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CSCI 3104, Algorithms **Due January 29, 2021**

Problem Set 2 (50 points) **Spring 2021, CU-Boulder**

Advice 1: For every problem in this class, you must justify your answer: show how you arrived at it and why it is correct. If there are assumptions you need to make along the way, state those clearly.

Advice 2: Verbal reasoning is typically insufficient for full credit. Instead, write a logical argument, in the style of a mathematical proof.

Instructions for submitting your solution:

- The solutions **should be typed** and we cannot accept hand-written solutions. Here's a short intro to Latex.
- You should submit your work through **Gradescope** only.
- The easiest way to access Gradescope is through our Canvas page. There is a Gradescope button in the left menu.
- Gradescope will only accept **.pdf** files.
- It is vital that you match each problem part with your work. Skip to 1:40 to just see the matching info.

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1. The following problems are a review of logarithm and exponent topics.

   (a) Solve for x.

      i. $3^{2x} = 81$
      ii. $3(5^{x-1}) = 375$
      iii. $\log_3 x^2 = 4$

   (b) Solve for x.

      i. $x^2 - x = \log_5 25$
      ii. $\log_{10}(x+3) - \log_{10} x = 1$

   (c) Answer each of the following with a TRUE or FALSE.

      i. $a^{\log_a x} = x$
      ii. $a^{\log_b x} = x$
      iii. $a = b^{\log_b a}$
      iv. $\log_a x = \frac{\log_b x}{\log_b a}$
      v. $\log b^m = m \log b$

Solution:

(a) i.

$$x = 2$$

$$9^x = 81$$

   ii. $x = 4$
   iii. $x = 9$

(b) i. $x = 2$
   ii.

$$x = \frac{1}{3}$$

$$x - \frac{3}{x} = 10$$

(c) i. True
   ii. False
   iii. True
   iv. True
   v. True

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2. Compute the following limits at infinity. Show all work and justify your answer.

   (a) $\lim_{x\to\infty} \frac{3x^3+2}{9x^3-2x^2+7}$

   (b) $\lim_{x\to\infty} \frac{x^3}{e^{x/2}}$

   (c) $\lim_{x\to\infty} \frac{\ln x^4}{x^3}$

   Solution:

   (a)

$$\lim_{x\to\infty} \frac{3x^3+2}{9x^3-2x^2+7} \overset{LH}{=} \lim_{x\to\infty} \frac{9x^2}{27x^2-4x}$$

$$\lim_{x\to\infty} \frac{9x^2}{27x^2-4x} \overset{LH}{=} \lim_{x\to\infty} \frac{18x}{54x-4}$$

$$\lim_{x\to\infty} \frac{18x}{54x-4} \overset{LH}{=} \lim_{x\to\infty} \frac{18}{54} = \frac{18}{54}$$

   (b)

$$\lim_{x\to\infty} \frac{x^3}{e^{x/2}} \overset{LH}{=} \lim_{x\to\infty} \frac{6x^2}{e^{x/2}}$$

$$\lim_{x\to\infty} \frac{6x^2}{e^{x/2}} \overset{LH}{=} \lim_{x\to\infty} \frac{24x}{e^{x/2}}$$

$$\lim_{x\to\infty} \frac{24x}{e^{x/2}} \overset{LH}{=} \lim_{x\to\infty} \frac{48}{e^{x/2}} = 0$$

   (c)

$$\lim_{x\to\infty} \frac{\ln x^4}{x^3} = \lim_{x\to\infty} \frac{4\ln x}{x^3}$$

   (d)

$$\lim_{x\to\infty} \frac{4\ln x}{x^3} \overset{LH}{=} \lim_{x\to\infty} \frac{1}{3x^3} = 0$$

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3. Compute the following limits at infinity. Show all work and justify your answer.

   (a) For real numbers $m, n > 0$ compute $\lim_{x\to\infty} \frac{x^m}{e^{nx}}$

   (b) What does this tell us about the rate at which e^{nx} approaches infinity relative to x^m? A brief explanation is fine for this part.

   (c) For real numbers $m, n > 0$ compute $\lim_{x\to\infty} \frac{(\ln x)^n}{x^m}$

   (d) What does this tell us about the rate at which $(\ln x)^n$ approaches infinity relative to x^m? A brief explanation is fine for this part.

Solution:

(a)

$$f(x) = x^m$$
$$f^{(m)}(x) = m!$$
$$g(x) = e^{nx}$$
$$g^{(m)}(x) = n^m e^{nx}$$

After m applications of L'Hopital's rule we arrive at

$$\lim_{x\to\infty} \frac{f^{(m)}(x)}{g^{(m)}(x)} = \lim_{x\to\infty} \frac{m!}{n^m e^{nx}} = 0$$

(b) In all cases e^{nx} grows more quickly than $f(x)$ is a constant after m derivatives.

(c)

$$f(x) = ln(x)^n$$
$$g(x) = x^m$$

$$\lim_{x\to\infty} \frac{(\ln x)^n}{x^m} \stackrel{LH}{=} \lim_{x\to\infty} \frac{f^{(n)}(x)}{g^{(n)}(x)} = \lim_{x\to\infty} \frac{n!}{m^n x^m} = 0$$

(d) After n applications of L'Hopital's rule we arrive at

$$\lim_{x\to\infty} \frac{f^{(n)}(x)}{g^{(n)}(x)} = \lim_{x\to\infty} \frac{n!}{m^n x^m} = 0$$

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4. The problems in this question deal with the Root and the Ratio Tests. Determine the convergence or divergence of the following series. State which test you used.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

   Solution:

   (a) This series converges absolutely by the root test.
   (b) This series converges absolutely by the ratio test.
   (c) This series converges absolutely by the ratio test.
   (d) This series converges absolutely by the root test.