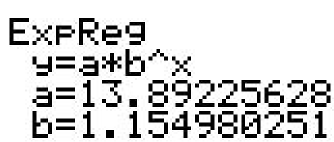
   (a) The plot of the data points is in the figure below. The horizontal axis is years since 1976, and the vertical axis is percent of homes with cable TV.

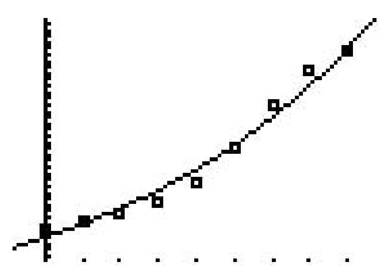
   

   The plot suggests the shape of exponential growth, so it is reasonable to approximate the data with an exponential function.

   (b) The regression parameters for the exponential model are calculated in the left-hand figure below. The exponential model is given by

   $$C = 13.892 \times 1.155^t.$$

   (c) We have added the graph of the exponential model in the right-hand figure above.

(d) Since the yearly growth factor is $1.155 = 1 + 0.155$, we have that $r = 0.155$. Thus the yearly percentage growth rate is 15.5%.

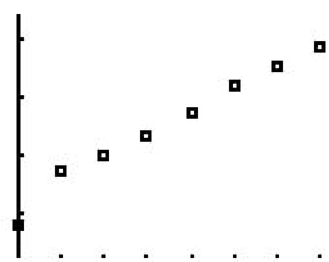
(e) Since 1987 is 11 years after 1976, we would predict that $13.892 \times 1.155^{11} = 67.79$ percent of the American households would have cable TV in 1987. So it appears that the executive's plan is reasonable.

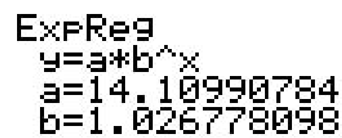
4. **Auto parts production workers**: Let $t$ be years since 1987 and $W$ the hourly wage in dollars.

(a) The plot of the data points is in the figure on the left below. The horizontal axis is years since 1987, and the vertical axis is hourly wage. Except for the initial point, the plot suggests the shape of exponential growth, so it is reasonable to approximate the data with an exponential function.

(b) The regression parameters for the exponential model are calculated in the right-hand figure below. The exponential model is given by

$$W = 14.110 \times 1.027^t.$$






(c) The yearly growth factor is $1.027 = 1 + 0.027$, so $r = 0.027$, and the yearly percentage growth rate for auto parts worker wages $W$ is 2.7%.

(d) If wages had increased by 3.8% every year, then $r$ would be 0.038 and $a$ would be 1.038. The hourly wages in 1994 would have been

$$13.79 \times 1.038^7 = 17.90 \text{ dollars.}$$

(e) To find the percentage raise, compute the ratio of the raise to the original wage:

$$\frac{\text{Amount of raise needed}}{\text{Original wage}} = \frac{17.90 - 16.85}{16.85} = 0.062.$$

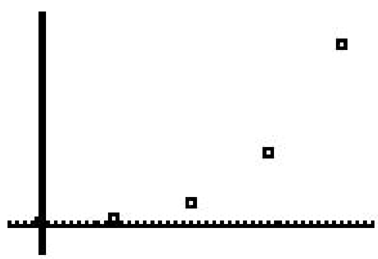
So a worker would need to receive about a 6.2% raise to keep up with inflation in 1994.

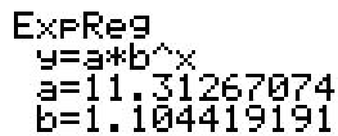
5. **National health care spending**: Let $t$ be years since 1950 and $H$ the costs in billions of dollars.

   (a) The plot of the data points is in the figure on the left below. The horizontal axis is years since 1950, and the vertical axis is cost. The plot suggests the shape of exponential growth, so it is reasonable to approximate the data with an exponential function.

   (b) The regression parameters for the exponential model are calculated in the right-hand figure below. The exponential model is given by

$$H = 11.313 \times 1.104^t.$$






   (c) The yearly growth factor is $1.104 = 1 + 0.104$, so $r = 0.104$. Thus the yearly percentage increase is 10.4%.

   (d) The year 2000 is 50 years after 1950, so $t = 50$. Thus the amount spent in 2000 on health care is expressed in functional notation as $H(50)$. We would estimate that

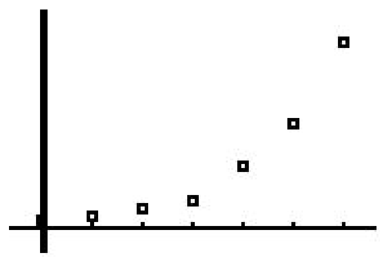
$$H(50) = 11.313 \times 1.104^{50} = 1592.32 \text{ billion dollars.}$$

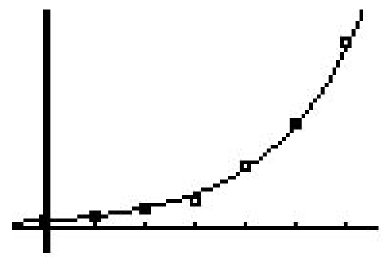
   Thus, the estimate is that about 1592 billion dollars, or 1.592 trillion dollars, was spent on health care in the year 2000.

6. **A bad data point**: If we plot the data we get the figure on the left below. From the picture it may be unclear which data point does not line up with the others. To clarify the situation we use regression to calculate the exponential model and find

$$N = 20.711 \times 1.760^t.$$

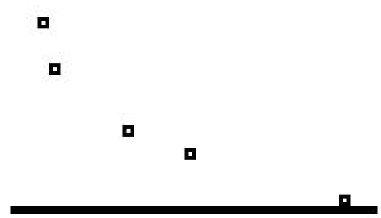
In the right-hand figure, we have added the graph of the exponential model to the data plot. The only point which is off is the one corresponding to $t = 3$, for which the value of $N$ is a bit small.





7. **Grazing rabbits**:

   (a) The plot of the data points is shown below. The horizontal axis is the vegetation level $V$, and the vertical axis is the difference $D$. The plot suggests the shape of exponential decay, so it appears that $D$ is approximately an exponential function of $V$.



   (b) We use regression to calculate the exponential model and find

$$D = 0.211 \times 0.988^V.$$

   (c) The satiation level for the rabbit is 0.18 pound per day, and thus we have

$$D = \text{satiation level} - A = 0.18 - A.$$

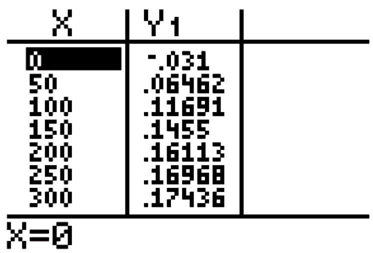
   Solving for $A$ yields $A = 0.18 - D$, and, using the formula from Part (b), we find that

$$A = 0.18 - 0.211 \times 0.988^V.$$

(d) Now 90% of the satiation level is $0.90 \times 0.18 = 0.162$, so we need to find when $A = 0.162$. By the formula in Part (c), this means

$$0.18 - 0.211 \times 0.988^V = 0.162.$$

We solve this using the crossing graphs method. The table of values for $A$ below shows that we reach 0.162 somewhere between 150 and 250. We use this for a horizontal span, and we use 0.1 to 0.2 for a vertical span. In the graph below, the horizontal axis is the vegetation level, and the vertical axis is the amount eaten. We have added the graph of 0.162 (thick line) and calculated the intersection point. Thus, the amount of food eaten by the rabbit will be 90% of satiation level when the vegetation level is $V = 203.89$ pounds per acre.



| X | Y1 |
|---|---|
| 0 | -.031 |
| 50 | .06462 |
| 100 | .11691 |
| 150 | .1455 |
| 200 | .16113 |
| 250 | .16968 |
| 300 | .17436 |

X=0


8. **Growth in length**:

(a) Now $D$ is the difference between the maximum length, which is 14.8 inches, and $L$, so $D = 14.8 - L$. Here's the table:

| $t$ = age | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $D$ = difference | 11.1 | 7.3 | 4.8 | 3.3 | 2.1 | 1.3 | 0.8 | 0.4 |

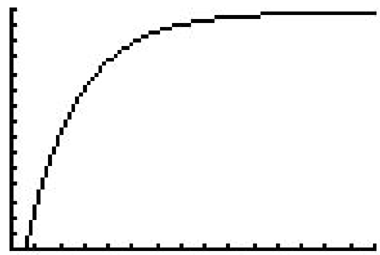
(b) We use regression to calculate the exponential model and find

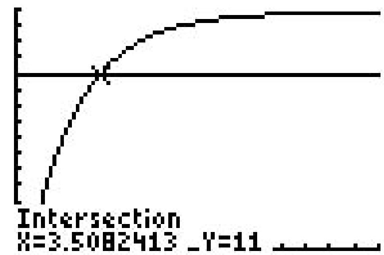
$$D = 19.113 \times 0.631^t.$$

(c) We know that $D = 14.8 - L$, and this equation can be solved for $L$ to yield $L = 14.8 - D$. Using the formula from Part (b), we find that

$$L = 14.8 - 19.113 \times 0.631^t.$$

(d) A table leads us to choose a horizontal span of 0 to 15 and a vertical span of 0 to 15. The graph is on the left below. The horizontal axis on the graph is age, and the vertical axis is length.





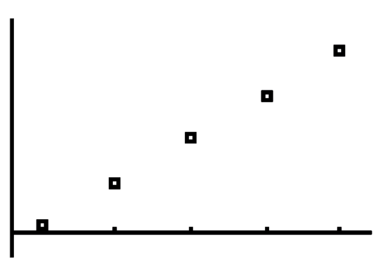

(e) If $L = 11$, then, from the formula in Part (c), $14.8 - 19.113 \times 0.631^t = 11$. We solve this for $t$ using the crossing-graphs method. In the figure on the right above, we have added the graph of 11 and calculated the intersection. We find that the sole is about 3.5 years old.

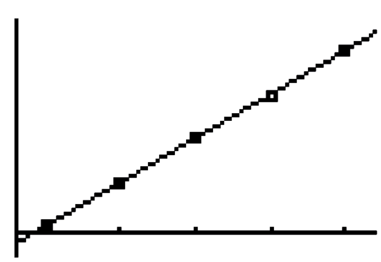
(f) Using the formula in Part (c) gives

$$L(9) = 14.8 - 19.113 \times 0.631^9 = 14.497.$$

This means that a sole which is 9 years old is about 14.5 inches long.

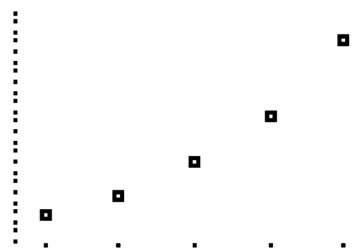
9. **Nearly linear or exponential data**: The left-hand figure below is the plot of the data from Table A. This appears to be linear. Computing the regression line gives $f = 19.842t - 16.264$, and adding it to the plot (right-hand figure) shows how very close to linear $f$ is. We conclude that Table A is approximately linear, with a linear model of $f = 19.842t - 16.264$.





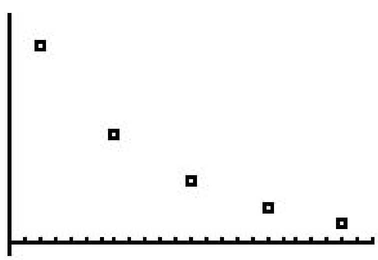
The plot of data from Table B in the figure below looks more like exponential data than linear data. We use regression to calculate the exponential model and find

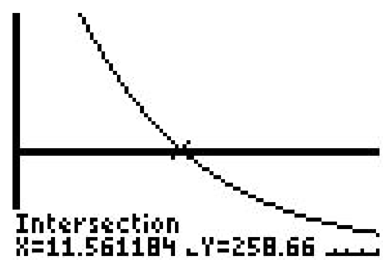
$$g = 2.330 \times 1.556^t.$$



10. **Atmospheric pressure**: Let $h$ be the altitude in kilometers and $P$ the atmospheric pressure in grams per square centimeter.

    (a) The plot of the data points is shown on the left below. The horizontal axis is the altitude $h$, and the vertical axis is the atmospheric pressure $P$.






    (b) We use regression to calculate the exponential model and find

$$P = 1034.651 \times 0.887^h.$$

    (c) An altitude of 30 kilometers corresponds to $h = 30$, so the atmospheric pressure is

$$P = 1034.651 \times 0.887^{30} = 28.35 \text{ grams per square centimeter.}$$

    (d) Earth's surface is at an altitude of 0 kilometers, so $h = 0$. The standard atmospheric pressure is therefore

$$P = 1034.651 \times 0.887^0 = 1034.651,$$

    or about 1034.65 grams per square centimeter.

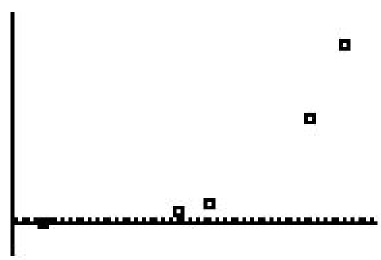
(e) Now 25% of standard atmospheric pressure is 25% of 1034.65, or $0.25 \times 1034.65 = 258.66$ grams per square centimeter. The altitude $h$ at which we find this pressure is the solution of the equation

$$1034.651 \times 0.887^h = 258.66.$$

We solve this using the crossing graphs method. The table of values given in the exercise suggests a horizontal span of 0 to 25 and a vertical span of 0 to 600. In the figure on the right above, we have graphed the pressure and the target pressure of 258.66 (thick line). The solution is an altitude of $h = 11.56$ kilometers.

11. **Sound pressure**:

(a) A plot of the data is shown below. The plot suggests the shape of exponential growth, so it appears that pressure is approximately an exponential function of loudness.

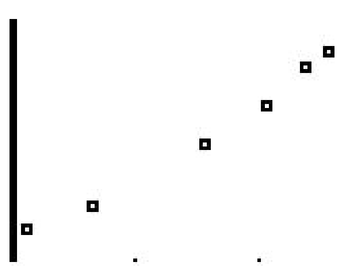


(b) We use regression to calculate the exponential model and find $P = 0.0003 \times 1.116^D$.

(c) When loudness $D$ is increased by one decibel, the pressure $P$ is multiplied by a factor of $1.116$, that is, it increases by 11.6%.

12. **Magnitude and distance**:

(a) The figure below shows a plot of $d$ against $m$. The plot suggests the shape of exponential growth, so it appears that distance is approximately an exponential function of magnitude difference.



(b) We use regression to calculate the exponential model and find $d = 33.236 \times 1.573^m$.

(c) If one star shows a magnitude difference of one greater than a second star, then $m$ is increased by 1. This causes $d$ to be multiplied by a factor of 1.573, so the star with the larger magnitude difference is 1.573 times as far away as the other star.

(d) If Alphecca shows a magnitude difference of 1.83, then $m = 1.83$ and so $d = 33.236 \times 1.573^{1.83} = 76.14$. Thus Alphecca is 76.14 light-years from Earth.

(e) If Alderamin is 52 light years from Earth, then $d = 52$ and so $52 = 33.236 \times 1.573^m$. Solving for $m$ we find that the magnitude difference $m$ is $0.99$. Since Alderamin has an apparent magnitude of 2.47, the absolute magnitude of Alderamin is $2.47 - 0.99 = 1.48$.

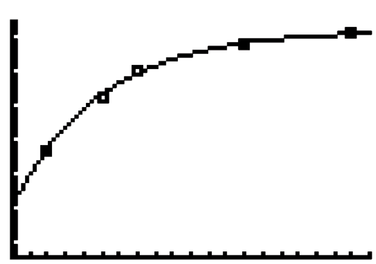
13. **Growth in length of haddock**:

(a) We use regression to calculate the exponential model and find $D = 40.909 \times 0.822^t$.

(b) The difference $D$ is the difference between 53 centimeters and the length at age $t$, $L = L(t)$. Thus $D = 53 - L$. Solving for $L$ and substituting the expression from Part (a) for $D$, we have

$$\begin{aligned} D &= 53 - L \\ D + L &= 53 \\ L &= 53 - D \\ L &= 53 - 40.909 \times 0.822^t. \end{aligned}$$

(c) The figure below shows a plot of the experimentally gathered data for the length $L$ at ages 2, 5, 7, 13, and 19 years along with the graph of the model for $L$ from Part (b). This graph shows that the 5-year-old haddock is a bit shorter than would be expected from the model for $L$.

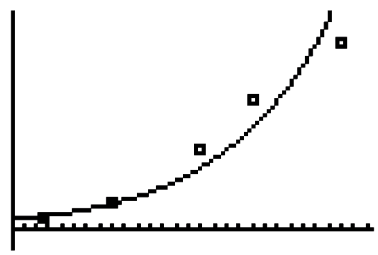


(d) If a fisherman has caught a haddock which measures 41 centimeters, then $L = 41$ and so its age $t$ satisfies the equation $41 = 53 - 40.909 \times 0.822^t$. Solving for $t$, we find that $t = 6.26$. Thus the haddock is about 6.3 years old.

14. **Caloric content versus shell length**:

(a) We use regression to calculate the exponential model and find $C = 42.944 \times 1.131^L$.

(b) The plot below shows the graph of the data together with the exponential model from Part (a). The data point lying below the curve shows a good deal less caloric content than the model would predict for the given length. This is the last data point, for which $C = 31$ and $L = 1480$.



(c) If length $L$ is increased by one millimeter, then the caloric content $C$ is multiplied by a factor of 1.131, so it is increased by 13.1%.

15. **Injury versus speed**:

(a) We use regression to calculate the exponential model and find $N = 16.726 \times 1.021^s$.

(b) Using the formula from Part (a), we find that $N(70) = 16.726 \times 1.021^{70} = 71.65$. In practical terms this means that if vehicles are traveling at 70 miles per hour, then we expect $71.65$ persons injured (on average) per 100 vehicles involved in accidents.

(c) An increase in 1 mile per hour of speed $s$ causes the number of people injured per 100 accident-involved vehicles $N$ to be multiplied by 1.021, and so to increase by 2.1%.

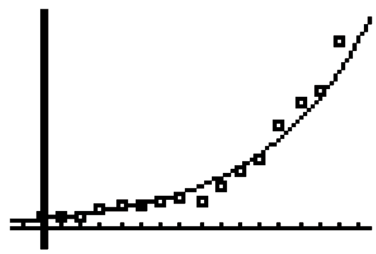
16. **Gray wolves in Wisconsin**:

(a) Since the wolves in Wisconsin have unlimited space and potential food, we expect their population to show exponential growth. Also, a plot of the data (see below) suggests the shape of exponential growth.

(b) This data are not exactly exponential. This may be seen by calculating ratios of New/Old values or, more simply, by noticing that while exponential growth functions are always increasing, the population was generally increasing, but from 1992 to 1993 the population decreased.

(c) We use regression to calculate the exponential model and find $W = 13.058 \times 1.209^t$.

(d) The figure below shows a plot of the data together with the exponential model from Part (c).



(e) The graph above is a nice fit to the data, although the data points for the early 1990's are below, and those of the late 1990's above, the number predicted by the exponential model.

17. **Walking in Seattle**:

(a) We use regression to calculate the exponential model and find $P = 84.726 \times 0.999^D$.

(b) The percentage of pedestrians who walk at least $D = 200$ feet from parking facilities is $P = 84.726 \times 0.999^{200} = 69.36\%$.

(c) For $D = 0$, the exponential model indicates that $P = 84.726$ and so only 84.726% of pedestrians walk at least 0 feet. The correct percentage is 100%. Thus this model is not appropriate to use for very small distances $D$.

18. **Traffic in the Lincoln Tunnel**:

(a) We use regression to calculate the exponential model and find $D = 233.206 \times 0.942^s$.

(b) The density of traffic flow when the average speed is 28 miles per hour is expressed in functional notation as $D(28)$. The value of $D(28)$ is, using the formula from Part (a), $D = 233.206 \times 0.942^{28} = 43.77$, or about 44 cars per mile.

(c) If the average speed $s$ increases by one mile per hour, then the density $D$ is multiplied by 0.942 and so it decreases by 5.8%.

19. **Frequency of earthquakes**:

    (a) We use regression to calculate the exponential model and find $N = 24{,}670{,}000 \times 0.125^{M}$.

    (b) To find the number of earthquakes per year of magnitude at least 5.5, we put $M = 5.5$ in the formula from Part (a) and find $N = 24{,}670{,}000 \times 0.125^{5.5} = 266.18$, or about 266 earthquakes per year.

    (c) The number of earthquakes per year of magnitude 8.5 or greater predicted by the model is $N = 24{,}670{,}000 \times 0.125^{8.5} = 0.52$, or about 0.5 earthquake per year.

    (d) The limiting value for $N$ is seen to be $0$ using a table or graph. In practical terms this means that earthquakes of large magnitude are very rare.

    (e) In the situation described, the number of earthquakes predicted is multiplied by 0.125, which is a reduction of 87.5%. Thus there are 87.5% fewer earthquakes of magnitude $M + 1$ or greater than of magnitude $M$ or greater.

20. **Laboratory experiment**: Answers will vary. Please go to the website for instructions and details.

21. **Laboratory experiment**: Answers will vary. Please go to the website for instructions and details.

22. **Laboratory experiment**: Answers will vary. Please go to the website for instructions and details.

23. **Research project**: Answers will vary. Please go to the website for instructions and details.

## 4.4 LOGARITHMIC FUNCTIONS

E-1. **Solving equations with logarithms**: In each case we apply the common logarithm to each side of the equation and use the properties of logarithms to solve for $t$.

(a)

$$\begin{aligned}
5 \times 4^t &= 7 \\
\log(5 \times 4^t) &= \log 7 \\
\log 5 + \log(4^t) &= \log 7 \qquad \textbf{Product law} \\
\log 5 + t\log 4 &= \log 7 \qquad \textbf{Power law} \\
t &= \frac{\log 7 - \log 5}{\log 4} = 0.24.
\end{aligned}$$

(b)

$$\begin{aligned}
2^{t+3} &= 7^{t-1} \\
\log(2^{t+3}) &= \log(7^{t-1}) \\
(t+3)\log 2 &= (t-1)\log 7 \qquad \textbf{Power law} \\
t\log 2 + 3\log 2 &= t\log 7 - \log 7 \\
t(\log 2 - \log 7) &= -\log 7 - 3\log 2 \\
t &= \frac{-\log 7 - 3\log 2}{\log 2 - \log 7} = 3.21.
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{12^t}{9^t} &= 7 \times 6^t \\
\log(\frac{12^t}{9^t}) &= \log(7 \times 6^t) \\
\log(12^t) - \log(9^t) &= \log 7 + \log(6^t) \qquad \textbf{Quotient and Product laws} \\
t\log 12 - t\log 9 &= \log 7 + t\log 6 \qquad \textbf{Power law} \\
t(\log 12 - \log 9 - \log 6) &= \log 7 \\
t &= \frac{\log 7}{\log 12 - \log 9 - \log 6} = -1.29.
\end{aligned}$$

(d)

$$
\begin{aligned}
a^{2t} &= 3a^{4t} \\
\log(a^{2t}) &= \log(3a^{4t}) \\
2t\log a &= \log 3 + 4t\log a \qquad \textbf{Power and Product laws} \\
t(2\log a - 4\log a) &= \log 3 \\
t &= \frac{\log 3}{2\log a - 4\log a} = -\frac{\log 3}{2\log a}.
\end{aligned}
$$

This assumes that $a \neq 1$, so that $\log a \neq 0$.

E-2. **Solving equations using the natural logarithm**: Since the basic rules for the natural logarithm (power, quotient, and power laws) are the same as those for the common logarithm, *mutatis mutandis*, the equations from Exercise E-1 are solved in exactly the same way, everywhere replacing $\log$ with $\ln$.

E-3. **Spectroscopic parallax again**: Exercise 13 from Exercise Set 4.4 below relates the spectroscopic parallax $S$ to the distance $D$ from Earth by the equation

$$S = 5\log D - 5.$$

(a) If $D = 38.04$, then $S = 5\log 38.04 - 5 = 2.90$. (This part does not use the laws of logarithms.)

(b) If $S = 1.27$, then $1.27 = 5\log D - 5$. Now we solve for $D$:

$$
\begin{aligned}
5\log D &= 6.27 \qquad \textbf{Adding 5} \\
\log D &= \frac{6.27}{5} \qquad \textbf{Dividing by 5} \\
D &= 10^{6.27/5} = 17.95. \qquad \textbf{By definition of log}
\end{aligned}
$$

Thus the distance from Earth is 17.95 parsecs.

(c) If $D$ is multiplied by 10, then $\log(10D) = \log 10 + \log D = 1 + \log D$ and so the spectroscopic parallax $S$ is increased by $5 \times 1 = 5$ units.

(d) If $D$ is multiplied by 3.78, then $\log(3.78D) = \log 3.78 + \log D = 0.577 + \log D$ and so the spectroscopic parallax $S$ is increased by $5 \times 0.577 = 2.89$ units.

E-4. **Calculating with logarithms**: Following the directions, we calculate the logarithm of the expression:

$$\log\left(22.7^{0.7} \times 17.6^{1.1}\right) = \log\left(22.7^{0.7}\right) + \log\left(17.6^{1.1}\right) = 0.7\log 22.7 + 1.1\log 17.6.$$

Using the calculator, we find that $\log 22.7 = 1.3560$ and $\log 17.6 = 1.2455$ and so the logarithm of the original expression equals $0.7 \times 1.3560 + 1.1 \times 1.2455 = 2.3193$. The original expression therefore equals $10^{2.3193} = 208.59$.

E-5. **Substitutions**:

(a) Substituting $y = 2^x$ into $4^x - 5 \times 2^x + 6 = 0$ yields $y^2 - 5y + 6 = 0$ since $(2^x)^2 = 2^{2x} = (2^2)^x = 4^x$. Now $y^2 - 5y + 6$ factors as $(y-2)(y-3)$ and so $y = 2$ or $y = 3$. Substituting for $y$, we see that $2^x = 2$ or $2^x = 3$. The first gives $x = 1$, and the second, using logarithms, gives $x = \dfrac{\log 3}{\log 2} = 1.58$.

(b) Substituting $y = 5^x$ into $25^x - 7 \times 5^x + 12 = 0$ yields $y^2 - 7y + 12 = 0$ since $(5^x)^2 = 5^{2x} = (5^2)^x = 25^x$. Now $y^2 - 7y + 12$ factors as $(y-3)(y-4)$ and so $y = 3$ or $y = 4$. Substituting for $y$, we see that $5^x = 3$ or $5^x = 4$. Using logarithms, the first gives $x = \dfrac{\log 3}{\log 5} = 0.68$ and the second gives $x = \dfrac{\log 4}{\log 5} = 0.86$.

(c) Substituting $y = 3^x$ into $9^x - 3^{x+1} + 2 = 0$ yields $y^2 - 3y + 2 = 0$ since $(3^x)^2 = 3^{2x} = (3^2)^x = 9^x$ and $3^{x+1} = 3^x 3^1 = 3 \times 3^x$. Now $y^2 - 3y + 2$ factors as $(y-1)(y-2)$ and so $y = 1$ or $y = 2$. Substituting for $y$, we see that $3^x = 1$ or $3^x = 2$. Using logarithms, the first gives $x = \dfrac{\log 1}{\log 3} = 0$ and the second gives $x = \dfrac{\log 2}{\log 3} = 0.63$.

E-6. **Solving exponential equations**: In each case, we will apply $\ln$ twice to bring the $x$ down from the exponent. These problems could equally well be done using $\log$.

(a)

$$\begin{aligned}
3^{(2^x)} &= 5 \\
\ln\left(3^{(2^x)}\right) &= \ln 5 \\
2^x \ln 3 &= \ln 5 && \textbf{Power law} \\
2^x &= \frac{\ln 5}{\ln 3} \\
\ln\left(2^x\right) &= \ln\left(\frac{\ln 5}{\ln 3}\right) \\
x \ln 2 &= \ln\left(\frac{\ln 5}{\ln 3}\right) && \textbf{Power law} \\
x &= \frac{\ln\left(\frac{\ln 5}{\ln 3}\right)}{\ln 2} = 0.55.
\end{aligned}$$

(b)

$$\begin{aligned}
3^{(4^x)} &= 7^{(2^x)} \\
\ln\left(3^{(4^x)}\right) &= \ln\left(7^{(2^x)}\right) \\
4^x \ln 3 &= 2^x \ln 7 \qquad \textbf{Power law} \\
\ln(4^x \ln 3) &= \ln(2^x \ln 7) \\
x \ln 4 + \ln\ln 3 &= x \ln 2 + \ln\ln 7 \qquad \textbf{Product and Power laws} \\
x(\ln 4 - \ln 2) &= \ln\ln 7 - \ln\ln 3 \\
x &= \frac{\ln\ln 7 - \ln\ln 3}{\ln 4 - \ln 2} = 0.82.
\end{aligned}$$

(c)

$$\begin{aligned}
2^{\left(5^{x+1}\right)} &= 4^{(3^x)} \\
\ln\left(2^{\left(5^{x+1}\right)}\right) &= \ln\left(4^{(3^x)}\right) \\
5^{x+1} \ln 2 &= 3^x \ln 4 \qquad \textbf{Power law} \\
\ln\left(5^{x+1} \ln 2\right) &= \ln(3^x \ln 4) \\
(x+1)\ln 5 + \ln\ln 2 &= x \ln 3 + \ln\ln 4 \qquad \textbf{Power law} \\
x(\ln 5 - \ln 3) &= \ln\ln 4 - \ln 5 - \ln\ln 2 \\
x &= \frac{\ln\ln 4 - \ln 5 - \ln\ln 2}{\ln 5 - \ln 3} = -1.79.
\end{aligned}$$

E-7. **Solving logarithmic equations**: To solve these logarithmic equations, we exponentiate each of the equation and utilize the identity $e^{\ln x} = x$.

(a)

$$\begin{aligned}
\ln(x+2) &= 5 \\
e^{\ln(x+2)} &= e^5 \\
x + 2 &= e^5 \\
x &= e^5 - 2 = 146.41
\end{aligned}$$

(b)

$$\begin{aligned}
\ln(x+5) &= 2\ln(x-1) = \ln((x-1)^2) \\
e^{\ln(x+5)} &= e^{\ln(x-1)^2} \\
x + 5 &= (x-1)^2 = x^2 - 2x + 1 \\
0 &= x^2 - 3x - 4 = (x-4)(x+1)
\end{aligned}$$

Thus $x = 4$ or $x = -1$. Unfortunately $x = -1$ is not actually a solution since $2\ln(x-1)$ has no value when $x = -1$. The only solution is $x = 4$.

(c)

$$\begin{aligned}
\ln(3x-10) &= \ln(x+2)+\ln(x-5)\\
e^{\ln(3x-10)} &= e^{\ln(x+2)+\ln(x-5)}\\
3x-10 &= e^{\ln(x+2)}e^{\ln(x-5)}\\
3x-10 &= (x+2)(x-5) = x^2-3x-10\\
0 &= x^2-6x = x(x-6)
\end{aligned}$$

Thus $x=0$ or $x=6$. Unfortunately $x=0$ is not actually a solution since $\ln(3x-10)$ has no value when $x=0$. The only solution is $x=6$.

E-8. **Moore's law**:

(a) Applying the common logarithm, we solve for $k$:

$$\begin{aligned}
10^k &= 2^{1/1.5}\\
\log\left(10^k\right) &= \log\left(2^{1/1.5}\right)\\
k &= \log\left(2^{1/1.5}\right) = 0.2.
\end{aligned}$$

(b) If the yearly growth factor is $a = 2^{1/1.5}$, then the 1.5-year growth factor is $a^{1.5} = \left(2^{1/1.5}\right)^{1.5} = 2^{1.5/1.5} = 2$, so the function doubles every 1.5 years.

(c) According to Part (b), every year the speed of computer chips increases by a factor of $a = 2^{1/1.5}$, which equals $10^k$ if $k$ is the solution of the equation in Part (a).

S-1. **The Richter scale**: If one earthquake reads 4.2 on the Richter scale and another reads 7.2, then the 7.2 earthquake is $10^{7.2-4.2} = 10^3 = 1000$ times as powerful as the 4.2 earthquake.

S-2. **The Richter scale again**: The second earthquake is 100 times as strong as the earthquake with a Richter scale reading of 6.5. Since $100 = 10^2$, the Richter scale reading of the more powerful earthquake is $6.5+2 = 8.5$.

S-3. **The decibel scale**: If one sound has a relative intensity of 1000 times that of another, then the common logarithm of the more intense sound is 3 more than that of the other, since $1000 = 10^3$. Since the decibel level is 10 times the logarithm, the decibel level is increased by $10\times 3 = 30$ decibels. Thus the more intense sound has a decibel level 30 decibels more than the other.

S-4. **More decibels**: If one sound has a decibel reading of 5.5, and another has a decibel reading of 7.5, then the difference in the decibel levels is 2. Since the decibel level is 10 times the logarithm of the relative intensity, then the difference in the logarithms is $\frac{2}{10} = 0.2$. Thus the relative intensity of the higher decibel sound is $10^{0.2} = 1.58$ times that of the lower decibel sound.

S-5. **Calculating logarithms**:

(a) Since $1000 = 10^3$, $\log 1000 = 3$.

(b) Since $1 = 10^0$, $\log 1 = 0$.

(c) Since $\frac{1}{10} = 10^{-1}$, $\log \frac{1}{10} = -1$.

S-6. **Solving logarithmic equations**:

(a) If $\log x = 2$, then $x = 10^2 = 100$.

(b) If $\log x = 1$, then $x = 10^1 = 10$.

(c) If $\log x = -2$, then $x = 10^{-2} = \frac{1}{100}$, or 0.01.

S-7. **How the logarithm increases**:

(a) Now $\log(10x) = \log 10 + \log x = 1 + 6.6 = 7.6$.

(b) Now $\log(1000x) = \log 1000 + \log x = 3 + 6.6 = 9.6$.

(c) Now $\log \frac{x}{10} = \log x - \log 10 = 6.6 - 1 = 5.6$.

S-8. **How the logarithm increases**: If $\log x = 8.3$ and $\log y = 10.3$, then $\log y$ is 2 more than $\log x$, so $y$ is 100 times $x$, that is, $y = 100x$.

S-9. **Solving exponential equations**:

(a) If $10^t = 5$, then $\log 10^t = \log 5$. Now $\log 10^t = t$, so $t = \log 5$, and so $t = 0.70$.

(b) If $10^t = a$, then $\log 10^t = \log a$. Now $\log 10^t = t$, so $t = \log a$.

(c) If $a^t = b$, then $\log a^t = \log b$. Now $\log a^t = t \log a$. Thus $t = \frac{\log b}{\log a}$.

S-10. **The graph of the logarithm**: We want to know why the graph of the logarithm $y = \log x$ eventually crosses the line $y = K$ for every value of $K$. It crosses when $K = \log x$, that is, when $x = 10^K$, since $\log(10^K) = K$. For example, the graph of $y = \log x$ crosses the horizontal line $y = 37.8$ when $x = 10^{37.8} = 6.31 \times 10^{37}$.

S-11. **Solving logarithmic equations**: If $t = \log(2x)$, then $10^t = 2x$ and so $x = 0.5 \times 10^t$.

S-12. **Solving exponential equations**: If $t = 10^{4x}$, then $\log t = \log 10^{4x}$. Now $\log 10^{4x} = 4x$, so $\log t = 4x$. Hence $x = 0.25 \log t$.

1. **Earthquakes in Alaska and Chile**:

   (a) The New Madrid earthquake of Example 4.8 had a Richter magnitude of 8.8, whereas the 1964 Alaska earthquake had a Richter magnitude of 8.4. The New Madrid earthquake is therefore $10^{8.8-8.4} = 10^{0.4} = 2.51$ times as strong as the Alaska earthquake.

   (b) The Chilean earthquake was 1.4 times as powerful as the Alaska quake. We want to write 1.4 as a power of 10. Solving the equation $1.4 = 10^t$, we find that $t = \log 1.4 = 0.15$, and so the Chilean earthquake was $10^{0.15}$ times as powerful as the Alaska quake. Thus the Richter scale reading for the Chilean earthquake was 0.15 more than that of the Alaska earthquake, so it was $8.4 + 0.15 = 8.55$.

2. **State quakes**:

   (a) The largest recorded earthquake centered in Montana was 3.16 times as powerful as the Idaho earthquake, so we want to write 3.16 as a power of 10. Solving the equation $3.16 = 10^t$, we find that $t = \log 3.16 = 0.50$. Thus the Richter scale reading of the Montana earthquake is 0.5 more than that of the Idaho earthquake, so it equals $7.2 + 0.5 = 7.7$.

   (b) The largest recorded earthquake centered in Arizona measured 5.6 on the Richter scale, whereas the Idaho earthquake measured 7.2. Thus the power of the Idaho quake is $10^{7.2-5.6} = 10^{1.6} = 39.81$ times that of the Arizona quake.

3. **Moore's law**:

   (a) If a chip were introduced in the year 2008, then that chip would be $10^{0.2\times 8} = 40$ times as fast as a chip introduced in 2000, such as the Pentium 4.

   (b) The speed of a chip will be $10{,}000$ times the speed of the Pentium 4 when $10{,}000 = 10^{0.2t}$, where $t$ is years since 2000, according to Moore's law. Now $10{,}000 = 10^4$, so $4 = 0.2t$, and so $t = \dfrac{4}{0.2} = 20$. Thus that limit will be reached in the year 2020.

   (c) If the fastest speed possible is about $10^{40}$ times that of the Pentium 4, then, according to Moore's law, this limit will be reached when $10^{40} = 10^{0.2t}$, where $t$ is years since 2000. Thus $40 = 0.2t$ and so $t = \dfrac{40}{0.2} = 200$, and so that limit will be reached in the year 2200.

4. **Brightness of stars**: For this exercise, $m = 2.5 \log I$, where $m$ is the apparent magnitude and $I$ is the relative intensity of light (relative to Vega).

   (a) If the light striking Earth from Vega is 2.9 times as bright as that of the star Fomalhaut, then $I = 2.9$. The apparent magnitude of Fomalhaut is therefore $m = 2.5 \log I = 2.5 \log 2.9 = 1.16$.

   (b) If Antares has an apparent magnitude of $m = 0.92$, then $0.92 = 2.5 \log I$, and so $\log I = 0.92/2.5 = 0.368$. Thus the intensity of light reaching Earth from Vega is $I = 10^{0.368} = 2.33$ times as intense as that from Antares.

   (c) If the intensity of light striking Earth from one star is twice that of another, then their magnitudes differ by $2.5 \log 2 = 0.75$. The dimmer star has a magnitude of 0.75 more than that of the brighter star (!).

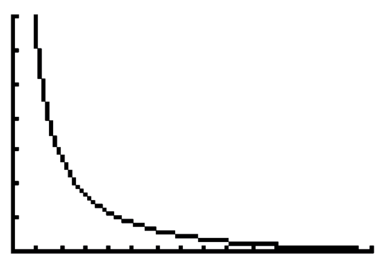
5. **The pH scale**:

   (a) Rain in the eastern United States has a pH level of 3.8, which is 1.8 less than the 5.6 pH level of normal rain. Thus the eastern United States rain is $10^{1.8} = 63.10$ times as acidic as normal rain.

   (b) A pH of 5 is 0.6 less than a pH of 5.6, so such water is $10^{0.6} = 3.98$ times as acidic as normal water.

6. **Dispersion models**:

   (a) The figure below shows a graph of $n$ against $r$ using a horizontal span of 0 to 15 and a vertical span of 0 to 7.

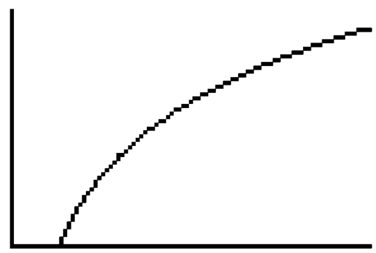


   (b) Using the formula, we see that the number of pill bugs to be found within 2 meters from the release point is $n = -0.772 + 0.297 \log 2 + \dfrac{6.991}{2} = 2.8$, so about 3 individuals. We could also have used a table or graph to solve this.

   (c) We would to find only a single individual when $n = 1$, that is, when $1 = -0.772 + 0.297 \log r + \dfrac{6.991}{r}$. Solving, for example using a table or graph, we find that $r = 4.42$ meters.

7. **Weight gain**:

(a) The figure below shows a graph of $G$ against $M$ using a horizontal span of 0 to 0.4 and a vertical span of 0 to 0.05.



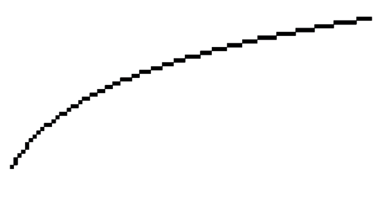
(b) If $M = 0.3$, then $G = 0.067 + 0.052 \log 0.3 = 0.04$ unit. This could also have been solved using a table or graph.

(c) A zookeeper wants $G = 0.03$, so $0.03 = 0.067 + 0.052 \log M$. This can be solved directly: $0.052 \log M = 0.03 - 0.067 = -0.037$, so $\log M = \frac{-0.037}{0.052}$ and therefore $M = 10^{-0.037/0.052} = 0.19$ unit should be the daily milk-energy intake.

(d) The graph is increasing and concave down; thus higher values of $M$ produce smaller and smaller effects on the value of $G$, supporting the quotation from the study.

8. **Reaction time**:

(a) The figure below shows a graph of $R$ versus $N$ using a horizontal span of 1 to 10 and a vertical span of 0 to 0.7.



(b) In functional notation the reaction time if there are 7 choices is expressed by $R(7)$. The value of $R(7)$ is given by the formula, for example, as $R(7) = 0.17 + 0.44 \log 7 = 0.54$ second.

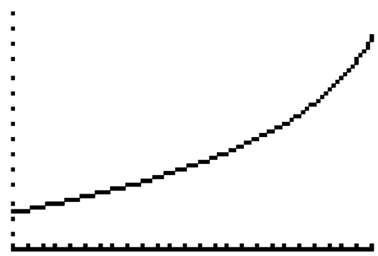
(c) If $R = 0.5$, then $0.5 = 0.17 + 0.44 \log N$. Solving for $N$, we find that $0.44 \log N = 0.5 - 0.17 = 0.33$, so $\log N = \frac{0.33}{0.44} = 0.75$ and therefore $N = 10^{0.75} = 5.62$. Thus if the reaction time is to be at most 0.5 second, then there can be at most 5 choices.

(d) If the number of choices increases by a factor of 10, then $\log N$ increases by 1, so the reaction time $R$ increases by $0.44$ second.

(e) The graph is increasing and concave down. Thus as the number of choices increases, then reaction time increases, but less and less with each additional choice.

9. **Age of haddock**:

(a) The figure below shows a graph of age $T$ versus length $L$ using a horizontal span of 25 to 50 and a vertical span of 0 to 15.

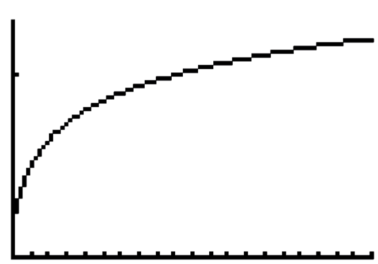


(b) In functional notation the age of a haddock that is 35 centimeters long is expressed by $T(35)$. The value of $T(35)$ is given by the formula, for example, as $T(35) = 19 - 5\ln(53 - 35) = 4.55$ years old.

(c) If a haddock is 10 years old, then $T = 10$ and so $10 = 19 - 5\ln(53 - L)$. Solving for $L$ directly, we see that $-5\ln(53 - L) = 10 - 19 = -9$, so $\ln(53 - L) = \frac{-9}{-5} = 1.8$. Thus $53 - L = e^{1.8}$, so $53 = e^{1.8} + L$ and finally $L = 53 - e^{1.8} = 46.95$ centimeters.

10. **Growth rate**:

(a) The figure below shows a graph of length $L$ versus age $T$ using a horizontal span of 0 to 20 and a vertical span of 0 to 1.3.



(b) In practical terms $L(15)$ is the length in feet of the animal when it is 15 years old. The value is obtained using the formula: $L(15) = 0.6\log(2 + 5 \times 15) = 1.13$ feet.

(c) If the animal is 1 foot long, then $1 = 0.6 \log(2+5T)$. Solving for $T$, we calculate that $\log(2+5T) = \dfrac{1}{0.6}$ and so $2+5T = 10^{1/0.6}$. Thus $5T = 10^{1/0.6} - 2$ and so $T = \dfrac{10^{1/0.6}-2}{5} = 8.88$ years.

(d) The graph is increasing and concave down. Thus as the animal gets older, it grows longer, but less and less each year.

(e) We want to solve $L = 0.6\log(2+5T)$ for $T$:

$$\begin{aligned} L &= 0.6\log(2+5T) \\ \frac{L}{0.6} &= \log(2+5T) \\ 10^{L/0.6} &= 2+5T \\ 10^{L/0.6}-2 &= 5T \\ T &= \frac{10^{L/0.6}-2}{5}. \end{aligned}$$

11. **Stand density**:

(a) If the stand has $N = 500$ trees per acre, and the diameter of a tree of average size is $D = 7$ inches, then, according to the formula, $\log SDI = \log 500 + 1.605\log 7 - 1.605 = 2.45$ and therefore the stand-density index is $SDI = 10^{2.45} = 281.84$, or about 282.

(b) If $D$ remains the same and $N$ is increased by a factor of 10, then $\log N$ will increase by 1 and so the logarithm of the stand-density index will increase by 1. Since the logarithm increases by 1, the $SDI$ will increase by a factor of $10^1 = 10$.

(c) If the diameter of a tree of average size is 10 inches, then

$$\log SDI = \log N + 1.605\log 10 - 1.605 = \log N + 1.605\times 1 - 1.605 = \log N,$$

and therefore $SDI = N$.

12. **Fully stocked stands**:

(a) If the constant $k$ is 4.1 for loblolly pines, and the diameter of a tree of average size is $D = 8$ inches, then $\log N = -1.605\log 8 + 4.1 = 2.65$. Thus there are $N = 10^{2.65} = 446.68$, or about 447, trees per acre in a fully stocked stand.

(b) If the average size of a tree $D$ is multiplied by a factor of 2, then $\log D$ is increased by $\log 2$. Thus $\log N$ is increased by $-1.605\times\log 2 = -0.482$, and therefore $N$ is multiplied by a factor of $10^{-1.605\times\log 2} = 0.33$.

(c) If the constant $k$ increases by 1 while $D$ is unchanged, then $\log N$ increases by 1 and therefore $N$ is multiplied by a factor of 10.

13. **Spectroscopic parallax**:

(a) If the distance to the star Kaus Astralis is $D = 38.04$ parsecs, then its spectroscopic parallax is $S = 5 \log 38.04 - 5 = 2.90$.

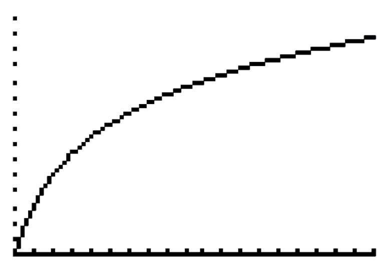
(b) If the spectroscopic parallax for the star Rasalhague is $S = 1.27$, then $1.27 = 5 \log D - 5$, so $5 \log D = 1.27 + 5 = 6.27$, and therefore $\log D = \frac{6.27}{5}$. Thus the distance to Rasalhague is $D = 10^{6.27/5} = 17.95$ parsecs.

(c) If distance $D$ is multiplied by 10, then $\log D$ is increased by 1, and so the spectroscopic parallax $S$ is increased by $5 \times 1 = 5$ units.

(d) Because the star Shaula is 3.78 times as far away as the star Atria, the distance $D$ for Atria in multiplied by 3.78. As in Part (c), the spectroscopic parallax $S$ is increased by $5 \log 3.78 = 2.89$, so that the spectroscopic parallax for Shaula is 2.89 units more than that of Atria.

14. **Rocket flight**:

(a) The figure below is a graph of $v = 4.6 \ln R$ versus $R$ using a horizontal span of 1 to 20 and a vertical span of 0 to 15.



(b) Since the graph in Part (a) is increasing, an increase in the mass ratio will increase the velocity attained, which is surely desirable.

(c) A velocity of $v = 7.8$ kilometers per second is attained when $7.8 = 4.6 \ln R$, so $\ln R = \frac{7.8}{4.6}$. Thus the smallest mass ratio for which this velocity is attained is $R = e^{7.8/4.6} = 5.45$.

15. **Rocket staging**:

(a) Since $R_1 = R_2 = 3.4$ and $c = 3.7$, the total velocity attained by this two-stage craft is $v = 3.7 \times \ln 3.4 + 3.7 \times \ln 3.4 = 9.06$ kilometers per second.

(b) Since the velocity $v$ from Part (a) is greater than 7.8 kilometers per second, this craft can achieve a stable orbit.

16. **Stereo speakers**:

(a) Since there are four speakers, the relative intensity is multiplied by 4. Now $10 \log 4 = 6.02$, so the effect of the four speakers together on the decibel level is to raise it from that of one speaker, 40 decibels, by an additional 6.02 decibels, for a total of 46.02, or about 46, decibels.

(b) The decibel level $D$ is given by $D = 10 \log I$, where $I$ is the relative intensity. We can solve for $I$ by first dividing by 10: $\frac{D}{10} = \log I$. Now we raise $10$ to that power: $10^{D/10} = 10^{\log I} = I$. Thus the formula is $I = 10^{D/10}$.

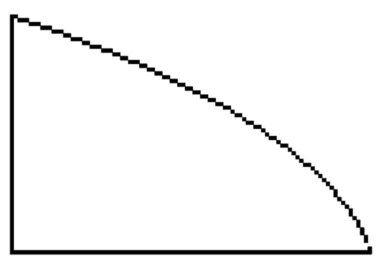
(c) The relative intensity $I = 10^{D/10} = \left(10^{1/10}\right)^D$, so $I$ as exponential function of decibels $D$ has a growth factor of $10^{1/10} = 1.26$.

(d) If the decibel level $D$ increases by 3 units, then the relative intensity $I$ increases by a factor of $1.26^3 = 2$.

17. **Relative abundance of species**:

(a) Since $S = 197$ and $N = 6814$, $\frac{S}{N} = \frac{197}{6814} = 0.0289$ for this collection.

(b) The figure below shows a graph of the function $\frac{x-1}{x} \ln(1-x)$ using a horizontal span of 0 to 1 and a vertical span of 0 to 1.

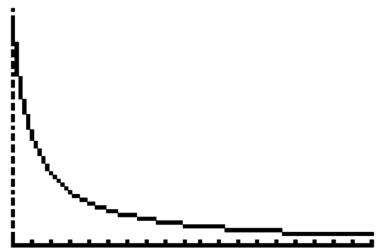


(c) Using crossing graphs, we can determine the value of $x$ for which $\frac{x-1}{x} \ln(1-x) = 0.0289$. We find that $x = 0.9945$.

(d) For this collection $\alpha = \frac{N(1-x)}{x} = \frac{6814(1-0.9945)}{0.9945} = 37.68$.

(e) The number of species of moth in the collection represented by 5 individuals is $\alpha\left(\frac{x^5}{5}\right) = 37.68\left(\frac{0.9945^5}{5}\right) = 7.33$, or about 7 species of moths.

(f) The figure below shows a graph of the number of species represented by $n$ individuals, that is $\alpha\left(\frac{x^n}{n}\right) = 37.68\left(\frac{0.9945^n}{n}\right)$ against $n$, using a horizontal span of 0 to 20 and a vertical span of 0 to 40.



## 4.5 CONNECTING EXPONENTIAL AND LINEAR DATA

E-1. **Verifying the quotient law**: To verify the quotient law for logarithms $\ln\frac{x}{y} = \ln x - \ln y$, suppose that $\ln x = p$ and $\ln y = q$. Then $x = e^p$ and $y = e^q$. Hence

$$\frac{x}{y} = \frac{e^p}{e^q} = e^{p-q}.$$

Thus $p - q$ is the power of $e$ that gives $\frac{x}{y}$, and therefore

$$\ln\frac{x}{y} = p - q = \ln x - \ln y.$$

E-2. **Exponentiating linear functions**: If we exponentiate a linear function $y = mx + b$, then we get (using the properties of exponents)

$$e^y = e^{mx+b} = e^{mx}e^b = (e^m)^x e^b = e^b \times (e^m)^x .$$

This is now in the form of an exponent function with initial value $P = e^b$ and growth or decay factor $a = e^m$.

E-3. **An important fact about logarithms**: Since $\ln x$ is the power of $e$ which gives $x$, $\ln e^x$ is the power of $e$ which gives $e^x$ and that is $x$ (!).

E-4. **Calculating using the laws of logarithms**:

(a)

$$\begin{aligned}\ln\left(\frac{x^4}{y^2}\right) &= \ln(x^4)-\ln(y^2)\\ &= 4\ln x - 2\ln y = 4\times 4 - 2\times 5 = 6\end{aligned}$$

(b)

$$\ln x^3y^5 = \ln(x^3)+\ln(y^5) = 3\ln x + 5\ln y = 3\times 4 + 5\times 5 = 37$$

(c)

$$\ln\left(\frac{1}{x}\right) = \ln 1 - \ln x = 0 - 4 = -4$$

(d)

$$\ln\sqrt{xy} = \ln\left((xy)^{1/2}\right) = \frac{1}{2}\ln(xy) = \frac{1}{2}(\ln x + \ln y) = \frac{1}{2}(4+5) = 4.5$$

E-5. **More calculations using laws of logarithms**:

(a) $\ln e^3 = 3\ln e = 3\times 1 = 3$

(b)

$$\ln\left(\frac{\sqrt{e}}{e^4}\right) = \ln(\sqrt{e}) - \ln(e^4) = \ln(e^{1/2} - 4\ln e = \frac{1}{2}\ln e - 4\ln e = \frac{1}{2} - 4 = -\frac{7}{2} \text{ or } -3.5$$

(c) $\ln\dfrac{1}{e} = \ln 1 - \ln e = 0 - 1 = -1$

(d)

$$\begin{aligned}\ln\ln\left(e^{(e^4)}\right) = \ln\left(\ln\left(e^{(e^4)}\right)\right)\\ &= \ln\left(e^4(\ln e)\right)\\ &= \ln(e^4) = 4\ln e = 4\times 1 = 4\end{aligned}$$

E-6. **Logarithms to other bases**:

(a) Since $64 = 2^6$, $\log_2 64 = 6$.

(b) Since $\frac{1}{4} = 2^{-2}$, $\log_2\dfrac{1}{4} = -2$.

(c) Since $27 = 3^3$, $27^{1/3} = 3$, and therefore $\log_{27} 3 = \dfrac{1}{3}$

(d) Since $8 = 8^1$, $\log_8 8 = 1$.

(e) Since, for any $a > 0$, $1 = a^0$, $\log_a 1 = 0$.

(f) Now $\log_a x$ is the power of $a$ which gives $x$. Thus $a^{\log_a x} = x$.

(g) Similarly, since $\log_a x$ is the power of $a$ which gives $x$, $\log_a a^x$ is the power of $a$ which gives $a^x$ and that is $x$ (!).

E-7. **Change of base**: Consider $p = \log_a b$ and $q = \log_b x$. This means that $a^p = b$ and $b^q = x$. Thus, substituting $a^p$ for $b$, we have $(a^p)^q = x$, and so $a^{pq} = x$. By the definitions of $p$ and $q$, this means that

$$a^{(\log_a b)(\log_b x)} = x,$$

as we needed to show. By definition of $\log_a$, the equation above means that $\log_a x = (\log_a b)(\log_b x)$. Dividing each side by $\log_a b$ gives the change-of-base formula.

E-8. **Using the change of base formula**: Using $a = e$ in E-7, we have $\log_b x = \dfrac{\ln x}{\ln b}$.

(a) $\log_7 12 = \dfrac{\ln 12}{\ln 7} = 1.28$

(b) $\log_{3.2} 5.8 = \dfrac{\ln 5.8}{\ln 3.2} = 1.51$

(c) $\log_5 e = \dfrac{\ln e}{\ln 5} = 0.62$

S-1. **Exponential transformation**: The second table consists of the same input values, while the function values are the exponentials of the function values of linear data; therefore the second table consists of exponential data.

S-2. **Logarithmic conversion**: The second table consists of the same input values, while the function values are the logarithms of the function values of exponential data; therefore the second table consists of linear data.

S-3. **Slope and growth factor**: If the slope of the regression line for the natural logarithm of exponential data is $m = -0.77$, then the decay factor for the exponential function is $a = e^m = e^{-0.77} = 0.463$.

S-4. **Regression line to exponential model**: If the regression line for the natural logarithm of exponential data is $y = 3x + 4$, then the exponential model for the data is $P \times a^x$, where $P = e^4 = 54.60$ and $a = e^3 = 20.09$. Thus the exponential model is $54.60 \times 20.09^x$.

S-5. **Exponential regression**: Calculating the natural logarithm of the data, we obtain the table

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\ln y$ | 1.411 | 2.163 | 2.955 | 3.353 | 4.170 |

The regression line is calculated as $\ln y = 0.671x + 0.798$.

S-6. **Exponential regression**: Calculating the natural logarithm of the data, we obtain the table

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\ln y$ | $-0.357$ | $-1.204$ | $-2.303$ | $-2.996$ | $-4.605$ |

The regression line is calculated as $\ln y = -1.029x + 0.794$.

S-7. **Exponential regression**: Calculating the natural logarithm of the data, we obtain the table

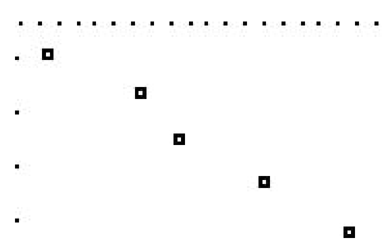
| $x$ | 4.2 | 7.9 | 10.8 | 15.5 | 20.2 |
|---|---|---|---|---|---|
| $\ln y$ | 2.015 | 2.092 | 2.140 | 2.322 | 2.510 |

The regression line is calculated as $\ln y = 0.031x + 1.849$. The exponential regression model is therefore $y = P \times a^x$, where $P = e^{1.849} = 6.35$ and $a = e^{0.031} = 1.03$. Thus the exponential model is $y = 6.35 \times 1.03^x$.

S-8. **Exponential regression**: Calculating the natural logarithm of the data, we obtain the table

| $x$ | 22.4 | 27.3 | 29.4 | 34.1 | 38.6 |
|---|---|---|---|---|---|
| $\ln y$ | $-2.937$ | $-3.689$ | $-4.510$ | $-5.298$ | $-6.215$ |

A plot of the natural logarithm of the data is shown below.



The regression line is calculated as $\ln y = -0.206x + 1.721$.

S-9. **Exponential regression**: Calculating the natural logarithm of the data, we obtain the table

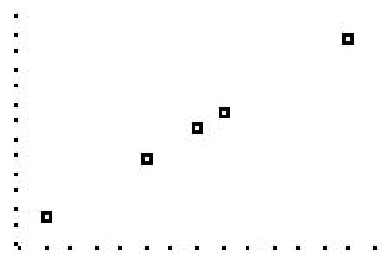
| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\ln y$ | 1.308 | 1.459 | 1.808 | 2.208 | 2.610 |

The regression line is calculated as $\ln y = 0.335x + 0.873$. The exponential regression model is therefore $y = P \times a^x$, where $P = e^{0.873} = 2.39$ and $a = e^{0.335} = 1.40$. Thus the exponential model is $y = 2.39 \times 1.40^x$.

S-10. **Exponential regression**: Calculating the natural logarithm of the data, we obtain the table

| $x$ | 3 | 7 | 9 | 10 | 15 |
|---|---|---|---|---|---|
| $\ln y$ | 3.512 | 6.896 | 8.607 | 9.460 | 13.703 |

A plot of the natural logarithm of the data is shown below.



The regression line is calculated as $\ln y = 0.850x + 0.959$.

S-11. **Exponential regression**: Calculating the natural logarithm of the data, we obtain the table

| $x$ | 2 | 5 | 6 | 9 | 10 |
|---|---|---|---|---|---|
| $\ln y$ | 1.435 | 1.856 | 2.001 | 2.425 | 2.557 |

The regression line is calculated as $\ln y = 0.141x + 1.155$. The exponential regression model is therefore $y = P \times a^x$, where $P = e^{1.155} = 3.17$ and $a = e^{0.141} = 1.15$. Thus the exponential model is $y = 3.17 \times 1.15^x$.

S-12. **Exponential regression**: Calculating the natural logarithm of the data, we obtain the table

| $x$ | 4 | 11 | 15 | 17 | 18 |
|---|---|---|---|---|---|
| $\ln y$ | 0 | 2.851 | 4.473 | 5.284 | 5.689 |

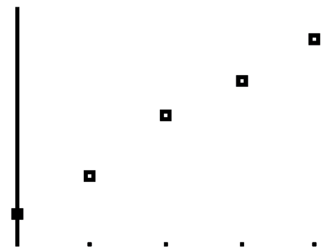
The regression line is calculated as $\ln y = 0.406x - 1.623$. The exponential regression model is therefore $y = P \times a^x$, where $P = e^{-1.623} = 0.20$ and $a = e^{0.406} = 1.50$. Thus the exponential model is $y = 0.20 \times 1.50^x$.

1. **Population**: Because the logarithm of exponential data is linear, a plot of the logarithm of the population values should be linear.

2. **Radioactive decay**: If a plot of the logarithm of the amount is linear, then the scientist should use an exponential function to model the original data.

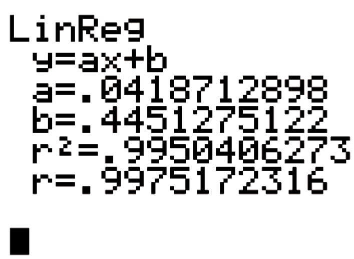
3. **Population growth**: Let $t$ be years since 2001 and $N$ the population (in thousands).

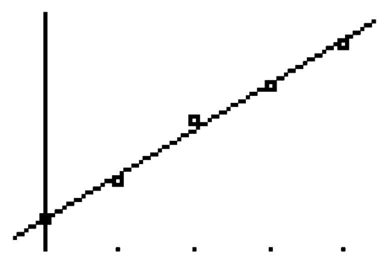
   (a) The plot of the natural logarithm of the data points is in the figure below.



   Since these points are close to being on a line, it is reasonable to approximate the original data with an exponential function.

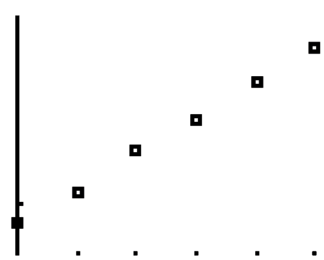
   (b) The regression line parameters for the natural logarithm of the data are calculated in the left-hand figure below. We find that $\ln N = 0.042t + 0.445$. We have added this graph to the data plot in the right-hand figure below.



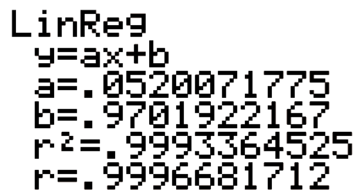
4. **Magazine circulation**: Let $t$ be years since 2001 and $C$ the circulation (in thousands).

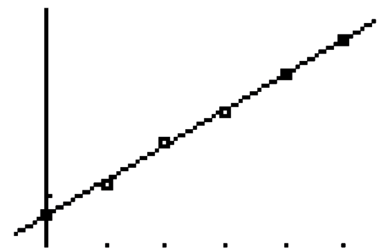
   (a) The plot of the natural logarithm of the data points is in the figure below.



   Since these points are close to being on a line, it is reasonable to approximate the original data with an exponential function.

(b) The regression line parameters for the natural logarithm of the data are calculated in the left-hand figure below. We find that $\ln C = 0.052t + 0.970$. We have added this graph to the data plot in the right-hand figure below.



5. **Population**: Since $\ln N = 0.039t - 0.693$, an exponential model for $N$ is $P \times a^t$, where $P = e^{-0.693} = 0.50$ and $a = e^{0.039} = 1.04$. Thus an exponential model is $N = 0.50 \times 1.04^t$.

6. **Radioactive decay**: Since $\ln A = -0.073t + 2.336$, an exponential model for $A$ is $P \times a^t$, where $P = e^{2.336} = 10.34$ and $a = e^{-0.073} = 0.93$. Thus an exponential model is $A = 10.34 \times 0.93^t$.

7. **Sales growth**:

   (a) Since $\ln S = 0.049t + 2.230$, an exponential model for $S$ is $P \times a^t$, where $P = e^{2.230} = 9.30$ and $a = e^{0.049} = 1.05$. Thus an exponential model is $S = 9.30 \times 1.05^t$.

   (b) By Part (a), the growth factor is 1.05, so sales grow by 5% each year.

   (c) Using the formula from Part (a), $S(6) = 9.30 \times 1.05^6 = 12.46$. In practical terms, this means 6 years after 2005, that is, at the start of 2011, sales will be $12.46$ thousand dollars.

   (d) We expect sales to reach a level of 12 thousand dollars when $12 = 9.30 \times 1.05^t$. Solving, for example by using a table or graph, we find that $t = 5.22$. Thus we expect sales to reach \$12,000 when $t = 5.22$, or about mid-March of 2010.

8. **Population decline**:

   (a) Since $\ln N = -0.051t + 1.513$, an exponential model for the population $N$ is $P \times a^t$, where $P = e^{1.513} = 4.54$ and $a = e^{-0.051} = 0.95$. Thus an exponential model is $N = 4.54 \times 0.95^t$.

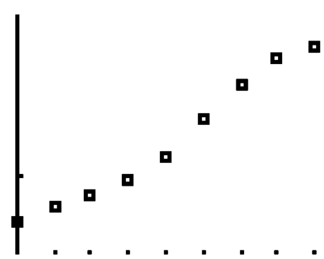
   (b) Since the decay factor is 0.95, which is $1 - 0.05$, the population is decreasing by 5% each year.

(c) Since $t$ is years since the start of 2005, the start of 2008 is represented by $t = 3$. In functional notation, population at the start of 2003 is expressed by $N(3)$. The value of $N(3)$ is $4.54 \times 0.95^3 = 3.89$ thousand.

(d) The population will fall to a level of 3 thousand when $3 = 4.54 \times 0.95^t$. Solving for $t$, for example using a table or graph, shows that $t = 8.08$, so we expect the population to decline to 3 thousand when $t = 8.08$, or about the start of February of 2013.

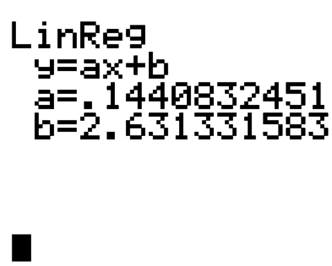
9. **Cable TV**: Let $t$ be years since 1976 and $C$ the percent with cable.

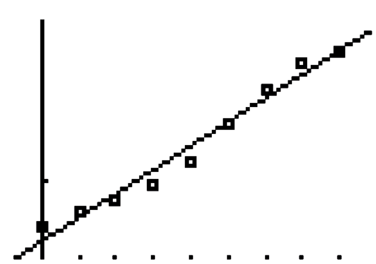
(a) The plot of the natural logarithm of the data points is in the figure below.



Since these points are close to being on a line, it is reasonable to approximate the original data with an exponential function.

(b) The regression line parameters for the natural logarithm of the data are calculated in the left-hand figure below. We find that $\ln C = 0.144t + 2.631$. We have added this graph to the data plot in the right-hand figure below.



(c) The slope of the regression line is 0.144, and so the yearly growth factor is

$$a = e^{0.144} = 1.155.$$

The exponential initial value is

$$P = e^{2.631} = 13.888.$$

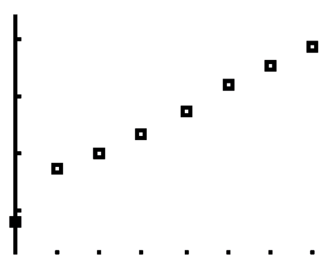
The exponential model is given by

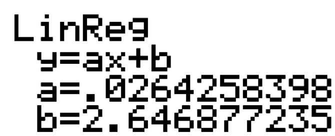
$$C = 13.888 \times 1.155^t.$$

10. **Auto parts production workers**: Let $t$ be years since 1987 and $W$ the hourly wage in dollars.

(a) As seen in the left-hand figure below, the plot of the natural logarithm of the data points looks linear except for the initial point. Since this plot is close to linear, it is reasonable to model the original data with an exponential function.

(b) As is shown in the right-hand figure below, the equation of the regression line is $\ln W = 0.026t + 2.647$.






(c) The yearly growth factor is

$$a = e^{0.026} = 1.026,$$

while the initial value is

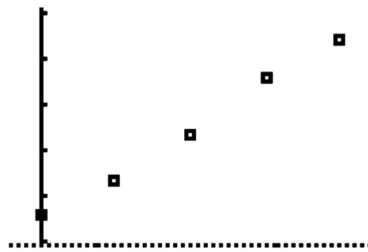
$$P = e^{2.647} = 14.112.$$

Thus the exponential model is given by

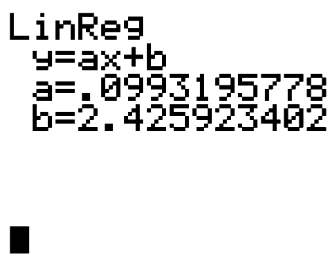
$$W = 14.112 \times 1.026^t.$$

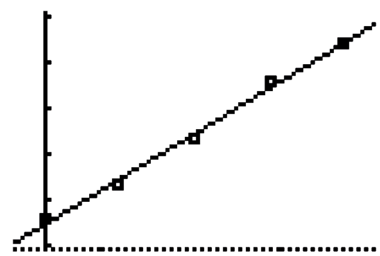
11. **National health care spending**: Let $t$ be years since 1950 and $H$ the costs in billions of dollars.

    (a) The plot of the natural logarithm of the data is shown below. Here we used the number of years since 1950 for the horizontal axis. The data points look like they lie on a straight line, so it is reasonable to model the original data with an exponential function.

    (b) The left-hand figure below shows that the equation of the regression line is $\ln H = 0.099t + 2.426$. The graph is added to the data plot in the right-hand figure below.

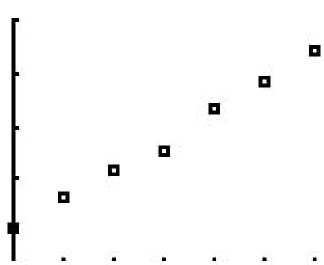

    

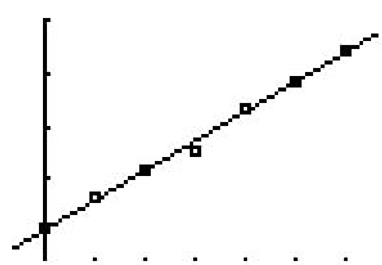
    (c) Since the slope of the line is 0.099, the yearly growth factor is

    $$a = e^{0.099} = 1.104.$$

    Therefore $r = 0.104$, so the yearly percentage increase is 10.4%.

12. **A bad data point**: If we plot the natural logarithm of the data we get the picture on the left below.

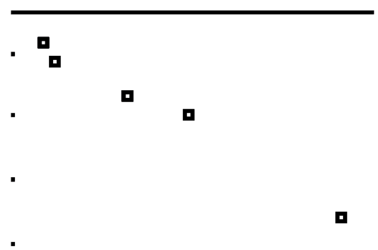




From the picture it may be unclear which data point does not line up with the others. To clarify the situation we calculate the regression line parameters and find $\ln N = 0.565t+3.031$. In the right-hand figure above we have added the graph of the regression line to the data plot. The only point which is off is the one corresponding to $t = 3$, for which the value of $N$ is a bit small.

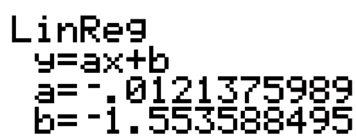
13. **Grazing rabbits**:

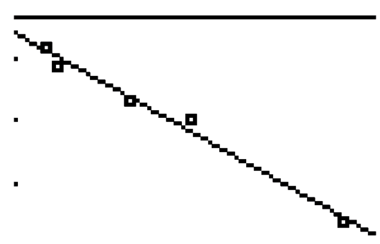
(a) Below is a plot of $\ln D$ against $V$.



Since the points roughly fall on a straight line, it appears that $D$ is approximately an exponential function of $V$.

(b) From the left-hand figure below, the regression line is $\ln D = -0.012V - 1.554$. We have added it to the data plot in the right-hand figure.



(c) The exponential function which approximates $D$ is found from the regression line. The decay factor is $a = e^{-0.012} = 0.988$, and the initial value is $P = e^{-1.554} = 0.211$, so the function is

$$D = 0.211 \times 0.988^V.$$

(d) The satiation level for the rabbit is 0.18 pound per day, and thus we have

$$D = \text{satiation level} - A = 0.18 - A.$$

Solving for $A$ yields $A = 0.18 - D$, and, using the formula from Part (c), we find that

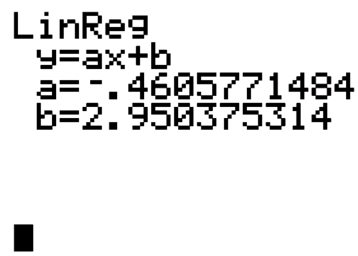
$$A = 0.18 - 0.211 \times 0.988^V.$$

14. **Growth in length**:

(a) Now $D$ is the difference between the maximum length, which is 14.8 inches, and $L$, so $D = 14.8 - L$. Here's the table:

| $t =$ age | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $D =$ difference | 11.1 | 7.3 | 4.8 | 3.3 | 2.1 | 1.3 | 0.8 | 0.4 |

(b) From the figure below, we see that the regression line is $\ln D = -0.461t + 2.950$.



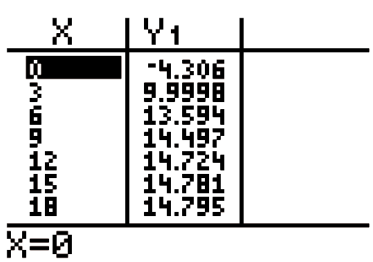

We find the exponential function by computing the decay factor and initial value from the regression line. The decay factor is $a = e^{-0.461} = 0.631$, and the initial value is $P = e^{2.950} = 19.106$. Thus $D$ is approximated by

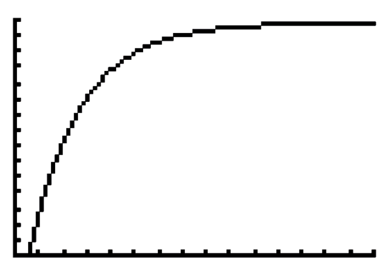
$$19.106 \times 0.631^t.$$

(c) We know that $D = 14.8 - L$, and this equation can be solved for $L$ to yield $L = 14.8 - D$. Using the formula from Part (b), we find that

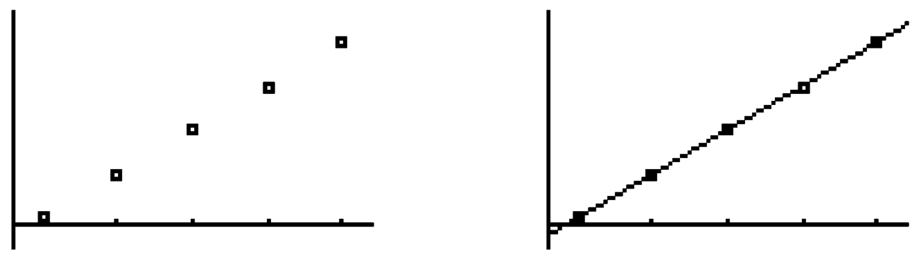
$$L = 14.8 - 19.106 \times 0.631^t.$$

(d) The table below leads us to choose a horizontal span of 0 to 15 and a vertical span of 0 to 15. The horizontal axis on the graph is age, and the vertical axis is length.



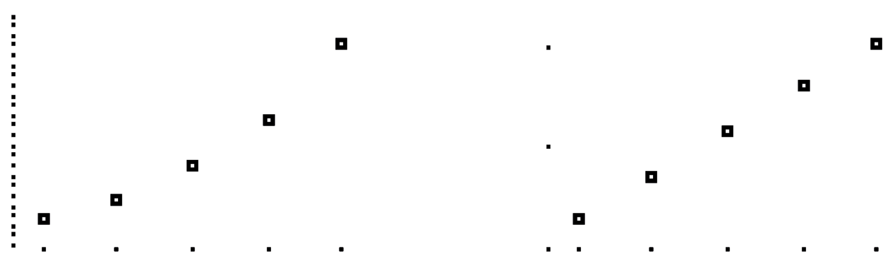



15. **Nearly linear or exponential data**: The left-hand figure below is the plot of the data from Table A. This appears to be linear. Computing the regression line $f = 19.84t - 16.26$ and adding it to the plot (right-hand figure) shows how very close to linear $f$ is.



We conclude that Table A is approximately linear, with a linear model of $f = 19.84t - 16.26$.

The plot of data from Table B in the left-hand figure below looks more like exponential data than linear data. We plot the logarithm of the data in the right-hand figure, and this falls on a straight line, which indicates that the data is indeed exponential.

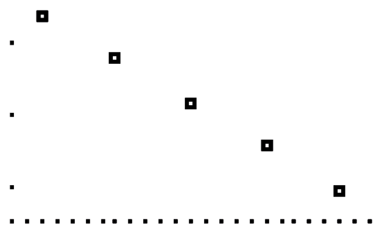


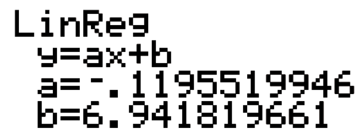
The equation of the regression line is $\ln g = 0.442t + 0.846$. The exponential model therefore has a growth factor of $a = e^{0.442} = 1.56$ and an initial value of $P = e^{0.846} = 2.33$, which gives the formula

$$g = 2.33 \times 1.56^t.$$

16. **Atmospheric pressure**:

(a) The plot of the natural logarithm of the data is in the left-hand figure below. Let $h$ be the altitude in kilometers and $P$ the atmospheric pressure in grams per square centimeter. From the right-hand figure below, the equation of the regression line is $\ln P = -0.120h + 6.942$.






(b) The exponential model for the data is computed from the regression line. The decay factor is $a = e^{-0.120} = 0.887$, and the initial value is $e^{6.942} = 1034.84$. The equation for the exponential model for pressure is therefore

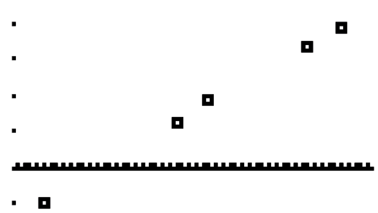
$$P = 1034.84 \times 0.887^h \text{ grams per square centimeter.}$$

(c) An altitude of 30 kilometers corresponds to $h = 30$, so the atmospheric pressure is

$$P = 1034.84 \times 0.887^{30} = 28.35 \text{ grams per square centimeter.}$$

17. **Sound pressure**:

(a) A plot the natural logarithm of the data is shown below. Since a plot of the natural logarithm of $P$ against $D$ appears linear, it is reasonable to model pressure as an exponential function of loudness.

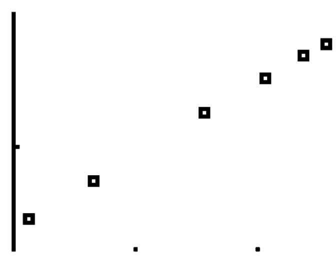


(b) To find an exponential model of $P$ as a function of $D$, first we find a linear regression model for $\ln P$. Calculating from the logarithm of the data, we find the regression model is $\ln P = 0.109D - 8.072$. For the exponential model, we use an initial value of $e^{-8.072} = 0.0003$ and a growth factor of $e^{0.109} = 1.12$; therefore the exponential model is $P = 0.0003 \times 1.12^D$.

(c) When loudness $D$ is increased by one decibel, the pressure $P$ is multiplied by a factor of $1.12$, that is, it increases by 12%.

18. **Magnitude and distance**:

(a) The figure below shows a plot of $\ln d$ against $m$. Since the graph appears linear, it is reasonable to model distance as an exponential function of magnitude difference.



(b) To find an exponential model of $d$ as a function of $m$, first we find a linear regression model for $\ln d$. Calculating from the logarithm of the data, we find the regression model is $\ln d = 0.453m + 3.504$. For the exponential model, we use an initial value of $e^{3.504} = 33.25$ and a growth factor of $e^{0.453} = 1.57$; therefore the exponential model is $d = 33.25 \times 1.57^m$.

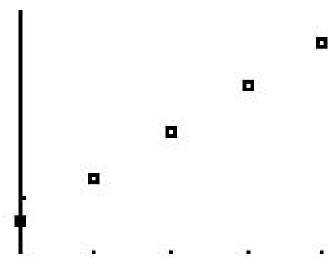
(c) If one star shows a magnitude difference of one greater than a second star, then $m$ is increased by 1. This causes $d$ to be multiplied by a factor of 1.57, so the star with the larger magnitude difference is 1.57 times further away than the other star.

(d) If Alphecca shows a magnitude difference of 1.83, then $m = 1.83$ and so $d = 33.25 \times 1.57^{1.83} = 75.91$. Thus Alphecca is 75.91 light-years from Earth.

(e) If Alderamin is 52 light years from Earth, then $d = 52$ and so $52 = 33.25 \times 1.57^m$. Solving for $m$ we find that the magnitude difference $m$ is $0.99$. Since Alderamin has an apparent magnitude of 2.47, the absolute magnitude of Alderamin is $2.47 - 0.99 = 1.48$.

19. **Catalogue sales**: Let $t$ be years since 2004 and $C$ catalogue sales in billions of dollars.

    (a) The plot of the natural logarithm of the data is shown below. We find the regression model to be $\ln C = 0.065t + 4.963$.



    (b) For the exponential model we use an initial value of $e^{4.963} = 143.022$ and a growth factor of $e^{0.065} = 1.067$. Therefore the exponential model is $C = 143.022 \times 1.067^t$.

20. **Internet sales**:

    (a) By Part (b) of the preceding exercise, the growth factor for the function $C$ is 1.067. Thus the percentage growth rate for catalogue sales is 6.7%. From the formula for $I$ we see that the growth factor is 1.27. Thus the percentage growth rate for Internet sales is 27%.

    (b) We want to find the smallest whole number $t$ for which $I$ is greater than $C$. Using a table of values or a graph gives $t = 6$. Thus Internet sales will overtake catalogue sales 6 years after 2004, or in 2010.

## Chapter 4 Review Exercises

1. **Percentage growth**: The population grows by 4.5% each year, so the yearly growth factor is $a = 1 + r = 1 + 0.045 = 1.045$. Since a month is $1/12$th of a year, the monthly growth factor is $1.045^{1/12} = 1.0037$. That is a growth of 0.37% per month.

2. **Percentage decline**: The population declines by 0.5% each year, so the yearly decay factor is $a = 1 - r = 1 - 0.005 = 0.995$. Since a decade is 10 years, the decade decay factor is $0.995^{10} = 0.9511$. Since $1 - 0.9511 = 0.0489$, that is a decline of 4.89% per decade.

3. **Credit cards**:

   (a) The decay factor is $a = (1 + 0.02)(1 - 0.05) = 0.969$.

   (b) Because the decay factor is $a = 0.969$, we have $r = 1 - 0.969 = 0.031$. Hence the balance decreases by 3.1% each month.

   (c) Because 3 years is 36 months, the balance after 3 years of payments will be $1000 \times 0.969^{36} = 321.85$ dollars.

4. **More on credit cards**:

   (a) We have $r = 0.015$ and $m = 0.04$, so the base is $a = (1 + 0.015)(1 - 0.04) = 0.9744$.

   (b) Because the decay factor is $a = 0.9744$, we have $r = 1 - 0.9744 = 0.0256$. Hence the balance decreases by 2.56% each month.

   (c) We have $r = 0.015$ and $m = 0.01$, so the base is $a = (1+0.015)(1-0.01) = 1.00485$. This number is greater than 1, so the balance will increase each month. This is not a realistic situation.

5. **Testing exponential data**: Calculating the ratios of successive terms of $y$, we get $\frac{30.0}{25.0} = 1.2$, $\frac{36.0}{30.0} = 1.2$, and $\frac{43.2}{36.0} = 1.2$. Since the $x$ values are evenly spaced and these ratios show a constant value of 1.2, the table does show exponential data.

6. **Modeling exponential data**: The data from Exercise 5 show a constant ratio of 1.2 for $x$ values increasing by 1's, and so the growth factor is $a = 1.2$. The initial value of $P = 25.0$ comes from the first entry in the table. Thus an exponential model for the data is $y = 25 \times 1.2^x$.

7. **Credit card balance**:

   (a) Each successive ratio of new/old is 0.97, so this is exponential data. Because $n$ increases by single units, the decay factor is $a = 0.97$. The initial value of 500.00 comes from the first entry in the table. Thus an exponential model for the data is $B = 500 \times 0.97^n$.

   (b) From the first entry in the table the initial charge is $500.00.

   (c) Because 2 years is 24 months, the balance after 2 years of payments will be $500 \times 0.97^{24} = 240.71$ dollars.

8. **Inflation**:

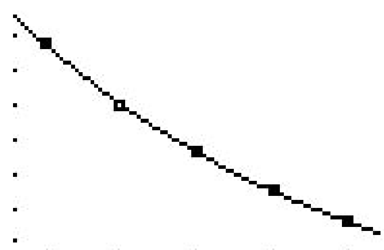
   (a) Each successive ratio of new/old is 1.03, so this is exponential data.

   (b) Let $t$ be the time in years since the start of 2002 and $P$ the price in dollars. Because $t$ increases by single units, by Part (a) the growth factor is $a = 1.03$. The initial value of $265.50$ comes from the first entry in the table. Thus an exponential model for the data is $P = 265.5 \times 1.03^t$.

   (c) We want to find the first whole number $t$ for which $P$ is greater than 325. From a table of values or a graph we find the number to be $t = 7$. Thus at the beginning of 2009 the price will surpass $325.

9. **Exponential regression**: The exponential model is $y = 20.97 \times 1.34^x$.

10. **Exponential regression**: The exponential model is $y = 10.96 \times 0.80^x$. A plot of the exponential model, together with the original data, is shown below.



11. **Sales**:

    (a) The exponential model for $A$ is $A = 8.90 \times 1.02^t$, and the exponential model for $B$ is $B = 2.30 \times 1.27^t$.

    (b) By Part (a) the growth factor for $A$ is 1.02, so the sales for *Alpha* are growing by 2% per year. By Part (a) the growth factor for $B$ is 1.27, so the sales for *Beta* are growing by 27% per year.

    (c) We want to find the first whole number $t$ for which $B$ is greater than $A$. From a table of values or a graph we find the number to be $t = 7$. Thus at the beginning of 2009 sales for *Beta* will overtake sales for *Alpha*.

12. **Credit card payments**:

    (a) The exponential model for $B$ is $B = 520 \times 0.9595^n$.

    (b) From Part (a) the initial charge was \$520.00.

    (c) From Part (a) the decay factor is 0.9595. Because $m = 0.05$ we know that

    $$(1 + r)(1 - 0.05) = 0.9595,$$

    and thus

    $$1 + r = \frac{0.9595}{1 - 0.05}.$$

    Hence

    $$r = 1 - \frac{0.9595}{1 - 0.05} = 0.01.$$

    Thus the monthly finance charge is 1%.

13. **Solving logarithmic equations**: If $\log x = -1$ then $x = 10^{-1} = 0.1$.

14. **Comparing logarithms**: If $\log x = 3.5$ and $\log y = 2.5$ then $\log y$ is 1 less than $\log x$. Hence $y$ is $10^{-1}$ times $x$, so $y = 0.1x$.

15. **Comparing earthquakes**: The Double Spring Flat quake was $6.0 - 5.5 = 0.5$ higher on the scale than the Little Skull Mountain quake. Hence the Double Spring Flat quake was $10^{0.5} = 3.16$ times as powerful as the Little Skull Mountain quake.

16. **Population growth**:

    (a) We put $N = 200$ in the formula and get $T = 25 \log 200 - 50 = 7.53$. Thus the population reaches a size of 200 after 7.53 years.

    (b) We want to find the value of $N$ for which $T = 0$. Thus we need to solve the equation $25 \log N - 50 = 0$ for $N$. We find $25 \log N = 50$ or $\log N = \frac{50}{25} = 2$. Hence $N = 10^2 = 100$. Thus the initial population size was 100.

17. **Exponential regression**: Calculating the natural logarithm of the data, we obtain the table

| $x$ | 1 | 5 | 7 | 10 | 11 |
|---|---|---|---|---|---|
| $\ln y$ | 3.336 | 4.508 | 5.093 | 5.971 | 6.264 |

The regression line is calculated as $\ln y = 0.293x + 3.043$. The exponential regression model is therefore $y = P \times a^x$, where $P = e^{3.043} = 20.97$ and $a = e^{0.293} = 1.34$. Thus the exponential model is $y = 20.97 \times 1.34^x$.

18. **Exponential regression**: Calculating the natural logarithm of the data, we obtain the table

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\ln y$ | 2.175 | 1.946 | 1.723 | 1.504 | 1.281 |

The regression line is calculated as $\ln y = -0.223x + 2.395$. The exponential regression model is therefore $y = P \times a^x$, where $P = e^{2.395} = 10.97$ and $a = e^{-0.223} = 0.80$. Thus the exponential model is $y = 10.97 \times 0.80^x$.

19. **Economic output**: If a linear model fits the logarithm of the data for economic output, then the economist should use an exponential function to fit the original data.

20. **Account balance**: Let $t$ be the time in months since the start of the year and $B$ the balance in dollars. For example, the beginning of February corresponds to $t = 1$.

(a) The regression line for the natural logarithm of the data is calculated as $\ln B = 0.0025t + 7.1309$.

(b) The slope of the regression line is 0.0025, and so the yearly growth factor is $a = e^{0.0025} = 1.0025$. The exponential initial value is $P = e^{7.1309} = 1250.00$. The exponential model is given by $B = 1250 \times 1.0025^t$.

(c) The monthly growth factor is 1.0025, so the monthly interest rate is 0.25%.

(d) To estimate the balance at the beginning of March we put $t = 2$ in the formula and obtain

$$B = 1250 \times 1.0025^2 = 1256.26 \text{ dollars.}$$