Relations & Functions

- **OUTLINE**
  - Cartesian Product and Relations
  - Properties of Relations
  - Types of Relations
    - Equivalence Relations
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  - Operations on Relations
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  - Functions
  - **The Principle of Mathematical Induction**
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Mathematical Induction

- **Principle of Mathematical Induction**
  - Let n_0 be some fixed integer.
  - Suppose that for some integer $n \geq n_0$, a statement $P(n)$ is either true or false.
  - Suppose now that:
    - $P(n_0)$ is true.
    - if $P(n)$ is true for all $n \leq k$,
      then $P(n)$ is true for $n=k+1$.
  - Then $P(n)$ is true for *every* integer $n \geq n_0$.

Mathematical Induction

- **Principle of Mathematical Induction**
  - ***An Analogy:* Domino Toppling**

    Let P(n): the nth domino is knocked over

    - **If** P(1) is true

      (The 1st domino is knocked over.)

    AND

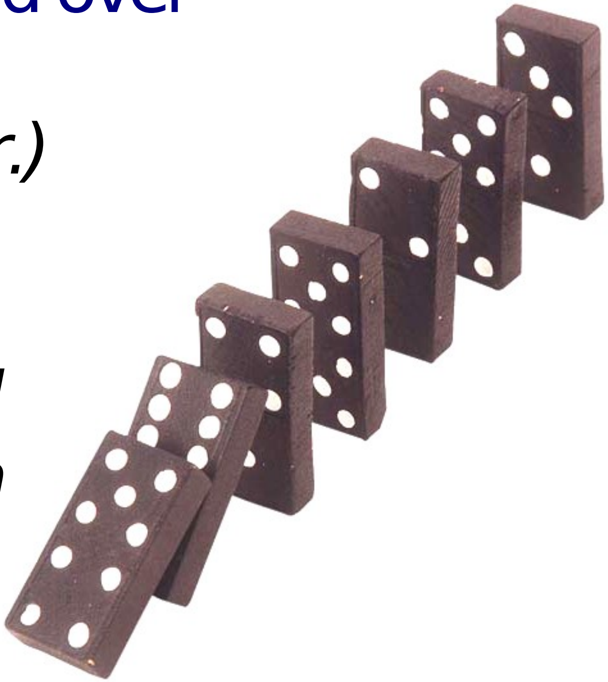
    - **If** $P(k) \rightarrow P(k+1)$ is true

      (The kth domino, when knocked over, will knock over the (k+1)th domino.)

    THEN

      P(n) is true for *all* n.

      (All dominoes are knocked over.)



Mathematical Induction

- **Proof by mathematical induction:**

  To show that P(n) is true for *every* integer $n \geq n_0$, we need to do the following:

  - BASIS STEP:

    Show that **P(n) is true for $n = n_0$** or

    for $n = n_0, n_0 + 1, n_0 + 2, n_0 + 3, \ldots, n_1$.

  - INDUCTIVE HYPOTHESIS:

    Assume that **P(n) is true for $n \leq k$**

    or $n = n_0, \ldots, k - 1, k$.

  - INDUCTIVE STEP:

    Show now that **P(n) is true for n=k+1**

    (Make use of inductive hypothesis when doing so).

Mathematical Induction

- **Example:**

  Use mathematical induction to prove that

  P(n): **$1 + 2 + 3 + 4 + \ldots + n$ = $n(n+1)/2$** holds for all integers $n \geq 1$.

  Proof:

  - BASIS STEP: **Show** that P(n) is true for $n = n_0$

    when $n = 1$

    $1 = 1(1+1)/2$??

    $1 = 1$

    when $n = 2$

    $1+2 = 2(2+1)/2$??

    $3 = 3$

  - INDUCTIVE HYPOTHESIS: **Assume** that P(n) is true for $n \leq k$

    Assume $1 + 2 + 3 + 4 + \ldots + k = k(k+1)/2$

Mathematical Induction

- **Example:**
  Use mathematical induction to prove that
  P(n): **$1 + 2 + 3 + 4 + \ldots + n = n(n+1)/2$** holds for all integers $n \geq 1$.
  Proof: (cont'n)
  - INDUCTIVE STEP: **Show** now that P(n) is true for n=k+1.
    Need to show that
    $1 + 2 + 3 + 4 + \ldots + k + (k+1) = (k+1)(k+2)/2$
    We thus start: $1 + 2 + 3 + 4 + \ldots + k + (k+1) = ?$
    $1 + 2 + 3 + 4 + \ldots + k + (k+1)$
    $= [\, k(k+1)/2 \,] + (k+1)$ (by our assumption)
    $= [\, k(k+1) + 2(k+1) \,] / 2$
    $= (k+1)(k+2)/2$
  - CONCLUSION: Thus, since P(1) is true and since $P(k) \rightarrow P(k+1)$, then
  P(n): $1 + 2 + 3 + 4 + \ldots + n = n(n+1)/2$ holds for *all* integers $n \geq 1$.

Mathematical Induction

- **Example:**

  Use mathematical induction to prove that

  P(n): **$n^3 - n$ is divisible by 3** for integers $n \geq 1$

  Proof:

  - BASIS STEP: **Show** that P(n) is true for $n = n_0$

    when $n = 1$

    $1^3 - 1$ is divisible by 3??

    0 *is* divisible by 3

    when $n = 2$

    $2^3 - 2$ is divisible by 3??

    6 *is* divisible by 3

  - INDUCTIVE HYPOTHESIS: **Assume** that P(n) is true for $n \leq k$

    Assume $k^3 - k$ is divisible by 3.

Mathematical Induction

- **Example:**

  Use mathematical induction to prove that

  P(n): **$n^3 - n$ is divisible by 3** for integers $n \geq 1$.

  Proof: (cont'n)

  - INDUCTIVE STEP: **Show** now that P(n) is true for n=k+1.

    Need to show that $(k+1)^3 - (k+1)$ is divisible by 3.

    $(k+1)^3 - (k+1)$

    $= (k^3 + 3k^2 + 3k + 1) - (k+1)$

    $= k^3 + 3k^2 + 3k - k$

    $= k^3 - k + 3(k^2 + k)$

    Since $k^3 - k$ is divisible by 3 (by our assumption)

    AND $3(k^2 + k)$ is divisible b 3

    then $k^3 - k + 3(k^2 + k)$ must be divisible by 3.

  - CONCLUSION: Thus, since P(1) is true and since $P(k) \rightarrow P(k+1)$, then

    P(n): $n^3 - n$ is divisible by 3 for all integers $n \geq 1$

Mathematical Induction

- **Example:**

  Use mathematical induction to prove that

  P(n): $\mathbf{n! \geq 2^{n-1}}$ for integers $n \geq 1$

  [<u>Note</u>: $n! = n \cdot (n-1) \ldots 3 \cdot 2 \cdot 1$ and $0! = 1$]

  Proof:

  - BASIS STEP: **Show** that P(n) is true for $n = n_0$

    when $n = 1$

    $1! \geq 2^0$??

    $1 \geq 1$

    when $n = 2$

    $2! \geq 2^1$??

    $2 \geq 2$

  - INDUCTIVE HYPOTHESIS: **Assume** that P(n) is true for $n \leq k$

    Assume $k! \geq 2^{k-1}$

Mathematical Induction

- **Example:**

  Use mathematical induction to prove that

  P(n): $\mathbf{n! \geq 2^{n-1}}$ for integers $n \geq 1$.

  Proof: (cont'n)

  - INDUCTIVE STEP: **Show** now that P(n) is true for n=k+1.

    Need to show that $(k+1)! \geq 2^{(k+1)-1}$

    $(k+1)!$

    $= (k+1)k!$

    $\geq (k+1)2^{k-1}$ (by our assumption)

    Now, $(k+1)2^{k-1} \geq 2 \cdot 2^{k-1}$

    since $k+1 \geq 2$ (as it is known that $k \geq 1$)

    We have thus shown that $(k+1)! \geq 2^k$

  - CONCLUSION: Thus, since P(1) is true and since $P(k) \rightarrow P(k+1)$, then

    P(n): $\mathbf{n! \geq 2^{n-1}}$ for all integers $n \geq 1$.

Mathematical Induction

- **Other examples/exercises:**

  Use mathematical induction to prove the following:

  1) P(n): $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

  holds for all integers $n \geq 1$.

  2) P(n): $n^3 - n$ is divisible by 3 for $n \geq 1$

  3) P(n): $a \cdot r^0 + a \cdot r^1 + a \cdot r^2 + \ldots + a \cdot r^n = a(r^{n+1} - 1)/(r - 1)$

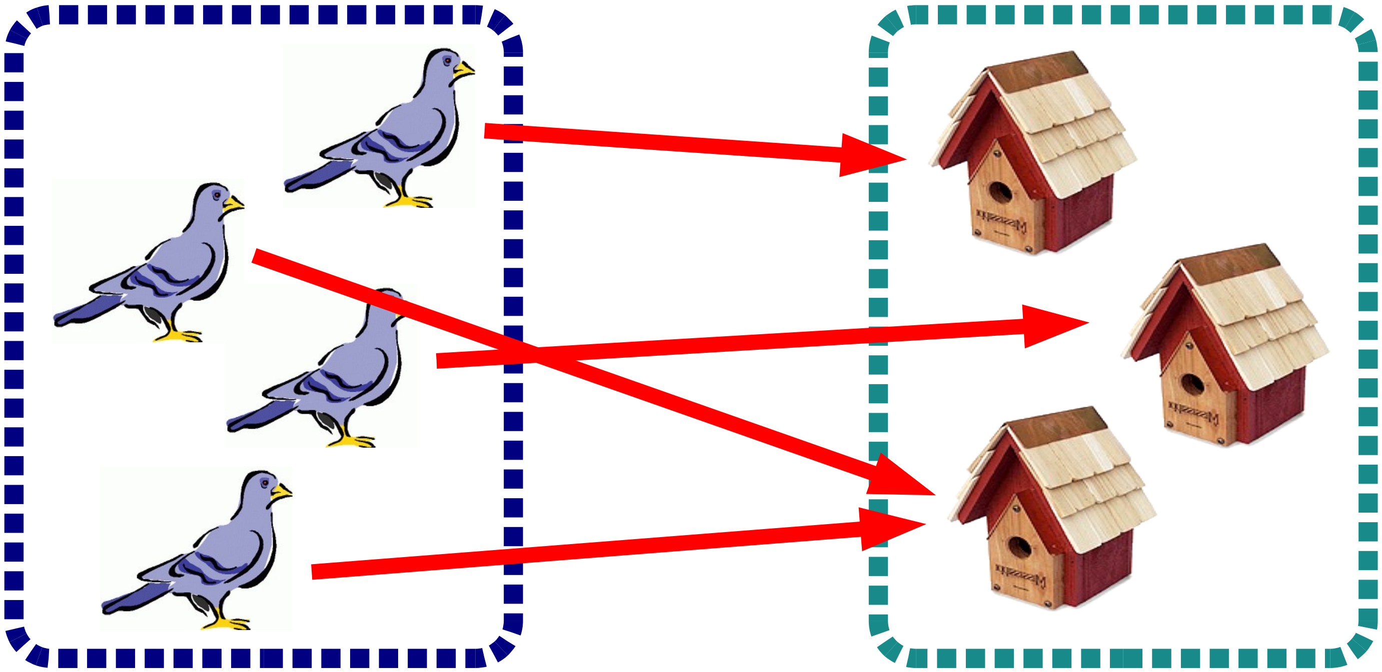
  for all integers $n \geq 0$ if $r \neq 1$

  4) P(n): $1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} + 1$ for $n \geq 0$

  5) P(n): $n^2 \geq 2n + 1$ for $n \geq 3$

Pigeonhole Principle

- **FIRST FORM**
  If k *pigeons* fly into n *pigeonholes* and $k > n$, then some pigeonhole will contain at least two pigeons.



Pigeonhole Principle

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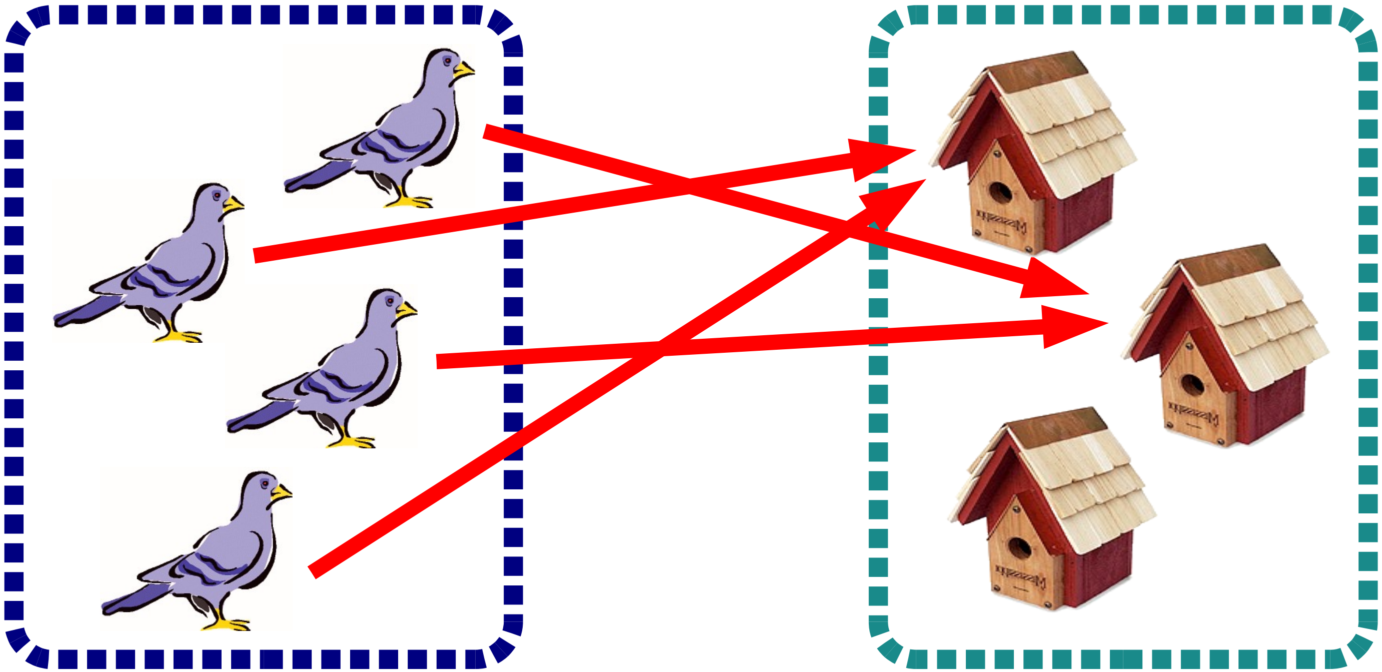


Pigeonhole Principle

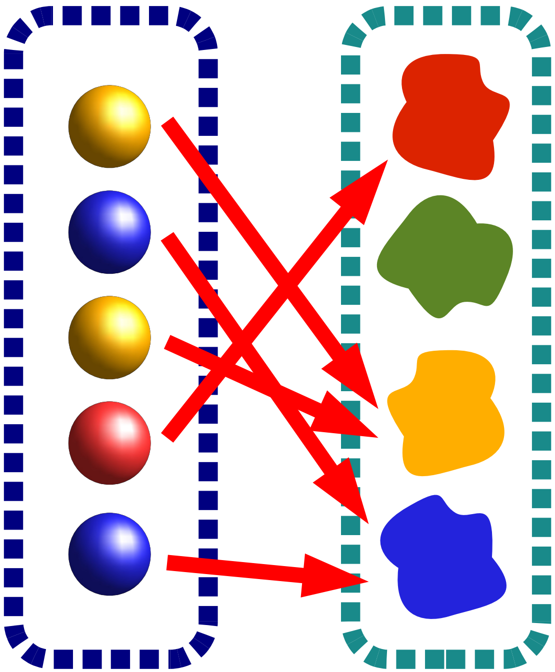
- ***Examples:***

  Suppose you are to take five balls from a box containing red, green, blue and yellow balls. Show that at least two of the balls you select will have the same color.

Solution:

- Let
  - *pigeons* ≡ five balls selected (k = 5)
  - *pigeonholes* ≡ four colors of the balls in the box (n=4)
- Since there are more balls than colors available (i.e., $k > n$)

  $\Rightarrow$ at least two balls selected will be of the same color.



Pigeonhole Principle

- ***Examples:***

  Fifteen students took a multiple-choice exam. Jose made 13 errors while each of the other students made less than that number. Prove that at least two students made the same number of errors.

Solution:

- Let
  - *pigeons* ≡ fifteen students selected ($k = 15$)
  - *pigeonholes* ≡ {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13}

    ≡ fourteen # of errors ($n=14$)
- Since there are more students than the number of number of errors (i.e., $k > n$)

  $\Rightarrow$ at least two students will have made the same number of errors

Pigeonhole Principle

- ***Examples:***

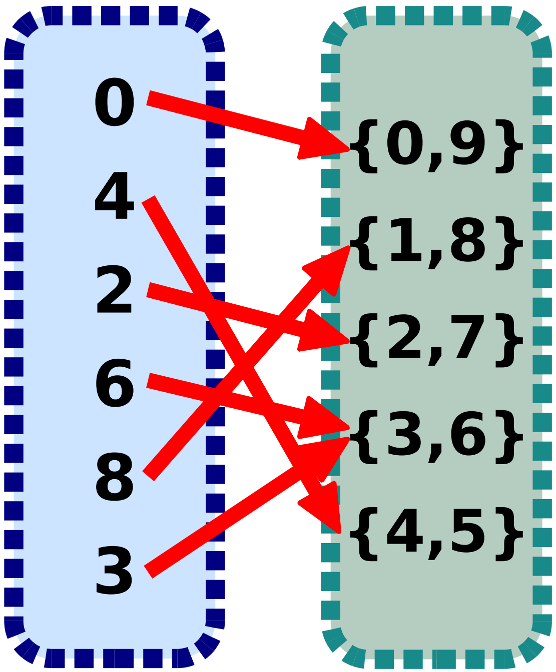
  Show that if any six(6) numbers are chosen from $\{0, 1, \ldots, 9\}$ then two of them will add up to 9.

Solution:

- Let
  - *pigeons* $\equiv$ six numbers chosen (k = 6)
  - *pigeonholes* $\equiv$ five pairs numbers that add up to 9 (n = 5)

    $\equiv \{\{0,9\},\{1,8\},\{2,7\},\{3,6\},\{4,5\}\}$
- Since there are more numbers than pairs/groups (i.e., $k > n$)

  $\Rightarrow$ at least two numbers chosen will add up to 9



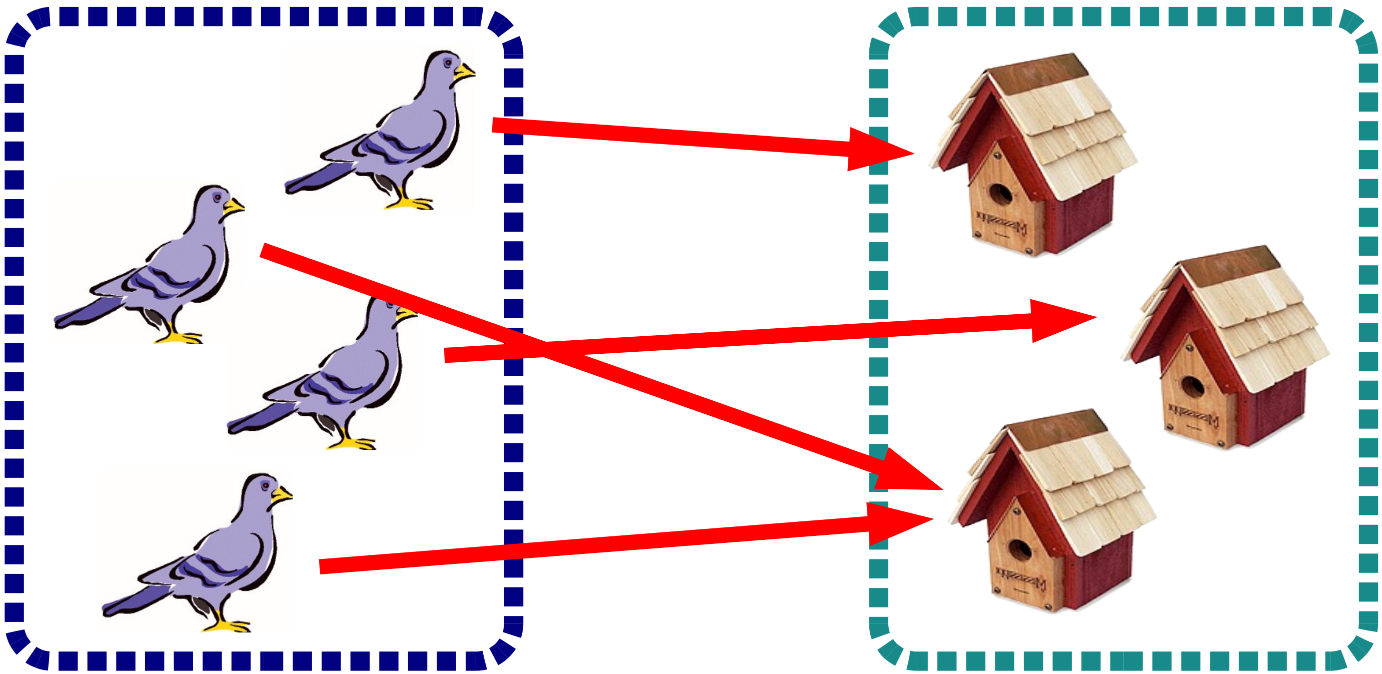
Pigeonhole Principle

- ***Other examples:***
  - Suppose it is pitch dark and you are rummaging your closet for a pair of socks. If you have a blue pair, a black pair, a white pair, a pink pair and a grey pair of socks in your closet, how many individual socks should you pick out to ensure that you have a matching pair?
  - Consider choosing any 11 positive integers. Suppose you divide each by 10 and take the remainder. For example, 152/10 = 15 remainder 2. Show that at least two of the chosen 11 positive integers will have the same remainder.
  - Consider the set of students taking CMSC 56 this semester and suppose we are to select students at random to enter a quiz contest. At least how many students should be selected so that at least two will come from the same lab section?

Pigeonhole Principle

- **SECOND FORM**

  If A and B are non-empty sets and $|A| > |B|$, and f is a function from A to B, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in A$, and $x_1 \neq x_2$. That is, we *cannot* define a one-to-one function from A to B.



Pigeonhole Principle

- ***Examples:***

  Show that if any six(6) numbers chosen from $\{0, 1, \ldots, 9\}$ then two of them will add up to 9.

Solution:

- Let
  - A = set of numbers chosen = $\{x_1, x_2, \ldots, x_6\}$
  - $B = \{\{0,9\}, \{1,8\}, \{2,7\}, \{3,6\}, \{4,5\}\}$
- Now define the function $f : A \rightarrow B$ as

  $f(x_i) = \{ Y \mid Y \in B \text{ where } x_i \in Y \}$
- Since $|A|=6$ and $|B|=5$ and therefore $|A|>|B|$,

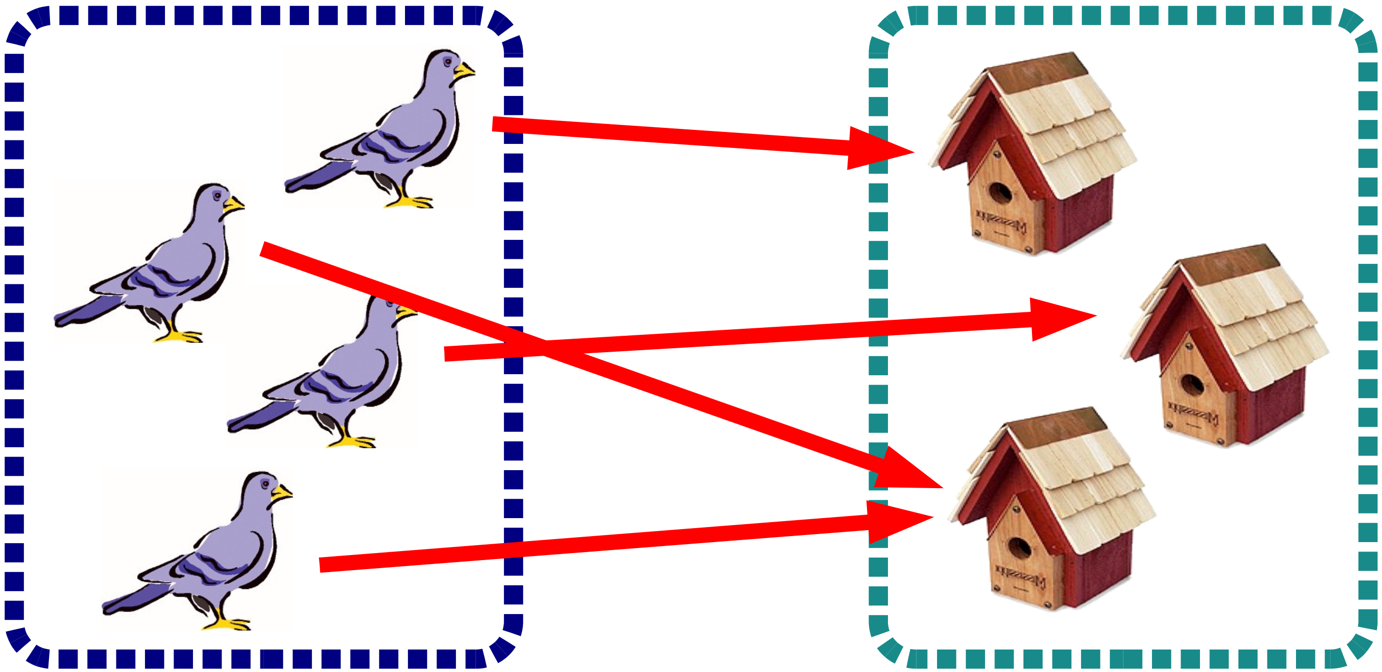
  the function $f : A \rightarrow B$ cannot be one-to-one.

  (*Or that there will exist two elements of A x_i and x_j where $i \neq j$ where $f(x_i) = f(x_j)$*).

Pigeonhole Principle

- **THIRD/EXTENDED FORM**

If k pigeons are assigned to n pigeonholes, then one of the pigeonholes will contain at least $\left\lfloor \frac{k-1}{n} \right\rfloor + 1$ pigeons.



Pigeonhole Principle

- ***Examples:***
  Show that if any 30 people are selected, then we can choose a subset of five people such that all five were born on the same day of the week.

Solution:

- Let
  - *pigeons* ≡ thirty people selected (k = 30)
  - *pigeonholes* ≡ seven days of the week (n = 7)

$$\left\lfloor \frac{k-1}{n} \right\rfloor + 1 = \left\lfloor \frac{30-1}{7} \right\rfloor + 1 = 4 + 1 = 5$$

$\Rightarrow$ at least five people will have been born on the same day of the week.

Pigeonhole Principle

- ***Examples:***

  Suppose a group of thirty students took a multiple-choice exam. Jose made 13 errors and each of the rest made less than 13 errors. There will be a group of at least how many students who made the same number of errors?

Solution:

- Let
  - *pigeons* ≡ thirty students (k = 30)
  - *pigeonholes* ≡ fourteen # of errors (n=14)

$$\left\lfloor \frac{k-1}{n} \right\rfloor + 1 = \left\lfloor \frac{30-1}{14} \right\rfloor + 1 = 2 + 1 = 3$$

$\Rightarrow$ there will be group of at least three students who made an equal number of errors.

Pigeonhole Principle

- ***Examples:***

  Consider the set of students taking CMSC 56 this semester and suppose we are to select students at random to enter a quiz contest. At least how many students should be selected so that the group will include at least four will come from the same lab section?

Solution:

- Let
  - *pigeons* ≡ k students (k = ??)
  - *pigeonholes* ≡ six lab sections (n=6)

$$\left\lfloor \frac{k-1}{n} \right\rfloor + 1 = \left\lfloor \frac{k-1}{6} \right\rfloor + 1 = 4$$

$$\Rightarrow \left\lfloor \frac{k-1}{6} \right\rfloor = 3 \quad \Rightarrow \frac{k-1}{6} \geqslant 3$$

⇒ at least **k=19** students should be selected

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