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CSCI 3104, Algorithms **Due THURSDAY, APRIL 29, 2021**
Problem Set 10 (50 points) **Spring 2021, CU-Boulder**

Advice 1: For every problem in this class, you must justify your answer: show how you arrived at it and why it is correct. If there are assumptions you need to make along the way, state those clearly.
Advice 2: Verbal reasoning is typically insufficient for full credit. Instead, write a logical argument, in the style of a mathematical proof.

Instructions for submitting your solution:

- The solutions **should be typed** and we cannot accept hand-written solutions. Here's a short intro to Latex.
- You should submit your work through **Gradescope** only.
- The easiest way to access Gradescope is through our Canvas page. There is a Gradescope button in the left menu.
- Gradescope will only accept **.pdf** files.
- It is vital that you match each problem part with your work. Skip to 1:40 to just see the matching info.

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1. Let P_1, P_2 be two problems such that $P_1 \leq_p P_2$. That is, we have a polynomial-time reduction $r : P_1 \to P_2$. If we assume $P_2 \in P$, explain why this implies that $P_1 \in P$.

2. Recall the k-Colorability problem.

   - Input: Let G be a simple, undirected graph.
   - Decision: Can we color the vertices of G using exactly k colors, such that whenever u and v are adjacent vertices, u and v receive different colors?

   It is well known that k-Colorability belongs to NP, and 3-Colorability is NP-complete. Our goal is to show that the 4-Colorability problem is also NP-complete. To do so, we reduce the 3-Colorability problem to the 4-Colorability problem. We provide the reduction below. The questions following the reduction will ask you to verify its correctness.

   Reduction: Let G be a simple, undirected graph. We construct a new simple, undirected graph H by starting with a copy of G. We then add a new vertex t to H, and for each vertex $v \in V(G)$ we add the edge tv to $E(H)$.

   Your job is to verify the correctness of the reduction by completing parts (a)–(c) below, which completes the proof that 4-Colorability is NP-complete. For the rest of this problem, let G be a graph, and let H be the result of applying the reduction to G.

   (a) Suppose that G is colorable using exactly 3 colors. Argue that H is colorable using exactly 4 colors.

   (b) Suppose that H is colorable using exactly 4 colors. Argue that G is colorable using exactly 3 colors.

   (c) Let n be the number of vertices in G. Carefully explain why H can be constructed in time polynomial in n. [**Hint:** Count the number of vertices and edges we add to G in order to obtain H.]

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1. By the nature of a polynomial-time reduction, we can guarantee that P_1 is no harder than P_2, as is stated by $P_1 \leq_p P_2$. If $P_2 \in P$, then $P_1 \in P$ as well, because if P_1 belonged to any other class then $P_1 \leq_p P_2$ would not hold true, and would instead be $P_1 \geq_p P_2$

2. Recall the k-Colorability problem.

   (a) If G can be colored with exactly 3 colors, adding a new vertex with edges to every other node will be colorable with 4 colors. If the new vertex takes the 4th color, then since the color is unique it will not have any neighbors of the same color.

   (b) If H is composed of one uniquely colored node attached to all other nodes, and the other nodes are colored using 3 colors (as defines H), then removal of the one unique node will yield a graph G that is colorable using only the remaining three colors.

   (c) Given n is the number of vertices in G, then H will contain $n+1$ vertices and n more edges. Given the new construction only takes $n + 1$ steps in adding the new vertex and the new vertices, which is done in polynomial time.