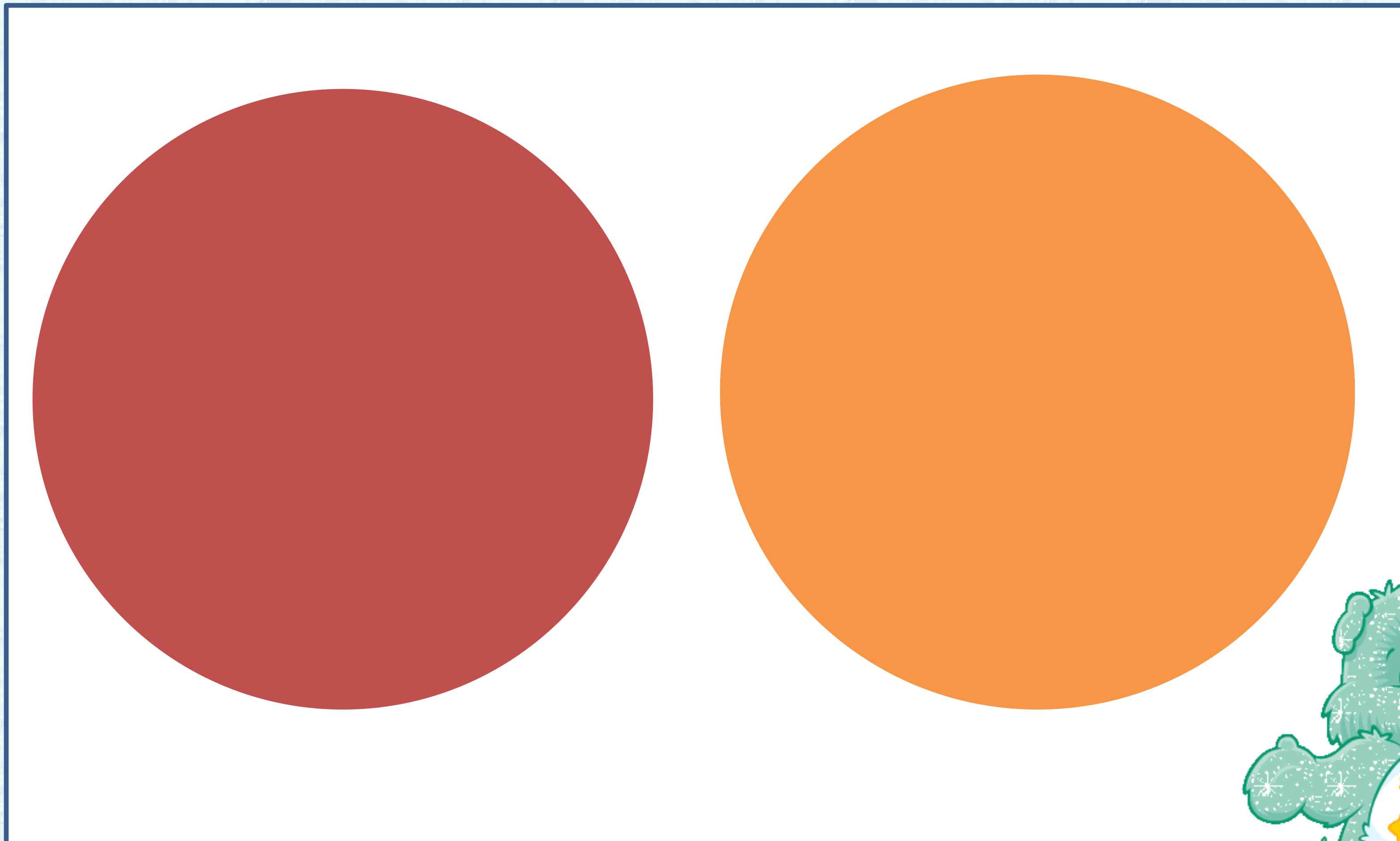
Inclusion Exclusion Principle

Theorem 1.1

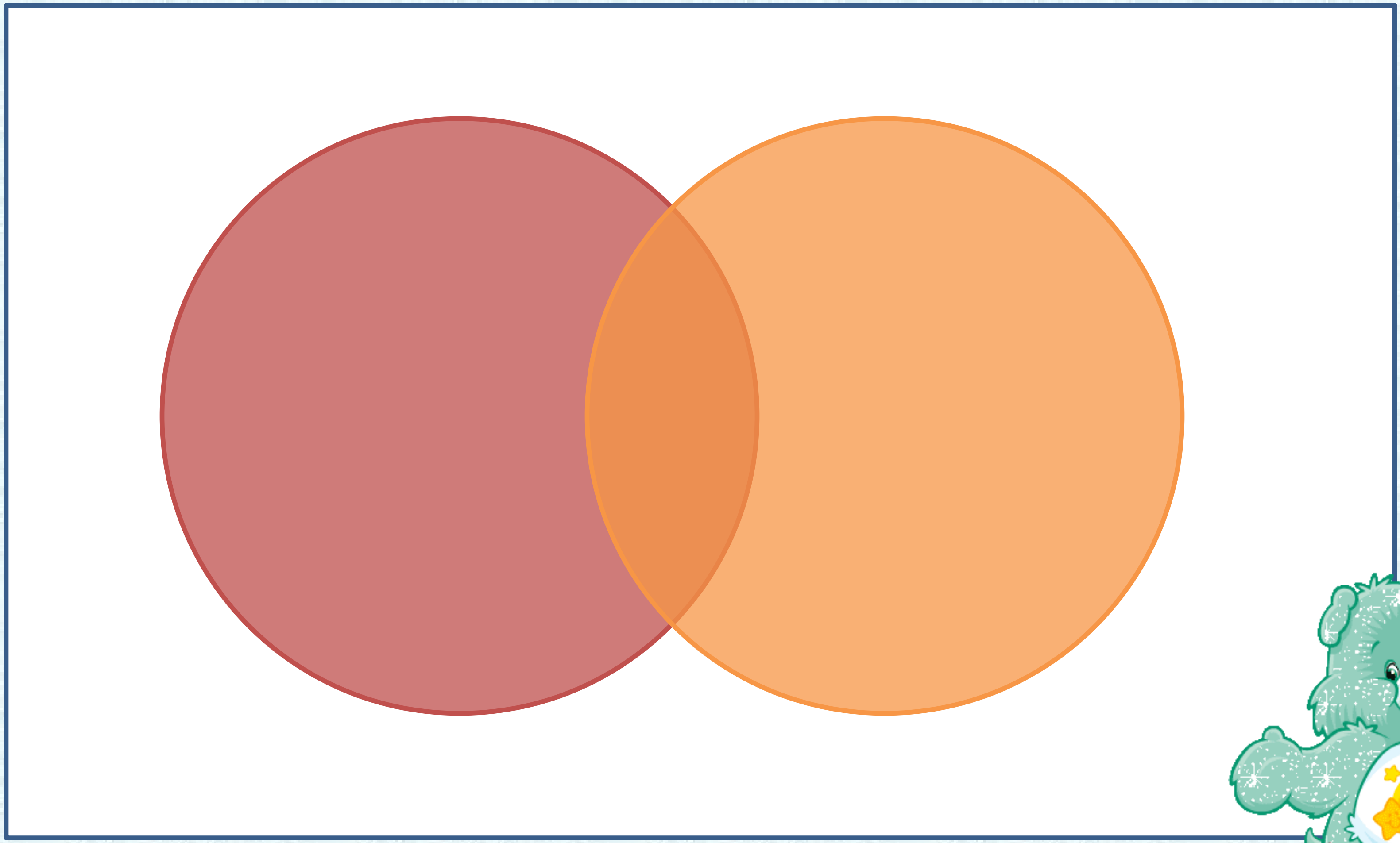
If *A and B are disjoint finite sets, then*

$$|A \cup B| = |A| + |B|.$$

Theorem 1.1 is true for disjoint sets



But what about non-disjoint sets?



Theorem 1.2

General form of Theorem 1.1 (that is, applicable to both disjoint and non-disjoint sets)

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

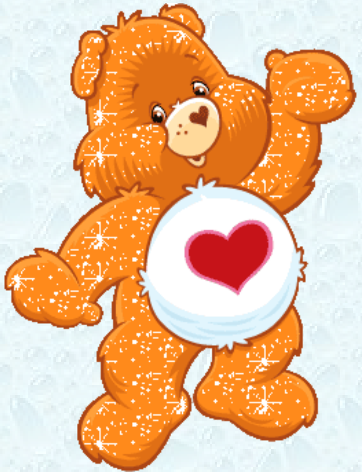
Example 1

Find the number of integers from the set {1, 2, . . . , 1000} which are divisible by 3 or 5.

Example 1

The integers which we are looking for are 3, 5, 6, 9, 10, 12, 15, 18, 20, *. . . , 999, 1000.*

How many are there?



Let's present the problem more formally

- $S = \{1, 2, . . . , 1000\}$,
- $A = \{x \in S \mid x \text{ is divisible by } 3\}$ *and*
- $B = \{x \in S \mid x \text{ is divisible by } 5\}$.

It is now clear that our task is to evaluate $|A \cup B|$ *as* $A \cup B$ *is the* set of numbers in *S which are divisible by 3 or 5.*

Before applying (1.2) to evaluate $|A \cup B|$, *we recall a useful* notation:

For a real number *r, let* $\lfloor r \rfloor$ *denote the greatest integer that is* smaller than or equal to *r.*

Thus $\lfloor 3.15 \rfloor = 3$, $\lfloor 20/3 \rfloor = 6$, $\lfloor 7 \rfloor = 7$ *and so on.*

How many integers in *{1, 2, . . . , 10} are there which are divisible* by 3?

There are three (namely, 3, 6, 9) and note that “three” can be expressed as $\lfloor 10/3 \rfloor$.

The number of integers in *{1, 2, . . . , 10} which* are divisible by 5 is two (namely, 5, 10) and note that “two” can be expressed as $\lfloor 10/5 \rfloor$.

In general…

For any two natural numbers n, k with $k \le n$, the number of integers in the set $\{1, 2, . . . , n\}$ which are divisible by k can be expressed as

$$\lfloor n/k \rfloor.$$

We now return to our original problem of evaluating $|A \cup B|$.

- $|A| = \lfloor 1000/3 \rfloor = 333$ *and*
- $|B| = \lfloor 1000/5 \rfloor = 200$

It remains to find $|A \cap B|$

- What does $A \cap B$ *represent?*

Well, $A \cap B$ is the set of integers in S which are divisible by both 3 and 5.

- How to evaluate $|A \cap B|$?

We can make use of the LCM

$A \cap B = \{x \in S \mid x \text{ is divisible by } 15\}$

- Thus $|A \cap B| = \lfloor 1000/15 \rfloor = 66.$

- Finally, by (1.2), we have

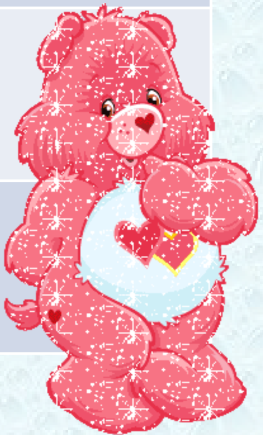
$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 333 + 200 - 66 = 467.$$

Example 2

Find the number of positive divisors of at least one of the numbers 5400 and 18000.

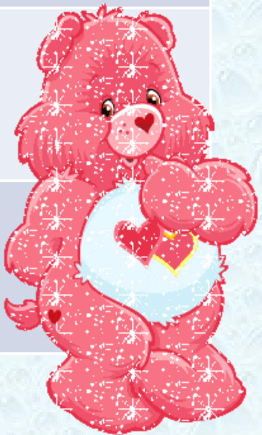
Before going to our example…

number	positive divisors	# of pos. divisors		
20	1, 20, 2, 10, 4, 5	6		
25	1, 25, 5	3		
18	1, 18, 2, 9, 3, 6	6		
17	1, 17	2		
6	1, 6, 2, 3	4		



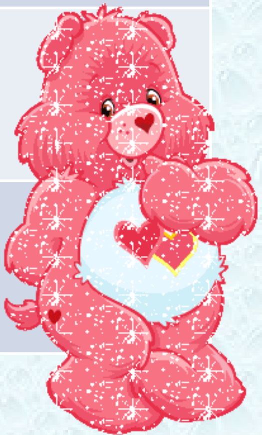
representing the numbers as multiplication of **distinct** primes

number	positive divisors	# of pos. divisors	multiplication of primes	
20	1, 20, 2, 10, 4, 5	6	$2^2 * 5^1$	
25	1, 25, 5	3	5^2	
18	1, 18, 2, 9, 3, 6	6	$2^1 * 3^2$	
17	1, 17	2	17^1	
6	1, 6, 2, 3	4	$2^1 * 3^1$	



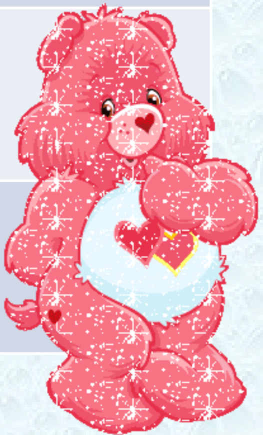
Now let's just focus on the powers of the primes

number	positive divisors	# of pos. divisors	multiplication of primes	
20	1, 20, 2, 10, 4, 5	6	$2^2 * 5^1$	
25	1, 25, 5	3	5^2	
18	1, 18, 2, 9, 3, 6	6	$2^1 * 3^2$	
17	1, 17	2	17^1	
6	1, 6, 2, 3	4	$2^1 * 3^1$	



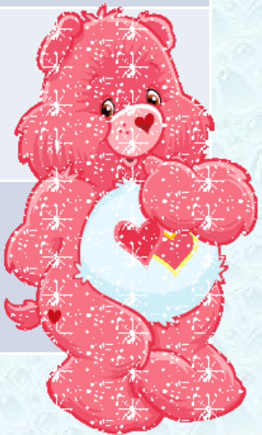
Let's remove the primes totally and try to see some connection with the # of positive divisors

number	positive divisors	# of pos. divisors	powers of primes	
20	1, 20, 2, 10, 4, 5	6	2 * 1	
25	1, 25, 5	3	2	
18	1, 18, 2, 9, 3, 6	6	1 * 2	
17	1, 17	2	1	
6	1, 6, 2, 3	4	1 * 1	



Here's the connection…

number	positive divisors	# of pos. divisors	powers of primes + 1	# of pos. divisors
20	1, 20, 2, 10, 4, 5	6	(2+1)*(1+1)	6
25	1, 25, 5	3	(2+1)	3
18	1, 18, 2, 9, 3, 6	6	(1+1)*(2+1)	6
17	1, 17	2	(1+1)	2
6	1, 6, 2, 3	4	(1+1)*(1+1)	4



Summing up…

- *Let $n \geq 2$ be a natural number.*
  If $n = p_1^{m1} p_2^{m2} \cdot \cdot \cdot p_k^{mk}$,
  *where $p_1, p_2, . . . , p_k$ are **distinct** primes, then the number of positive divisors of n is*
  $(m_1 +1)(m_2 + 1) \cdot \cdot \cdot (m_k + 1)$.

Again, formalizing our problem

Let *A* = $\{x \in \mathbf{N} \mid x$ *is a divisor of 5400*$\}$ *and*

B = $\{x \in \mathbf{N} \mid x$ *is a* divisor of 18000$\}$.

Clearly, our task is to evaluate $|A \cup B|$. *To apply* (1.2), we need to count $|A|$, $|B|$ *and* $|A \cap B|$.

Observe that

$5400 = 2^3 \cdot 3^3 \cdot 5^2$ *and*

$18000 = 2^4 \cdot 3^2 \cdot 5^3.$

Thus, by applying the result stated above, we have

$|A| = (3 + 1)(3 + 1)(2 + 1) = 48$ *and*

$|B| = (4 + 1)(2 + 1)(3 + 1) = 60.$

What does $A \cap B$ *represent?*

By definition, $A \cap B$ *is the set of* common positive divisors of 5400 and 18000, and so it is the set of positive divisors of the Greatest Common Divisor (GCD) of 5400 and 18000. Since

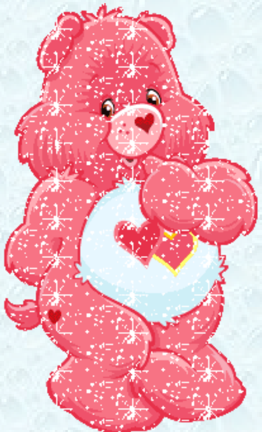
$$\text{GCD}\{5400, 18000\} =$$

$$GCD\{2^3 \cdot 3^3 \cdot 5^2, 2^4 \cdot 3^2 \cdot 5^3\}$$

$$= 2^3 \cdot 3^2 \cdot 5^2$$

it follows that

$$|A \cap B| = (3 + 1)(2 + 1)(2 + 1) = 36.$$



Hence, by (1.2), we have

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 48 + 60 - 36$$
$$= 72$$

(1.2) is for 2 sets, what about three?

Formula (1.2) provides an expression for $|A \cup B|$. *We shall now apply it to derive an expression for* $|A \cup B \cup C|$, *where* A,B *and* C are any three finite sets. Observe that

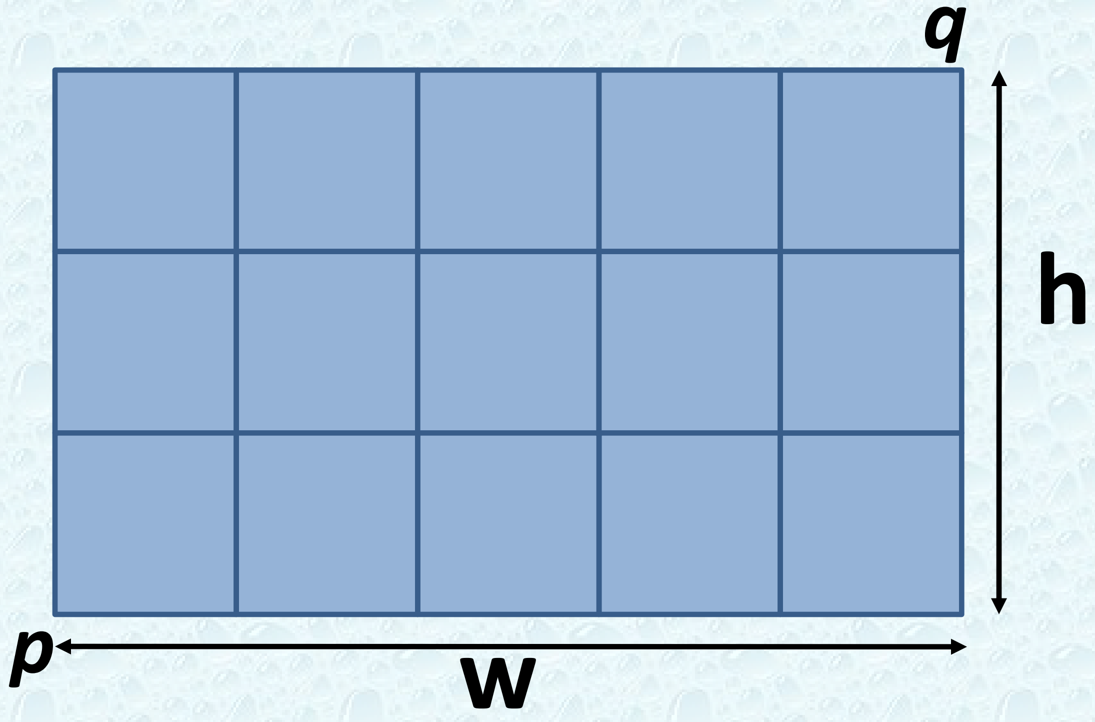
$$
\begin{aligned}
&|A \cup B \cup C| \\
&= |A \cup (B \cup C)| \\
&= |A| + |B \cup C| - |A \cap (B \cup C)| \text{ (by (1.2)} \\
&= |A| + |B \cup C| - |(A \cap B) \cup (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| \\
&-(|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|) \text{ (by(1.2)} \\
&= |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) \\
&\quad + |A \cap B \cap C|.
\end{aligned}
$$

Theorem 1.3

$$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$$

Before we move to our example, a little introduction to the problem…

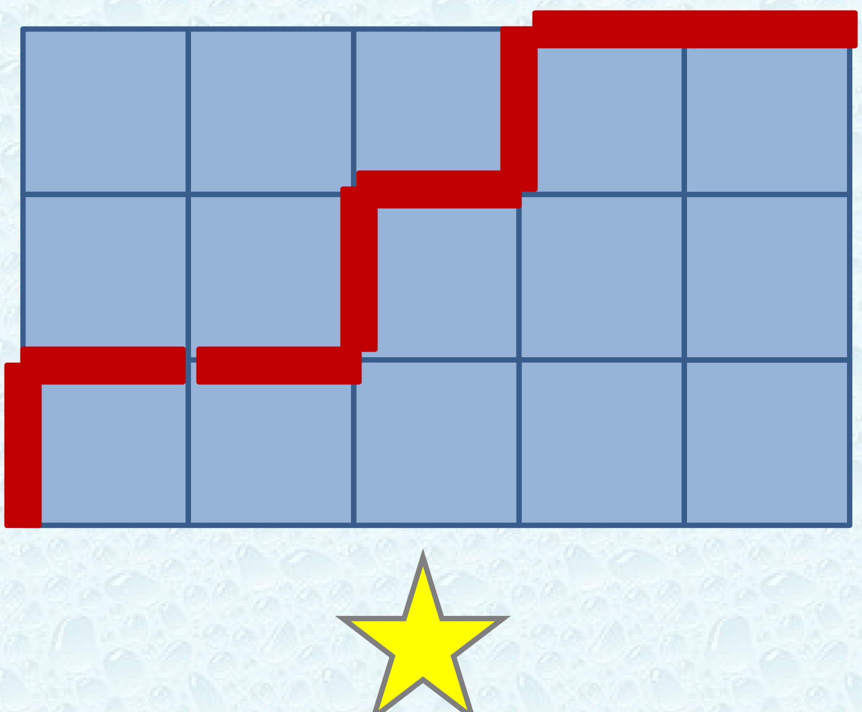
If we have an h x w grid, how many shortest paths are there from point p to point q?

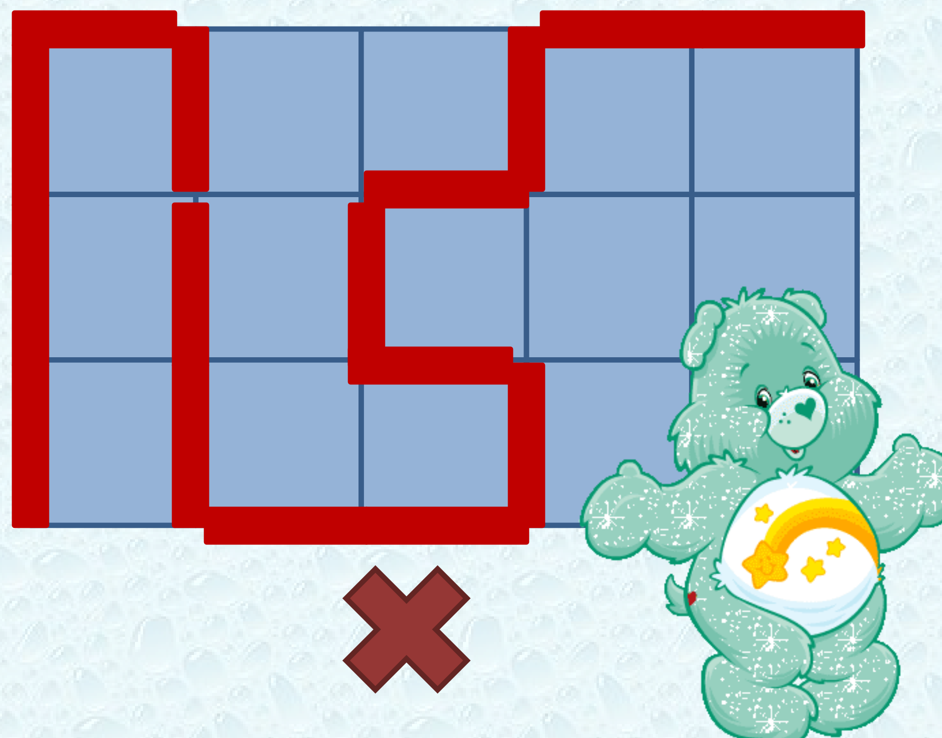


Since we are talking about shortest paths…

We are talking about right and up turns only.

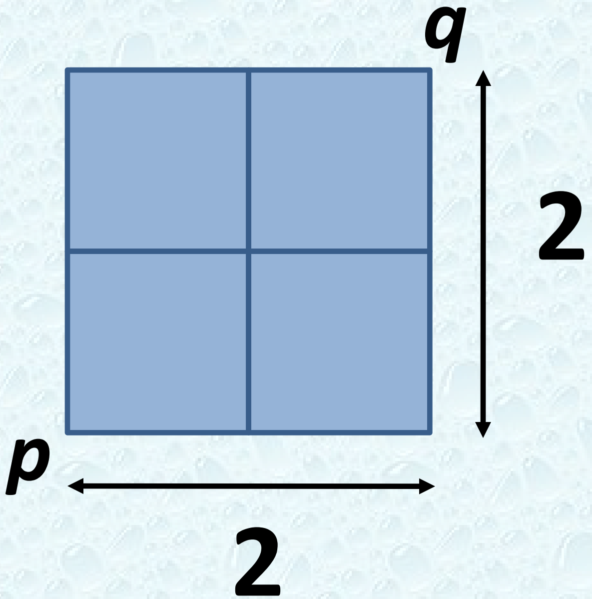
Going ‘back’ by moving downwards and left will not make it one of the shortest paths.



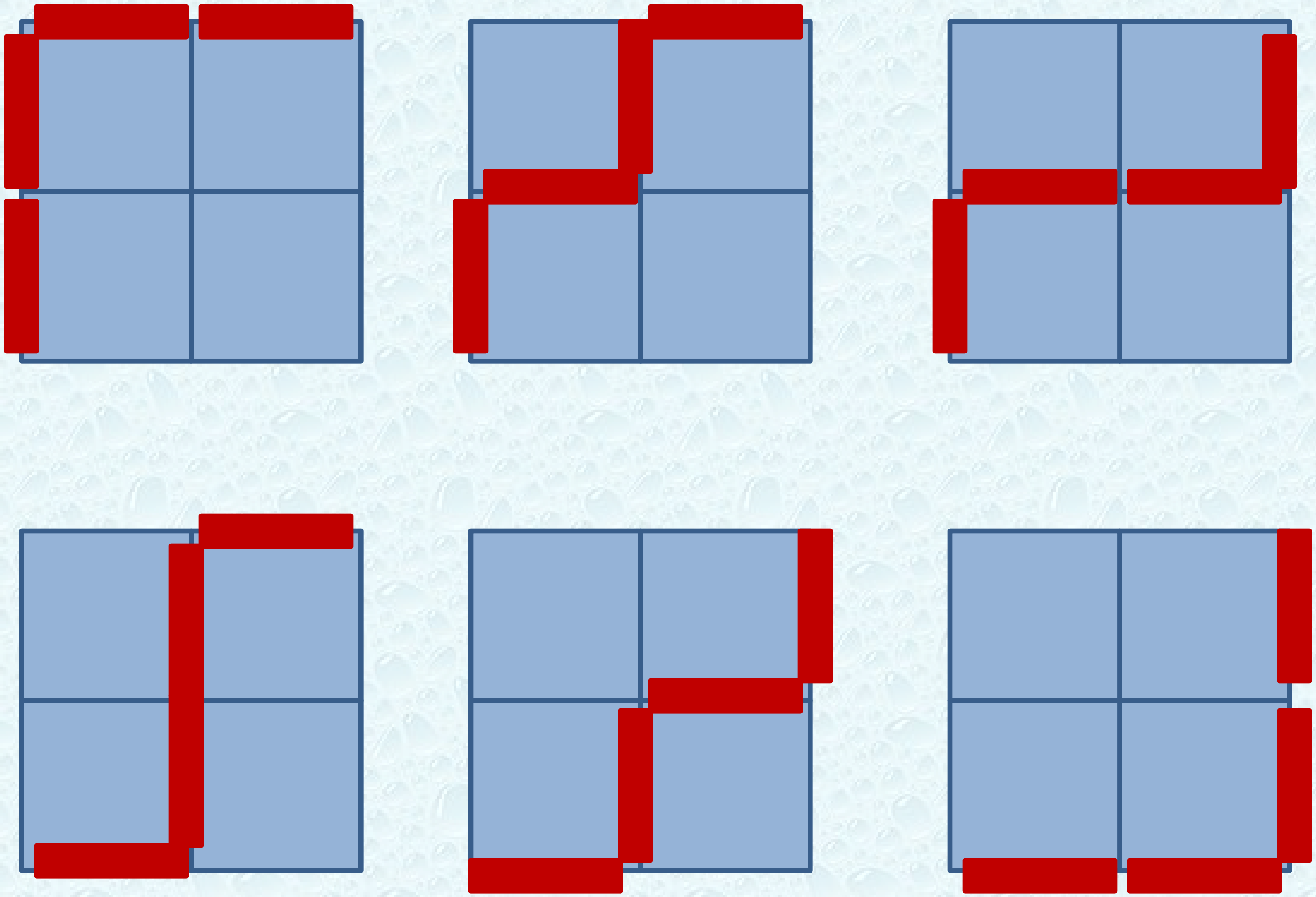


Let's try to count with simple examples first!

How many shortest paths are there from point p to q in 2 x 2 grid?



There are 6



To stop you from guessing…

There are **C(h+w, h)** shortest paths from point p to point q in our problem.

It is actually the same as C(h+w, w) because of the nature of the formula for Combination.

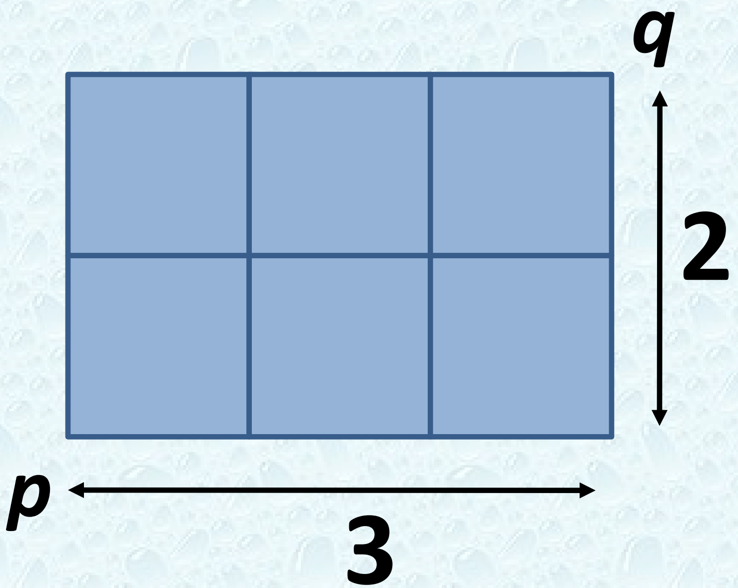
C(h+w, h) = (h+w)!/(h+w-h)!h!

C(h+w, w) = (h+w)!/(h+w-w)!w!

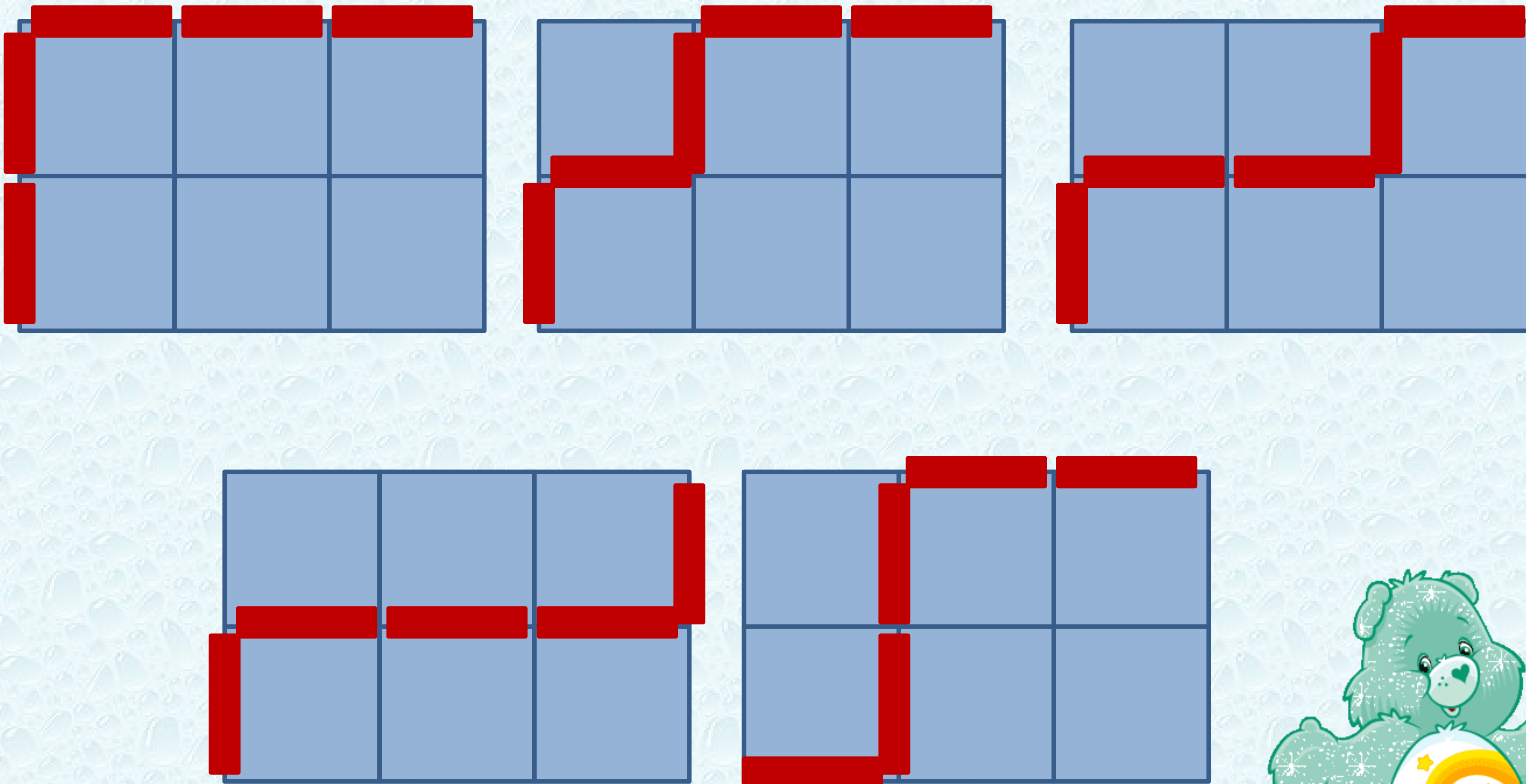
But just to make it uniform for all of us, we'll stick to the first formula.

Let's try our formula with a slightly bigger example

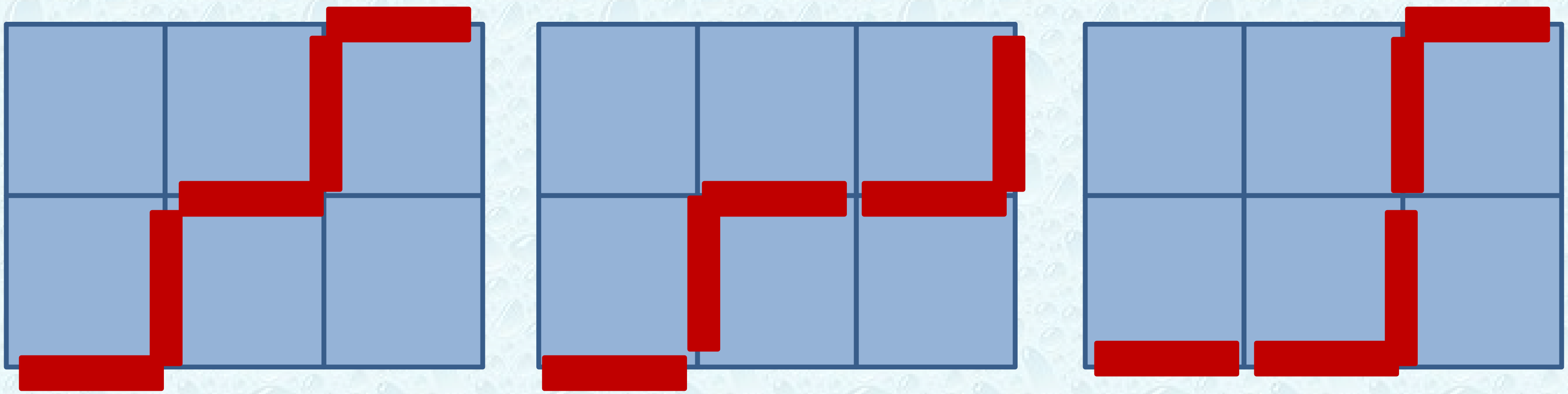
How many shortest paths are there from point p to q in 2 x 3 grid?

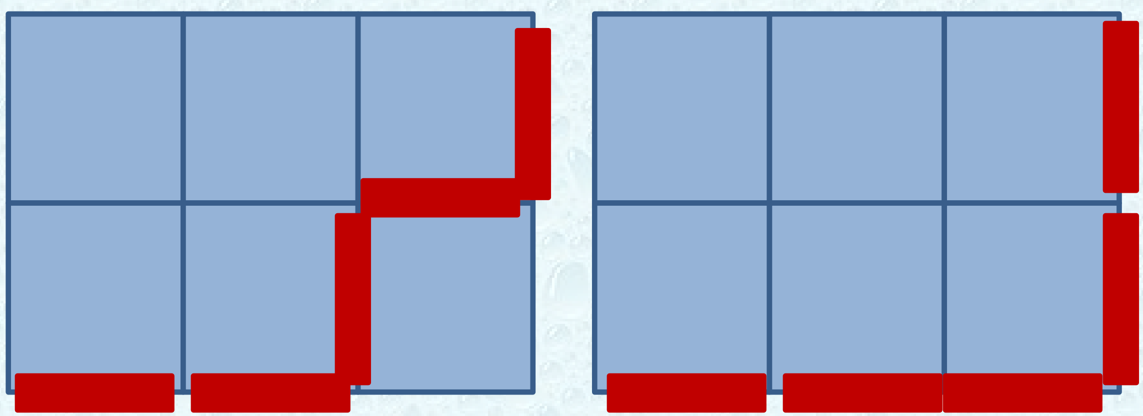


There should be C(2+3,2) ways



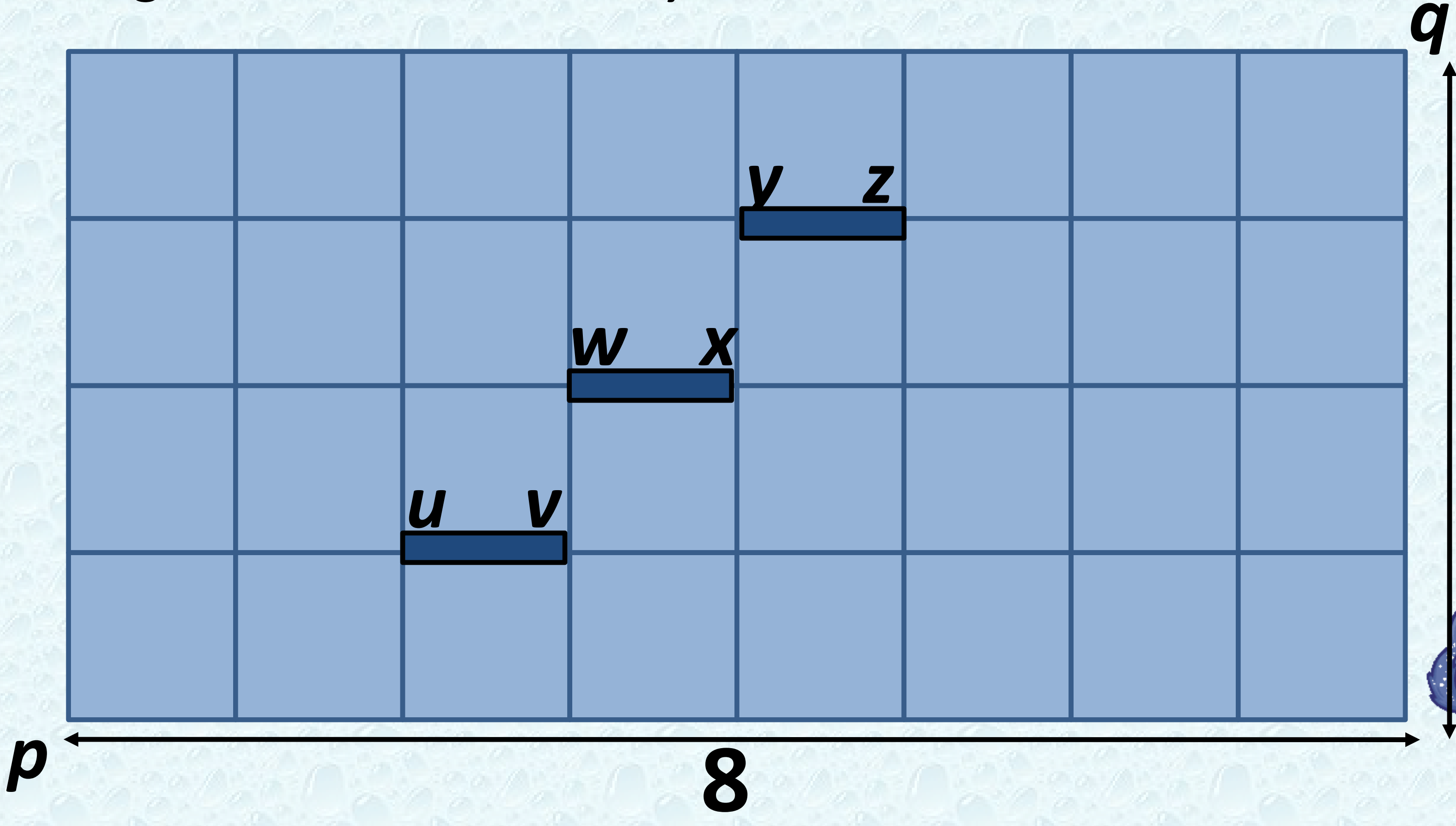
Yes, C(2+3,2) = 10





Example 3

The figure below shows a 4 by 8 rectangular grid with two specified corners p and q and three specified segments uv, wx and yz.



Example 3

Find in the grid

(i) *the number of shortest p–q routes;*

(ii) *the number of shortest p–q routes which pass through wx;*

(iii) *the number of shortest p–q routes which pass through at least one of the segments uv, wx and yz;*

(iv) *the number of shortest p–q routes which do not pass through any of the segments uv, wx and yz.*

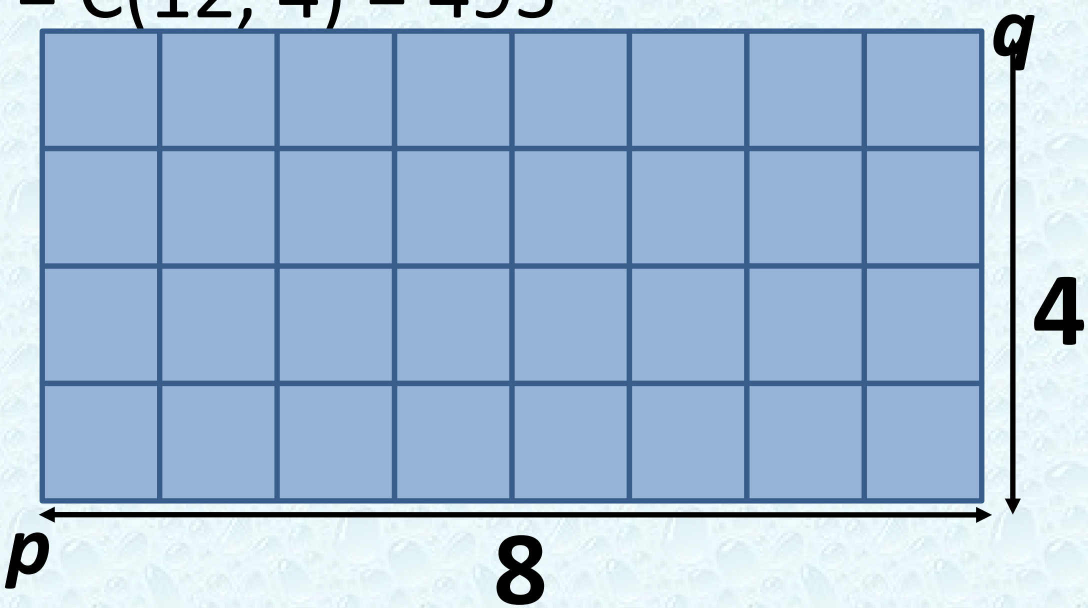
Example 3.*i*

Find in the grid

(i) *the number of shortest p–q routes;*

Employing the idea in our mini-examples, we have

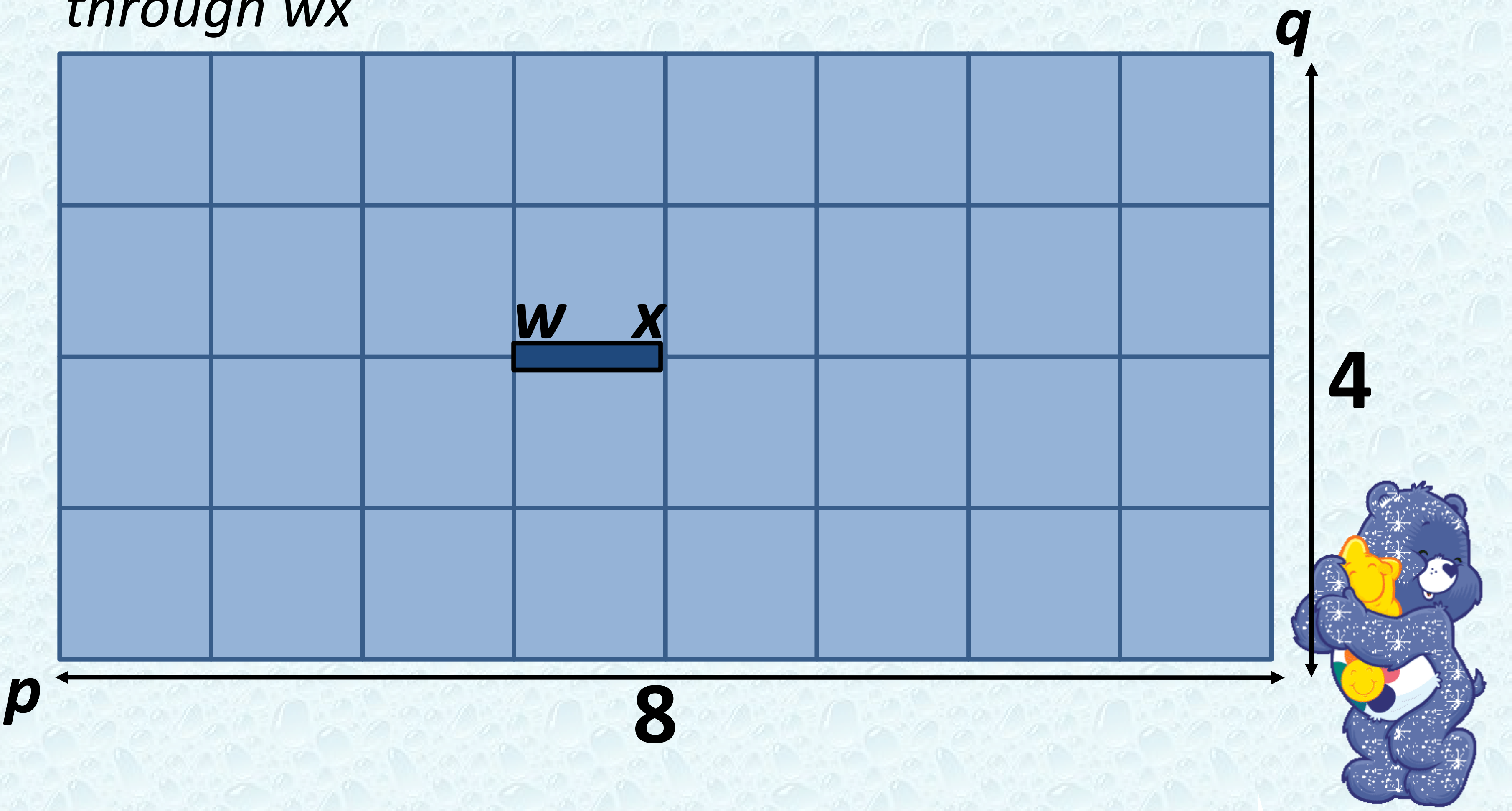
C(4+8, 4) = C(12, 4) = 495



Example 3.*ii*

Find in the grid

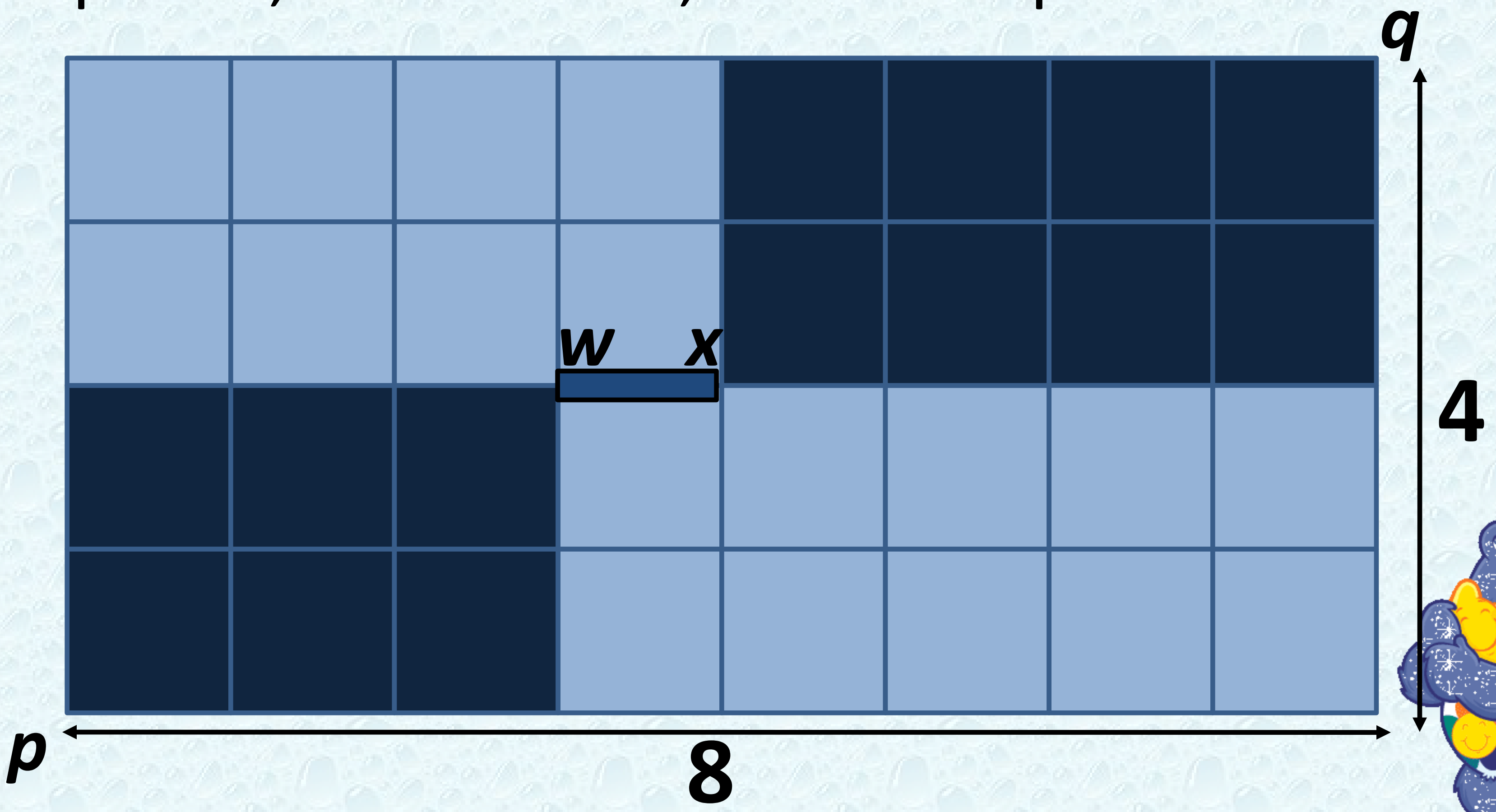
(ii) *the number of shortest p–q routes which pass through wx*



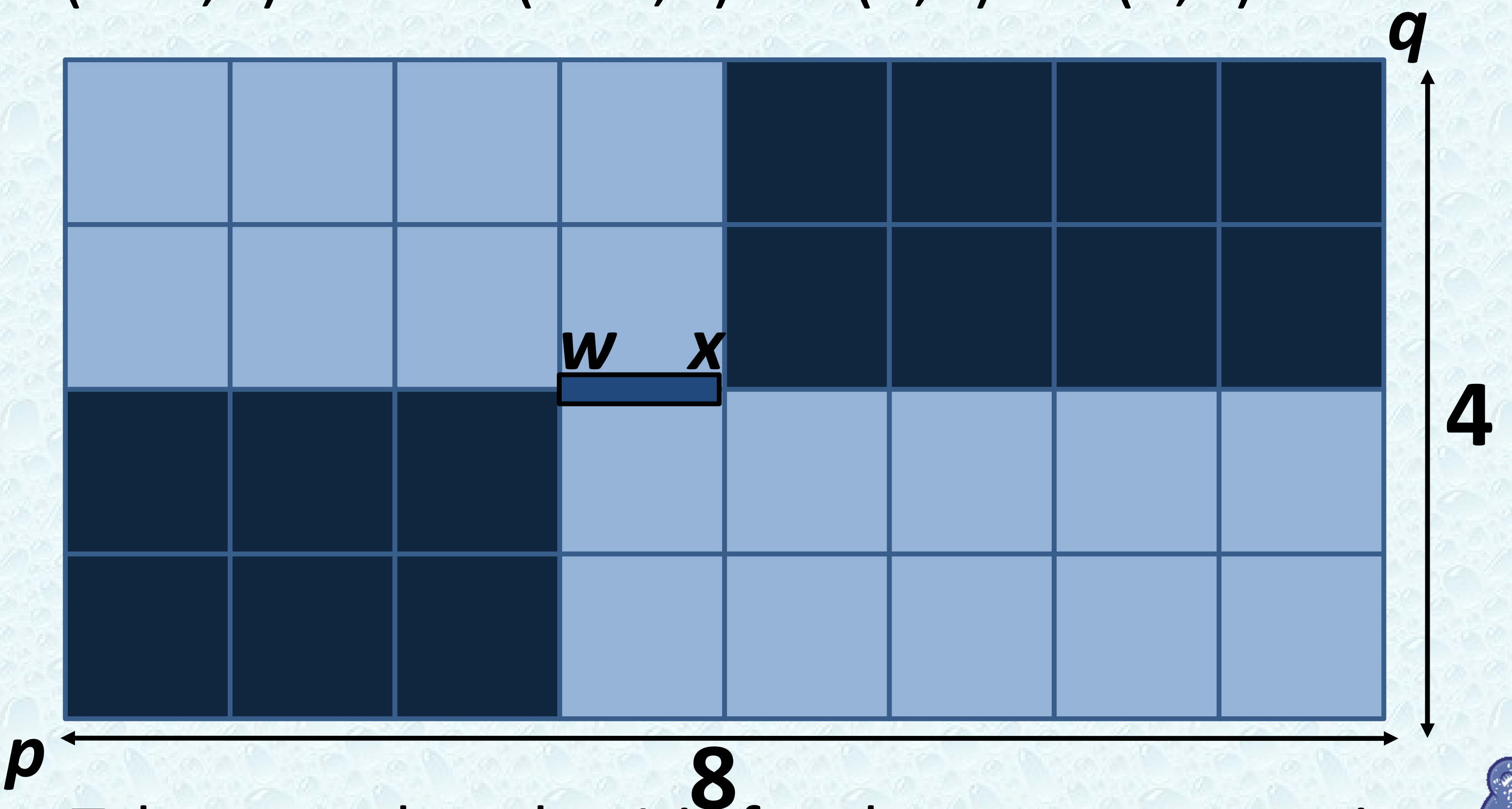
we need to pass through w-x!

This breaks our grid from

p to w;   w to x and;   x to q



$C(2+3,2) * 1 * C(2+4,2) = C(5,2) * C(6,2) = 10*15$



Take note that the 1 is for the segment wx since there is only 1 way to go from w to x.

Example 3.iii

Find in the grid

(iii) *the number of shortest p–q routes which pass through at least one of the segments uv, wx and yz;*

The counting is more complicated in this case. We introduce three subsets of the set of shortest *p–q routes below.* Let

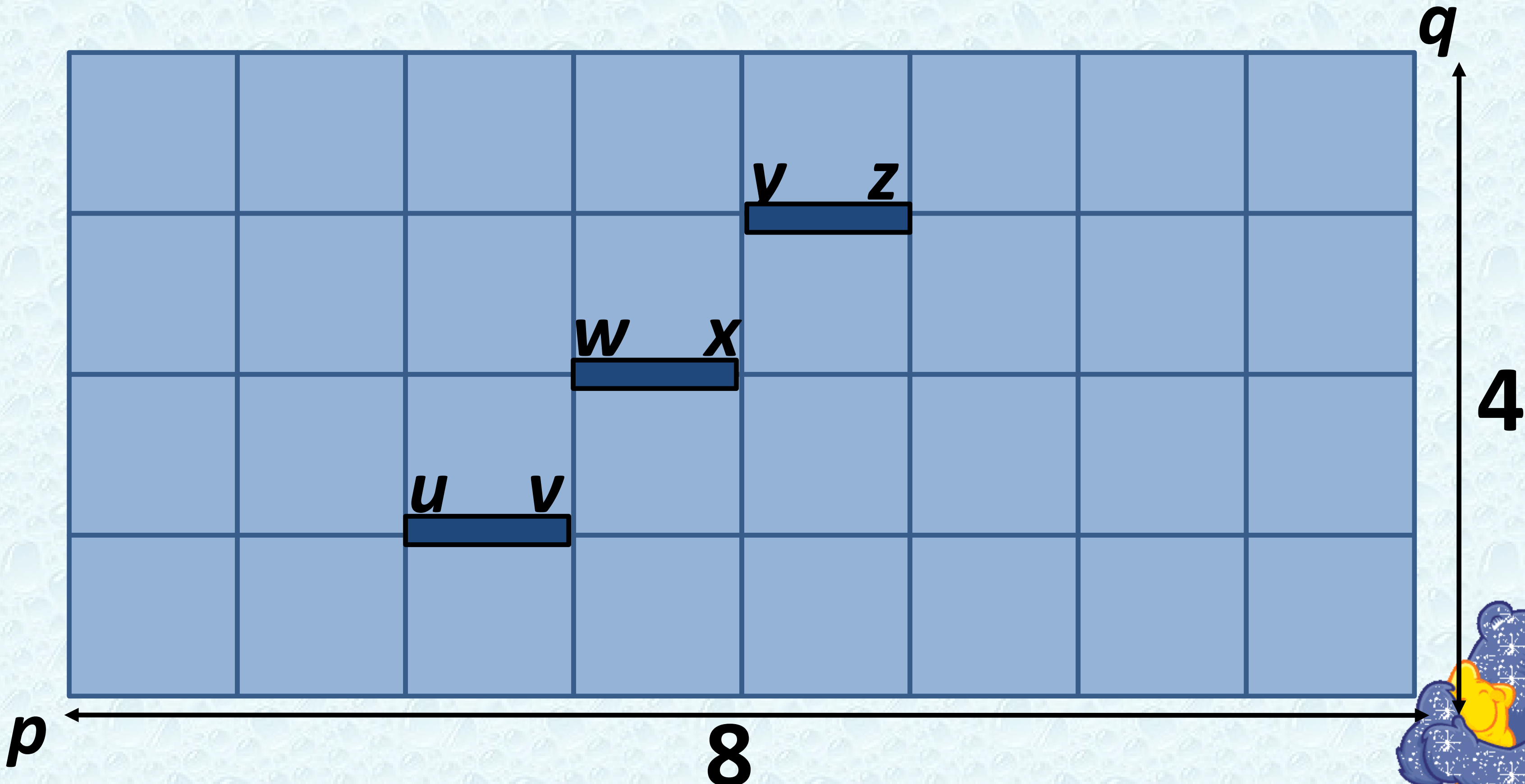
A be the set of shortest p–q routes which pass through uv,

B be the set of shortest *p–q routes which pass through wx, and*

C be the set of shortest *p–q routes which pass through yz.*

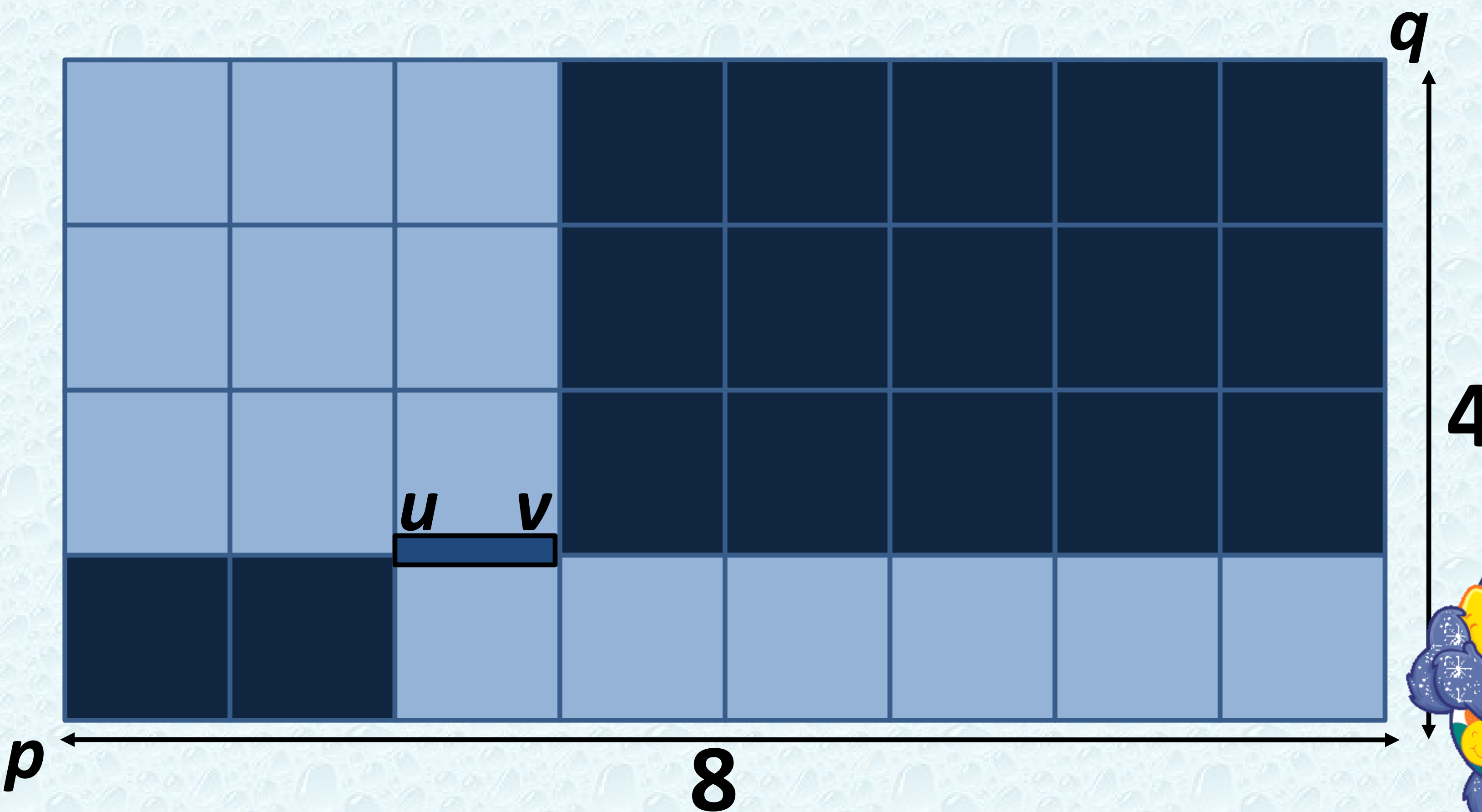
We note that the answer we are looking for **is not** $|A|+|B|+|C|$ *as the sets A, B, C* are **not pairwise disjoint**. The desired answer **should be** $|A \cup B \cup C|$, and this gives us a chance to apply (1.3).

To apply (1.3), we need to evaluate each term on the right-hand side of (1.3).



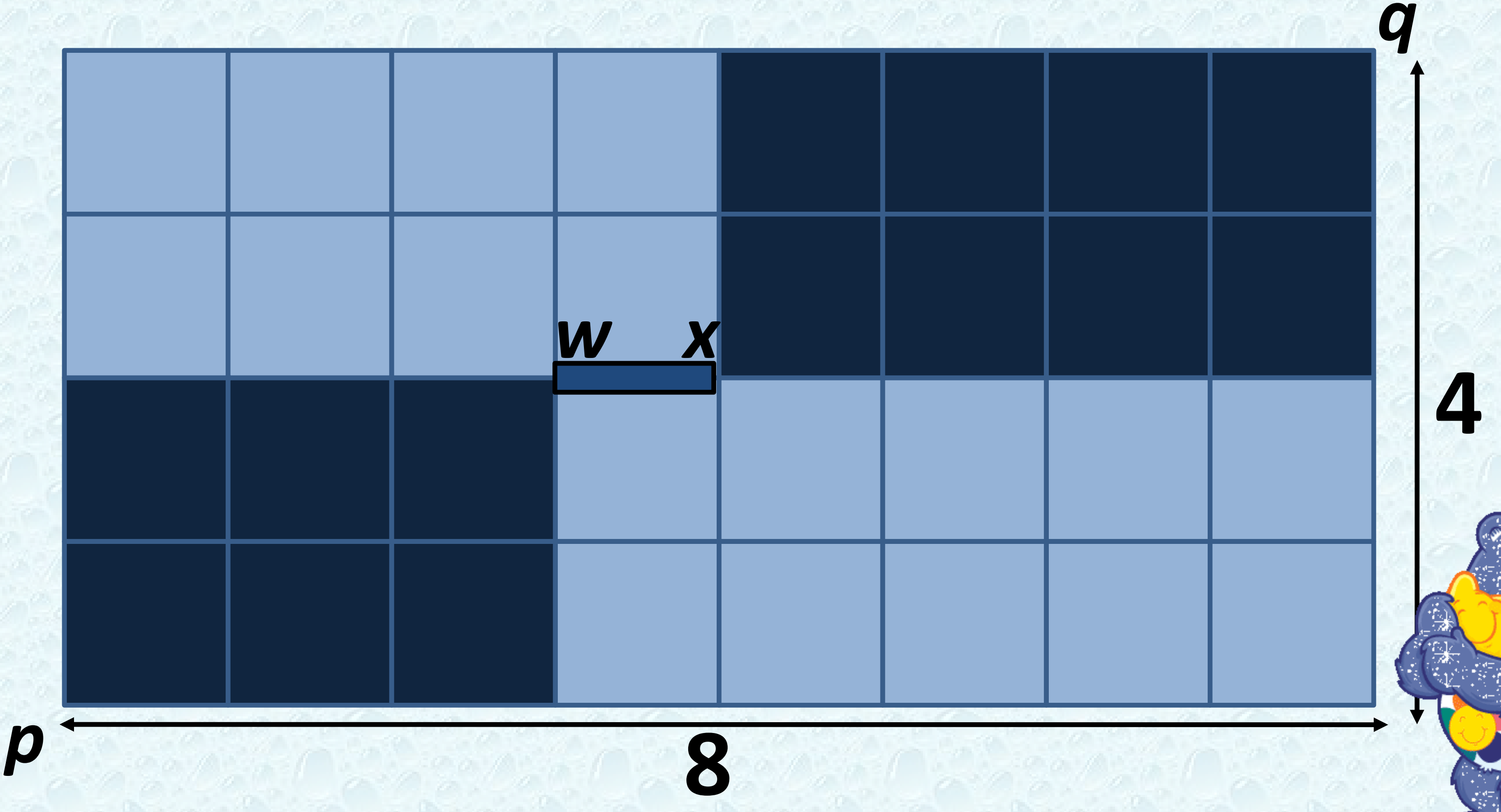
A = the set of shortest p–q routes which pass through uv

$|A| = C(3,2)*1*C(8,3) = 3*56 = 168$



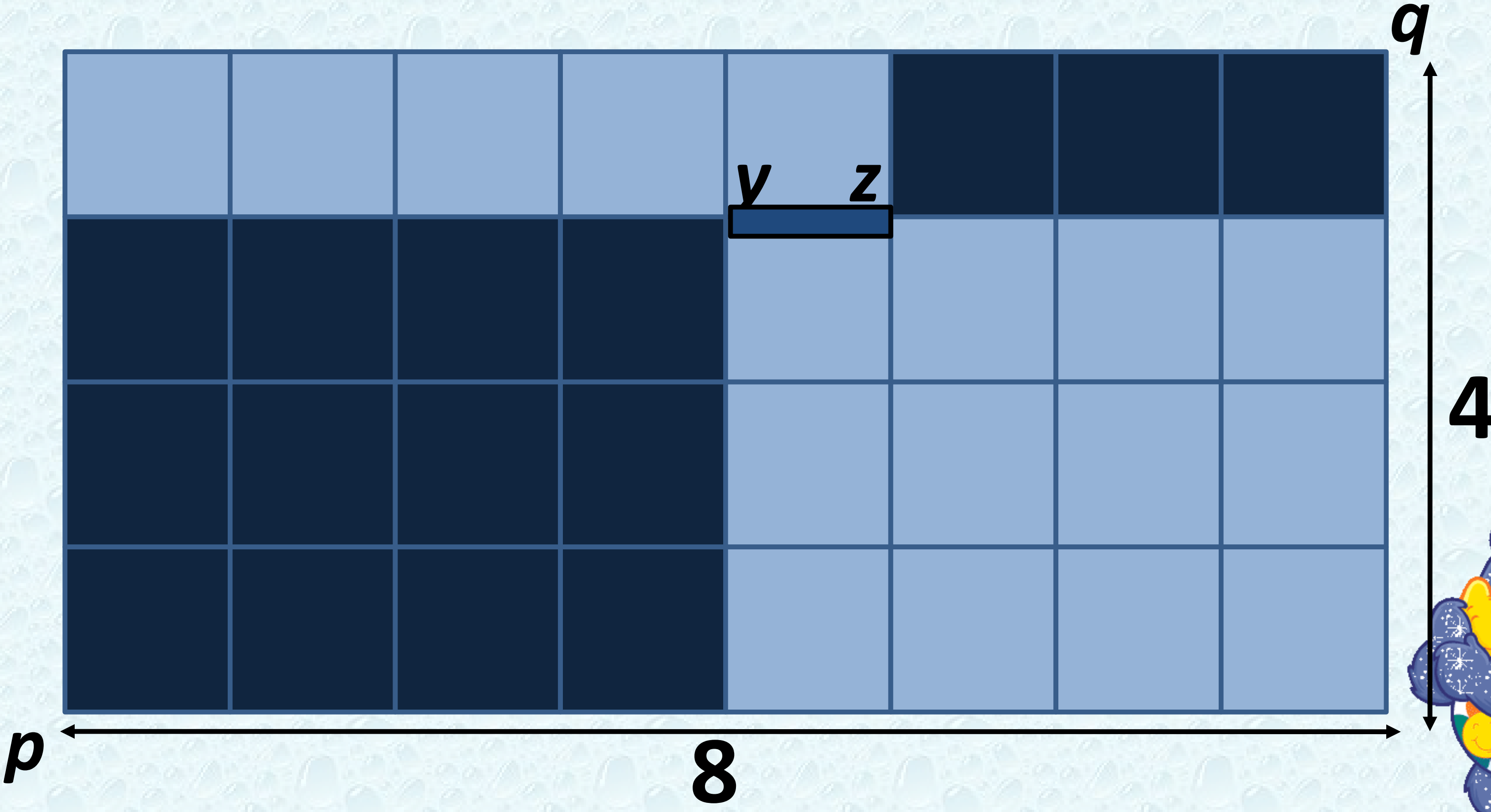
B= the set of shortest *p–q routes which pass through wx*

|B| = C(5,2)*1*C(6,2) = 10*15 = 150



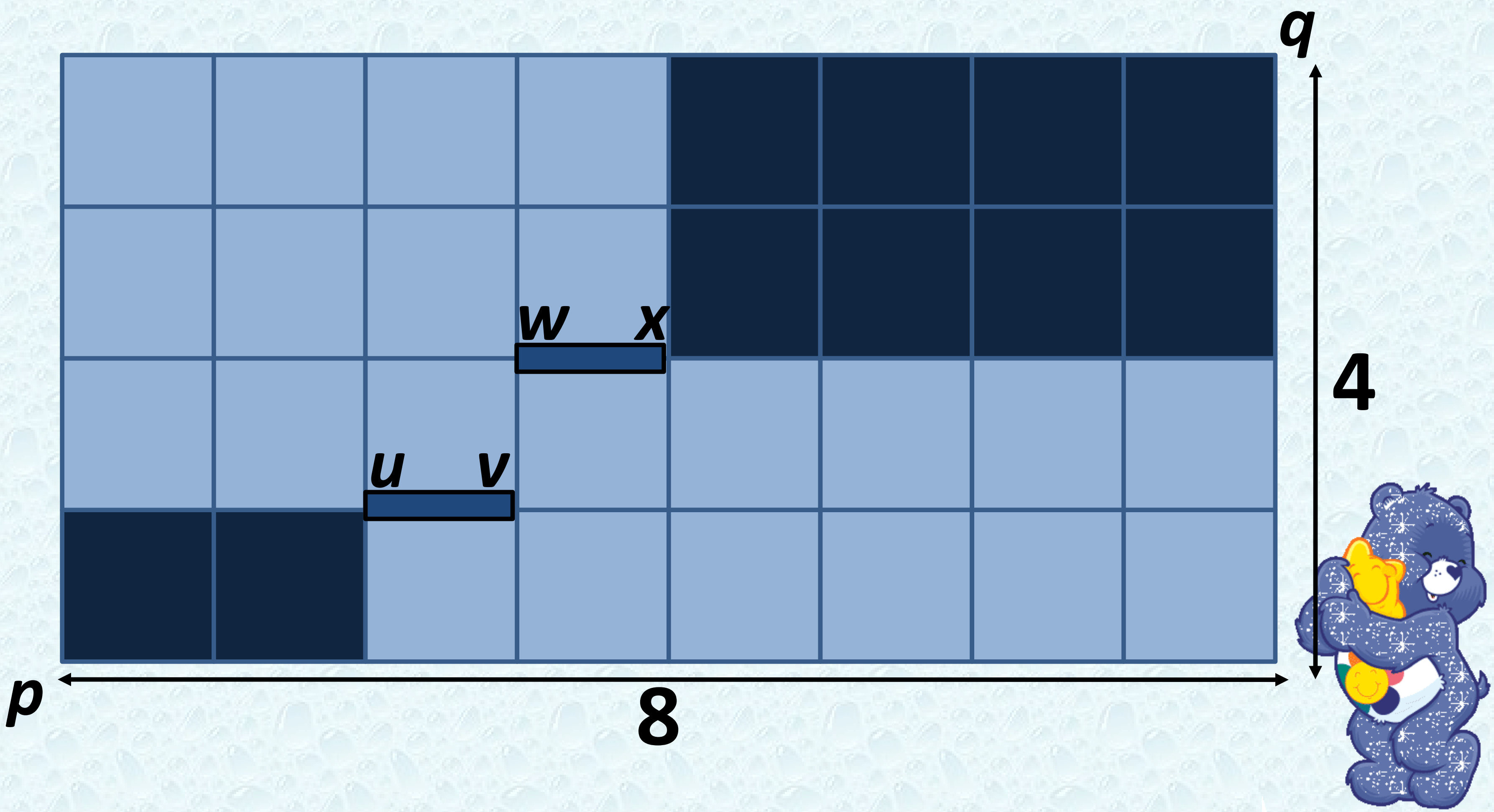
C = the set of shortest *p–q routes which pass through yz.*

|C| = C(7,3)*1*C(4,1) = 35*4 = 140



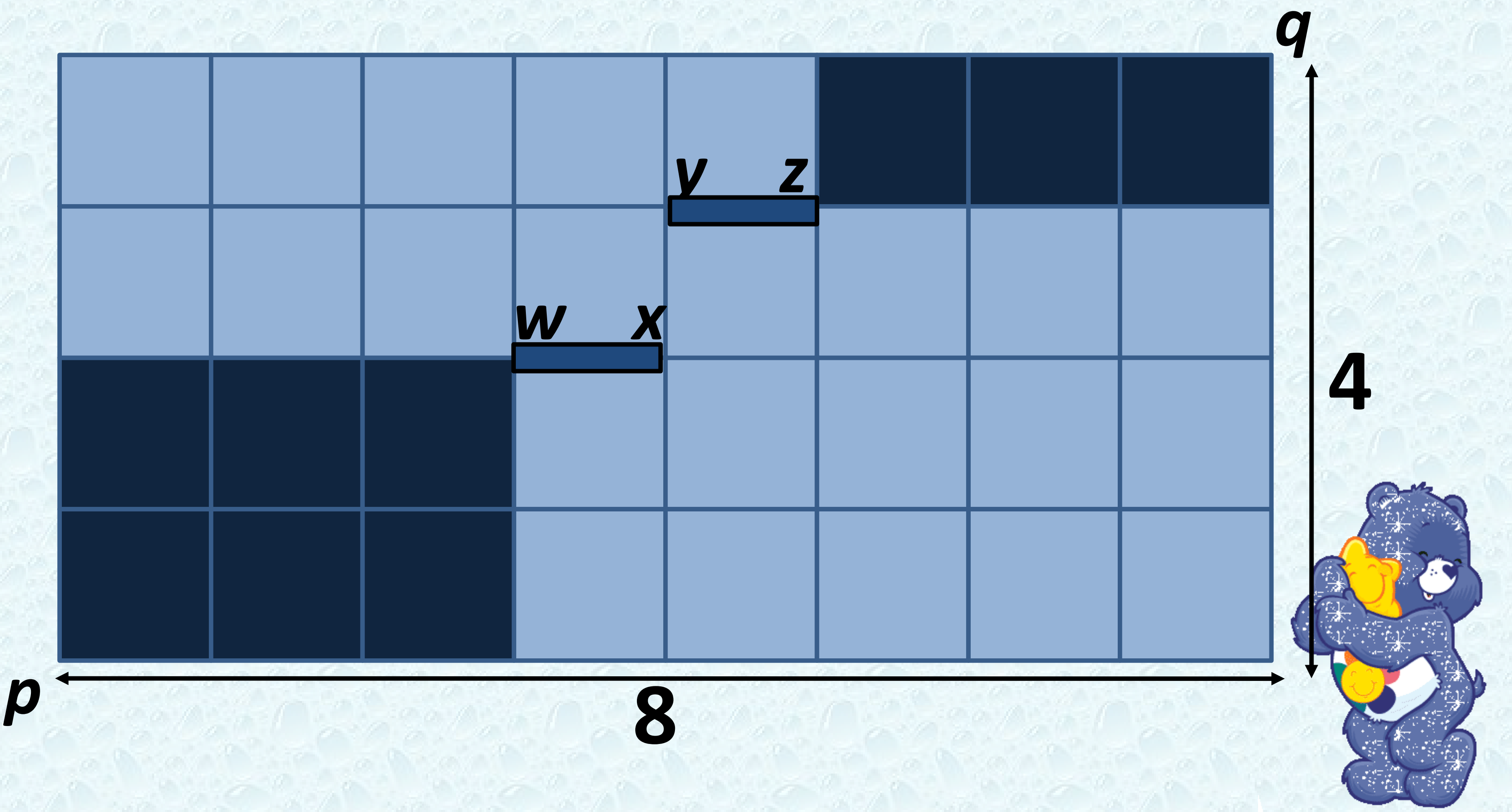
A ∩ B = the set of shortest p–q routes which pass through uv and wx

$|A \cap B| = C(3,1)*1*C(6,4) = 3*15 = 45$



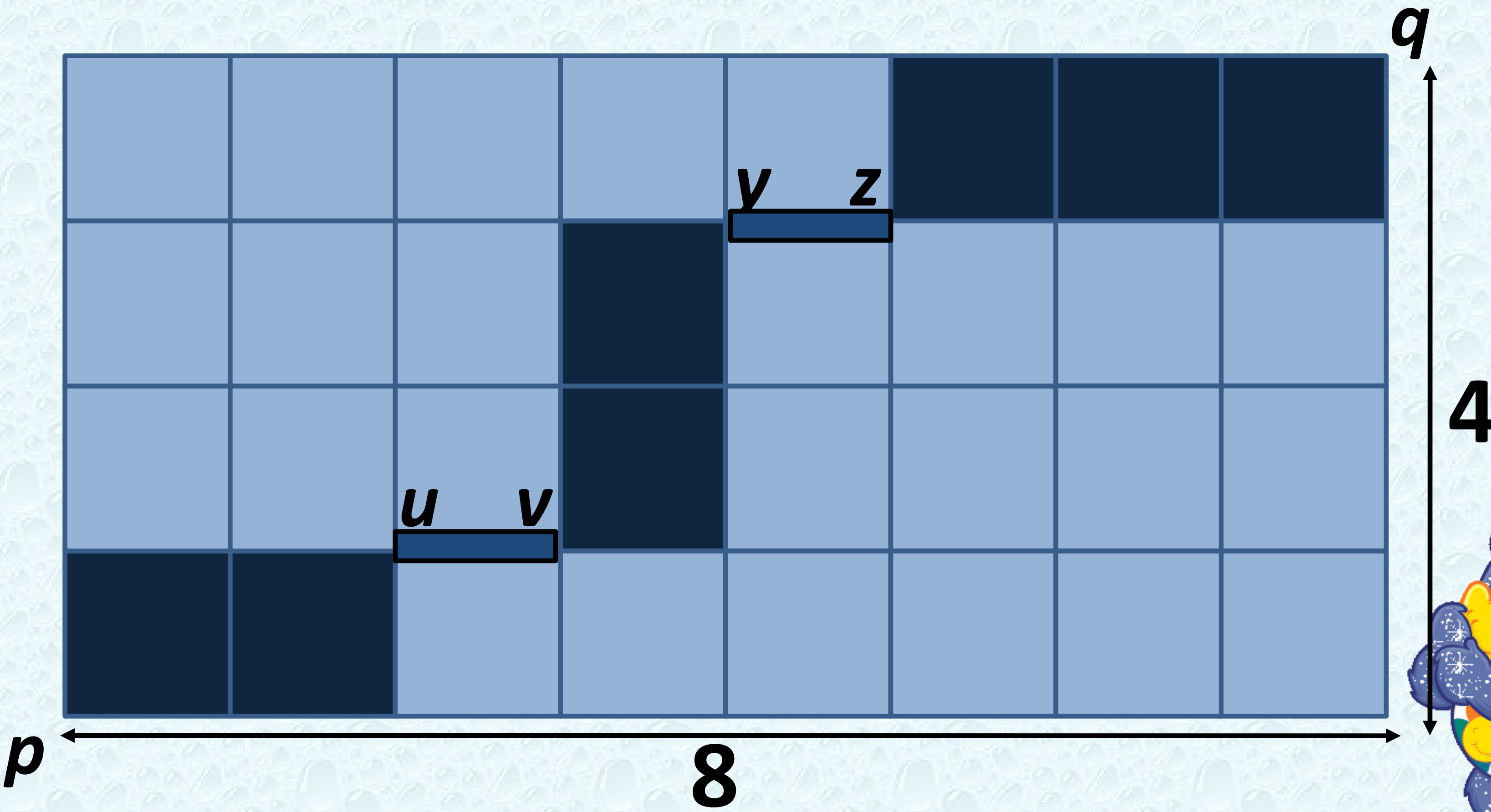
B ∩ C = the set of shortest p–q routes which pass through wx and yz

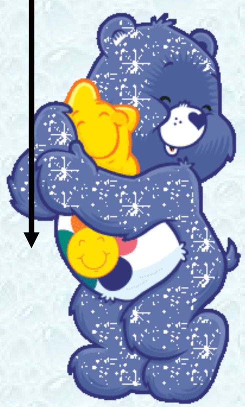
$|B \cap C| = C(5,2)*1*C(4,1) = 10*4 = 40$



A ∩ C = the set of shortest p–q routes which pass through uv and yz

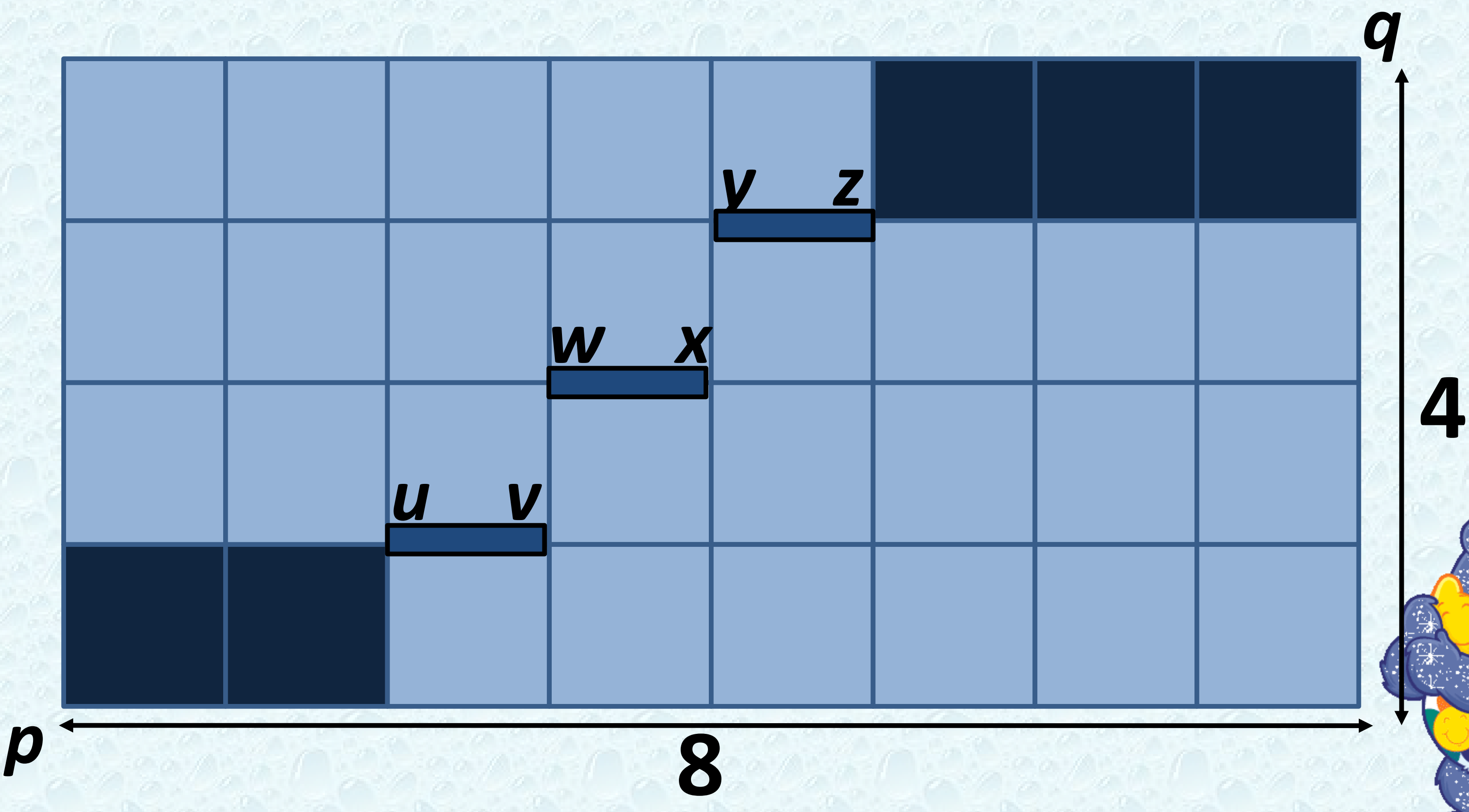
$|A \cap C| = C(3,1)*C(3,2)*C(4,1) = 3*3*4 = 36$





$A \cap B \cap C$ = *set of shortest p–q routes which pass through uv, wx, and yz*

$|A \cap B \cap C| = C(3,1)*1*C(4,1) = 3*4 = 12$



So finally our answer…

$$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$$

$$= 168 + 150 + 140 - (45 + 40 + 36) + 12$$

$$= 349$$

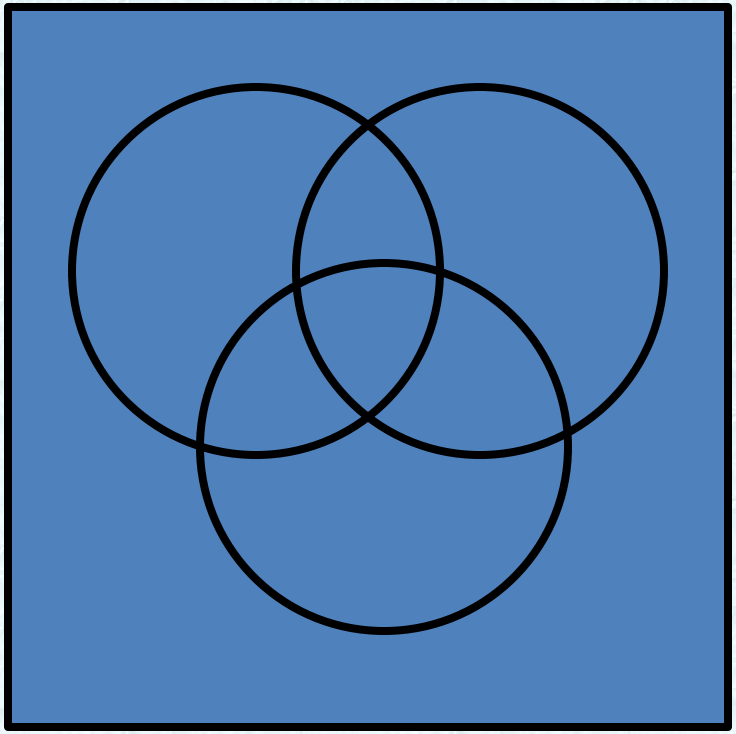
Example 3.iv

Find in the grid

(iv) *the number of shortest p–q routes which do* ***not*** *pass through any of the segments uv, wx and yz.*

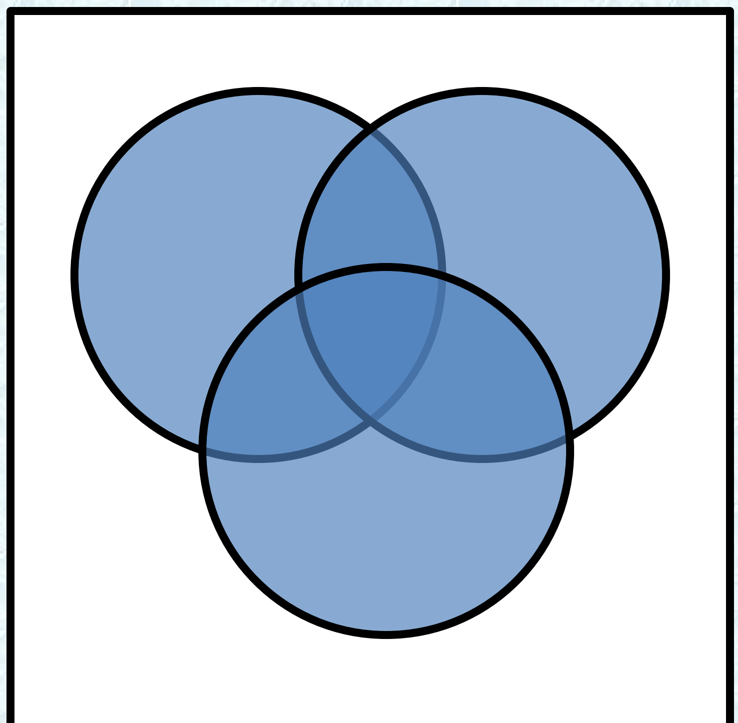
We are getting the reverse of the union!!!

In example 3.i, we got the whole universe of the problem, the general case without restrictions.



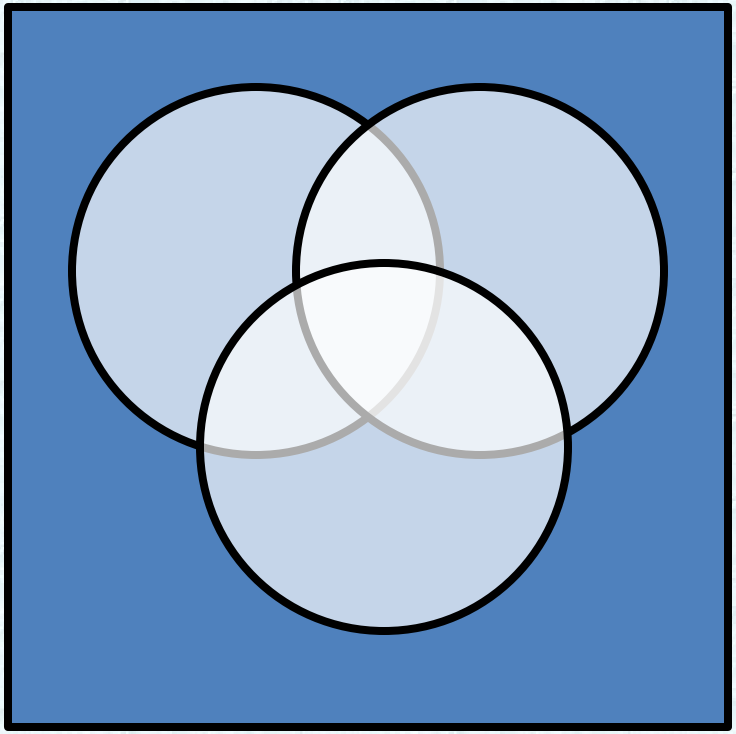
In the previous example…

In example 3.iii, we got the union of the 3 sets.

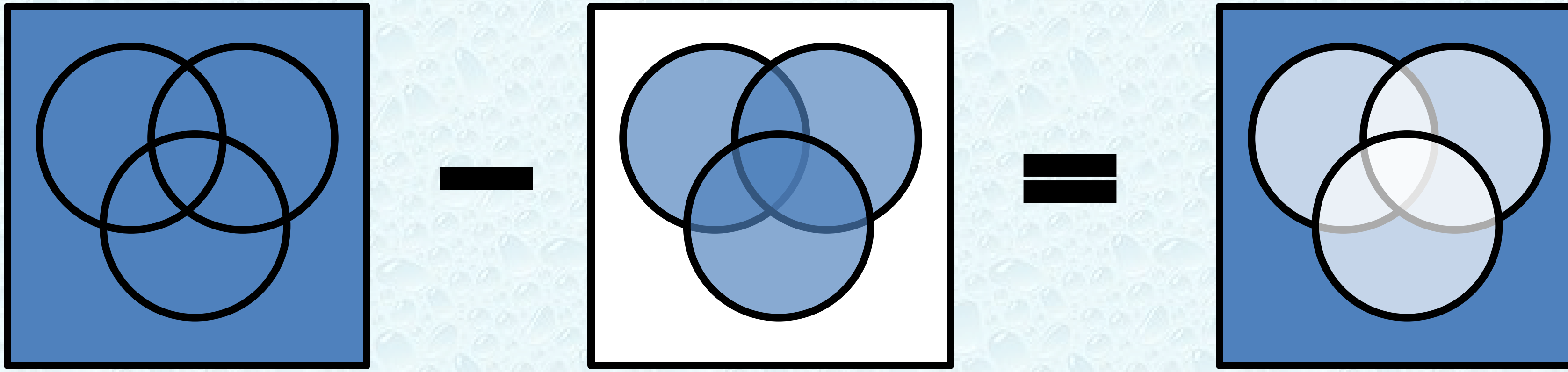


Again, we are getting the reverse of the union!!!

We are interested in getting to point q from point p without passing through any of the segments uv, wx, and yz.



So we simply subtract our answer in 3.iii from 3.i to get 3.iv



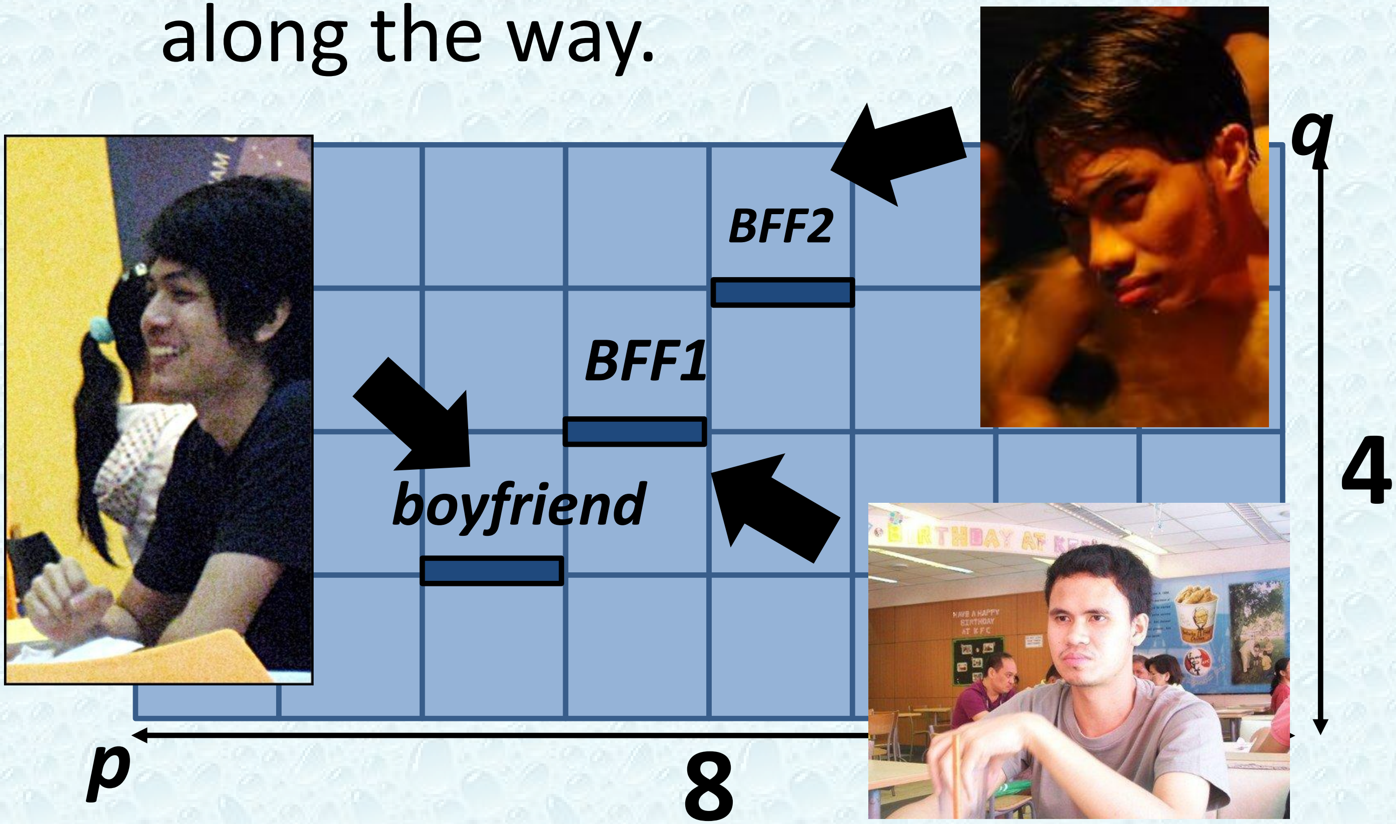
495 - 349 = 146

So where do we apply this problem?

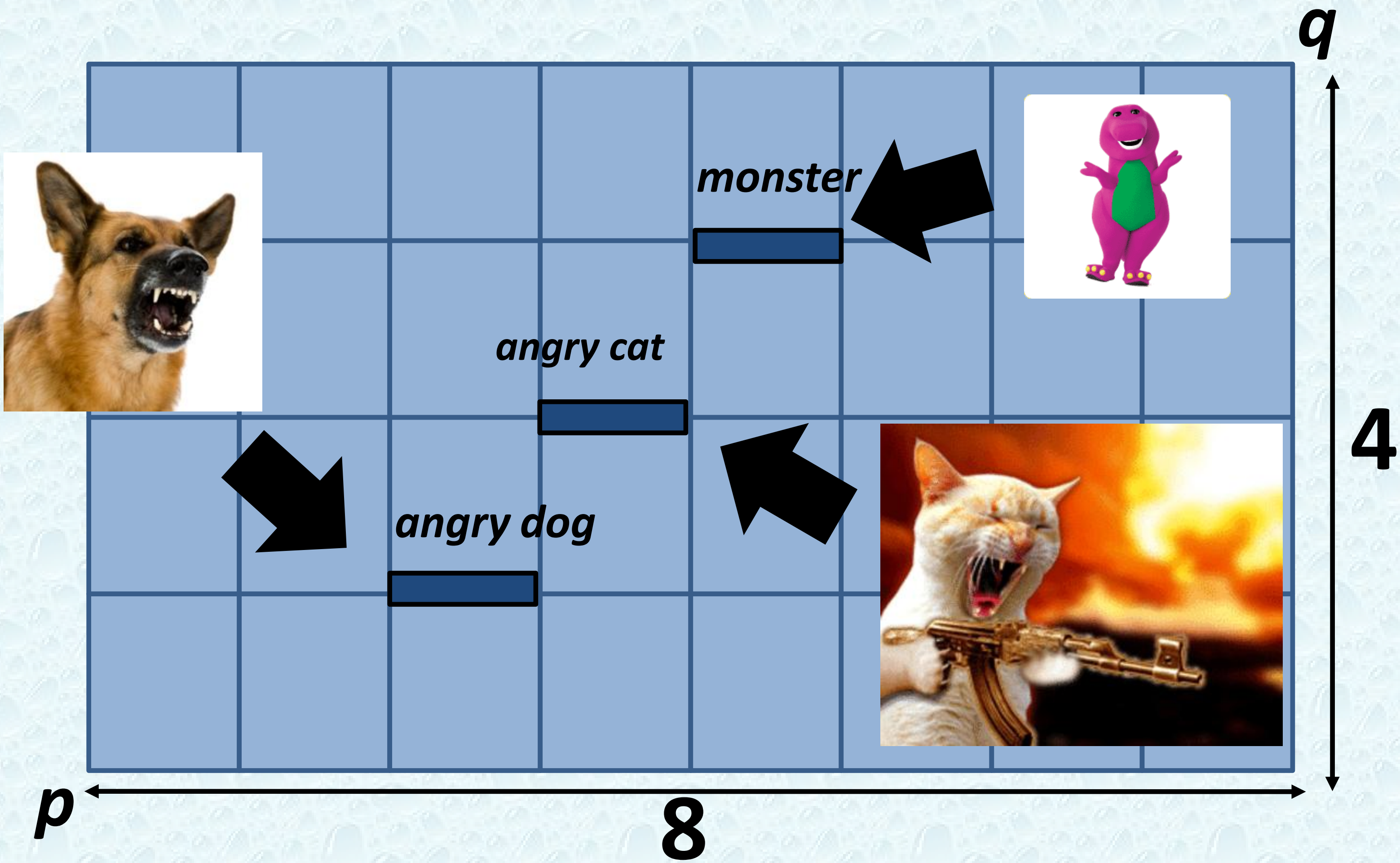
Streets are mostly designed in grids. It is just like getting from your house to your school in the shortest possible way, but you want to take different routes everyday so that you are not bored.

What about those with restrictions?

Just imagine you want to go to school, but you need to fetch your boyfriend and your BFF's along the way.



And for the reverse of that, the streets you avoid…



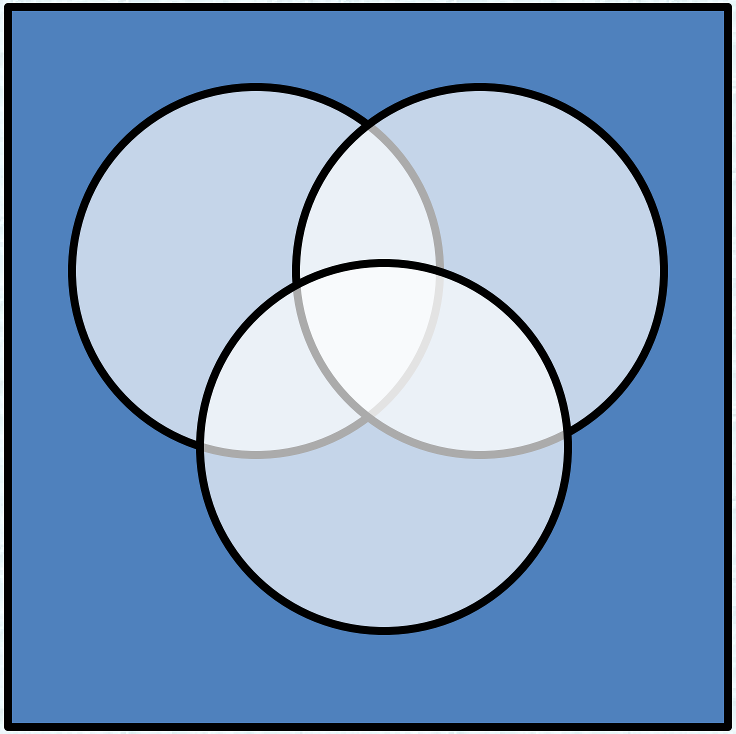
Theorem 1.4

$$|\neg A \cap \neg B \cap \neg C|$$

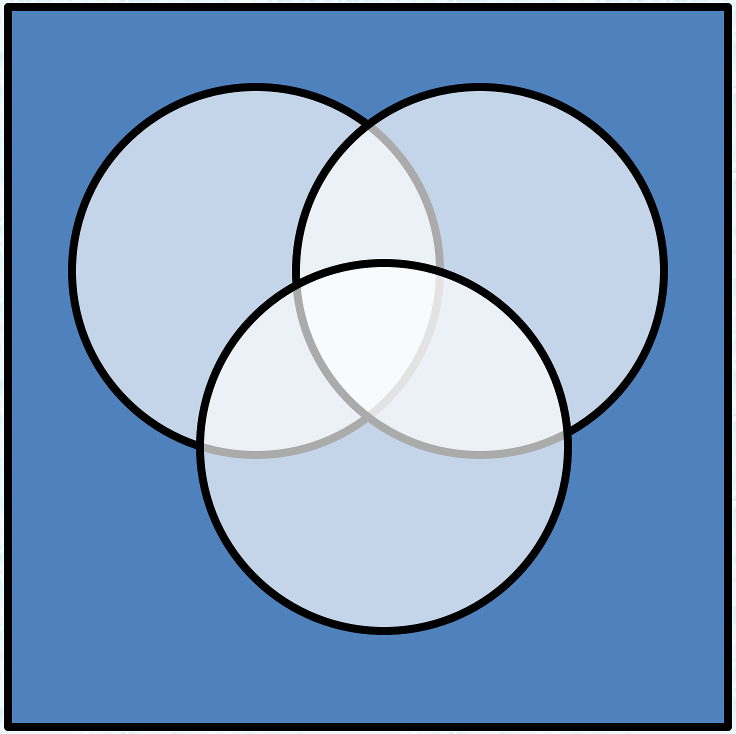
$$= |U| - \Big[(|A| + |B| + |C|) - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|\Big]$$

$$= |U| - (|A| + |B| + |C|) + (|A \cap B| + |A \cap C| + |B \cap C|) - |A \cap B \cap C|$$

Yep, Theorem 1.4 is just stating that the intersection of the "not" areas is the colored area in the following figure

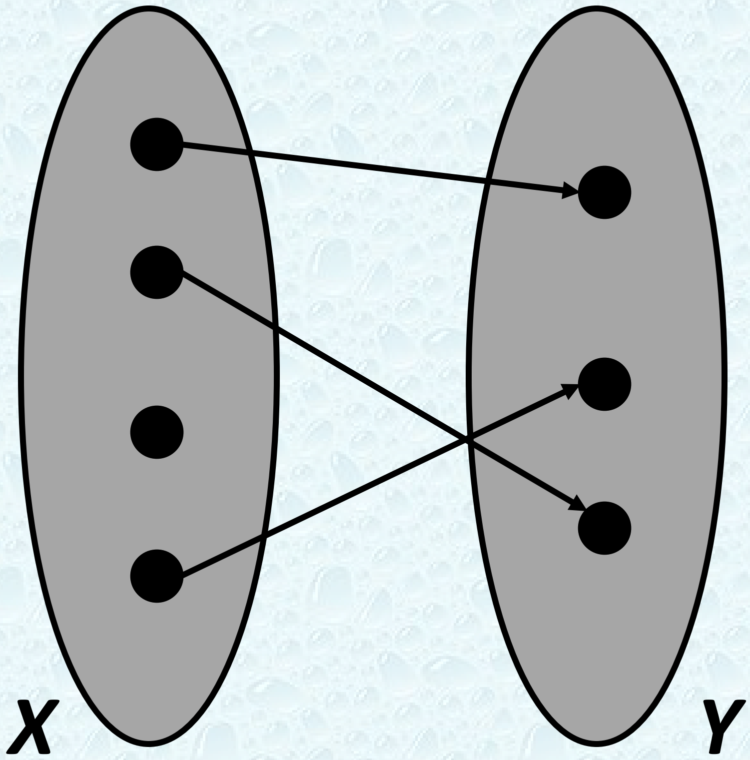


Sometimes, it is better to state the sets in the negative form as in our next example. But before that…



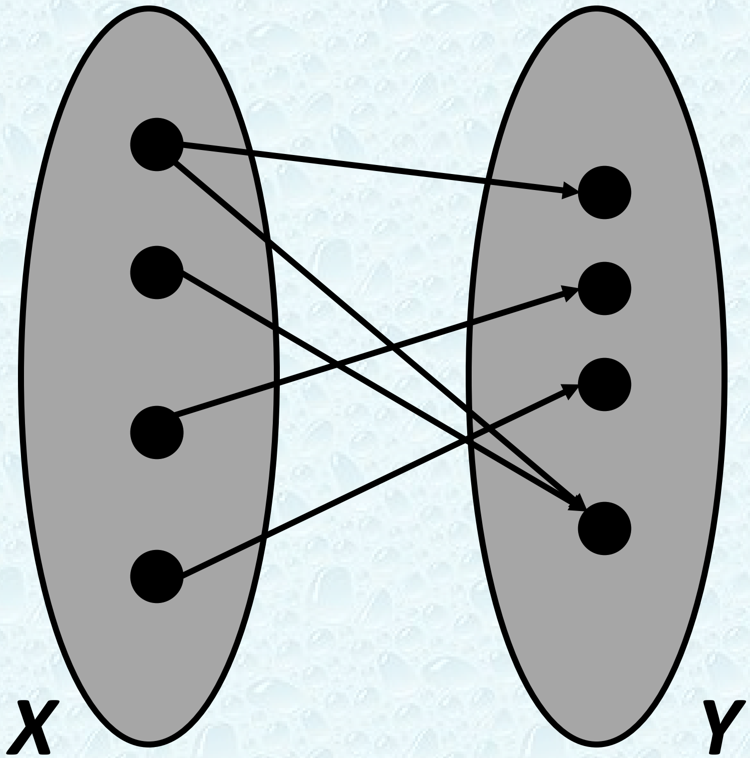
…let's go back to not too distant memory lane. (CMSC 56)

This is not a function since not all elements of X are mapped to Y



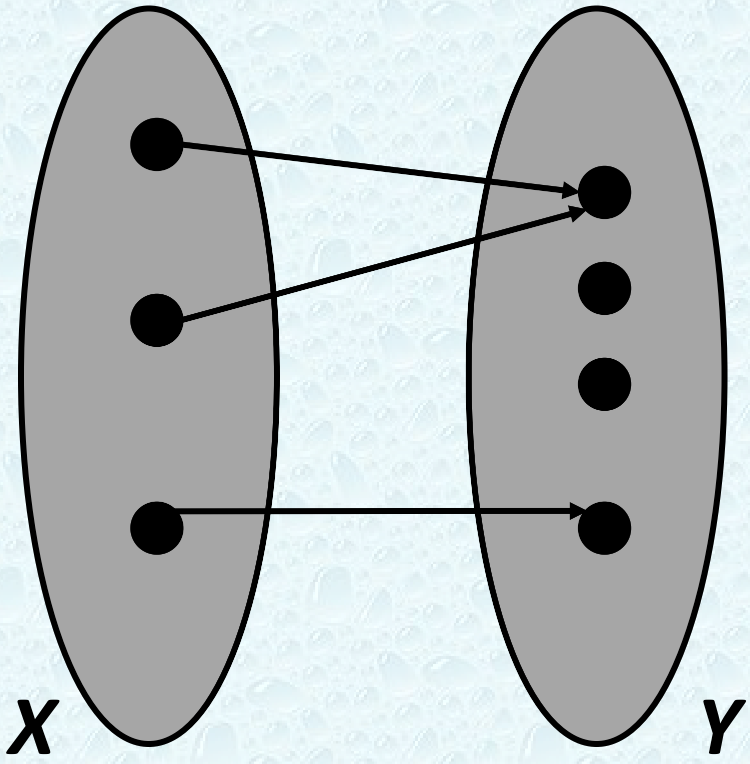
The figure below is still not a function…

…because some elements of X are mapped more than once to the elements of Y



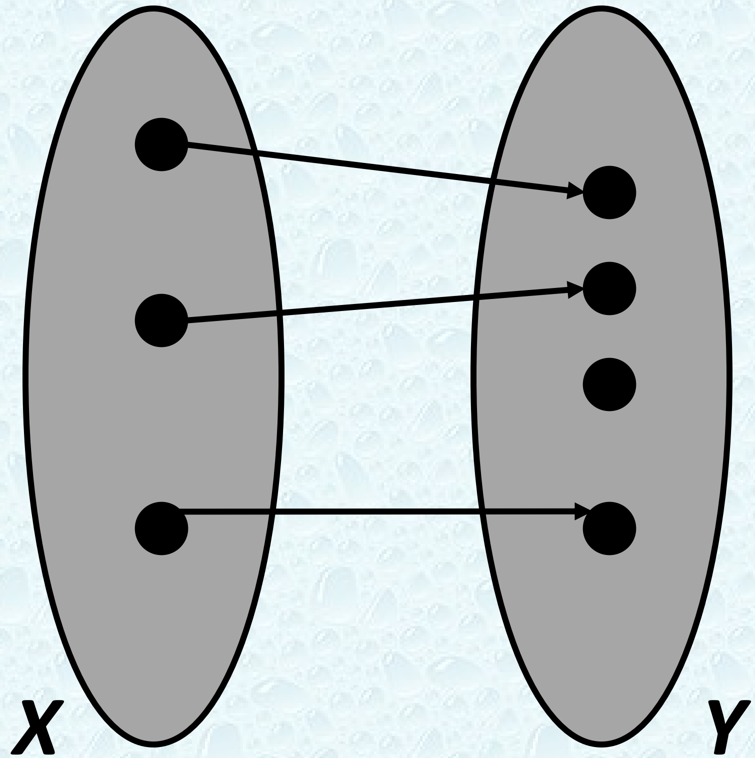
Finally, the figure below is a function

BUT it is **NOT an injection nor a surjection**.



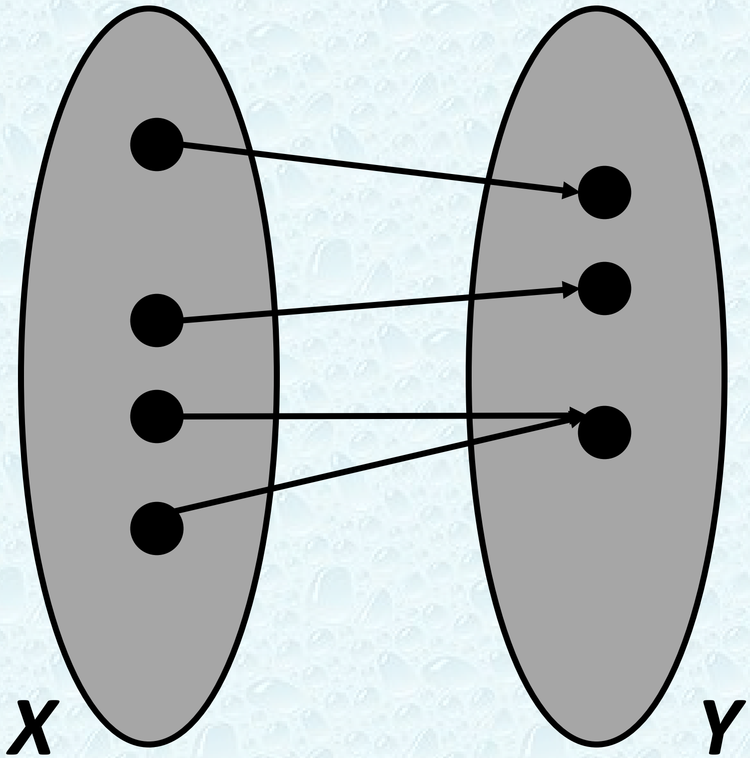
The function below is an INJECTION

All elements of Y are mapped **at most once**.



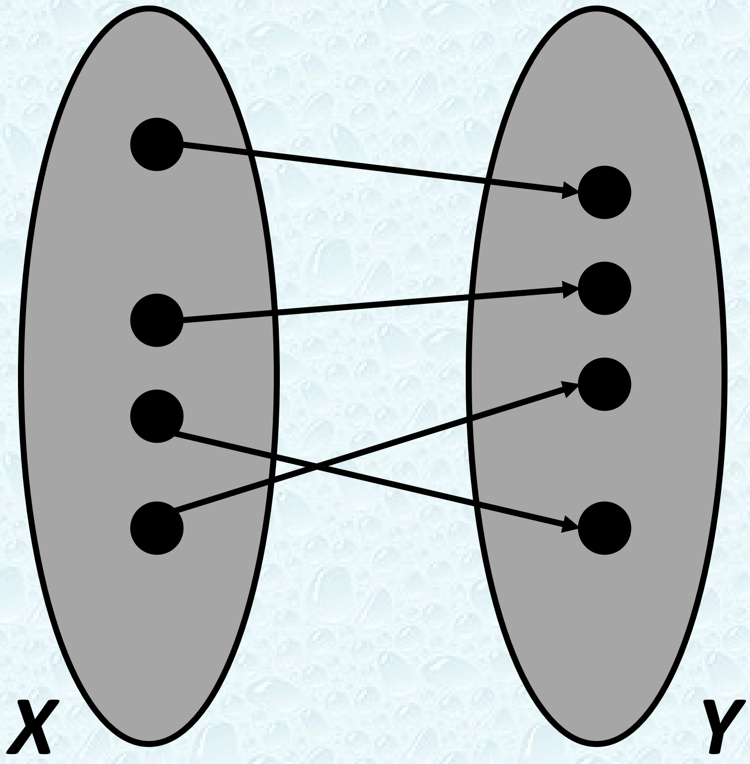
The function below is a SURJECTION

All elements of Y are mapped **at least once**.



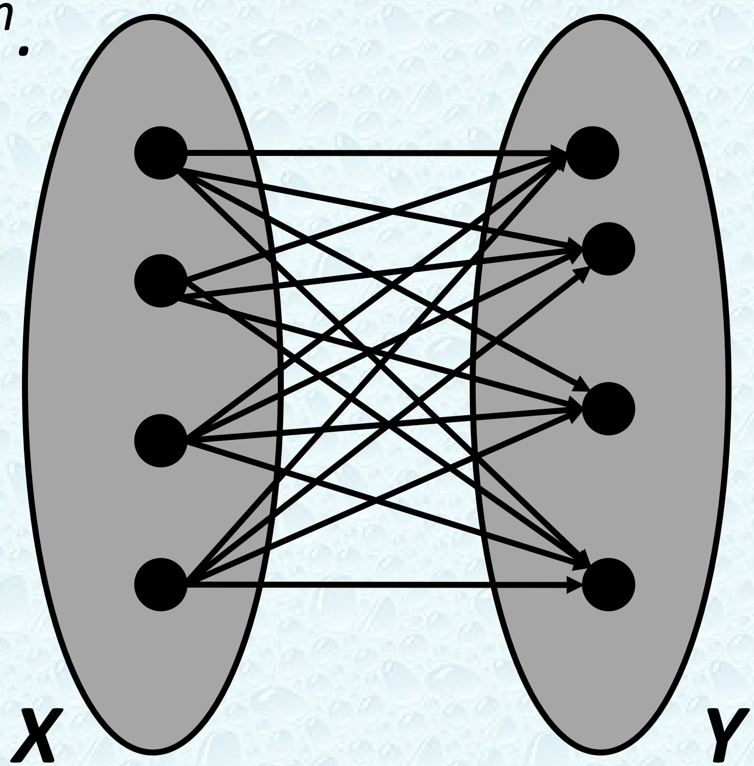
The function below is a BIJECTION

It is an INJECTION and a SURJECTION at the same time.



Once again, another generalization

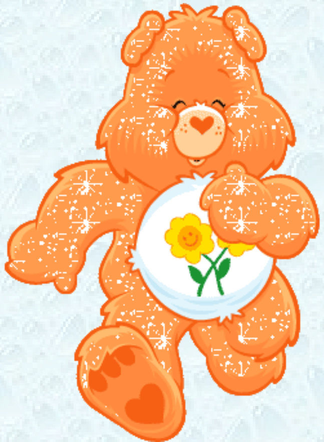
Suppose $X = \{1, 2, . . . ,m\}$ *and* $Y = \{1, 2, . . . , n\}$. *Then* the number of mappings from X *to* Y *is given by* n^m.



Example 4

Let $X = \{1, 2, 3, 4, 5\}$ and
$Y = \{1, 2, 3\}$.

Find the number of ***onto mappings*** *from X to Y.*



How do we solve the problem?

Onto mappings mean that each element of Y (in this example, {1,2,3}) should be mapped **at least once**. And since |X|>|Y| (X={1,2,3,4,5}), we are sure that an onto mapping is possible. Again, we just need to make sure that each element in Y is mapped **at least once**.

How de we relate 1.4 to this problem?

It is easier to state the negative. We let

A = {mappings that do not map 1 in Y}

B = {mappings that do not map 2 in Y}

C = {mappings that do not map 3 in Y}

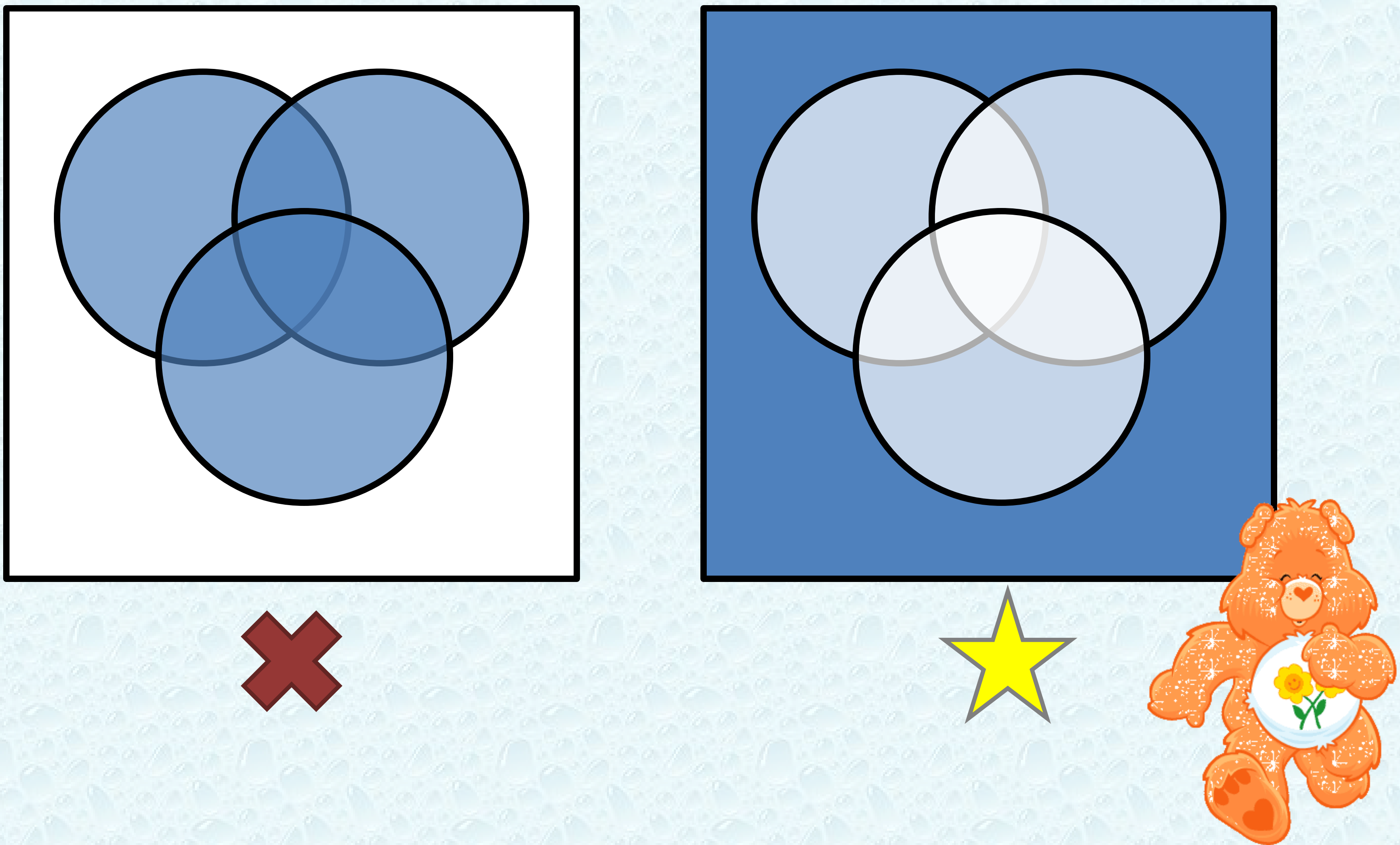
$A \cap B$ = {mappings that do not map 1 and 2 in Y}

$A \cap C$ = {mappings that do not map 1 and 3 in Y}

$B \cap C$ = {mappings that do not map 2 and 3 in Y}

$A \cap B \cap C$ = {mappings that do not map 1, 2, and 3 in Y}

So, we are interested in getting the reverse of the union.



Let's get the |U| first…

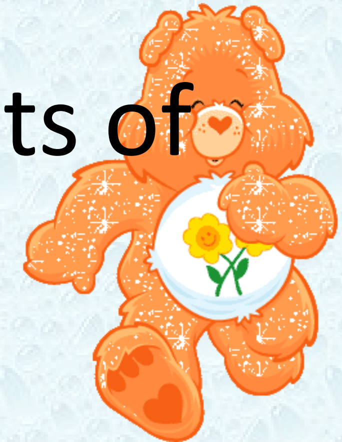
Recall this generalization:

Suppose *X = {1, 2, . . . ,m} and Y = {1, 2, . . . , n}. Then* the number of mappings from *X to Y is given by* n^m.

So for this problem,

$|U| = 3^5$

That is, we can map the 5 elements of X to the 3 elements of Y.



A = {mappings that do not map 1 in Y}

Since we are just leaving 1 out, we can still use the generalization on m x n mappings. We are simply taking 1 out of the possible elements to be mapped to.

$|A| = 2^5$

$|B|$ and $|C|$ are equal to 2^5 as well.

A ∩ B = {mappings that does not map 1 and 2 in Y}

We can still use the generalization because we are just leaving out two elements in Y that are not to be mapped.

$|A \cap B| = 1^5 = 1$

$|B \cap C|$ and $|A \cap C|$ are equal to 1 as well.

A ∩ B ∩ C = {mappings that do not map 1, 2, and 3 in Y}

In this case we can not use the generalization.

$$| A \cap B \cap C | = 0$$

If we do not map the elements in X to any elements in Y, the mapping will no longer be a function!

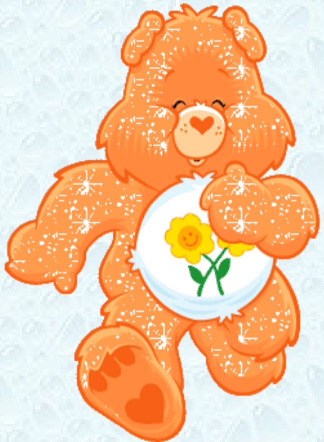
It all boils down to 1.4

$| \neg A \cap \neg B \cap \neg C|$

$= |U| - (|A| + |B| + |C|) + (|A \cap B| + |A \cap C| + |B \cap C|) - |A \cap B \cap C|$

$= 3^5 - (2^5 + 2^5 + 2^5) + (1 + 1 + 1) - 0$

$= 150$



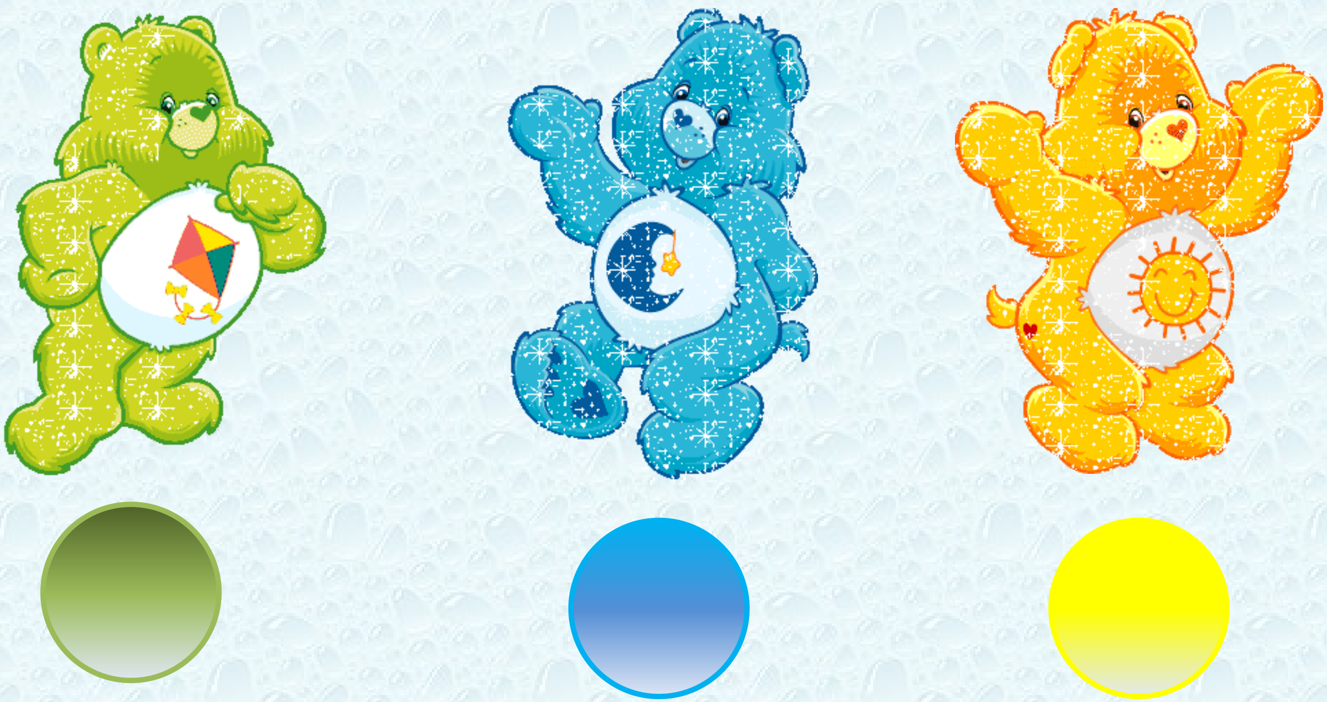
Derangements

$$D(n,k) = \frac{n!}{k!}\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-k}}{(n-k)!}\right]$$

Where k objects are in their original positions

Relating Derangements w/ IE Principle

Suppose we have three care bears and they have their own bouncing balls which are colored after them.



Let us now consider making a Venn diagram of their ownership

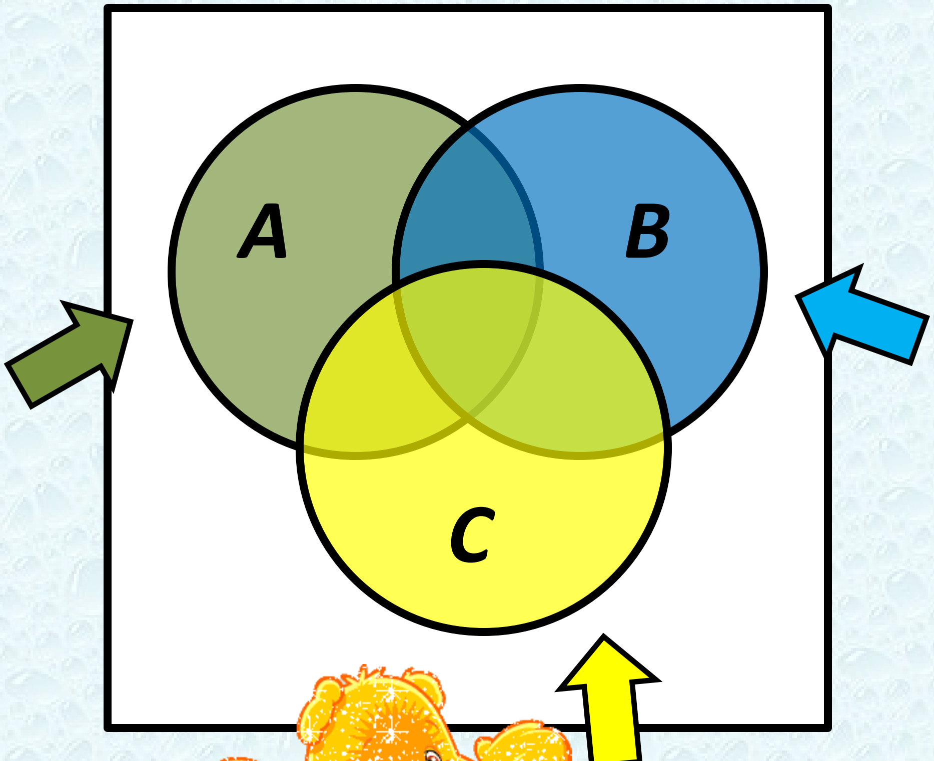
Let

A = {Distribution of balls to the bears where the green ball is distributed to the green carebear}

B = {Distribution of balls to the bears where the blue ball is distributed to the blue carebear}

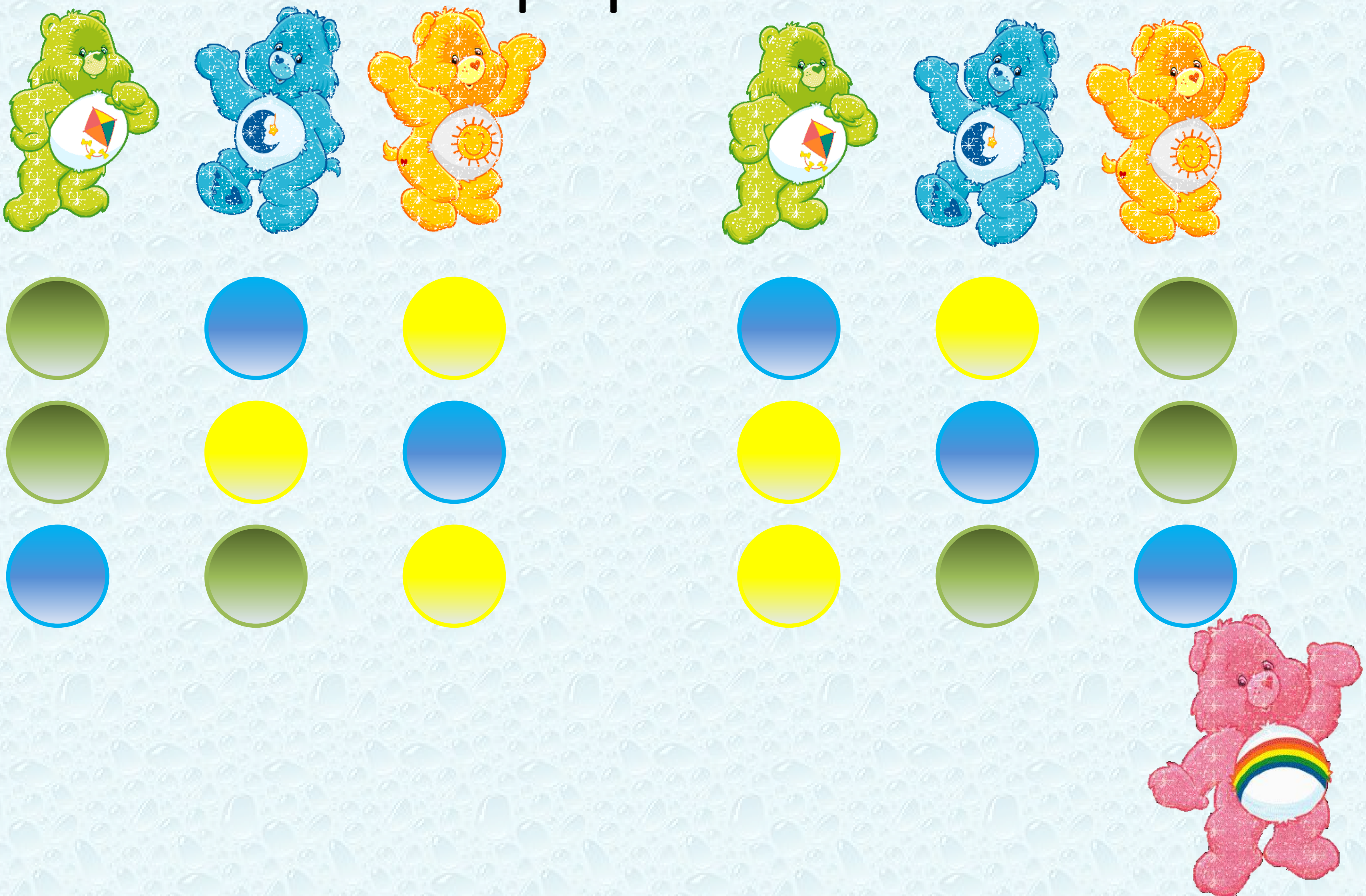
C = {Distribution of balls to the bears where the yellow ball is distributed to the yellow carebear}

To visualize…

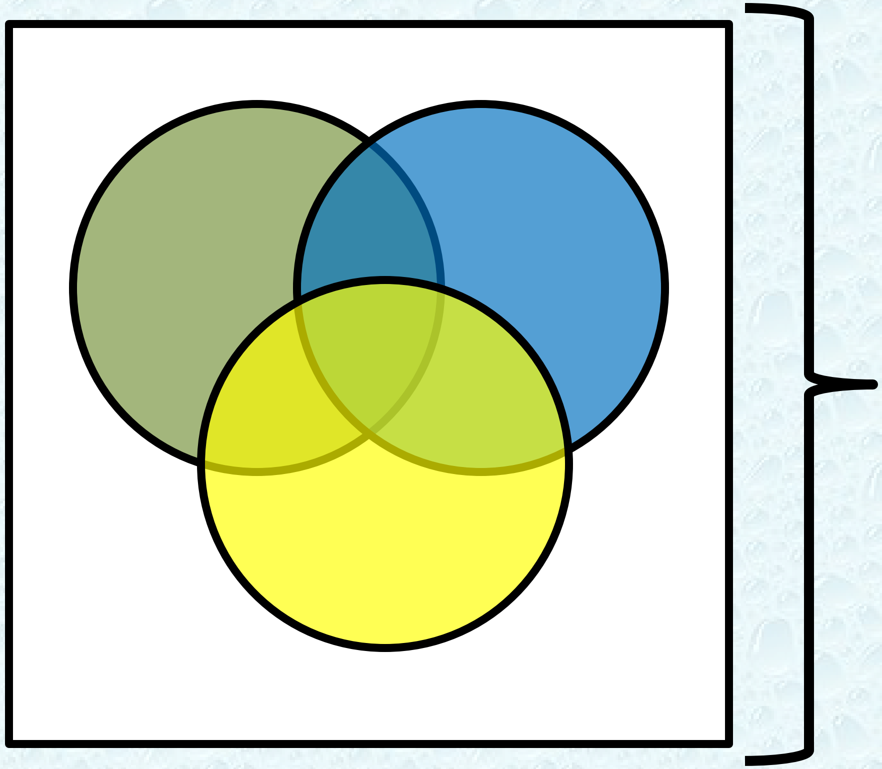





From basic counting we know that the $|U| = 6$

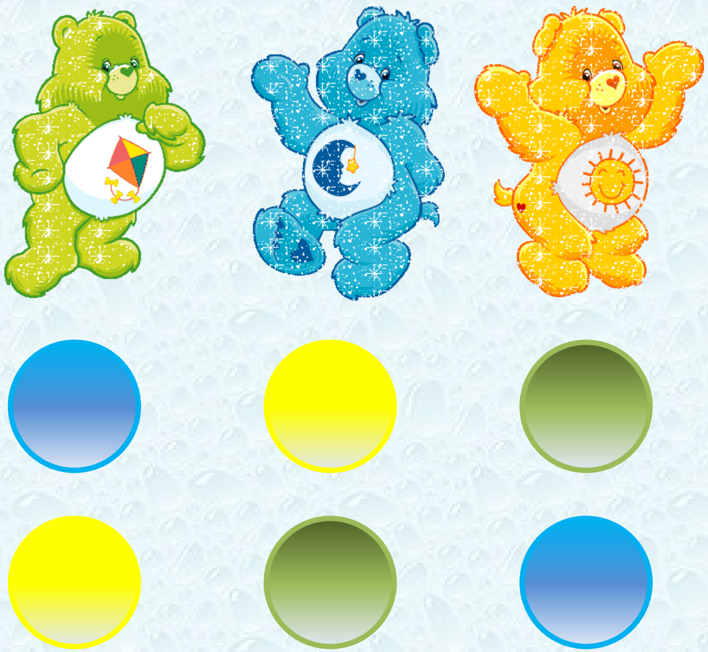


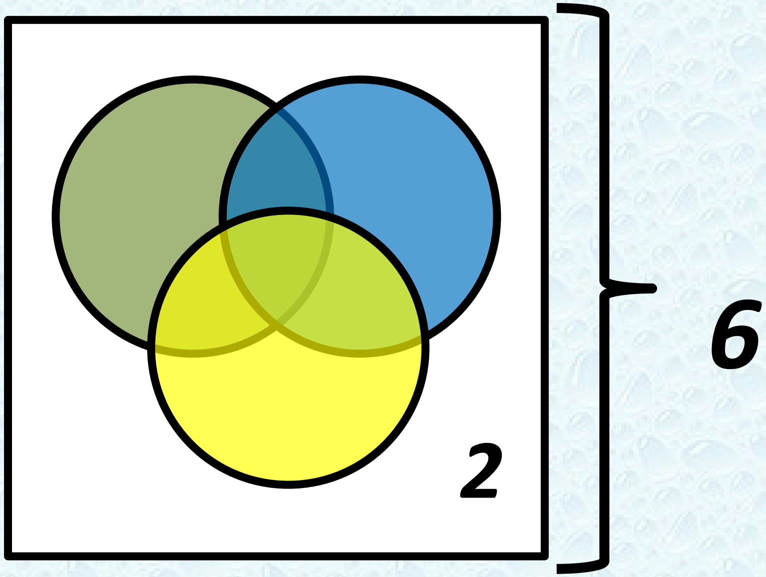
So we know that $|U|=6$



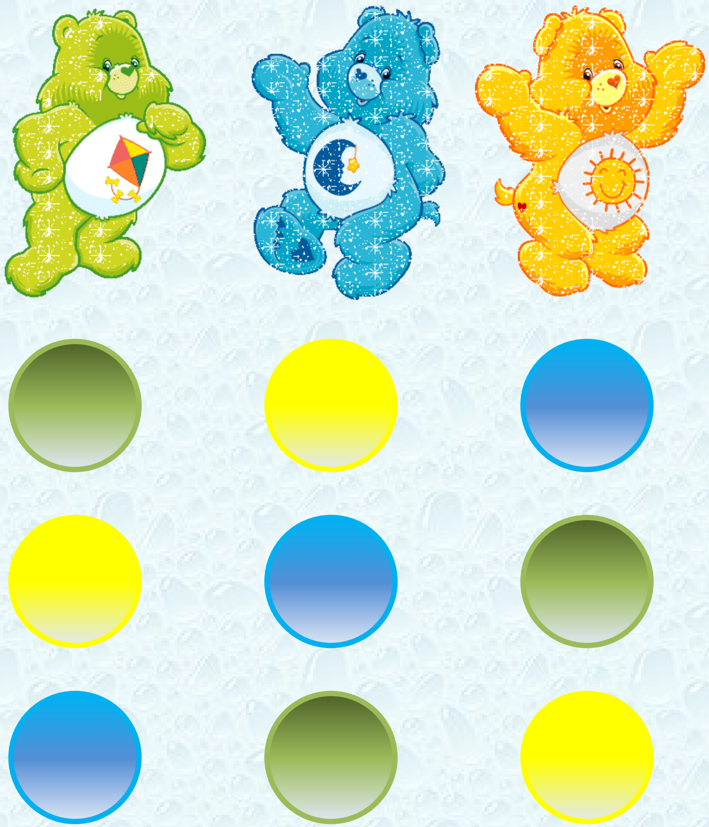
6

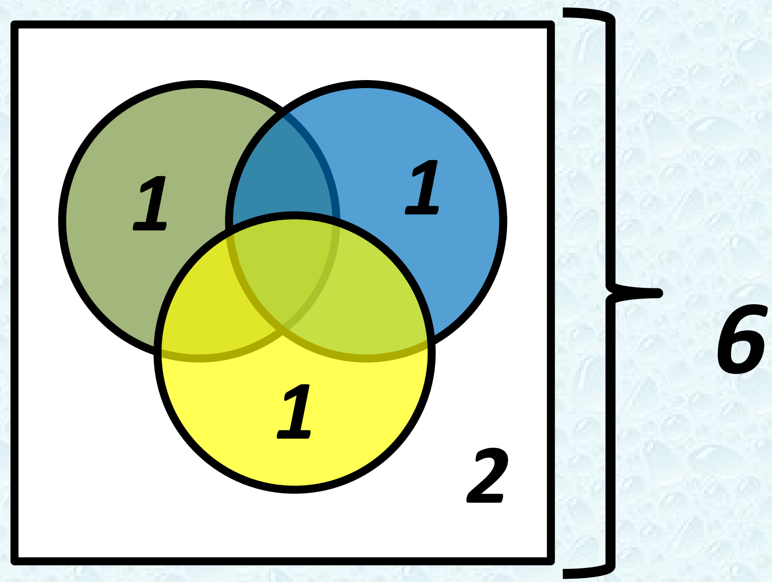
There are 2 arrangement where all the balls are not in order



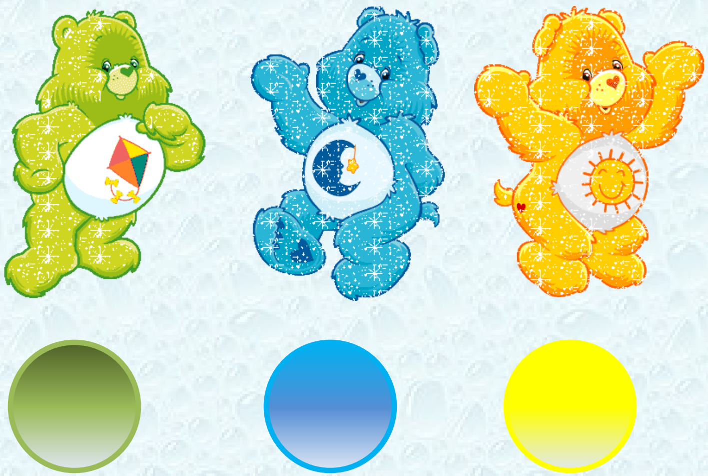


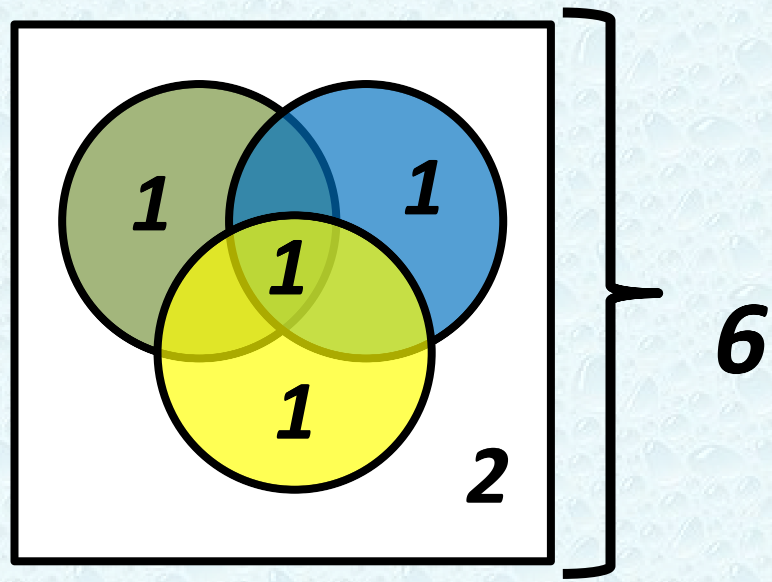
1 arrangement each where only 1 ball is in order and the other two are not.





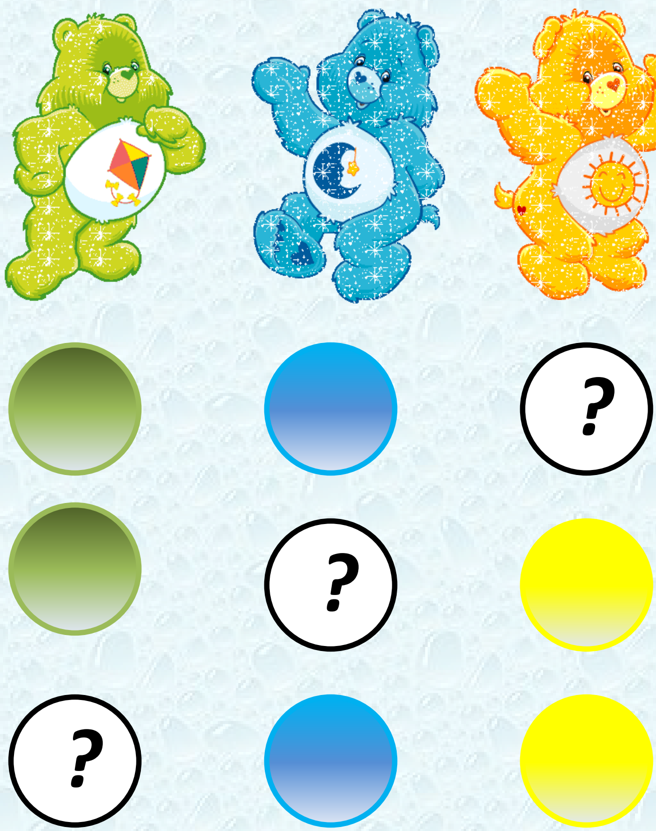
And only 1 arrangement where all balls are in order

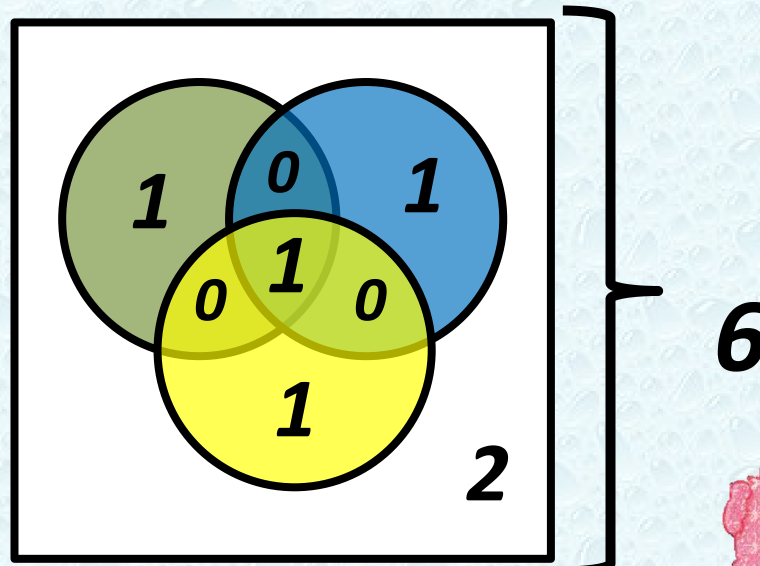




$|A \cap B|$, $|A \cap C|$, and $|B \cap C|$ are all 0

because if two balls are arranged, there is no way the third ball is not arranged.





On to the presentation with much less slides…