STATE SPACE FORMULA SHEET

STATE SPACE FORM

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

VECTORS: $\dot{x}, u, x, y$

$\dot{x} = \frac{d}{dt}$ STATE VARIABLES
x = STATE VARIABLES
A = SYSTEM CHARACTERISTICS
B = INPUTS
y = OUTPUT EQUATION
x = OUTPUTS

OF STATES = ORDER OF D.E.

ELEMENT EQNS

$$\dot{V}_m = \frac{1}{m} F_m \qquad \dot{V}_c = \frac{1}{C} i_c$$
$$\dot{F}_k = \frac{1}{K} V_k \qquad \dot{i}_L = \frac{1}{L} V_L$$
$$F_B = B\, V_B \qquad V_R = R\, i_R$$

SOLVED STATE SPACE FORM

$$\vec{y}(t) = Ce\,\vec{x}(0) + C\int e^{A(t-\tau)} B\vec{u}(\tau)\, d\tau + D\vec{u}(t)$$

SOLVED STATE SPACE LAPLACE

$$\vec{X}(s) = (sI - A)^{-1}\left[\vec{x}(0) + B\vec{u}(s)\right]$$
$$\vec{Y}(s) = C(sI - A)^{-1}\left[\vec{x}(0) + B\vec{u}(s)\right] + D\vec{u}(s)$$

D.E. TO STATE SPACE

1. $x_1 = x$; $x_2 = \dot{x}_1 = \dot{x}$, $x_3 = \dot{x}_2 = \ddot{x}$; ...
2. REWRITE D.E. TO HAVE HIGHEST ORDER TERM ON ONE SIDE + SOLVE
3. BUILD MATRIX WITH STEP 2 SOLUTION

EX: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

T.F. TO STATE SPACE

$$T(s) = \frac{X(s)}{U(s)} = \frac{N(s)}{D(s)} \rightsquigarrow s\,X(s)D(s) = s\,U(s)N(s)$$

a. MULTIPLY $U(s)$ BY $N(s)$ AND $X(s)$ BY $D(s)$
b. SEPARATE HIGHEST ORDER s TERM FROM $s\,X(s)D(s)$, AND SUBTRACT ALL TERMS OVER TO GET SOMETHING LIKE

$$s^3X(s) = -a_1 s^2 X(s) - a_2 s X(s) - a_3 X(s) + s\,U(s)N(s)$$

$x_1 = x$
... $x_4(t) = \dddot{x}$

c. $\mathcal{L}^{-1}[\ \ ]$ & TURN INTO A & B

- IF $\dot{U}(t)$ IN EQN
  $\Rightarrow$ USE $\frac{X(s)}{U(s)} = \frac{X(s)}{Z(s)} \frac{Z(s)}{U(s)} = \left(\frac{1}{D(s)}\right)N(s)$
- PRETEND $\frac{X(s)}{Z(s)} = \frac{1}{D(s)} = Y(s)$
  - SOLVE FOR $Y(s)$ HIGHEST TERM
  - OUTPUT C MATRIX SHOULD REFLECT

$\frac{Z(s)}{U(s)} = N(s)$ (USUALLY $x_1 + x_2 = y$)

USE $\mathcal{L}^{-1}[Z(s)] = z(t) = x_1$
NOW ITS IN x's

STATE SPACE TO TRANSFER FUNCTION

1. $G_p(s) = \frac{Y(s)}{U(s)}$, SO $\mathcal{L}\{\ \}$ BOTH SIDES (MULTIPLY BY s)
2. REWRITE TO $(sI - A)X(s) = B\vec{u}(s)$
   THEN $X(s) = (sI - A)^{-1} B\vec{u}(s)$
3. PUT STEP 2 SOLUTION INTO OUTPUT STATE EQN

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$
$$Y = C(sI - A)Bu + Du$$

LINEAR ALGEBRA

DIAGONAL MATRIX

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ I = IDENDITY MATRIX (3x3 HERE) $|I| = 1$

TRANSPOSE

$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ $A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$ $A^T = [A_{ij}]^T = A_{ji}$

DETERMINANT
- ASSOCIATED W/SQUARE MATRICIES
- THE FACTOR BY WHICH THE AREA OF A MATRIX IS SQUISHED OR STRETCHED
  - IF $\det(\sim) = 0 \Rightarrow$ 2D SPACE COMPRESSED TO A LINE, NO UNIQUE SOLUTION
  - $\det(\sim) < 0 \Rightarrow$ SPACE IS INVERTED

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $|A| = ad - bc$ · IF det = 0 $\Rightarrow$ SINGULAR MATRIX $\Rightarrow$ NO UNIQUE SOLUTION

$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $|B| = a(ei - hf) - b(di - fg) + c(dh - ge)$ · SIMILAR TO SQUARING A # (MULTIPLY THEN SUBTRACT DIAGONAL ELEMENTS)

MATRIX MULTIPLICATION IF $\underset{A}{A}\,\underset{B}{B} = C$ DOT PRODUCT OF ROW + COLUMN OF $A_{2:}$ & $B_{:1}$ FOR C_{21}

TAKING AN INVERSE BY HAND

MINOR MATRIX $M_{ij} = [m_{ij}]$ · m IS THE DETERMINANT IF THE i^{th} & j^{th} ROW ARE REMOVED FROM ORIGINAL MATRIX

COFACTOR $C_{ij} = (-1)^{i+j} m_{ij}$

THE TRANSPOSE OF THE COFACTOR IS THE ADJOINT

ADJOINT TRANSPOSE OF COFACTOR

INVERSE $A^{-1} = \frac{C^T}{|A|}$ $A^{-1} = \frac{\left[(-1)^{i+j}(M_{ij})\right]^T}{|A|}$ IF SQUARE $AA^{-1} = I$

IF $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ AND $A^{-1} = B$ $B_{21} = \frac{-1\, m_{12}}{|A|}$

$m_{12} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$ $\therefore$

IMPORTANT: B_{21}, NOT B_{12}

INVERSE 2X2

IF $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

RANK: TRANSFORM TO ROW-ESCHELON FORM + COUNT # OF NON-ZERO ROWS

NULL VECTOR

VECTOR THAT SATISFIES EQN: $A\vec{x} = 0$

EIGENVECTORS/EIGENVALUES

ONLY WORKS ON LINEAR SYSTEMS

EIGENVECTOR: BASE FOR STRETCHING/TRANSPORMING A SYSTEM INTO A LOWER DIMENSION (3D → 2D) (AXIS FOR ROTATION OF A SYSTEM)

· INFINITE # OF BOTH

EIGENVALUES: FACTOR BY WHICH AN EIGENVECTOR IS STRETCHED OR SQUISHED (CAN BE NEGATIVE OR POSITIVE)

COMPLEX: IF EIGENVECTOR COMPLEX $\Rightarrow$ OSCILLATING
IF EIGENVALUE COMPLEX $\Rightarrow$ NO EIGENVECTOR (LIKELY A ROTATION)

EIGENBASIS: PERFORM A CHANGE OF BASE W/EIGENVECTORS AS THE BASE, THE REPRESENTED MATRIX WILL BE DIAGONALIZED WHICH MAKES CERTAIN OPERATIONS MUCH EASIER (TRANSFORM BACK AT END)

· IF MATRIX IS DIAGONAL $\Rightarrow$ BASIS VECTORS ARE EIGENVECTORS (W/EIGENVECTORS)

$$A\vec{v} = \lambda\vec{v}$$

$\vec{v}$: EIGENVECTOR
λ: EIGENVALUE
A: TRANSFORMATION MATRIX

· IF λ IS AN EIGENVALUE OF A, AND $\vec{x}$ IS AN EIGENVECTOR BELONGING TO $\lambda \Rightarrow$ ANY NONZERO MULTIPLE OF $\vec{x}$ WILL BE AN EIGENVECTOR

FINDING EIGENVECTORS/VALUES

$$A\vec{v} = \lambda v \Rightarrow A\vec{v} - (\lambda I)\vec{v} = 0 \Rightarrow (A - \lambda I)\vec{v} = 0 \Rightarrow \det(A - \lambda I) = 0$$

· WE WANT A NONZERO $\vec{v}$

· $\vec{v}$ WORKS BECAUSE IT SQUISHES SPACE INTO A LOWER DIMENSION
· THIS WILL MAKE THE $\det(A - \lambda I) = 0$ $\Rightarrow A\vec{v} = \lambda\vec{v}$

· BECAUSE $\det(\sim) = 0$ THERE ARE INFINITE SOLUTIONS (WE FOUND A NEW BASE?)

$|A - \lambda I|$ = CHARACTERISTIC EQUATION

SITUATIONS:

1. λ's OF A DISTINCT $A_i\vec{q}_i = \lambda_i\vec{q}_i$
   $A_i = \lambda_i \Rightarrow \hat{A}$ = DIAGONALIZED EIGENVALUES

· SELECTION OF λ_1 OR λ_2 IS ARBITRARY BUT STAY CONSISTENT

2. COMPLEX λ's
   SAME AS ABOVE

3. λ's NOT DISTINCT

JORDAN BLOCK $(A - \lambda I)^n\vec{v} = 0$

$$A\vec{x} = \lambda I\vec{x}$$

$J = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \Rightarrow$

$A\vec{v}_1 = \lambda\vec{v}_1$
$A\vec{v}_2 = \vec{v}_1 + \lambda\vec{v}_2$
$A\vec{v}_3 = \vec{v}_2 + \lambda\vec{v}_3$
$A\vec{v}_4 = \vec{v}_3 + \lambda\vec{v}_4$

STATE TRANSITION MATRIX

$$S(t) \equiv e^{at}$$

$S(t)$ = SOLUTION TO 1ST ORDER ODE

$$S(t) = I + At + \frac{1}{2}AAt^2 + \dots \frac{1}{n!}A^nt^n$$
$$S(t) = I + \sum_{n=1}^{\infty}\frac{1}{n!}A^nt^n = e^{At}$$

PROPERTIES:

$S(t + \tau) = S(t)S(\tau)$
$S(-t) = S^{-1}(t)$
$S(nt) = [S(t)]^n$
$\frac{d}{dt}[S(t)] = \frac{d}{dt}[e^{At}] = AS(t)$

PARTICULAR SOLUTION

$x_p(t) = e^{At}\vec{p}(t) = S(t)\vec{p}(t)$
$x_p(t) = \xi$
$\vec{x}_p(t) = \int_0^t S(t - \tau)Bu(\tau)\, d\tau$

$$\vec{x}(t) = \vec{x}_p + \vec{x}_H = S(t)\vec{x}(0) + \int_0^t S(t - \tau)B\vec{u}(\tau)\, d\tau$$

APPROACHES

EIGENVALUES

$$S(t) = e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$
$$S(t) = I + \sum_{n=1}^{\infty}\frac{1}{n!}A^nt^n$$

LAPLACE — EASIEST WAY

$$S(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$

1. FIND $sI - A$ MATRIX
2. $|sI - A|$
3. $(sI - A)^{-1}$
4. $\mathcal{L}^{-1}$ EACH ELEMENT

INFINITE SERIES

$$S(t) = I + \sum_{n=1}^{\infty}\frac{1}{n!}A^nt^n$$

SYLVESTER'S THEOREM ← IF YOU HAVE COMPLEX #'s, PUT THEM IN AND KEEP UNTIL YOU HAVE COMPLEX EXPONENTS

$$S(t) = \sum_i^n e^{\lambda_i t}F_i$$

WHERE $F_i = \prod_{\substack{j=1 \\ i \neq j}}^{n}\frac{\lambda_j I - A}{\lambda_j - \lambda_i}$

Π = PRODUCT (LIKE SUM)

DIGITAL CONTROL

RESOLUTION $= \frac{V}{2^n - 1}$

n = # OF BITS
V = VOLTAGE RANGE

$\frac{\omega_s}{2}$ = NYQUIST FREQ $= f_N$

ω_s = SAMPLING FREQ (RATE)
f_N = MAX FREQUENCY THAT CAN BE MEASURED

MATRIX FORMS/TRANSFORMS

CONTROLLABLE FORM 1

$$\underset{\sim}{A} = \begin{bmatrix} 0 & & \\ \vdots & \underset{\sim}{I} & \\ 0 & & \\ -a_n & -a_{n-1} & \cdots -a_1 \end{bmatrix} \quad \underset{\sim}{B} = \begin{bmatrix} b_n \\ \vdots \\ b_1 \end{bmatrix} \quad \underset{\sim}{C} = [0 \dots 0\ 1] \quad \underset{\sim}{D} = [0]$$

a_n = NEGATIVE ELEMENT OF CHAR EQN

OBSERVABLE FORM

LOCATION OF COLUMN CAN BE ON LEFT OR RIGHT

$$\underset{\sim}{A} = \begin{bmatrix} 0 & \cdots & 0 & -a_n \\ & & & -a_{n-1} \\ & I & & \vdots \\ & & & a_1 \end{bmatrix} \quad \underset{\sim}{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \underset{\sim}{C} = [b_n\ b_{n-1} \dots b_1] \quad \underset{\sim}{D} = [0]$$

TRANSFORMING BETWEEN FORMS

CONTROLLABILITY TRANSFORMATION

FULLY CONTROLLABLE $\Rightarrow$ INPUT CAN INDEPENDENTLY AFFECT EACH STATE

n = SIZE OF $\underset{\sim}{A}$ MATRIX

CONTROLLABILITY MATRIX

$$\underset{\sim}{M} = [\underset{\sim}{B}\ \ \underset{\sim}{A}\underset{\sim}{B} \dots \underset{\sim}{A}^{n-1}\underset{\sim}{B}]$$ ← IF ROWS ARE NOT INDEPENDENT $\Rightarrow$ NOT CONTROLLABLE

FIND CHARACTERISTIC EQN: $|s\underset{\sim}{I} - \underset{\sim}{A}| = s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n$

$$\underset{\sim}{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & & \\ a_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

IF 2x2 $\Rightarrow$ $\underset{\sim}{W} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$

3x3 $\Rightarrow$ $\underset{\sim}{W} = \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\underset{\sim}{P} = \underset{\sim}{M}\,\underset{\sim}{W}$$

$\underset{\sim}{W}$ = CONTROLLABILITY GRAMIAN

$\underset{\sim}{P}$ = CONTROLLABILITY TRANSFORMATION MATRIX

OBSERVABLE TRANSFORMATION ← IF OBSERVABLE, ALL STATES CAN BE DETERMINED FROM OUTPUT

$$\underset{\sim}{N} = [\underset{\sim}{C}^T\ \ \underset{\sim}{A}^T\underset{\sim}{C}^T \dots (\underset{\sim}{A}^T)^{n-1}\underset{\sim}{C}^T]$$

IF ROWS NOT INDEPENDENT $\Rightarrow$ NOT OBSERVABLE

$\underset{\sim}{W}$ = (SAME AS CONTROLLABLE $\underset{\sim}{W}$)

$$\underset{\sim}{Q} = (\underset{\sim}{W}\underset{\sim}{N}^T)^{-1}$$

OBSERVABLE TRANSFORMATION MATRIX

$$\Rightarrow \left.\begin{aligned} \underset{\sim}{A}_c &= \underset{\sim}{P}^{-1}\underset{\sim}{A}\underset{\sim}{P} \\ \underset{\sim}{B}_c &= \underset{\sim}{P}^{-1}\underset{\sim}{B} \\ \underset{\sim}{C}_c &= \underset{\sim}{C}\underset{\sim}{P} \end{aligned}\right\} \longleftrightarrow \left\{\begin{aligned} \underset{\sim}{A}_c &= \underset{\sim}{A}_o^T \\ \underset{\sim}{B}_c &= \underset{\sim}{C}_o^T \\ \underset{\sim}{C}_c &= \underset{\sim}{B}_o^T \end{aligned}\right\} \longleftrightarrow \left\{\begin{aligned} \underset{\sim}{A}_o &= \underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q} \\ \underset{\sim}{B}_o &= \underset{\sim}{Q}^{-1}\underset{\sim}{B} \\ \underset{\sim}{C}_o &= \underset{\sim}{C}\underset{\sim}{Q} \end{aligned}\right.$$

CANNONICAL FORM NEEDS A B C D MATRICIES

· CHANGE BASE TO $\vec{q}$

$\underset{\sim}{Q} = [\vec{q}_1, \vec{q}_2, \vec{q}_3 \dots \vec{q}_n]$

$$\left.\begin{aligned} \underset{\sim}{A}\underset{\sim}{x} &= y \\ \hat{\underset{\sim}{A}}\hat{\underset{\sim}{x}} &= \hat{y} \end{aligned}\right| \rightarrow \begin{aligned} x &= Q\hat{x} \\ y &= Q\hat{y} \end{aligned}$$

$$\hat{\underset{\sim}{A}} = \underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q}$$
$$\underset{\sim}{A} = \underset{\sim}{Q}\hat{\underset{\sim}{A}}\underset{\sim}{Q}^{-1}$$

WHY THESE TRANSFORMATIONS WORK:

$\underset{\sim}{A}\underset{\sim}{Q} = \underset{\sim}{Q}\hat{\underset{\sim}{A}}$ $\hat{A}$ = $\underset{\sim}{A}$ WRT $\vec{q}$

$\underset{\sim}{Q} = [\underset{\sim}{B} \vdots \underset{\sim}{A}\underset{\sim}{B} \vdots \underset{\sim}{A}\underset{\sim}{A}\underset{\sim}{B} \vdots\vdots \underset{\sim}{A}^n\underset{\sim}{B}]$ CONTROLLABILITY MATRIX

$\underset{\sim}{A}\underset{\sim}{Q}\hat{\underset{\sim}{x}} = \underset{\sim}{Q}\hat{\underset{\sim}{y}}$ OR $\underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q}\hat{\underset{\sim}{x}} = \hat{\underset{\sim}{y}}$

$\underset{\sim}{A}$ = ORIGINAL MATRIX

$\hat{\underset{\sim}{A}}$ = MATRIX $\underset{\sim}{A}$ REPRESENTED IN A NEW BASE

$\underset{\sim}{Q}$ = CONTROLLABILITY MATRIX (NONSINGULAR MAP)

$$\hat{\underset{\sim}{A}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \beta_1 \\ 1 & 0 & \cdots & 0 & \beta_2 \\ 0 & 1 & \cdots & 0 & \beta_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \beta_{n-1} \\ 0 & 0 & \cdots & 1 & \beta_n \end{bmatrix}$$

· $\hat{\underset{\sim}{A}}$ CAN BE TRANSPOSED (β's ON BOTTOM ROW)

COMPANION FORM

EQUIVALENCE TRANSFORM

$$\bar{\underset{\sim}{x}} = \underset{\sim}{P}\underset{\sim}{x}$$

TRANSFORM BETWEEN CONTROLLABLE OR OBSERVABLE FORMS

$$\underset{\sim}{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\underset{\sim}{x}$ = COMPLETELY OBSERVABLE SYS

$\bar{\underset{\sim}{x}}$ = COMPLETELY CONTROLLABLE SYS

IF SYS IS COPRIME

CLOSED FORM

$$s(t) = e^{\lambda t}$$

PLUG IN t

ROW-ESCHELON FORM

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \text{ OR } \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ROWS CAN BE CHANGED AS LONG AS ACCOMPANYING VECTORS/MATRICIES ARE ALSO SWITCHED

· TO TRANSFORM TO THIS:
· MULTIPLY TOP ROW BY SCALAR TO GET 1
· ADD TOP ROW TO NEXT ROW TO REDUCE LEADING COEFF

DIAGONAL FORM

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$ ← EIGENVAL'S

MINIMUM REALIZATIONS

· ALL MINIMUM REALIZATIONS ARE EQUAL

COPRIME = NO COMMON FACTORS
· BASICALLY, NO ZERO-POLE CANCELLATIONS IN TRANSFER EQN

FIND MINIMUM SIZE OF A SYSTEM/MATRIX

PUT 1'S ON THE DIAGONAL

COMPLETELY OBSERVABLE $\begin{cases} \text{RANK}(N) = \text{FULL} \\ \text{RANK}(M) = \text{FULL} \end{cases} \Rightarrow$ A COMPLETELY CONTROLLABLE + OBSERVABLE MATRIX

TO COMPUTE THIS, GO TO TRANSFER FUNCTION + ALGEBRA

NOT COPRIME $\Rightarrow$ NOT MUCH

COPRIME $\Rightarrow$ OBSERVABLE OR CONTROLLABLE <u>NOT</u> FULL RANK

BIBO = BOUNDED INPUT BOUNDED OUTPUT

GLOBALLY STABLE: SYSTEM DOESN'T GO TO ∞ AS $t \to \infty$

UNDAMPED OSCILLATION IS GLOBALLY STABLE, NOT ASYMPTOTICALLY

ASYMPTOTICALLY STABLE: SYSTEM GOES TO ORIGIN AS $t \to \infty$

LYAPUNOV STABILITY

· DERIVED FROM THE IDEA THAT IF THE ENERGY OF A SYSTEM IS DECREASING ($dE/dt < 0$), THEN SYSTEM IS STABLE). INSTEAD OF ENERGY WE USE AN ARBITRARY SCALAR

· MAKES NON-LINEAR LINEAR SO WE CAN FIND STABILITY

$\underset{\sim}{R}$ = POSITIVE DEFINITE SET OF CONSTANTS

$$v(\vec{x}, t) = \vec{x}^T \underset{\sim}{R} \vec{x}$$

$$\begin{aligned} \dot{v}(x,t) &= \dot{\vec{x}}^T \underset{\sim}{R} \vec{x} + \vec{x}^T \underset{\sim}{R} \dot{\vec{x}} \\ &= \vec{x}^T \underset{\sim}{A}^T \underset{\sim}{R} \vec{x} + \vec{x}^T \underset{\sim}{R}\underset{\sim}{A} \vec{x} \\ &= \vec{x}^T [\underset{\sim}{A}^T \underset{\sim}{R} + \underset{\sim}{R}\underset{\sim}{A}] \vec{x} \end{aligned}$$

$$\dot{v} = -\vec{x}^T [\underset{\sim}{Q}] \vec{x}$$
$$Q = -[\underset{\sim}{A}^T \underset{\sim}{R} + \underset{\sim}{R}\underset{\sim}{A}]$$

PROCEDURE:

1. CHOOSE A Q WHICH IS SYMMETRIC + POSITIVE DEFINITE (det > 0) OR SEMIDEFINITE (det = 0)
2. SOLVE FOR R
3. USE SYLVESTER'S THEOREM TO MAKE R POSITIVE DEFINITE

START W/ $V(x)$ THAT IS:
· SIMPLE
· $V(0) = 0$
· WHEN YOU TAKE DERIVATIVE, TERMS WILL CANCEL
· REMEMBER $\frac{d}{dt}(x^2) = 2x\dot{x}$

$\nu(x)$ ← ARBITRARY SCALAR

IFF: $\left.\begin{aligned} \nu(0,t) &= 0 \\ \nu(x,t) &\geq 0 \\ \dot{\nu}(x,t) &\leq 0 \end{aligned}\right\} \Rightarrow$ STABLE

MUST BE OPPOSITE FOR STABILITY

· IF $\nu(x,t)$ & $\dot{\nu}(x,t)$ ARE SAME SIGN $\Rightarrow$ UNSTABLE

$$v = r^2 + f^2$$
$$\dot{v} = 2\dot{r}r + 2f\dot{f}$$

$\underset{\sim}{R}$ = POSITIVE DEFINITE

$$\nu(\vec{x}, t) = \vec{x}^T \underset{\sim}{R} \vec{x}$$
$$\dot{\nu}(\vec{x}, t) = \dot{\vec{x}}^T \underset{\sim}{R} \vec{x} + \vec{x}^T \underset{\sim}{R} \dot{\vec{x}} = \vec{x}^T [\underset{\sim}{A}^T \underset{\sim}{R} + \underset{\sim}{R}\underset{\sim}{A}] \vec{x}$$

CAYLEY-HAMILTON THEORY

CHARACTERISTIC EQN: $\det(\lambda \underset{\sim}{I} - \underset{\sim}{A}) = \lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_n$

SINCE $\underset{\sim}{A}$ & λ SIMILAR

$$\underset{\sim}{A}^n + \alpha_1 A^{n-1} + \dots + \alpha_{n-1}\underset{\sim}{A} + \alpha_n \underset{\sim}{I} = 0$$

SYLVESTER'S THEOREM FOR POSITIVE DEFINITENESS

$$v(x,t) = \vec{x}^T \underset{\sim}{R} \vec{x}$$

· IS POSITIVE DEFINITE IF ALL $i \times i$ SUBSECTIONS HAVE DETERMINANTS > 0
· IF ANY SUBSECTIONS HAVE det = 0 THEN THEY ARE SEMI-DEFINITE

$$\dot{\nu} = -\vec{x}^T [\underset{\sim}{Q}] x$$
$$\underset{\sim}{Q} = -[\underset{\sim}{A}^T \underset{\sim}{R} + \underset{\sim}{R}\underset{\sim}{A}]$$

INVERSE OF NON-SQUARE MATRIX

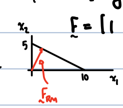
· NO UNIQUE SOLUTION
· WE TRY TO FIND THE SHORTEST SOLUTION VECTOR

$$\underset{\sim}{F}_{RM} = \underset{\sim}{F}^T (\underset{\sim}{F}\underset{\sim}{F}^T)^{-1}$$

F_{RM} = MINIMUM RH INVERSE

IF: $1x_1 + 2x_2 = y = 10$

$\underset{\sim}{F} = [1\ \ 2]$



MORE EQUATIONS THAN UNKNOWNS (REGRESSION)

$$\underset{\sim}{H}\underset{\sim}{u} = \underset{\sim}{z}$$

z = KNOWN n-VECTOR
u = m-VECTOR
H = $n \times m$ MATRIX

CONTROLLERS

CHAR EQN: SOLUTION TO HOMOGENEOUS EQN

1ST ORDER ← τ

$$H(s) = \frac{1}{s+\sigma} \qquad h(t) = e^{-\sigma t}\,1(t)$$

$$\tau = \frac{1}{\sigma}$$

2ND ORDER ← ω_n ξ

$$T(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

OS IS NOT %. $(0 < OS < 1)$

$$\xi = \frac{-\ln(OS)}{\sqrt{\pi^2 + \ln^2(OS)}}$$

$$s = -\omega_n\xi + j\omega_d$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

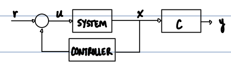
$$t_r = \frac{1.8}{\omega_n}$$

$$t_{s,5\%} = \frac{3}{\xi\omega_n}$$

$$t_{s,2\%} = \frac{4}{\xi\omega_n}$$

$$t_{s,1\%} = \frac{4.6}{\xi\omega_n}$$

STATE FEEDBACK



$$u = -kx \rightsquigarrow \dot{\vec{x}} = \underset{\sim}{A}\vec{x} + \underset{\sim}{B}\vec{u} \rightsquigarrow \dot{\vec{x}} = (A - Bk)\vec{x}$$

$$\underset{\sim}{A}_{cl} = \underset{\sim}{A} - \underset{\sim}{B}\vec{k}$$

k = ROW VEC

FINDING K GAINS

CLOSED LOOP CHAR EQN: $\left| s\underset{\sim}{I} - (\underset{\sim}{A} - \underset{\sim}{B}\vec{k}) \right|$

BRUTE FORCE

- IF YOU KNOW WHERE S_1 & S_2 SHOULD BE, SOLVE USING CLOSED LOOP CHAR EQN

ACKERMANN'S FORMULA ← NEEDS CONTROLLABLE FORM INPUTS

- FINDS CONTROLLER GAINS

$$\vec{k} = [0 \dots 0\ 1]\left[\underset{\sim}{B}\ \ \underset{\sim}{A}\underset{\sim}{B}\ \dots\ \underset{\sim}{A}^{n-1}\underset{\sim}{B}\right]^{-1}\alpha_c(A)$$

$$= [0 \dots 0\ ]\,[M]^{-1}\alpha_c(A)$$

$$\alpha_c(\underset{\sim}{A}) = \underset{\sim}{A}^n + \alpha_{n-1}\underset{\sim}{A}^{n-1} + \dots\ \alpha_1$$

$\alpha_c(A)$ = COEFF OF DESIRED CHAR EQN

IF DESIRED CHAR EQN IS... $s^2 + 5s + 6 = 0 \rightsquigarrow \alpha_c(\underset{\sim}{A}) = [\underset{\sim}{A}\,\underset{\sim}{A}] + 5[\underset{\sim}{A}] + 6\underset{\sim}{I}$

OBSERVERS

- GREAT WHEN YOU CAN'T MEASURE DATA
- ESTIMATES + FILTERS STATE DATA
  - DOES THIS BY GUESSING STATE VALUES
- AVOIDS DIFFERENTIATION (STATE NOISE GONE)
- MINIMUM STATE OBSERVER (ESTIMATES ONLY A FEW STATES)

HATS "^" ARE ESTIMATES

$\hat{A}$ $\hat{B}$ $\hat{C}$ = GUESSES

u & y ARE NOT ESTIMATES

$$\dot{\hat{x}} = \hat{\underset{\sim}{A}}\hat{x} + \hat{\underset{\sim}{B}}u + K_e(y - \hat{y})$$

K_e = WEIGHTING FACTOR

ESTIMATE STATES W/ STATE EQN + CORRECTION TERM (K_e)

$$e = x - \hat{x}$$

e = ERROR BETWEEN ACTUAL + ESTIMATE STATES

ERROR BETWEEN ESTIMATE + ACTUAL

$$\dot{x} - \dot{\hat{x}} = Ax - A\hat{x} + K_e(Cx - C\hat{x})$$

$$\dot{e} = Ae + \hat{B}u + K_eCe$$

IF STABLE, $A - K_eC$ WILL GO TO ZERO ⇒ e GOES TO ZERO ⇒ $\hat{x} \to x$

$$A_o = \underbrace{(A - K_eC)}_{\text{CLOSED LOOP A MATRIX}} =$$

TO HAVE DESIRED POLES...

$$\left.\begin{aligned} &\left|sI - (A - K_eC)\right| = 0 \\ &\left|sI - (A - LC)\right| = 0 \end{aligned}\right\}\ \text{SOLVE IN THE SAME WAY AS CONTROLLER DESIGN}$$

L = COLUMN VEC

$$\dot{\vec{x}} = (A - BK - LC)\vec{x} + Ly$$

$$u = -k\hat{x}$$

MINIMUM STATE OBSERVER

$$\begin{bmatrix}\dot{x}_a \\ \dot{x}_b\end{bmatrix} = \begin{bmatrix}A_{aa} & A_{ba} \\ A_{ba} & A_{bb}\end{bmatrix}\begin{bmatrix}x_a \\ x_b\end{bmatrix} + \begin{bmatrix}B_a \\ B_b\end{bmatrix}u$$

a = MEASURED
b = UNMEASURED

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_a \\ x_b\end{bmatrix}$$

FULL STATE	MINIMUM STATE
$\hat{x}$	x_b
A	A_{bb}
Bu	$A_{ba}x_a + B_bu$
y	$\dot{x}_a - A_{aa}x_a - B_au$
C	A_{ab}

$$\dot{e} = (A_{bb} - K_eA_{ab})e$$

CHAR EQN $\left|sI - A_{bb} - K_eA_{ab}\right|$

EXAMPLE:

$$A = \begin{bmatrix}-1 & 1 \\ 1 & -2\end{bmatrix}$$

$\left|sI - A_{aa} + k_eA_{ab}\right|$

$$\begin{bmatrix}A_{aa} = -1 & A_{ab} = 1 \\ A_{ba} = 1 & A_{bb} = -2\end{bmatrix}$$

$\mu_{DES} = -5$

x_1 MEASURED
x_2 OBSERVE

$\left|sI - A_{bb} + k_eA_{ab}\right| = s - (-2) + k_e \rightsquigarrow s + k_e + 2 = s + 5$

$k_e = 3$

PARTITION 3RD ORDER SYSTEM

$$A = \left[\begin{array}{c:cc} 0 & 1 & 0 \\ \hdashline 0 & 0 & 1 \\ -6 & -11 & -6 \end{array}\right] \qquad B = \left[\begin{array}{c} 0 \\ \hdashline 0 \\ 1 \end{array}\right]$$

$A_{aa} = 0$ $\quad A_{ab} = [1\ \ 0]$ $\quad B_a = 0$

$A_{ba} = \begin{bmatrix}0 \\ -6\end{bmatrix}$ $\quad A_{bb} = \begin{bmatrix}0 & 1 \\ -11 & -6\end{bmatrix}$ $\quad B_b = \begin{bmatrix}0 \\ 1\end{bmatrix}$

REGULATOR STATE SPACE EQUATIONS

$$\begin{bmatrix}\dot{x} \\ \dot{e}\end{bmatrix} = \begin{bmatrix}A - BK & BK \\ 0 & A - LC\end{bmatrix} + \begin{bmatrix}B \\ 0\end{bmatrix}u$$

$$y = \begin{bmatrix}C & 0\end{bmatrix}\begin{bmatrix}x \\ e\end{bmatrix}$$

LQR

$$Q = \begin{bmatrix}1/\Delta^2 & 0 \\ 0 & 1/\Delta^2\end{bmatrix} \qquad R = \begin{bmatrix}1/\Delta^2\end{bmatrix}$$

REFRESH

Δ = ALLOWABLE VALUES

$\pm 5° \rightsquigarrow 1/25°$