Discrete Probability

- **Outline**:
  - Basic Concepts
  - Axioms and Theorems on Probability
  - Mutually Exclusive Events
  - Independent Events
  - Conditional Probability
  - **Random Variables**
  - **Discrete Probability Distributions**

Random Variables

- **Random variable**
  - A *function* that assigns a real number to ________ ________________of the sample space.
  - *NOTATION*:
    - Random variables ⇒ capital letters (X, Y, Z)
    - Actual values of random variables ⇒ lowercase letters
- ***Examples***:
  - Tossing three coins:

    X = number of tails obtained

    so that, {HHT} ⇒

    or {TTT} ⇒

    Note that if E = event of obtaining two tails then E ⇔

    It follows that

Random Variables

- ***Examples***:
  - Rolling two dice until we obtain the same number on both dice

    Y = number of attempts before same number is obtained

    Z = 0 if the number of attempts needed is even;
    1 if the number of attempts are more than 10
  - Rolling two dice

    Z = 0 if the total number of dots is even;
    1 otherwise

    X = 0 if both numbers obtained are even
    1 if exactly one number obtained is odd

Random Variables

- **Discrete sample space**
  - contains a finite number of possibilities

  OR contains an unending sequence with as many sample points there are as whole numbers
- **Discrete random variable**
  - Random variable defined over a discrete sample space
- *Examples*:
  - Tossing three coins:

    S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

    let X =
  - Tossing a coin until a head appears

    S = {H, TH, TTH, TTTH, TTTTH, ... }

    let X =

Discrete Probability Dist'ns

- **(Discrete) Probability distribution** on a random variable X
  - Function f(x) that satisfies the following for each possible outcome x:
    - **f(x) ≥ 0**
    - $\sum_{all\ x} f(x) = 1$
    - **P(X = x) = f(x)**

Discrete Probability Dist'ns

- ***Examples***:

  Consider placing three balls number 1, 2 and 3 into boxes likewise numbered 1, 2 and 3. Find the probability distribution of X = number of balls placed into a box with the same label.

Solution:

1	2	3	X=*x*
1	2	3	
1	3	2	
2	1	3	
2	3	1	
3	1	2	
3	2	1	

Note that:

Discrete Probability Dist'ns

- ***Examples***:
  Consider placing three balls number 1, 2 and 3 into boxes likewise numbered 1, 2 and 3. Find the probability distribution of X = number of balls placed into a box with the same label.

  Solution:

Note that:

Probability distribution is given by

X=x	P(X=x)

Discrete Probability Dist'ns

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  Solution:

Probability distribution is given by

X=x	P(X=x)

Discrete Probability Dist'ns

- ***Examples***:
  Consider tossing a coin three times and suppose the coin is biased so that one is twice likely to obtain a head than a tail. Find the probability distribution of X = the number of heads obtained.

  Solution:

  S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

Discrete Probability Dist'ns

- ***Examples***:
  Consider tossing a coin three times and suppose the coin is biased so that one is twice likely to obtain a head than a tail. Find the probability distribution of X = the number of heads obtained.

  Solution: (cont'n)

Discrete Probability Dist'ns

- ***Examples***:
  Consider tossing a coin three times and suppose the coin is biased so that one is twice likely to obtain a head than a tail. Find the probability distribution of X = the number of heads obtained.

  Solution: (cont'n)

  Probability distribution given by:

Discrete Probability Dist'ns

- **Selected Discrete Probability Dist'ns**
  - Binomial
  - Multinomial
  - Geometric
  - Hypergeometric

Binomial Probability Dist'n

- **Binomial experiment**
  - Consists of n ______________________________trials
  - Each trial results in two outcomes that may be ________________________
  - Probability of success, p, *remains constant* from trial to trial
- ***Example***:
  - Tossing a (fair) coin n times:
    - n independent tosses
    - classify result of each toss as head(success) or tail(failure)
    - Probability of head/success, $p = 1/2$

Binomial Probability Dist'n

- **Binomial probability dist'n**
  - Given that a binomial trial can result in:
    - A success with probability p
    - A failure with probability 1-p
  - X = number of _________ obtained out of the n trials
  - Probability that X = k is

$$B(X=k;n,p)=\binom{n}{k}p^k(1-p)^{n-k}$$

Binomial Probability Dist'n

- ***Examples***:

  Consider drawing a ball from an urn 6 times with replacement. If the urn contains three red balls and five blue balls, what is the probability that exactly four red balls are drawn?

  Solution:

Binomial Probability Dist'n

- ***Examples***:

  Consider rolling a die seven times. What is the probability of obtaining a number less than three four times?

  Solution:

Multinomial Probability Dist'n

- **Multinomial experiment**
  - Similar to binomial experiment except that each trial can result in _________________________.
- ***Example***:
  - Rolling a (fair) die n times:
    -
    - may classify result of each rolls as
      -
      -
    - Probability of outcomes
      -
      -

Multinomial Probability Dist'n

- **Multinomial probability dist'n**
  - Given that a trial can result in:
    - t outcomes $E_1, E_2, E_3, \ldots, E_t$
    - outcome E_i with probability p_i
  - If n such trials are performed, the probability that ____ ____________ is given by

$$M(n_1,n_2,\ldots,n_t;n;p_1,p_2,\ldots,p_t)=\frac{n!}{n_1!n_2!\ldots n_t!}\cdot p_1^{n_1}p_2^{n_2}\ldots p_t^{n_t}$$

where $n_1 + n_2 + \ldots + n_t = n$ and $p_1 + p_2 + \ldots p_3 = 1$.

Multinomial Probability Dist'n

- ***Examples***:

  Consider rolling a die seven times. If obtaining an even number is twice that of obtaining an odd number, what is the probability of obtaining a one twice, a two thrice, and a four twice?

  Solution:

Geometric Probability Dist'n

- **Geometric experiment**
  - Consists of *identical* and *independent* trials
  - Each trial results in an outcome that *may be classified as success or failure*
  - Probability of success, *p*, remains constant
  - Each trial is ______________________________
- ***Example***:
  - Rolling a (fair) die *until* a six is obtained
    -
    -
    -

Geometric Probability Dist'n

- **Geometric probability dist'n**
  - Given that a binomial trial can result in:
    - A success with probability *p*
    - A failure with probability 1-*p*
  - X = number of trials _______________________
  - A trial is repeated *until a success occurs*
  - Probability that X = *k* is

$$\boldsymbol{G(X=k\,;p)=(1-p)^{k-1}p}$$

Geometric Probability Dist'n

- ***Examples***:

  Consider drawing a balls with replacement in succession from an urn containing three black balls and two white balls. What is the probability that white balls are always drawn *until* a black ball is finally drawn on the sixth draw?

  Solution:

Geometric Probability Dist'n

- ***Examples***:

  Roger plays a series of tennis matches *until* he loses one. Assume that the results of each match are independent of the other and that the probability that he wins a match is 9/10. What is the probability that he wins at most three matches before losing?

Hypergeometric Prob Dist'n

- **Hypergeometric experiment**
  - A random sample of n objects is selected _______ ___________ from *N* objects
    - *s* of the *N* items are of one type (successes)
    - *N-s* of the *N* items are of another type (failures)
- ***Example***:
  - Drawing *n* balls from a bag of *s* black balls and *N-s* white balls without replacement

Hypergeometric Prob Dist'n

- **Hypergeometric probability dist'n**
  - Sample of *n* objects selected from a set of *N* objects
    - *s* of the *N* items are of one type (successes)
    - *N-s* of the *N* items are of another type (failures)
  - X = number of ______________________________
  - Probability that X = *k* is

$$H(X=k;N,n,s)=\frac{\binom{s}{k}\binom{N-s}{n-k}}{\binom{N}{n}}$$

Hypergeometric Prob Dist'n

- ***Examples***:
  A box of 15 cookies is labeled "raisin cookies" although only ten cookies actually contain at least one raisin. If you are to pick ten cookies at random to eat as snacks during the day, what is the probability that all ten have at least one raisin?
  Solution:

Hypergeometric Prob Dist'n

- ***Examples***:
  Suppose you are to select at random five miniature bars of chocolate from a bag containing three bars of milk chocolate, six bars of dark chocolate, three bars of white chocolate and two bars of chocolate with nuts. What is the probability that you will get two bars of dark chocolate and of milk chocolate and one bar of white chocolate?
  Solution:

Hypergeometric Prob Dist'n

- ***Examples***:
  Suppose you are to select at random five miniature bars of chocolate from a bag containing three bars of milk chocolate, six bars of dark chocolate, three bars of white chocolate and two bars of chocolate with nuts. What is the probability that you will get two bars of dark chocolate and of milk chocolate and one bar of white chocolate?
  Solution: (cont'n)

Hypergeometric Prob Dist'n

- **Hypergeometric probability dist'n**
  - Sample of n objects selected from a set of N objects of which there are t types
    - s_i of the *N* objects are of ith type
  - Probability of obtaining x_i of the ith type for all i types

$$H(n_1,n_2,\ldots,n_t;N,n,s_1,s_2,\ldots,s_t)=\frac{\binom{s_1}{n_1}\binom{s_2}{n_2}\cdots\binom{s_t}{n_t}}{\binom{N}{n}}$$

  where $x_1 + x_2 + \ldots + x_t = n$