1

1. $$\frac{dT}{dt} = -k(T - T_s)$$

temp of surroundings

assume room temp 20°

$$\frac{d(T-T_s)}{dt} = -k(T-T_s)$$

$$(T-T_s)(t) = (T-T_s)(0)e^{-kt}$$

$t=0$: 5pm $\quad T = 40^\circ \quad \therefore (T-T_s)(0) = 20^\circ$

$t=30$ mins : 5.30pm $\quad T = 30^\circ$

$$\therefore 10 = 20e^{-k30}$$

$$\frac{1}{2} = e^{-30k}$$

$$-\ln 2 = -30k \longrightarrow k = \frac{\ln 2}{30} \text{ mins}^{-1}$$

②

Assume $T = 100^{\circ}$ when made

$$100 - 20 = 20e^{-kt}$$

$$4 = e^{-kt}$$

$$\ln 4 = -kt$$

$$t = -\frac{\ln 4}{k} = -\frac{2\ln 2}{\ln 2} \cdot 30 = -60$$

$\boxed{4pm}$

3

3(ii) $y' - y = 2xe^{2x}$

I.F : e^{-x}

$e^{-x}y' - e^{-x}y = 2xe^{x}$

$\frac{d}{dx}(e^{-x}y) = 2xe^{x}$

$e^{-x}y = \int 2xe^{x}dx$

$= 2xe^{x} - \int 2e^{x}dx$

$= 2xe^{x} - 2e^{x} + C$

$y = 2xe^{2x} - 2e^{2x} + Ce^{x}$.

$y(0) = 1$ $\therefore$ $C = 3$.

5(iii) $\frac{dy}{dx} = \frac{1}{e^{y} - x}$

$\frac{dx}{dy} = e^{y} - x$

$\frac{dx}{dy} + x = e^{y}$

I.f. e^{y}

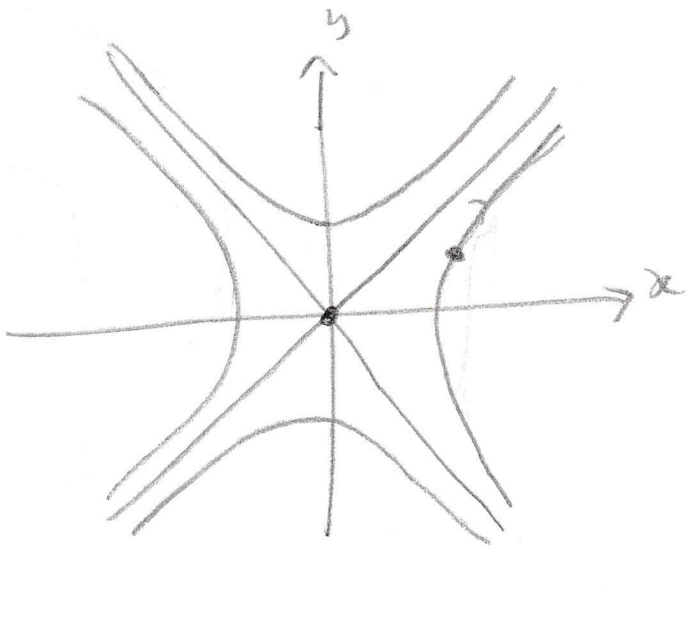
$\frac{d}{dy}(e^{y}x) = e^{2y}$

$e^{y}x = \frac{1}{2}e^{2y} + C$

$x = \frac{1}{2}e^{y} + Ce^{-y}$

7, $y^2 - x^2 = C$

hyperbolae

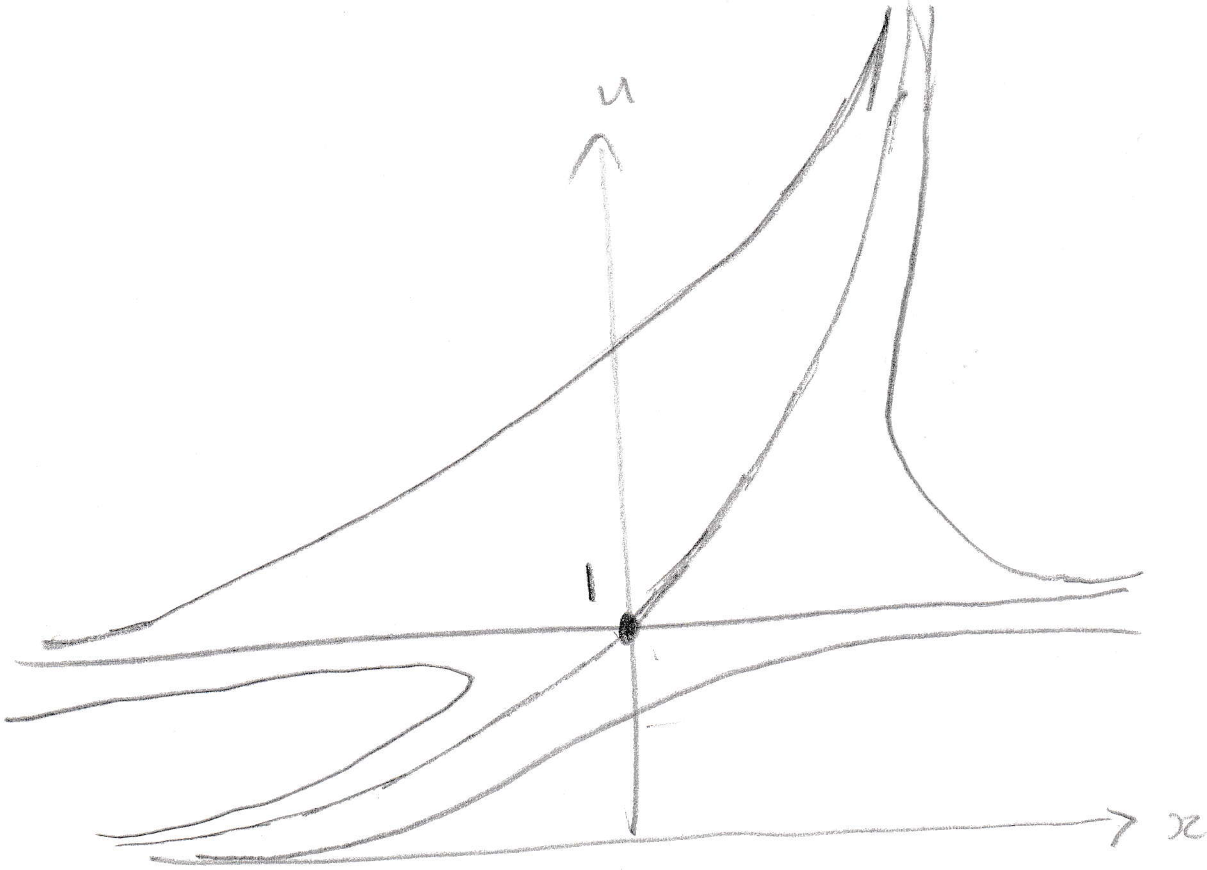


$(x,y) \longrightarrow (x,u)$

$\log u = y + x \qquad u = e^{x+y} > 0$

$y = -x \longrightarrow u = 1.$

$y = +x \longrightarrow u = e^{2x}.$



$$\frac{dy}{dx} = f(x, y)$$

If f well-defined then y' is uniquely.

$$\frac{dy}{dx} = \frac{y}{x}. \longleftarrow \text{bad at } (0,0).$$

$$\frac{dx}{dt} = x^{\frac{1}{3}}.$$

$x(0) = 0.$

$$\frac{dx}{x^{\frac{1}{3}}} = dt$$

$$-\frac{3}{2} x^{\frac{2}{3}} = t + C$$

$x(0) = 0 \rightarrow C = 0.$

$$x = \left(-\frac{2t}{3}\right)^{\frac{3}{2}}$$

but $x(t) \equiv 0$ is also a soln

$f(x)$

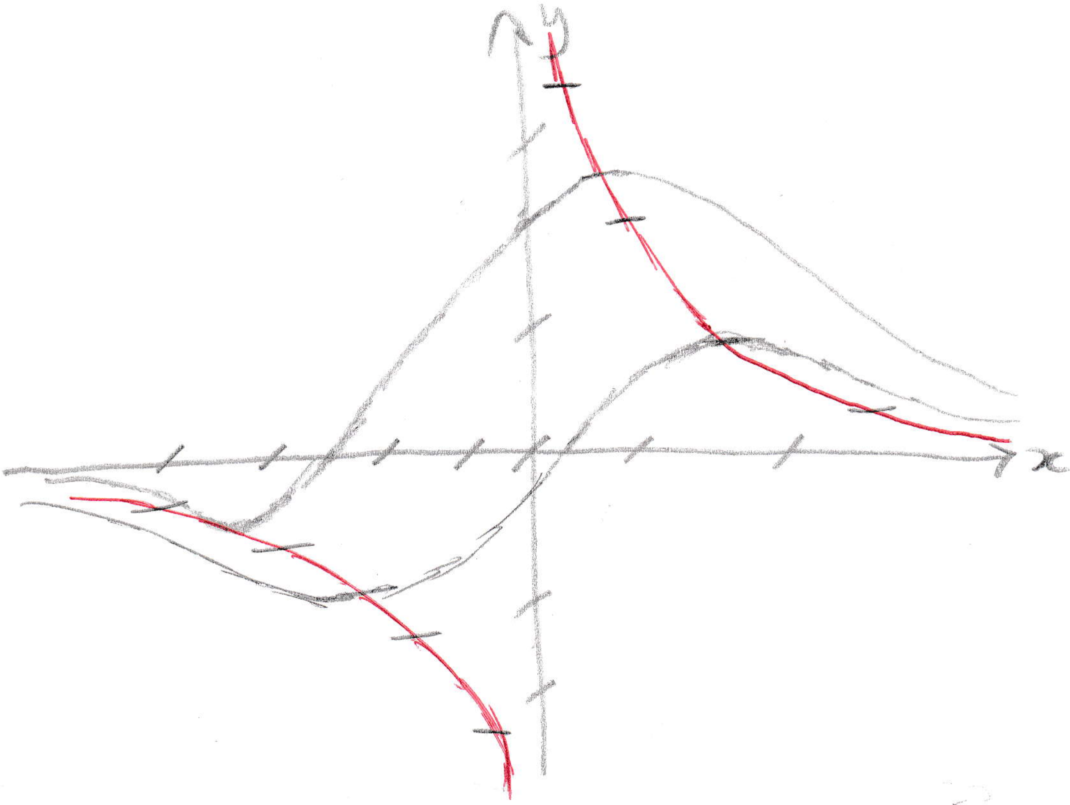
$|f(x) - f(y)| < C|x - y|.$

non-Lipschitz

8(i) $y' = 1 - xy$

$xy = 1 \rightarrow y' = 0$ max/min.

on axes $xy = 0 \rightarrow y' = 1$.



large xy $\qquad y' \approx -xy$

$\frac{y'}{y} \approx -x \qquad \ln|y| \approx -\frac{1}{2}x^2 + C$

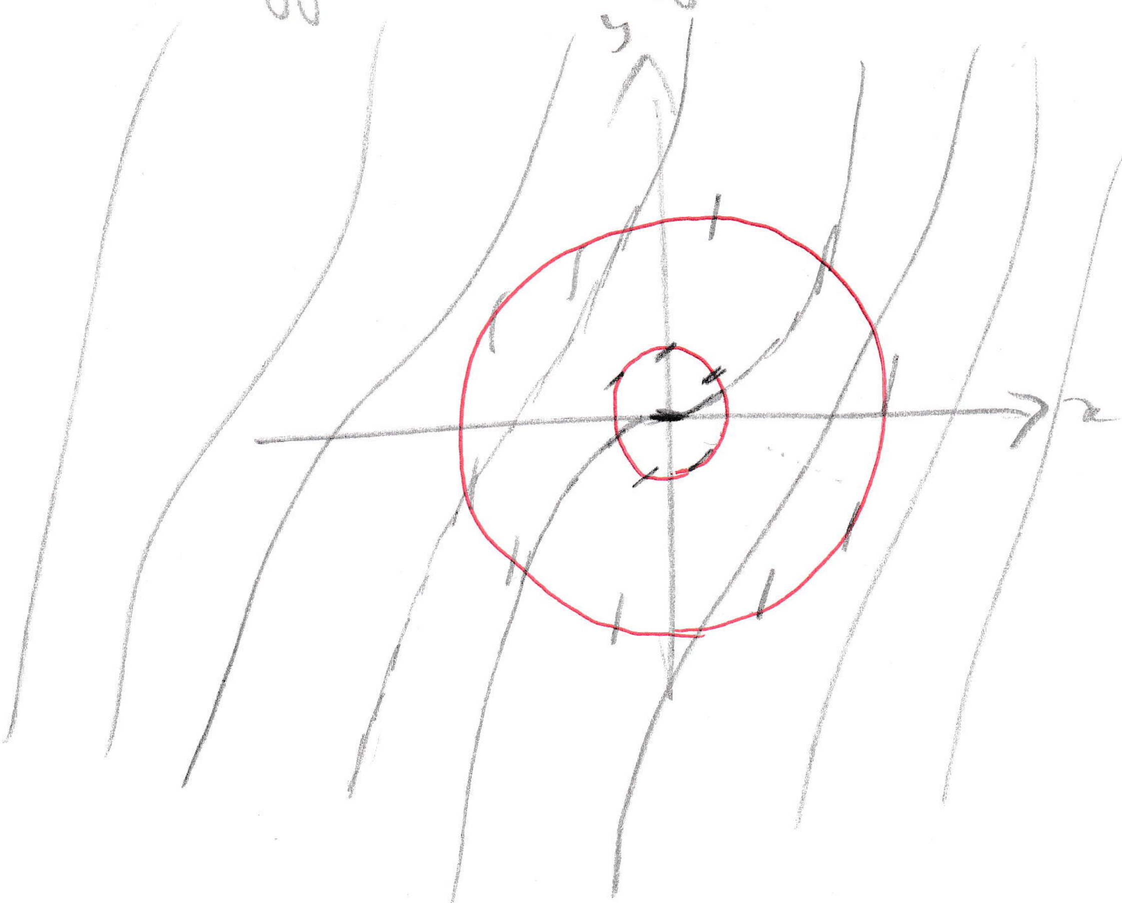
$|y| = e^c e^{-\frac{1}{2}x^2} \qquad y = \pm e^c e^{-\frac{1}{2}x^2}$

$= Ae^{-\frac{1}{2}x^2}.$

(iii) $y' = x^2 + y^2$

y' is constant on circles centre $(0,0)$

bigger on bigger circles



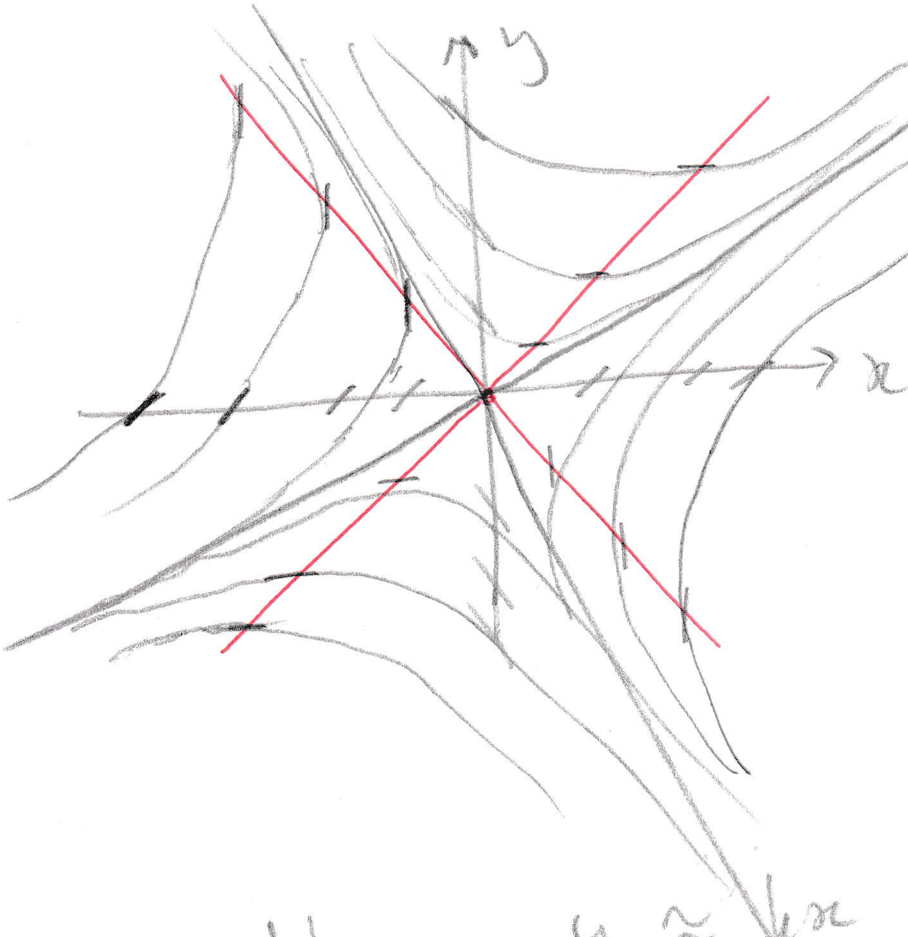
9(ii) $\dfrac{dy}{dx} = \dfrac{x-y}{x+y}$

- $y' = 0$ at $y = x$

$y' = \infty$ at $y = -x$.

$y' = 1$ at $y = 0$

$y' = -1$ at $x = 0$



asymptotes $\quad y \approx kx$

$$k \approx \frac{x-kx}{x+kx} = \frac{1-k}{1+k} \quad \text{exact.}$$

$k^2+k = 1-k$

$k^2+2k = 1$

$(k+1)^2 = 2 \qquad k = -1 \pm \sqrt{2}.$

$y^2+2xy-x^2 = C$

$(y+x)^2 - 2x^2 = C \qquad$ hyperbolae.

1st order <u>homogeneous</u> ODE

$$\frac{dy}{dx} = f(x,y)$$

$$f(x,y) = f(\lambda x, \lambda y) \qquad \forall \lambda \neq 0.$$

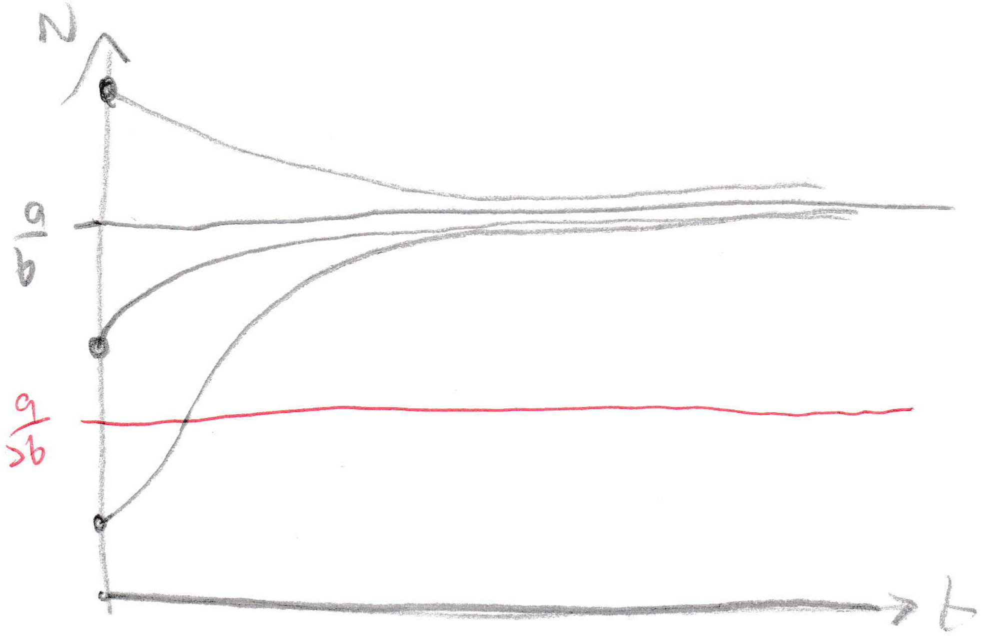
try $y = x\,u(x)$

$$x\frac{du}{dx} + u = f(x, xu) = f(1,u)$$

$$x\frac{du}{dx} = \underbrace{f(1,u) - u}_{g(u)}$$

$$\int \frac{du}{g(u)} = \int \frac{dx}{x}$$

10)



$$\frac{dN}{aN-bN^2} = dt$$

$$\frac{1}{aN-bN^2} = \frac{1}{a}\left[\frac{b}{a-bN} + \frac{1}{N}\right]$$

$$\frac{1}{a}\left[-\ln|a-bN| + \ln N\right] = t + C.$$

$$\frac{N}{|a-bN|} = e^{at}e^{aC}.$$

$$\frac{N}{a-bN} = \overbrace{\left(\pm e^{aC}\right)}^{A} e^{at} = Ae^{at}$$

$$N = (a - bN)Ae^{at}$$

$$N(1 + bAe^{at}) = aAe^{at}$$

$$N = \frac{aAe^{at}}{1 + bAe^{at}} = \frac{a}{b + A^{-1}e^{-at}}.$$

$$t \to \infty : \; N \to a/b.$$

$\cancel{A}$.

$$A > 0 \iff N < \frac{a}{b}$$

$$A < 0 \iff N > a/b.$$

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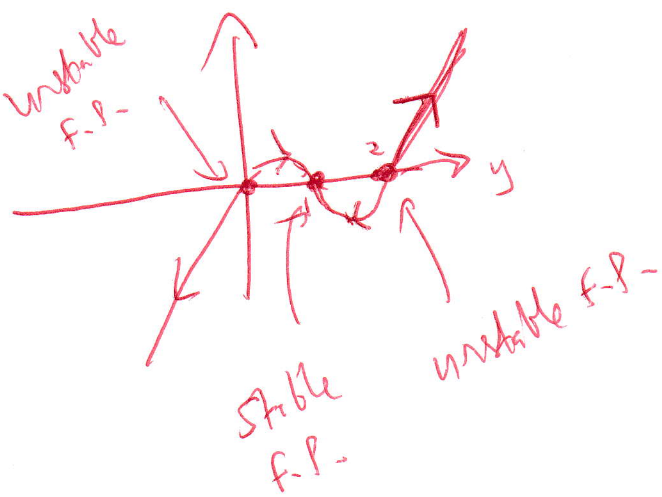
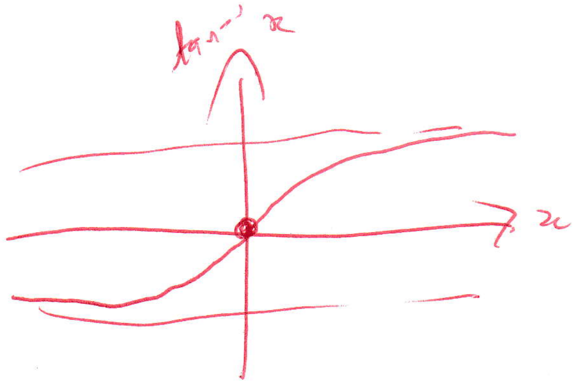
12 (i') $\dfrac{dy}{dt} = f(y) = y(y-1)(y-2)$



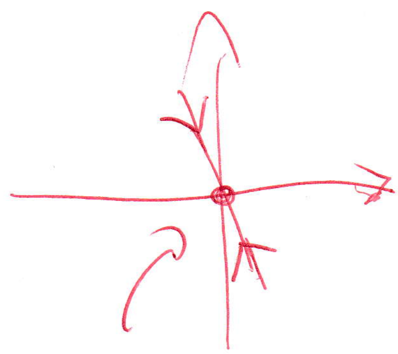
(ii) $\frac{dy}{dt} = f(y) = -2\tan^{-1}\frac{y}{1+y^2}$ 13



$f(y) = 0 \iff \frac{y}{1+y^2} = 0 \iff y = 0.$

Small y: $\frac{y}{1+y^2} \approx y.$

$f(y) \approx -2\tan^{-1} y \approx -2y.$

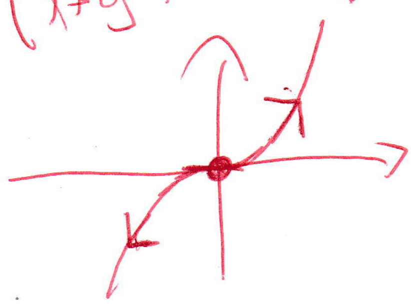


Stable F.P.

(iii) $\frac{dy}{dt} = f(y) = y^3(e^y - 1)^2$

$y = 0$ only F.P.

Small y $\quad f(y) \approx y^3(\cancel{1}+y+\dots-\cancel{1})^2 \approx y^5.$



$\boxed{\text{unstable F.P.}}$

$\theta = i\tau$

$u_0 = (i \sinh \tau)^2 = -\sinh^2 \tau$

$u_0 = \cosh^2 \theta .$

$u_1 = 4 \cosh^2 \theta \underbrace{(1 - \cosh^2 \theta)}_{-\sinh^2 \theta}$

$= - \sinh^2 2\theta$

$u_2 = 4 (-\sinh^2 2\theta) \underbrace{(1 + \sinh^2 2\theta)}_{\cosh^2 2\theta}$

$= - \sinh^2 4\theta .$

etc. $\quad u_n = - \sinh^2 2^n \theta \qquad n \geq 1 .$