1(ii)

$y'' + 9y = \cos 3x$

$m^2 + 9 = 0$

$m = \mp 3i$

C.F. $\quad A\cos 3x + B\sin 3x.$

P.I. try
$ax\cos 3x + bx\sin 3x.$

$y' = a\cos 3x + b\sin 3x \}$

$$t e^{-t^2/2}(1-t)^{-2}$$

$$= t\left(1 - \frac{t^2}{2} + O(t^4)\right)\left(1 + 2t + \frac{1}{2}(-2)(-3)t^2 + O(t^3)\right)$$

$$= t\left(1 + 2t - \frac{t^2}{2} + 3t^2 + O(t^3)\right)$$

$y_2(0) = 0 \Rightarrow x_0 = 0.$

$$= t + 2t^2 + \frac{5}{2}t^3 + O(t^4)$$

$$y_2 = (1-x)W_0\int_0^x \left[t + 2t^2 + \frac{5}{2}t^3 + O(t^4)\right]dt$$

$$= W_0(1-x)\left(\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{5}{8}x^4 + O(x^5)\right)$$

$$y_2 = W_0\left[\frac{1}{5}x^2 + \frac{3}{5}x^3 + \frac{5}{8}x^4 - \frac{1}{5}x^3 - \frac{2}{5}x^4 + O(x^5)\right]$$

(2)

$$y_2''(0) = 1 \quad \Rightarrow \quad W_0 = 1.$$

3(iii) $$y_n = (An+B)2^n + \frac{a^n}{(a-2)^2}.$$

$$a^n = (2+a-2)^n = \sum_{r=0}^{n} \binom{n}{r} 2^{n-r}(a-2)^r$$

$$\frac{a^n}{(a-2)^2} = \sum_{r=0}^{n} \binom{n}{r} 2^{n-r}(a-2)^{r-2}$$

$$= \frac{2^n}{(a-2)^2} + \frac{n\,2^{n-1}}{(a-2)} + \frac{1}{8}n(n-1)\,2^{n-2} + O(a-2)$$

$$y_n = (An+B)2^n + \frac{2^n}{(a-2)^2} + \frac{n\,2^{n-1}}{(a-2)}$$

$$+ \frac{1}{8}n(n-1)2^n + O(a-2)$$

$$= (A'n+B')2^n + \frac{1}{8}n(n-1)2^n + O(a-2)$$

let $a \to 2$ with A', B' fixed $\qquad y_n \to \left(A'n + B' + \frac{1}{8}n(n-1)\right)2^n.$

$$\frac{2(h^2+1)^{-2}\sqrt{(h^2+1)^2-1+\frac{1}{4}h^2}}{2-h}$$

$$\sqrt{} = \sqrt{h^4+2h^2+\cancel{1}-\cancel{1}+\frac{1}{4}h^2}$$

$$= h\sqrt{h^2+\frac{9}{4}}$$

$$= h\cdot\frac{3}{2}\sqrt{1+\frac{4}{9}h^2}$$

$$= \frac{3}{2}h\left(1+\frac{1}{\cancel{4}}\cdot\frac{4}{9}h^2+O(h^4)\right)$$

$$= \frac{3}{2}h+\frac{1}{6}h^3+O(h^5).$$

$$\left[2(h^2+1)-2\left(\frac{3}{2}h+O(h^3)\right)\right](2-h)^{-1}$$

$$= \left(2+2h^2-3h+O(h^3)\right)\frac{1}{2}\left(1-\frac{h}{2}\right)^{-1}$$

$$= \left(1-\frac{3}{2}h+h^2+O(h^3)\right)\left(1+\frac{h}{2}+\frac{h^2}{4}\dots\right)$$

$$= 1-\frac{3}{2}h+h^2+h-\frac{3}{4}h^2+\frac{h^2}{4}+O(h^3)$$

$$= 1-\frac{1}{2}h+\frac{1}{2}h^2+O(h^3)$$

④

$$x_n = x_0 + nh$$

$$y(x_{n+1}) - y(x_{n-1})$$

$$= y(x_n + h) - y(x_n - h)$$

$$= \cancel{y(x_n)} + h y'(x_n) + \cdots$$

$$- \left(\cancel{y(x_n)} - h y'(x_n) + \cdots\right)$$

$$= 2h y'(x_n) + O(h^2)$$

$$y'(x_n) = \frac{y(x_n+h) - y(x_n-h)}{2h} + O(h)$$

$$y(x_{n+1}) - 2y(x_n) + y(x_{n-1})$$

$$= \cancel{y(x_n)} + \cancel{h y'(x_n)} + \frac{1}{2}h^2 y''(x_n) + \cdots$$

$$- \cancel{2y(x_n)}$$

$$+ \cancel{y(x_n)} - \cancel{h y'(x_n)} + \frac{1}{2}h^2 y''(x_n) + \cdots$$

$$= h^2 y''(x_n) + O(h^3)$$

$$y''(x_n) = \frac{y(x_{n+1}) - 2y(x_n) + y(x_{n-1})}{h^2} + O(h)$$ (5)

Let $y_n = y(x_n)$

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} - \frac{y_{n+1} - y_{n-1}}{2h} - 2y_n = o(h)$$

$$y_{n+1} - 2y_n + y_{n-1} - \tfrac{1}{2}h(y_{n+1} - y_{n-1}) - 2h^2 y_n = O(h^3)$$

expect sol^{n}s of this eq will converge to sol^{n}s of differential eq as $h \to 0$ with x_n fixed.

~~Ex~~: Is this true?

Here $x_0 = 0$ $x_n = nh$

$$y_n = \left(1 - \frac{x_n}{n} + \frac{x_n^2}{2n^2}\right)^n$$

$h \to 0$ x_n fixed $\Rightarrow n \to \infty$

$\to e^{-x_n}$ as $n \to \infty$.

the sol^n of the differential eq ✓

6, $$\frac{x^2}{x+1} = \frac{x^2-1}{x+1} + \frac{1}{x+1}$$

$$= x-1 + \underbrace{\frac{1}{x+1}}_{y_1}$$

$$y_2 = x-1$$

$$y_2 = e^{-x^2}.$$

7, $$W_{ij} = y_j^{(i-1)}$$

$$W'_{ij} = y_j^{(i)}$$

$i<n$ $$W'_{ij} = W_{i+1,j}$$

$i=n$ $$W'_{nj} = y_j^{(n)} = -p_1 y_j^{(n-1)} - p_2 y_j^{(n-2)} - \cdots - p_n y_j^{(0)}$$

$$\underline{\underline{W}}' = \underline{\underline{A}}\,\underline{\underline{W}}$$

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -p_n & -p_{n-1} & \cdots & & -p_1 \end{pmatrix}$$

$$W = W_0 e^{-\int p \, dx}$$

$$y'' + p y' + q y = f$$

8. $$\underline{Y} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$\underline{Y}' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -p y' - q y + f \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}}_{\underline{\underline{M}}} \begin{pmatrix} y \\ y' \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$\boxed{\underline{Y}' = \underline{\underline{M}}\,\underline{Y} + \begin{pmatrix} 0 \\ f \end{pmatrix}}$$

$$\underline{Y}_1 = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} \qquad \underline{Y}_2 = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$$

$$\underline{Y}_1' = \underline{\underline{M}}\,\underline{Y}_1 \qquad \underline{Y}_2' = \underline{\underline{M}}\,\underline{Y}_2$$

$\underline{Y}_1, \underline{Y}_2$ linearly indep?

$$\underline{Y} = u\,\underline{Y}_1 + v\,\underline{Y}_2$$

$$\underline{Y}' = u'\underline{Y}_1 + v'\underline{Y}_2 + u\underline{Y}_1' + v\underline{Y}_2'$$

$$= u'\underline{Y}_1 + v'\underline{Y}_2 + u\underline{\underline{M}}\,\underline{Y}_1 + v\underline{\underline{M}}\,\underline{Y}_2$$

$$= u'\underline{Y}_1 + v'\underline{Y}_2 + \underline{\underline{M}}\,(\underbrace{u\underline{Y}_1 + v\underline{Y}_2}_{\underline{Y}})$$

$$\underline{Y}' - \underline{\underline{M}}\,\underline{Y} = u'\underline{Y}_1 + v'\underline{Y}_2$$

$$\begin{pmatrix} 0 \\ f \end{pmatrix} = \begin{pmatrix} u'y_1 + v'y_2 \\ u'y_1' + v'y_2' \end{pmatrix} = \underbrace{\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}}_{\underline{\underline{W}}} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \underline{\underline{W}}^{-1} \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

$$\underline{\underline{W}}^{-1} = \frac{1}{y_1y_2' - y_2y_1'} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{y_1y_2' - y_2y_1'} \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} f.$$

$u, v \longleftrightarrow$ 2 consts of integration.

$z =$ C.F. $+$ P.I.

transient (exponential decay)

fixed by forcing $\sim \cos \omega t$.

- underdamped
- critical
- overdamped

11,

$$y'' - 4y = \delta(x-a)$$

Let soln be $G(x,a) = -\frac{1}{4} e^{-2|x-a|}$

Green function.

Now consider

$$y'' - 4y = f(x)$$

write $f(x) = \int_{-\infty}^{\infty} da\, f(a)\, \delta(x-a)$

Superposn of δ-fns.

linearity $\rightarrow$

$$y(x) = \int_{-\infty}^{\infty} da\, f(a)\, G(x,a).$$

$C^\infty(\mathbb{R})$: space of smooth functions on $\mathbb{R}$.

distribution : linear map from $C^\infty(\mathbb{R})$ to $\mathbb{R}$.

e.g. if $g(x)$ is integrable

define $\Phi_g : C^\infty(\mathbb{R}) \to \mathbb{R}$.

$$\Phi_g : f \longmapsto \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

$$\delta : f \longmapsto f(0)$$

$$\text{"} f(0) = \int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx \text{"}$$

$$\int_{-\infty}^{\infty} f(x)\,(y'' - 4y)\, dx = f(a). \quad \underline{\text{for any } f}.$$

$$\int_{-\infty}^{\infty} (-f'y' - 4fy)\, dx = f(a).$$