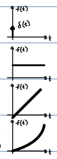
Classical Controls Formula Sheet

Inputs

Impulse	$\delta(t)$	$\int_{0-}^{0+} \delta(t) = 1$	Transient response modeling
Step	$u(t)$	$u(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$	Transient response, steady state error
Ramp	$t\,u(t)$	$t\,u(t) = \begin{cases} t & t>0 \\ 0 & t<0 \end{cases}$	Steady state error
Parabola	$\frac{1}{2}t^2 u(t)$	$\frac{1}{2}t^2 u(t) = \begin{cases} \frac{1}{2}t^2 & t>0 \\ 0 & t<0 \end{cases}$	Steady state error



Absolute step vs magnitude step
Go to "M" — Move by "M"

Elements

Mass — Spring — Damper

$F = m\ddot{x}$; $F = k(x_2 - x_1)$; $F = c(\dot{x}_2 - \dot{x}_1)$

Resistor — Cap — Inductor

$Ri_R = V_1 - V_2$; $V_1 - V_2 = \frac{1}{C}\int i_c\,dt$; $V_1 - V_2 = L\frac{di_L}{dt}$

$\frac{V(s)}{I(s)} = R$ $\quad \frac{V(s)}{I(s)} = Cs$ $\quad \frac{V(s)}{I(s)} = \frac{1}{Ls}$

Laplace Transform Theories/Properties

$\mathcal{L}[kf(t)] = kF(s)$	Linearity
$\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity
$\mathcal{L}[e^{at}f(t)] = F(s-a)$	Frequency shift
$\mathcal{L}[f(t-T)] = e^{-sT}F(s)$	Time shift
$\mathcal{L}[f(at)] = \frac{1}{a}F(\frac{s}{a})$	Scaling
$\mathcal{L}[\frac{df}{dt}] = sF(s) - f(0-)$	Differentiation
$\mathcal{L}[\frac{d^2f}{dt^2}] = s^2F(s) - sf(0-) - f'(0-)$	
$\mathcal{L}[\frac{d^nf}{dt^n}] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	
$\mathcal{L}[\int_{0-}^{t} f(\tau)d\tau] = F(s)/s$	Integration

Laplace Transform

$f(t)$	Without $u(t)$	$F(s)$
$\delta(t)$	$\delta(t)$	1
$u(t)$	1	$\frac{1}{s}$
$t\,u(t)$	t	$\frac{1}{s^2}$
$t^n u(t)$	t^n	$\frac{n!}{s^{n+1}}$
$e^{at}u(t)$	e^{at}	$\frac{1}{s-a}$
$\sin(\omega t)\,u(t)$	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)\,u(t)$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

Conditions: Poles of $sF(s)$ lie in open left half plane

Final value

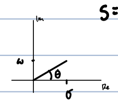
$f(\infty) = \lim_{s\to 0} sF(s)$

Initial value

$f(0+) = \lim_{s\to 0}$

$s = \sigma \pm j\omega$ $\quad j = \sqrt{-1}$

$\theta = \tan^{-1}\left(\frac{\omega}{\sigma}\right)$



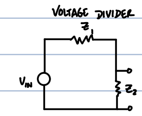
Note: multiplying by imaginary will rotate 90°

Elements

Electrical

Resistor	$V_R(s) = Z_R I_R(s) = RI_R(s)$
Inductor	$V_L(s) = Z_L I_L(s) = LsI_L(s)$
Capacitor	$V_C(s) = Z_C I_C(s) = \frac{1}{Cs}I_C(s)$

Voltage divider simplification



$V_0 = \frac{Z_2}{Z_1 + Z_2}V_{in}$

Applies to nearly everything

Transfer Functions

$G(s) = \frac{Y(s)}{U(s)}$ $\quad Y(s)$ = output, $U(s)$ = input

If $G(s) = \frac{N(s)}{D(s)}$ $\quad N(s) = 0 \Rightarrow$ zeros; $D(s) = 0 \Rightarrow$ poles

- Poles affect stability
  - & transient response if 2nd order
- Zeros affect transient response

Final value theorem

$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$

Only applies if all poles of $sY(s)$ are in open left half plane

Initial value

$\lim_{s\to\infty} sF(s) = f(t=0+)$

Natural vs forced response

$\underbrace{\dot{v}(t) + \frac{1}{\tau}v(t)}_{\text{free/natural}} = \underbrace{\frac{K}{\tau}f(t)}_{\text{forced}}$

- From initial conditions
- From forcing function

$v(t)$ = free + forced

First-Order Systems

$G(s) = \bar{K}\frac{\sigma}{s+\sigma}$ $\quad \sigma = 1/\tau$ $\quad \tau$ = time constant

$\bar{K}$ = DC gain

Poles: $s = -\sigma$

Behavior

$g(t\to\infty) = K$

$G(s) = \frac{b_0}{a_0}\frac{a_0}{s+a_0}$ $\quad \frac{b_0}{a_0} = \bar{K}$ $\quad \frac{1}{a_0} = \tau$

$t_r \approx \frac{2.2}{a_0}$ $\quad t_r$ = rise time (0.1 F.V → 0.9 F.V)

$t_s \approx \frac{4}{a_0}$ $\quad t_s$ = settling time (± 2% of F.V.)

Example: temp of a cooling drink

Second Order Systems (s^2 in bottom of T.F.)

$T(s) = \bar{K}\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

ω_n = natural frequency
$\bar{K}$ = DC gain
ξ = damping coefficient, $0 \le \xi < \infty$

$T(s) = \frac{b_0}{a_2(s^2 + \frac{a_1}{a_2}s + \frac{a_0}{a_2})} = \frac{b_0}{a_0}\frac{a_0/a_2}{s + \frac{a_1}{a_2}s + \frac{a_0}{a_2}}$

Damping

$\xi = 0 \Rightarrow$ no damping
$0 < \xi < 1 \Rightarrow$ underdamped
$\xi > 1 \Rightarrow$ overdamped

If $U(s) = \frac{M}{s}$ & $T(s) = \frac{Y(s)}{U(s)}$ $\Rightarrow y(t) = \bar{K}M\left(1 - \frac{1}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\cos(\omega_n\sqrt{1-\xi^2}\,t - \phi)\right)$

$\phi = \tan^{-1}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)$

Only works if system is stable

Behavior

$t_r \approx \frac{1.8}{\omega_n}$ $\quad t_r$ = rise time

$t_s \approx \frac{4}{\xi\omega_n} = \frac{4}{\sigma}$ $\quad t_s$ = settling time

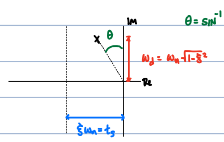
$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$ $\quad t_p$ = time to peak

$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$ $\quad 0 < \xi < 1$ $\quad M_p$ = overshoop

$\%OS = M_p \cdot 100$

Poles on s plane

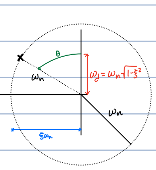
$\theta = \sin^{-1}(\xi)$



S-plane

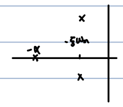
$s = \sigma \pm j\omega$

Poles = x
Zeros = o



$\xi = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$ $\quad \xi$ from %OS

3RD OR HIGHER ORDER SYSTEM



- IF $\alpha > 5\xi\omega_n \Rightarrow$ 2ND-ORDER DOMINANT
- IF $\xi\omega_n > 5\alpha \Rightarrow$ 1ST-ORDER DOMINANT

ADDING A ZERO TO 2ND ORDER T.F.

$$T(s) \longrightarrow (s+a)T(s)$$

OUTPUT: IF $C(s)$ = RESPONSE =>

$$(s+a)C(s) = \underbrace{sC(s)}_{\text{DERIVATIVE OF } C(s)} + \underbrace{aC(s)}_{\text{SCALED VERSION OF } C(s)}$$

NONMINIMUM PHASE

IF $a<0$ => DERIVATIVE TERM COULD BEGIN AS NEGATIVE (BECAUSE $a<0$)
- ONLY OCCURS IF a IS LARGE ENOUGH

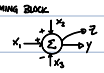
=> RESPONSE TAKES INITIAL NEGATIVE TURN EVEN THOUGH FINAL OUTPUT IS NEGATIVE

BLOCK DIAGRAMS

1. BASIC BLOCK

$Y(s) = G(s)U(s)$ $\quad U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$

2. SUMMING BLOCK



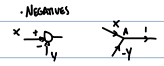
$Y = X_1 + X_2 - X_3$ · ONLY ONE OUTPUT

$Y = Z$

MASON'S RULE

- GAINS BECOME BRANCHES
- SIGNALS = NODES

SUMMATION JUNCTION
- BECOMES A NODE W/OUTPUT 1
  - NEGATIVES



PATH = SEQUENCE OF CONNECTED NODES & BRANCHES

FORWARD PATH = PATH FROM INPUT VARIABLE TO OUTPUT VARIABLE

LOOP = ANY CLOSED PATH

PATH GAINS = EVERYTHING ALONG BRANCHES MULTIPLIED TOGETHER

LOOP GAIN = PRODUCT OF GAINS FOUND BY TRAVERSING A PATH THAT STARTS + ENDS AT SAME NODE

TOUCHING LOOPS/PATHS = TWO OR MORE LOOPS/PATHS THAT HAVE A COMMON COMPONENT/BRANCH OR NODE

NON-TOUCHING LOOPS = LOOPS WITH NO NODES IN COMMON

$$\frac{Y(s)}{R(s)} = \frac{1}{\Delta}\sum_i F_i \, \Delta_i$$

F_i = FORWARD PATH GAIN

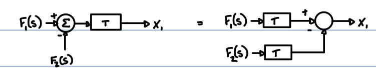
$\Delta = 1 - \Sigma$ (ALL INDIVIDUAL LOOP GAINS) $- \Sigma$ (ALL GAIN PRODUCTS OF ALL POSSIBLE COMBOS OF TWO LOOPS THAT DON'T TOUCH)
$\quad - \Sigma$ (GAIN PRODUCTS OF ALL POSSIBLE 3 LOOP + ... COMBOS THAT DON'T TOUCH)

$\Delta_i = \Delta - \Sigma$ (LOOP GAINS THAT TOUCH IN THE i^{TH} FORWARD PATH)
$\quad = \Delta -$ LOOP GAINS THAT TOUCH i^{TH} FORWARD PATH

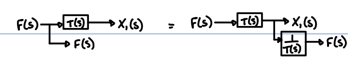
ALGEBRA

1. CASCADING BLOCKS

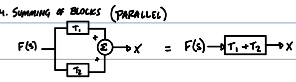
2. MOVING A SUMMING JUNCTION



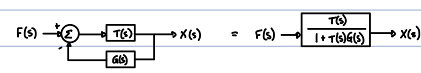
3. MOVING PICK OFF POINTS



4. SUMMING OF BLOCKS (PARALLEL)



5. FEEDBACK LOOP



REMEMBER: $\dfrac{\text{FORWARD PATH}}{1 + \text{LOOP}}$ ← SIGN OPPOSITE OF SUMMATION

STABILITY ← DEPENDS ON POLES OF CLOSED LOOP T.F.

- WHEN $\sigma > 0$ => UNSTABLE
  - $\sigma = 0$ => MARGINALLY STABLE
  - $\sigma < 0$ => STABLE

ROUTH-HURWITZ STABILITY CRITERION

MAKE $d(s)$ A MONIC POLYNOMIAL
- HIGHEST ORDER s TERM COEFFICIENT = 1
- $d(s)$ = TRANSFER FUNCTION

1. IF A SIGN CHANGE IN COEFFICIENTS OF $d(s)$ THEN THERE WILL BE AT LEAST ONE POLE NOT IN OLHP ← $s^3 - 4s + 6 = 0$ (SIGN CHANGE 1, SIGN CHANGE 2)
2. IF ALL COEFFICIENTS OF $d(s)$ ARE POSITIVE WE CAN'T SAY ANYTHING

ROUTH-HURWITZ APPROACH

1. PUT $d(s)$ IN MONIC POLYNOMIAL
2.

$$\begin{array}{c|ccc} s^n & 1 & a_2 & a_4 \\ s^{n-1} & a_1 & a_3 & a_5 \\ \hline s^{n-2} & b_1 & b_2 & b_3 \\ \vdots & c_1 & c_2 & c_3 \\ s^0 & & & \end{array}$$

$$b_1 = \frac{-\begin{vmatrix} 1 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1} \qquad c_1 = \frac{-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1}$$

$$b_2 = \frac{-\begin{vmatrix} 1 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1} \qquad c_2 = \frac{-\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}}{b_1}$$

$$b_3 = \frac{-\begin{vmatrix} 1 & a_6 \\ a_1 & a_7 \end{vmatrix}}{a_1}$$

3. # OF SIGN CHANGES IN 1ST COLUMN CORRESPONDS WITH HOW MANY POLES ARE IN ORHP

SOMETIMES YOU'LL GET A PROBLEM LIKE:

SOLVE FOR K TO MAKE SYSTEM STABLE

$$s^3 + 2s^2 + s + \underbrace{10 + K}_{\text{PUT AS A } s^0 \text{ TERM}}$$

IF FIRST COLUMN HAS A ZERO, SUBSTITUTE ε IN AND LEAVE $\lim_{\varepsilon \to 0}$, STILL COUNT SIGN CHANGES. DOESN'T MATTER IF ∞, ONLY PAY ATTENTION TO SIGNS

IF THERE AREN'T TERMS FOR HIGHER COEFFICIENT VALUES, THEY'RE ZERO

I.E.

$s^3 + 2s^2 + 5s + 10$ $\quad a_1 = 2$, $a_3 = 5$, $a_4 = 10$ $\quad a_5 = a_6 = a_7 = 0$

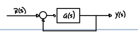
STEADY STATE ERROR

$E(s) = R(s) - Y(s)$ ← ALWAYS TRUE

WE TYPICALLY WANT: $e_{ss} = 0$

STEP INPUT VS ABSOLUTE STEP
JUMP BY "M" SPOTS — JUMP TO "M"

NEGATIVE UNITY FEEDBACK



$$E(s) = R(s) - \frac{G(s)}{1+G(s)} R(s)$$

$$= \frac{R(s)}{1+G(s)}$$

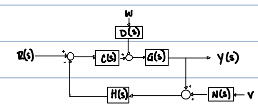
$R(s)$ = INPUT
$G(s)$ = GAIN OF UNITY FEEDBACK
$E(s)$ = ERROR

NOTE: $G(s)$ CAN INCLUDE SYSTEM AND CONTROLLER

NEGATIVE FEEDBACK NON-UNITY

$$Y_T = Y_R + Y_D + Y_N = \frac{Y_R}{R} + \frac{Y_W}{W} + \frac{Y_V}{V}$$

$$Y_T = \frac{CG}{1+CGH} R + \frac{DG}{1+CGH} W + \frac{-NHCG}{1+CGH} V$$



$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[\frac{R(s)}{1+G(s)}\right]$$

SYSTEM TYPE
- HOW MANY POLES AT ZERO (IF UNITY FEEDBACK)
- TYPE 0 → CONSTANT ERROR W/ STEP
- TYPE 1 → CONSTANT ERROR W/ RAMP
- TYPE 2 → CONSTANT ERROR W/ PARABOLIC
- IF NOT UNITY FEEDBACK:
  - IF 2 UNIT STEP & $e_{ss}=0$ ⇒ AT LEAST SYSTEM 1

$E = R - Y_T$

SUBSTITUTE Y_T IN FOR Y IF MULTIPLE INPUTS OR DISTURBANCES

NON-UNITY FEEDBACK ERROR

$$E(s) = [1 - T(s)]R(s)$$

$$e_{ss}(t) = \lim_{s\to 0} s[(1-T(s))R(s)]$$

NOTE: $G(s)$ INCLUDES CONTROLLER

STEP INPUT $\quad R(s) = \frac{A}{s} \quad r(t) = A\,u(t)$

0 $\quad e_{ss} = \frac{1}{1+K_p} A \quad K_p = \lim_{s\to 0} \hat{G}(s)$

1-3 $\quad e_{ss} = 0$

RAMP INPUT $\quad R(s) = \frac{A}{s^2} \quad r(t) = A\,t\,u(t)$

0 $\quad e_{ss} = \infty$

1 $\quad e_{ss} = \frac{1}{1+K_v} A \quad K_v = \lim_{s\to 0} s\hat{G}(s)$ ← ONLY $G(s)$ NOT CLOSED LOOP TRANSFER FUNCTION

2-3 $\quad e_{ss} = 0$

PARABOLIC $\quad R(s) = \frac{A}{s^3} \quad r(t) = A\frac{1}{2}t^2 u(t)$

0-1 $\quad e_{ss} = \infty$

2 $\quad e_{ss} = \frac{1}{1+K_a} A \quad K_a = \lim_{s\to 0} s^2\hat{G}(s)$

3 $\quad e_{ss} = 0$

SENSITIVITY

- DEGREE TO WHICH CHANGES IN SYSTEM PARAMETERS AFFECT SYSTEM T.F.

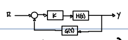
$$S_P^F = \frac{P}{F}\underbrace{\frac{\delta F}{\delta P}}$$

DERIVATIVE OF F WRT P

F = FUNCTION OF INTEREST
P = PARAMETER OF INTEREST

IF $S_P^F = 0$ ⇒ 0% SENSITIVE
$S_P^F = 1.5$ ⇒ 150% SENSITIVE
- CAN BE NEGATIVE, BUT ITS RARE

ROOT LOCUS



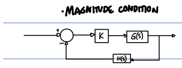
IF DRAWING FOR K

R.L. CHARACTERISTIC EQN

CHARACTERISTIC EQN SOLUTION ARE POLES AND ZEROS

$$1 + KG(s)H(s) = 0$$ (DENOMINATOR OF CLTF)

- A ROOT LOCUS IS A REPRESENTATION OF THE PATHS OF THE CLOSED-LOOP POLES AS THE GAIN (OR PARAMETER OF INTEREST) IS VARIED
  - WHEN K=0, ITS THE OPEN-LOOP TRANSFER FUNCTION (NO FEEDBACK)
- ROOT LOCUS MUST SATISFY CHARACTERISTIC EQUATION THROUGH BOTH:
  - ANGLE CONDITION
  - MAGNITUDE CONDITION



ANGLE CONDITION: $\angle[G(s)H(s)] = 180° + 360° l$

$$\hat{G}(s_o) = \angle(G(s_o)H(s_o)) = \Sigma\psi_i - \Sigma\phi_j = 180 + 360\,l$$

$l = 0, \pm1, \pm2 \quad S_o$ = TEST POINT $\quad \psi_i$ = ZERO ANGLE $\quad \phi$ = POLE ANGLE

MAGNITUDE CONDITION: $K = \frac{1}{|G(s)|\,|H(s)|} \qquad -\frac{1}{K} = |G(s)H(s)|$

$$K = \frac{|s+z_1||s+z_2|\ldots}{|s+p_1||s+p_2|\ldots}$$

- IF NOT DRAWING FOR K, MANIPULATE SYSTEM TO LOOK LIKE $1 + PT(s) = 0$

$T(s)$ = CLOSED LOOP T.F.
P = PARAMETER

- IF P ON DENOMINATOR OF $T(s)$, MULTIPLY TO GET RID OF DENOMINATORS, THEN DIVIDE BY NON P TERMS

m = # ZEROS
n = # POLES

SKETCHING ROOT LOCUS PLOT

1. NUMBER OF BRANCHES EQUAL TO NUMBER OF POLES IN CHARACTERISTIC EQN $\qquad$ # BRANCHES $= n$
2. SYMMETRICAL ON REAL AXIS
3. BRANCHES START AT ZEROS AND END AT POLES
   - IF # POLES > # ZEROS, CALCULATE ASYMPTOTES
   - ASYMPTOTE ANGLE $= \phi_i \quad i = 1,2,3,\ldots,n-m$
   - ASYMPTOTE CENTERED @ α

$$\alpha = \frac{\Sigma Re(p_i) - \Sigma Re(z_i)}{n-m}$$

$$\phi_i = \frac{180° + 360°(h_i - 1)}{n-m} \qquad h = 1,2,3,\ldots,n-m$$

4. ANGLE OF DEPARTURE FOR POLES

$$q\,\phi_{dep} = \underbrace{\Sigma\psi}_{\text{ANGLE OF ALL ZEROES TO POLE OF INTEREST}} - \underbrace{\Sigma\phi}_{\text{ANGLE OF ALL POLES TO POLE OF INTEREST}} - 180 - 360(h-1) \qquad h = 1,2,\ldots q$$

ANGLE OF DEPARTURE FOR iTH POLE

5. FIND $j\omega$ AXIS CROSSING
   - USE STABILITY CRITERION TO FIND K BY SUBSTITUTING: $s = j\omega_d$
   - ROUTH-HURWITZ
6. FIND BREAKAWAY POINTS (WHERE R.L. LEAVES REAL AXIS FOR ASYMPTOTES)
   - TAKE DERIVATIVE OF K & $-\frac{1}{G(s)}$ WRT $s \,\big|_{s=s_o}$

$$\frac{d}{ds}(K) = \frac{d}{ds}\left(-\frac{1}{G(s)}\right)\bigg|_{s=s_o} = 0$$

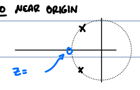
   - TAKE $\frac{d}{ds}\left(-\frac{1}{G(s)}\right)$, SOLVE WHEN TOP OF RESULT IS EQUAL TO ZERO
   - PICK ANSWER THAT MAKES SENSE

$$q\,\psi_{ARRIVAL} = \Sigma\phi - \Sigma\psi + 180 + 360(h-1)$$

ROOT LOCUS DESIGN

LEAD CONTROLLER DESIGN ← DOMINANT ZERO NEAR ORIGIN

1. PUT z CLOSE TO CLOSED LOOP ω_n
   - ω_n COMES FROM DESIRED POLES
2. CHECK ANGLE CONDITION TO FIND LOCATION OF P
   - P SHOULD BE 3X→20X BIGGER THAN z (RULE OF THUMB)
3. FIND K BY MAGNITUDE CONDITION



$$C(s) = K\frac{s+z}{s+p}$$

LAG CONTROLLER DESIGN ← DOMINANT POLE NEAR ORIGIN

1. PICK SMALL P
2. MAKE $z > p$ & $\frac{z}{p} \simeq 3 \to 10$
3. MAKE SURE IT WORKS W/ ANGLE CONDITION
4. FIND K WITH MAGNITUDE CONDITION

IF DESIGNING BY SETTLING TIME

$$t_s = \frac{4}{\zeta\omega_n}$$

ζ DETERMINED BY OVERSHOOT

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

PID TUNING

$$C(s) = K_p + \frac{K_i}{s} + K_d s \longrightarrow C(s) = K_p + \frac{K_p}{T_i}\frac{1}{s} + K_p T_d s$$

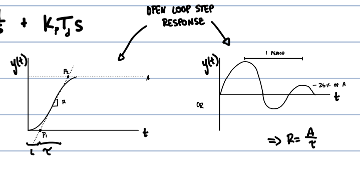
25% DECAY METHOD

$R = \frac{A}{\tau}$

P	$K_p = \frac{1}{RL}$		
PI	$K_p = \frac{0.9}{RL}$	$T_i = \frac{L}{0.3}$	
PID	$K_p = \frac{1.2}{RL}$	$T_i = 2L$	$T_d = 0.5L$

IF SYSTEM IS 1ST ORDER, MAKE L VERY SMALL (0.001)

OPEN LOOP STEP RESPONSE



⇒ $R = \frac{A}{\tau}$

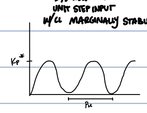
P CONTROL: • WHEN R.L. SPLITS, YOU BASICALLY JUST INCREASE OSCILLATIONS
- INCREASES OSCILLATION, DECREASES DAMPING, INCREASES SPEED AS K_p ↑

I CONTROL: • DECREASES DAMPING → MORE OSCILLATIONS
- CAN GO UNSTABLE IF TOO HIGH (CAN ALSO SATURATE SYSTEM)
- IMPROVES S.S. ERROR

D CONTROL: • USUALLY DESIGNED AS $C(s) = K_p + K_d s$
- AFFECTS DAMPING

2. Ultimate Sensitivity Test

SYS RESPONSE
UNIT STEP INPUT
W/CL MARGINALLY STABLE



- Tune closed loop system until system is marginally stable

P $\quad K_p = 0.5\,K_p^*$

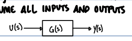
PI $\quad K_p = 0.45\,K_p^* \qquad T_i = \dfrac{P_u}{1.2}$

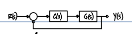
PID $\quad K_p = 0.6\,K_p^* \qquad T_i = 0.5\,P_u \qquad T_d = 0.125\,P_u$

FREQUENCY RESPONSE

- A way of looking at stuff as a function of s (frequency)
- Assume all inputs and outputs are sinusoidal

$$dB = 20\log(|G(j\omega)|)$$





$R(s) = \sin(\omega t) \longrightarrow Y(s) = M\sin(\omega t + \phi)$

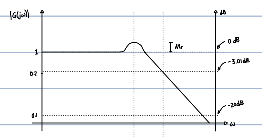
FREQUENCY RESPONSE REASONING:

2^{nd} order sys

$$G(s) = \frac{K\omega_n^2}{\left(\frac{s}{\omega_n}\right)^2 + \left(\frac{2\zeta}{\omega_n}\right)s + 1} \longrightarrow G(j\omega) = \frac{1}{\underbrace{\left[1 - \frac{\omega^2}{\omega_n}\right]^2}_{\text{REAL}} + \underbrace{\left[\frac{2\zeta\omega}{\omega_n}\right]j}_{\text{IMAGINARY}}}$$

- As $\omega \to \omega_n$, real cancels leaving imaginary component
  - @ ω_n, magnitude $\to \infty$ (no damping)

TYPE 0 2^{ND} ORDER SYSTEM



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\omega_p = \omega_n\sqrt{1 - 2\zeta^2}$$

(valid when: $\zeta < 0.707$)

M_r = resonant peak $\qquad \omega_p$ = frequency for M_r

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\omega_{BW} = \omega_n\sqrt{1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

ω_{BW} = bandwidth frequency
- When system is 70% effective, which is ½ power
- When magnitude plot crosses −3.01 dB
  - If offset start (not zero), ω_{BW} is when plot crosses −3.01 dB from original

CLOSED-LOOP BANDWIDTH
- The closed loop bandwidth can be estimated from open-loop frequency response

WHEN

$-7.5\text{ dB} < \omega_{open} < -6\text{dB}$ is where $\omega_{BW,closed}$

If t_p & t_s available:

$$\omega_{BW} = \frac{4}{t_s\zeta}\sqrt{(1-2\zeta^2) - \sqrt{4\zeta^4 - 4\zeta^2 + 2}} = \frac{4}{t_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2) - \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

BODE PLOTS ← We can look at open loop bode plots and determining things about closed loop unity feedback

CASE 1: $K_0(j\omega)^n \qquad n = 0, \pm1, \pm2 \ldots \qquad \omega_{BREAK} = 1$

$|M|$
1. Find $\omega = 1$ and plot $20\log K_0$
2. Draw slope of n or $20n$ dB/dec passing through $20\log_{10}(|K_0|)$ w/ slope $20n$ dB/dec (or n slope if not dB)

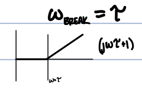
ϕ
1. $\phi = 90^\circ \cdot n$

SYSTEM TYPE

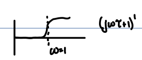
Type 0: 0 initial slope
Type 1: −20 dB/dec initial slope
Type 2: −40 dB/dec initial slope
Type 3: −60 dB/dec initial slope

CASE 2: $(j\omega\tau + 1)^{\pm1} \qquad \omega_{BREAK} = \tau$

$|M|$
1. 0 dB gain for $\omega < \omega\tau$
2. ±20 dB/dec for $\omega\tau < \omega$



ϕ
1. $\omega\tau < 1 \to \phi = 0^\circ$
2. $\omega\tau = 1 \to \phi = \pm45^\circ$
3. $1 < \omega\tau \to \phi = \pm90^\circ$



MULTIPLE

$$KG(s)\Big|_{s=j\omega} = K\frac{(j\omega+z)(j\omega+z)\ldots}{(j\omega+p)(j\omega+p)\ldots}$$

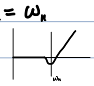
$\tau_1 = \frac{1}{z_1} \qquad \tau_2 = \frac{1}{z_2}$

$\tau_a = \frac{1}{p_1} \qquad \tau_b = \frac{1}{p_2}$

DRAWING BODE PLOTS:
- Separate system into components (case 1, 2, or 3)
  - Make as many type 3 for convenience
- Draw break frequencies for each component
- Mark magnitude points for $\omega = \omega_B$ on case 1 components
- Separate into regions
- Add $|M|$ and ϕ from each component

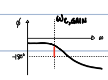
CASE 3. $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1\right]^{\pm1} \qquad \omega_{BREAK} = \omega_n$

1. Same as case 2 w/break @ $\omega = \omega_n$
   - <u>However</u>, slight jump in opposite direction
     - Height depends on ζ (ζ small ⇒ big jump)
   - <u>However</u>, ±40 dB/dec



ϕ
1. Same as case 2 w/break @ $\omega = \omega_n$
   - <u>However</u> ±180°

PHASE MARGIN
- If feedback of system has a positive dB value when out of phase, system can go unstable
  ⇒ You can lose stability by increasing gain
- PM = 60° means you can only slow system by 60° before instability



$\omega_{c,gain}$ = <u>gain</u> crossover frequency (frequency when $|M| = 0$ dB)

(red) = phase margin (graphically)

- Rule of thumb: PM ≈ 45°
  - Good: 40 < PM < 45°
  - OK: 30 < PM < 45°
- Phase margin connected to damping in 2^{nd} order systems
  - Closed loop 2^{nd} order sys $\quad T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$

PHASE MARGIN EQUATION (FROM R.L.)

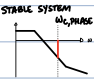
$$\angle K\,H(s)\,G(s) = 180^\circ$$

$$PM = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}\right)$$

If $0 < PM < 60^\circ$, $\quad PM \approx 100\zeta$

GAIN MARGIN
- Factor by which the gain is below 0 dB when $\phi = -180^\circ$

STABLE SYSTEM



$\omega_{c,phase}$ = phase crossover frequency (frequency when $\phi = -180^\circ$)

(red) = gain margin
- positive when $|M|$ below 0dB

- If $M(\omega_{c,phase}) > 0$ dB ⇒ <u>system is unstable</u>
  - if sys crosses 0dB <u>once</u>

$$GM = \frac{K_{MS}}{K}$$

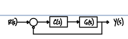
K = K of interest

K = K for marginal stability

<u>GAIN MARGIN EQN:</u>

$$|K\,G(s)\,H(s)| = 1 = 0\text{dB}$$

FREQUENCY RESPONSE DESIGN



- Derivatives add 90° of phase
- Integrals subtract 90° of phase
- Adding sensor adds phase

$$t_s = \frac{4}{\omega_{BW}\zeta}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

- Reduce %OS, by increasing PM

$$e_{ss} = \frac{1}{1+K_p} \qquad K_p = K_0$$

- Improve e_{ss} by increasing low-frequency $|M|$

P CONTROL $\quad D(s) = K$

As $K \uparrow$
- ζ affected
- PM affected
- You can shift $|M|$ by varying K which won't affect phase

PD CONTROL $\quad D(s) = K + K_d s = \left[\frac{K_d}{K}(j\omega) + 1\right]K$
- Don't do it (gives noise high gain)

1. Draw Bode plots for convenient K
2. Compute required PM from desired ζ or %OS
   - $PM = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}\right)$
3. Find frequency ω_{ϕ_m} on Bode phase plot that yields desired PM
   - $\phi = -180 + PM_{des}$
   - Get $\omega_{\phi m_{des}}$ when phase Bode crosses ϕ
4. Compute new gain K
   - Find additional gain needed for $|M|$ to be zero at $\omega_{\phi m,des}$
   - K_{ADD}

$$K = K_1 K_2$$

$K_2 = K_{ADD} =$

K_1 = gain Bode plotted at

$$K_2 = 10^{(K_{ADD}/20)}$$

STEADY STATE ERROR

K_p in a type 0 system:

$$|G(j\omega)|\Big|_{\omega=0} = K_p$$

PI CONTROL
- Reduce bandwidth
- Minimize steady state E
- Better to use lag compensator

$$D(s) = K + K_1\left(\frac{1}{s}\right)$$

STABILITY FROM BODE PLOT

K G unity feedback $\quad$ stability conditions: $\quad |K\hat{G}(s)| < 1$ at $\angle K\hat{G}(s) = -180^\circ$

$|K\hat{G}(j\omega)| = 0$ dB

Marginal stability: when $|K\hat{G}(s)| = 1$ & $\angle K\hat{G}(s) = -180^\circ$

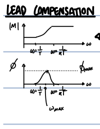
Closed loop B.W. when Bode crosses −3 dB

DESIGN STEPS

LEAD COMPENSATION

$$D(s) = K\frac{Ts+1}{\alpha Ts+1} \qquad \alpha < 1$$

- $\alpha < 1$ ⇒ pole further in LHP than zero



- Adds phase
  - Max phase addition of 90° (adds to PM, subtracts from ϕ)
- Adds damping
- Adds gain

$$\omega_{max} = \frac{1}{T\sqrt{\alpha}}$$

$$D(s) = \alpha\frac{Ts+1}{\alpha Ts+1} \qquad \alpha > 1$$

$$\phi = \tan^{-1}(\omega T) - \tan^{-1}(\alpha T \omega)$$

OR $D(s) = K \frac{s+z}{s+p}$ $\quad z = \frac{1}{T}$ $\quad p = \frac{1}{\alpha T}$

$$\omega_{max} = \sqrt{|z||p|}$$

GUIDELINES: 1. AMOUNT OF PHASE AT ω_{max} ONLY DEPENDS ON α

2. ONLY PROVIDE UP TO 60° PER LEAD CONTROLLER (MAX IS 90° BUT ONLY DO 60°)

LAG COMPENSATION

- APPROXIMATES PI CONTROLLER
- IMPROVES STATIC ERROR
- INCREASE PHASE TO YIELD RIGHT TRANSIENT RESPONSE (SUBTRACTS FROM PM, ADDS TO ϕ)
- REDUCES HIGH FREQUENCY GAIN

1. FIND K (OPEN-LOOP) THAT WILL SATISFY e_{ss} + PLOT BODE DIAGRAMS OF SYS W/ GAIN K
2. FIND FREQUENCY WHERE PHASE MARGIN IS 5°→12° GREATER THAN PM THAT WILL YIELD IDEAL TRANSIENT RESPONSE
3. FIND WHERE UNCOMPENSATED SYS'S BODE PLOT'S MAGNITUDE GOES THROUGH 0 dB IN STEP 2

DESIGN FOR CLOSED-LOOP BANDWIDTH

1. DETERMINE OPEN-LOOP CROSS-OVER FREQUENCY TO BE A FACTOR OF 2 BELOW BANDWIDTH
2. FIND PM OF UNCOMPENSATED SYSTEM W/GAIN K
3. FIND ϕ_{max} PLUS SMALL AMOUNT (5° − 12°)
4. DETERMINE α
5. FIND $20\log_{10}(\sqrt{\alpha})$ ON UNCOMPENSATED MAG PLOT. THE FREQUENCY WHERE $M = 20\log_{10}\sqrt{\alpha}$ IS EQUAL TO ω_{max}
6. SOLVE FOR ZEROS + POLES
7. PLOT BODE DIAGRAM FOR NEW SYSTEM AND CHECK PM & e_{ss}
8. DRAW CLOSED-LOOP BODE PLOT TO GET CLOSED-LOOP BANDWIDTH
9. ITERATE THE DESIGN

OR

DESIGN FOR LOW-FREQUENCY GAIN.

1. DETERMINE K TO SATISFY e_{ss}

$$\phi_{max} = PM_{DES} + EXTRA - PM_{\text{UNCOMPENSATED SYS}}$$

$$\alpha = \frac{1 - \sin(\phi_{max})}{1 + \sin(\phi_{max})} \qquad \phi_{max} = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)$$

$M_{dB} = 20\log_{10}(\sqrt{\alpha})$ } ON UNCOMPENSATED PLOT.
- FIND M_{dB} ON PLOT, AND ω_c IS $M(\omega_c) = M_{dB}$

ω_c = CROSSOVER FREQUENCY (WHEN $|M| = 0$ dB)

ω_{max} = FREQUENCY WHEN PHASE CONTRIBUTION FROM COMPENSATOR IS HIGHEST

$D(s) = K \frac{1}{\alpha} \frac{s + 1/T}{s + 1/\alpha T}$ IF $D(s) = K \frac{s+z}{s+p}$

$$\frac{1}{T} = \omega_{max}\sqrt{\alpha} = z$$

$$\frac{1}{\alpha T} = \frac{\omega_{max}}{\sqrt{\alpha}} = p$$

10.55

$$\omega_{BW} = \frac{4}{T_s \zeta}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}\right)$$

LEAD/LAG

1. FIND $\omega_{BW,\text{CLOSED LOOP}}$ REQUIRED TO MEET t_s, t_p, t_r
2. SET GAIN K TO VALUE REQUIRED BY e_{ss}
3. PLOT BODE FOR K GAIN
4. CALCULATE PM REQUIREMENT FROM DAMPING RATIO OR %OS
5. SELECT NEW PM NEAR ω_{BW}
6. AT NEW PM FREQUENCY, FIND ADDITIONAL PHASE LEAD REQUIRED TO MEET PM. ADD SMALL CONTRIBUTION THAT WILL BE REQUIRED AFTER LAG
7. DESIGN LAG BY SELECTING HIGHER BREAK POINT FREQUENCY ONE DECADE BELOW NEW PM FREQUENCY

DIGITAL CONTROL

Z-TRANSFORM

- TAKES CONTINUOUS TO DISCRETE
  - ONLY WORKS FOR
- A DERIVATIVE IN Z MEANS GOING BACK 1 TIME STEP

$$\mathcal{Z}\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k} \qquad k = 0, 1, 2, \ldots$$

$f(k)$ = SAMPLED VERSION OF $f(t)$

$\mathcal{Z}\{f(k-1)\} = z^{-1}F(z)$ ← DERIVATIVE IN Z-DOMAIN

$$F(z) = \mathcal{Z}\{e^{-akT}\} = \frac{z}{z - e^{-aT}}$$

FINAL VALUE THEOREM

$$\lim_{k\to\infty} x(k) = x_{ss} = \lim_{z\to 1}(1 - z^{-1})X(z)$$

- ONLY APPLICABLE IF POLES OF $(1-z^{-1})X(z)$ ARE INSIDE UNIT CIRCLE

DIGITIZATION:

T = SAMPLE PERIOD

k = INTEGER

$t(k) = kT$

$x(k)$ = VALUE OF $x(t)$ AT t_k

FINDING A DIFFERENCE EQUATION

1. TAKE TRANSFER FUNCTION IN S
2. REWRITE IN TIME ($sY(s) = \dot{y}(t)$)
3. Z TRANSFORM USING ZOH OR TUSTIN'S

IF INPUTTING TO CODE:

4. SOLVE FOR $u(k+1)$ OR $u(k)$

LONG DIVISION IN Z

Z TRANSFORM TABLE

$X(s)$	$X(z)$
$\frac{1}{s}$	$\frac{1}{1-z^{-1}}$
$\frac{1}{s+a}$	$\frac{1}{1-e^{-aT}z^{-1}}$
$\frac{1}{s^2}$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
$\frac{a}{s(s+a)}$	$\frac{(1-e^{aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$

$z = z^{-1} \qquad z^2 = z^{-2}$

DESIGNING Z BY EMULATION

1. DESIGN CONTINUOUS COMPENSATOR $D(s)$
2. CONVERT $D(s) \to D(z)$
3. SIMULATE

ZERO-ORDER HOLD

- HOLDS OUTPUT CONSTANT ON CONTINUOUS TIME

$$D(s) = (1 - z^{-1})\mathcal{Z}\left\{\frac{D(s)}{s}\right\}$$

IF $D(s)$

TUSTIN'S METHOD

- TRAPEZOIDAL

$D(s) \qquad s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$