ME EN 5230/6230, CS 6330

Intro to Robot Control – Spring 2023

Problem Set #3 Solutions: Drive Train and Motor Dynamics

1. For the planar 2-DOF robot from PS#2:

a. Derive the transmission Jacobian as a compounding of gear and pulley effects:

Derive a gear Jacobian:

$$\dot{\boldsymbol{\Phi}} = \mathbf{J_g}\dot{\boldsymbol{\Phi}}_\mathbf{m}$$

$$\mathbf{J_g} = \begin{bmatrix} \frac{1}{N_1} & 0 \\ 0 & \frac{1}{N_2} \end{bmatrix}$$

Derive a pulley Jacobian:

$$\theta_1 = \phi_1$$

$$\theta_2 = -\frac{r_1}{r_2}\phi_1 + \frac{r_1}{r_2}\phi_2$$

$$\dot{\boldsymbol{\Theta}} = \mathbf{J_p}\dot{\boldsymbol{\Phi}}$$

$$\mathbf{J_p} = \begin{bmatrix} 1 & 0 \\ -\frac{r_1}{r_2} & \frac{r_1}{r_2} \end{bmatrix}$$

Combine the Jacobians to get the compound transmission Jacobian to go from $\dot{\Phi}_\mathrm{m}$ to $\dot{\Theta}$:

$$\dot{\boldsymbol{\Theta}} = \mathbf{J_p}\dot{\boldsymbol{\Phi}} = \mathbf{J_p}\,\mathbf{J_g}\dot{\boldsymbol{\Phi}}_\mathbf{m}$$

$$\mathbf{J_t} = \mathbf{J_p}\,\mathbf{J_g}$$

$$\mathbf{J_t} = \begin{bmatrix} \frac{1}{N_1} & 0 \\ -\frac{r_1}{N_1\, r_2} & \frac{r_1}{N_2\, r_2} \end{bmatrix}$$

b. When the pulley radii are equal:

$$\mathbf{J_t} = \begin{bmatrix} \frac{1}{N_1} & 0 \\ -\frac{r_1}{N_1\, r_2} & \frac{r_1}{N_2\, r_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{N_1} & 0 \\ -\frac{1}{N_1} & \frac{1}{N_2} \end{bmatrix}$$

c. To map from motor velocities to end-effector velocities:

$$\dot{\boldsymbol{d}}_{\mathbf{02}} = \mathbf{J}\dot{\boldsymbol{\Theta}} = \mathbf{J}\mathbf{J_t}\dot{\boldsymbol{\Phi}}_{\mathbf{m}}$$

where J is the manipulator Jacobian from PS#1.

$$\mathbf{JJ_t} = \begin{bmatrix} -a_1\sin\theta_1 - a_2\sin(\theta_1+\theta_2) & -a_2\sin(\theta_1+\theta_2) \\ a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) & a_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \frac{1}{N_1} & 0 \\ -\frac{1}{N_1} & \frac{1}{N_2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{a_1\sin\theta_1}{N_1} & -\frac{a_2\sin(\theta_1+\theta_2)}{N_2} \\ \frac{a_1\cos\theta_1}{N_1} & \frac{a_2\cos(\theta_1+\theta_2)}{N_2} \end{bmatrix} = \begin{bmatrix} -\frac{a_1}{N_1}\sin\left(\frac{\phi_{m1}}{N_1}\right) & -\frac{a_2}{N_2}\sin\left(\frac{\phi_{m2}}{N_2}\right) \\ \frac{a_1}{N_1}\cos\left(\frac{\phi_{m1}}{N_1}\right) & \frac{a_2}{N_2}\cos\left(\frac{\phi_{m2}}{N_2}\right) \end{bmatrix}$$

d. Use duality to write a matrix equation relating motor torques to end-effector forces:

$$\boldsymbol{\tau_m} = \mathbf{J}_{\boldsymbol{t}}^{\mathbf{T}}\,\mathbf{J}^{\mathbf{T}}\boldsymbol{F} = (\mathbf{JJ_t})^{\boldsymbol{T}}\boldsymbol{F}$$

$$(\mathbf{JJ_t})^{\boldsymbol{T}} = \begin{bmatrix} -\frac{a_1}{N_1}\sin\left(\frac{\phi_{m1}}{N_1}\right) & \frac{a_1}{N_1}\cos\left(\frac{\phi_{m1}}{N_1}\right) \\ -\frac{a_2}{N_2}\sin\left(\frac{\phi_{m2}}{N_2}\right) & \frac{a_2}{N_2}\cos\left(\frac{\phi_{m2}}{N_2}\right) \end{bmatrix}$$

2. For the 2-DOF Quanser Robot from PS#2, combine arm and transmission dynamics

a. Combine the dynamics in motor space:

$$\dot{\boldsymbol{\Theta}} = \mathbf{J_t}\dot{\boldsymbol{\Phi}}_\mathbf{m}$$

$$\boldsymbol{\tau}_\mathbf{m} = \mathbf{J_t^T}\boldsymbol{\tau}_{joint}$$

$$\boldsymbol{\tau_m} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{bmatrix} \ddot{\phi}_{m1} \\ \ddot{\phi}_{m2} \end{bmatrix} + \mathbf{J_t^T}\boldsymbol{H}(\boldsymbol{\Theta})\mathbf{J_t}\begin{bmatrix} \ddot{\phi}_{m1} \\ \ddot{\phi}_{m2} \end{bmatrix} + \mathbf{J_t^T}\boldsymbol{V}(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}}) + \mathbf{J_t^T}\boldsymbol{G}(\boldsymbol{\Theta}) + \boldsymbol{F}(\dot{\boldsymbol{\Phi}}_\mathbf{m})$$

$$\boldsymbol{\tau_m} = \boldsymbol{H}'(\boldsymbol{\Phi}_\mathbf{m})\ddot{\boldsymbol{\Phi}}_\mathbf{m} + \mathbf{V}(\boldsymbol{\Phi}_\mathbf{m}, \dot{\boldsymbol{\Phi}}_\mathbf{m}) + \boldsymbol{G}(\boldsymbol{\Phi}_\mathbf{m}) + \boldsymbol{F}(\dot{\boldsymbol{\Phi}}_\mathbf{m})$$

$$\boldsymbol{H}'(\boldsymbol{\Phi}_\mathbf{m}) = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} + \mathbf{J_t^T}\boldsymbol{H}(\boldsymbol{\Theta})\,\mathbf{J_t}$$

$$= \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{N_1} & -\frac{1}{N_1} \\ 0 & \frac{1}{N_2} \end{bmatrix}\begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix}\begin{bmatrix} \frac{1}{N_1} & 0 \\ -\frac{1}{N_1} & \frac{1}{N_2} \end{bmatrix}$$

$$= \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} + \begin{bmatrix} \frac{H_{11}}{N_1} - \frac{H_{12}}{N_1} & \frac{H_{12}}{N_1} - \frac{H_{22}}{N_1} \\ \frac{H_{12}}{N_2} & \frac{H_{22}}{N_2} \end{bmatrix}\begin{bmatrix} \frac{1}{N_1} & 0 \\ -\frac{1}{N_1} & \frac{1}{N_2} \end{bmatrix}$$

$$= \begin{bmatrix} J_1 + \frac{H_{11} - 2H_{12} + H_{22}}{N_1^2} & \frac{H_{12} - H_{22}}{N_1 N_2} \\ \frac{H_{12} - H_{22}}{N_1 N_2} & J_2 + \frac{H_{22}}{N_2^2} \end{bmatrix}$$

$$\boldsymbol{H}'(\boldsymbol{\Phi}_\mathbf{m}) = \begin{bmatrix} J_1 + \frac{I_1 + m_2 a_1^2}{N_1^2} & \frac{a_1 r_{12} m_2}{N_1 N_2} cos\left(\frac{\phi_{m2}}{N_2} - \frac{\phi_{m1}}{N_1}\right) \\ \frac{a_1 r_{12} m_2}{N_1 N_2} cos\left(\frac{\phi_{m2}}{N_2} - \frac{\phi_{m1}}{N_1}\right) & J_2 + \frac{I_2}{N_2^2} \end{bmatrix}$$

$$\boldsymbol{V}(\boldsymbol{\Phi}_\mathbf{m}, \dot{\boldsymbol{\Phi}}_\mathbf{m}) = \begin{bmatrix} \frac{1}{N_1} & -\frac{1}{N_1} \\ 0 & \frac{1}{N_2} \end{bmatrix}\begin{bmatrix} 0 & -2h & -h \\ h & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{h}{N_1} & -\frac{2h}{N_1} & -\frac{h}{N_1} \\ \frac{h}{N_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{\dot{\phi}_{m1}}{N_1}\right)^2 \\ \frac{\dot{\phi}_{m1}}{N_1}\left(\frac{\dot{\phi}_{m2}}{N_2} - \frac{\dot{\phi}_{m1}}{N_1}\right) \\ \left(\frac{\dot{\phi}_{m2}}{N_2} - \frac{\dot{\phi}_{m1}}{N_1}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{h}{N_1} & -\frac{2h}{N_1} & -\frac{h}{N_1} \\ \frac{h}{N_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{\dot{\phi}_{m1}}{N_1}\right)^2 \\ \frac{\dot{\phi}_{m1}}{N_1}\left(\frac{\dot{\phi}_{m2}}{N_2} - \frac{\dot{\phi}_{m1}}{N_1}\right) \\ \left(\frac{\dot{\phi}_{m2}}{N_2} - \frac{\dot{\phi}_{m1}}{N_1}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{h\,\dot{\phi}_{m1}^2}{N_1^3} - \frac{2h}{N_1^2 N_2}\dot{\phi}_{m1}\dot{\phi}_{m2} + \frac{2h}{N_1^3}\dot{\phi}_{m1}^2 - \frac{h\dot{\phi}_{m2}^2}{N_1 N_2^2} + \frac{2h}{N_1^2 N_2}\dot{\phi}_{m1}\dot{\phi}_{m2} - \frac{h}{N_1^3}\dot{\phi}_{m1}^2 \\ \frac{h\,\dot{\phi}_{m1}^2}{N_1^2 N_2} \end{bmatrix}$$

$$\boldsymbol{V}(\boldsymbol{\Phi}_\mathbf{m}, \dot{\boldsymbol{\Phi}}_\mathbf{m}) = \begin{bmatrix} 0 & -\frac{h}{N_1 N_2^2} \\ \frac{h}{N_1^2 N_2} & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_{m1}^2 \\ \dot{\phi}_{m2}^2 \end{bmatrix}$$

Where:

$$h = a_1 r_{12} m_2 sin(\theta_2) = a_1 r_{12} m_2 sin\left(\frac{\phi_{m2}}{N_2} - \frac{\phi_{m1}}{N_1}\right)$$

$$\boldsymbol{G}(\boldsymbol{\Phi}_\mathbf{m}) = \begin{bmatrix} \frac{1}{N_1} & -\frac{1}{N_1} \\ 0 & \frac{1}{N_2} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} \frac{G_1 - G_2}{N_1} \\ \frac{G_2}{N_2} \end{bmatrix} = \begin{bmatrix} (r_{01} m_1 + a_1 m_2)\frac{g}{N_1}\, cos\left(\frac{\phi_{m1}}{N_1}\right) \\ r_{12}\, m_2\, \frac{g}{N_2}\, cos\left(\frac{\phi_{m2}}{N_2}\right) \end{bmatrix}$$

$$\boldsymbol{F}(\dot{\boldsymbol{\Phi}}_\mathbf{m}) = \begin{bmatrix} b_1\dot{\phi}_{m1} + c_1 * sgn(\dot{\phi}_{m1}) \\ b_2\dot{\phi}_{m2} + c_2 * sgn(\dot{\phi}_{m2}) \end{bmatrix}$$

b. Combine the dynamics in joint space:

$$\boldsymbol{\tau}_{joint} = \mathbf{J_t^{-T}} \boldsymbol{\tau}_\mathbf{m}$$
$$\dot{\boldsymbol{\Phi}}_\mathbf{m} = \mathbf{J_t^{-1}} \boldsymbol{\Theta}$$

$$\mathbf{J_t^{-1}} = \begin{bmatrix} \frac{1}{N_1} & 0 \\ -\frac{1}{N_1} & \frac{1}{N_2} \end{bmatrix}^{-1} = \begin{bmatrix} N_1 & 0 \\ N_2 & N_2 \end{bmatrix}$$

$$\boldsymbol{\tau}_{joint} = \mathbf{J_t^{-T}} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \mathbf{J_t^{-1}} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \mathbf{J_t^{-T}} \boldsymbol{F}\left(\dot{\boldsymbol{\Phi}}_\mathbf{m}\right) + \mathbf{H}(\boldsymbol{\Theta})\ddot{\boldsymbol{\Theta}} + \mathbf{V}\left(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}}\right) + \mathbf{G}(\boldsymbol{\Theta})$$

$$\boldsymbol{\tau}_{joint} = \boldsymbol{H}'(\boldsymbol{\Theta})\ddot{\boldsymbol{\Theta}} + \mathbf{V}\left(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}}\right) + \boldsymbol{G}(\boldsymbol{\Theta}) + \boldsymbol{F}(\dot{\boldsymbol{\Theta}})$$

$$\boldsymbol{H}'(\boldsymbol{\Theta}) = \mathbf{J_t^{-T}} \boldsymbol{\tau}_{\boldsymbol{J}} \mathbf{J_t^{-1}} + \boldsymbol{H}(\boldsymbol{\Theta})$$

$$= \begin{bmatrix} N_1 & N_2 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} N_1 & 0 \\ N_2 & N_2 \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix}$$

$$= \begin{bmatrix} N_1^2 J_1 + N_2^2 J_2 & N_2^2 J_2 \\ N_2^2 J_2 & N_2^2 J_2 \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix}$$

$$= \begin{bmatrix} H_{11} + N_1^2 J_1 + N_2^2 J_2 & H_{12} + N_2^2 J_2 \\ H_{12} + N_2^2 J_2 & H_{22} + N_2^2 J_2 \end{bmatrix}$$

$$\boldsymbol{F}\left(\dot{\boldsymbol{\Theta}}\right) = \mathbf{J_t^{-T}} \boldsymbol{F}\left(\dot{\boldsymbol{\Phi}}_\mathbf{m}\right) = \begin{bmatrix} N_1 & N_2 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} b_1 N_1 \dot{\theta}_1 + c_1 * sgn(N_1 \dot{\theta}_1) \\ b_2\left(N_2 \dot{\theta}_1 + N_2 \dot{\theta}_2\right) + c_2 * sgn(N_2 \dot{\theta}_1 + N_2 \dot{\theta}_2) \end{bmatrix}$$

$$= \begin{bmatrix} b_1 N_1^2 \dot{\theta}_1 + b_1 N_2^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_1 N_1 sgn\left(\dot{\theta}_1\right) + c_2 N_2 sgn(\dot{\theta}_1 + \dot{\theta}_2) \\ b_2 N_2^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_2 N_2 sgn(\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix}$$

V and G are same as originally derived in PS#2 because they are already in joint space.

$$\mathbf{V}\left(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}}\right) = \begin{bmatrix} 0 & -2h & -h \\ h & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_2^2 \end{bmatrix}$$

$$\boldsymbol{G}(\boldsymbol{\Theta}) = \begin{bmatrix} (r_{01} m_1 + a_1 m_2) g \cos(\theta_1) + r_{12}\, m_2 g \cos(\theta_1 + \theta_2) \\ r_{12}\, m_2 g \cos(\theta_1 + \theta_2) \end{bmatrix}$$

c. Use gear Jacobian to reflect dynamics to encoder space

$$\boldsymbol{\tau_m} = \boldsymbol{H}'(\boldsymbol{\Phi}_\mathbf{m})\ddot{\boldsymbol{\Phi}}_\mathbf{m} + \mathbf{V}(\boldsymbol{\Phi}_\mathbf{m}, \dot{\boldsymbol{\Phi}}_\mathbf{m}) + \boldsymbol{G}(\boldsymbol{\Phi}_\mathbf{m}) + \boldsymbol{F}(\dot{\boldsymbol{\Phi}}_\mathbf{m})$$

$$\dot{\boldsymbol{\Phi}}_\mathbf{m} = \mathbf{J}_\mathbf{g}^{-1}\dot{\boldsymbol{\Phi}} \quad \text{where} \quad \mathbf{J}_\mathbf{g}^{-1} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

$$\boldsymbol{\tau_e} = \mathbf{J}_\mathbf{g}^{-T}\boldsymbol{\tau_m} = \mathbf{J}_\mathbf{g}^{-T}\boldsymbol{H}'(\boldsymbol{\Phi}_\mathbf{m})\mathbf{J}_\mathbf{g}^{-1}\ddot{\boldsymbol{\Phi}} + \mathbf{J}_\mathbf{g}^{-T}\mathbf{V}(\boldsymbol{\Phi}_\mathbf{m}, \dot{\boldsymbol{\Phi}}_\mathbf{m}) + \mathbf{J}_\mathbf{g}^{-T}\boldsymbol{G}(\boldsymbol{\Phi}_\mathbf{m}) + \mathbf{J}_\mathbf{g}^{-T}\boldsymbol{F}(\dot{\boldsymbol{\Phi}}_\mathbf{m})$$

$$\boldsymbol{\tau_e} = \boldsymbol{H}'(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \mathbf{V}(\boldsymbol{\Phi}, \dot{\boldsymbol{\Phi}}) + \boldsymbol{G}(\boldsymbol{\Phi}) + \boldsymbol{F}(\dot{\boldsymbol{\Phi}})$$

$$\boldsymbol{H}'(\boldsymbol{\Phi}) = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} J_1 + \dfrac{I_1 + m_2 a_1^2}{N_1^2} & \dfrac{a_1 r_{12} m_2}{N_1 N_2} cos\left(\dfrac{\phi_{m2}}{N_2} - \dfrac{\phi_{m1}}{N_1}\right) \\ \dfrac{a_1 r_{12} m_2}{N_1 N_2} cos\left(\dfrac{\phi_{m2}}{N_2} - \dfrac{\phi_{m1}}{N_1}\right) & J_2 + \dfrac{I_2}{N_2^2} \end{bmatrix} \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

$$= \begin{bmatrix} N_1^2 J_1 + I_1 + m_2 a_1^2 & a_1 r_{12} m_2\, cos(\phi_2 - \phi_1) \\ a_1 r_{12} m_2 cos(\phi_2 - \phi_1) & N_2^2 J_2 + I_2 \end{bmatrix}$$

$$\boldsymbol{V}(\boldsymbol{\Phi}, \dot{\boldsymbol{\Phi}}) = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} 0 & -\dfrac{h}{N_1 N_2^2} \\ \dfrac{h}{N_1^2 N_2} & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_{1m}^2 \\ \dot{\phi}_{2m}^2 \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{h}{N_2^2} \\ \dfrac{h}{N_1^2} & 0 \end{bmatrix} \begin{bmatrix} N_1^2 \dot{\phi}_1^2 \\ N_2^2 \dot{\phi}_2^2 \end{bmatrix} = \begin{bmatrix} 0 & -h \\ h & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1^2 \\ \dot{\phi}_2^2 \end{bmatrix}$$

where

$$h = a_1 r_{12} m_2 sin(\phi_2 - \phi_1)$$

$$\boldsymbol{G}(\boldsymbol{\Phi}) = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} (r_{01} m_1 + a_1 m_2) \dfrac{g}{N_1} cos\left(\dfrac{\phi_{m1}}{N_1}\right) \\ \dfrac{r_{12}\, m_2\, g}{N_2} cos\left(\dfrac{\phi_{m2}}{N_2}\right) \end{bmatrix} = \begin{bmatrix} (r_{01} m_1 + a_1 m_2) g\, cos(\phi_1) \\ r_{12}\, m_2\, g\, cos(\phi_2) \end{bmatrix}$$

$$\boldsymbol{F}(\boldsymbol{\Phi}) = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} + \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} sgn\left(\begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} N_1^2 b_1 & 0 \\ 0 & N_2^2 b_2 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} + \begin{bmatrix} N_1 c_1 * sgn(\dot{\phi}_1) \\ N_2 c_2 * sgn(\dot{\phi}_2) \end{bmatrix}$$

3. Include motor dynamics:

a. Current control:

$$\mathbf{J_g^T}\boldsymbol{\tau_m} = \boldsymbol{H}'(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \mathbf{V}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}}) + \boldsymbol{G}(\boldsymbol{\Phi}) + \boldsymbol{F}(\dot{\boldsymbol{\Phi}})$$

$$\tau_{mi} = k_{ti}\, i_i(t)$$

$$\begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}\begin{bmatrix} k_{t_1} i_1(t) \\ k_{t_2} i_2(t) \end{bmatrix} = \boldsymbol{H}'(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \mathbf{V}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}}) + \boldsymbol{G}(\boldsymbol{\Phi}) + \boldsymbol{F}(\dot{\boldsymbol{\Phi}})$$

So $$\begin{bmatrix} A_1 i_1(t) \\ A_2 i_2(t) \end{bmatrix} = \boldsymbol{H}'(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \mathbf{V}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}}) + \boldsymbol{G}(\boldsymbol{\Phi}) + \boldsymbol{F}(\dot{\boldsymbol{\Phi}})$$

Where $$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} N_1 k_{t_1} \\ N_2 k_{t_2} \end{bmatrix}$$

b. Voltage control:

$$\mathbf{J_g^T}\boldsymbol{\tau_m} = \boldsymbol{H}'(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \mathbf{V}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}}) + \boldsymbol{G}(\boldsymbol{\Phi}) + \boldsymbol{F}(\dot{\boldsymbol{\Phi}})$$

$$V_i(t) = R_i\,\frac{\tau_{mi}}{k_{ti}} + k_{ti}\,\dot{\phi}_{mi}$$

$$\tau_{mi} = \frac{k_{ti}}{R_i}\,V_i(t) - \frac{k_{ti}^2}{R_i}\,N_i\dot{\phi}_i$$

$$\begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}\begin{bmatrix} \frac{k_{t_1}}{R_1}V_1(t) - \frac{k_{t_1}^2}{R_1}N_1\dot{\phi}_1 \\ \frac{k_{t_2}}{R_2}V_2(t) - \frac{k_{t_2}^2}{R_2}N_2\dot{\phi}_2 \end{bmatrix} = \boldsymbol{H}'(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \mathbf{V}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}}) + \boldsymbol{G}(\boldsymbol{\Phi}) + \boldsymbol{F}(\dot{\boldsymbol{\Phi}})$$

So $$\begin{bmatrix} A_1 V_1(t) \\ A_2 V_2(t) \end{bmatrix} = \boldsymbol{H}'(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \mathbf{V}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}}) + \boldsymbol{G}(\boldsymbol{\Phi}) + \boldsymbol{F}'(\dot{\boldsymbol{\Phi}})$$

where

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} N_1\frac{k_{t_1}}{R_1} \\ N_2\frac{k_{t_2}}{R_2} \end{bmatrix}$$

and $$\boldsymbol{F}'(\dot{\boldsymbol{\Phi}}) = \begin{bmatrix} N_1^2\left(b_1 + \frac{k_{t_1}^2}{R_1}\right) & 0 \\ 0 & N_2^2\left(b_2 + \frac{k_{t_2}^2}{R_2}\right) \end{bmatrix}\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} + \begin{bmatrix} N_1 c_1 * sgn(\dot{\phi}_1) \\ N_2 c_2 * sgn(\dot{\phi}_2) \end{bmatrix}$$

Note that with voltage control, the back EMF adds to the viscous damping terms