Dynamics Formula Sheet

Base Basics

CHANGE BASE

$$e = \underbrace{(e_1 \cdot e_i')}_{\text{DOT PRODUCT}} e_i'$$

ORTHOGONAL (90° BETWEEN BASE ANGLES)

δ_{ij} = KRONECKER DELTA = WAY TO DESCRIBE

$$e_i \cdot e_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \Rightarrow \text{CAN FORM A BASE}$$

NORMAL BASES

$$E_1 \cdot E_1 = E_2 \cdot E_2 = E_3 \cdot E_3 = 1$$

$$E_1 \cdot E_2 = E_2 \cdot E_3 = E_3 \cdot E_1 = 0$$

RIGHT HAND BASES

$$e_1 \times e_2 = e_3$$

$$e_2 \times e_3 = e_1$$

$$e_3 \times e_1 = e_2$$

CIRCLE DIAGRAMS
- DRAW 3RD VECTOR ALWAYS COMING OUT OF SCREEN

Vector Conventions

$\dot{E} = \frac{d}{dt} E$ $\quad$ $\| \ \|$ = MAGNITUDE $\quad$ $| \ |$ = DETERMINANT

$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_{\{E_i\}} = aE_1 + bE_2 + cE_3$ $\quad$ $\{ \ \}$ = 3 COMPONENTS

$\sim$ = VECTOR $\quad$ $\overset{o}{}$ = ABOUT ORIGIN "O" $\quad$ $Q\big|_{e_i \otimes e_j}$ = MATRIX COMPONENTS OF Q ON $e_i \otimes e_j$

E = FIXED BASE $\quad$ e = DYNAMIC BASE

p = PROJECTION OF p IN DIRECTION OF e_1, e_2, & e_3 SUMMED

$$p_i E_i = p_1 E_1 + p_2 E_2 + p_3 E_3$$

Vector/Matrix Math

DOT PRODUCT
- NEEDS TO BE IN SAME BASE
- VECTORS MUST BE IN SAME BASE

$$p \cdot q = p_1 q_1 + p_2 q_2 + p_3 q_3 = \|p\| \|q\| \cos\theta$$

$= p_i q_i$ ← IMPLIES SUMMATION

θ = ANGLE BETWEEN VECTORS p & q

CROSS PRODUCT
- RIGHT FINGERS IN DIRECTION OF a, CURL TOWARDS b, THUMB TO c

$$a \times b = c$$

$p \times q = \|p\| \|q\| \sin\theta$ $\quad$ θ = ANGLE BETWEEN p & q

$$p \times q = \begin{vmatrix} e_1 & e_2 & e_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}$$

$\{e_i\}$ = COMMON BASE FOR p & q

DERIVATIVE OF VECTOR/BASE/OPERATION

$$\dot{p} = \dot{p}_1 e_1 + p_1 \dot{e}_1 + \dot{p}_2 e_2 + p_2 \dot{e}_2 + \dot{p}_3 e_3 + p_3 \dot{e}_3$$

$$\frac{d}{dt}(p \cdot q) = \dot{p} \cdot q + p \cdot \dot{q}$$

$$\frac{d}{dt}(p \times q) = \dot{p} \times q + p \times \dot{q}$$

$\frac{d}{dt} \cos\theta = -\sin\theta$ $\quad$ $\frac{d}{dt} \sin\theta = \cos\theta$

$\frac{d}{dt} -\cos\theta = \sin\theta$ $\quad$ $\frac{d}{dt} -\sin\theta = -\cos\theta$

COROTATIONAL DERIVATIVE
- RATE OF CHANGE OF p WRT $\{e_i\}$ OBSERVED BY OBSERVER FIXED ON $\{e_i\}$

$\overset{o}{p}\big|_{e_i}$ = COROTATIONAL DERIVATIVE OF p WRT $\{e_i\}$

BASICALLY ASSUME THE BASE IS NOT MOVING

$$\dot{p} = \dot{p}_i e_i + p_i \dot{e}_i = \underbrace{\overset{o}{p}\big|_{e_i}}_{\text{COROTATIONAL DERIVATIVE}} + \omega \times (p_i e_i) = \overset{o}{p}\big|_{e_i} + \omega \times p$$

$\overset{o}{p}\big|_{e_i} = \dot{p}_i e_i$ ← BASE VECTORS NO LONGER CHANGE WRT TIME (OBSERVER ON BASIS)

MAGNITUDE

$\|p\| = \sqrt{p_1^2 + p_2^2 + p_3^2}$ ← COMPONENTS ON SAME BASE (NO MIXED)

$$\|p\|^2 = p \cdot p$$

PROJECTION

PROJECTION OF p ON q $= \frac{p \cdot q}{\|q\|} \frac{q}{\|q\|}$

MATRIX MULTIPLICATION

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad$ $AB = C$ $\quad$ $\overset{i}{\text{ROW}}\ A \cdot \overset{j}{\text{COLUMN}}\ B = C_{ij}$

IF: $C = AB$

$\Rightarrow C^T = B^T A^T$

Tensors

TENSOR: LINEAR MAPPING OF VECTORS INTO VECTORS
- REPRESENTED AS COMPONENTS OF A MATRIX RELATIVE TO A SPECIFIC BASIS
- ELEMENT OF THE TENSOR PRODUCT OF TWO OR MORE VECTOR SPACES
- DOT PRODUCT OF b & c MAPPED IN DIRECTION OF a

← VECTORS ARE TENSORS OF RANK 1, SCALARS TENSOR OF RANK ZERO

RANK: HOW MANY VECTORS YOU NEED TO DESCRIBE THE TENSOR

2 → (9 COMPONENTS, 9 SETS OF 2 BASE VECTORS)

3 → 27 COMPONENTS, 27 SETS OF BASE VECTORS, 3 SETS OF BASE VECTORS

General Tensor Equation

$$A = e_i \otimes e_i' = A_{ij}\, e_i \otimes e_j$$

A = TENSOR THAT HAS A RULE ACCORDING TO SOME RULE

A_{ij} = MATRIX THAT REPRESENTS A ON SOME GIVEN BASIS (EX: $e_i \otimes e_i'$) — INPUT BASE, OUTPUT BASE

TENSOR (LINEAR MAPPING) INPUT + OUTPUT

WHERE... $A_{ij} = e_i \cdot A e_j$ — TYPICALLY OUTPUT BASE

MATRIX REPRESENTING A ON CHOSEN BASIS (SUM OF 9 COMPONENT TENSORS)
- EACH TERM IS A TENSOR

BASE TENSORS $(e_i \otimes e_j)$

Tensor Notation + Properties

$$Sc = (A+B)c = Ac + Bc$$

IF $P = AB$

$$Pc = (AB)c = A(Bc)$$

GENERALLY: $AB \neq BA$

A = TENSOR

$A_{ij} = \underline{A}$ = COMPONENTS OF A ON A GIVEN BASIS

SYMMETRIC TENSOR: $A = A^T$

SKEW-SYMMETRIC MATRIX: $A^T = -A$

Tensor Product/Bun Operator

DEFINES A 2D BASIS:
- NO BASE CHANGE: $(e_i \otimes e_j)$ ← INPUT VECTOR u ON $\{e_j\}$ → OUTPUT $\{e_i\}$ $\quad$ $Au = z$
- BASE CHANGE: $(e_i \otimes E_j)$ ← INPUT VECTOR u ON $\{e_j\}$ → OUTPUT $\{E_i\}$

$$\underbrace{(a \otimes b)}_{\text{TENSOR}}\, c = a\,(b \cdot c)$$

VECTORS

REMEMBER: $(e_i \cdot e_j)$ IS A SCALAR
- IT CAN BE MOVED AROUND BUN OPERATORS

Tensor Derivatives

$$\dot{A} = \overset{o}{A}\big|_{e_i \otimes e_j} + \Omega' A - A \Omega'$$

COROTATIONAL DERIVATIVE: $\overset{o}{A}\big|_{e_i \otimes e_j} = \dot{A}_{ij}\, e_i' \otimes e_j'$

Index Math

KRONECKER DELTA
- WE'RE LOOKING FOR THE SAME BASE TO DOT PRODUCT ITSELF TO MAKE δ
- USE δ'S TO REPLACE + GET RID OF INDICES

$$\delta_{ji} = \delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & \text{IF } i=j \\ 0 & \text{IF } i \neq j \end{cases}$$

$$\delta_{ij} = e_i \cdot e_j$$

TENSOR PROPERTIES/HELPFUL INDICES

BUILDING COMPONENT MATRIX W/ TENSOR INDICES:

$V_i^2 = V_i V_i$

$A_{ij} = B_{ik} C_{kj}$
- k WON'T CHANGE, i & j WILL
- $B_{11}C_{11} + B_{12}C_{21} =$

$A_i = B_{ijj}$ ← $A_i = B_{i11} + B_{i22} + B_{i33}$

$A_{ij}\delta_{jr} = A_{ir}$ ←
- REPLACE 1ST INDEX IN KRONECKER W/ 2ND INDEX
- GET RID OF j IN ENTIRE PROBLEM

TRANSPOSE:

$[A_{ij}]^T = A_{ji}$

$A^T u = (A_{ij})^T u_i = A_{ji} u_i$

$A^T = A_{ji}\, e_i \otimes e_j$

$(C + D)^T = C^T + D^T$

BUN OPERATOR:

$(a \otimes b)^T = (b \otimes a)$

$(e_i \otimes e_j)(e_k \otimes e_m) = \delta_{jk}(e_i \otimes e_m)$

MATRIX MATH

$AB = A_{jm} B_{mk}$

$A^T B = A_{ij} B_{ik}$

$AB^T = A_{jm} B_{km}$

NOTE: YOU CAN ONLY MULTIPLY TENSORS IN THE SAME BASIS

Inertia Tensor

$H = J\omega$ $\quad$ $J^T = J$ SYMMETRIC TENSOR

$$J_{ij} = E_i \cdot J E_j$$

DIAGONALS ARE MOMENTS OF INERTIA $(J_{11}, J_{22} \ldots)$

PRODUCTS OF INERTIA $(J_{12}\ J_{13}\ J_{23})$

- PUT INTO BODY-FIXED FRAME TO REMOVE TIME DEPENDENT TERMS
- EIGENVECTORS OF J (u_i) ARE AXES OF ROTATION

Rotation + ∢ Velocity Tensors

ROTATION

DEFINE: $Q = e_i \otimes e_i'$ AND $e_i = Q e_i'$

$$Q_{ij} = e_i \cdot Q e_j$$

PROPERTIES: $Q = e_i \otimes e_i'$ OR $Q^T = e_i' \otimes e_i$

$Q^{-1} = Q^T$ ← SYMMETRIC MATRIX

- Q IS EQUAL + UNIQUE ON ALL BASES WHICH IT IS DEFINED $(Q_{ij})_{e_i \otimes e_j,\ E_i \otimes E_j}$

COROTATIONAL DERIVATIVE: $\overset{o}{Q}\big|_{e_i \otimes e_j} = \dot{Q}\, e_i \otimes e_j$

MULTIPLE ROTATIONS: $Q = Q_2 Q_1$ ← IF $Q_1 = e_i' \otimes E_i$, $Q_2 = e_i \otimes e_i'$ → $Q = e_i \otimes E_i$

VELOCITY

DEFINE: $\Omega = \dot{Q} Q^T$

Ω = ∢ VELOCITY TENSOR OF $\{e_i\}$

ω = ∢ VELOCITY VECTOR OF $\{e_i\}$ $\quad$ Ω_{REL} = ∢ VELOCITY TENSOR OF $\{e_i\}$ RELATIVE TO $\{e_i'\}$

PROPERTIES: $\Omega^T = -\Omega$ ← SKEW SYMMETRIC (ZEROS ON DIAGONALS)

BASE VELOCITIES: $\dot{e}_i = \Omega e_i = \omega \times e_i$

RELATIVE VELOCITY: $\Omega_{REL} = \overset{o}{Q}\big|_{e_i \otimes e_j} Q'$ $\quad$ WHERE $Q' = e_i \otimes e_i'$

∢ VELOCITY $\quad$ $\dot{\omega} = \overset{o}{\omega}\big|_{\{e_i'\}} + \omega \times \omega$

∢ VELOCITY VECTOR: $\Omega p = \omega \times p$

$$\omega = -\Omega_{23} e_1 + \Omega_{13} e_2 - \Omega_{12} e_3$$

- NO INDICES MATCH (I.E. $\Omega_{23}\, e_1$ ← 1, 2, 3 NO MATCHES) NEGATIVE e_1, e_3

RIGID BODY

POSITION: $\underset{\sim}{r}(t) = \underset{\sim}{r}_p(t) + \underset{\sim}{Q}\left(\underset{\sim}{R} - \underset{\sim}{R}_p\right)$

$\underset{\sim}{r}$ = POSITION TO ARBITRARY POINT ON RB
$\underset{\sim}{Q}$ = ROTATION VEC ABOUT POINT P
$\underset{\sim}{r}_p$ = POSITION TO POINT P @ TIME T
ORIGINAL CONFIGURATION @ $t = 0$
$R - R_p = r(0) - r_p(0)$

VELOCITY: $\dot{\underset{\sim}{r}}(t) = \dot{\underset{\sim}{r}}_p + \underset{\sim}{\omega} \times \left(\underset{\sim}{r} - \underset{\sim}{r}_p\right)$

or $\dot{\underset{\sim}{\pi}} = \underset{\sim}{\Omega}\underset{\sim}{\pi} = \underset{\sim}{\omega} \times \underset{\sim}{\pi}$

$\underset{\sim}{\omega}$ = ∡ VELOCITY VECTOR FOR ANY POINT
$\underset{\sim}{\pi}$ = POSITION OF ARBITRARY PARTICLE IN CURRENT CONFIG
$\underset{\sim}{\omega}$ = TOTAL ∡ VELOCITY VECTOR (AXIAL VECTOR OF $\underset{\sim}{\Omega}$)

ACCELERATION $\ddot{\underset{\sim}{r}}(t) = \ddot{\underset{\sim}{r}}_p + \dot{\underset{\sim}{\omega}} \times \left(\underset{\sim}{r} - \underset{\sim}{r}_p\right) + \underset{\sim}{\omega} \times \left[\underset{\sim}{\omega} \times \left(\underset{\sim}{r} - \underset{\sim}{r}_p\right)\right]$

ROTATION VECTOR: $\underset{\sim}{e}_i = \underset{\sim}{Q}\underset{\sim}{E}_i$

$\underset{\sim}{p}$ = CORRESPONDING UNIT EIGENVECTOR

- EIGENVAL OF 1
- BE CAREFUL W/ORDER OF ROTATIONS

∡ VELOCITY TENSOR

$$\underset{\sim}{\Omega} = \dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T$$

2 WAYS TO FIND $\underset{\sim}{\omega}$:
- FIND $\underset{\sim}{\Omega} = \dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T$, W/ $\underset{\sim}{Q}$ AS $\underset{\sim}{Q} = \underset{\sim}{Q}_a\underset{\sim}{Q}_b\underset{\sim}{Q}_c$ (ALL ROTATIONS)
- DETERMINE COMPONENTS OF $\underset{\sim}{\Omega}$, AND GET AXIAL VECTOR

TRANSFORM FROM $\{\underset{\sim}{E}_i\}$ TO $\{\underset{\sim}{e}_i\}$ EXAMPLE:

$$\underset{\sim}{Q} = \underset{\sim}{Q}_\phi\underset{\sim}{Q}_\theta\underset{\sim}{Q}_\psi$$

$\underset{\sim}{e}_i = \underset{\sim}{Q}_\phi\underset{\sim}{e}_i''$
IFF $\underset{\sim}{e}_i'' = \underset{\sim}{Q}_\theta\underset{\sim}{e}_i'$
$\underset{\sim}{e}_i' = \underset{\sim}{Q}_\psi\underset{\sim}{E}_i$

EXAMPLE:

$$\underset{\sim}{\Omega}_\theta = \dot{\underset{\sim}{Q}}_\theta\underset{\sim}{Q}_\theta^T = \Omega_{\theta ij}\,\underset{\sim}{E}_i \otimes \underset{\sim}{E}_j$$

$$\underset{\sim}{\omega}_\theta = \dot{\theta}\underset{\sim}{E}_3 \qquad \underset{\sim}{\omega}_\phi = \dot{\phi}\underset{\sim}{e}_1'$$

$$\underset{\sim}{\omega} = \underset{\sim}{\omega}_\theta + \underset{\sim}{\omega}_\phi$$

∡ ACCELERATION TENSOR

$$\dot{\underset{\sim}{\Omega}} = \frac{d}{dt}\left(\underset{\sim}{\Omega}\right)$$

$\underset{\sim}{\omega}$ IS AXIAL VEC IN TERMS OF $e_i \otimes e_j$ IN ∡ VEL TENSOR

GENERAL MOTION

$\underset{\sim}{r} = \ddot{\bar{\underset{\sim}{r}}} + Q\left(R - R\right)$

TRANSLATION: $\underset{\sim}{F} = m\ddot{\bar{\underset{\sim}{r}}}$

ANGULAR: $\underset{\sim}{M}^o = \dot{\underset{\sim}{H}}^o$

$\underset{\sim}{F}$ = RESULTANT OF ALL FORCES ACTING ON RB
$\ddot{\bar{r}}$ = ACCELERATION OF RB C.M.
$\underset{\sim}{H}$ = ∡ MOMENTUM ABOUT "O"
$\underset{\sim}{M}$ = RESULTANT MOMENT ABOUT "O"

$\underset{\sim}{F} = \dot{\underset{\sim}{G}}$

ROTATION ABOUT FIXED POINT

$\underset{\sim}{r} = \underset{\sim}{Q}\underset{\sim}{R}$

$\underset{\sim}{F} = m\ddot{\underset{\sim}{r}}$

$\underset{\sim}{M}^o = \dot{\underset{\sim}{H}}^o$

$\underset{\sim}{F}$ = REACTION FORCES

IMPORTANT EQNS

$\dot{\underset{\sim}{e}}_i = \underset{\sim}{\omega} \times \underset{\sim}{e}_i$ $\qquad$ $\dot{\underset{\sim}{e}}_i = \underset{\sim}{\Omega}\underset{\sim}{e}_i = \underset{\sim}{\omega} \times \underset{\sim}{e}_i$ $\qquad$ $\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^T$

$\underset{\sim}{Q} = \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j'$ $\qquad$ $\underset{\sim}{\Omega} = \dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T$ $\qquad$ $\underset{\sim}{\Omega}^T = -\underset{\sim}{\Omega}$

$Q_{ij} = \underset{\sim}{e}_i \cdot \underset{\sim}{Q}\underset{\sim}{e}_j$ $\qquad$ $\underset{\sim}{r} = \underset{\sim}{r}_p + \underset{\sim}{Q}\left(\underset{\sim}{R} - \underset{\sim}{R}_p\right)$

$(a \otimes b)c = a\underbrace{(b \cdot c)}_{\text{SCALAR}}$ $\qquad$ $\underset{\sim}{Q} = \underset{\sim}{Q}_2\underset{\sim}{Q}_1$ $\qquad$ $\dot{\underset{\sim}{\omega}} = \dot{\underset{\sim}{\omega}}\big|_{\{e_i'\}} + \underset{\sim}{\omega}' \times \underset{\sim}{\omega}$

PARTIAL DERIVATIVES

$f = f(x, y)$

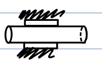
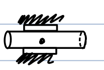
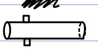
$\frac{\partial f}{\partial x}(xy) = y$

$\frac{\partial f}{\partial x}(x^2y) = 2xy$

$\frac{\partial f}{\partial x}(\sin(x)) = \cos(x)$

$\frac{\partial f}{\partial x}(\cos^2(x)) = -2\cos(x)\sin(x)$

BEARING/FIXED FORCES

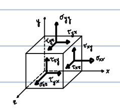
	FORCE COMPONENTS	MOMENT COMPONENTS	
ROLLER:	$\underset{\sim}{F} \to 2$	$\underset{\sim}{M} \sim 2$	
THRUST BEARING:	$\underset{\sim}{F} \sim 3$	$\underset{\sim}{M} \sim 2$	
BALL BEARING	$\underset{\sim}{F} \sim 2$	$\underset{\sim}{M} \sim 0$	

1ST ↓ 2ND → 3RD ← INDICES DIRECTIONS

$$\text{DYAD} \quad \begin{array}{c} x \\ y \\ z \end{array}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3 COMPONENTS, EACH W/A VECTOR
9 COMPONENTS

$$\begin{array}{l} T_x \text{ (TOP ROW)} \to \\ T_y \text{ (MID ROW)} \to \\ T_z \text{ (BOT ROW)} \to \end{array}\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$



INERTIA TENSOR

$$\underset{\sim}{T} = \underset{\sim}{I}\,\underset{\sim}{\alpha}$$

$$\underset{\sim}{F} = m\ddot{\underset{\sim}{r}}$$

$$\dot{\underset{\sim}{W}} = \underset{\sim}{\alpha}$$

T = TORQUE
α = ∡ ACCELERATION
$\underset{\sim}{I} = m\underset{\sim}{R}^2$
$\underset{\sim}{\omega}$ = ∡ VELOCITY

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{zy} & I_{zz} \end{bmatrix}\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$

NEWTONIAN POSTULATES OF DYNAMICS

·RULES FOR POSTULATES:

LINEAR MOMENTUM

$\underset{\sim}{G}_i = \sum m_i \underset{\sim}{r}_i$ $\qquad \underset{\sim}{F}_i = m \ddot{\underset{\sim}{r}}_i$ OR $\underset{\sim}{F} = \int_\rho \rho \underset{\sim}{b}\, dV + \int_{d\rho} \underset{\sim}{t}\, dA$

G_i = LINEAR MOMENTUM

$\underset{\sim}{b}$ = BODY FORCE/UNIT MASS
$\underset{\sim}{t}$ = CONTACT FORCE/UNIT AREA

$\dot{r}$ = VELOCITY
V = VOLUME
ρ = DENSITY
m = MASS

REMEMBER: $\underset{\sim}{r}_i \neq r_i$ $\quad r_i$ HAS NO DIRECTION, ONLY MAGNITUDE

$\underset{\sim}{F}_i$ = EXTERNAL FORCES

1. WRITE POSITION VECTOR

$$\underset{\sim}{r} = r_i \underset{\sim}{e}_i$$

·CAN BE A MIXED BASIS

REMEMBER: $\ddot{\underset{\sim}{r}}_i \neq \omega \times r_i$

2. DERIVE VELOCITY VECTOR

$$\dot{\underset{\sim}{r}}_i = \frac{d}{dt}\left(r_i \underset{\sim}{e}_i\right) = \dot{r}_i \underset{\sim}{e}_i + r_i \dot{\underset{\sim}{e}}_i$$

$$\dot{\underset{\sim}{e}}_i = \underset{\sim}{\omega} \times \underset{\sim}{e}_i$$

$\underset{\sim}{\omega}$ = ∡ VELOCITY OF BASE SET $\{\underset{\sim}{e}_i\}$ WHERE $\underset{\sim}{e}_i$ RESIDES
$\underset{\sim}{e}_i$ = BASE VECTOR SET $\{e_i\}$

3. DERIVE ACCELERATION VECTOR

$$\ddot{\underset{\sim}{r}}_i = \frac{d}{dt}\left(\dot{r}_i\right) \underset{\sim}{e}_i + \dot{r}\, \underset{\sim}{e}_i$$

$$\dot{\underset{\sim}{e}}_i = \underset{\sim}{\omega} \times \underset{\sim}{e}_i$$ ← SAME AS ABOVE

POSTULATE RULES + VARIABLES

1. POSTULATES ARE INDEPENDENT OF EACH OTHER
2. APPLY TO BODY OR SYSTEM AS A WHOLE, AND ANY ARBITRARY SUBSET OF SYSTEM
3. IT'S USEFUL TO DECOMPOSE MOTION INTO TRANSLATION + ROTATION ABOUT THE CENTER OF MASS

MASS

$$\left.\frac{dm}{dt}\right|_{\text{FIXED BODY}} = 0$$

m = MASS

·BODY HELD FIXED

ANGULAR MOMENTUM

$\underset{\sim}{H}_i^o$ = ∡ MOMENTUM ABOUT ORIGIN "O"
$\underset{\sim}{M}^o$ = MOMENTS ABOUT ORIGIN "O"

$$\underset{\sim}{M}_i^o = \dot{\underset{\sim}{H}}_i^o$$

$$\underset{\sim}{M}^o = \int_\rho \rho \underset{\sim}{r} \times \underset{\sim}{b}\, dV + \int_{d\rho} \underset{\sim}{r} \times \underset{\sim}{t}\, d\underset{\sim}{A}$$

r = POSITION VECTOR RELATIVE TO A FIXED POINT

$$\underset{\sim}{H}^o = \int_\rho \underset{\sim}{r} \times (\rho \dot{\underset{\sim}{r}})\, dV = \int_\rho \rho \underset{\sim}{r} \times \dot{\underset{\sim}{r}}\, dV$$

OR $\underset{\sim}{M}^o = \sum_{i=0}^{n} r_i \times F_i$ $\qquad$ OR $\underset{\sim}{H}^o = \sum_{i=1}^{n} m_i \underset{\sim}{r}_i \times \dot{\underset{\sim}{r}}_i$

LAGRANGIAN

$$\underset{\sim}{r}_j = r_j\left(q_1, q_2, \ldots, q_n\right)$$

$n = 3N - M$
$j = 1, 2, \ldots, n$

N = # OF PARTICLES
M = # OF CONSTRAINTS

LAGRANGE STEPS

$j = 1, 2, \ldots, N$
$i = 1, 2, \ldots, n$

$\frac{\partial \underset{\sim}{r}_j}{\partial \dot{q}_i}$ = COEFFICIENT OF $\dot{q}_i$ IN $\dot{\underset{\sim}{r}}_j$ — USED IN Q_i

$\frac{\partial \underset{\sim}{\omega}_j}{\partial \dot{q}_i}$ = COEFFICIENT OF $\dot{q}_i$ IN $\underset{\sim}{\omega}_j$

$\frac{\partial \underset{\sim}{r}_j}{\partial q_i}$

$$\underbrace{\dot{\underset{\sim}{r}}_j = \sum_{i=1}^{n} \frac{\partial \underset{\sim}{r}_j}{\partial q_i} \dot{q}_i + \frac{\partial \underset{\sim}{r}_j}{\partial t}}_{\text{WEIRD CHAIN RULE THAT IS USED WHEN CALCULATING } \frac{\partial \dot{\underset{\sim}{r}}_j}{\partial \dot{q}_i}}$$

1. BASIS + CIRCLE DIAGRAMS
2. ∡ VELOCITY OF EACH RB
3. POSITION VECTORS $\underset{\sim}{r}_j$ FOR EACH PARTICLE
   - WRITE ONLY IN TERMS OF GENERALIZED COORDINATES + THEIR DERIVATIVES
     - MAKE SURE TO EXPAND ANY ROTATING BASIS WHICH HAS GENERALIZED COORDINATE DEPENDANCE
   - IF RB., WRITE $\underset{\sim}{r}_j$ OF ORIGINS + INCLUDE $\underset{\sim}{\omega}_j$ PART OF KINETIC ENERGY FORMULA
4. TOTAL SYSTEM ENERGY (KINETIC ENERGY)
5. DRAW FBD FOR EACH j & FIND:
   1. $\underset{\sim}{F}_j$
      - APPLIED FORCES (GRAVITY, EXTERNAL, SPRINGS)
      - CONSTRAINT (NON-HOLONOMIC AND STUFF NOT DESCRIBING $\underset{\sim}{r}_j$ OR $\underset{\sim}{\omega}_j$)
   2. $\underset{\sim}{M}_j$
      - SAME AS ABOVE, BUT MOMENTS
6. FIND Q_i's (NOT A ROTATION TENSOR)
7. APPLY LAGRANGE'S EQNS TO OBTAIN n EQNS OF MOTION, ONE FOR EACH q_i

$$T = \frac{1}{2} \sum_{j=1}^{N} m_j \left(\dot{\underset{\sim}{r}}_j \cdot \dot{\underset{\sim}{r}}_j\right) + \underbrace{\underset{\sim}{\omega} \cdot J_j \underset{\sim}{\omega}_j}_{\text{ONLY FOR RB}}$$

RIGID BODIES:

$$Q = \sum_{i=1}^{n} \left(\underset{\sim}{F}_j \cdot \frac{\partial \dot{\underset{\sim}{r}}_j}{\partial \dot{q}_i} + \underset{\sim}{M}_j \cdot \frac{\partial \underset{\sim}{\omega}_j}{\partial \dot{q}_i}\right) \qquad i = 1, 2, \ldots, n$$

PARTICLES:

$$Q_i = \sum_{j=1}^{N} \underset{\sim}{F}_j \cdot \frac{\partial \underset{\sim}{r}_j}{\partial q_i} = \sum_{j=1}^{N} \underset{\sim}{F}_j \cdot \frac{\partial \dot{\underset{\sim}{r}}_j}{\partial \dot{q}_i} \qquad j = 1, 2, \ldots, N \quad i = 1, 2, \ldots, n$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i(t) \qquad i = 1, 2, \ldots, n$$

DETERMINING GENERALIZED COORDINATES

1. IF YOU WANT A PARTICULAR CONSTRAINT FORCE, INCLUDE THE PARAMETER + ITS BASIS VECTOR
2. IF THERE IS ANY FORCE DEPENDANT ON VELOCITY, INCLUDE IT $\left(\dot{\underset{\sim}{r}}_P \cdot \underset{\sim}{e}_2 = 0\right)$
   - NO-SLIP CONDITIONS, KNIFE EDGE, DRAG
3. IF ∡ VELOCITY OF BASIS IS CONSTANT, YOU CAN IGNORE IT