FORMULA SHEET ROBOT CONTROL

$$a \times b = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

CHAIN RULE $\frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)\,g(x) + f(x)\,g'(x)$

PRODUCT RULE $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\,g'(x)$

QUOTIENT RULE $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

DH PARAMETERS

a_i : FROM z_{i-1} TO z_i ALONG x_i

d_i : FROM x_{i-1} TO x_i ALONG z_{i-1}

α_i : $\measuredangle$ FROM z_{i-1} TO z_i ABOUT x_i

θ_i : $\measuredangle$ FROM x_{i-1} TO x_i ABOUT z_{i-1}

IF ROTARY, $x_i = x_{i-1}$ IN ZERO $\measuredangle$

JOINT i: CONNECTS LINK i TO LINK $i-1$

z_{i-1} : LOCATED AT JOINT i

O_i : INTERSECTION OF x_i FROM z_i

x_i : $\parallel$ TO n_i IN DIRECTION FROM z_{i-1} TO z_i

$${}^0R_1 = \begin{bmatrix} C\theta_1 & -S\theta_1 C\alpha_1 & S\theta_1 S\alpha_1 \\ S\theta & C\theta_1 C\alpha_1 & -C\theta_1 S\alpha_1 \\ 0 & S\alpha_1 & C\alpha_1 \end{bmatrix}$$

$${}^1T_0 = \begin{bmatrix} {}^0R_1 & -{}^0R_1^T\,{}^0d_{01} \\ 0 & 1 \end{bmatrix}$$

$${}^{i-1}T_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

JACOBIANS

JACOBIAN: MATRIX REPRESENTING DERIVATIVES OF A VECTOR FUNCTION W.R.T. ITS INPUTS

- JACOBIAN OF f FROM R^n TO R^m IS SIZE $m \times n$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$\dot{\Theta}$ = STATE VECTOR (JOINT $\measuredangle$'s)

${}^0\dot{d}_{on}$ = VELOCITY END EFFECTOR

${}^0\omega_{on}$ = $\measuredangle$ VELOCITY OF END EFFECTOR

MANIPULATOR

$$\dot{d}_{on} = J\dot{\theta}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \cdots & \frac{\partial x}{\partial \theta_n} \\ \frac{\partial y}{\partial \theta_1} & \cdots & \frac{\partial y}{\partial \theta_n} \\ \frac{\partial z}{\partial \theta_1} & \cdots & \frac{\partial z}{\partial \theta_n} \end{bmatrix}$$

J = MANIPULATOR JACOBIAN

- LOTS OF COMPUTATION, ONLY POSITION

TRANSMISSION:

VELOCITY:

$$\begin{bmatrix} {}^0\dot{d}_{on} \\ {}^0\omega_{on} \end{bmatrix} = J_v \dot{\Theta}$$

TYPICALLY 6×1 — TYPICALLY $6\times n$ — $n\times1$

J_v = VELOCITY JACOBIAN

$$\tau = J_v^T W_{n+1}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} \qquad W = \begin{bmatrix} f_{n,n+1} \\ n_{n,n+1} \end{bmatrix} \begin{matrix} \text{FORCES} \\ \text{TORQUES} \end{matrix}$$

WRENCH

$$J_v = \begin{bmatrix} {}^0z_0 \times {}^0d_{on} & {}^0z_1 \times {}^0d_{1n} & \cdots & {}^0z_{n-1} \times {}^0d_{n-1,n} \\ {}^0z_0 & {}^0z_1 & \cdots & {}^0z_{n-1} \end{bmatrix}$$

- LESS COMPUTATION, PLUS IT GIVES YOU $\measuredangle$ VELOCITY

JACOBIAN'S USES

INVERSE VELOCITY:
- YOU CAN INVERT J_v TO GIVE VELOCITIES
- ONLY WORKS IF J_v IS SQUARE, AND THERE AREN'T SINGULARITIES $(\det(J_v) = 0)$
- SINGULARITIES OCCUR AT EDGE OF ROBOT'S REACH, OR OUTSIDE OF ITS CAPABILITIES
- IF YOU HAVE REDUNDANT DOF, YOU USE A PSEUDO-INVERSE TO CANCEL REDUNDANCIES

EXAMPLE FULL DYNAMICS

$$\tau = H(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta)$$

H = INERTIAL MATRIX

V = VELOCITY TERMS (A.K.A CORIOLIS)

G = GRAVITY

TRANSMISSION

- LETS YOU REWRITE FROM JOINT SPACE INTO MOTOR SPACE

$$\Theta = J_t \Phi \qquad \Phi = \text{MOTOR SPEEDS}$$

$$\tau_m = J_t^{-1}\tau_{joint} \longleftarrow \text{DOESN'T WORK IF } J_t \text{ NOT SQUARE}$$

$$J_t = \begin{bmatrix} \frac{1}{N_1} & 0 & \cdots \\ 0 & \frac{1}{N_2} & \\ \vdots & & \ddots \\ & & & \frac{1}{N_n} \end{bmatrix}$$

$\longleftarrow$ NOT A HARD/FAST RULE. WRITE EQNS OF JOINT VELOCITIES, & HOW THEY RELATE TO MOTOR VELOCITIES. USE THAT TO BUILD J_t

FULL DYNAMICS W/ TRANSMISSION

$$\tau_{joint} = H'(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta)$$

$$\tau_m = J_t^T\tau_{joint} = J_t^T H'(\Theta) J_t \ddot{\Phi} + J_t^T V(\Theta,\dot{\Theta}) + J_t^T G(\Theta)$$

FIRST J_t^T FROM $\tau_m = J_t^T \tau_{joint}$

SECOND J_t^T TO DEAL W/ $\ddot{\Theta}$

$H(\Theta)$ = INERTIA MATRIX WRT Θ

$H'(\Theta)$ = INERTIA MATRIX W/ MOTOR INERTIA INCLUDED

$H'(\Phi)$ = INERTIA MATRIX WRT Φ_m

DRIVETRAIN DYNAMICS

$$N_i = \frac{r_B}{r_A} \qquad N_i = \text{GEAR RATIO}$$

$$\dot{\phi}_i = N\dot{\theta}_i$$

$\dot{\theta}_i$ = JOINT VELOCITY

$\dot{\phi}_i$ = MOTOR VELOCITY

$$\tau_B = N_i \tau_A$$

DC MOTOR DYNAMICS

$$V_b = K_b\dot{\phi} \qquad \phi = \text{MOTOR SPEED}$$

$$\tau_m = K_t i_a \qquad K_b = \text{BACK EMF}$$

IF IN S.I. UNITS: $K_t = K_b$

$$V_a = R_a i_a + K_t\dot{\phi}$$

V_a = ARMATURE VOLTAGE

R_a = ARMATURE RESISTANCE

i_a = ARMATURE CURRENT

$$\tau_m = \frac{K_t V_a}{R_a} - \frac{K_t}{R_a}\dot{\phi}$$

$$\tau_m = H(\Phi)\ddot{\Phi} + V(\Phi,\dot{\Phi}) + G(\Phi)$$

τ_m = MOTOR TORQUES

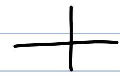
$$\text{WHERE } H' = \begin{bmatrix} J_1 + \frac{H_{11}}{N_1^2} & \frac{H_{12}}{N_1N_2} & \cdots \\ \frac{H_{21}}{N_1N_2} & J_2 + \frac{H_{22}}{N_2^2} & \\ & & \ddots & J_n + \frac{H_{nn}}{N_n^2} \end{bmatrix}$$

NOT SURE IF THAT'S RIGHT (MAYBE $J_n + \frac{H_{nn}}{H_n^2}$)

$$\Omega_{xyz} = \dot{\phi}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \dot{\theta}\begin{bmatrix} -S\phi \\ C\phi \\ 0 \end{bmatrix} + \dot{\psi}\begin{bmatrix} C\phi S\theta \\ S\phi S\theta \\ C\theta \end{bmatrix}$$

VELOCITY + ACCELERATION

$\dot{R} = \omega R$ $\quad$ ${}^0\dot{p}_1 = {}^0\dot{R}_1 {}^1p_1 = S(\underline{\omega})\,{}^0R_1 {}^1p_1 = {}^0\underline{\omega}_{01} \times {}^0p_1$



CONTROL

P:

$\frac{1}{a} = \tau$ $\quad$ a = DIS TO 1ST POLE (OR DOMINANT POLE)

PD:

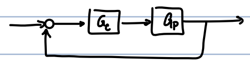
$\zeta\omega_n$ = DIS TO 1ST POLE $\quad$ 5x BIGGER OR SMALLER TO BE DOMINANT

PID:

ROOT LOCUS DESIGN

OLT.F = WHEN YOU DON'T CONSIDER FEEDBACK

$\hookrightarrow$ OLTF = $G_c G_p$



PID DESIGN

PD

1. FIND DESIRED POLE LOCATIONS (USE %OS, t_s ...)
2. FIND WHERE A ZERO SHOULD BE PLACED ON R.L

$$\underbrace{\Sigma\theta_z - \Sigma\theta_p = -180^\circ}_{\text{GRAPH METHOD (R.L.)}} \quad \text{OR} \quad \text{CHARACTERISTIC EQN}$$

3. FIND WHERE A POLE SHOULD BE PLACED

$$K_{OVERALL} = \frac{\Pi l_p}{\Pi l_z} \quad \text{OR} \quad K = \frac{1}{\|GH\|}$$

PI

4. SATISFY $\measuredangle$ CONDITION
5. AN OPEN LOOP POLE CANCELS AN OPEN LOOP ZERO

Formula Sheet Robot Control

$$a \times b = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Chain rule $\frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)\,g(x) + f(x)\,g'(x)$

Product rule $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\,g'(x)$

Quotient rule $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

DH Parameters

a_i : from z_{i-1} to z_i along x_i

d_i : from x_{i-1} to x_i along z_{i-1}

α_i : ∡ from z_{i-1} to z_i about x_i

θ_i : ∡ from x_{i-1} to x_i about z_{i-1}

If rotary, $x_i = x_{i-1}$ in zero ∡

Joint i : connects link i to link $i-1$

z_{i-1} : located at joint i

θ_i : intersection of n_i from z_i

x_i : // to n_i in direction from z_{i-1} to z_i

$${}^{i-1}T_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}R_1 = \begin{bmatrix} C\theta_1 & -S\theta_1 C\alpha_1 & S\theta_1 S\alpha_1 \\ S\theta & C\theta_1 C\alpha_1 & -C\theta_1 S\alpha_1 \\ 0 & S\alpha_1 & C\alpha_1 \end{bmatrix}$$

$${}^{1}T_0 = \begin{bmatrix} {}^{0}R_1 & -{}^{0}R_1^T\,{}^{0}d_{01} \\ 0 & 1 \end{bmatrix}$$

Jacobians

Jacobian: matrix representing derivatives of a vector function w.r.t. its inputs

- Jacobian of f from R^n to R^m is size $m \times n$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Manipulator

$${}^{0}\dot{d}_{0n} = J\dot{\theta}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \cdots & \frac{\partial x}{\partial \theta_n} \\ \frac{\partial y}{\partial \theta_1} & \cdots & \frac{\partial y}{\partial \theta_n} \\ \frac{\partial z}{\partial \theta_1} & \cdots & \frac{\partial z}{\partial \theta_n} \end{bmatrix}$$

J = manipulator Jacobian

- lots of computation, only position

Velocity:

$$\underbrace{\begin{bmatrix} {}^{0}\dot{d}_{0n} \\ {}^{0}\omega_{0n} \end{bmatrix}}_{\text{typically } 6\times1} = \underbrace{J_v}_{\text{typically } 6\times n}\, \underbrace{\dot{\Theta}}_{n\times1}$$

J_v = velocity Jacobian

$$\tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix}$$

$$\underbrace{W}_{\text{wrench}} = \begin{bmatrix} f_{n,n+1} \\ n_{n,n+1} \end{bmatrix} \begin{matrix} \text{forces} \\ \text{torques} \end{matrix}$$

$${}^{0}z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad {}^{0}z_1 = {}^{0}R_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_v = \begin{bmatrix} {}^{0}z_0 \times {}^{0}d_{0n} & {}^{0}z_1 \times {}^{0}d_{1n} & \cdots & {}^{0}z_{n-1} \times {}^{0}d_{n-1,n} \\ {}^{0}z_0 & {}^{0}z_1 & \cdots & {}^{0}z_{n-1} \end{bmatrix}$$

- less computation, plus it gives you ∡ velocity

If prismatic...

$$\begin{bmatrix} {}^{0}z_{i-1} \\ 0 \end{bmatrix}$$

When calculating these, write them out in matrix form

$$\tau = J_v^T\, W_{n+1}$$

$\dot{\Theta}$ - state vector (joint ∡'s, or d^*)

${}^{0}\dot{d}_{0n}$ = velocity end effector

${}^{0}\omega_{0n}$ = ∡ velocity of end effector

Jacobians uses

Inverse velocity:

- You can invert J_v to give velocities
- Only works if J_v is square, and there aren't singularities ($\det(J_v) = 0$)
- Singularities occur at edge of robot's reach, or outside of its capabilities
- If you have redundant DoF, you use a pseudo-inverse to cancel redundancies

Transmission

- Lets you rewrite from joint space into motor space

$$\dot{\Theta} = J_t\, \dot{\Phi}$$

$\dot{\Phi}$ = motor speeds

$$\tau_m = J_t^{-1}\, \tau_{joint}$$

← doesn't work if J_t not square

$$J_t = \begin{bmatrix} \frac{1}{N_1} & 0 & \cdots \\ 0 & \frac{1}{N_2} & \\ \vdots & & \ddots & \frac{1}{N_n} \end{bmatrix}$$

← Not a hard/fast rule. Write eqns of joint velocities & how they relate to motor velocities. Use that to build J_t

Dynamics

$$\tau = H(\Theta)\,\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

H = inertial matrix

V = velocity terms (a.k.a. coriolis)

G = gravity

Full dynamics w/ transmission

$$\tau_{joint} = H'(\Theta)\,\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

$$\tau_m = J_t^T\,\tau_{joint} = J_t^T H'(\Theta) J_t\, \ddot{\Phi} + J_t^T V(\Theta, \dot{\Theta}) + J_t^T G(\Theta)$$

First J_t^T from $\tau_m = J_t^T \tau_{joint}$

Second J_t^T to deal w/ $\ddot{\Phi}$

$H(\Theta)$ = inertia matrix wrt Θ

$H'(\Theta)$ = inertia matrix w/ motor inertia included

$H'(\Phi)$ = inertia matrix wrt Φ_m

Drivetrain Dynamics

$$N_i = \frac{r_B}{r_A}$$

N_i = gear ratio

$$\dot{\phi}_i = N\,\dot{\theta}_i$$

$\dot{\theta}_i$ = joint velocity

$\dot{\phi}_i$ = motor velocity

$$\tau_B = N_i\,\tau_A$$

DC Motor Dynamics

$$V_b = K_b\,\dot{\phi}$$

ϕ = motor speed

$$\tau_m = K_t\, i_a$$

K_b = back EMF

If in S.I. units: $K_t = K_b$

$$V_a = R_a i_a + K_t \dot{\phi}$$

V_a = armature voltage

R_a = armature resistance

i_a = armature current

$$\tau_m = \frac{K_t V_a}{R_a} - \frac{K_t}{R_a}\dot{\phi}$$

$$\tau_m = H'(\Phi)\,\ddot{\Phi} + V(\Phi, \dot{\Phi}) + G(\Phi)$$

τ_m = motor torques

where
$$H' = \begin{bmatrix} J_1 + \frac{H_{11}}{N_1^2} & \frac{H_{12}}{N_1 N_2} & \cdots \\ \frac{H_{21}}{N_1 N_2} & J_2 + \frac{H_{22}}{N_2^2} & \\ & & \ddots & J_n + \frac{H_{nn}}{N_n^2} \end{bmatrix}$$

Not sure if that's right (maybe $J_n + \frac{H_{nn}}{H_n^2}$)

IMPEDANCE MATCHING