1. $o(r) = 5$. $o(s) = 2$.

$|G/\langle r\rangle| = 2 \Rightarrow \langle r\rangle \triangleleft G$

$\Rightarrow srs^{-1} \in \langle r\rangle$.

Let $srs^{-1} = r^i$.

$$\cancel{s}\, r = s^2 r s^{-2} = s(srs^{-1})s^{-1}$$
$$= sr^is^{-1} = (srs^{-1})^i = (r^i)^i = r^{i^2}$$

$\therefore i^2 \equiv 1 \mod 5$

$\Rightarrow i = \pm 1 \mod 5$.

$\therefore$ either $sr = rs \implies G_{10}$ $(o(sr) = 10)$.

or $sr = r^{-1}s \Rightarrow D_{10}$.

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2. $G \curvearrowright G$ by conjugation. $|G| = p^2$.

orbit sizes: $1, p, \cancel{p^2}$.

$\{g \in G \mid |cc\ell(g)| = 1\} = Z(G) = \{g \in G \mid \forall h \in G, gh = hg\}$.

Note: $p^2 = |Z(G)| + p \,\#$ orbits of size p

$\Rightarrow p \mid |Z(G)|$.

$Z(G) \trianglelefteq G$.

~~Pos~~

Let $Q \equiv \cancel{Z}G/Z(G) \Rightarrow |Q| = 1$ or p.

$\Rightarrow$ Q cyclic.

$\therefore$ ~~if g~~ all $g \in G$ can be written as

$a^i z$ for some fixed $a \in G$, $z \in Z(G)$.

Now $a^i z . a^j z' = a^{i+j} z z' = a^j z' a^i z$.

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6. $C_n = \langle \zeta \rangle$.

$H = \langle \zeta^m \rangle$.

$C_n / H \cong C_m = \langle \eta \rangle$.

Let $\theta : C_n \longrightarrow C_m$

$\zeta^i \longmapsto \eta^i$

Note: θ is a surjective hom.

& $\ker \theta = H$.

$\therefore$ by Iso Thm, $C_n / H \cong C_m$.

9. $\mathbb{C} \xrightarrow{\theta} S^1_t \times S^1_1$

$x+iy \longmapsto (e^{2\pi i x}, e^{2\pi i y})$

$\vdots$ $\ker\theta = \Gamma$.

<u>Iso Th:</u> $\mathbb{C}/\Gamma \cong S^1\times S^1$.

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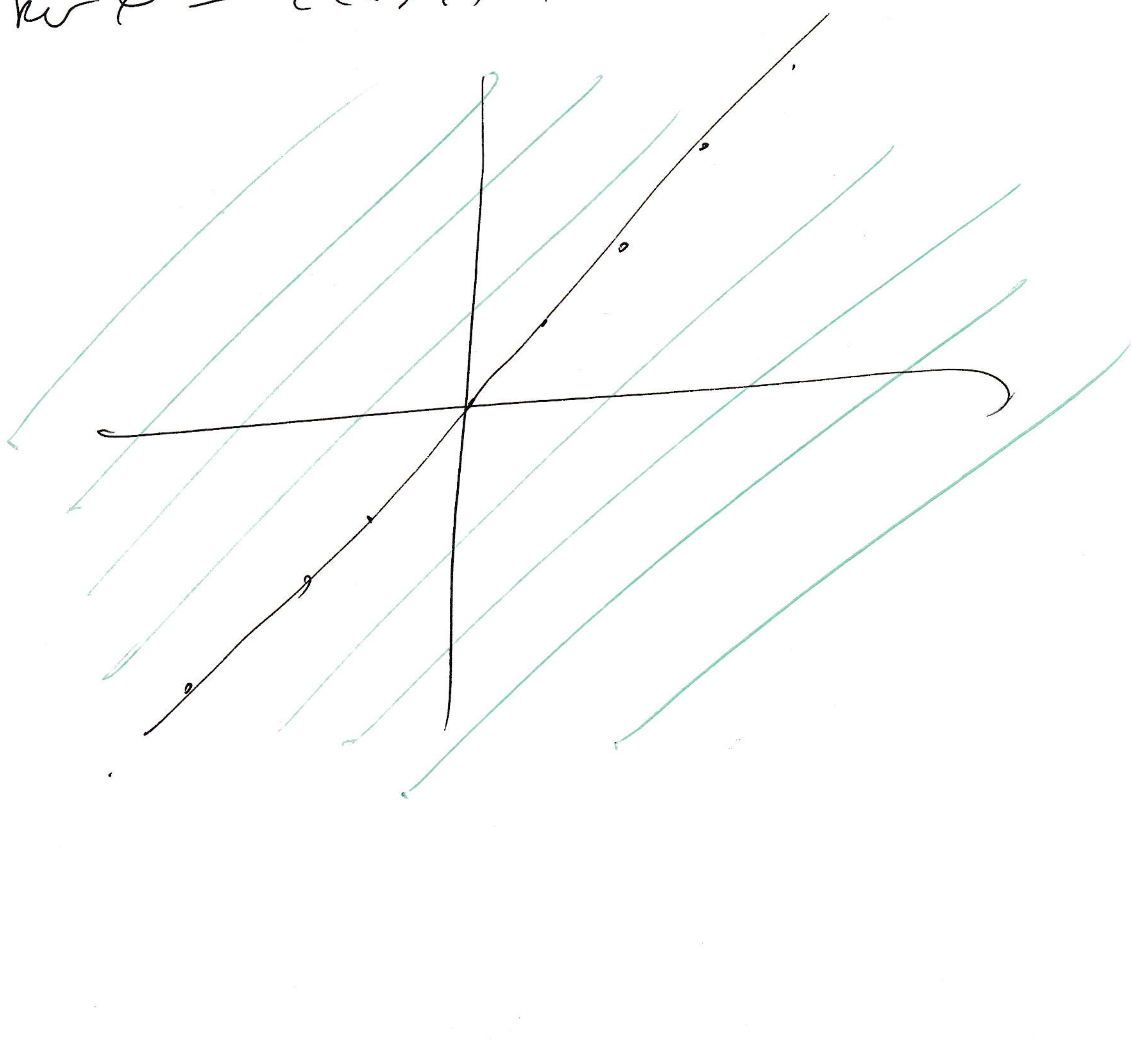
10. $\mathbb{Z}^2 \xrightarrow{\phi} \mathbb{Z}$

$(x,y) \longmapsto ax+by$.

$(a,b) \neq (0,0)$. $\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}$

$\ker\phi = \{(x,y) \in \mathbb{Z}^2 \mid ax+by = 0\}$.



14. (a) WLOG: $j < k$

$$(j\ k) = (j\ j+1)(j+1\ j+2)\cdots(k-1\ k)$$
$$(k-2\ k-1)\cdots(j+1\ j).$$

(b) Pf:

$$(i\ j) = (1\ i)(1\ j)(1\ i)$$

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17. Pf:

$G \curvearrowright A$ on the left.

B^A $= \{f : A \longrightarrow B\}$.

$G \curvearrowright B^A$ by "left translation".

$g \cdot f(a) = f(g a)$.

$\cancel{f((gh)a) = f(g(h(a))) = f}$

What is $f((gh)(a))$?

Write $f'z(a)$ for $f(g(a))$.

$$f((gh)(a)) = f(g(h(a))) = f'(h(a))$$

i.e. If I write f^g for the action of g on f,

we have

$$(f^g)^h = f^{(gh)}$$

ie $G \curvearrowright B^A$ is a <u>right</u> action.

$S_4 \curvearrowright \{1, 2, 3, 4\}$. <u>left</u> action

$\leadsto$

$S_4 \curvearrowright \mathbb{R}^4 = \{(x_1, \dots, x_4) \mid x_i \in \mathbb{R}\}$

$= \mathbb{R}^{\{1, \dots, 4\}}$ on <u>right</u> action.

$\leadsto$

$S_4 \curvearrowright (\mathbb{R}^4)^{\mathbb{R}} \supseteq \{\text{polys}\}$ <u>left</u> action

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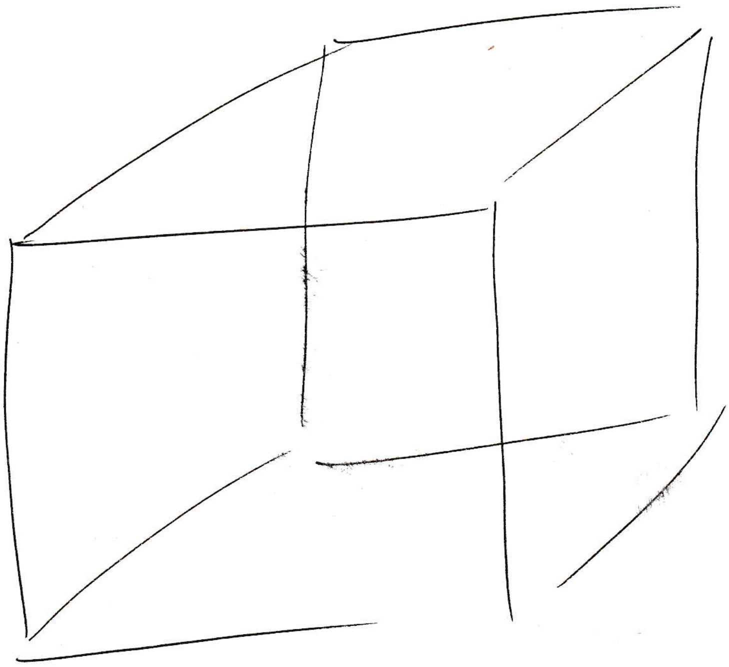
Question:



Colour the faces ~~each~~ ~~to~~ R, G or B.

How many different ways are there, up to symmetry of the cube?