# System Dynamics Formula Sheet

EFFORT × FLOW $P(t) = e(t)\,f(t)$

| DOMAIN | EFFORT $e(t)$ | FLOW $f(t)$ |
|---|---|---|
| | FORCE $F(t)$ | VELOCITY $V(t)$ |
| | TORQUE $T(t)$ | ANGULAR VELOCITY $\omega(t)$ |
| | PRESSURE $P(t)$ | VOLUME FLOW RATE $Q(t)$ |
| | VOLTAGE $e(t)$ | CURRENT $i(t)$ |

MOMENTUM $p(t)$

$$p(t) = \int^t e(t)\,dt = p_0 + \int_{t_0}^{t} e(t)\,dt \qquad dp = e\,dt$$

DISPLACEMENT $q(t)$

$$q(t) = \int^t f(t)\,dt = q_0 + \int_{t_0}^{t} f(t)\,dt \qquad dq = f\,dt$$

## STATE VARIABLE FORM

$$\{\dot{x}\} = [A]\{x\} + \{B\}u$$

- USES MATRICIES
- CAN HANDLE MULTIPLE INPUTS

$\{\dot{x}\}$ = STATE VECTOR

$$x = \begin{Bmatrix} x \\ dx/dt \\ \cdots \\ d^{n-1}x/dt^{n-1} \end{Bmatrix} \qquad \dot{x} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdots \\ \dot{x}_n \end{Bmatrix} = \begin{Bmatrix} dx/dt \\ d^2x/dt^2 \\ \cdots \\ d^nx/dt^n \end{Bmatrix}$$

$\{B\}$ · CAN BE $\{B\}f(t)$

$$\{y\} = [C]\{x\} + \{D\}u$$

$\{y\}$ = SOLUTION TO ODE
↖ $y$ CAN BE EQUAL TO ODE
i.e. $y = x_1$

### SOLVING FOR STATE VARIABLE FROM ODE

1. MOVE EVERYTHING BUT HIGHEST ORDER DERIVATIVE TO ONE SIDE
2. DIVIDE BY $d^n x/dt^n$'s COEFFICIENT
3. MAKE $\{\dot{x}\} = [A]\{x\} + \{B\}$
   - $\dot{x}_1 = x_2$

### STATE FORM TO T.F.

1. $G_p(s) = \dfrac{Y(s)}{U(s)}$ SO, $\mathcal{L}$ BOTH SIDES
2. REWRITE TO GET RID OF $X(s)$
   - $\vec{y} = \vec{C}\,(s[I] - [A])^{-1}\vec{B}\,U(s) + \vec{D}U(s)$
3. PERFORM INVERSE OPERATIONS

## TRANSFER FUNCTION

$$T(s) = \frac{\text{INPUT}}{\text{OUTPUT}}$$

- GIVES OUTPUT FOR A GIVEN INPUT

## MATRIX PROPERTIES

### INVERSE

$$A^{-1}A = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \left(\frac{((-1)^{i+j}M_{ij})^T}{\det A}\right)$$

IF $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$M_{12} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{23}a_{23} - a_{23}a_{31}$$

$$\therefore A^{-1}_{21} = \frac{(-1)M_{12}}{\det(A)}$$

### 2×2 INVERSE

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ c & a \end{bmatrix}$$

### 3×3 INVERSE

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det(A)}\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

### DETERMINANT

2×2 DETERMINANT

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad |A| = ad - bc$$

3×3 DETERMINANT

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad |A| = a(ei - hf) - b(di - fg) + c(dh - eg)$$

HIGHER ORDER
- ALTERNATE + & −

IF A IS 4×4

$$|A| = a\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & p \end{vmatrix} - b\begin{vmatrix} d & f & g \\ i & j & k \\ m & n & p \end{vmatrix} + \ldots$$

## ENERGETIC VARIABLES

| | EFFORT OR ACROSS | FLOW OR THROUGH |
|---|---|---|
| ELECTRIC: | VOLTAGE | CURRENT |
| MECHANICAL: | VELOCITY | FORCE |
| HYDRAULIC: | PRESSURE | FLOW RATE |
| ROTATIONAL: | ANGULAR VELOCITY | TORQUE |

| | EFFORT STORAGE | FLOW STORAGE |
|---|---|---|
| ELECTRIC: | CAPACITOR | INDUCTOR |
| MECHANICAL: | SPRING | MASS |
| HYDRAULIC: | FLUID CAPACITANCE | FLUID INERTANCE |
| ROTATIONAL: | | |

## LINEAR GRAPH

RULES:

DIRECTION OF ELEMENT ARROWS
- SOURCE ACROSS GOES FROM POSITIVE TO NEGATIVE
- SOURCE CURRENT IN SAME DIRECTION
- ELEMENTS ARBITRARY

TREE PRIORITY
1. ACROSS VARIABLES OF ACROSS SOURCES
2. ACROSS VARIABLES OF ACROSS ENERGY STORAGE
3. ACROSS VARIABLES OF DISSIPATORS
4. ACROSS VARIABLES OF THROUGH ENERGY STORAGE
5. MEDVSE: PUT IN THROUGH SOURCES

TRANSDUCERS
- TRANSFORMER: PICK ONE FOR TREE
- GYRATOR: BOTH OR NEITHER IN TREE

- TREE CONNECTS NODES (VOLTAGES)
- TREE CAN'T HAVE LOOPS
- DRAW TREES UNTIL YOU'D MAKE A LOOP
- MASS ALWAYS CONNECTS TO GROUND
- TREE MUST CONNECT TO GROUND AT SAME POINT

A-TYPE:
PRIMARY VARIABLE: ACROSS VARIABLES IN TREE, OR THROUGH VARIABLE IN LINK

T-TYPE:
SECONDARY VARIABLE: OTHER

STATE: ACROSS VARIABLES OF ACROSS STORAGE IN TREE
THROUGH VARIABLES OF THROUGH STORAGE IN LINK

SYSTEM ORDER: NUMBER OF INDEPENDENT STATE VARIABLES

(NO $(V_2 - V_1)$, ONLY $F_{R_1} = K_1\dot{V}_1$)

### LINEAR GRAPH TO STATE EQNS

1. FIND STATE VARIABLES
2. WRITE ELEMENT EQUATIONS (PRIMARY = CONSTANT · SECONDARY) ↷ YOU'LL USUALLY NEED A DIF ON PRIMARY OR SECONDARY
3. SECONDARY EQUATION USING KVL OR KCL
4. SUBSTITUTE KVL OR KCL INTO PRIMARY EQUATION (MAKE IT ONLY IN STATE VARIABLES)
5. SUBSTITUTE TO ONLY HAVE STATE VARIABLES
6. WRITE MATRIX

# BOND GRAPHS

| EFFORT | FLOW |
|---|---|
| · VOLTAGE | · CURRENT |
| · FORCE | · VELOCITY |
| · PRESSURE | · FLOW |
| · TORQUE | · ANGULAR VELOCITY |

IF POWER CAUSALITY => STATE VARIABLES

CAUSAL STROKE MEANING: DIRECTION IN WHICH EFFORT SIGNAL IS DIRECTED

PLACEMENT OF EFFORT/FLOW
- EFFORT ABOVE OR LEFT
- FLOW BELOW OR RIGHT

HALF-ARROW = POWER FLOW

FULL-ARROW = SIGNAL FLOW @ LOW POWER

DESIRED CAUSAL STROKES:

| EFFORT STORAGE | FLOW STORAGE |
|---|---|
| C | L |
| k | M |
| K | $\frac{J}{I_f}$ |
| $C_f$ | |

TRANSFORMERS
· ONE CAUSAL STROKE

TF OR TF — GY OR GY

SOURCES:

$S_e$ —|

$S_f$ |—

JUNCTIONS

| EFFORT JUNCTION | FLOW JUNCTION |
|---|---|
| COMMON FLOW JUNCTION | COMMON EFFORT JUNCTION |
| 1 | 0 |
| · EVERYTHING CAN BE DEFINED IN TERMS OF THE FLOW VARIABLE | · EVERYTHING CAN BE DEFINED BY EFFORT VARIABLE |
| · ONLY 1 CAUSAL STROKE OUT | · ONLY 1 NON-CAUSAL STROKE OUT |
| $e_1 + e_2 + e_3 = 0$ | $e_1 = e_2 = e_3$ |
| $f_1 = f_2 = f_3$ | $f_1 + f_2 + f_3 = 0$ |

IF ALL ARROWS POINT INWARD TO JUNCTION

SIMPLIFICATIONS:

IF 1 JUNCTION & 0 JUNCTION IN SERIES, YOU CAN REDUCE THEM TO A SINGLE 1 OR 0 JUNCTION

DRAWING BOND GRAPHS
- DRAW ELEMENTS BEFORE DETERMINING JUNCTION TYPE
- IF CAUSALITY TOWARDS JUNCTION THEN WRITE ELEMENT EQN IN TERMS OF EFFORT VARIABLE

SOLVING BOND GRAPHS
- $\Sigma$ UNKNOWNS = 0 UNKNOWN = NOT COMMON VARIABLE

# ELEMENT EQUATIONS

ELECTRIC:
$\dot{V}_c = \frac{1}{C} I_c$
$I_c = \frac{1}{L} V_L$
$V_R = I_R R$
EFFORT → V, FLOW → I, ACROSS = V, THROUGH = I

ROTATIONAL
$\dot{\omega}_J = \frac{1}{J} T_J$
$\dot{T}_k = k \omega_k$
$T_B = \omega_B B$
EFFORT → T, FLOW → ω, ACROSS → ω, THROUGH → T

HYDRAULIC
$\dot{P}_c = \frac{1}{C_f} Q_c$
$\dot{Q}_I = \frac{1}{I_f} P_I$
$P_R = R Q_R$
EFFORT → P, FLOW → Q, ACROSS → P, THROUGH → Q

MECHANIC
$\dot{V}_m = \frac{1}{m} F_m$
$\dot{F}_k = K V_k$
$F_B = B V_B$
EFFORT → F, FLOW → v, ACROSS → v, THROUGH → F

THERMAL
$\dot{T}_C = \frac{1}{C_t} q_t$
$q = \frac{1}{R_T} T$
ACROSS → T, THROUGH → q

# TRANSDUCERS

· DO NOT WORRY ABOUT MINUS SIGN WITH BOND GRAPHS

RACK + PINION:
$V = r\omega$
$F = -\frac{1}{r} T$
$\begin{bmatrix} V \\ F \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & -1/r \end{bmatrix} \begin{bmatrix} \omega \\ T \end{bmatrix}$
LINEAR → TF, BOND → TF

PISTON CYLINDER:
$V = -\frac{1}{A} Q$
$F = AP$
$\begin{bmatrix} v \\ F \end{bmatrix} = \begin{bmatrix} 0 & -1/A \\ A & 0 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix}$
LINEAR → GY, BOND → TF

GEAR TRAIN:
$\omega_1 = -\frac{1}{N} \omega_2$
$T_1 = N T_2$
$\begin{bmatrix} \omega_1 \\ T_1 \end{bmatrix} = \begin{bmatrix} -N_2/N_1 & 0 \\ 0 & N_1/N_2 \end{bmatrix} \begin{bmatrix} \omega_2 \\ T_2 \end{bmatrix}$
LINEAR → TF, BOND → TF

BELT DRIVE
$\omega_1 = R \omega_2$
$T_1 = -\frac{1}{R} T_2$
$R = r_2/r_1$
$\begin{bmatrix} \omega_1 \\ T_1 \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} \omega_2 \\ T_2 \end{bmatrix}$
LINEAR → TF, BOND → TF

ELECTRIC MOTOR:
$V = K \omega$
$I = -\frac{1}{K} T$
$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & -1/K \end{bmatrix} \begin{bmatrix} \omega_2 \\ T_2 \end{bmatrix}$
LINEAR → TF, BOND → GY

PUMP RESERVOIR
$\omega = -\frac{1}{D} Q$
$T = DP$
$\begin{bmatrix} \omega_1 \\ T_1 \end{bmatrix} = \begin{bmatrix} 0 & -1/D \\ D & 0 \end{bmatrix} \begin{bmatrix} P_2 \\ Q_2 \end{bmatrix}$
LINEAR → GY, BOND → TF

LEVER:
$V_1 = -\frac{1}{L} V_2$
$F_1 = L F_2$

THREADED SHAFT
$\omega = \frac{1}{M} V$
$T = -M F$
N = THREADS/m
$M = 2\pi(N)$ [RAD/m]
BOND → TF, LINEAR → TF

# MODULATED TRANSDUCERS

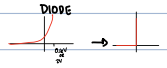
· ARROW ABOVE SHOWS HOW ITS MODULATED

MGY: $( )e_1 = e_2$, $f_1 = ( ) f_2$

MTF: $e_1 = ( ) f_2$, $( ) f_1 = e_2$

NONLINEAR ELEMENTS

DIODE



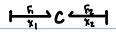
# MULTIPORT FIELD

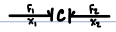
- EFFORT STORAGE → C FIELD
- FLOW STORAGE → I FIELD
- DISSIPATING FIELD → R FIELD

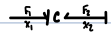
EXPLICIT FIELD => ALL RELATIONS SET

$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ OR $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}^{-1} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ OR $\xrightarrow[X_1]{F_1}$ C $\xleftarrow[X_2]{F_2}$

INPUT OUTPUT

INTEGRAL: FLOW DEFINES EFFORT 

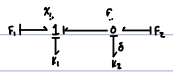
DERIVATIVE: EFFORT DEFINES FLOW 

MIXED: HALF INTEGRAL, HALF DERIVATIVE 

IMPLICIT FIELD

SOMETHING LIKE: $F_1 = k_1 x_1 - k_2 \delta$

$\delta$ = DISPLACEMENT OF $k_2$




## SUBSTITUTION TIPS

- IF YOU CAN'T GET RID OF A DERIVATIVE:
  - IF YOU'RE SOLVING FOR $\dot{v}_1$, YOU CAN HAVE $\dot{v}_1$ IN RHS (DIVIDE@END)
  - IT'S A STATE (DON'T WORRY ABOUT $\dot{F}_1$ IF $F_1$ IS STATE)