

Proof nets for first-order additive linear logic

Willem Heijltjes

University of Bath



Dominic Hughes

UC Berkeley



Lutz Straßburger

Inria Saclay



3rd MLA Workshop — Nancy — March 11–14, 2019

Additive linear logic

Additive linear logic

Formulas

$$A ::= a \mid \bar{a} \mid A + A \mid A \times A$$

Additive linear logic

Formulas

$$A ::= a \mid \bar{a} \mid A + A \mid A \times A$$

Sequents

$$\vdash A, B$$

Additive linear logic

Formulas

$$A ::= a \mid \bar{a} \mid A + A \mid A \times A$$

Sequents

$$\vdash A, B$$

Proofs

$$\frac{}{\vdash a, \bar{a}} \text{ax} \qquad \frac{\vdash A, B_i}{\vdash A, B_1 + B_2} \text{+R}, i \qquad \frac{\vdash A, B \quad \vdash A, C}{\vdash A, B \times C} \text{xR}$$

Features

Compact: Proof search on $\vdash A, B$ is $\mathcal{O}(|A| \times |B|)$ propositionally¹,
and *NP-complete* for first-order²

¹[Galmiche & Marion 1995] ²[Heijltjes & Hughes 2015]

Features

Compact: Proof search on $\vdash A, B$ is $\mathcal{O}(|A| \times |B|)$ propositionally¹,
and *NP-complete* for first-order²

Rich: Additives $+/\times$ (binary choice),
Duality $\overline{\overline{A}} = A$ (player/opponent),
meta-level linear implication/par $\vdash A, B$ (paralellism)

¹[Galmiche & Marion 1995] ²[Heijltjes & Hughes 2015]

Features

Compact: Proof search on $\vdash A, B$ is $\mathcal{O}(|A| \times |B|)$ propositionally¹,
and *NP-complete* for first-order²

Rich: Additives $+/\times$ (binary choice),
Duality $\overline{\overline{A}} = A$ (player/opponent),
meta-level linear implication/par $\vdash A, B$ (parallelism)

Interesting: Two-player parallel games,
Communication along a two-way channel,
the Blass problem (sequential strategies don't compose
associatively)³

¹[Galmiche & Marion 1995] ²[Heijltjes & Hughes 2015] ³[Abramsky 2003]

Features

Compact: Proof search on $\vdash A, B$ is $\mathcal{O}(|A| \times |B|)$ propositionally¹,
and *NP-complete* for first-order²

Rich: Additives $+/\times$ (binary choice),
Duality $\overline{\overline{A}} = A$ (player/opponent),
meta-level linear implication/par $\vdash A, B$ (paralellism)

Interesting: Two-player parallel games,
Communication along a two-way channel,
the Blass problem (sequential strategies don't compose associatively)³

Well-behaved: Canonical proof nets⁴, also with units⁵,
Fixed-point formulae (mu-lattice hierarchy)⁶

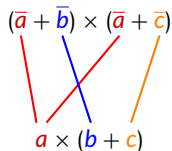
¹[Galmiche & Marion 1995] ²[Heijltjes & Hughes 2015] ³[Abramsky 2003]

⁴[Hughes & Van Glabbeek 2005] ⁵[Heijltjes 2011] ⁶[Santocanale 2002]

Proof nets and cut elimination

Proof nets and cut elimination

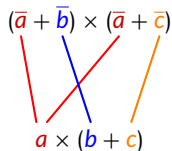
Proof net = sequent + linking



$$\begin{array}{c}
 \frac{\frac{\frac{\overline{\vdash \bar{a}, a}^{\text{ax}}}{\vdash \bar{a} + \bar{b}, a}^{+R,1}}{\vdash \bar{a} + \bar{b}, a \times (b + c)}^{\times R} \quad \frac{\frac{\frac{\overline{\vdash \bar{b}, b}^{\text{ax}}}{\vdash \bar{b}, b + c}^{+R,1}}{\vdash \bar{a} + \bar{b}, b + c}^{+R,2}}{\vdash \bar{a} + \bar{b}, a \times (b + c)}^{\times R} \\
 \frac{\frac{\frac{\overline{\vdash \bar{a}, a}^{\text{ax}}}{\vdash \bar{a} + \bar{c}, a}^{+R,1}}{\vdash \bar{a} + \bar{c}, a \times (b + c)}^{\times R} \quad \frac{\frac{\frac{\overline{\vdash \bar{c}, c}^{\text{ax}}}{\vdash \bar{c}, b + c}^{+R,2}}{\vdash \bar{a} + \bar{c}, b + c}^{+R,2}}{\vdash \bar{a} + \bar{c}, a \times (b + c)}^{\times R} \\
 \hline
 \vdash (\bar{a} + \bar{b}) \times (\bar{a} + \bar{c}), a \times (b + c) \quad \times R
 \end{array}$$

Proof nets and cut elimination

Proof net = sequent + linking

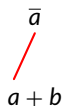


$$\frac{\frac{\frac{\overline{\vdash \bar{a}, a}^{\text{ax}}}{\vdash \bar{a} + \bar{b}, a}^{+R,1}}{\vdash \bar{a} + \bar{b}, a \times (b + c)}^{\times R} \quad \frac{\frac{\frac{\overline{\vdash \bar{b}, b}^{\text{ax}}}{\vdash \bar{b}, b + c}^{+R,1}}{\vdash \bar{a} + \bar{b}, b + c}^{+R,2}}{\vdash \bar{a} + \bar{b}, a \times (b + c)}^{\times R} \quad \frac{\frac{\frac{\overline{\vdash \bar{a}, a}^{\text{ax}}}{\vdash \bar{a} + \bar{c}, a}^{+R,1}}{\vdash \bar{a} + \bar{c}, a \times (b + c)}^{\times R} \quad \frac{\frac{\frac{\overline{\vdash \bar{c}, c}^{\text{ax}}}{\vdash \bar{c}, b + c}^{+R,2}}{\vdash \bar{a} + \bar{c}, b + c}^{+R,2}}{\vdash \bar{a} + \bar{c}, a \times (b + c)}^{\times R}}{\vdash (\bar{a} + \bar{b}) \times (\bar{a} + \bar{c}), a \times (b + c)}^{\times R}$$

More examples:



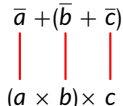
diagonal



injection



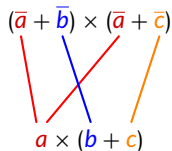
symmetry



associativity

Proof nets and cut elimination

Proof net = sequent + linking



$$\begin{array}{c}
 \frac{\frac{\frac{\overline{a}, a}{}^{ax} \quad \frac{\overline{b}, b}{}^{ax}}{\vdash \overline{a} + \overline{b}, a}^{+R,1} \quad \frac{\overline{b}, b + c}{}^{+R,1}}{\vdash \overline{a} + \overline{b}, b + c}^{+R,2} \quad \frac{\frac{\overline{a}, a}{}^{ax} \quad \frac{\overline{c}, c}{}^{ax}}{\vdash \overline{a} + \overline{c}, a}^{+R,1} \quad \frac{\overline{c}, b + c}{}^{+R,2}}{\vdash \overline{a} + \overline{c}, b + c}^{+R,2} \\
 \hline
 \frac{\vdash \overline{a} + \overline{b}, a \times (b + c)}{\vdash \overline{a} + \overline{b}, a \times (b + c)}^{xR} \quad \frac{\vdash \overline{a} + \overline{c}, a \times (b + c)}{\vdash \overline{a} + \overline{c}, a \times (b + c)}^{xR} \\
 \hline
 \vdash (\overline{a} + \overline{b}) \times (\overline{a} + \overline{c}), a \times (b + c) \quad xR
 \end{array}$$

More examples:



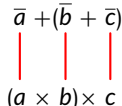
diagonal



injection



symmetry



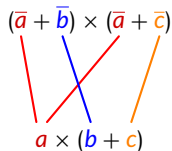
associativity

These generate all
additive proof nets.



Proof nets and cut elimination

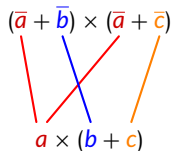
Proof net = sequent + linking



$$\frac{
 \frac{
 \frac{\overline{\vdash \bar{a}, a}^{\text{ax}}}{\vdash \bar{a} + \bar{b}, a}^{+R,1}
 \quad
 \frac{
 \frac{\overline{\vdash \bar{b}, b}^{\text{ax}}}{\vdash \bar{b}, b + c}^{+R,1}
 }{\vdash \bar{a} + \bar{b}, b + c}^{+R,2}
 }{\vdash \bar{a} + \bar{b}, a \times (b + c)}^{\times R}
 \quad
 \frac{
 \frac{
 \frac{\overline{\vdash \bar{c}, c}^{\text{ax}}}{\vdash \bar{c}, b + c}^{+R,2}
 }{\vdash \bar{a} + \bar{c}, b + c}^{+R,2}
 \quad
 \frac{\overline{\vdash \bar{a}, a}^{\text{ax}}}{\vdash \bar{a} + \bar{c}, a}^{+R,1}
 }{\vdash \bar{a} + \bar{c}, a \times (b + c)}^{\times R}
 }{\vdash (\bar{a} + \bar{b}) \times (\bar{a} + \bar{c}), a \times (b + c)}^{\times R}$$

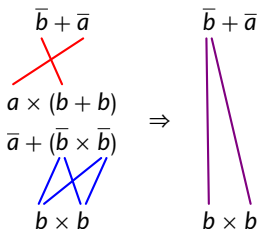
Proof nets and cut elimination

Proof net = sequent + linking



$$\begin{array}{c}
 \frac{\frac{\frac{\overline{\vdash a, a}^{\text{ax}}}{\vdash \bar{a} + \bar{b}, a}^{+R,1}}{\vdash \bar{a} + \bar{b}, a \times (b + c)}^{\times R} \quad \frac{\frac{\frac{\overline{\vdash b, b}^{\text{ax}}}{\vdash \bar{b}, b + c}^{+R,1}}{\vdash \bar{a} + \bar{b}, b + c}^{+R,2}}{\vdash \bar{a} + \bar{b}, a \times (b + c)}^{\times R} \\
 \frac{\frac{\frac{\overline{\vdash a, a}^{\text{ax}}}{\vdash \bar{a}, a}^{+R,1}}{\vdash \bar{a} + \bar{c}, a}^{+R,1} \quad \frac{\frac{\frac{\overline{\vdash c, c}^{\text{ax}}}{\vdash \bar{c}, b + c}^{+R,2}}{\vdash \bar{a} + \bar{c}, b + c}^{+R,2}}{\vdash \bar{a} + \bar{c}, a \times (b + c)}^{\times R} \\
 \hline
 \vdash (\bar{a} + \bar{b}) \times (\bar{a} + \bar{c}), a \times (b + c) \quad \times R
 \end{array}$$

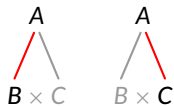
Cut elimination = relational composition



Correctness

Correctness

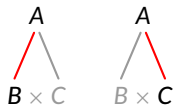
(1) Slicing:



[Hughes & Van Glabbeek 2003]

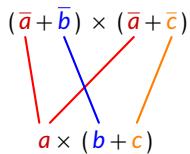
Correctness

(1) Slicing:



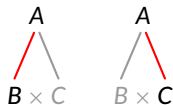
[Hughes & Van Glabbeek 2003]

Example:



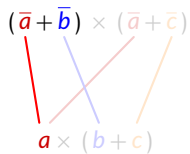
Correctness

(1) Slicing:



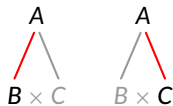
[Hughes & Van Glabbeek 2003]

Example:



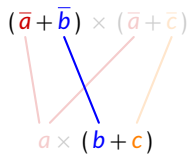
Correctness

(1) Slicing:



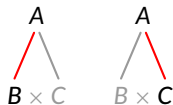
[Hughes & Van Glabbeek 2003]

Example:



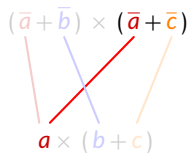
Correctness

(1) Slicing:



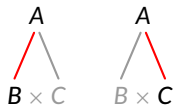
[Hughes & Van Glabbeek 2003]

Example:



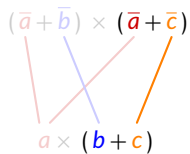
Correctness

(1) Slicing:



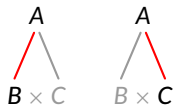
[Hughes & Van Glabbeek 2003]

Example:



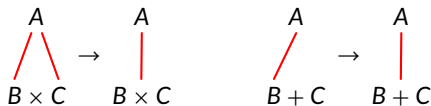
Correctness

(1) Slicing:



[Hughes & Van Glabbeek 2003]

(2) Coalescence:



[Heijltjes & Hughes 2015]

additive linear logic

Formulas

$$A ::= a \mid \bar{a} \mid A + A \mid A \times A$$

Sequents

$$\vdash A, B$$

Proofs

$$\frac{}{\vdash a, \bar{a}} \text{ax} \qquad \frac{\vdash A, B_i}{\vdash A, B_1 + B_2} \text{+R}, i \qquad \frac{\vdash A, B \quad \vdash A, C}{\vdash A, B \times C} \text{xR}$$

First-order additive linear logic

Formulas

$$A ::= a \mid \bar{a} \mid A + A \mid A \times A \mid \exists x.A \mid \forall x.A$$

$$a ::= P(t_1, \dots, t_n)$$

$$t ::= f(t_1, \dots, t_n) \mid x$$

Sequents

$$\vdash A, B$$

Proofs

$$\frac{}{\vdash a, \bar{a}} \text{ax} \quad \frac{\vdash A, B_i}{\vdash A, B_1 + B_2} \text{+R,i} \quad \frac{\vdash A, B \quad \vdash A, C}{\vdash A, B \times C} \text{xR}$$

$$\frac{\vdash A, B[t/x]}{\vdash A, \exists x.B} \exists\text{R,t} \quad \frac{\vdash A, B}{\vdash A, \forall x.B} \forall\text{R} \ (x \notin \text{FV}(A))$$

What is first-order quantification in proofs?

What is first-order quantification in proofs?

Commonly: *expansion + substitution*¹

$$\exists x.A \quad \Rightarrow \quad A[t_1/x] + \dots + A[t_n/x]$$

¹For classical logic: [Herbrand 1930, Miller 1987, Heijltjes 2010]

What is first-order quantification in proofs?

Commonly: *expansion + substitution*¹

$$\exists x.A \quad \Rightarrow \quad A[t_1/x] + \dots + A[t_n/x]$$

$$\frac{\frac{\overline{\vdash \bar{P}(s), P(s)}}{\vdash \exists x.\bar{P}(x), P(s)} \exists R,s \quad \frac{\overline{\vdash \bar{P}(t), P(t)}}{\vdash \exists x.\bar{P}(x), P(t)} \exists R,t}{\vdash \exists x.\bar{P}(x), P(s) \times P(t)} \times R$$

¹For classical logic: [Herbrand 1930, Miller 1987, Heijltjes 2010]

What is first-order quantification in proofs?

Commonly: *expansion + substitution*¹

$$\exists x.A \quad \Rightarrow \quad A[t_1/x] + \dots + A[t_n/x]$$

$$\frac{\frac{\overline{\vdash \bar{P}(s), P(s)}}{\vdash \exists x.\bar{P}(x), P(s)} \exists R, s \quad \frac{\overline{\vdash \bar{P}(t), P(t)}}{\vdash \exists x.\bar{P}(x), P(t)} \exists R, t}{\vdash \exists x.\bar{P}(x), P(s) \times P(t)} \times R$$

Problem: Incompatible with *sequent + linking* and *relational composition*

Both *expansion* and *substitution* destroy the original formula

¹For classical logic: [Herbrand 1930, Miller 1987, Heijltjes 2010]

Two solutions

$$\frac{\frac{\overline{\vdash \bar{P}(s), P(s)}}{\vdash \exists x. \bar{P}(x), P(s)} \exists R, s \quad \frac{\frac{\overline{\vdash \bar{P}(t), P(t)}}{\vdash \exists x. \bar{P}(x), P(t)} \exists R, t}{\vdash \exists x. \bar{P}(x), P(s) \times P(t)} \times R$$

Witness nets: explicit substitution

$$\begin{array}{c} \exists x. \bar{P}(x) \\ \swarrow [s/x] \quad \searrow [t/x] \\ P(s) \times P(t) \end{array}$$

Unification nets: first-order unification¹

$$\begin{array}{c} \exists x. \bar{P}(x) \\ \swarrow \quad \searrow \\ P(s) \times P(t) \end{array}$$

¹For MLL: [Hughes 2018]

Proof identity

If *any* witness will do, is the choice significant?

$$\frac{\frac{}{\vdash P(s), \bar{P}(s)}}{} \quad \stackrel{?}{\equiv} \quad \frac{\frac{}{\vdash P(t), \bar{P}(t)}}{} \\ \hline \vdash \exists x.P(x), \exists y.\bar{P}(y) \quad \quad \quad \vdash \exists x.P(x), \exists y.\bar{P}(y)$$

Proof identity

If *any* witness will do, is the choice significant?

$$\frac{\frac{}{\vdash P(s), \bar{P}(s)}}{} \quad \stackrel{?}{\equiv} \quad \frac{\frac{}{\vdash P(t), \bar{P}(t)}}{} \\ \hline \vdash \exists x.P(x), \exists y.\bar{P}(y) \quad \quad \quad \vdash \exists x.P(x), \exists y.\bar{P}(y)$$

What if the quantifier is *vacuous*?

$$\frac{\frac{}{\vdash P, \bar{P}}}{\vdash \exists x.P, \bar{P}} \exists R, s \quad \stackrel{?}{\equiv} \quad \frac{\frac{}{\vdash P, \bar{P}}}{\vdash \exists x.P, \bar{P}} \exists R, t$$

Proof identity

If *any* witness will do, is the choice significant?

$$\frac{\frac{}{\vdash P(s), \overline{P(s)}}}{\vdash \exists x.P(x), \exists y.\overline{P(y)}} \quad \stackrel{?}{\equiv} \quad \frac{\frac{}{\vdash P(t), \overline{P(t)}}}{\vdash \exists x.P(x), \exists y.\overline{P(y)}}$$

What if the quantifier is *vacuous*?

$$\frac{\frac{}{\vdash P, \overline{P}}}{\vdash \exists x.P, \overline{P}} \stackrel{\exists R, s}{=} \frac{\frac{}{\vdash P, \overline{P}}}{\vdash \exists x.P, \overline{P}} \stackrel{\exists R, t}{=}$$

What if a quantified variable occurs only in a *weakened* formula?

$$\frac{\frac{\frac{}{\vdash P, \overline{P}}}{\vdash P+Q(s), \overline{P}} \stackrel{+R, 1}{=}}{\vdash \exists x.P+Q(x), \overline{P}} \stackrel{\exists R, s}{=} \frac{\frac{\frac{}{\vdash P, \overline{P}}}{\vdash P+Q(t), \overline{P}} \stackrel{+R, 1}{=}}{\vdash \exists x.P+Q(x), \overline{P}} \stackrel{\exists R, t}{=}$$

Generality

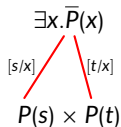
Witness nets: make *none* of these identifications (like sequent calculus)

Unification nets: make *all* of these identifications

Witness nets and Unification nets

Witness nets and Unification nets

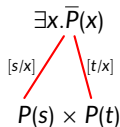
Witness net = sequent + linking with substitution



$$\frac{\frac{\overline{\vdash \bar{P}(s), P(s)}}{\vdash \exists x. \bar{P}(x), P(s)} \exists R, s \quad \frac{\overline{\vdash \bar{P}(t), P(t)}}{\vdash \exists x. \bar{P}(x), P(t)} \exists R, t}{\vdash \exists x. \bar{P}(x), P(s) \times P(t)} \times R$$

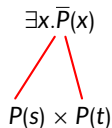
Witness nets and Unification nets

Witness net = sequent + linking with substitution



$$\frac{\frac{\overline{\vdash \bar{P}(s), P(s)}}{\vdash \exists x. \bar{P}(x), P(s)} \exists R, s \quad \frac{\overline{\vdash \bar{P}(t), P(t)}}{\vdash \exists x. \bar{P}(x), P(t)} \exists R, t}{\vdash \exists x. \bar{P}(x), P(s) \times P(t)} \times R$$

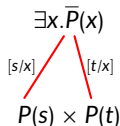
Unification net = sequent + linking



$$\frac{\frac{\overline{\vdash \bar{P}(s), P(s)}}{\vdash \exists x. \bar{P}(x), P(s)} \exists R, s \quad \frac{\overline{\vdash \bar{P}(t), P(t)}}{\vdash \exists x. \bar{P}(x), P(t)} \exists R, t}{\vdash \exists x. \bar{P}(x), P(s) \times P(t)} \times R$$

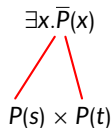
Witness nets and Unification nets

Witness net = sequent + linking with substitution



$$\frac{\frac{\overline{\vdash \bar{P}(s), P(s)}}{\vdash \exists x. \bar{P}(x), P(s)} \exists R, s \quad \frac{\overline{\vdash \bar{P}(t), P(t)}}{\vdash \exists x. \bar{P}(x), P(t)} \exists R, t}{\vdash \exists x. \bar{P}(x), P(s) \times P(t)} \times R$$

Unification net = sequent + linking



$$\frac{\frac{\overline{\vdash \bar{P}(s), P(s)}}{\vdash \exists x. \bar{P}(x), P(s)} \exists R, s \quad \frac{\overline{\vdash \bar{P}(t), P(t)}}{\vdash \exists x. \bar{P}(x), P(t)} \exists R, t}{\vdash \exists x. \bar{P}(x), P(s) \times P(t)} \times R$$

(such that “sequent + linking with mgu” forms a witness net)

Coalescence for witness nets

Coalescence for witness nets (rewrite rules)

$$\begin{array}{c} C \\ \sigma \diagup \\ D_1 + D_2 \end{array} \rightarrow \begin{array}{c} C \\ \sigma \diagdown \\ D_1 + D_2 \end{array}$$

$$\begin{array}{c} C \\ \sigma \diagdown \\ \exists x.D \end{array} \xrightarrow{x \in \text{DOM}(\sigma)} \begin{array}{c} C \\ \sigma/x \diagdown \\ \exists x.D \end{array}$$

$$\begin{array}{c} C \\ \sigma \diagup \quad \sigma \diagdown \\ D_1 \times D_2 \end{array} \rightarrow \begin{array}{c} C \\ \sigma \diagdown \\ D_1 \times D_2 \end{array}$$

$$\begin{array}{c} C \\ \sigma \diagdown \\ \forall x.D \end{array} \xrightarrow{x \notin \sigma} \begin{array}{c} C \\ \sigma \diagup \\ \forall x.D \end{array}$$

Coalescence for witness nets (example)

$$\begin{array}{ccc} \forall x. \exists y. \bar{P}(x,y) & & \\ \textcolor{red}{\downarrow} [x/z, s/y] & & \textcolor{blue}{\downarrow} [x/z, t/y] \\ \exists z. P(z,s) \times P(z,t) & & \end{array}$$

Coalescence for witness nets (example)

$$\begin{array}{c} \forall x. \exists y. \bar{P}(x,y) \\ \text{[x/z,s/y]} \quad \text{[x/z,t/y]} \\ \exists z. P(z,s) \times P(z,t) \end{array}$$

$$\overline{\vdash \bar{P}(x,s), P(x,s)}$$

$$\overline{\vdash \bar{P}(x,t), P(x,t)}$$

Coalescence for witness nets (example)

$$\begin{array}{c}
 \forall x. \exists y. \bar{P}(x,y) \\
 \text{[x/z]} \quad \text{[x/z,t/y]} \\
 \exists z. P(z,s) \times P(z,t)
 \end{array}$$

$$\frac{\overline{\vdash \bar{P}(x,s), P(x,s)}}{\vdash \exists y. \bar{P}(x,y), P(x,s)} \exists R, s$$

$$\overline{\vdash \bar{P}(x,t), P(x,t)}$$

Coalescence for witness nets (example)

$$\begin{array}{c} \forall x. \exists y. \bar{P}(x,y) \\ \textcolor{red}{\nearrow} \text{ [x/z] } \quad \textcolor{blue}{\searrow} \text{ [x/z] } \\ \exists z. P(z,s) \times P(z,t) \end{array}$$

$$\frac{\overline{\vdash \bar{P}(x,s), P(x,s)}}{\vdash \exists y. \bar{P}(x,y), P(x,s)} \textcolor{red}{\exists R,s}$$

$$\frac{\overline{\vdash \bar{P}(x,t), P(x,t)}}{\vdash \exists y. \bar{P}(x,y), P(x,t)} \textcolor{blue}{\exists R,t}$$

Coalescence for witness nets (example)

$$\begin{array}{c} \forall x. \exists y. \bar{P}(x,y) \\ \quad \quad \quad \swarrow [x/z] \\ \exists z. P(z,s) \times P(z,t) \end{array}$$

$$\frac{\frac{\overline{\vdash \bar{P}(x,s), P(x,s)}}{\vdash \exists y. \bar{P}(x,y), P(x,s)} \exists R,s \quad \frac{\overline{\vdash \bar{P}(x,t), P(x,t)}}{\vdash \exists y. \bar{P}(x,y), P(x,t)} \exists R,t}{\vdash \exists y. \bar{P}(x,y), P(x,s) \times P(x,t)} \times R$$

Coalescence for witness nets (example)

$$\begin{array}{c} \forall x. \exists y. \bar{P}(x,y) \\ \diagup \\ \exists z. P(z,s) \times P(z,t) \end{array}$$

$$\frac{\frac{\overline{\vdash \bar{P}(x,s), P(x,s)}}{\vdash \exists y. \bar{P}(x,y), P(x,s)} \exists R,s \quad \frac{\overline{\vdash \bar{P}(x,t), P(x,t)}}{\vdash \exists y. \bar{P}(x,y), P(x,t)} \exists R,t}{\vdash \exists y. \bar{P}(x,y), P(x,s) \times P(x,t)} \times R$$

$$\frac{\vdash \exists y. \bar{P}(x,y), P(x,s) \times P(x,t)}{\vdash \exists y. \bar{P}(x,y), \exists z. P(z,s) \times P(z,t)} \exists R,x$$

Coalescence for witness nets (example)

$$\frac{}{\forall x. \exists y. \bar{P}(x,y)} \quad \frac{}{\exists z. P(z,s) \times P(z,t)}$$

$$\frac{\frac{\frac{}{\vdash \bar{P}(x,s), P(x,s)}}{\vdash \exists y. \bar{P}(x,y), P(x,s)} \exists R,s \quad \frac{\frac{\frac{}{\vdash \bar{P}(x,t), P(x,t)}}{\vdash \exists y. \bar{P}(x,y), P(x,t)} \exists R,t}{\vdash \exists y. \bar{P}(x,y), P(x,s) \times P(x,t)} \times R}{\vdash \exists y. \bar{P}(x,y), \exists z. P(z,s) \times P(z,t)} \exists R,x}{\vdash \forall x. \exists y. \bar{P}(x,y), \exists z. P(z,s) \times P(z,t)} \forall R$$

Slicing for witness nets

Slicing for witness nets

Correctness conditions:

Slicing for witness nets

Correctness conditions:

- *local eigenvariables*

Slicing for witness nets

Correctness conditions:

- *local eigenvariables*

The diagram illustrates a logical derivation. At the top is the formula $\forall x. \exists y. \bar{P}(x,y)$. At the bottom is the formula $\exists z. P(z,s) \times P(z,t)$. A red line connects the top formula to the left part of the bottom formula, $P(z,s)$, with the substitution $[x/z, s/y]$ written next to it. A blue line connects the top formula to the right part of the bottom formula, $P(z,t)$, with the substitution $[x/z, t/y]$ written next to it.

$$\forall x. \exists y. \bar{P}(x,y)$$

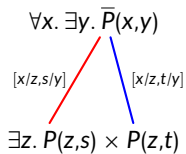
$[x/z, s/y]$ $[x/z, t/y]$

$$\exists z. P(z,s) \times P(z,t)$$

Slicing for witness nets

Correctness conditions:

- *local eigenvariables*
- *exact coverage*


$$\begin{array}{c} \forall x. \exists y. \bar{P}(x,y) \\ \begin{array}{cc} [x/z,s/y] & [x/z,t/y] \\ \textcolor{red}{\diagdown} & \textcolor{blue}{\diagdown} \\ \exists z. P(z,s) \times P(z,t) \end{array} \end{array}$$

Slicing for witness nets

Correctness conditions:

- *local eigenvariables*
- *exact coverage*
- *slice-correctness* (every slice is a singleton)

$$\begin{array}{c} \forall x. \exists y. \bar{P}(x,y) \\ \begin{array}{cc} [x/z,s/y] & [x/z,t/y] \\ \textcolor{red}{/} & \textcolor{blue}{\backslash} \\ \exists z. P(z,s) \times P(z,t) \end{array} \end{array}$$

Slicing for witness nets

Correctness conditions:

- *local eigenvariables*
- *exact coverage*
- *slice-correctness* (every slice is a singleton)
- *dependency-correctness* (dependency relation for each link is a partial order)

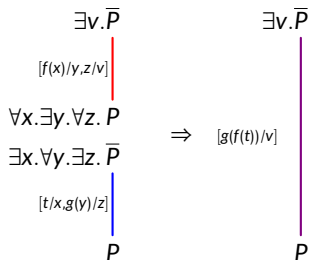
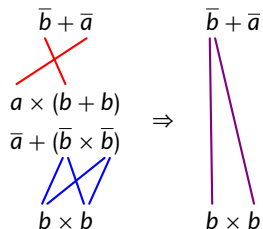
$$\begin{array}{ccc} \forall x. \exists y. \bar{P}(x,y) & & \\ \text{[x/z,s/y]} \swarrow & & \searrow \text{[x/z,t/y]} \\ \exists z. P(z,s) \times P(z,t) & & P(z,t) \end{array}$$

Composition / cut elimination

relational composition

+

composition of substitution



Summary

	Monomial nets ¹	Witness nets ²	Unification nets ²
Canonicity	?	✓	✓
Generality	✗	✗	✓
Direct composition	✗	✓	✓
Coalescence	?	✓	✓
Slicing	✓	✓	?

¹[Girard 1987, 1996] ²[Inria RR-9201]

Summary

	Monomial nets ¹	Witness nets ²	Unification nets ²
Canonicity	?	✓	✓
Generality	✗	✗	✓
Direct composition	✗	✓	✓
Coalescence	?	✓	✓
Slicing	✓	✓	?



¹[Girard 1987, 1996] ²[Inria RR-9201]