Relations & Functions

- **OUTLINE**
  - **Cartesian Product and Relations**
  - **Properties of Relations**
  - Types of Relations
    - Equivalence Relations
    - Ordering Relations
  - Operations on Relations
  - Hasse Diagrams and Directed Graphs
  - Functions
  - The Principle of Mathematical Induction
  - The Pigeonhole Principle

Relations & Functions

- **Introduction**
  - Relations **represent relationships** between elements of two or more sets
    - Student name $\Leftrightarrow$ student number
    - Student name $\Rightarrow$ course
    - Business $\Leftrightarrow$ phone number
    - Integer $\Rightarrow$ its divisors
  - Relations in computer science
    - Relational databases

Relations & Functions

- **Cartesian Product (Cross Product)**

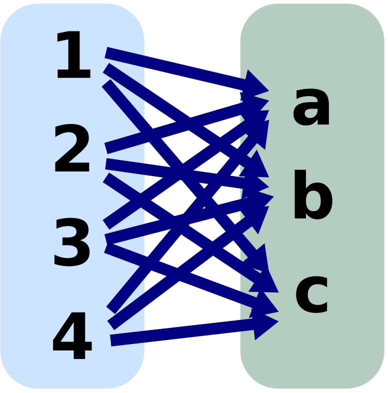
Given two non-empty sets A and B:

$$\mathbf{A \times B = \{ (x, y) \mid (x \in A) \wedge (y \in B) \}}$$

Example:

Let A = {1,2,3,4} and B = {a,b,c}

$\mathbf{A \times B}$ = {(1,a),(1,b),(1,c)
(2,a),(2,b),(2,c)
(3,a),(3,b),(4,c)
(4,a),(4,b),(4,c)}



Relations & Functions

- **Cartesian Product (Cross Product)**

***Example*:**

Let $A = \{1,2\}$ and $B = \{a,b,c\}$. What is $\mathbf{B} \times \mathbf{A}$?

$\mathbf{B \times A = \{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}}$

- ***What is*** $| A \times B |$***?***

  $| A \times B | = |A| \cdot |B|$

- ***Is*** $A \times B = B \times A$***?***

  No. Only when $A = B$***.***

Relations & Functions

- **n-fold Cartesian Product of n sets A_i**

$$A_1 \times A_2 \times \ldots \times A_n$$
$$= \{ (x_1, x_2, \ldots x_n) \mid (x_1 \in A_1), \ldots, (x_n \in A_n)\}$$

Example:

Let $A = \{1, 2\}$, $B = \{a, b\}$ and $C = \{x, y\}$

What is $\mathbf{A} \times \mathbf{C} \times \mathbf{B}$?

$$A \times C \times B$$
$$= \{(1,x,a),(1,y,a),(1,x,b),(1,y,b)$$
$$(2,x,a),(2,y,a),(2,x,b),(2,y,a)\}$$

Relations & Functions

- **RELATION** from set A to set B
  - *any subset* of $A \times B$.
  - $(x, y) \in R \Rightarrow$ **x is related to y by R**,
    $\Rightarrow$ also denoted by $_{\mathbf{x}}\mathbf{R}_{\mathbf{y}}$.
  - **relation on a set A** $\equiv$ relation from A to A

  - A **n-ary relation** is *any subset* R_n of the n-fold cartesian product $A_1 \times A_2 \times \ldots \times A_n$.

Relations & Functions

- ***Examples:***
  - Let A = {1,2,3} and B = {a,b}

    We can define:

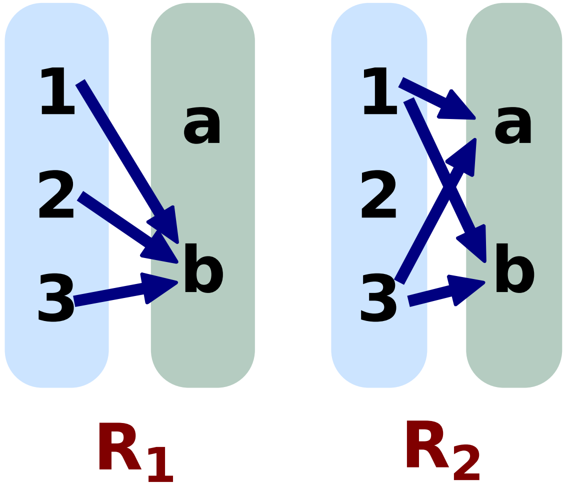
    $R_1 = \{(1,b),(2,b),(3,b)\}$

    $R_2 = \{(1,a),(1,b),(3,a),(3,b)\}$
  - Let A, B = Z (set of integers)

    We can define:

    $R_3 = \{(x,y) \mid x = y + 1\}$

    $R_4 = \{(x,y) \mid x \geq y\}$



Relations & Functions

- **Definitions**
  - The **domain** of a relation R, ***D*(R)**
    set of all *first coordinates* in R.
  - The **range** of a relation R, ***R*(R)**
    set of all *second coordinates* in R.
  - The **field** of a relation R, ***F*(R)**
    union of its domain and range.

Relations & Functions

- ***Examples:***

  Give the domain and range of each of the following relations on $A = \{0,1,2,3\}$

  $R_1 = \{(x,y) \mid x < y\}$

  $R_2 = \{(x,y) \mid |x - y| = 2\}$

  $R_3 = \{(x,y) \mid x + y < 5\}$

  $R_4 = \{(x,y) \mid x + y = 5\}$

  $R_5 = \{(x,y) \mid x = y\}$

  $R_6 = \{(x,y) \mid x \geq y\}$

Properties of Relations

- **Definitions**

  Given relation R on set A

  - R is **reflexive** iff

    $(\forall x)(x \in A \rightarrow (x,x) \in R)$
  - R is **irreflexive** iff

    $(\forall x)(x \in A \rightarrow (x,x) \notin R)$
  - R is **symmetric** iff

    $(\forall x)(\forall y)\{x,y \in A \rightarrow [(x,y) \in R \rightarrow (y,x) \in R]\}$
  - R is **asymmetric** iff

    $(\forall x)(\forall y)\{x,y \in A \rightarrow [(x,y) \in R \rightarrow (y,x) \notin R]\}$

Properties of Relations

- **Definitions**

  Given relation R on set A

  - R is **antisymmetric** iff

    $(\forall x)(\forall y)$

    $\{x,y \in A \to [((x,y) \wedge (y,x) \in R) \to (x = y)]\}.$

  - R is **transitive** iff

    $(\forall x)(\forall y)(\forall z)$

    $\{x,y,z \in A \to [((x,y) \wedge (y,z) \in R) \to ((x,z) \in R)]\}.$

  - R is **intransitive** iff

    $(\forall x)(\forall y)(\forall z)$

    $\{x,y,z \in A \to [((x,y) \wedge (y,z) \in R) \to ((x,z) \notin R)]\}.$

Properties of Relations

- ***Examples 1:*** Consider the following relations on the set $A = \{1,2,3\}$:

  $R_1 = \{(1, 1), (1, 2), (2, 2), (3, 3)\}$

  $R_2 = \{(1, 1), (2, 1), (1, 2), (3, 3)\}$

  $R_3 = \{(1, 2), (1, 3), (2, 3)\}$

  $R_4 = \{(1, 1), (2, 3), (2, 2), (3, 2),(3, 3)\}$

  $R_5 = \{(1, 1), (1, 2), (2, 3), (1, 3)\}$

  $R_6 = \{(1, 2), (2, 3), (3, 3)\}$

  $R_7 = \{(1, 1), (2, 2), (3, 3)\}$

  $R_8 = \{(1, 2), (2, 3), (3, 1)\}$

Properties of Relations

- ***Examples 2:***

  Consider the following relations:

  $R_1 = \{(x,y) \mid x \text{ is the sister of } y\}$

  $R_2 = \{(x,y) \mid x \text{ is the cousin of } y\}$

  $R_3 = \{(x,y) \mid x \text{ is the uncle of } y\}$

  $R_4 = \{(x,y) \mid x \text{ and } y \text{ have the same last name}\}$

  $R_5 = \{(a,b) \mid a \geq b \text{ and } a,b \in Z\}$

  $R_6 = \{(a,b) \mid a < b \text{ and } a,b \in Z\}$

  $R_7 = \{(a,b) \mid b \text{ is divisible by } a\}$

  $R_8 = \{(a,b) \mid b \text{ is divisible by } a\} \text{ on } A = \{2,3,4,6,8\}$

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Types of Relations

- **Compatibility relation R**

  $\Leftrightarrow$ R is both *reflexive and symmetric*.

Examples:

Which of the following are compatibility relations?

$R_1 = \{(1, 1), (1, 3), (2, 2), (3, 1),(3, 3)\}$ on $A=\{1,2,3\}$

$R_2 = \{(1, 1), (1, 3), (2, 2), (2, 3), (3, 1),(3, 2),(3, 3)\}$ on $A=\{1,2,3\}$

$R_3 = \{(x,y) \mid x \text{ is an ancestor of } y\}$

$R_4 = \{(x,y) \mid x \text{ and } y \text{ have the same last name}\}$

$R_5 = \{(a,b) \mid a = b \text{ and } a,b \in Z\}$

Types of Relations

- **Equivalence relation R**
  $\Leftrightarrow$ R is *symmetric, reflexive and transitive*.

- ***Examples:***
  Which of the following are equivalence relations?

  $R_1 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$ on $A=\{1,2,3\}$

  $R_2 = \{(1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ on $A=\{1,2,3\}$

  $R_3 = \{(x,y) \mid x$ is an ancestor of $y\}$

  $R_4 = \{(x,y) \mid x$ and y have the same last name$\}$

  $R_5 = \{(a,b) \mid a = b$ and $a,b \in Z\}$

Types of Relations

- A **partition** of a non-empty set A

  $\equiv$ collection of non-empty sets $\{A_1, A_2, \ldots, A_n\}$ such that:
  - $\mathbf{A_1 \cup A_2 \cup \ldots \cup A_n = A}$
  - $\mathbf{A_i \cap A_j = \varnothing}$ **for all** $\mathbf{i \neq j}$
- ***Examples:***

  Which of the following are *valid partitions* of the set A = {1,2,3,4,5}?

  1) {{1,2},{3,4},{4,5}}

  2) {{1,2,4},{5}}

  3) {{1,5},{2,3},{4}}

  4) {{1,4,5},{2,3}}

  5) {{1,2,3,4,5}}

Types of Relations

- **Equivalence class** of an element a with respect to an equivalence relation R

  $[a]_R = \{ y \mid (a,y) \in R \}$
- **Class of all equivalence classes**

  $\mathcal{D}(R)/R = \{ [a]_R \mid a \in \mathcal{D}(R) \}$

Types of Relations

- ***Examples:***

  Define the equivalence classes and the class of all equivalence classes of the $R = \{(1,1), (1,3), (2,2), (2,4), (3,1), (3,3), (4,2), (4,4), (5,5)\}$ on $X = \{1,2,3,4,5\}$.

Solution:

Equivalence classes:

$[1]_R = \{1,3\}$

$[2]_R = \{2,4\}$

$[3]_R = \{1,3\}$ $(= [1]_R)$

$[4]_R = \{2,4\}$ $(= [2]_R)$

$[5]_R = \{5\}$

Class of all equivalence classes:

$D(R)/R = \{\{1,3\},\{2,4\},\{5\}\}$

Types of Relations

- ***Examples:***

  Find the equivalence classes and the class of all equivalence classes of S = {(1,1),(1,2),(1,5),(2,1),(2,2),(2,5),(5,1),(5,2),(5,5),(3,3),(3,4),(4,3),(4,4)} on X={1,2,3,4,5}.

Solution:

Equivalence classes:

$[1]_S = \{1,2,5\} = [2]_S = [5]_S$

$[3]_S = \{3,4\} = [4]_S$

Class of all equivalence classes:

$D(R)/R = \{\{1,2,5\},\{3,4\}\}$

Types of Relations

- **Remarks:**
  - Each equivalence relation on a given set partitions the set, and each relation which partitions a given set is an equivalence relation.
  - The class of all equivalence classes is also a partition of the set A where A = D(R).

Examples:

For each of given class of equivalence classes, extract the corresponding equivalence relation.

1) Let X = {a,b,c,d} and D(R)/R = {{a,c},{b,d}}.

2) Let X = {1,2,3,4,5} and D(R)/R = {{1,3,4},{2,5}}.

Types of Relations

- A relation R on a set A is a **partial ordering** in A $\Leftrightarrow$ R is *reflexive, antisymmetric and transitive*.

- ***Examples:***
  Which of the following are partial orderings?
  $R_1 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$ on $A=\{1,2,3\}$
  $R_2 = \{(1, 1), (1, 3), (2, 2), (2, 3), (3, 1),(3, 2),(3, 3)\}$ on $A=\{1,2,3\}$
  $R_3 = \{(x,y) \mid x$ is as old as or older than $y\}$
  $R_4 = \{(x,y) \mid x$ and y have the same last name$\}$
  $R_5 = \{(x,y) \mid x \leq y$ where $x,y \in Z\}$
  $R_6 = \{(x,y) \mid x$ is divisible by y where $x,y \in Z\}$

Types of Relations

- If R is a partial order and either **(x, y) $\in$ R** <u>or</u> **(y, x) $\in$ R**
  $\Leftrightarrow$ **x and y are comparable**.
- If R is a partial order and **(x, y) $\notin$ R** <u>and</u> **(y, x) $\notin$ R**
  $\Leftrightarrow$ **x and y are <u>in</u>comparable**.

Examples:

Consider the following ordering relations defined on the set of integers. Give examples of pairs that are comparable/incomparable in each?

1) $R = \{(x,y) \mid y$ is divisible by x and $x,y \in Z\}$.

2) $R = \{(x,y) \mid y$ is divisible by x and $x,y \in \{2,3,6,18\}\}$.

3) $R = \{(x,y) \mid x \leq y$ and $x,y \in Z\}$

Types of Relations

- R is a **total ordering** in a set A if and only if
  - R is a *partial ordering*
  - either $(x, y) \in R$ or $(y, x) \in R$ *for every* $x, y \in A$ (that is, x and y are comparable).

Examples:

Which of the following are total orderings?

$R_1 = \{(x,y) |$ y is divisible by x$\}$ on $A = \{1,3,6,18\}$

$R_2 = \{(x,y) |$ y is divisible by x$\}$ on $A = \{1,3,6,9,18\}$

$R_3 = \{(x,y) |$ y is divisible by x$\}$ on Z (integers)

$R_4 = \{(1, 1), (1, 3), (3, 1), (2, 2), (3, 3)\}$ on $A=\{1,2,3\}$

$R_5 = \{(x,y) | x \leq y \text{ where } x,y \in Z\}$

Operations on Relations

- **Inverse** of a relation R

  $$R^{-1} = \{ (y, x) \mid (x, y) \in R \}$$

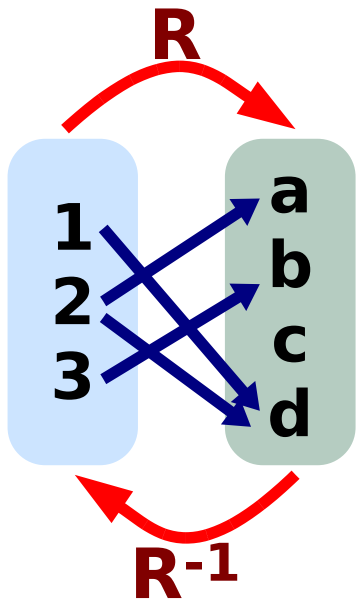
Examples:

- Find R^{-1} given $R = \{(3,b),(2,a),(1,d),(2,d)\}$.

  $R^{-1} = \{(b,d),(a,2),(d,1),(d,2)\}$

- Find R^{-1} given $R = \{(x,y) \mid x = y + 1\}$

  $R^{-1} = \{(x,y) \mid x = y - 1\}$



Operations on Relations

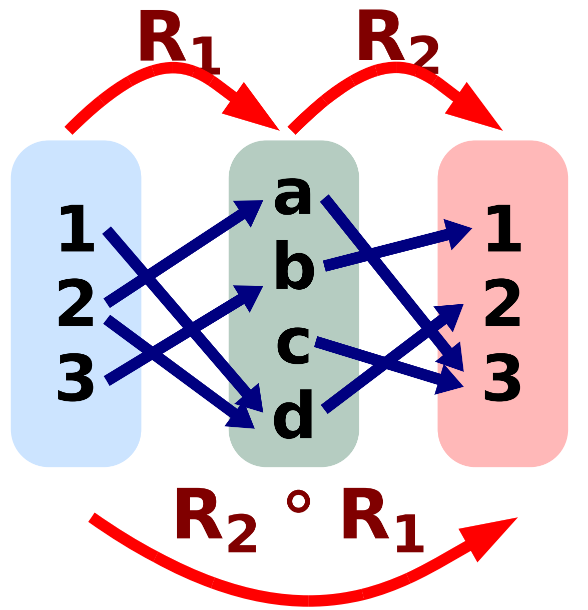
- **Composite** or **relative product**
  $R \circ S = \{ (x, y) \mid (x, z) \in S \text{ and } (z,y) \in R \;\; \exists z \in A \}$

Example:

Find $R_2 \circ R_1$ given

$R_1 = \{(3,b),(2,a),(1,d),(2,d)\}$ and

$R_2 = \{(b,1),(d,2),(a,3),(c,3)\}$



$R_2 \circ R_1 = \{(3,1),(2,3),(1,2),(2,2)\}$

Operations on Relations

- **Composite** or **relative product**

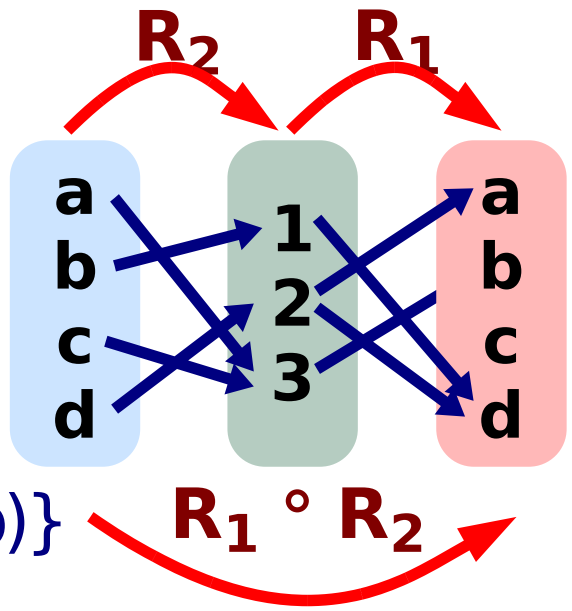
  $R \circ S = \{ (x, y) \mid (x, z) \in S \text{ and } (z,y) \in R \ \exists z \in A \}$

Example:

Find $R_1 \circ R_2$ given

$R_1=\{(3,b),(2,a),(1,d),(2,d)\}$ and

$R_2=\{(b,1),(d,2),(a,3),(c,3)\}$



$R_1 \circ R_2 = \{(b,d),(d,a),(d,d),(a,b),(c,b)\}$

Operations on Relations

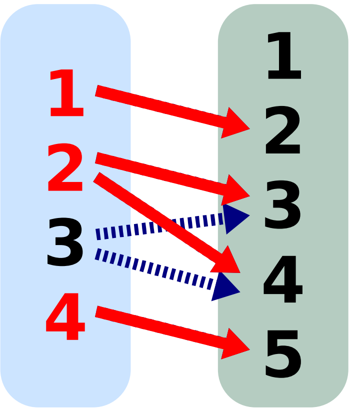
- **R restricted to the set X**

  $R|X$ = { (u, v) | u $\in$ X and (u, v) $\in$ R }

  Example:

  Find **$R|X$** given R = {(1,2),(2,3),(2,4),(3,3),(3,4),(4,5)} and X = {1,2,4,5}

  $R|X$ = {(1,2),(2,3),(2,4),(4,5)}



Operations on Relations

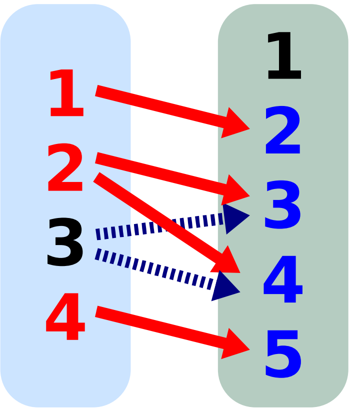
- **Image of set X under the relation R**

  R''X = { v | (∃u) (u ∈ X and (u,v) ∈ R }

  Example:

  Find **R''X** given R = {(1,2),(2,3),(2,4),(3,3),(3,4),(4,5)} and X = {1,2,4,5}.

  R''X = {2,3,4,5}



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Representations of Relations

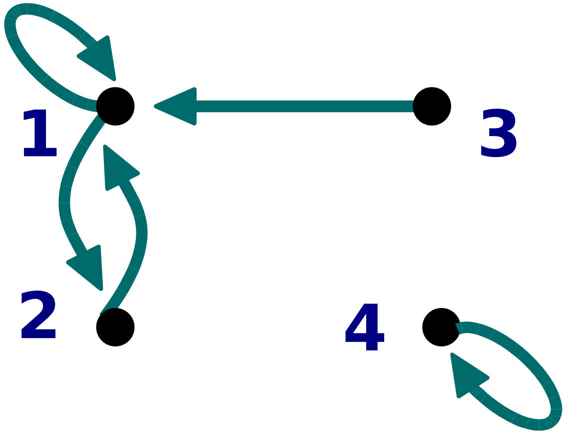
- **Directed Graphs**

A *directed graph* of a relation R consists of

- **points** ⇔ each element in the set; and
- **rays/arrows** ⇔ ordered pairs (x, y) in R.

Example:

Let A = {1,2,3,4}. Draw the graph corresponding to the R = {(1,1),(1,2),(2,1),(3,1),(4,4)}



Representations of Relations

- **Hasse Diagrams**
  - Used to represent *partial orders* on a finite set.
  - Read from top to bottom

How to draw a Hasse diagram of a partial ordering R

- **Draw** the directed graph representation of the relation.
- **Remove** all loops from the digraph.
- **Delete edges** implied by the transitive property.
- **Rearrange the nodes** so that all directed edges point upwards.
- **Ignore** the directions of the directed edges.

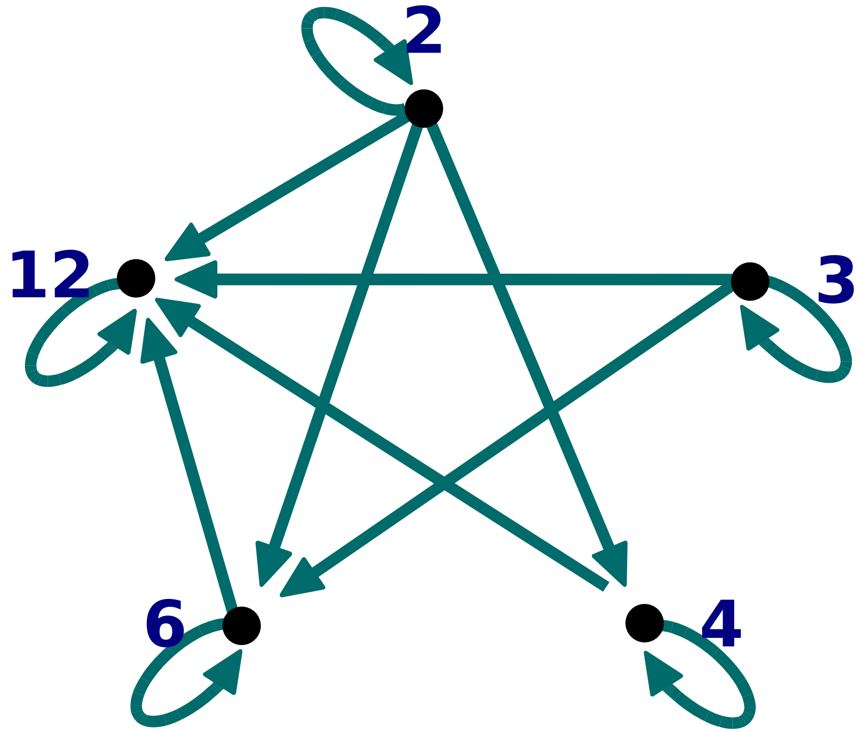
NOTE: Total orderings have linear Hasse diagrams.

Representations of Relations

- **Hasse Diagrams**

Example:

Let $A = \{2,3,4,6,12\}$ and $R = \{(x,y) \mid x \text{ divides } y\}$



Draw the directed graph representation of the relation.

Representations of Relations

- **Hasse Diagrams**

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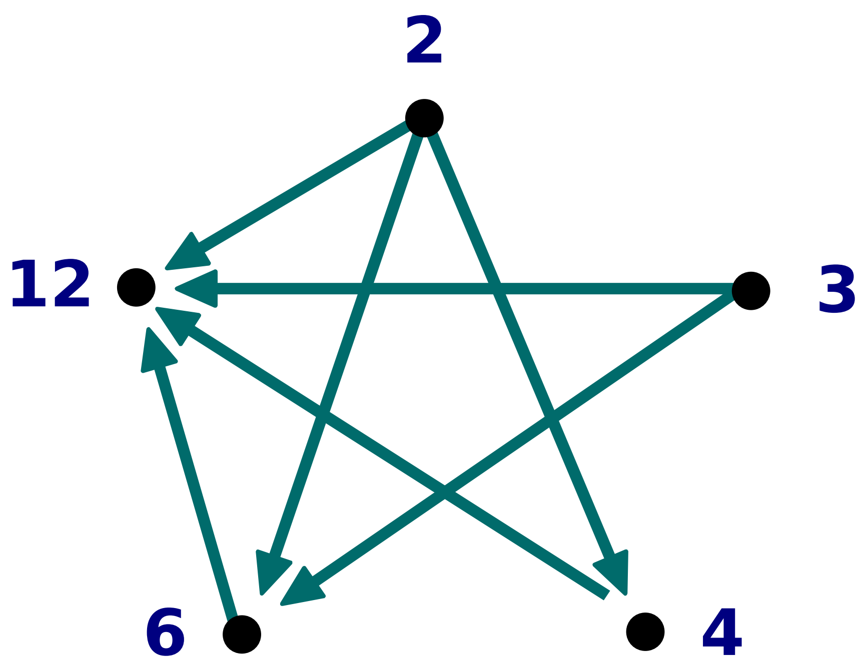

Remove all loops from the digraph.

Representations of Relations

- **Hasse Diagrams**

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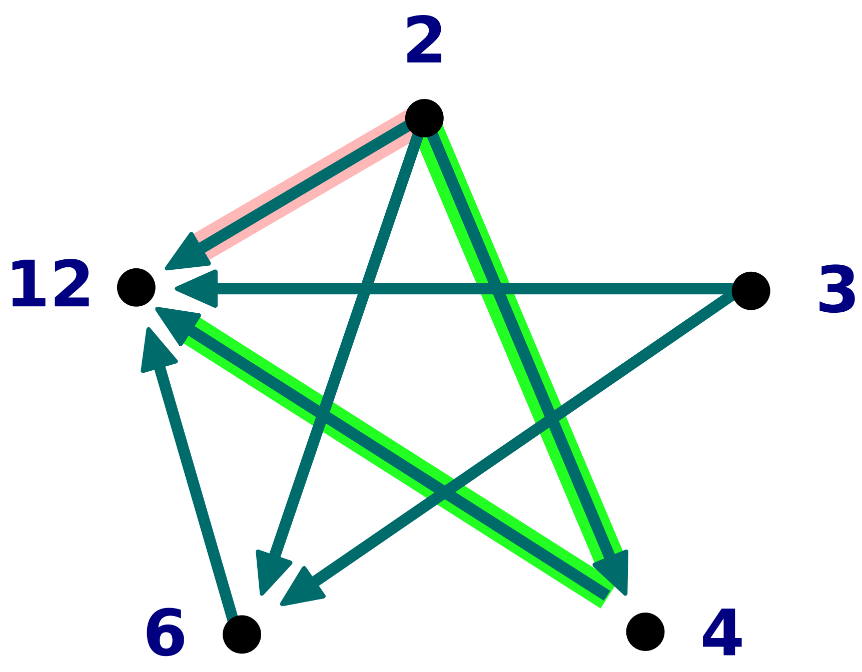

Delete edges implied by the transitive property.

Representations of Relations

- **Hasse Diagrams**

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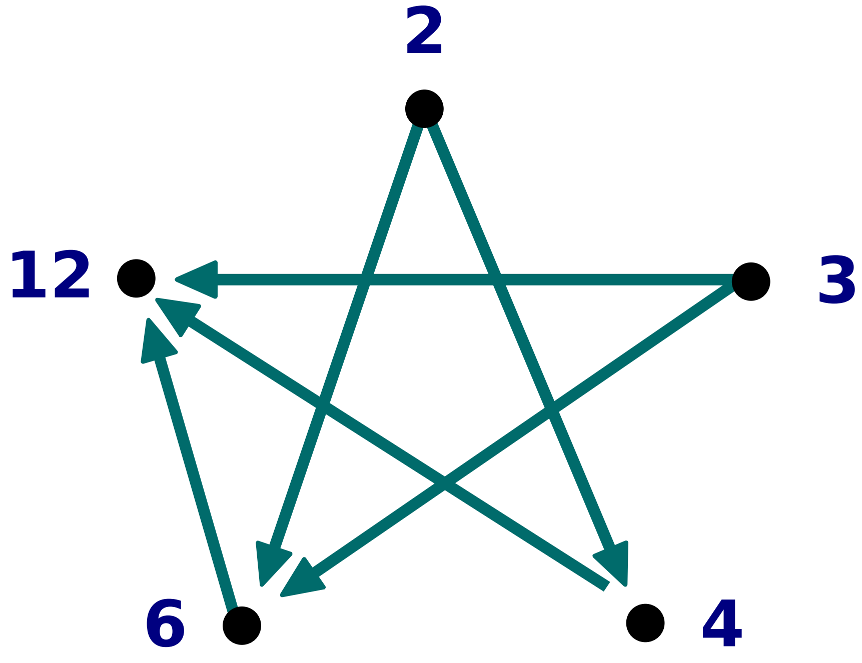

Delete edges implied by the transitive property.

Representations of Relations

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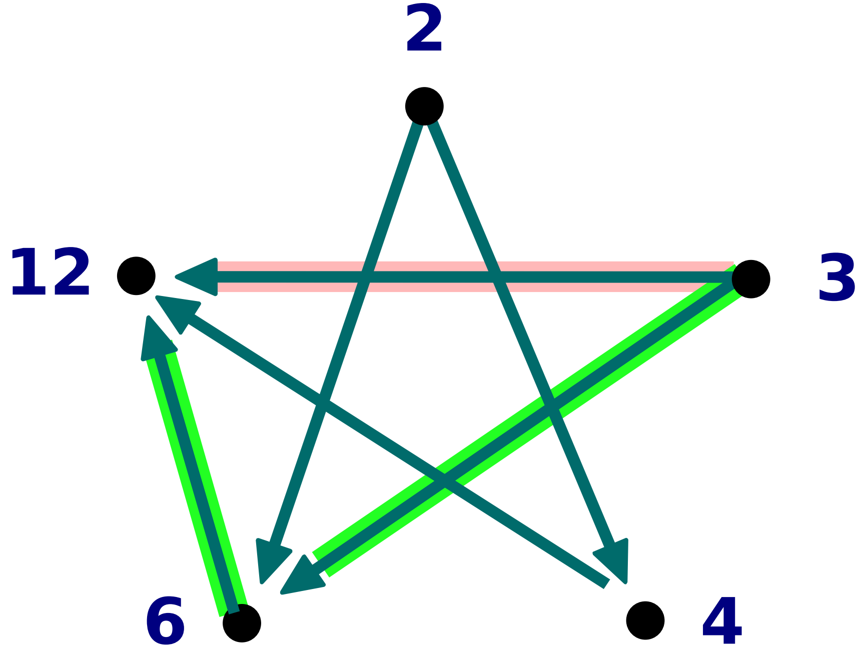

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Representations of Relations

- **Hasse Diagrams**

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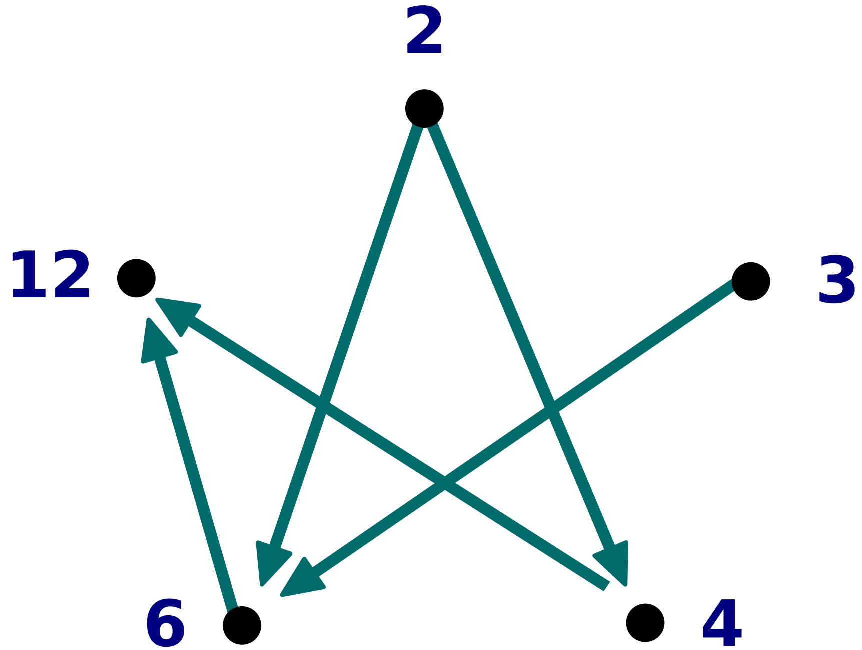

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Representations of Relations

- **Hasse Diagrams**

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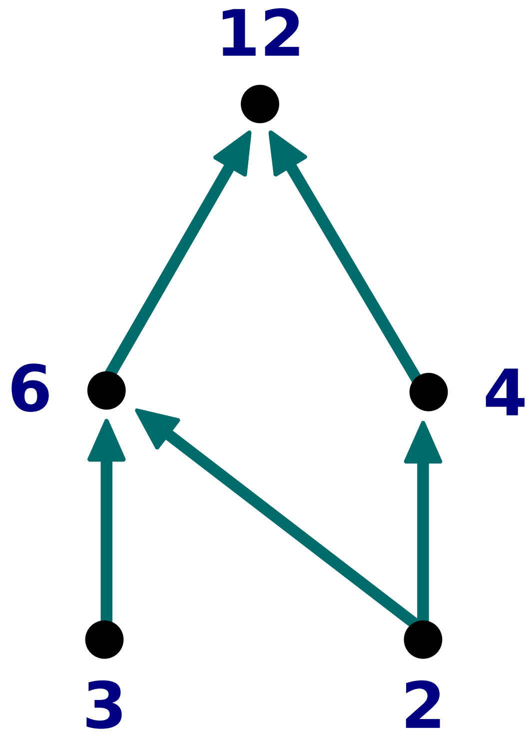


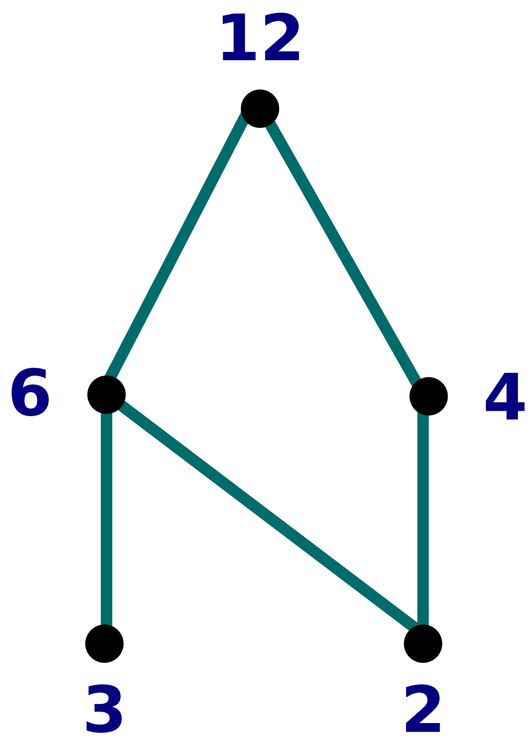
Rearrange the nodes so that all directed edges point upwards.

Representations of Relations

- **Hasse Diagrams**

Example:

Let $A = \{2,3,4,6,12\}$ and $R = \{(x,y) \mid x \text{ divides } y\}$



Ignore the directions of the directed edges

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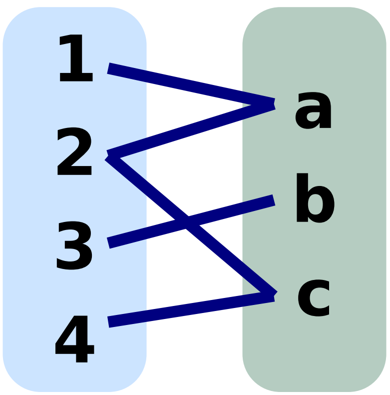
Functions

- **DEFINITION**
  - Let A and B be non-empty sets.
  - A **function** or **mapping** ***f*** from A to B, is
    - a relation from A to B
    - every element of A *appears exactly once* as the first coordinate of an ordered pair in the relation.

  - **f is a function from set A to set B:**
    - **f : A $\rightarrow$ B**
    - **f(x) = y** where $(x, y) \in f$ and $x \in A$ and $y \in B$.

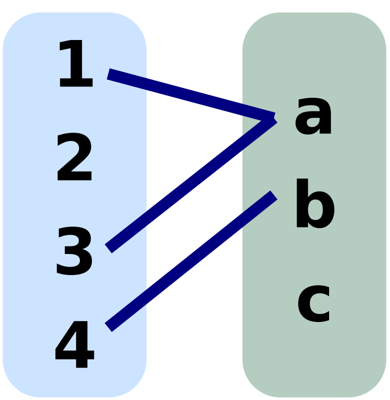
Functions

- **Examples:** Let A = {1,2,3,4} and B = {a,b,c}



f = {(1,a),(2,a),(2,c),(3,b),(4,c)}

NOT A FUNCTION

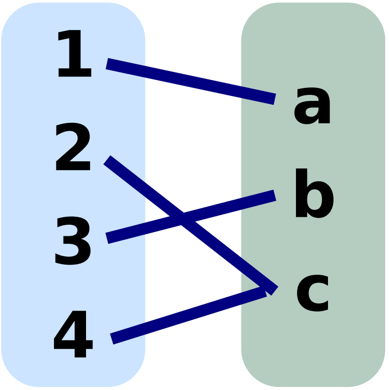


f = {(1,a),(3,a),(4,b)}

NOT A FUNCTION

Functions

- **Example:** Let A = {1,2,3,4} and B = {a,b,c}



f = {(1,b),(2,d),(3,c),(4,c)}

IS A FUNCTION

- **Examples:**

  Let A = {1,2,3} and B = {4,5,6,7,8}.

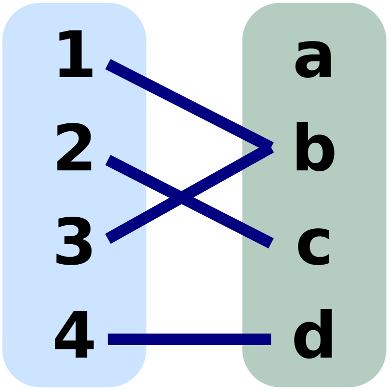
  Define f = {(1,4), (2,5), (3,6)}.

  may also be defined as

  f(x) = x + 3 where **x=1,2,3**

Functions

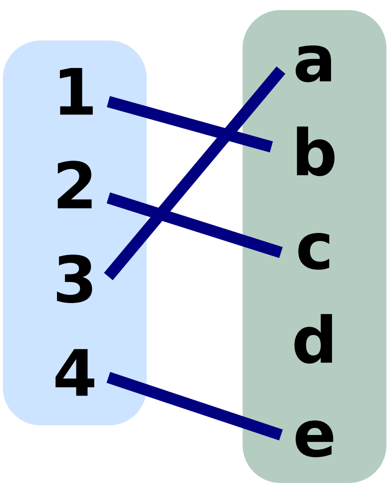
- **Other terms:**
  - Let f: A $\rightarrow$ B is a function from A to B and f(x) = y
    - y $\Leftrightarrow$ **image** of x under f.
    - x $\Leftrightarrow$ **pre-image** of y.
    - Set A $\Leftrightarrow$ **domain** of f.
    - Set B $\Leftrightarrow$ **codomain** of f.
    - *Actual* second coordinates in f $\Leftrightarrow$ **range** of f
  - ***Example***:



Domain: {1,2,3,4}
Codomain: {a,b,c,d}
Range: {b,c,d}

Types of Functions

- **1-1 Function** or **Injection**
  - if and only if each element of B appears *at most once* as a second coordinate in f.
  - ***Example***:



f = {(1,b),(2,c),(3,a),(4,e)}

Types of Functions

- **1-1 Function** or **Injection**
  - ***Examples***:

    Are the following 1-1 functions?

    1) Let $A = \{1,2,3\}$ and $B = \{a,b,c,d\}$

    a) $f = \{(1,a),(2,b),3,d)\}$ **1-1**

    b) $h = \{(1,b),(2,a),(3,a)\}$ **not 1-1**

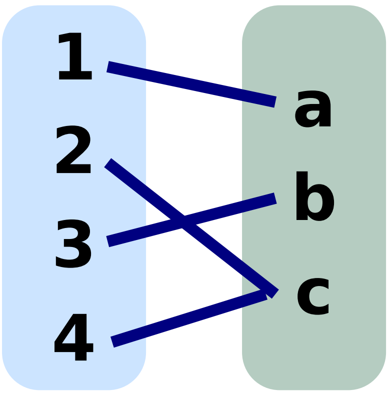
    c) $g = \{(1,c),(2,a),(3,d),(1,b)\}$ **not a f'n**

    2) $f: Z \to Z$ where $f(x)= 2x$ **1-1**

    3) $g: Z \to Z$ where $f(x) = x^2$ **not 1-1**

Types of Functions

- **Onto Function** or **Surjection**
  - if and only if each element of B appears *at least once* as a second coordinate in f.
  - ***Example***:



f = {(1,b),(2,d),(3,c),(4,c)}

Types of Functions

- **Onto Function** or **Surjection**
  - ***Examples***:

    Are the following onto functions?

    1) Let $A = \{1,2,3,4\}$ and $B = \{a,b,c\}$

    a) $f = \{(1,a),(2,c),(3,a),(4,b)\}$ **onto**

    b) $h = \{(1,c),(2,a),(3,c),(4,a)\}$ **not onto**

    2) $f: Z \to Z$ where $f(x)= 2x$ **not onto**

    3) $g: Q \to Q$ where $f(x) = 2x$ **onto**

    4) $h: R \to R$ where $f(x) = x^2$ **not onto**

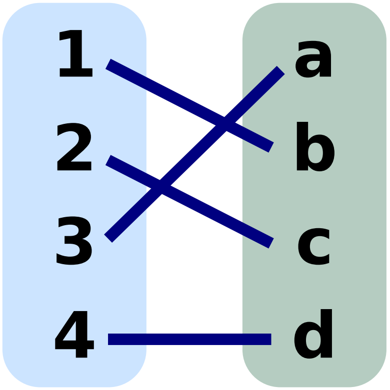
Types of Functions

- **One-to-one correspondence** or **Bijection**
  - if and only if f is *both 1-1 and onto*.

OR

  - if and only if each element of B appears *exactly once* as a second coordinate in f

  - ***Example***:



f = {(1,b),(2,c),(3,a),(4,d)}

Types of Functions

- **One-to-one correspondence** or **Bijection**
  - ***Examples***:

    Is each of the following a one-to-one correspondence?

    1) Let $A = \{1,2,3,4\}$ and $B = \{a,b,c,d\}$

    a) $f = \{(1,c),(2,a),(3,c),(4,b)\}$ **not 1-1**

    b) $h = \{(1,b),(2,a),(3,d),(4,c)\}$ **one-to-one**

    2) $f: Q \to Q$ where $f(x)= 2x$ **one-to-one**

    3) $g: Z \to Q$ where $f(x) = 2x$ **not onto**

    4) $h: Z \to Z^+$ where $f(x) = |x|$ **not 1-1**

Relations & Functions

- **OUTLINE**
  - Cartesian Product and Relations
  - Properties of Relations
  - Types of Relations
    - Equivalence Relations
    - Ordering Relations
  - Operations on Relations
  - Hasse Diagrams and Directed Graphs
  - Functions
  - **The Principle of Mathematical Induction**
  - **The Pigeonhole Principle**