Proof nets for first-order additive linear logic

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Abstract

We extend proof nets for additive linear logic with first-order quantification.

We present two versions of our proof nets. One, witness nets, retains explicit witnessing information to existential quantification. For the other, unification nets, this information is absent but can be reconstructed through unification. Unification nets embody a central contribution of the paper: first-order witness information can be left implicit, and reconstructed as needed.

Witness nets are canonical for first-order additive sequent calculus. Unification nets in addition factor out any inessential choice for existential witnesses. Both notions of proof net are defined through coalescence (the additive version of multiplicative contractibility), and for witness nets a geometric correctness criterion is provided. Both feature a global composition operation.

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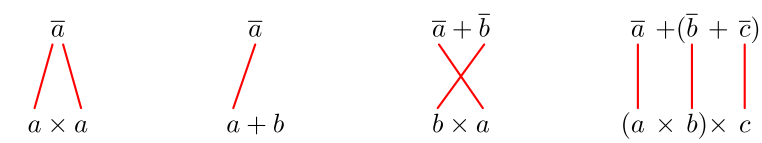
1 Introduction

Additive linear logic (ALL) is the logic of product ($\times$) and sum ($+$). Proofs, over a sequent $A \vdash B$ or, here, $\vdash \overline{A}, B$, are built from the canonical morphisms: projections, injections, and diagonals. These are captured naturally in *proof nets* [6, 16], which consist of a set of *links* between subformulas in $\overline{A}$ and in B, satisfying a simple geometric correctness condition.



diagonal injection symmetry associativity

The logic is combinatorially rich, yet well-behaved and tractable: proof search [5, 4] and proof net correctness [11] are linear in $|A| \times |B|$ (with $|A|$ the size of the syntax tree of A); proof nets remain canonical and equally tractable when extended with the two units [10, 11]; and the first-order case is merely NP-complete [11].

At the same time, the ubiquity of its main operations, product and sum, mean the ideas and results it garners are widely applicable. It serves as a microcosmos for important ideas and observations: it fostered the *connections method* of proof search [4]; it demonstrated the *Blass problem* [1] of game semantics, that sequential strategies do not in general have associative composition; and it is at the root of the study of fixed points in linear logic [20]. The logic describes two-way communication [2], and is a core part of *session types* [13].

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ALL is of course part of MALL (multiplicative-additive linear logic), and its lessons are clearly visible in the second author's canonical proof nets [16] for MALL, as well as the first and second author's locally canonical *conflict nets* [15]. Its proof nets also appear as the *skew fibrations* in the second author's *combinatorial proofs* for classical logic [14], and in the third author's study of the *medial rule* for classical logic [21]. To prepare the ground for the cut elimination of first-order *combinatorial proofs* is one motivation for the current presentation.

We present proof nets for first-order additive linear logic (ALL1). The main content of first-order proof, and our central challenge, is the *witness assignment* to existential quantifiers. The standard approach to evaluate a formula $\exists x.A$ is by a substitution $A[t/x]$, which destroys the subformula A.[1] But the current, successful formulation of additive proof nets, as a set of links over a two-formula sequent, requires subformulas to remain intact. Moreover, the same quantification $\exists x.A$ may occur in different *slices* (branches of a proof above a product), with different witnesses, as below. But the *formulas + links* design has only one instance of $\exists x.A$.

$$\dfrac{\dfrac{\vdash A[s/x], B}{\vdash \exists x.A, B}\,{\scriptstyle \exists\mathrm{R},s} \qquad \dfrac{\vdash A[t/x], C}{\vdash \exists x.A, C}\,{\scriptstyle \exists\mathrm{R},t}}{\vdash \exists x.A, B \times C}\,{\scriptstyle \times\mathrm{R}}$$

We advance two solutions. ***Witness nets*** replace the *implicit* substitutions of sequent proofs with *explicit* substitutions, recorded at each link. ***Unification nets***, introduced by the second author in [17] for first-order multiplicative linear logic, omit existential witnesses and reconstruct them by unification; we extend them to ALL1. Respective examples are below.

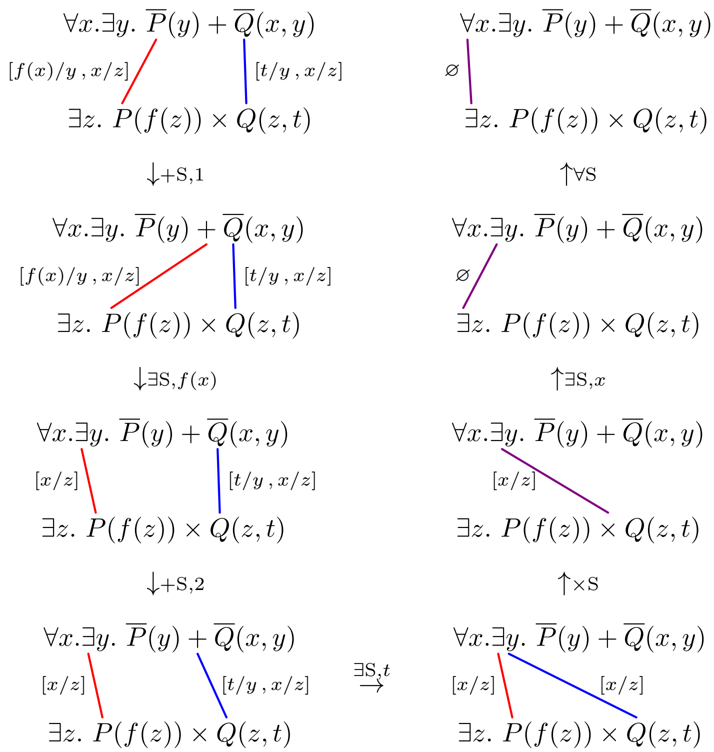
$$\begin{array}{c} \forall x.\exists y.\ \overline{P}(y) + \overline{Q}(x,y) \\ {\scriptstyle [f(x)/y\,,\,x/z]} \qquad\qquad {\scriptstyle [t/y\,,\,x/z]} \\ \exists z.\ P(f(z)) \times Q(z,t) \end{array} \qquad\qquad \begin{array}{c} \forall x.\exists y.\ \overline{P}(y) + \overline{Q}(x,y) \\ \\ \exists z.\ P(f(z)) \times Q(z,t) \end{array}$$

Correctness and sequentialization for ALL1 proof nets will be through ***coalescence*** [11, 15], the additive counterpart to ***contractibility*** for multiplicative linear logic [3]. It is a simple graph rewrite relation that pushes links from the leaves towards the roots of a sequent, that is effectively top-down sequentialization (like its multiplicative counterpart [8]). Figure 1 illustrates the process for the witness net above. In the example, a link between subformulas A and B carrying a substitution map σ represents a sequent $\vdash A\sigma, B\sigma$. Each coalescence step $(\star S)$ corresponds to a proof rule $(\star R)$, by which the sequence generates the sequent proof.

In unification nets, existential witnesses are reconstructed during coalescence. The initial substitution map assigned to a link on atomic formulas $a = P(s_1, \ldots, s_n)$ and $b = \overline{P}(t_1, \ldots, t_n)$ is the most general unifier (MGU) of a and $\bar{b}$ (the first step in Figure 2). Coalescence then proceeds as for witness nets, except when joining two links from different slices with a product step, ($\times$U) in Figure 2. Here, witness nets require both links to carry the same substitution map, while for unification nets a common, more general map is generated by unification.

We include the following results for our ALL1 proof nets. Witness nets are *canonical* for the permutations of the sequent calculus, and we provide a geometric correctness condition. *Composition* of unification nets (over $\vdash A, B$ and $\vdash \overline{B} C$, into one over $\vdash A, C$) is remarkedly simple, as the relational composition of both sets of links. For composition of witness nets, in addition the witness assignments of links must be composed, which is done through a simple process of *interaction + hiding* similar to those of game semantics [19].

[1] Given $A[t/x]$ and t, it is not possible to unambiguously recover A, which may contain other occurrences of t not originating in the substitution $[t/x]$.



$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash \overline{P}(f(x)), P(f(x))}\;\mathrm{ax}}{\vdash \overline{P}(f(x)) + \overline{Q}(x, f(x)), P(f(x))}\;{+}\mathrm{R},1}{\vdash \exists y.\overline{P}(y) + \overline{Q}(x, y), P(f(x))}\;\exists\mathrm{R},f(x) \qquad \dfrac{\dfrac{\overline{\vdash \overline{Q}(x,t), Q(x,t)}\;\mathrm{ax}}{\vdash \overline{P}(t) + \overline{Q}(x,t), Q(x,t)}\;{+}\mathrm{R},2}{\vdash \exists y.\overline{P}(y) + \overline{Q}(x, y), Q(x,t)}\;\exists\mathrm{R},t}{\dfrac{\vdash \exists y.\overline{P}(y) + \overline{Q}(x, y), P(f(x)) \times Q(x,t)}{\vdash \exists y.\overline{P}(y) + \overline{Q}(x, y), \exists z.P(f(z)) \times Q(z,t)}\;\exists\mathrm{R},x}\;\times\mathrm{R}}{\vdash \forall x.\exists y.\overline{P}(y) + \overline{Q}(x, y), \exists z.P(f(z)) \times Q(z,t)}\;\forall\mathrm{R}$$

Figure 1 A coalescence and sequentialization example

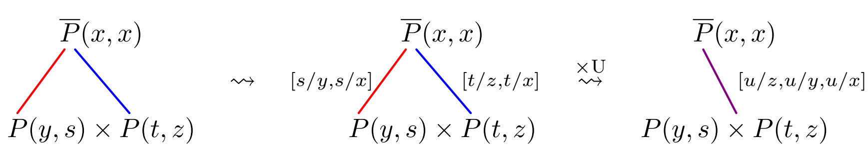


Figure 2 Coalescence with unification; u is the smallest term unifying s and t

1.1 Proof identity

At the heart of a theory of proof nets is the question of *proof identity*: when are two proofs equivalent? The answer determines which proofs should map onto the same proof net. The introduction of quantifiers creates an interesting issue: if two proofs differ by an immaterial choice of existential witness, should they be equivalent? For example, to prove the sequent $\vdash \exists x.P(x), \exists y.\overline{P}(y)$ both quantifiers must receive *the same* witness, as in the following two proofs, but any witness will do.

$$\dfrac{\overline{\vdash P(s), \overline{P}(s)}}{\vdash \exists x.P(x), \exists y.\overline{P}(y)} \quad \stackrel{?}{\equiv} \quad \dfrac{\overline{\vdash P(t), \overline{P}(t)}}{\vdash \exists x.P(x), \exists y.\overline{P}(y)}$$

The issue is more pronounced where quantifiers are *vacuous*, $\exists x.A$ where x is not free in A. The proofs below left can only be distinguished even syntactically because the $\exists R$-rule makes the instantiating witness explicit. Below right is an interesting intermediate variant: the witness s or t can be observed without explicit annotation in the $\exists R$-rule, but the choice is equally immaterial to the content of the proof as when the quantifier were vacuous.

$$\dfrac{\overline{\vdash P, \overline{P}}}{\vdash \exists x.P, \overline{P}}\,{\scriptstyle \exists\mathrm{R},s} \quad \stackrel{?}{\equiv} \quad \dfrac{\overline{\vdash P, \overline{P}}}{\vdash \exists x.P, \overline{P}}\,{\scriptstyle \exists\mathrm{R},t} \qquad \dfrac{\dfrac{\overline{\vdash P, \overline{P}}}{\vdash P + Q(s), \overline{P}}\,{\scriptstyle +\mathrm{R},1}}{\vdash \exists x.P + Q(x), \overline{P}}\,{\scriptstyle \exists\mathrm{R},s} \quad \stackrel{?}{\equiv} \quad \dfrac{\dfrac{\overline{\vdash P, \overline{P}}}{\vdash P + Q(t), \overline{P}}\,{\scriptstyle +\mathrm{R},1}}{\vdash \exists x.P + Q(x), \overline{P}}\,{\scriptstyle \exists\mathrm{R},t}$$

In this paper we will not attempt to settle the question of proof identity. Rather, our two notions of proof net each represent a natural and coherent perspective, at either end of the spectrum. *Witness nets* make all existential witnesses explicit, including those to vacuous quantifiers, rejecting all three equivalences above. *Unification nets* leave all witnesses implicit, thus identifying all proofs modulo witness assignment, and validating all three equivalences.

1.2 Monomial nets and Expansion Trees

Proof nets with additives and quantifiers exist as *monomial nets* [7], where slices are managed explicitly with *monomial weights*. These nets are not generally canonical: they admit the permutation (and duplication) of proof rules past implicit *contractions*. We believe (though have not technically verified) that additive monomial nets include, and could be restricted to, canonical forms. These would be equivalent to the following approach.

Expansion tree proofs for classical logic [18, 9] provide another approach to canonicity for first-order quantification. In this formulation, a formula $\exists x.A$ is interpreted as the sum (or classically, *disjunction*) over a fixed number of instantiations $A[t_1/x] + \ldots + A[t_n/x]$. The idea can be traced to Herbrand's Theorem [12]: $\exists x.A$ is equivalent to the infinite sum over $A[t/x]$ for all terms t in the language, but for any given proof a finite set of terms suffices.

We use this approach implicitly for our geometric correctness condition for witness nets (Section 3), which generalizes the existing *slice-based* criterion for additive proof nets [16]. In constructing the slices over a formula, we interpret $\exists x.A$ by an *expansion* $A[t_1/x] + \ldots + A[t_n/x]$, where the witnesses t_1 through t_n are collected from the substitution maps of the links in the proof net.

For the definition of proof nets, we did not pursue this direction, and consider the present choice preferable. Witness nets and unification nets retain the original structure of *formulas + links*, without requiring expansion trees. More importantly, as in the propositional case, they enable direct (non-inductive) composition by a relational composition of links. Finally, available space does not permit a detailed exposition, which we defer to a journal version.

$$\frac{}{\vdash a, \overline{a}}\text{ax} \qquad \frac{\vdash A, B_i}{\vdash A, B_1 + B_2}{+}\text{R},i \qquad \frac{\vdash A, B \quad \vdash A, C}{\vdash A, B \times C}{\times}\text{R} \qquad \frac{\vdash A, B[t/x]}{\vdash A, \exists x.B}\exists\text{R},t \qquad \frac{\vdash A, B}{\vdash A, \forall x.B}\forall\text{R}\ (x \notin \text{FV}(A))$$

Figure 3 A sequent calculus for ALL1

2 Proof nets for first-order additive linear logic

2.1 First-order additive linear logic

First-order terms and the formulas of first-order ALL are generated by the following grammars.

$$t \ ::= \ x \ \mid \ f(t_1, \dots, t_n)$$

$$a \ ::= \ P(t_1, \dots, t_n) \ \mid \ \overline{P}(t_1, \dots, t_n)$$

$$A \ ::= \ a \ \mid \ A + A \ \mid \ A \times A \ \mid \ \exists x.A \ \mid \ \forall x.A$$

Negation $(\overline{\cdot})$ is applied to predicate symbols, $\overline{P}$ as a matter of convenience. The ***dual*** $\overline{A}$ of an arbitrary formula A is given by DeMorgan. We use the following notational conventions.

x, y, z	$\in$ VAR	first-order variables
f, g, h	$\in \Sigma_f$	n-ary ($n \geq 0$) function symbols from a fixed alphabet Σ_f
P, Q, R	$\in \Sigma_p$	n-ary ($n \geq 0$) predicate symbols from a fixed alphabet Σ_p
s, t, u	$\in$ TERM	first-order terms over VAR and Σ_f
a, b, c	$\in$ ATOM	atomic propositions
A, B, C	$\in$ FORM	ALL1 formulas

A ***sequent*** $\vdash A, B$ is a pair of formulas A and B. A sequent calculus for ALL1 is given in Figure 3, where each rule has a symmetric counterpart for the first formula in the sequent. We write $\pi \vdash A, B$ for a proof π with conclusion sequent $\vdash A, B$. Two proofs are ***equivalent*** $\pi \sim \pi'$ if one is obtained from the other by rule permutations (Appendix B, Figure 10).

By a ***subformula*** we will mean a subformula ***occurrence***. For instance, a formula $A \times A$ has two subformulas A, one on the left and one on the right. The ***subformulas*** $\text{SUB}(A)$ of a formula are defined as follows; we write $B \leq A$ if B is a subformula of A, i.e. if $B \in \text{SUB}(A)$.

$$\text{SUB}(A) = \{A\} \cup \begin{cases} \text{SUB}(B) \uplus \text{SUB}(C) & \text{if } A = B + C \text{ or } A = B \times C \\ \text{SUB}(B) & \text{if } A = \exists x.B \text{ or } A = \forall x.B \end{cases}$$

A ***link*** (C, D) on a sequent $\vdash A, B$ is a pair of subformulas $C \leq A$ and $D \leq B$. A ***linking*** λ on the sequent $\vdash A, B$ is a set of links on it.

▶ **Definition 1.** A ***pre-net*** $\lambda \triangleright A, B$ is a sequent $\vdash A, B$ with a linking λ on it.

2.2 Witness maps

A ***witness map*** $\sigma\colon \text{VAR} \rightharpoonup \text{TERM}$ is a substitution map which assigns terms to variables, given as a (finite) partial function. We define it as $\sigma = [t_1/x_1, \dots, t_n/x_n]$, where its ***domain*** $\text{DOM}(\sigma)$ is $\{x_1, \dots, x_n\}$. We abbreviate by $y \in \sigma$ that a variable y occurs free in the range of σ, i.e., $y \in \text{FV}(t_i)$ for some $i \leq n$. The map $\sigma /\!\!/ x$ is undefined on x and otherwise as σ, we write $\sigma|_V$ for the restriction of σ to a set of variables V, and $\varnothing$ for the empty witness map. We write $A\sigma$ for the application of the substitutions in σ to the formula A, and $\pi\sigma$ for its application to the proof π, where it is applied to each formula in the proof, and to each existential witness t recorded with a rule $\exists\mathrm{R}, t$. The ***composition*** of two maps is written $\sigma\tau$, where $A(\sigma\tau) = (A\sigma)\tau$.

A ***witness linking*** λ_Σ is a linking λ with a ***witness labelling*** $\Sigma\colon \lambda \to \mathrm{VAR} \rightharpoonup \mathrm{TERM}$ that assigns each link (C, D) a witness map. We may use and define λ_Σ as a set of ***witness links*** $(C, D)_\sigma$, where $(C, D) \in \lambda$ and $\Sigma(C, D) = \sigma$.

▸ **Definition 2.** A ***witness pre-net*** $\lambda_\Sigma \rhd A, B$ is a sequent $\vdash A, B$ with a witness linking λ_Σ.

We will assume the following variable naming conventions.

Barendregt's convention All quantifiers in a sequent $\vdash A, B$ have a distinct binding variable, and no bound variable shares a name with a free one.

Eigenvariables not free For a link $(C, D)_\sigma$ over $\vdash A, B$, if a variable x in the range of σ is universally quantified as $\forall x.X$ in $\vdash A, B$ (an *eigenvariable*), then $C \leq X$ or $D \leq X$.

Freshness For a link $(C, D)_\sigma$ over $\vdash A, B$, other variables in the range of σ are *fresh*: distinct from existentially quantified variables in $\vdash A, B$, and from those of other links.

▸ **Definition 3.** The ***de-sequentialization*** $[\pi]$ of a sequent proof $\pi \vdash A, B$ is the witness pre-net $[\pi]^{A,B}_{\varnothing} \rhd A, B$ where the function $[-]^{A,B}_{\sigma}$ is defined inductively as follows.

$$\left[\frac{}{\vdash a, \bar{a}}\,\mathrm{ax}\right]^{b\,,\,c}_{\sigma} = \{(b, c)_\sigma\}$$

$$\left[\frac{\begin{array}{c}\pi\\ \vdash A, B_i\end{array}}{\vdash A, B_1 + B_2}\,{+}\mathrm{R}, i\right]^{A'\,,\,B_1'+B_2'}_{\sigma} = \left[\begin{array}{c}\pi\\ \vdash A, B_i\end{array}\right]^{A'\,,\,B_i'}_{\sigma}$$

$$\left[\frac{\begin{array}{c}\pi\\ \vdash A, B\end{array} \quad \begin{array}{c}\pi'\\ \vdash A, C\end{array}}{\vdash A, B \times C}\,\times\mathrm{R}\right]^{A'\,,\,B'\times C'}_{\sigma} = \left[\begin{array}{c}\pi\\ \vdash A, B\end{array}\right]^{A'\,,\,B'}_{\sigma} \cup \left[\begin{array}{c}\pi'\\ \vdash A, C\end{array}\right]^{A'\,,\,C'}_{\sigma}$$

$$\left[\frac{\begin{array}{c}\pi\\ \vdash A, B[t/x]\end{array}}{\vdash A, \exists x.B}\,\exists\mathrm{R}, t\right]^{A'\,,\,\exists x.B'}_{\sigma} = \left[\begin{array}{c}\pi\\ \vdash A, B[t/x]\end{array}\right]^{A'\,,\,B'}_{\sigma[t/x]}$$

$$\left[\frac{\begin{array}{c}\pi\\ \vdash A, B\end{array}}{\vdash A, \forall x.B}\,\forall\mathrm{R}\right]^{A'\,,\,\forall x.B'}_{\sigma} = \left[\begin{array}{c}\pi\\ \vdash A, B\end{array}\right]^{A'\,,\,B'}_{\sigma}$$

A function call $[\pi \vdash A, B]^{A',B'}_{\sigma}$ expects that $A = A'\sigma$ and $B = B'\sigma$: the translation separates a sequent $\vdash A, B$ into subformulas A', B' of the ultimate conclusion of the proof, and the accumulated existential witnesses σ. For an example, we refer to Figure 1: the de-sequentialization of the sequent proof is the first pre-net of the coalescence sequence.

We will conclude this section by discussing two further essential notions. First, a witness link $(a, b)_\sigma$ on two atomic formulas a and b is an ***axiom*** link if $a\sigma = \bar{b}\sigma$. A witness ***axiom*** linking is one where every link is an axiom link. Observe that the de-sequentialization $[\pi]$ is a witness pre-net with axiom linking $[\pi]^{A,B}_{\varnothing}$.

Second, given a link $(C, D)_\sigma$ in a pre-net $\lambda \rhd A, B$, we expect the domain of σ to be exactly the existentially quantified variables in A and B in whose scope C and D occur. For a subformula C of A, let the ***existential variables*** $\textsc{ev}_A(C)$ of C in A be the set

$$\textsc{ev}_A(C) = \{\, x \mid C < \exists x.X \leq A \,\} \ .$$

▸ **Definition 4.** A witness pre-net $\lambda_\Sigma \triangleright A, B$ has ***exact coverage*** if for every link $(C,D)_\sigma$ in λ_Σ the domain of σ is exactly the existential variables of C and D:

$$\textsc{dom}(\sigma) = \textsc{ev}_A(C) \cup \textsc{ev}_B(D) \ .$$

The witness map σ of a link $(C,D)_\sigma$ then has two natural, disjoint components, $\sigma|_{\textsc{ev}_A(C)}$ and $\sigma|_{\textsc{ev}_B(D)}$, which we will abbreviate by σ_C and σ_D respectively.

Finally, note that a de-sequentialization $[\pi]$ has exact coverage.

2.3 Correctness and sequentialization

For sequentialization, the links in a pre-net will be labelled with a sequent proof. An axiom link will carry an axiom, and each coalescence step introduces one proof rule. Formalizing this, a ***proof linking*** λ^Π_Σ is a witness linking λ_Σ with a ***proof labelling*** $\Pi\colon \lambda \to \textsc{proof}$ assigning a sequent proof to each link. We will use and define λ^Π_Σ as a set of ***proof links*** $(C,D)^\pi_\sigma$, where we require that $\pi \vdash C\sigma, D\sigma$, i.e. that π proves the conclusion $\vdash C\sigma, D\sigma$. A ***labelled pre-net*** $\lambda^\Pi_\Sigma \triangleright A, B$ is a witness pre-net $\lambda_\Sigma \triangleright A, B$ with a proof labelling Π on λ_Σ. If λ_Σ is an axiom linking, we assign an ***initial proof labelling*** $\lambda^\star_\Sigma$ as follows.

$$\lambda^\star_\Sigma = \{\ (a,b)^\pi_\sigma \ \mid \ (a,b)_\sigma \in \lambda_\Sigma \ , \ \pi = \overline{\vdash a\sigma, b\sigma} \ \}$$

For correctness we may coalesce a pre-net directly, without constructing a proof. Then to recap, we have accumulated the following further notational conventions.

π,ϕ,ψ	$\in$	$\textsc{proof}$	$\textsc{all1}$ sequent proofs
κ,λ	$\subset$	$\textsc{form} \times \textsc{form}$	linkings (sets of pairs of formulas)
ρ,σ,τ	:	$\textsc{var} \rightharpoonup \textsc{term}$	witness maps
Σ,Θ	:	$\lambda \to \textsc{var} \rightharpoonup \textsc{term}$	witness labellings on a linking λ
Π,Φ,Ψ	:	$\lambda \to \textsc{proof}$	proof labellings on a linking λ

▸ **Definition 5.** ***Strict sequentialization*** $(\to)$ is the rewrite relation on labelled pre-nets generated by the following rules, that replace one or two links by another in a pre-net $\lambda^\Pi_\Sigma \triangleright A, B$ (where B has a subformula $D_1{+}D_2$, $D_1{\times}D_2$, $\exists x.D$, and $\forall x.D$ respectively).

$$(C,D_i)^\pi_\sigma \to (C,D_1{+}D_2)^\psi_\sigma \qquad \psi = \cfrac{\overset{\pi}{\vdash C\sigma, D_i\sigma}}{\vdash C\sigma, D_1\sigma + D_2\sigma}\,{+}\mathrm{R},i \tag{$+$S, i}$$

$$\left.\begin{matrix}(C,D_1)^\pi_\sigma \\ (C,D_2)^\phi_\sigma\end{matrix}\right\} \to (C,D_1{\times}D_2)^\psi_\sigma \qquad \psi = \cfrac{\overset{\pi}{\vdash C\sigma, D_1\sigma} \quad \overset{\phi}{\vdash C\sigma, D_2\sigma}}{\vdash C\sigma, D_1\sigma \times D_2\sigma}\,{\times}\mathrm{R} \tag{$\times$S}$$

$$(C,D)^\pi_\sigma \to (C,\exists x.D)^\psi_{\sigma/\!\!/x} \quad {\scriptstyle (x \in \textsc{dom}(\sigma))} \qquad \psi = \cfrac{\overset{\pi}{\vdash C\sigma, D\sigma}}{\vdash C(\sigma/\!\!/x), \exists x.D(\sigma/\!\!/x)}\,\exists\mathrm{R},\sigma(x) \tag{$\exists$S}$$

$$(C,D)^\pi_\sigma \to (C,\forall x.D)^\psi_\sigma \quad {\scriptstyle (x \notin \sigma)} \qquad \psi = \cfrac{\overset{\pi}{\vdash C\sigma, D\sigma}}{\vdash C\sigma, \forall x.D\sigma}\,\forall\mathrm{R} \tag{$\forall$S}$$

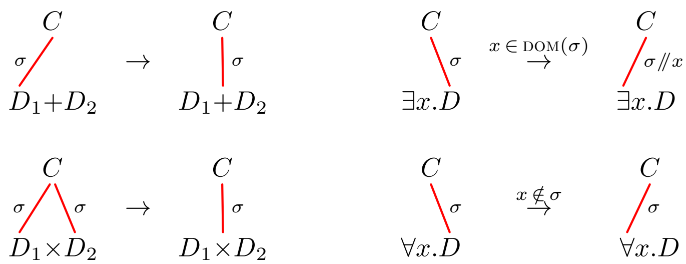


Figure 4 Coalescence rules

Strict coalescence is the same relation on witness pre-nets, ignoring proof labels, illustrated in Figure 4. A witness pre-net $\lambda_\Sigma \triangleright A, B$ ***strict-coalesces*** if it reduces to $\{(A,B)_\varnothing\} \triangleright A, B$. It ***strongly*** strict-coalesces if any coalescence path terminates at $\{(A,B)_\varnothing\} \triangleright A, B$.

For an example of coalescence, see Figure 1 in the introduction.

▸ **Definition 6.** An ALL1 ***witness proof net*** or ***witness net*** is a witness pre-net $\lambda_\Sigma \triangleright A, B$ with λ_Σ an axiom linking, that strict-coalesces. It ***sequentializes*** to a proof π if its initial labelling $\lambda^\star_\Sigma \triangleright A, B$ reduces in $(\rightarrow)$ to $\{(A,B)^\pi_\varnothing\} \triangleright A, B$.

We conclude this section by establishing that sequentialization and de-sequentialization for witness nets are inverses.

▸ **Theorem 7.** *For any* ALL1 *proof π, the witness net $[\pi]$ sequentializes to π.*

Proof. It follows by induction on π that if $\lambda_\Sigma = [\pi \vdash A, B]^{A',B'}_\sigma$ where $A'\sigma = A$ and $B'\sigma = B$, then $\lambda^\star_\Sigma \triangleright A, B$ reduces in $(\rightarrow)$ to $\{(A', B')^\pi_\sigma\} \triangleright A, B$. The statement is the case $\sigma = \varnothing$. ◂

▸ **Theorem 8.** *If $\lambda_\Sigma \triangleright A, B$ sequentializes to π, then $[\pi]$ is $\lambda_\Sigma \triangleright A, B$.*

Proof. By induction on the sequentialization path $\lambda^\star_\Sigma \triangleright A, B \rightarrow^* \{(A,B)^\pi_\varnothing\} \triangleright A, B$ it follows that in every pre-net $\kappa^\Phi_\Theta \triangleright A, B$ on this path, λ_Σ is equal to the union over the de-sequentialization of all proof labels ϕ in Φ:

$$\lambda_\Sigma = \bigcup \{ [\phi]^{C,D}_\sigma \mid (C,D)^\phi_\sigma \in \kappa^\Phi_\Theta \} .$$

The statement is then the case $\kappa^\Phi_\Theta = \{(A,B)^\pi_\varnothing\}$. ◂

3 Geometric correctness

A ***slice*** is the fraction of a proof that depends on a given choice of one branch (or projection) on each product formula $A \times B$. Important to additive proof theory is that many operations can be performed on a per-slice basis, such as normalization, or proof net correctness. We will here use slices for the latter purpose.

As in the propositional case [16], we define a ***slice*** of a sequent $\vdash A, B$ as a set of potential links, of which exactly one must be realized in a proof net $\lambda_\Sigma \triangleright A, B$. We extend the propositional criterion in two ways:

- When defining slices, we interpret an existential quantification $\exists x.A$ as a sum over all witnesses t_i to x that occur in the pre-net, $A[t_1/x] + \ldots + A[t_n/x]$. This captures the non-permutability of a product rule over distinct existential instantiations, as below left.

- We define a ***dependency*** relation between a universal quantification $\forall x.A$ and an instantiation $B[t/y]$ of $\exists x.B$ where the ***eigenvariable*** x occurs free in t (a standard approach to first-order quantification [18, 7, 9, 19]). For each link, we require the dependency to be *acyclic*, which amounts to *slice-wise* first-order correctness. It captures the non-permutability of universal and existential sequent rules due to the *eigenvariable condition* of the former, as below right.

$$\dfrac{\dfrac{\vdash A[s/x], B}{\vdash \exists x.A, B}\,\exists\mathrm{R},s \qquad \dfrac{\vdash A[t/x], C}{\vdash \exists x.A, C}\,\exists\mathrm{R},t}{\vdash \exists x.A, B \times C}\,\times\mathrm{R} \qquad\qquad \dfrac{\dfrac{\vdash A, B[t/y]}{\vdash A, \exists y.B}\,\exists\mathrm{R},t}{\vdash \forall x.A, \exists y.B}\,\forall\mathrm{R} \quad \text{where } x \in \mathrm{FV}(t)$$

For a witness linking λ_Σ and variable x, write $\Sigma(x) \subseteq \textsc{term}$ for the ***witness set*** of x, which collects the terms assigned to x by the witness maps in Σ for every link in λ:

$$\Sigma(x) = \{\ \sigma(x)\ \mid\ (C,D)_\sigma \in \lambda_\Sigma\ \}\ .$$

▸ **Definition 9** (Slice)**.** Given a witness linking λ_Σ, a ***slice*** S of a formula A and a witness map σ is a set of pairs (A',σ'), where $A' \leq A$ and $\sigma' \supseteq \sigma$, given by $S = \{(A,\sigma)\} \cup S'$ where:

- If $A = a$ then $S' = \varnothing$.
- If $A = B + C$ then $S' = S_B \uplus S_C$ with S_B a slice of B and σ, and S_C one of C and σ.
- If $A = B \times C$ then S' is a slice of B and σ or a slice of C and σ.
- If $A = \exists x.B$ then $S' = \biguplus_{t\,\in\,\Sigma(x)} S_t$ where each S_t is a slice of B and $\sigma[t/x]$.
- If $A = \forall x.B$ then S' is a slice of B and σ.

A ***slice*** of a sequent $\vdash A, B$ is a set of links

$$\{\ (C,D)_{\sigma\cup\tau}\ \mid\ (C,\sigma) \in S_A\ ,\ (D,\tau) \in S_B\ \}$$

where S_A is a slice of A and $\varnothing$, and S_B a slice of B and $\varnothing$. A ***slice*** of a witness pre-net $\lambda_\Sigma \triangleright A, B$ is the intersection $\lambda_\Sigma \cap S$ of λ_Σ with a slice S of $\vdash A, B$.

As in the propositional case, for correctness we will require that each slice is a singleton. We will further define a ***dependency*** condition to ensure that the order in which quantifiers are instantiated is sound, corresponding to the *eigenvariable condition* on the $\forall$R-rule of sequent calculus. For simplicity, we define the condition on individual links rather than slices.

▸ **Definition 10.** In a pre-net $\lambda_\Sigma \triangleright A, B$, let the ***column*** of a link $(C,D)_\sigma$ be the set of pairs

$$\{\ (X, \sigma|_{\mathrm{EV}_A(X)})\ \mid\ C \leq X \leq A\ \} \cup \{\ (Y, \sigma|_{\mathrm{EV}_B(Y)})\ \mid\ D \leq Y \leq B\ \}\ ,$$

with a ***dependency*** order $(\preccurlyeq)$: $(X,\rho) \preccurlyeq (Y,\tau)$ if $X \leq Y$ or Y occurs as $\forall x.Y$ and $x \in \rho$.

▸ **Definition 11.** A witness pre-net is ***correct*** if

- it has exact coverage,
- it is ***slice-correct***: every slice is a singleton, and
- it is ***dependency-correct***: every column is a partial order (i.e. is acyclic/antisymmetric).

In the remainder of this section we will establish that the two correctness conditions, by coalescence and by slicing, are equivalent. The geometric condition further gives *strong* coalescence: since it is preserved, no coalescence step yields an incorrect, and thus non-coalescing, pre-net. Finally, canonicity follows by inspecting the critical pairs of sequentialization.

▸ **Lemma 12.** *Strict coalescence preserves and reflects correctness.*

Proof. See the appendix. ◂

▸ **Lemma 13.** *To a correct witness pre-net $\lambda_\Sigma \triangleright A, B$ a coalescence step applies, unless it is fully coalesced already, $\lambda_\Sigma = \{(A,B)_\varnothing\}$.*

Proof. See the appendix. ◂

▸ **Theorem 14.** *A witness pre-net is correct if and only if it strict-coalesces.*

Proof. From right to left, we proceed by induction on the coalescence path from $\lambda_\Sigma \triangleright A, B$ to $\{(A,B)_\varnothing\} \triangleright A, B$, with the end result as the base case. It is slice-correct: every slice of $\vdash A, B$ contains $(A,B)_\varnothing$, so every slice of $\{(A,B)_\varnothing\} \triangleright A, B$ is the singleton $\{(A,B)_\varnothing\}$. It is also dependency-correct: the column of $(A,B)_\varnothing$ is the set $\{(A,\varnothing),(B,\varnothing)\}$, where A and B are unrelated in $(\preccurlyeq)$. For the inductive step, by Lemma 12 coalescence reflects correctness, so that any pre-net along the coalescence path is correct, in particular $\lambda_\Sigma \triangleright A, B$.

From left to right, let $\lambda_\Sigma \triangleright A, B$ be correct. By Lemma 13 either the net has coalesced, or a coalescence step applies. By Lemma 12 the result of this coalescence step is again correct. It follows that the pre-net $\lambda_\Sigma \triangleright A, B$ strict-coalesces. ◂

▸ **Corollary 15.** *A correct witness pre-net with axiom linking is a witness proof net.*

▸ **Corollary 16.** *A correct witness pre-net strongly strict-coalesces.*

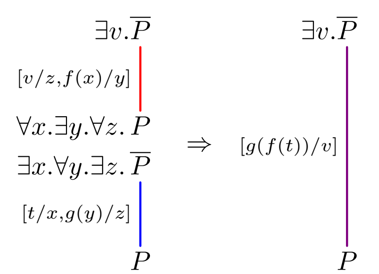
Proof. By Theorem 14 a correct pre-net coalesces, and by Lemma 12 any coalescence step preserves correctness. ◂

▸ **Theorem 17.** *Proof nets are canonical: $[\pi] = [\phi]$ if and only if $\pi \sim \phi$.*

Proof. From left to right is by inspection of the critical pairs of sequentialization $(\to)$. From right to left is by inspection of the rule permutations in Figure 10 in Appendix B. ◂

4 Composition

We will describe the composition of two witness nets by a global operation. It consists of the relational composition of both linkings, as in the propositional case, where for each pair of links that are being connected, their witness maps are composed. As links correspond to slices, the operation is effectively first-order composition [19] applied slice-wise.



Cut-elimination rules for ALL1 are given in Figure 5; the needed permutations are in Appendix B, Figure 9.

We use the example on the right to illustrate the composition of links. To eliminate the central cut, on $\forall x.\exists y.\forall z.P$ and $\exists x.\forall y.\exists z.\overline{P}$, the explicit substitutions for both formulas must be effectuated. An inductive procedure, as in sequent calculus, could apply them from outside in: first $[t/x]$, then $[f(t)/y]$ (previously $[f(x)/y]$), then $[g(f(t))/z]$ (previously $[g(y)/z]$).

For a direct definition, to compose two links $(a,b)_\sigma$ and $(\overline{b},c)_\tau$, the substitutions into the cut-formula σ_b and $\tau_{\overline{b}}$ must be applied as often as needed, up to the depth of quantifiers above b, to the terms in the range of the remaining substitutions, σ_a and τ_c. To formalize this, we will use the following notions:

- The ***domain-preserving composition*** of two witness maps $\sigma \cdot \tau$ is the map $(\sigma\tau)|_{\mathrm{DOM}(\sigma)}$.
- The ***least fixed point*** $\vec{\sigma}$ of a witness map σ is the least map ρ satisfying $\rho = \rho\sigma$.

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash A, B_1} \quad \overset{\pi_2}{\vdash A, B_2}}{\vdash A, B_1 \times B_2}\times\mathrm{R} \qquad \dfrac{\overset{\phi}{\vdash B_i, C}}{\vdash \overline{B}_1 + \overline{B}_2, C}+\mathrm{R},i}{\vdash A, C}\mathrm{cut} \quad \Rightarrow \quad \dfrac{\overset{\pi_i}{\vdash A, B_i} \quad \overset{\phi}{\vdash \overline{B}_i, C}}{\vdash A, C}\mathrm{cut}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, B[t/x]}}{\vdash A, \exists x.B}\exists\mathrm{R},t \qquad \dfrac{\overset{\phi}{\vdash \overline{B}, C}}{\vdash \forall x.\overline{B}, C}\forall\mathrm{R}}{\vdash A, C}\mathrm{cut} \quad \Rightarrow \quad \dfrac{\overset{\pi}{\vdash A, B[t/x]} \quad \overset{\phi[t/x]}{\vdash \overline{B}[t/x], C}}{\vdash A, C}\mathrm{cut}$$

Figure 5 ALL1 cut-elimination steps

The latter is the shortest sequence $\vec{\sigma} = \sigma\sigma\ldots\sigma$ such that no variable is both in the domain and range of $\vec{\sigma}$. This is not necessarily finite; in our composition operations, finiteness is ensured by the correctness conditions on proof nets.

▸ **Definition 18.** The ***composition*** $(A,B)^\pi_\sigma\,;(\overline{B},C)^\phi_\tau$ of two proof links is $(A,C)^\psi_\rho$ where

$$\rho = \sigma_A\tau_C \cdot \overrightarrow{\sigma_B\tau_{\overline{B}}} \qquad \text{and} \qquad \psi = \dfrac{\left(\overset{\pi}{\vdash A\sigma, B\sigma}\right)\overrightarrow{\sigma_B\tau_{\overline{B}}} \quad \left(\overset{\phi}{\vdash \overline{B}\tau, C\tau}\right)\overrightarrow{\sigma_B\tau_{\overline{B}}}}{\vdash A\rho, C\rho}\mathrm{cut}\ .$$

The ***composition*** $\lambda^\Pi_\Sigma\,;\kappa^\Phi_\Theta$ of two linkings is the linking

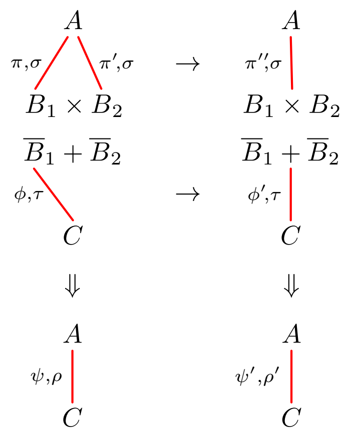
$$\{\ (X,Y)^\pi_\sigma\,;(\overline{Y},Z)^\phi_\tau \ \mid\ (X,Y)^\pi_\sigma \in \lambda^\Pi_\Sigma\ ,\ (\overline{Y},Z)^\phi_\tau \in \kappa^\Phi_\Theta\ \}$$

The ***composition*** $(\lambda^\Pi_\Sigma \triangleright A, B)\,;(\kappa^\Phi_\Theta \triangleright \overline{B}, C)$ of two pre-nets is the pre-net $(\lambda^\Pi_\Sigma\,;\kappa^\Phi_\Theta) \triangleright A, C$. These compositions may omit proof annotations and witness annotations.

The composition of two links is strongly related to composition of strategies in game semantics. There, two strategies on $\vdash A, B$ and $\vdash \overline{B}, C$ are composed by *interaction* on the interface of B and $\overline{B}$, and subsequently *hiding* that interaction.

In the following we will demonstrate that composition gives the desired result: if a net L sequentializes to π and R to ϕ, then $L\,;R$ sequentializes to a normal form of the composition of π and ϕ with a cut. To this end we will explore how composition and sequentialization interact. We will consider the critical pairs of sequentialization ($\rightarrow$) with composition ($\Rightarrow$) given in Figures 6–8, and demonstrate how they are resolved.

- $\vdash A, B_1{\times}B_2\ ;\vdash \overline{B}_1{+}\overline{B}_2, C$ (Figure 6)
  Since the existential covers of B and B_1 are the same, $\sigma_B\tau_{\overline{B}} = \sigma_{B_1}\tau_{\overline{B_1}}$ and $\rho = \rho'$. It then follows that ψ' cut-eliminates in one step to ψ.
- $\vdash A, \exists x.B\ ;\vdash \forall x.\overline{B}, C$ (Figure 7)
  Since x is not free in the range of τ, and (by the freshness convention) nor in the range of σ, we have that $\overrightarrow{\sigma_B\tau_{\overline{B}}}$ is $\overrightarrow{(\sigma_B /\!\!/ x)\tau_{\overline{B}}}$ plus the substitution $[\sigma(x)/x]$. Then $\rho = \rho'$ (as x does not occur in the range of $\sigma_A\tau_C$) and ψ' reduces to ψ in a single cut-elimination step.
- $\vdash A, B\ ;\vdash \overline{B}, \exists x.C$ (Figure 8)
  Observe that since x occurs in C but not B, it is not in the domain of τ_B, so that $\tau_B /\!\!/ x$ is just τ_B. Then $\rho' = \rho /\!\!/ x$, and the diagram is closed by a sequentialization step (from left



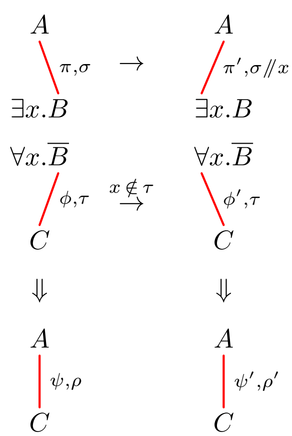
$$\rho \;=\; \sigma_A \tau_C \cdot \overrightarrow{\sigma_B \tau_{\overline{B}}}$$

$$\rho' \;=\; \sigma_A \tau_C \cdot \overrightarrow{\sigma_{B_1} \tau_{\overline{B_1}}}$$

$$\psi \;=\; \dfrac{\left(\begin{array}{c}\pi\\ \vdash A\sigma, B_1\sigma\end{array}\right)\overrightarrow{\sigma_{B_1}\tau_{\overline{B_1}}} \qquad \left(\begin{array}{c}\phi\\ \vdash \overline{B}_1\tau, C\tau\end{array}\right)\overrightarrow{\sigma_{B_1}\tau_{\overline{B_1}}}}{\vdash A\rho, C\rho}\,\text{cut}$$

$$\psi' = \dfrac{\left(\dfrac{\begin{array}{c}\pi\\ \vdash A\sigma, B_1\sigma\end{array} \quad \begin{array}{c}\pi'\\ \vdash A\sigma, B_2\sigma\end{array}}{\vdash A\sigma, B_1\sigma \times B_2\sigma}\right)\overrightarrow{\sigma_B\tau_{\overline{B}}} \qquad \left(\dfrac{\begin{array}{c}\phi\\ \vdash \overline{B}_1\tau, C\tau\end{array}}{\vdash \overline{B}_1\tau + \overline{B}_2\tau, C\tau}\right)\overrightarrow{\sigma_B\tau_{\overline{B}}}}{\vdash A\rho', C\rho'}\,\text{cut}$$

Figure 6 The critical pair $\vdash A, B_1 \times B_2$; $\vdash \overline{B}_1 + \overline{B}_2, C$



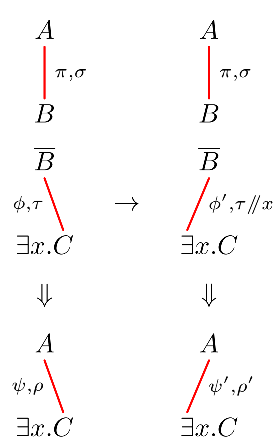
$$\rho \;=\; \sigma_A \tau_C \cdot \overrightarrow{\sigma_B \tau_{\overline{B}}}$$

$$\rho' \;=\; \sigma_A \tau_C \cdot \overrightarrow{(\sigma_B /\!\!/ x)\tau_{\overline{B}}}$$

$$\psi \;=\; \dfrac{\left(\begin{array}{c}\pi\\ \vdash A\sigma, B\sigma\end{array}\right)\overrightarrow{\sigma_B\tau_{\overline{B}}} \qquad \left(\begin{array}{c}\phi\\ \vdash \overline{B}\tau, C\tau\end{array}\right)\overrightarrow{\sigma_B\tau_{\overline{B}}}}{\vdash A\rho, C\rho}\,\text{cut}$$

$$\psi' = \dfrac{\left(\dfrac{\begin{array}{c}\pi\\ \vdash A\sigma, B\sigma\end{array}}{\vdash A\sigma, \exists x.B\sigma /\!\!/ x}\right)\overrightarrow{(\sigma_B /\!\!/ x)\tau_{\overline{B}}} \qquad \left(\dfrac{\begin{array}{c}\phi\\ \vdash \overline{B}\tau, C\tau\end{array}}{\vdash \forall x.\overline{B}\tau, C\tau}\right)\overrightarrow{(\sigma_B /\!\!/ x)\tau_{\overline{B}}}}{\vdash A\rho', C\rho'}\,\text{cut}$$

Figure 7 The critical pair $\vdash A, \exists x.B$; $\vdash \forall x.\overline{B}, C$



$$\rho \;=\; \sigma_A \tau_C \cdot \overrightarrow{\sigma_B \tau_{\overline{B}}}$$

$$\rho' \;=\; \sigma_A (\tau_C /\!\!/ x) \cdot \overrightarrow{\sigma_B \tau_{\overline{B}}}$$

$$\psi \;=\; \dfrac{\left(\begin{array}{c}\pi\\ \vdash A\sigma, B\sigma\end{array}\right)\overrightarrow{\sigma_B\tau_{\overline{B}}} \qquad \left(\begin{array}{c}\phi\\ \vdash \overline{B}\tau, C\tau\end{array}\right)\overrightarrow{\sigma_B\tau_{\overline{B}}}}{\vdash A\rho, C\rho}\,\text{cut}$$

$$\psi' = \dfrac{\left(\begin{array}{c}\pi\\ \vdash A\sigma, B\sigma\end{array}\right)\overrightarrow{\sigma_B\tau_{\overline{B}}} \qquad \left(\dfrac{\begin{array}{c}\phi\\ \vdash \overline{B}\tau, C\tau\end{array}}{\vdash \overline{B}\tau, \exists x.C(\tau /\!\!/ x)}\right)\overrightarrow{\sigma_B\tau_{\overline{B}}}}{\vdash A\rho', C\rho'}\,\text{cut}$$

Figure 8 The critical pair $\vdash A, B$; $\vdash \overline{B}, \exists x.C$

to right) that extends ψ with an existential introduction rule, to a proof equivalent to ψ':

$$\dfrac{\begin{array}{c}\psi\\ \vdash A\rho, C\rho\end{array}}{\vdash A\rho', \exists x.C\rho'}\;\exists\mathrm{R},\rho(x)$$

There are three further critical pairs, for a proof net on $\vdash A, B$ composed with one on $\vdash \overline{B}, C_1{+}C_2$, one on $\vdash \overline{B}, C_1{\times}C_2$, and one on $\vdash \overline{B}, \forall x.C$. These converge as the one above.

Resolving these critical pairs gives the soundness of the composition operation, per the following theorem. We abbreviate a cut on proofs $\pi \vdash A, B$ and $\phi \vdash \overline{B}, C$ by $\pi\,;\phi$.

▸ **Theorem 19.** *If proof nets $\lambda_\Sigma \rhd A, B$ and $\kappa_\Theta \rhd \overline{B}, C$ sequentialize to π and ϕ respectively, then their composition $(\lambda_\Sigma \rhd A, B)\,;(\kappa_\Theta \rhd \overline{B}, C)$ sequentializes to a normal form ψ of $\pi\,;\phi$.*

Proof. See the appendix. ◂

5 Unification nets

In this final section we explore a second notion of ALL1 proof net: ***unification nets*** omit any witness information, which is then reconstructed by coalescence. This yields a natural notion of *most general* proof net, where every other proof net is obtained by introducing more witness information. Conversely, every witness net has an underlying unification net, that sequentializes to a *most general* proof.

We consider a proof $\pi \vdash A, B$ ***more general*** than $\pi' \vdash A, B$, written $\pi \leq \pi'$, if there is a substitution map ρ such that $\pi\rho = \pi'$. Unlike for proof nets, this notion is not so natural for sequent proofs: in the permutation of existential and product rules below, from left to right u must be generated as the least term more general than s and t; from right to left, s and t cannot be reconstructed from u, and must be retrieved from their respective subproofs.

$$\dfrac{\dfrac{\vdash A, C}{\vdash A, \exists x.C}\;\exists\mathrm{R},s \qquad \dfrac{\vdash B, C}{\vdash B, \exists x.C}\;\exists\mathrm{R},t}{\vdash A\times B, \exists x.C}\;\times\mathrm{R} \quad\sim\quad \dfrac{\dfrac{\vdash A, C \quad \vdash B, C}{\vdash A\times B, \exists x.C}\;\times\mathrm{R}}{\vdash A\times B, C}\;\exists\mathrm{R},u$$

To reconstruct witnesses by unification, we define the following operations.

$\sigma \leq \tau$: A witness map σ is ***more general*** than τ if there is a map ρ such that $\sigma\rho = \tau$.
$\sigma \frown \tau$: Two witness maps σ and τ are ***coherent*** if there is a map ρ such that $\sigma\rho = \tau\rho$.
$\sigma \vee \tau$: The ***join*** of coherent witness maps is the least map ρ such that $\sigma \leq \rho$ and $\tau \leq \rho$.

A link (a, b) on two atomic formulas is an ***axiom*** link if there exists a witness map σ such that $a\sigma = \overline{b}\sigma$. To an axiom link (a, b) over $\vdash A, B$ we assign an ***initial witness map***, which is the least witness map σ over the domain $\mathrm{EV}_A(a) \cup \mathrm{EV}_B(b)$ such that $a\sigma = \overline{b}\sigma$. In other words, σ is the most general unifier of a and $\overline{b}$, over the given domain, written $\mathrm{MGU}(a, b)$. For an axiom linking λ over $\vdash A, B$ the ***initial witness pre-net*** $\lambda_\star \rhd A, B$ is given by

$$\lambda_\star = \{\ (a,b)_\sigma \ \mid\ (a,b) \in \lambda\ ,\ \sigma = \mathrm{MGU}(a,b)\ \}\ .$$

Observe that the initial witness pre-net satisfies the *exact coverage* condition.

▸ **Definition 20.** ***Unifying sequentialization*** ($\rightsquigarrow$) is the rewrite relation on labelled pre-nets generated by the rules $(+\mathrm{U}, i)$, $(\exists\mathrm{U})$, $(\forall\mathrm{U})$, which are respectively as $(+\mathrm{S}, i)$, $(\exists\mathrm{S})$, and $(\forall\mathrm{S})$, and the rule

$$\left.\begin{array}{l}(C, D_1)^{\pi}_{\sigma}\\ (C, D_2)^{\phi}_{\tau}\end{array}\right\} \rightsquigarrow (C, D_1{\times}D_2)^{\psi}_{\sigma\vee\tau=\sigma\rho=\tau\rho} \quad {\scriptstyle(\sigma\frown\tau)} \qquad \psi = \dfrac{\left(\begin{array}{c}\pi\\ \vdash A\sigma, B\sigma\end{array}\right)\rho \quad \left(\begin{array}{c}\phi\\ \vdash A\tau, C\tau\end{array}\right)\rho}{\vdash A(\sigma{\vee}\tau), B\sigma\rho\times C\tau\rho}\;(\times\mathrm{U})$$

Unifying coalescence is the relation ($\rightsquigarrow$) on witness pre-nets, ignoring proof labels. A witness pre-net $\lambda_\Sigma \triangleright A, B$ ***unifying-coalesces*** if it reduces to $\{(A,B)_\varnothing\} \triangleright A, B$ and ***strongly*** unifying-coalesces if any coalescence path terminates at $\{(A,B)_\varnothing\} \triangleright A, B$.

▸ **Definition 21.** An ALL1 ***unification proof net*** or ***unification net*** is a pre-net $\lambda \triangleright A, B$ with axiom linking λ such that the initial witness pre-net $\lambda_\star \triangleright A, B$ unifying-coalesces. It ***sequentializes*** to π if $\lambda_\star^\star \triangleright A, B$ reduces in ($\rightsquigarrow$) to $\{(A,B)^\pi_\varnothing\} \triangleright A, B$.

In the above definition, note that $\lambda^\star_\star = (\lambda_\star)^*$ is the initial proof labelling of $\lambda_\star$, which assigns an axiom rule to each axiom link. For a minimal example, see Figure 2 in the introduction. Observe also that unifying coalescence includes strict coalescence, $(\to) \subseteq (\rightsquigarrow)$. The following two lemmata relate sequentialization for witness nets and unification nets.

▸ **Lemma 22.** *In ($\rightsquigarrow$), if $\lambda_\Sigma \triangleright A, B$ sequentializes to π then $\lambda_\star \triangleright A, B$ sequentializes to $\pi' \leq \pi$.*

Proof. The sequentialization path $\lambda^\star_\Sigma \triangleright A, B = L_1 \rightsquigarrow L_2 \rightsquigarrow \ldots \rightsquigarrow L_n = (A,B)^\pi_\varnothing \triangleright A, B$ has a corresponding path $\lambda^\star_\star \triangleright A, B = R_1 \rightsquigarrow R_2 \rightsquigarrow \ldots \rightsquigarrow R_n = (A,B)^{\pi'}_\varnothing \triangleright A, B$ where the same links (but with potentially different witness maps) are coalesced. It follows by induction on this path (where the base case is L_1 and R_1) that for every corresponding pair of links $(C,D)^\phi_\sigma$ in L_i and $(C,D)^\psi_\tau$ in R_i we have $\tau \leq \sigma$ and $\psi \leq \phi$. ◂

▸ **Lemma 23.** *If $\lambda_\star \triangleright A, B$ unifying-sequentializes to π then there exists a witness assignment Σ and substitution ρ such that $\lambda_\Sigma \triangleright A, B$ strict-sequentializes to π and $\lambda_\Sigma = \lambda_\star \rho$.*

Proof. By induction on the sequentialization path $\lambda_\star \triangleright A, B \rightsquigarrow^* (A,B)^\pi_\varnothing \triangleright A, B$. For the end result, the statement holds with $\rho = \varnothing$. For the inductive step, consider a step $L \rightsquigarrow R$. We show the case ($\times$U); the other cases are immediate.

- $(C, D_1)_\sigma, (C, D_2)_\tau \rightsquigarrow (C, D_1 \times D_2)_{\sigma \vee \tau}$
  By the inductive hypothesis, $R\rho'$ strict-sequentializes to π. Let $\sigma \vee \tau = \sigma\rho'' = \tau\rho''$ and let $\rho = \rho''\rho'$. Then $L\rho$ strict-sequentializes to π by

$$(C, D_1)_{\sigma\rho}, (C, D_2)_{\tau\rho} \to (C, D_1 \times D_2)_{(\sigma \vee \tau)\rho'} \ .$$ ◂

We can then show that sequentialization and de-sequentialization for unification nets are inverses up to generality, and that composition is sound.

▸ **Theorem 24.** *If $[\pi \vdash A, B]$ is $\lambda_\Sigma \triangleright A, B$ then $\lambda \triangleright A, B$ unifying-sequentializes to $\pi' \leq \pi$.*

Proof. By Theorem 7, $\lambda_\Sigma \triangleright A, B$ sequentializes to π in ($\to$), and hence also in ($\rightsquigarrow$). Then by Lemma 22 $\lambda_\star \triangleright A, B$ sequentializes to $\pi' \leq \pi$. ◂

▸ **Theorem 25.** *If $\lambda \triangleright A, B$ sequentializes to π, then $[\pi] = \lambda_\Sigma \triangleright A, B$ for some Σ.*

Proof. By Lemma 23, since $\lambda \triangleright A, B$ sequentializes to π there is a net $\lambda_\Sigma \triangleright A, B$ that sequentializes to π. By Theorem 8, $[\pi] = \lambda_\Sigma \triangleright A, B$. ◂

▸ **Theorem 26.** *If $\lambda \triangleright A, B$ sequentializes to π and $\kappa \triangleright \overline{B}, C$ to ϕ then their composition $\lambda \,;\, \kappa \triangleright A, C$ sequentializes to a proof $\psi' \leq \psi$ where ψ is a normal form of $\pi \,;\, \phi$.*

Proof. By Lemma 23 there are witness labellings Σ and Θ such that $\lambda_\Sigma \triangleright A, B$ strict-sequentializes to π and $\kappa_\Theta \triangleright \overline{B}, C$ to ϕ. By Theorem 19 their composition $(\lambda_\Sigma \,;\, \kappa_\Theta) \triangleright A, C$ strict-sequentializes to a normal form ψ of $\pi \,;\, \phi$. By Lemma 22 the net $(\lambda \,;\, \kappa)_\star \triangleright A, C$ unifying-sequentializes to $\psi' \leq \psi$. ◂

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A Postponed proofs

▸ **Lemma 12** (Restatement). *Strict coalescence preserves and reflects correctness.*

Proof. For a strict coalescence step $L \to R$, we will show that the witness pre-net L is correct if and only if R is. Let $L = \lambda_\Sigma \triangleright A, B$ and $R = \kappa_\Theta \triangleright A, B$. In each case, exact coverage is immediately preserved and reflected. For slice-correctness, we will demonstrate that the left-hand side and right-hand side of each rule belong to the same slice of $\vdash A, B$, or in the case of $\exists$S, naturally corresponding slices. For dependency-correctness, we will briefly show how acyclicity of the columns of the involved links is preserved.

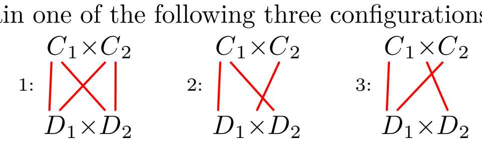
- $(C, D_i)_\sigma \to (C, D_1{+}D_2)_\sigma$
  A slice S_B of B and $\varnothing$ containing one of (D_1, τ), (D_2, τ), and $(D_1 + D_2, \tau)$ must also contain the other two. A slice S of $\vdash A, B$ then contains all three of $(C, D_1)_\sigma$, $(C, D_2)_\sigma$, and $(C, D_1{+}D_2)_\sigma$, or none. It follows that $S \cap \lambda_\Sigma$ is a singleton if and only if $S \cap \kappa_\Theta$ is. Since other slices are unaffected, L is slice-correct if and only if R is.
  For dependency-correctness, the column of $(C, D_i)_\sigma$ is that of $(C, D_1{+}D_2)_\sigma$ plus the pair $(D_i, \sigma|_{\mathrm{EV}_B(D_i)})$ itself, which is minimal in the order $\preccurlyeq$.
- $(C, D_1)_\sigma, (C, D_2)_\sigma \to (C, D_1{\times}D_2)_\sigma$
  A slice S of $\vdash A, B$ contains $(C, D_1{\times}D_2)_\sigma$ if and only if it contains either of $(C, D_1)_\sigma$ or $(C, D_2)_\sigma$, and cannot contain both. Then $S \cap \lambda_\Sigma$ is a singleton if and only if $S \cap \kappa_\Theta$ is.
  Dependency-correctness is immediate, as above.
- $(C, D)_\sigma \to (C, \exists x.D)_{\sigma /\!\!/ x}$
  The witness sets $\Sigma(x)$ and $\Theta(x)$ for L and R need not be the same, since L has σ where R has $\sigma /\!\!/ x$. Let $\Sigma(x) = \{t, t_1, \ldots, t_n\}$ and $\Theta(x) = \{t_1, \ldots, t_n\}$, where $\sigma(x) = t$. For every slice S_B of B over Σ there is a corresponding slice S'_B over Θ, both (or neither) containing $(\exists x.D, \tau)$ and $(D, \tau[t_i/x])$ for $i \leq n$, but the former in addition having $(D, \tau[t/x])$. Then $S \cap \lambda_\Sigma$ is the singleton $\{(C, D)_\sigma\}$ if and only if $S \cap \kappa_\Theta$ is $\{(C, \exists x.D)_{\sigma /\!\!/ x}\}$, where it should be observed that if t is also a witness to x in some other slice of R, then $t \in \Theta(x)$ and in fact $S = S'$.
  For dependency-correctness, the column of $(C, D)_\sigma$ is that of $(C, \exists x.D)_{\sigma /\!\!/ x}$ plus a pair (D, τ), which is minimal in $(\preccurlyeq)$.
- $(C, D)_\sigma \to (C, \forall x.D)_\sigma$
  A slice S of $\vdash A, B$ contains $(C, D)_\sigma$ if and only if it contains also $(C, \forall x.D)_\sigma$, and hence $S \cap \lambda_\Sigma$ is a singleton if and only if $S \cap \kappa_\Theta$ is.
  For dependency-correctness, the column of $(C, D)_\sigma$ is that of $(C, \forall x.D)_\sigma$ plus a pair (D, τ). The side-condition of the coalescence step is that $x \notin \sigma$; then x does not occur free in any (X, ρ), and (D, τ) is minimal in $(\preccurlyeq)$. ◂

▸ **Lemma 13** (Restatement). *To a correct witness pre-net $\lambda_\Sigma \triangleright A, B$ a coalescence step applies, unless it is fully coalesced already, $\lambda_\Sigma = \{(A, B)_\varnothing\}$.*

Proof. Let the ***depth*** of a link $(C, D)_\sigma$ be a pair of integers (n, m), where n is the distance from C to the root of A, and m that from D to B. We order link depth in the product order: $(i, j) \leq (n, m)$ if and only if $i \leq n$ and $j \leq m$. We will demonstrate that a link at maximal depth may always be coalesced, unless it is the unique link $(A, B)_\varnothing$ at $(0, 0)$.

To see that a maximally deep link coalesces, first note that a link $(C, D_i)_\sigma$ where D_i occurs in $D_0{+}D_1$ may always coalesce, as may a link $(C, D)_\sigma$ where D occurs in $\exists x.D$. This leaves the following cases:

- $(A, D_i)_\sigma$ with D_i occurring in $D = D_1 \times D_2$.
  Without loss of generality, let $i = 1$. A slice S_1 of $\vdash A, B$ containing $(A, D_1)_\sigma$ has a counterpart S_2 containing $(A, D_2)_\sigma$. The depth of $(A, D_2)_\sigma$ is the same as that of $(A, D_1)_\sigma$. By correctness $S_2 \cap \lambda_\Sigma$ is a singleton; by the assumption of maximality it may not contain a deeper link than $(A, D_2)_\sigma$; and it may not contain a shallower one since that would be shared with $S_1 \cap \lambda_\Sigma$. Then $\lambda_\Sigma \triangleright A, B$ contains both $(A, D_1)_\sigma$ and $(A, D_2)_\sigma$, and these contract to $(A, D)_\sigma$.
- $(A, D)_\sigma$ with D in $\forall x.D$.
  The step $(A, D)_\sigma \to (A, \forall x.D)_\sigma$ applies if $x \notin \sigma$. By way of contradiction, assume $x \in \sigma$. The column of $(A, D)_\sigma$ contains (D, σ_D) and $(\forall x.D, \tau)$ where $\tau = \sigma|_{\mathrm{EV}_B(\forall x.D)}$. By the exact coverage condition, $\sigma = \sigma_A \cup \sigma_D$, and since the existential variables in D and $\forall x.D$ are the same, $\mathrm{EV}_B(D) = \mathrm{EV}_B(\forall x.D)$, so that $\tau = \sigma_D$. (Note that since $\sigma_A = \varnothing$, we get $\sigma = \sigma_D = \tau$, but this is not essential to the argument.) Since $x \in \sigma$ we have $x \in \tau$, and in the column of $(A, D)_\sigma$ we have $(\forall x.D, \tau) \preccurlyeq (D, \tau)$ since D occurs as $\forall x.D$. But we already have $(D, \tau) \preccurlyeq (\forall x.D, \tau)$ because $D \leq \forall x.D$, contradicting antisymmetry of $(\preccurlyeq)$. Then $x \notin \sigma$, and the step $(A, D)_\sigma \to (A, \forall x.D)_\sigma$ applies.
- $(C_i, D_j)_\sigma$ in $C = C_1 \times C_2$ and $D = D_1 \times D_2$.
  Without loss of generality, let $i = j = 1$. By minimal depth and using similar reasoning to the first case above, the pre-net must contain one of the following three configurations.
  1. $(C_1, D_1)_\sigma, (C_1, D_2)_\sigma, (C_2, D_1)_\sigma, (C_2, D_2)_\sigma$
  2. $(C_1, D_1)_\sigma, (C_1, D_2)_\sigma, (C_2, D)_\sigma$
  3. $(C_1, D_1)_\sigma, (C_2, D_1)_\sigma, (C, D_2)_\sigma$

  In the second case, the step $(C_1, D_1)_\sigma, (C_1, D_2)_\sigma \to (C_1, D)_\sigma$ applies; in the third case, $(C_1, D_1)_\sigma, (C_2, D_1)_\sigma \to (C, D_1)_\sigma$; and in the first case, both.
- $(C_i, D)_\sigma$ in $C = C_1 \times C_2$ and $\forall x.D$.
  Without loss of generality let $i = 1$. If $x \notin \sigma$ the rewrite step $(C_1, D)_\sigma \to (C_1, \forall x.D)_\sigma$ applies. Otherwise, let $x \in \sigma$. The slice S_1 of $\vdash A, B$ containing $(C_1, D)_\sigma$ has a counterpart S_2 containing $(C_2, D)_\sigma$, which must include exactly one link of λ_Σ. By the assumption of minimal depth, it cannot have greater depth than $(C_2, D)_\sigma$. It cannot be $(C, D)_\sigma$ or any shallower link, since that would be shared with the slice S_1 which already contains $(C_1, D)_\sigma$. It cannot be $(C_2, \forall x.D)_\sigma$ or any shallower link $(C_2, X)_\tau$ (i.e. with $\forall x.D \leq X$) because $x \in \sigma$. This would mean either $x \in \tau$ which contradicts the *eigenvariables not free* convention, or $x \in \mathrm{FV}(\sigma(y))$ where $\forall x.D < \exists y.Y \leq X$ which creates a cyclic column, as in the second case above. It follows that $S_2 \cap \lambda_\Sigma = \{(C_2, D)_\sigma\}$, so that the rewrite step $(C_1, D)_\sigma, (C_2, D)_\sigma \to (C, D)_\sigma$ applies.
- $(C, D)_\sigma$ in $\forall x.C$ and $\forall y.D$.
  A rewrite step $(C, D)_\sigma \to (\forall x.C, D)_\sigma$ or $(C, D)_\sigma \to (C, \forall y.D)_\sigma$ applies unless $x, y \in \sigma$. But that would generate a cycle in the column of $(C, D)_\sigma$, in one of three ways. If $x \in \sigma_C$ or $y \in \sigma_D$ then, since $\sigma_C = \sigma_{\forall x.C}$ and $\sigma_D = \sigma_{\forall y.D}$, respectively:

  $$(C, \sigma_C) \preccurlyeq (\forall x.C, \sigma_C) \preccurlyeq (C, \sigma_C) \qquad (D, \sigma_D) \preccurlyeq (\forall y.D, \sigma_D) \preccurlyeq (D, \sigma_D)\ .$$

  Otherwise, if $x \in \sigma_D$ and $y \in \sigma_C$ then

  $$(C, \sigma_C) \preccurlyeq (\forall x.C, \sigma_C) \preccurlyeq (D, \sigma_D) \preccurlyeq (\forall x.D, \sigma_D) \preccurlyeq (C, \sigma_C)\ .$$ ◂

▸ **Theorem 19** (Restatement). *If proof nets $\lambda_\Sigma \triangleright A, B$ and $\kappa_\Theta \triangleright \overline{B}, C$ sequentialize to π and ϕ respectively, then their composition $(\lambda_\Sigma \triangleright A, B)\,;(\kappa_\Theta \triangleright \overline{B}, C)$ sequentializes to a normal form ψ of $\pi\,;\phi$.*

Proof. By Corollary 16 the proof nets $L = \lambda_\Sigma \triangleright A, B$ and $R = \kappa_\Theta \triangleright \overline{B}, C$ strongly coalesce. We may then interleave their coalescence sequences as follows: if a synchronized step in L and R on the interface B and $\overline{B}$ is available, apply it; otherwise perform steps in L on A and in R on C until it is. This gives the following combined sequence.

$$\begin{array}{ccccccccc}
L & = & L_1 & \to^? & L_2 & \to^? & \dots & \to^? & L_n \\
R & = & R_1 & \to^? & R_2 & \to^? & \dots & \to^? & R_n \\
\Downarrow & & \Downarrow & & \Downarrow & & & & \Downarrow \\
L\,;R & = & L_1\,;R_1 & \to & L_2\,;R_2 & \to & \dots & \to & L_n\,;R_n
\end{array}$$

(Here, $(\to^?)$ is the relation $(\to)\cup(=)$, but we assume that at least $L_i \to L_{i+1}$ or $R_i \to R_{i+1}$.) The path along the top and right of this diagram sequentializes L to π' and R to ϕ' (equivalent to π and ϕ respectively), and then composes to $L_n\,; R_n = \{(A,C)^{\pi';\phi'}_{\varnothing}\} \triangleright A, C$.

Each square of the diagram converges as one of the critical pairs of sequentialization and composition discussed above. Then each path along the diagram from top left (L and R) to bottom right ($L_n\,; R_n$) gives a sequentialization, with cuts, of $L_n\,; R_n$. Let the path taking the vertical step from L_i and R_i to $L_i\,; R_i$ sequentialize to ψ_i, so that $\psi_n = \psi'$. By the way each square converges, we have that ψ_i is reached from ψ_{i+1} by a cut-elimination or permutation step.

Finally, in L and R every link is an axiom link. Any link in $L\,; R$ is composed from two links $(a,b)_\sigma$ in L and $(\overline{b},c)_\tau$ in R, which yields $(a,c)_\rho$ where $\rho = \sigma_a\tau_c \cdot \overrightarrow{\sigma_b\tau_{\overline{b}}}$. This sequentializes to the axiom $\overline{\vdash a\rho, c\rho}$, which is in normal form. Then $L\,; R$ is a proof net (it has an axiom linking and it coalesces), and it sequentializes to a normal form of ψ. ◂

B Permutations

$$\dfrac{\dfrac{\vdash A,B}{\vdash \forall x.A,B}\,{\scriptstyle\forall\mathrm{R}} \qquad \vdash \overline{B},C}{\vdash \forall x.A,C}\,{\scriptstyle\mathrm{cut}} \quad\sim\quad \dfrac{\dfrac{\vdash A,B \qquad \vdash \overline{B},C}{\vdash A,C}\,{\scriptstyle\mathrm{cut}}}{\vdash \forall x.A,C}\,{\scriptstyle\forall\mathrm{R}}$$

$$\dfrac{\dfrac{\vdash A[t/x],B}{\vdash \exists x.A,B}\,{\scriptstyle\exists\mathrm{R},t} \qquad \vdash \overline{B},C}{\vdash \exists x.A,C}\,{\scriptstyle\mathrm{cut}} \quad\sim\quad \dfrac{\dfrac{\vdash A[t/x],B \qquad \vdash \overline{B},C}{\vdash A[t/x],C}\,{\scriptstyle\mathrm{cut}}}{\vdash \exists x.A,C}\,{\scriptstyle\exists\mathrm{R},t}$$

$$\dfrac{\dfrac{\vdash A,B \qquad \vdash \overline{B},C}{\vdash A,C}\,{\scriptstyle\mathrm{cut}} \qquad \vdash \overline{C},D}{\vdash A,D}\,{\scriptstyle\mathrm{cut}} \quad\sim\quad \dfrac{\vdash A,B \qquad \dfrac{\vdash \overline{B},C \qquad \vdash \overline{C},D}{\vdash \overline{B},D}\,{\scriptstyle\mathrm{cut}}}{\vdash A,D}\,{\scriptstyle\mathrm{cut}}$$

$$\dfrac{\dfrac{\vdash A_i,B}{\vdash A_1{+}A_2,B}\,{\scriptstyle+\mathrm{R},i} \qquad \vdash \overline{B},C}{\vdash A_1{+}A_2,C}\,{\scriptstyle\mathrm{cut}} \quad\sim\quad \dfrac{\dfrac{\vdash A_i,B \qquad \vdash \overline{B},C}{\vdash A_i,C}\,{\scriptstyle\mathrm{cut}}}{\vdash A_1{+}A_2,C}\,{\scriptstyle+\mathrm{R},i}$$

$$\dfrac{\dfrac{\vdash A_1,B \qquad \vdash A_2,B}{\vdash A_1{\times}A_2,B}\,{\scriptstyle\times\mathrm{R}} \qquad \vdash \overline{B},C}{\vdash A_1{\times}A_2,C}\,{\scriptstyle\mathrm{cut}} \quad\sim\quad \dfrac{\dfrac{\vdash A_1,B \qquad \vdash \overline{B},C}{\vdash A_1,C}\,{\scriptstyle\mathrm{cut}} \qquad \dfrac{\vdash A_2,B \qquad \vdash \overline{B},C}{\vdash A_2,C}\,{\scriptstyle\mathrm{cut}}}{\vdash A_1{\times}A_2,C}\,{\scriptstyle\times\mathrm{R}}$$

Figure 9 Cut-permutations

$$\dfrac{\dfrac{\vdash A, B}{\vdash A, \forall y.B}{\scriptstyle \forall\mathrm{R}}}{\vdash \forall x.A, \forall y.B}{\scriptstyle \forall\mathrm{R}} \quad\sim\quad \dfrac{\dfrac{\vdash A, B}{\vdash \forall x.A, B}{\scriptstyle \forall\mathrm{R}}}{\vdash \forall x.A, \forall y.B}{\scriptstyle \forall\mathrm{R}}$$

$$\dfrac{\dfrac{\vdash A, B[t/y]}{\vdash A, \exists y.B}{\scriptstyle \exists\mathrm{R},t}}{\vdash \forall x.A, \exists y.B}{\scriptstyle \forall\mathrm{R}} \quad\sim\quad \dfrac{\dfrac{\vdash A, B[t/y]}{\vdash \forall x.A, B[t/y]}{\scriptstyle \forall\mathrm{R}}}{\vdash \forall x.A, \exists y.B}{\scriptstyle \exists\mathrm{R},t}$$

$$\dfrac{\dfrac{\vdash A, B_i}{\vdash A, B_1+B_2}{\scriptstyle +\mathrm{R},i}}{\vdash \forall x.A, B_1+B_2}{\scriptstyle \forall\mathrm{R}} \quad\sim\quad \dfrac{\dfrac{\vdash A, B_i}{\vdash \forall x.A, B_i}{\scriptstyle \forall\mathrm{R}}}{\vdash \forall x.A, B_1+B_2}{\scriptstyle +\mathrm{R},i}$$

$$\dfrac{\dfrac{\vdash A, B \quad \vdash A, C}{\vdash A, B\times C}{\scriptstyle \times\mathrm{R}}}{\vdash \forall x.A, B\times C}{\scriptstyle \forall\mathrm{R}} \quad\sim\quad \dfrac{\dfrac{\vdash A, B}{\vdash \forall x.A, B}{\scriptstyle \forall\mathrm{R}} \quad \dfrac{\vdash A, C}{\vdash \forall x.A, C}{\scriptstyle \forall\mathrm{R}}}{\vdash \forall x.A, B\times C}{\scriptstyle \times\mathrm{R}}$$

$$\dfrac{\dfrac{\vdash A[s/x], B[t/y]}{\vdash A[s/x], \exists y.B}{\scriptstyle \exists\mathrm{R},t}}{\vdash \exists x.A, \exists y.B}{\scriptstyle \exists\mathrm{R},s} \quad\sim\quad \dfrac{\dfrac{\vdash A[s/x], B[t/y]}{\vdash \exists x.A, B[t/y]}{\scriptstyle \exists\mathrm{R},s}}{\vdash \exists x.A, \exists y.B}{\scriptstyle \exists\mathrm{R},t}$$

$$\dfrac{\dfrac{\vdash A[t/x], B_i}{\vdash A[t/x], B_1+B_2}{\scriptstyle +\mathrm{R},i}}{\vdash \exists x.A, B_1+B_2}{\scriptstyle \exists\mathrm{R},t} \quad\sim\quad \dfrac{\dfrac{\vdash A[t/x], B_i}{\vdash \exists x.A, B_i}{\scriptstyle \exists\mathrm{R},t}}{\vdash \exists x.A, B_1+B_2}{\scriptstyle +\mathrm{R},i}$$

$$\dfrac{\dfrac{\vdash A[t/x], B \quad \vdash A[t/x], C}{\vdash A[t/x], B\times C}{\scriptstyle \times\mathrm{R}}}{\vdash \exists x.A, B\times C}{\scriptstyle \exists\mathrm{R},t} \quad\sim\quad \dfrac{\dfrac{\vdash A[t/x], B}{\vdash \exists x.A, B}{\scriptstyle \exists\mathrm{R},t} \quad \dfrac{\vdash A[t/x], C}{\vdash \exists x.A, C}{\scriptstyle \exists\mathrm{R},t}}{\vdash \exists x.A, B\times C}{\scriptstyle \times\mathrm{R}}$$

$$\dfrac{\dfrac{\vdash A_i, B_j}{\vdash A_i, B_1+B_2}{\scriptstyle +\mathrm{R},j}}{\vdash A_1+A_2, B_1+B_2}{\scriptstyle +\mathrm{R},i} \quad\sim\quad \dfrac{\dfrac{\vdash A_i, B_j}{\vdash A_1+A_2, B_j}{\scriptstyle +\mathrm{R},i}}{\vdash A_1+A_2, B_1+B_2}{\scriptstyle +\mathrm{R},j}$$

$$\dfrac{\dfrac{\vdash A_i, B \quad \vdash A_i, C}{\vdash A_i, B\times C}{\scriptstyle \times\mathrm{R}}}{\vdash A_1+A_2, B\times C}{\scriptstyle +\mathrm{R},i} \quad\sim\quad \dfrac{\dfrac{\vdash A_i, B}{\vdash A_1+A_2, B}{\scriptstyle +\mathrm{R},i} \quad \dfrac{\vdash A_i, C}{\vdash A_1+A_2, C}{\scriptstyle +\mathrm{R},i}}{\vdash A_1+A_2, B\times C}{\scriptstyle \times\mathrm{R}}$$

$$\dfrac{\dfrac{\vdash A, C \quad \vdash A, D}{\vdash A, C\times D}{\scriptstyle \times\mathrm{R}} \quad \dfrac{\vdash B, C \quad \vdash B, D}{\vdash B, C\times D}{\scriptstyle \times\mathrm{R}}}{\vdash A\times B, C\times D}{\scriptstyle \times\mathrm{R}} \quad\sim\quad \dfrac{\dfrac{\vdash A, C \quad \vdash B, C}{\vdash A\times B, C}{\scriptstyle \times\mathrm{R}} \quad \dfrac{\vdash A, D \quad \vdash B, D}{\vdash A\times B, D}{\scriptstyle \times\mathrm{R}}}{\vdash A\times B, C\times D}{\scriptstyle \times\mathrm{R}}$$

Figure 10 Cut-free rule permutations