

# Padé Reformulations and Proof of Monotonicity

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The goal of this short memo is to reformulate the equations for Padé splines with continuous second derivatives to be a function of the endpoints and control point as well as to prove the monotonicity these functions guarantee.

From (1.18), the restrictions leading to (1.29) and (1.30) (notably  $s_1(0) = 0$ ,  $s_2(1) = 1$ ), and enforcing a continuous second derivative at  $(x_c, y_c)$  we can write the Padé spline with fixed endpoints  $(0, 0)$ ,  $(1, 1)$  as a function of the parameters  $(x_c, y_c)$  by substituting in (1.32) and simplifying.

In requiring  $s_1^i(x_c) = s_2^i(x_c)$   $i \in 0, 1, 2$  we create a system of three equations for each function which can be solved for the three unknowns  $a_{0i}, a_{1i}, b_{bi}$  yielding the solution  $s_1(x) \equiv s_2(x)$ , a single Padé approximant. This spline is *consistent* with the solution of the system found by requiring a single Padé approximant to have the 2 endpoints and midpoint (a single Padé approximant has continuous derivatives).

Thus we find the following equations over  $[0, 1]$ :

$$s(x) = \frac{(x_c - 1)y_c x}{x_c(y_c - 1) + x(x_c - y_c)}$$
$$s'(x) = \frac{(x_c - 1)x_c(y_c - 1)y_c}{[(y_c - x_c)x - x_c(y_c - 1)]^2}$$

The signs of the derivatives are entirely dependent on the sign of the numerator. Noting again that here we chose  $(0, 0)$  and  $(1, 1)$  as endpoints, it is clear that we have a positive derivative when  $x \in (0, 1) \implies s(x)$  monotonically increasing.

Now we can perform a similar analysis, but without the assumptions on the endpoints of our Padé spline. We use the following 6 constraints to write our Padé spline as a function of the parameters  $\{x_0, y_0, x_c, y_c, x_f, y_f\}$ , where  $(x_0, y_0)$  and  $(x_f, y_f)$  are our endpoints. We again enforce continuity of the second derivative at our control point.

$$s_1(x_0) = y_0 \quad (1)$$

$$s_2(x_f) = y_f \quad (2)$$

$$s_1(x_c) = y_c \quad (3)$$

$$s_2(x_c) = y_c \quad (4)$$

$$s'_1(x_c) = s'_2(x_c) \quad (5)$$

$$s''_1(x_c) = s''_2(x_c) \quad (6)$$

As before, our solution in a single Padé approximant with the equivalent conditions:

$$s(x_0) = y_0 \quad (7)$$

$$s(x_c) = y_c \quad (8)$$

$$s(x_f) = y_f \quad (9)$$

$$(10)$$

Plugging in these constraints and solving for the spline using our new set of parameters we arrive at the following formulation:

$$s(x) = \frac{x x_0 y_0 y_c - x x_0 y_0 y_f - x x_c y_0 y_c + x_0 x_c y_0 y_f - x_0 x_f y_0 y_c + x x_f y_0 y_f + x x_c y_c y_f - x_0 x_c y_c y_f + x_c x_f y_0 y_c - x x_f y_c y_f + x_0 x_f y_c y_f - x_c x_f y_0 y_f}{x x_0 y_c - x x_c y_0 + x_0 x_c y_0 - x x_0 y_f + x x_f y_0 - x_0 x_f y_0 - x_0 x_c y_c + x x_c y_f - x x_f y_c + x_0 x_f y_f + x_c x_f y_c - x_c x_f y_f}$$

$$s'(x) = \frac{(x_c - x_0)(x_f - x_0)(x_f - x_c)(y_c - y_0)(y_f - y_0)(y_f - y_c)}{((x_c - x_0)y_f - (x_f - x_0)y_c + (x_f - x_c)y_0)x - (x_c - x_0)x_f y_f + x_c \cdot (x_f - x_0)y_c - x_0 \cdot (x_f - x_c)y_0)^2}$$

As an aside, it is worth noting in both this case and the previous one that the splines are highly nonlinear in their coefficients.

From the explicit derivatives above, we see something very similar to the derivatives of the fixed-endpoint splines; The sign of the derivative is dependent only on the numerator. Let us assume the following sequences are monotonic:  $(x_0, x_c, x_f), (y_0, y_c, y_f)$ . Then given immediately is monotonicity of the Padé splines defined by these sequences.

Thus, if we assume the sequence  $(x_0, x_c, x_f)$  is defined to be monotonic increasing, Padé splines defined as in (1.18) are monotonic iff the sequence  $(y_0, y_c, y_f)$  is monotonic. ■

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<sup>1</sup>Fixed endpoint Padé spline visualization <https://www.desmos.com/calculator/xthgIwoqok>