STA2503 Project 2: Dynamic Hedging

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Abstract

This study aims to synthesize the theoretical constructs of continuous time financial markets, the Black Scholes model, the Greeks and portfolio neutrality in the context of European call options. A comprehensive analysis of delta hedging and delta-gamma hedging a European call option is described, once defining an options delta and gamma metrics. Moreover, the study employs computational implementation, facilitated through the Python programming language, to simulate a portfolio and its P&L when employing each of the described hedging strategies. The resulting P&L distributions are compared, and the effect on these distributions when altering different market parameters is described. Notably, we find that altering the drift of an underlying asset causes the variance of returns to decrease, when transaction costs are non-zero. We also find that when the realized volatility is higher than the volatility used to price and hedge an option, an option seller will average negative profits, and that the opposite is also true. Finally, we thoroughly describe the portfolio contents under both hedging schemes in both the cases where an option ends in the money, and out of the money.

Introduction

Financial derivatives are instruments that derive their value from some underlying price process. For example, the holder of a European call option can claim the maximum of the difference between the underlying asset price and some predefined strike price, and zero at a pre-defined time called the maturation date. Both the holder and writer of a European call option are exposed to numerous risk factors, including changes in price to the underlying asset, changes in asset volatility, and changes in rates. In practice, practitioners often seek to hedge the risks associated with their option position in order to reduce the chances of a large negative return.

Dynamic delta hedging and delta-gamma hedging are two methods an option holder or seller can use to reduce their variance of returns. Namely, they seek to minimize the impact of the underlying asset's price changes changes on their portfolio profit/loss (P&L). Dynamic delta-hedging is a method where at discrete time-steps, the first-order risk of changes in price to the underlying are neutralized, where in dynamic delta-gamma hedging, the second-order risk is also eliminated.

In this report, a continuous-time model of European call options is used to value options, and relevant risk measures. We investigate how to develop and implement hedging strategies, and the distribution of the P&L they generate under different conditions.

In particular, we investigate how to hedge the sale of 10,000 units of a European call option, g, on an asset S whose current price is $S_0 = 10$, with time to maturity $T_g = 0.25$ and strike $K_g = 10$. The remaining relevant parameters are as follows:

1. The asset S is assumed to follow the Black-Scholes model. Hence, $dS_t = \mu S_t dt + \sigma S_t dW_t$, where $W = (W_t)_{t \geq 0}$ is a $\mathbb{P}$-Brownian motion, μ is the asset's drift, and σ is its volatility.
2. We may trade in the bank account, B_t, which satisfies $dB_t = rB_t dt$, where r is the interest rate.
3. We may trade in a hedging option, h, on the same underlying asset S, with maturity $T_h = 0.3$

and strike $K_h = 10$.

4. Each purchase and sale in S and h results in a transaction cost of $\$0.005$ per option or share, and we may hedge once daily.

The report is organized as follows. First, we derive an equation for the delta and gamma of a European call option. Using these equations, we develop a strategy to dynamically delta hedge a position in g, as well as a method to dynamically delta-gamma hedge the option.

We then implement these hedging strategies and comment on the resulting P&L distributions. We investigate on how these distributions differ with different market parameters, and examine the positions we take in S and h as we hedge g under different possible paths S may take. Finally, we find a relationship between P&L distributions and when the real world volatility of S differs from the volatility which is used to price and hedge g.

Part 1: Theory

The Greeks

Consider the market environment described in the Introduction. Note that the European call must satisfy the Black-Scholes PDE, with terminal condition $G(S_T) = (S_T - K)_+$.

$$\frac{\partial g(t,S)}{\partial t} + rS\frac{\partial g(t,S)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 g(t,S)}{\partial S^2} = rg(t,S), \quad g(t,S) = G(S) \tag{1}$$

Using the Feynman-Kac theorem, we can express the solution to the PDE as:

$$g(t, S_t) = \mathbb{E}^{\mathbb{M}}[e^{-r(T-t)}G(S_T)|F_t] \tag{2}$$

where S_t is the price process described by, $dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{M}}$, where $W^{\mathbb{M}} = (W_t^{\mathbb{M}})_{t\geq 0}$ is an $\mathbb{M}$-Brownian motion.

Finally, using a change of numeraire and Girsanov's theorem, we may express the price of a European call option as:

$$g(t, S_t) = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-) \tag{3}$$

with

$$d_{\pm} = \frac{ln(\frac{S_t}{K}) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

If we would like to build a portfolio θ such that its value V^θ does not depend on changes in the underlying price S, we must have that the portfolio value's partial derivative with respect to S is zero, $\frac{\partial V^\theta}{\partial S} = 0$. Thus, it is necessary to determine the value of $\frac{\partial g(t,S)}{\partial S}$, which we call an option's delta. Define delta as the following:

$$\Delta_t^g = \frac{\partial g(t, S_t)}{\partial S_t} = \Phi(d_+) \tag{4}$$

This equation can be used to build delta-neutral portfolios to hedge an option position. However to improve hedging performance we must also define another risk-measure, gamma, which measures an option's sensitivity to second-order moves in the price of the underlying asset S. Define gamma, Γ_t^g, as follows:

$$\Gamma_t^g = \frac{\partial^2 g(t, S_t)}{\partial S_t^2} = \frac{\Phi(d_+)}{S_t \sigma \sqrt{T_g - t}} \tag{5}$$

Now that we have defined the Greek measures Delta and Gamma, and provided some motivation for why these are important, it is time to derive delta hedging and delta-gamma hedging strategies.

Delta Hedging

The aim of delta hedging is to construct a delta-neutral portfolio using the option g, the underlying asset S and the bank account B. Call this portfolio $\theta = (\theta_t)_{t\geq 0}$ with $\theta_t = (\alpha_t, \beta_t, -1)$, where α_t is the position in S_t and β_t is the position in B_t, and -1 is the position in g_t. If constructed properly at time t, we have $\Delta_t^\theta = \frac{\partial V_t^\theta}{\partial S_t} = 0$, i.e. the value of the portfolio is instantaneously constant with respect to changes in S. Then,

$$V_t^\theta = \alpha_t S_t + \beta_t B_t - g_t \tag{6}$$

$$\Delta_t^\theta = \alpha_t \Delta_t^S + \beta_t \Delta_t^B - \Delta_t^g \tag{7}$$

However, since $\Delta_t^S = 1$, and $\Delta_t^B = 0$, we have that:

$$\Delta_t^\theta = \alpha_t - \Delta_t^g \tag{8}$$

Hence, if we wish to build a self-financing delta-neutral portfolio at $t = 0$, we select $\alpha_0 = \Delta_0^g$, and $\beta_0 = g_0 - \alpha_0 S_0$ (assuming $B_0 = 1$).

Next, at time Δt, $S_0 \to S_{\Delta t}$ and $\Delta_0^g \to \Delta_{\Delta t}^g$. We must then adjust our position $\alpha_{\Delta t}$ such that the portfolio remains delta-neutral, a process which is called rebalancing. The proceeds or funds from this rebalance are added to the bank account position $\beta_{\Delta t}$. In other words,

$$\beta_{\Delta t} = \beta_0 e^{r\Delta t} - S_{\Delta t}(\alpha_{\Delta t} - \alpha_0) - k \mid \alpha_{\Delta t} - \alpha_0 \mid \tag{9}$$

Here we assume a transaction cost of k applies to each unit bought or sold of S.

More generally, at the discrete rebalancing time $n < T_g$, we rebalance our portfolio so it remains delta-neutral. In doing so, the bank account evolves as:

$$\beta_n = \beta_{n-1} e^{r\Delta t} - S_n(\alpha_n - \alpha_{n-1}) - k \mid \alpha_n - \alpha_{n-1} \mid \tag{10}$$

Finally at time T_g, the option matures, and we set our positions in S and B to 0. Then, the final P&L by applying our hedging strategy becomes:

$$\beta_{T_g} = \beta_{T_g-1}e^{r\Delta t} + S_{T_g}\alpha_{T_g-1} - (S_{T_g} - K_g)_+ - k \mid \alpha_{T_g-1} \mid \tag{11}$$

Delta hedging is an effective strategy of reducing the variance of returns when a market participant has a position in an option. However, if there is a substantial change in the price of S_t between two discrete re-balancing times, the portfolio will no longer be delta-neutral just before the second re-balancing is done. To improve this, we must hedge second-order changes in S_t. In other words, we must introduce gamma hedging , so that

$$\frac{\partial^2 V_t^\theta}{\partial S_t^2} = 0$$

at all re-balancing times t. Delta-gamma hedging is introduced in the following section.

Delta-Gamma Hedging

Note that $\Gamma_t^S = \Gamma_t^B = 0 \ \forall \ t$. Hence, to have a gamma-neutral strategy, we must introduce a hedging option h, which has non-zero gamma. Thus, we delta-gamma hedge with the portfolio $\theta = (\theta_t)_{t\geq 0}$ with $\theta_t = (\alpha_t, \beta_t, \gamma_t, -1)$, where at time t, α_t is the position in S, β_t is the position in B, γ_t is the position in h, and -1 is the position in g. Then,

$$V_t^\theta = \alpha_t S_t + \beta_t B_t + \gamma_t h_t - g_t \tag{12}$$

$$\Delta_t^\theta = \alpha_t \Delta_t^S + \beta_t \Delta_t^B + \gamma_t \Delta_t^h - \Delta_t^g \tag{13}$$

$$\Gamma_t^\theta = \alpha_t \Gamma_t^S + \beta_t \Gamma_t^B + \gamma_t \Gamma_t^h - \Gamma_t^g \tag{14}$$

However, since $\Gamma_t^S = \Gamma_t^B = 0$, we can simplify Equation 14 to:

$$\Gamma_t^\theta = \gamma_t \Gamma_t^h - \Gamma_t^g$$

Then, to construct a gamma-neutral portfolio, we select $\gamma_t = \frac{\Gamma_t^g}{\Gamma_t^h}$.

Finally, to keep the portfolio delta-neutral, we select $\alpha_t = \Delta_t^g - \frac{\Gamma_t^g \Delta_t^h}{\Gamma_t^h}$.

Hence, if we wish to build a self-financing delta-neutral portfolio at $t = 0$, we select $\alpha_0 = \Delta_0^g - \frac{\Gamma_0^g \Delta_0^h}{\Gamma_0^h}$.

Next, at time Δt, $S_0 \to S_{\Delta t}$, $\Delta_0^g \to \Delta_{\Delta t}^g$, $\Delta_0^h \to \Delta_{\Delta t}^h$, $\Gamma_0^g \to \Gamma_{\Delta t}^g$, and $\Gamma_0^h \to \Gamma_{\Delta t}^h$. We must then re-balance our positions $\alpha_{\Delta t}$ and $\gamma_{\Delta t}$ such that the portfolio remains delta-neutral and gamma-neutral. Once again, the proceeds or funds from this re-balance are added to the bank account position $\beta_{\Delta t}$, and a transaction cost of k applies to each unit bought or sold of S and h. Then,

$$\beta_{\Delta t} = \beta_0 e^{r\Delta t} - S_{\Delta t}(\alpha_{\Delta t} - \alpha_0) - h_{\Delta t}(\gamma_{\Delta t} - \gamma_0) - k \mid \alpha_{\Delta t} - \alpha_0 \mid -k \mid \gamma \Delta t - \gamma_0 \mid \tag{15}$$

More generally, at time $n < T_g$, we re-balance our portfolio such that it remains delta and gamma neutral. Then, the bank account evolves as:

$$\beta_n = \beta_{n-1} e^{r\Delta t} - S_n(\alpha_n - \alpha_{n-1}) - h_n(\gamma_n - \gamma_{n-1}) - k_1 \mid \alpha_n - \alpha_{n-1} \mid -k_2 \mid \gamma_n - \gamma_{n-1} \mid \tag{16}$$

Finally at time T_g, the option expires, and we set our positions in S, h and B to 0. The final P&L by applying our hedging strategy becomes:

$$\beta_{T_g} = \beta_{T_g-1} e^{r\Delta t} + S_{T_g}\alpha_{T_g-1} + h_{T_g}\gamma_{T_g-1} - (S_{T_g} - K_g)_+ - k \mid \alpha_{T_g-1} \mid -k \mid \gamma_{T_g-1} \mid \tag{17}$$

By applying these extra steps to keep the portfolio gamma-neutral at re-balancing times, hedging performance increases, as the variance of returns decreases (as will be shown in Section 2). This is because the portfolio is neutral to first and second order changes in S.

In Section 2, we describe the implementation of delta and delta-gamma hedging strategies, and compare the strategies under different market conditions.

Part 2: Description of Implementation and Interpretation

Comparison Between Delta Hedging and Delta-Gamma Hedging

Impact of Changing Drift

In Part 1, we described mathematically how to implement delta hedging and delta-gamma hedging. Once done, we implemented these strategies in Python. A detailed description of the algorithms implemented for delta hedging and delta-gamma hedging are shown in the Appendix. After implementation of these strategies was done, we conducted experiments to establish relationships between market parameters, and the resulting P&L distributions.

First, we investigated how the P&L distributions resulting from both hedging strategies compared as we varied μ, the drift of S. As shown in Part 1, the price, delta and gamma of an option do not depend on the value of μ. Therefore, it was expected that changing μ would not impact the P&L distributions. This was true when we set the transaction cost k equal to zero, as seen in Figure 1.

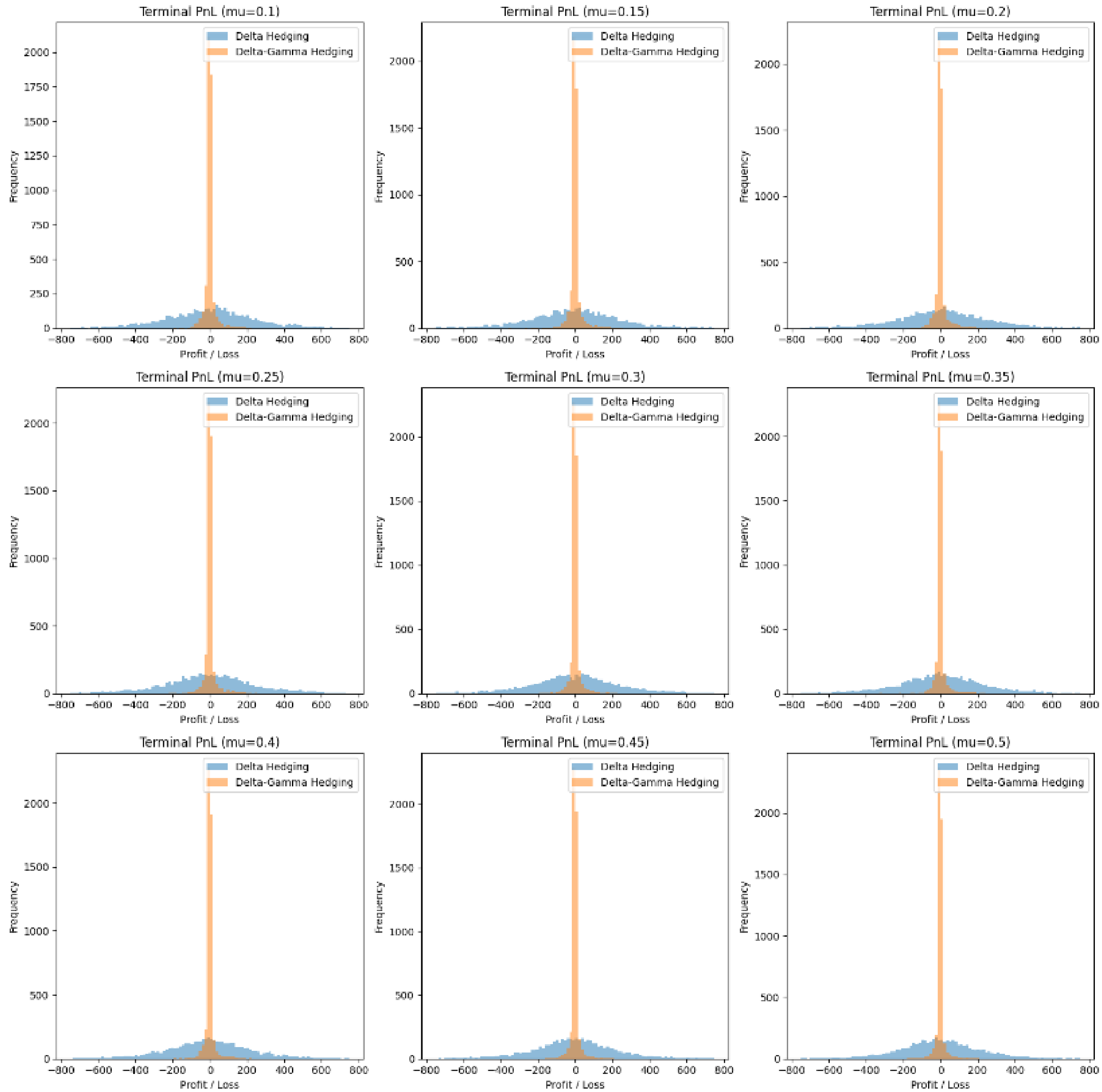

Figure 1: Comparison of Terminal P&L Distributions of Delta Hedging and Delta-Gamma Hedging with different drift (μ) with $k = \$0$

As Figure 1 shows, all the distributions are roughly identical, showing that the choice of μ does not impact the hedging strategy P&L distribution. We can also conclude that when transaction costs are zero, both strategies yield distributions centered around 0. However, the P&L distribution when using delta hedging is much wider, i.e. it has a much higher variance of returns. As described in Part 1, this is because we are ignoring the impact of second order changes in the price process S when we do not maintain gamma neutrality. Therefore, our portfolio does not remain delta-neutral in between rebalancing times. With delta-gamma hedging, we take extra steps to hedge this second-order risk, resulting in profit distributions with lower variances.

However, when we set $k = \$0.005$, we see a new trend emerge. The resulting P&L distributions are shown in Figure 2.

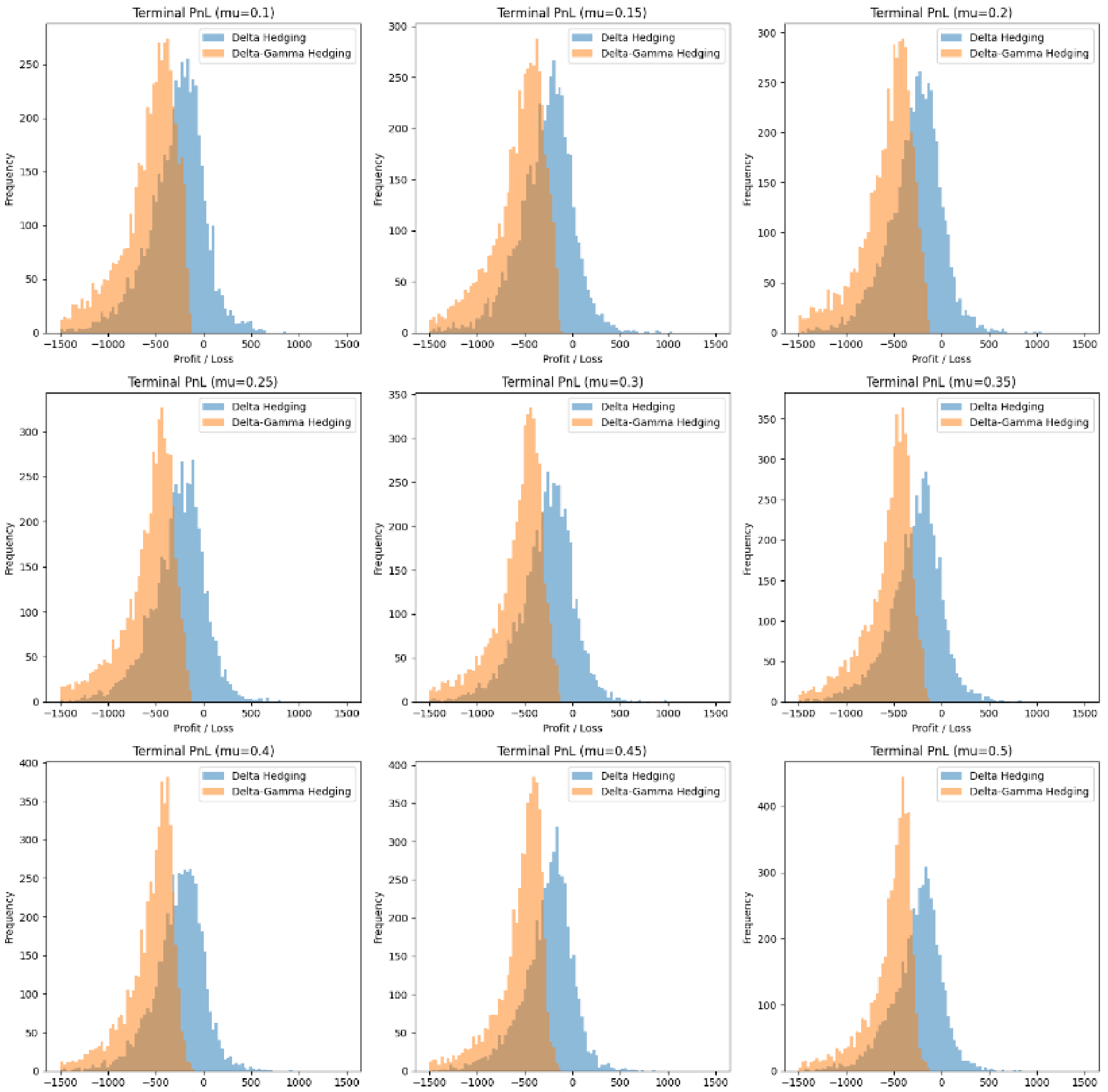

Figure 2: Comparison of Terminal P&L Distributions of Delta Hedging and Delta-Gamma Hedging with different drift (μ) when $k = \$0.005$

Here, we notice a few things. In all cases, the delta and delta-gamma hedging P&L distributions are shifted left, with a negative mean. This is due to the addition of non-zero transaction costs. We also note that the delta-gamma hedging distributions are shifted further to the left than the delta hedging distributions. This is because additional transaction costs apply to trading h in order to keep the portfolio gamma-neutral.

Unlike when $k = 0$, we also note that changing μ seems to impact the P&L distributions. In particular, as μ increases, the variance of all the distributions tends to decrease. Figure 3 shows this effect in more detail.

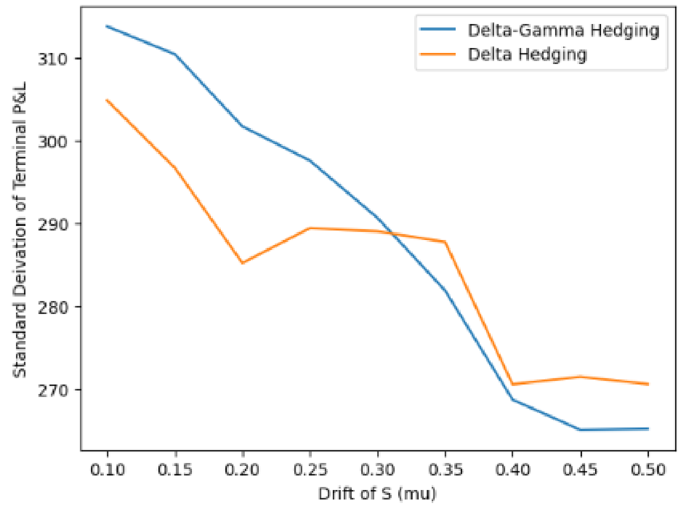

Figure 3: Standard Deviation of Returns Over Different Drifts of S (μ)

As μ increases, it is more likely that S increases in value, and therefore a higher proportion of sample paths of S will increase to a point such that Δ^g approaches 1. At values of Δ^g near 1, Γ^g tends to approach 0. Therefore, by gamma's definition, this results in a lower variance of Δ^g and Γ^g. Therefore, less transactions are needed to properly delta and gamma neutralize the portfolios. Hence, there is less variance in the number of transaction costs that are needed, reducing overall P&L variance. This explains why in Figure 2, the peaks of both the delta and delta-gamma hedging P&L distributions tend to become higher as μ increases.

Table 1 displays some key statistics of the Delta and Delta-Gamma hedging P&L distributions over different values of μ when σ is fixed. This summary provides analytical support to the intuition gained from P&L distributions in Figure 2. Firstly, this table presents that the variance of returns does in fact decline as μ increases. We also see that the skew becomes more negative with μ on average, meaning the distributions get heavier left tails.

		μ								
		0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
Delta Hedging	Mean	-153.7	-159.6	-161.1	-148.7	-161.6	-180.2	-177.0	-165.3	-187.0
	Std. Dev.	554.0	554.8	539.6	538.4	533.4	528.2	517.1	514.3	508.7
	Variance	306895.3	307783.3	291197.9	289919.5	284545.9	279008.9	267369.7	264537.8	258743.1
	Skewness	-0.3	-0.3	-0.3	-0.3	-0.4	-0.4	-0.4	-0.2	-0.5
	Kurtosis	1.6	2.2	1.8	2.0	1.8	2.6	2.3	2.1	2.5
Delta Gamma Hedging	Mean	-301.0	-308.9	-315.2	-314.3	-314.8	-309.6	-308.2	-311.5	-305.6
	Std. Dev.	159.4	158.4	155.8	152.4	146.6	145.0	142.5	135.3	127.5
	Variance	25399.4	25097.3	24285.9	23236.9	21500.6	21010.5	20304.8	18300.2	16261.8
	Skewness	-1.6	-1.8	-1.4	-1.3	-1.6	-1.8	-1.2	-1.9	-1.9
	Kurtosis	5.8	8.4	4.1	7.5	4.9	7.6	13.1	6.7	7.5

Table 1: P&L distribution statistics for different values of μ in case of Delta and Delta-Gamma hedging.

The other notable trend here is that in the case of delta hedging, mean P&L tends to decrease. This is because Δ_t^g is more likely to approach 1, meaning more delta hedging is likely to be needed, and hence more transaction costs which decreases P&L. In the case of delta-gamma hedging, there is no trend present in terms of mean P&L, because the increased transaction costs of Δ_t^g trending to 1 is offset by Γ_t^g trending to zero, reducing transaction costs associated with h.

As a final general note to this section, it is clear that when transaction costs are present, there is a clear cost to using delta-gamma hedging instead of delta hedging. The mean P&L is lower when delta-gamma hedging is used, as a result of more transaction costs. The benefit is that the variance of returns is lower when using delta-gamma hedging. As a financial practitioner, there is a clear trade-off between the two strategies, and a choice must be made if it is worth extra costs to reduce one's variance of returns.

Impact of Changing Real World Volatility

After investigating how the P&L distributions changed with different values of μ, we examined how the profit distributions differed when S experienced different levels of volatility. Unlike drift, volatility is a key input to price a European call option, and to compute its delta and gamma. Therefore, we expect to see significant changes in profit distributions as the volatility differs.

When pricing g, it is impossible to know how volatile S will be over the life of the trade. We must make our "best guess" to price and hedge the option. But in the real world, the $\mathbb{P}$-volatility that S experiences may be different which would cause the terminal profit distributions to shift to the left or right. In this experiment, it is assumed that the option sold is always priced and hedged assuming that $\sigma = 0.25$. Since increased volatility increases the value of European options, and we are selling the option, we would expect that if the realized $\mathbb{P}$ volatility is lower than 0.25, we should have a terminal P&L distributions with a higher mean, since we sold the option for more than it was truly worth based on the real-world volatility S experienced. If the realized $\mathbb{P}$ volatility is higher than 0.25, the opposite is true since we undervalued the option when we sold it.

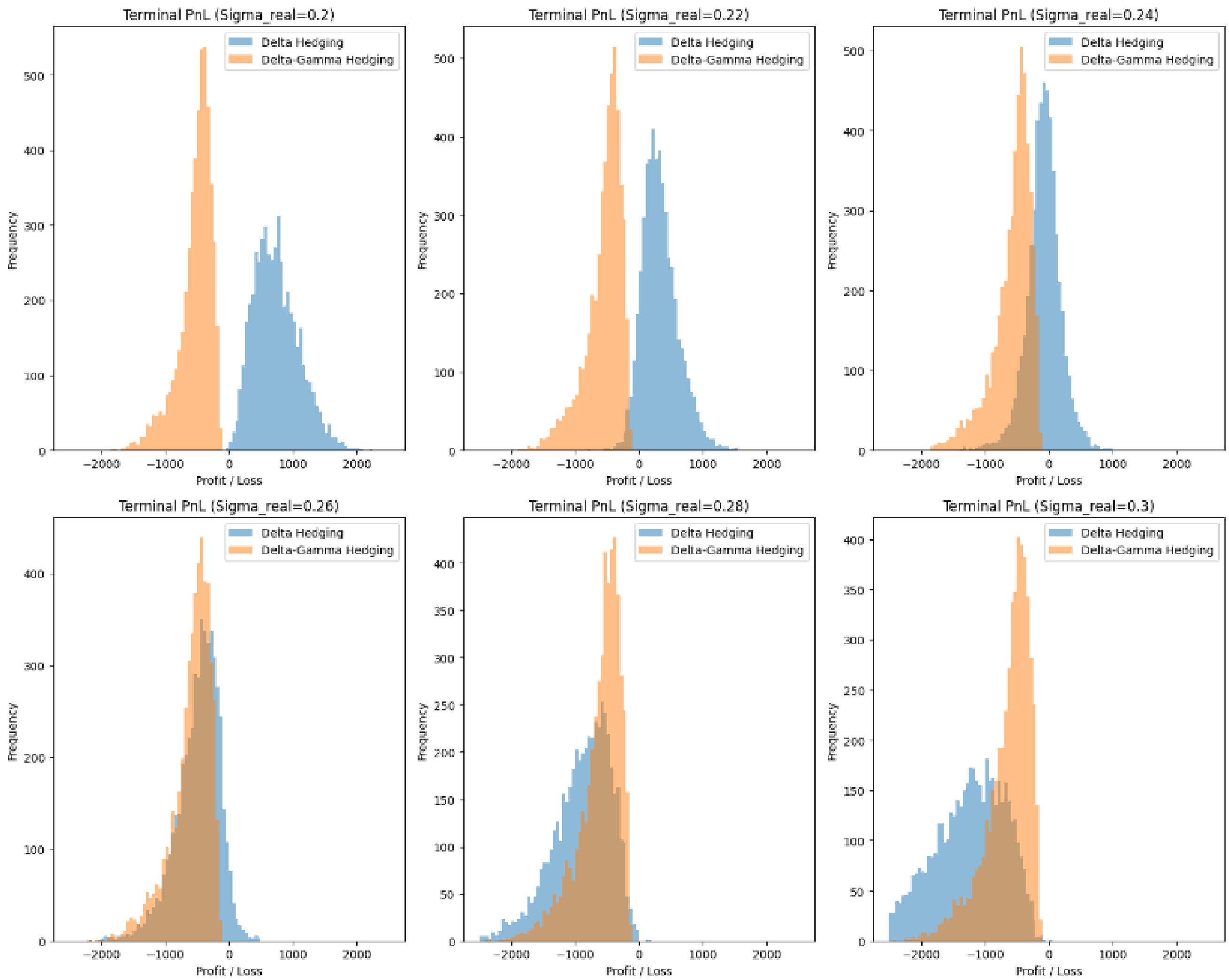

Figure 4: Comparison of Terminal P&L Distributions of Delta Hedging and Delta-Gamma Hedging with different real world volatilities (σ_{real})

Figure 4 shows the P&L distributions with different values of real world volatilities for S. As can be seen, as we expected, when delta hedging, the P&L distribution shifts to the left as σ_{real} increases. When the real world volatility is lower than the "best guess" 0.25, the delta hedging mean profit is positive, whereas when it is higher than 0.25, the mean profit is negative. However, the same trend does not apply when delta-gamma hedging. This is because when we hedge gamma by using h and $\sigma_{real} < 0.25$, the benefit of selling g at an overvalued price is offset by buying h at an overvalued price (since h is priced based on the same volatility as g). The opposite is also true when $\sigma_{real} > 0.25$. Therefore, the effect of changing the real world volatility on mean profit is much smaller when delta-gamma hedging. However, there is still a slightly negative trend as σ_{real} increases. These trends are shown more clearly in Figure 5.

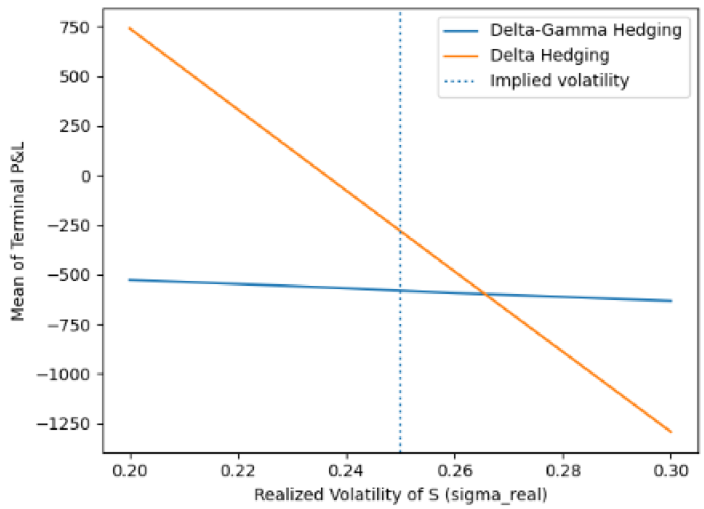

Figure 5: Mean Terminal P&L with different real world volatilities (σ_{real})

Table 2 contains further descriptive statistics of the Delta and Delta-Gamma hedging P&L distributions for $\mu = 0.1$ and different values of σ_{real}. Notably, the skew in the case of both hedging strategies decreases as σ_{real} increases. This shows that the distribution becomes more fat-tailed, and the chances of a large loss increases when the real world volatility increases. The standard deviation of terminal profits also increases with higher σ_{real} under both hedging strategies. This is somewhat obvious because as the volatility of S increases, larger changes in positions in S and h will be needed to hedge the portfolio, resulting a broader range of possible profits and losses.

This concludes this section describing and explaining the impact of market parameters (changing drift and real world volatility) on P&L distributions when both delta hedging and delta-gamma hedging strategies are used. Next, we look in more detail at the positions required to hedge the position in g when S follows different price paths.

		σ_{real}					
		0.2	0.22	0.24	0.26	0.28	0.30
Delta Hedging	Mean	863.7	448.6	40.6	-356.6	-768.0	-1155.9
	Std. Dev.	537.0	518.8	524.9	572.7	678.9	799.7
	Variance	288379.9	269116.4	275492.9	328036.3	460942.3	639541.7
	Skewness	0.5	0.5	0.1	-0.7	-0.9	-1.1
	Kurtosis	0.1	0.8	1.5	2.3	1.9	1.7
Delta Gamma Hedging	Mean	-283.9	-293.6	-301.1	-309.2	-316.0	-328.0
	Std. Dev.	126.4	139.1	150.7	164.1	187.3	203.7
	Variance	15979.9	19351.1	22721.4	26938.4	35097.2	41485.4
	Skewness	-1.0	-1.1	-1.1	-1.7	-2.1	-2.0
	Kurtosis	2.2	2.5	2.8	9.5	11.9	10.0

Table 2: P&L distribution statistics for different values of σ_{real} in case of Delta and Delta-Gamma hedging. $\mu = 0.1, \sigma_{imp} = 0.25$

Hedging Strategies Comparison - Two Sample Paths

In our analysis, we generated two sample paths to evaluate the performance of delta hedging and delta-gamma hedging strategies. These sample paths are shown in the left image of Figure 6. The middle chart illustrates the in-the-money (ITM) path where the blue line represents the asset position under a delta-only hedging strategy. This line adjusts in line with price movements in S to maintain a neutral delta. As the option approaches maturity, we observe that the number of shares required for hedging rises to approximately 10,000, correlating with a standard option contract that represents 10,000 shares with delta equals to 1. This increment is logical, as the option delves deeper into ITM status, which typically causes the delta to approach 1.

In the same ITM scenario, but using the delta-gamma hedging strategy, the asset position depicted by the orange line is less stable, particularly as the maturity date nears. Here, we notice a significant shift in the asset position from a negative position to a positive one. This is because as the option nears expiration, once again Δ_t^g approaches 1, but Γ_t^g approaches 0, causing a significant change in the position in h needed to retain gamma neutrality. This position in h is shown by the dashed green line, and undergoes substantial adjustments as expiration approaches, since the gamma of g drops to 0 much faster than that of h as we approach T_g, since h has a longer time until expiration. This is indicative of a significant rebalancing requirement, where we sell h to keep gamma neutral. When selling h, we are also decreasing the portfolio delta, so we must buy S during this period to maintain delta neutrality.

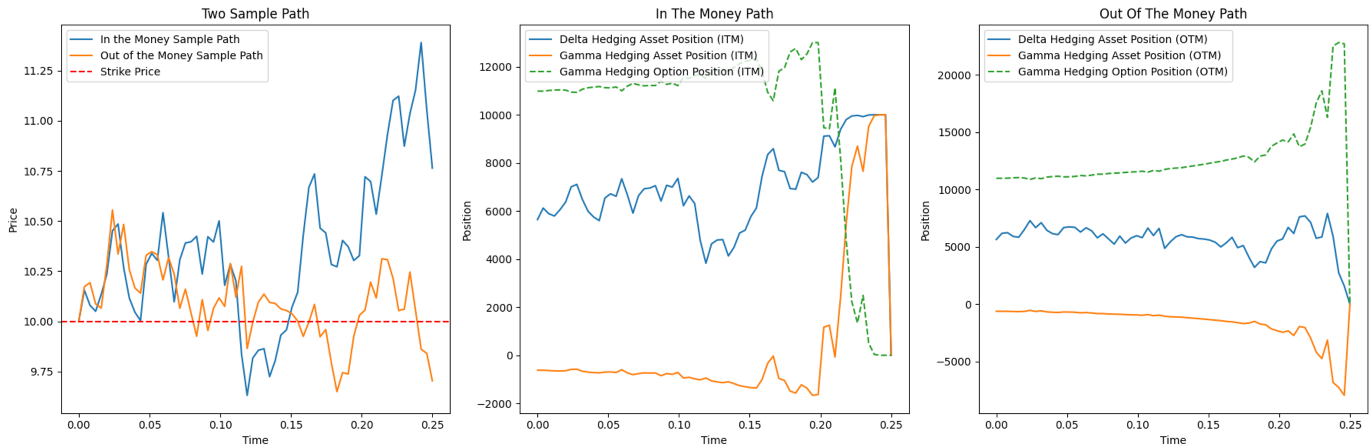

Figure 6: Two Sample Paths of S and the Corresponding Hedging Strategies

On the contrary, in the out-of-the-money (OTM) path, the delta hedging strategy reveals that as time approaches expiration, there is a significant reduction in the number of shares needed for hedging. This is consistent with the financial intuition we introduced above: as the time to maturity (TTM) nears zero, the option's delta approaches zero for an OTM option, resulting in a marked decline in the hedging position. A similar pattern is observed with delta-gamma hedging. As we near the maturity date, there is a substantial change in both the positions of S and h. This reflects the expected behavior, as the gamma of g converges to zero, so we offload most of our position in h. This causes portfolio delta to decrease, forcing us to purchase shares of S to maintain delta neutrality.

The primary distinction between delta and delta-gamma hedging strategies lies in the positioning of the hedging portfolios. With delta-only hedging, the approach involves purchasing S to neutralize the option's delta, meaning we carry a positive position in S. However, when delta-gamma hedging, we have the additional positive position in h to offset Γ_t^g, which carries its own delta Δ_t^h. Therefore the position in S is used to offset the $\Delta_t^h + \Delta_t^g$, and since $T_h > T_g$, we typically have a short position in S.

Part 3: Conclusions

In this report we first built up the theory needed to develop delta and delta-gamma hedging strategies. These strategies are described in mathematical detail, and the algorithms are shown (in the Appendix).

Next, we experimented to see how the profit distributions of each of these hedging strategies compared, and how they differed as we changed the drift and real-world volatility of S. We found that delta-gamma hedging tends to have an additional cost (i.e. a lower mean profit), for the benefit of a lower variance of returns. In the case that transaction costs are, we found that the drift of S has no impact on the distribution under either hedging strategy. However, when transaction costs are non-zero, we determined that increased μ causes a lower variance of profit, whether we are delta hedging, or delta-gamma hedging.

We also found that as real-world volatility of S increases, since we are sellers of g, mean profit and variance of returns decreases when delta hedging. When the real world volatility is lower than volatility we use to price and hedge the option, mean profit is positive, while when the opposite is true, mean profit shifts left to be negative. When we use delta-gamma hedging, this effect is much less pronounced, and the P&L distribution remains nearly constant, regardless of realized volatility experienced by S.

Finally, the we investigated the positions in S and h that required to hedge g when S follows two different price paths, one where S ends in the money, and one where it expires out of the money. When delta hedging, we found that the position in S is always used to offset Δ_t^g, meaning we carry a positive position in S. However, when delta-gamma hedging, we have the additional positive position in h, which causes us to typically carry a negative position in S.

Appendix

Please note that all theory references the STA2503 course notes.

0.0.1 Implemented Algorithms

Algorithm 1 Delta Hedging Simulation

Require: $S_0, r, T, \sigma, K_g, pos, k, num\_paths, N, dt$

$S \leftarrow$ SimulateAssetPaths$(S_0, r, T, \sigma, num_paths)$
procedure DeltaHedgeSim(S, num_paths)
  $\delta \leftarrow$ Get All Paths' Delta(S, num_paths)
  $M \leftarrow$ Initialize a Matrix of Size $((num\_paths, N \cdot dt))$
  $g_0 \leftarrow$ ComputeCallPrice$(S_0, K_g, \sigma, r, T_g)$
  $\alpha \leftarrow$ Initialize a Matrix of Size $((num\_paths, N \cdot dt))$
  for $i \leftarrow 0$ **to** $N \cdot dt - 1$ **do**
    $\alpha[:, i] \leftarrow$ *round*$(\delta[:, i] \times pos, 0)$
  end for
  $M[:, 0] \leftarrow pos \times g_0 - S[:, 0] \times \alpha[:, 0]$
  for $i \leftarrow 1$ **to** $N \cdot dt - 1$ **do**
    if $i < N \cdot dt - 1$ **then**
      $M[:, i] \leftarrow M[:, i-1] \times e^{(r \times dt)} - S[:, i] \times (\alpha[:, i] - \alpha[:, i-1]) - k \times |\alpha[:, i] - \alpha[:, i-1]|$
    else
      $M[:, i] \leftarrow M[:, i-1] \times e^{(r \times dt)} + \alpha[:, i-1] \times S[:, i] - pos \times \max(S[:, i] - K_g, 0) - k \times |\alpha[:, i-1]|$
    end if
  end for
  return α, M
end procedure

Algorithm 2 Delta-Gamma Hedging Simulation

Require: $S_0, r, T_g, T_h, K_g, K_h, \sigma_{imp}, num\_paths, N \cdot dt, dt, pos, k_1, k_2$

$S \leftarrow$ SIMULATEASSETPATHS$(S_0, r, T_g, sigma_imp, num_paths)$

procedure DELTAGAMMAHEDGESIM(S, num_paths)

  $\delta_g, \delta_h, \gamma_g, \gamma_h \leftarrow$ GET ALL PATHS' DELTA & GAMMA(S, num_paths)

  $h \leftarrow$ Initialize a Matrix of Size $((num\_paths, Ndt))$

  for $i \leftarrow 0$ **to** $N \cdot dt - 1$ **do**

    $h[:, i] \leftarrow$ COMPUTECALLPRICE$(S[:, i], K_h, \sigma_{imp}, r, T_h - t_i)$

  end for

  $M \leftarrow$ Initialize a Matrix of Size $((num\_paths, Ndt))$

  $g_0 \leftarrow$ COMPUTECALLPRICE$(S_0, K_g, \sigma_{imp}, r, T_g)$

  $\alpha \leftarrow$ Initialize a Matrix of Size $((num\_paths, Ndt))$

  $\omega \leftarrow$ Initialize a Matrix of Size $((num\_paths, Ndt))$

  for $i \leftarrow 0$ **to** $N \cdot dt - 2$ **do**

    $\alpha[:, i] \leftarrow$ ***round***$(pos \times (\delta_g[:, i] - \gamma_g[:, i]/\gamma_h[:, i] \times \delta_h[:, i]), 0)$

    $\omega[:, i] \leftarrow$ ***round***$(pos \times \gamma_g[:, i]/\gamma_h[:, i], 0)$

  end for

  $M[:, 0] \leftarrow pos \times g_0 - S[:, 0] \times \alpha[:, 0] - h[:, 0] \times \omega[:, 0]$

  for $i \leftarrow 1$ **to** $N \cdot dt - 1$ **do**

    if $i < N \cdot dt - 1$ **then**

      $M[:, i] \leftarrow M[:, i-1] \times e^{(r \times dt)} - S[:, i] \times (\alpha[:, i] - \alpha[:, i-1]) - h[:, i] \times (\omega[:, i] - \omega[:, i-1]) - |\alpha[:, i] - \alpha[:, i-1]| \times k_1 - |\omega[:, i] - \omega[:, i-1]| \times k_2$

    else

      $M[:, i] \leftarrow M[:, i-1] \times e^{(r \times dt)} + \alpha[:, i-1] \times S[:, i] + \omega[:, i-1] \times h[:, i] - pos \times \max(S[:, i] - K_g, 0) - |\alpha[:, i-1]| \times k_1 - |\omega[:, i-1]| \times k_2$

    end if

  end for

  return α, ω, M

end procedure

where in both algorithms above, S_0 represents the initial asset price, r is the risk-free interest rate, T is time to maturity, K_g represents the strike price of the option we short, K_h represents the strike of the option we used to gamma hedge the portfolio, k is the transaction cost rate associated with the underlying equity and hedging option, and N represents the number of period.

Contribution Attestation

We attest that all members in the group made fair contributions to the assignment. The contributions made by each member are as follows:

Willem: Wrote the code to simulate delta hedging and delta-gamma hedging. Wrote the abstract, introduction, conclusion, Part 1 and parts of Part 2. Edited the document.

Yaroslav: Wrote the code to keep track of key metrics of P&L distributions. Transferred the metrics into the tables and elaborated on important observations.

Xuanze: Wrote code to check results from Q1 to Q3. Wrote a draft of part 2 summarizing important results, and edited part 1 of the document.

Signatures:

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