Set Theory

- **OUTLINE**
  - **Sets**
  - **Set Membership and Set Containment**
  - Set Operations
  - Venn Diagrams
  - Laws of Set Theory
  - Proving in Set Theory
    - Venn Diagrams
    - Membership Table
    - Algebraic Method
    - Element arguments

Set Theory

- **BRIEF HISTORY**
  - Evolved during 19th and early 20th centuries
    - **George Boole** published a book in 1854 on the algebra dealing with sets and logic
    - **Georg Cantor** defined a set in 1895

Set Theory

- **Basic Concepts: Def'n of a set**
  - A **set** is a collection of *unordered*, *well-defined* and *distinct* objects. The objects that belong to a given set are called **elements**.

  - *Notation*:
    - **sets**: uppercase letters
    - **set groupings**: brackets { … }
  - *Illustration*:
    - **A** = { a, e, i, o, u }
      is the *same* as the set **B** = { u, i, o, a, e }
    - {1, 3, 4, 4, 7} can be rewritten as {1, 3, 4, 7}

Set Theory

- **Basic Concepts: Set membership**

  x belongs to a set A: $\mathbf{x \in A}$

  x is not an element of the set A: $\mathbf{x \notin A}$

Example:

Let B = {1, 2, {3}}. What are the elements of set B?

a) the number 1?

b) the number 2?

c) the number 3?

d) the set containing 3?

Set Theory

- **Basic Concepts**: **Defining sets**
  - **Roster method**
    - *list down* the elements of the set

    Examples:

    A = {green, maroon}
    B = {0, 2, 4, 6, 8, 10}
    C = {0, 1, 2, 3, 4, 5, …}
  - **Rule method**
    - *describe* the elements that belong to the set

    Examples:

    A = the set of official colors of U.P
    B = $\{ x \mid x = 2y \text{ and } 0 \le y \le 5 \}$
    C = the set of nonnegative integers or $\{ x \mid x \in Z^{\oplus} \}$

Set Theory

- **Basic Concepts**: **Defining sets**
  - **Remarks**:
    - Rule method $\Rightarrow$ more precise way to define a set.
    - Roster method $\Rightarrow$ can lead to ambiguity esp. for sets with infinite number of elements.
  - ***Example***:

    Consider $Y = \{2, 4, 8, 16, \ldots\}$
    - may be described as

      Y = { x | x = 2^n where n is a positive integer}.
    - may also be described by

      Y = { x | x = $(n^3 - 3n^2 + 8n)/3$ where $n \in Z^+$ }

Set Theory

- **Basic Concepts**: Other terms/concepts
  - **Universal set: U**

    the *totality of all elements* under consideration.

  - ***Examples***:
    - For set **A** = { a, e, i, o, u }

      U = set of all letters of the English alphabet.
    - For set **B** = {students enrolled in CMSC 56}

      U = set of all students in UPLB.

Set Theory

- **Basic Concepts**: Other terms/concepts
  - **Cardinality of a set A:** |A|
    the number of elements in the (finite) set A.

  - ***Example*:**
    What is the cardinality of each of the ff. sets?
    S =set of students enrolled in CMSC 56
    B =vowels in the English alphabet
    Z ={ x | x is an integer}
    A={ 0, 2, 4, 6, 8}

Set Theory

- **Basic Concepts**: Other terms/concepts
  - **Equal sets A = B**

    iff **a)** whenever $x \in A$ then $x \in B$; *and*

    b) whenever $x \in B$ then $x \in A$.

    In other words

    $$(\mathbf{A = B}) \leftrightarrow (\mathbf{x} \in \mathbf{A} \rightarrow \mathbf{x} \in \mathbf{B}) \wedge (\mathbf{x} \in \mathbf{B} \rightarrow \mathbf{x} \in \mathbf{A})$$

  - ***Example***:

    Which of the ff. sets are equal?

    $A = \{ x \mid x^2 + 2x - 8 = 0 \}$

    $B = \{ 2, -4 \}$

    $C = \{ -4, 2 \}$

    $D = \{ 4, -2 \}$

Set Theory

- **Basic Concepts**: **Set containment**
  - Set A is a **subset** of set B**:** $\mathbf{A \subseteq B}$
    - iff every element of set A is also an element of set B.

    In other words,

    $$(A \subseteq B) \leftrightarrow (\forall x)(x \in A \rightarrow x \in B)$$
  - Set A is a **proper subset** of set B: $\mathbf{A \subset B}$,
    - if and only if
      - A is a subset of B *and*
      - there is an element in set B that is *not* in set A.

    In other words,

    $$(A \subset B) \leftrightarrow (\exists x)(x \in B \rightarrow x \notin A)$$

    or $$(A \subset B) \leftrightarrow (A \subseteq B) \wedge (A \neq B)$$

Set Theory

- **Basic Concepts**: **Set containment**
  - ***Example***:

    Consider

    $A = \{ x \mid x \in Z \text{ and } x \leq 10 \}$

    $B = \{ x \mid x \in Z^{+} \text{ and } x \leq 10 \}$

    $C = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$

  - **Remark**:
    - Alternative definition for set equality:

      $\mathbf{(A = B) \leftrightarrow (A \subseteq B) \wedge (B \subseteq A)}$

Set Theory

- **Basic Concepts**: Other terms/concepts
  - **Theorems**
    - $A \subseteq A$
    - $(A \subset B) \rightarrow (A \subseteq B)$
    - $[\, (A \subseteq B) \wedge (B \subseteq C) \,] \rightarrow (A \subseteq C)$
    - $[\, (A \subset B) \wedge (B \subseteq C) \,] \rightarrow (A \subset C)$
    - $[\, (A \subseteq B) \wedge (B \subset C) \,] \rightarrow (A \subset C)$
    - $[\, (A \subset B) \wedge (B \subset C) \,] \rightarrow (A \subset C)$
  - ***Example***:

    Consider the sets

    $A = \{1, 3, 4, 5\}$,

    $B = \{1, 2, 3, 4, 5, 6\}$

    $C = \{1, 2, 3, 4, 5, 6\}$

Set Theory

- **Basic Concepts**: Other terms/concepts
  - **Null** or **empty set:** $\varnothing$ or **{}**
    does not contain any element.
    Note that $|\varnothing| = 0$ and that $\varnothing \neq \{\varnothing\}$.
  - ***Example***:
    The set $A = \{ x \mid x^2 - 6x + 9 = 0 \text{ and } x < 0 \}$ is an empty set.
  - **Theorem**: For any universe U, if $A \subseteq U$ then $\varnothing \subseteq \mathbf{A}$.
    And if $A \neq \varnothing$ then $\varnothing \subset \mathbf{A}$.

Set Theory

- **Basic Concepts**: Other terms/concepts
  - The **power set** of a set A, denoted by P(A), is the set of all subsets of A.
  - ***Examples***:

    What is the power set of each of the following?

    1. $\{a, b, c\}$
    2. $\{\varnothing\}$

What is the cardinality of the power set of a set A? That is what is $|P(A)|$ equal to?

Set Theory

- **Russell's Paradox**
  - **History**:
    - George Cantor's **naïve set theory** was inconsistent
      - definition of sets was unrestricted
    - Lord **Bertrand Russell** (1872-1970) and **Alfred Whitehead** (1861-1947)
      - developed a theory of types which avoided Russell's Paradox.
  - Let R be the set of all sets that are not elements of themselves. That is, $R = \{ S \mid S \notin S \}$ or if S is a set and $S \notin S$, then $S \in R$. *This set R does not exist*.

Set Theory

- **Russell's Paradox**
  - ***Another version***:

    A set is an ordinary set if it is not an element of itself. Otherwise, it is an extraordinary set.

    Now let R be the set of all ordinary sets, that is, $R = \{ S \mid S \notin S \}$.

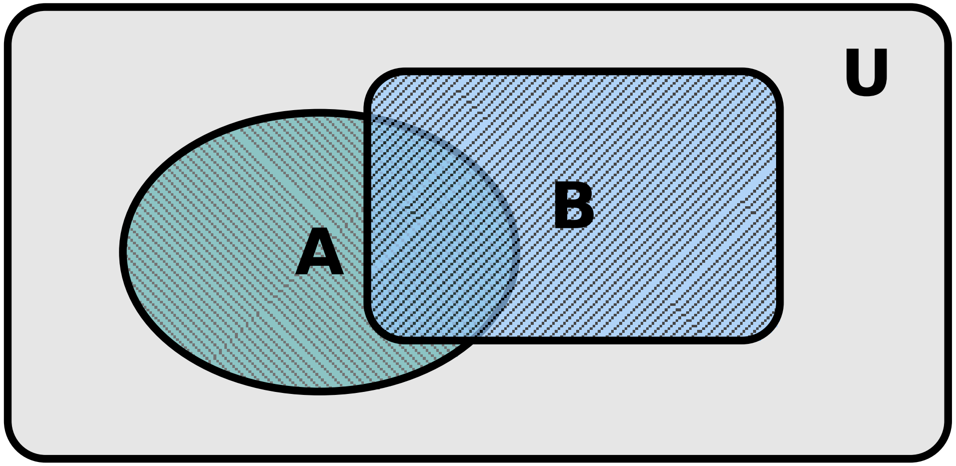
    What kind of set if R, ordinary or extraordinary?
  - ***The Barber's paradox***

    The barber of a certain village shaves everyone who do not shave themselves.

    Who shaves the barber?

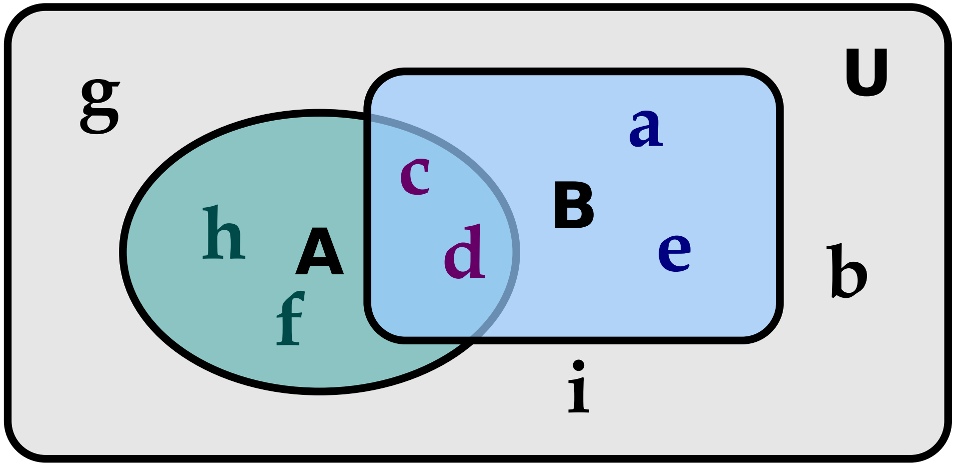
Set Theory

- **Venn Diagrams**
  - visual representations of sets.
  - developed by English mathematician, **John Venn** (1834-1923)
    - **U** $\Rightarrow$ *rectangle*
    - **other sets A, B …** $\in$ **U** $\Rightarrow$ *closed polygons*



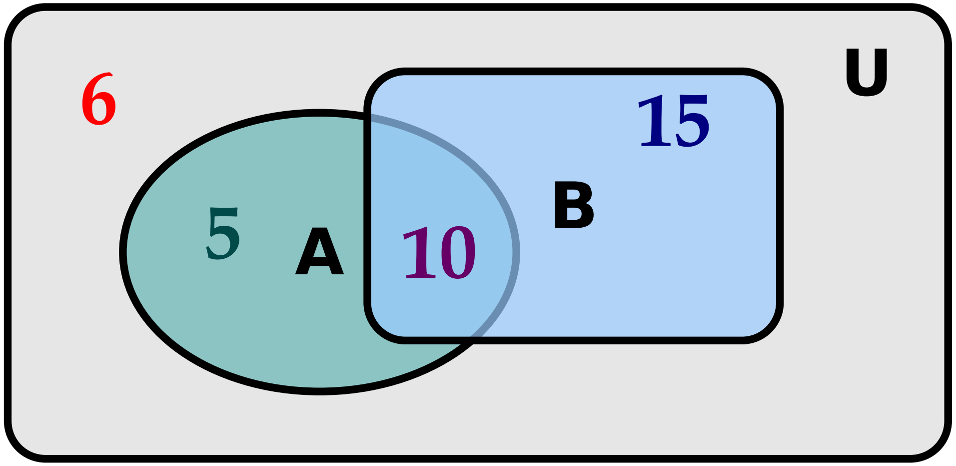
Set Theory

- **Venn Diagrams**
  - visual representations of sets.
  - developed by English mathematician, **John Venn** (1834-1923)
  - may include elements or cardinality of the sets



Set Theory

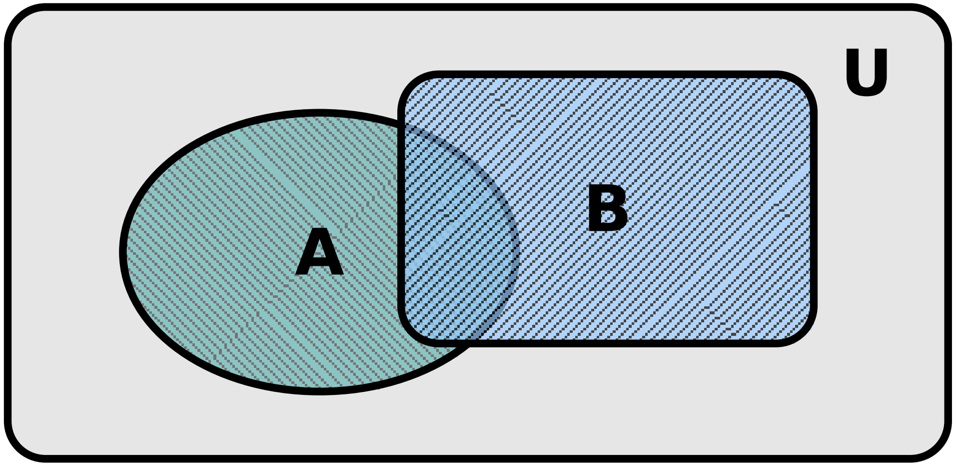
- **Venn Diagrams**
  - visual representations of sets.
  - developed by English mathematician, **John Venn** (1834-1923)
    - may include elements or cardinality of the sets
    - $|\mathbf{A}| = 15$, $|\mathbf{B}| = 25$, $|\mathbf{A} \cap \mathbf{B}| = 10$, $\mathbf{U} = 35$



$|(A \cup B)'| = ?$

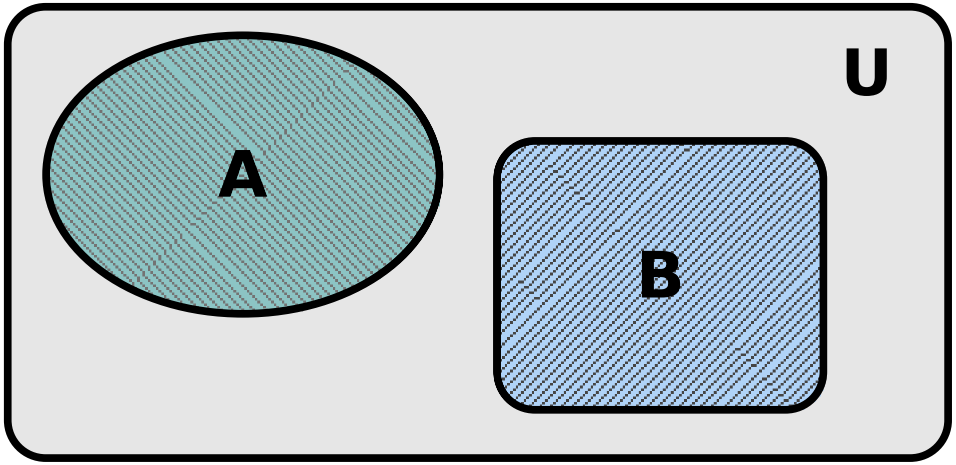
Set Theory

- **Venn Diagrams**
  - ***Example***:
    - Illustrate the following sets given that
      - **U = set of all UPLB students.**
      - **A = set of students enrolled in CMSC 56**
      - **B = set of students enrolled in MATH 26**



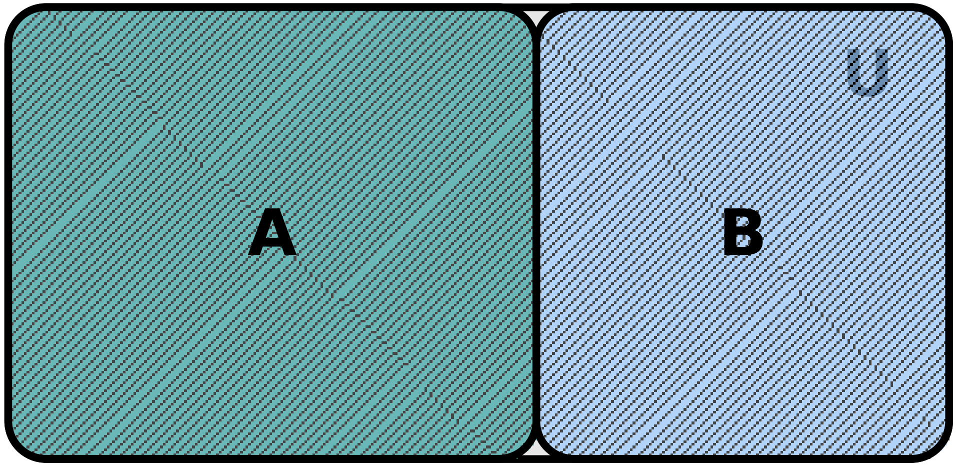
Set Theory

- **Venn Diagrams**
  - ***Example***:
    - Illustrate the following sets given that
      - **U = set of all UPLB students.**
      - **A = set of sophomores**
      - **B = set of juniors**



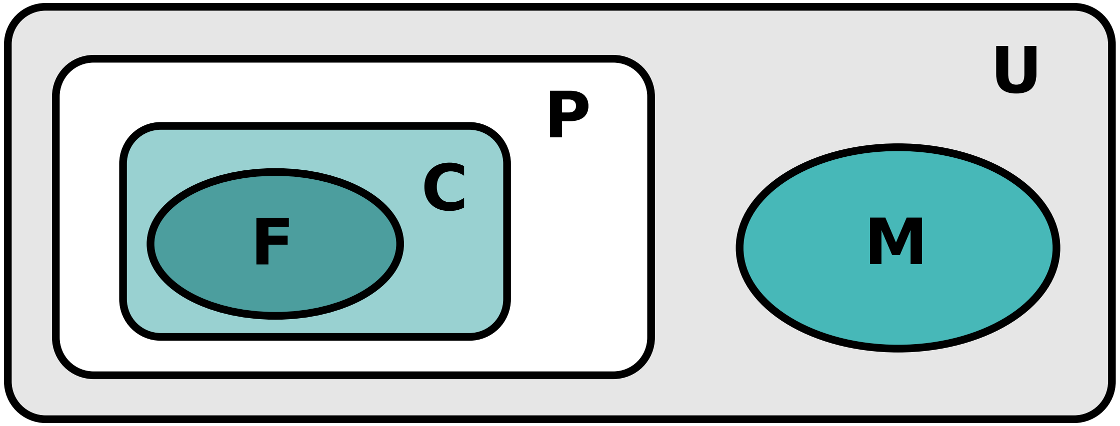
Set Theory

- **Venn Diagrams**
  - ***Example***:
    - Illustrate the following sets given that
      - **U = set of all UPLB students.**
      - **A = set of female students**
      - **B = set of male students**



Set Theory

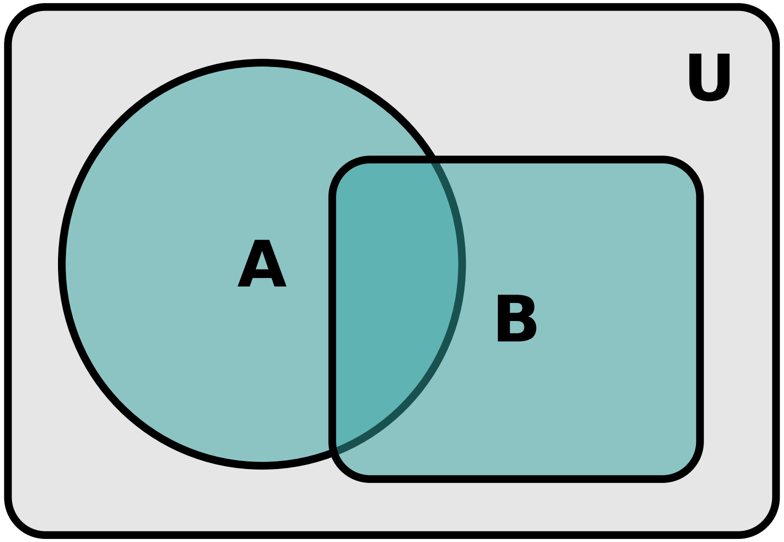
- **Venn Diagrams and Arguments**
  - ***Example***:
    - Check the validity of the following argument:
      - **C**MSC 56 students enjoy **P**rogramming.
      - My **F**riends are CMSC 56 students.
      - None of my class**M**ates enjoy programming.
      - *Therefore* my friends are not my classmates.



Set Theory

- **OPERATIONS ON SETS**
  - The **union** of sets A and B

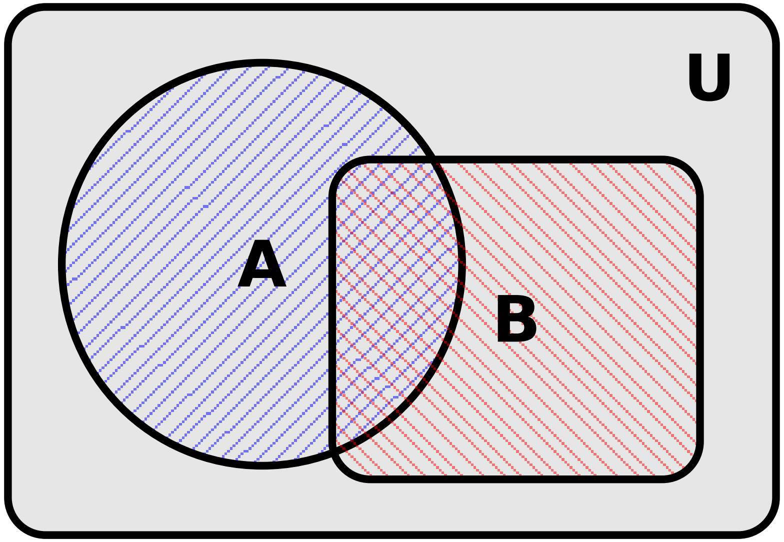
$$\mathbf{A \cup B = \{ x \mid (x \in A) \vee (x \in B)\}}$$



Set Theory

- **OPERATIONS ON SETS**
  - The **intersection** of sets A and B

    $$\mathbf{A \cap B = \{\, x \mid (x \in A) \land (x \in B) \}}$$



Set Theory

- **OPERATIONS ON SETS**
  - The **symmetric difference** of two sets A and B

$$\mathbf{A \oplus B = \{\, x \mid (x \in A) \lor (x \in B) \text{ and } x \notin (A \cap B) \,\}}$$

Set Theory

- **OPERATIONS ON SETS**
  - The **relative complement** of set B w/ respect to set A

    $$\mathbf{A - B = \{ x \mid (x \in A) \land (x \notin B)\}}$$

Set Theory

- **OPERATIONS ON SETS**
  - The **complement** of a set A, denoted by A’

$$A' = U - A = \{ x \mid (x \in U) \wedge (x \in A)\}$$

Set Theory

- ***Examples*:**
  - Define the sets $A = \{ a, c, d, e\}$, $B = \{ c, d, f, h\}$ and $U = \{ a, b, c, d, e, f, g, h, i \}$
  - What are the elements of the following sets?
    - **$A \cup B$**
      $= \{a, c, d, e, f, h\}$
    - **$A \cap B$**
      $= \{c, d\}$
    - **$A – B$**
      $= \{a, e\}$
    - **$B – A$**
      $= \{f, h\}$

Set Theory

- ***Examples*:**
  - Define the sets $A = \{a, c, d, e\}$, $B = \{c, d, f, h\}$ and $U = \{a, b, c, d, e, f, g, h, i\}$
  - What are the elements of the following sets?
    - **$A \oplus B$**
      $= \{a, e, f, h\}$
    - **B'**
      $= \{a, b, e, g, i\}$
    - **$(A \cap B)'$**
      $= (\{c, d\})'$
      $= \{a, b, e, f, g, h, i\}$

Set Theory

- ***Examples*:**
  - Define the sets $A = \{ a, c, d, e\}$, $B = \{ c, d, f, h\}$ and $U = \{ a, b, c, d, e, f, g, h, i \}$
  - What are the elements of the following sets?
    - **$A \cup B'$**

      $= \{a, c, d, e\} \cup \{a, b, e, g, i\}$

      $= \{a, b, c, d, e, g, i\}$
    - **$A \cup \varnothing$**

      $= \{a, c, d, e\}$

      $= A$

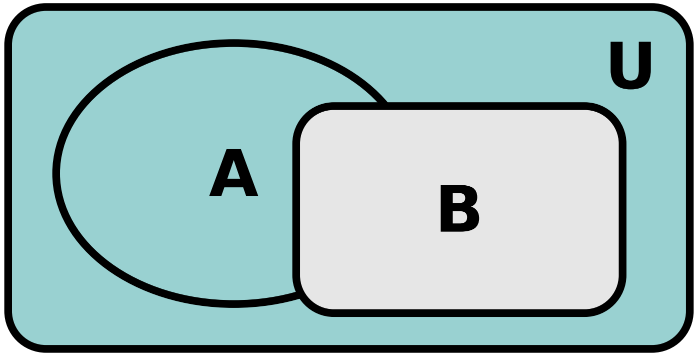
Set Theory

- **OPERATIONS ON SETS**
  - **Disjoint** sets A and B
    - if and only if $\mathbf{A \cap B = \varnothing}$.
  - ***Examples*:**
    - Which of the following sets are disjoint?

      A = { a, c, d, e},
      B = { c, d, f, h}
      C = {a, e, g, i}

Set Theory

- **Venn Diagrams and Set Operations**
  - Shade the area corresponding to set considered
  - ***Example***:
    - **A ∪ B**
    - **B'**



Set Theory

- **QUIZ (1/2 sheet)**
  - Draw a Venn diagram to represent the following facts:

    *Some **p**eople are **h**appy.*
    *Some **p**eople are **w**ealthy.*
    *No **w**ealthy person is **h**appy.*
    *All **p**oets are **h**appy people.*
    *All p**o**liticians are **w**ealthy.*
  - Then verify if the following conclusions are valid (state whether true or false):

    1) No poet is wealthy.
    2) Politicians are happy people.
    3) No person can be both a politician and a poet.

Set Theory

- **Laws of Set Theory**
  - Double Negation
    - $(A')' = A$
  - De Morgan's Laws
    - $(A \cup B)' = A' \cap B'$
    - $(A \cap B)' = A' \cup B'$
  - Commutative Laws
    - $A \cup B = B \cup A$
    - $A \cap B = B \cap A$
  - Associative Laws
    - $A \cup (B \cup C) = (A \cup B) \cup C$
    - $A \cap (B \cap C) = (A \cap B) \cap C$

Set Theory

- **Laws of Set Theory**
  - Distributive Laws
    - $\mathbf{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$
    - $\mathbf{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$
  - Idempotency Laws
    - $\mathbf{A \cup A = A}$
    - $\mathbf{A \cap A = A}$
  - Identity Laws
    - $\mathbf{A \cup \varnothing = A}$
    - $\mathbf{A \cap U = A}$

Set Theory

- **Laws of Set Theory**
  - Inverse Laws
    - $\mathbf{A \cup A' = U}$
    - $\mathbf{A \cap A' = \varnothing}$
  - Domination Laws
    - $\mathbf{A \cup U = U}$
    - $\mathbf{A \cap \varnothing = \varnothing}$
  - Absorption Laws
    - $\mathbf{A \cup (A \cap B) = A}$
    - $\mathbf{A \cap (A \cup B) = A}$

Set Theory

- **Theorems.** Let $A, B \subseteq U$
  1. $A - B = A \cap B'$
  2. $A \oplus B = (A \cup B) - (A \cap B)$
  3. $(A \cap B) \subseteq A \subseteq (A \cup B)$
  4. The ff statements are *equivalent* to each other:
     a) $A \subseteq B$
     b) $A \cup B = B$
     c) $A \cap B = A$
     d) $B' \subseteq A'$
  5. If $A, B \subseteq U$ then A and B are disjoint if and only if $A \cap B = A \oplus B$.

Methods of Proof

- **METHODS of PROOF**
  - **Venn Diagram**
    - Make Venn diagrams and shade the indicated sets.

      Example: Show that $(A - B)' = A' \cup B$
  - **Membership Table**
    - Similar to truth table
    - Columns correspond to sets:
      - **T**(rue) $\Leftrightarrow$ x is in the set
      - **F**(alse) $\Leftrightarrow$ x is not in the set
    - Two sets are equal if the contents of their corresponding columns are exactly the same.

      Example: Show that $(A - B)' = A' \cup B$

Methods of Proof

- **METHODS of PROOF**
  - **Algebraic Method**
    - Like chain of equivalence method
    - Cite theorems and laws of set theory

    Example: Show that $\mathbf{(A - B)' = A' \cup B}$

    Example: Show that $\mathbf{(A \oplus B) - A = B - A}$
  - **Proof by Element Arguments**
    - Use more *formal* definitions of set concepts and concepts of logic.

    Example: Show that $\mathbf{(A - B)' = A' \cup B}$

    Example: Show that $\mathbf{[(A \subseteq B) \wedge (B \subseteq C)] \rightarrow (A \subseteq C)}$