Proof nets for first-order additive linear logic

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Additive linear logic

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Formulas

$$A \;::=\; a \;\mid\; \bar{a} \;\mid\; A + A \;\mid\; A \times A$$

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Sequents

$$\vdash A, B$$

Additive linear logic

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Sequents

$$\vdash A, B$$

Proofs

$$\frac{}{\vdash a, \overline{a}}\,\mathrm{ax} \qquad \frac{\vdash A, B_i}{\vdash A, B_1 + B_2}\,{+}\mathrm{R},i \qquad \frac{\vdash A, B \quad \vdash A, C}{\vdash A, B \times C}\,{\times}\mathrm{R}$$

Features

Compact: Proof search on $\vdash A, B$ is $\mathcal{O}(|A| \times |B|)$ propositionally[1], and *NP-complete* for first-order[2]

[1] [Galmiche & Marion 1995] [2] [Heijltjes & Hughes 2015]

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Rich: Additives $+/\times$ (binary choice),
Duality $\overline{\overline{A}} = A$ (player/opponent),
meta-level linear implication/par $\vdash A, B$ (parallelism)

[1][Galmiche & Marion 1995] [2][Heijltjes & Hughes 2015]

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Interesting: Two-player parallel games,
Communication along a two-way channel,
the Blass problem (sequential strategies don't compose associatively)[3]

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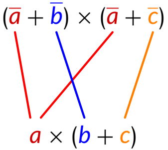
Well-behaved: Canonical proof nets[4], also with units[5],
Fixed-point formulae (mu-lattice hierarchy)[6]

[1][Galmiche & Marion 1995] [2][Heijltjes & Hughes 2015] [3][Abramsky 2003]
[4][Hughes & Van Glabbeek 2005] [5][Heijltjes 2011] [6][Santocanale 2002]

Proof nets and cut elimination

Proof nets and cut elimination

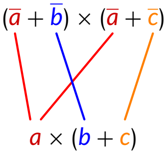
Proof net = sequent + linking



$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,\text{ax}}{\vdash \overline{a} + \overline{b}, a}\,{+}\text{R},1 \qquad \dfrac{\dfrac{\dfrac{}{\vdash \overline{b}, b}\,\text{ax}}{\vdash \overline{b}, b + c}\,{+}\text{R},1}{\vdash \overline{a} + \overline{b}, b + c}\,{+}\text{R},2}{\vdash \overline{a} + \overline{b}, a \times (b + c)}\,{\times}\text{R} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,\text{ax}}{\vdash \overline{a} + \overline{c}, a}\,{+}\text{R},1 \qquad \dfrac{\dfrac{\dfrac{}{\vdash \overline{c}, c}\,\text{ax}}{\vdash \overline{c}, b + c}\,{+}\text{R},2}{\vdash \overline{a} + \overline{c}, b + c}\,{+}\text{R},2}{\vdash \overline{a} + \overline{c}, a \times (b + c)}\,{\times}\text{R}}{\vdash (\overline{a} + \overline{b}) \times (\overline{a} + \overline{c}), a \times (b + c)}\,{\times}\text{R}$$

Proof nets and cut elimination

Proof net = sequent + linking



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}{\vdash \bar{a} + \bar{c}, a \times (b + c)}\,{\times}\text{R}
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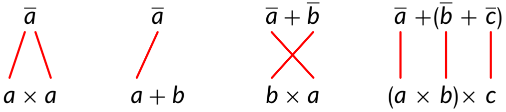
$$

More examples:



diagonal   injection   symmetry   associativity

Proof nets and cut elimination

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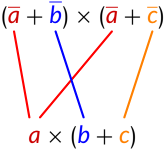

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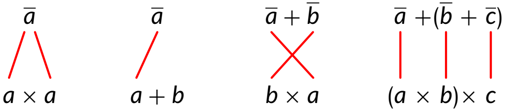
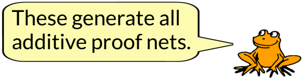
$$

More examples:

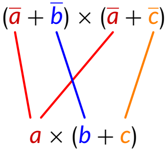


diagonal | injection | symmetry | associativity



Proof nets and cut elimination

Proof net = sequent + linking



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$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}\,{\times}\mathrm{R}$$

$$\vdash (\overline{a} + \overline{b}) \times (\overline{a} + \overline{c}), a \times (b + c)$$

Proof nets and cut elimination

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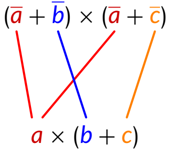

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\begin{prooftree}
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\RightLabel{+R,1}
\UnaryInfC{$\vdash \overline{a} + \overline{b}, a$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash \overline{b}, b$}
\RightLabel{+R,1}
\UnaryInfC{$\vdash \overline{b}, b + c$}
\RightLabel{+R,2}
\UnaryInfC{$\vdash \overline{a} + \overline{b}, b + c$}
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\BinaryInfC{$\vdash \overline{a} + \overline{b}, a \times (b + c)$}
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\end{prooftree}
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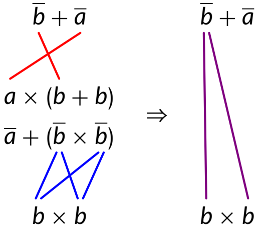
Cut elimination = relational composition



Correctness

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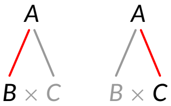
(1) Slicing:



[Hughes & Van Glabbeek 2003]

Correctness

(1) Slicing:

$$A \qquad\qquad A$$

$$B \times C \qquad B \times C$$

[Hughes & Van Glabbeek 2003]

Example:

$$(\overline{a}+\overline{b}) \times (\overline{a}+\overline{c})$$

$$a \times (b+c)$$

Correctness

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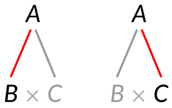

[Hughes & Van Glabbeek 2003]

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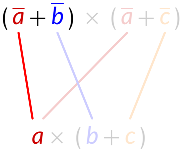

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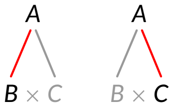

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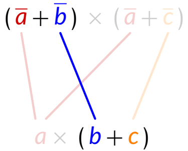

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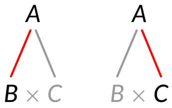

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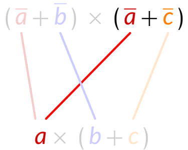

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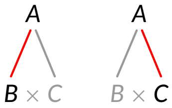
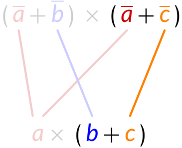


[Hughes & Van Glabbeek 2003]

Example:



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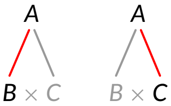
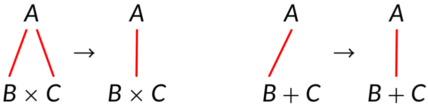


[Hughes & Van Glabbeek 2003]

(2) Coalescence:



[Heijltjes & Hughes 2015]

additive linear logic

Formulas

$$A \;::=\; a \;\mid\; \bar{a} \;\mid\; A + A \;\mid\; A \times A$$

Sequents

$$\vdash A, B$$

Proofs

$$\frac{}{\vdash a, \bar{a}}\,\mathrm{ax} \qquad \frac{\vdash A, B_i}{\vdash A, B_1 + B_2}\,{+}\mathrm{R}, i \qquad \frac{\vdash A, B \quad \vdash A, C}{\vdash A, B \times C}\,{\times}\mathrm{R}$$

First-order additive linear logic

Formulas

$$A \;::=\; a \;\mid\; \overline{a} \;\mid\; A + A \;\mid\; A \times A \;\mid\; \exists x.A \;\mid\; \forall x.A$$

$$a \;::=\; P(t_1, \ldots, t_n)$$

$$t \;::=\; f(t_1, \ldots, t_n) \;\mid\; x$$

Sequents

$$\vdash A, B$$

Proofs

$$\frac{}{\vdash a, \overline{a}}\,\text{ax} \qquad \frac{\vdash A, B_i}{\vdash A, B_1 + B_2}\,{+}\text{R},i \qquad \frac{\vdash A, B \quad \vdash A, C}{\vdash A, B \times C}\,{\times}\text{R}$$

$$\frac{\vdash A, B[t/x]}{\vdash A, \exists x.B}\,\exists\text{R},t \qquad \frac{\vdash A, B}{\vdash A, \forall x.B}\,\forall\text{R}\ (x \notin \textsc{fv}(A))$$

What is first-order quantification in proofs?

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Commonly: *expansion + substitution*[1]

$$\exists x.A \quad \Rightarrow \quad A[t_1/x] + \ldots + A[t_n/x]$$

[1] For classical logic: [Herbrand 1930, Miller 1987, Heijltjes 2010]

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$$\dfrac{\dfrac{\dfrac{}{\vdash \overline{P}(s), P(s)}}{\vdash \exists x.\overline{P}(x), P(s)}\,\exists\mathrm{R},s \qquad \dfrac{\dfrac{}{\vdash \overline{P}(t), P(t)}}{\vdash \exists x.\overline{P}(x), P(t)}\,\exists\mathrm{R},t}{\vdash \exists x.\overline{P}(x), P(s) \times P(t)}\,\times\mathrm{R}$$

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Problem: Incompatible with *sequent + linking* and *relational composition*

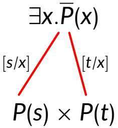
Both *expansion* and *substitution* destroy the original formula

[1] For classical logic: [Herbrand 1930, Miller 1987, Heijltjes 2010]

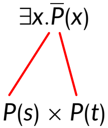
Two solutions

$$\dfrac{\dfrac{\dfrac{}{\vdash \overline{P}(s), P(s)}}{\vdash \exists x.\overline{P}(x), P(s)}\;{\scriptstyle \exists\mathrm{R},s} \qquad \dfrac{\dfrac{}{\vdash \overline{P}(t), P(t)}}{\vdash \exists x.\overline{P}(x), P(t)}\;{\scriptstyle \exists\mathrm{R},t}}{\vdash \exists x.\overline{P}(x), P(s) \times P(t)}\;{\scriptstyle \times\mathrm{R}}$$

Witness nets: explicit substitution



Unification nets: first-order unification[1]



[1] For MLL: [Hughes 2018]

Proof identity

If *any* witness will do, is the choice significant?

$$\dfrac{\dfrac{}{\vdash P(s), \overline{P}(s)}}{\vdash \exists x.P(x), \exists y.\overline{P}(y)} \quad \stackrel{?}{\equiv} \quad \dfrac{\dfrac{}{\vdash P(t), \overline{P}(t)}}{\vdash \exists x.P(x), \exists y.\overline{P}(y)}$$

Proof identity

If *any* witness will do, is the choice significant?

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What if the quantifier is *vacuous*?

$$\dfrac{\dfrac{}{\vdash P, \overline{P}}}{\vdash \exists x.P, \overline{P}}\,\exists\mathrm{R},s \quad \stackrel{?}{\equiv} \quad \dfrac{\dfrac{}{\vdash P, \overline{P}}}{\vdash \exists x.P, \overline{P}}\,\exists\mathrm{R},t$$

Proof identity

If *any* witness will do, is the choice significant?

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What if the quantifier is *vacuous*?

$$\dfrac{\dfrac{}{\vdash P, \overline{P}}}{\vdash \exists x.P, \overline{P}}\,\exists\mathrm{R},s \quad \overset{?}{\equiv} \quad \dfrac{\dfrac{}{\vdash P, \overline{P}}}{\vdash \exists x.P, \overline{P}}\,\exists\mathrm{R},t$$

What if a quantified variable occurs only in a *weakened* formula?

$$\dfrac{\dfrac{\dfrac{}{\vdash P, \overline{P}}}{\vdash P{+}Q(s), \overline{P}}\,{+}\mathrm{R},1}{\vdash \exists x.P{+}Q(x), \overline{P}}\,\exists\mathrm{R},s \quad \overset{?}{\equiv} \quad \dfrac{\dfrac{\dfrac{}{\vdash P, \overline{P}}}{\vdash P{+}Q(t), \overline{P}}\,{+}\mathrm{R},1}{\vdash \exists x.P{+}Q(x), \overline{P}}\,\exists\mathrm{R},t$$

Generality

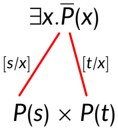
Witness nets: make *none* of these identifications (like sequent calculus)

Unification nets: make *all* of these identifications

Witness nets and Unification nets

Witness nets and Unification nets

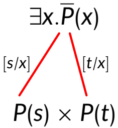
Witness net = sequent + linking with substitution



$$\dfrac{\dfrac{\dfrac{}{\vdash \overline{P}(s), P(s)}}{\vdash \exists x.\overline{P}(x), P(s)}\,{\scriptstyle \exists\mathrm{R},s} \qquad \dfrac{\dfrac{}{\vdash \overline{P}(t), P(t)}}{\vdash \exists x.\overline{P}(x), P(t)}\,{\scriptstyle \exists\mathrm{R},t}}{\vdash \exists x.\overline{P}(x), P(s) \times P(t)}\,{\scriptstyle \times\mathrm{R}}$$

Witness nets and Unification nets

Witness net = sequent + linking with substitution



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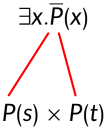
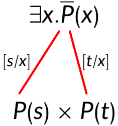
Unification net = sequent + linking



$$\dfrac{\dfrac{\dfrac{}{\vdash \overline{P}(s), P(s)}}{\vdash \exists x.\overline{P}(x), P(s)}\,{\scriptstyle \exists\mathrm{R},s} \qquad \dfrac{\dfrac{}{\vdash \overline{P}(t), P(t)}}{\vdash \exists x.\overline{P}(x), P(t)}\,{\scriptstyle \exists\mathrm{R},t}}{\vdash \exists x.\overline{P}(x), P(s) \times P(t)}\,{\scriptstyle \times\mathrm{R}}$$

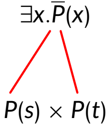
Witness nets and Unification nets

Witness net = sequent + linking with substitution



$$\dfrac{\dfrac{\dfrac{}{\vdash \overline{P}(s), P(s)}}{\vdash \exists x.\overline{P}(x), P(s)}\,{\scriptstyle \exists R,s} \qquad \dfrac{\dfrac{}{\vdash \overline{P}(t), P(t)}}{\vdash \exists x.\overline{P}(x), P(t)}\,{\scriptstyle \exists R,t}}{\vdash \exists x.\overline{P}(x), P(s) \times P(t)}\,{\scriptstyle \times R}$$

Unification net = sequent + linking



$$\dfrac{\dfrac{\dfrac{}{\vdash \overline{P}(s), P(s)}}{\vdash \exists x.\overline{P}(x), P(s)}\,{\scriptstyle \exists R,s} \qquad \dfrac{\dfrac{}{\vdash \overline{P}(t), P(t)}}{\vdash \exists x.\overline{P}(x), P(t)}\,{\scriptstyle \exists R,t}}{\vdash \exists x.\overline{P}(x), P(s) \times P(t)}\,{\scriptstyle \times R}$$

(such that "sequent + linking with mgu" forms a witness net)

Coalescence for witness nets

Coalescence for witness nets (rewrite rules)

$$
\begin{tikzcd}[column sep=small] C \arrow[d, dash, "\sigma"'] \\ D_1+D_2 \end{tikzcd}
\longrightarrow
\begin{tikzcd} C \arrow[d, dash, "\sigma"] \\ D_1+D_2 \end{tikzcd}
\qquad\qquad
\begin{tikzcd} C \arrow[d, dash, "\sigma"] \\ \exists x.D \end{tikzcd}
\xrightarrow{x \in \mathrm{DOM}(\sigma)}
\begin{tikzcd} C \arrow[d, dash, "\sigma/x"] \\ \exists x.D \end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small] C \arrow[d, dash, bend right, "\sigma"'] \arrow[d, dash, bend left, "\sigma"] \\ D_1\times D_2 \end{tikzcd}
\longrightarrow
\begin{tikzcd} C \arrow[d, dash, "\sigma"] \\ D_1\times D_2 \end{tikzcd}
\qquad\qquad
\begin{tikzcd} C \arrow[d, dash, "\sigma"] \\ \forall x.D \end{tikzcd}
\xrightarrow{x \notin \sigma}
\begin{tikzcd} C \arrow[d, dash, "\sigma"] \\ \forall x.D \end{tikzcd}
$$

Coalescence for witness nets (example)

$$\forall x.\ \exists y.\ \overline{P}(x,y)$$

$[x/z, s/y]$ $\quad$ $[x/z, t/y]$

$$\exists z.\ P(z,s) \times P(z,t)$$

Coalescence for witness nets (example)

$$\forall x.\ \exists y.\ \overline{P}(x,y)$$

$[x/z,s/y]$ $[x/z,t/y]$

$$\exists z.\ P(z,s) \times P(z,t)$$

$$\dfrac{}{\vdash \overline{P}(x,s),\ P(x,s)} \qquad \dfrac{}{\vdash \overline{P}(x,t),\ P(x,t)}$$

Coalescence for witness nets (example)

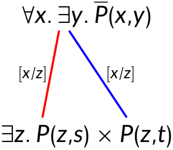
$$\forall x.\ \exists y.\ \overline{P}(x,y)$$

$[x/z]$ $\qquad$ $[x/z, t/y]$

$$\exists z.\ P(z,s) \times P(z,t)$$

$$\dfrac{\dfrac{}{\vdash \overline{P}(x,s), P(x,s)}}{\vdash \exists y.\overline{P}(x,y), P(x,s)}\ \exists\mathrm{R}, s \qquad\qquad \dfrac{}{\vdash \overline{P}(x,t), P(x,t)}$$

Coalescence for witness nets (example)



$$\dfrac{\dfrac{}{\vdash \overline{P}(x,s),\ P(x,s)}}{\vdash \exists y.\overline{P}(x,y),\ P(x,s)}\ \exists\mathrm{R},s \qquad \dfrac{\dfrac{}{\vdash \overline{P}(x,t),\ P(x,t)}}{\vdash \exists y.\overline{P}(x,y),\ P(x,t)}\ \exists\mathrm{R},t$$

Coalescence for witness nets (example)

$$\forall x.\ \exists y.\ \overline{P}(x,y)$$

$[x/z]$

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$$\cfrac{\cfrac{\cfrac{\cfrac{}{\vdash \overline{P}(x,s),\ P(x,s)}}{\vdash \exists y.\overline{P}(x,y),\ P(x,s)}\ \exists\mathrm{R},s \qquad \cfrac{\cfrac{}{\vdash \overline{P}(x,t),\ P(x,t)}}{\vdash \exists y.\overline{P}(x,y),\ P(x,t)}\ \exists\mathrm{R},t}{\vdash \exists y.\overline{P}(x,y),\ P(x,s) \times P(x,t)}\ \times\mathrm{R}}{\vdash \exists y.\overline{P}(x,y),\ \exists z.P(z,s) \times P(z,t)}\ \exists\mathrm{R},x$$

Coalescence for witness nets (example)

$$\forall x.\ \exists y.\ \overline{P}(x,y)$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overline{P}(x,s),\ P(x,s)}}{\vdash \exists y.\overline{P}(x,y),\ P(x,s)}\ {\scriptstyle \exists\mathrm{R},s} \qquad \dfrac{\dfrac{}{\vdash \overline{P}(x,t),\ P(x,t)}}{\vdash \exists y.\overline{P}(x,y),\ P(x,t)}\ {\scriptstyle \exists\mathrm{R},t}}{\vdash \exists y.\overline{P}(x,y),\ P(x,s) \times P(x,t)}\ {\scriptstyle \times\mathrm{R}}}{\vdash \exists y.\overline{P}(x,y),\ \exists z.P(z,s) \times P(z,t)}\ {\scriptstyle \exists\mathrm{R},x}}{\vdash \forall x.\exists y.\overline{P}(x,y),\ \exists z.P(z,s) \times P(z,t)}\ {\scriptstyle \forall\mathrm{R}}$$

Slicing for witness nets

Slicing for witness nets

Correctness conditions:

Slicing for witness nets

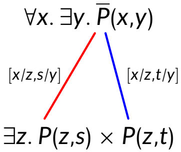
Correctness conditions:

- *local eigenvariables*

Slicing for witness nets

Correctness conditions:

- *local eigenvariables*



Slicing for witness nets

Correctness conditions:

- *local eigenvariables*
- *exact coverage*

$$\forall x.\ \exists y.\ \overline{P}(x,y)$$

$[x/z, s/y]$ $\quad$ $[x/z, t/y]$

$$\exists z.\ P(z,s) \times P(z,t)$$

Slicing for witness nets

Correctness conditions:

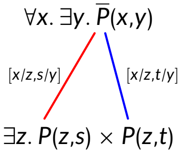
- *local eigenvariables*
- *exact coverage*
- *slice-correctness* (every slice is a singleton)

$$\begin{tikzcd}[column sep=small] & \forall x.\,\exists y.\,\overline{P}(x,y) \arrow[dl, dash, "{[x/z,s/y]}"'] \arrow[dr, dash, "{[x/z,t/y]}"] & \\ \exists z.\,P(z,s) & \times & P(z,t) \end{tikzcd}$$

Slicing for witness nets

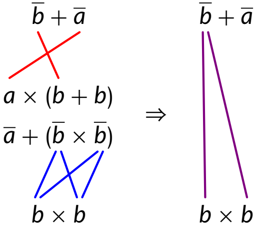
Correctness conditions:

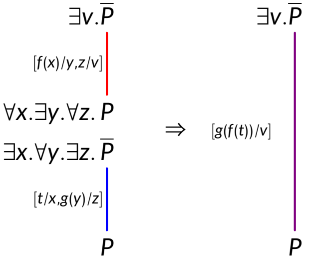
- *local eigenvariables*
- *exact coverage*
- *slice-correctness* (every slice is a singleton)
- *dependency-correctness* (dependency relation for each link is a partial order)



Composition / cut elimination

relational composition $\quad + \quad$ composition of substitution





Summary

	Monomial nets[1]	Witness nets[2]	Unification nets[2]
Canonicity	?	✓	✓
Generality	✗	✗	✓
Direct composition	✗	✓	✓
Coalescence	?	✓	✓
Slicing	✓	✓	?

[1][Girard 1987, 1996] [2][Inria RR-9201]

Summary

	Monomial nets[1]	Witness nets[2]	Unification nets[2]
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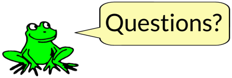
Direct composition	✗	✓	✓
Coalescence	?	✓	✓
Slicing	✓	✓	?



[1][Girard 1987, 1996]   [2][Inria RR-9201]