Problem Set # 4 Solutions

1. Impedance Matching

Maximize acceleration:

$$\ddot{\theta} = \frac{N}{(I_1 + N^2 J_m)} \tau \qquad \text{for } \theta = 0,\ \dot{\theta} = 0$$

set $\dfrac{d\ddot{\theta}}{dN} = 0$ and solve for N

$$\frac{d\ddot{\theta}}{dN} = \frac{(I_1 + N^2 J_m) - N(2NJ_m)}{(I_1 + N^2 J_m)^2} \tau = 0$$

$$I_1 + N^2 J_m - 2N^2 J_m = 0$$

$$I_1 = N^2 J_m$$

$$N = \sqrt{\frac{I_1}{J_m}} = \sqrt{\frac{0.83 \times 10^{-3}}{0.65 \times 10^{-6}}} = 36$$

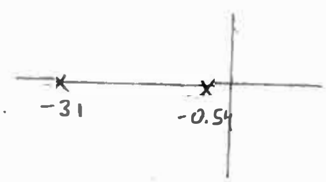
2.1

$$\frac{\Theta(s)}{V(s)} = \frac{N k_t / R_a}{(I_1 + N^2 J_m) s^2 + N^2\left(b + \frac{k}{R_a}\right) s + m g r_{01}}$$

$$= \frac{0.207}{0.0040 s^2 + 0.1275 s + 0.067} = \frac{52}{s^2 + 32 s + 17}$$

O.L. poles: $s^2 + 32s + 17 = 0$

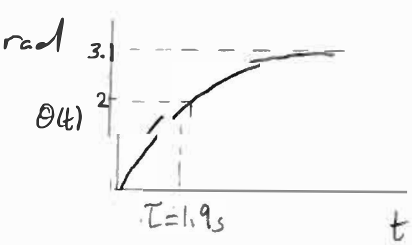
$$s = -0.54,\ -31$$



since faster pole is > 5x to left of slower pole, this system will behave like a 1st order system with time constant $\tau = \dfrac{1}{0.54} = 1.9$ sec

Step response will rise to 63% of final value in 1.9 sec

Final value $= \lim\limits_{s \to 0} \dfrac{52}{s^2 + 32s + 17} = 3.1$ rad

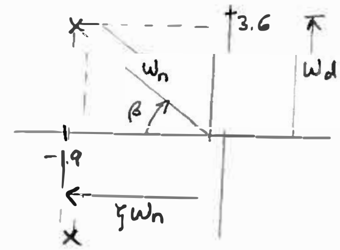


2.2

$$\frac{\Theta(s)}{i(s)} = \frac{N k_t}{(I_1 + N^2 J_m)s^2 + N^2 b s + m g r_{01}} = \frac{134}{s^2 + 3.7s + 17}$$

O.L. poles: $s^2 + 3.7s + 17 = 0$

$$s = -1.9 \pm 3.6j$$



Since we have a complex pair of poles, system will behave like a 2nd order underdamped system

$\omega_n^2 = 17 \Rightarrow \omega_n = 4.1$ rad/s

$2\zeta\omega_n = 3.7 \qquad \zeta = 0.45$

Peak time: $T_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.6} = 0.87s$

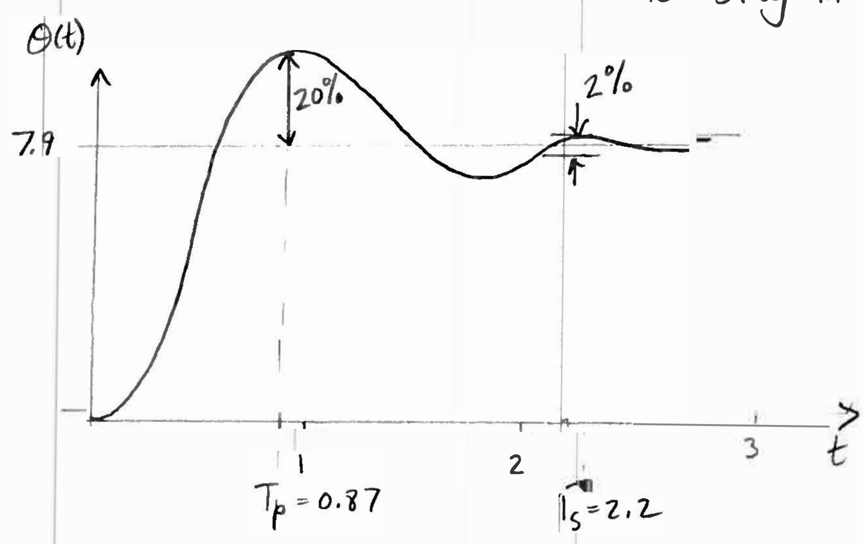
2% settling time: $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{1.9} = 2.2$ s

$$\%OS = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = e^{-\pi\zeta\omega_n/\omega_d} = e^{-\pi(1.9)/3.6} = 0.19$$

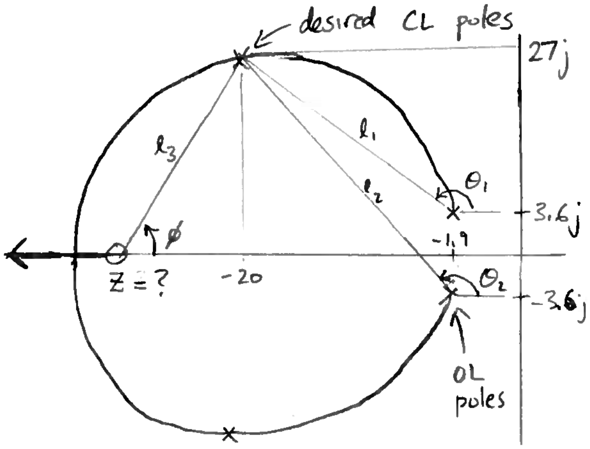
(19%)

Final value $= \lim_{s\to 0} \frac{134}{s^2+3.7s+17} = 7.9$ rad

in practice, we would have to use a smaller step to stay in linear range



3.1 PD Control : Design for 10% OS and $T_s = 0.2\,s$



$$\text{Desired } \zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = \frac{-\ln(0.1)}{\sqrt{\pi^2 + \ln^2(0.1)}}$$

$$\zeta = 0.59$$

$$\text{desired } T_s = \frac{4}{\zeta\omega_n} = 0.2 \text{ sec}$$

so $\zeta\omega_n = 20$

$$\omega_n = 34$$

$$G_c G_p = \frac{(K_P + K_D s)\,134}{s^2 + 3.7s + 17}$$

$$= \frac{134K_D \left(s + \frac{K_P}{K_D}\right)}{s^2 + 3.7s + 17}$$

loop gain

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 27$$

want closed loop poles at $s = -20 \pm 27j$

need to place zero to get root locus to go through desired closed-loop pole location

use Angle Condition: $\phi - \theta_1 - \theta_2 = \pm 180°$

$$\phi = \pm 180° + \tan^{-1}\left(\frac{23.4}{-18.1}\right) + \tan^{-1}\left(\frac{30.6}{-18.1}\right) = \pm 180° + 128° + 121° = 69°$$

$$\frac{27}{-20 - Z} = \tan 69° \qquad Z = -\frac{27}{\tan 69°} - 20 = -30$$

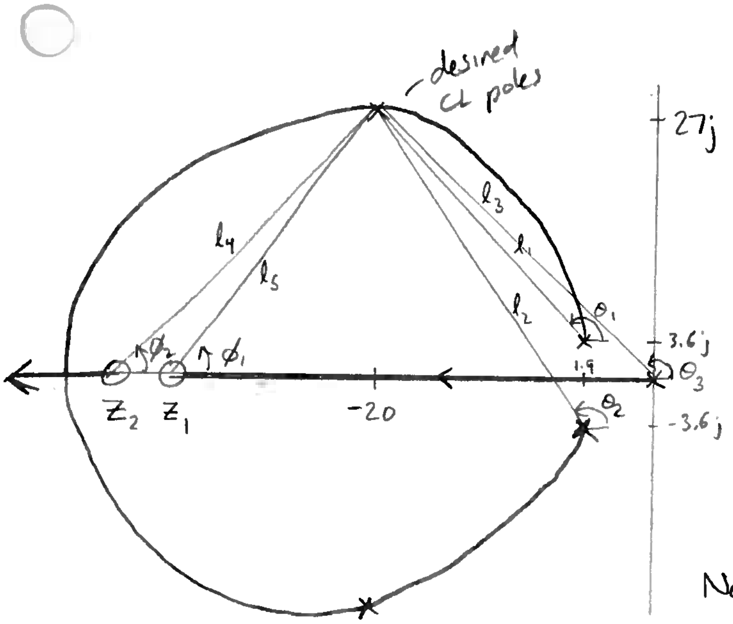
so place zero at $s = -30 \Rightarrow \frac{K_P}{K_D} = 30$

use magnitude condition to find K_D

$$134\,K_D = \frac{l_1 l_2}{l_3} = \frac{\sqrt{23.4^2 + 18.1^2}\,\sqrt{30.6^2 + 18.1^2}}{\sqrt{10^2 + 27^2}} = 36.5$$

$$\boxed{\begin{aligned} K_D &= 0.27 \\ K_P &= (30)(0.27) = 8.2 \end{aligned}}$$

3.2 PID Control: Design for 10% OS, $T_s = 0.2s$, 0 s.s. error



$$G_c = K_P + K_D s + \frac{K_I}{S}$$

$$G_c G_p = \frac{134 K_D \left(s^2 + \frac{K_P}{K_D} s + \frac{K_I}{K_D} \right)}{S(s^2 + 3.7s + 17)}$$

loop gain

The O.L. pole at $s=0$ guarantees zero steady-state error. The closed-loop poles need to be at $s = -20 \pm 27j$ as before to get desired %OS and T_s

Need to place two zeros using angle cond.

One strategy would be to place both zeros at same location such that $\phi_1 = \phi_2 = \phi$
$l_4 = l_5 = l_z$

Angle Condition: $2\phi - \theta_1 - \theta_2 - \theta_3 = \pm 180°$

$$2\phi = \pm 180° + 128° + 121° + \underbrace{\tan^{-1}\left(\frac{27}{-20}\right)}_{127°}$$

$$\phi = 98°$$

$$Z = \frac{-27}{\tan 98°} - 20 = -16$$ so place two zeros at $s = -16$

now use magnitude condition to find K_D

$$134 K_D = \frac{l_1 l_2 l_3}{l_z^2} = \frac{\sqrt{23.4^2 + 18.1^2}\sqrt{30.6^2 + 18.1^2}\sqrt{27^2 + 20^2}}{4^2 + 27^2} = 47.4$$

$$\boxed{K_D = 0.35}$$

$$s^2 + \frac{K_P}{K_D} s + \frac{K_I}{K_D} = (s + 16)^2 = s^2 + 32s + 256$$

location of zeros

$$\frac{K_P}{K_D} = 32 \Rightarrow \boxed{K_P = 11}$$

$$\frac{K_I}{K_D} = 256 \Rightarrow \boxed{K_I = 90}$$

Solutions to Problem Set #4: 1-DOF Linear Control

Problems 3.3 and 3.4

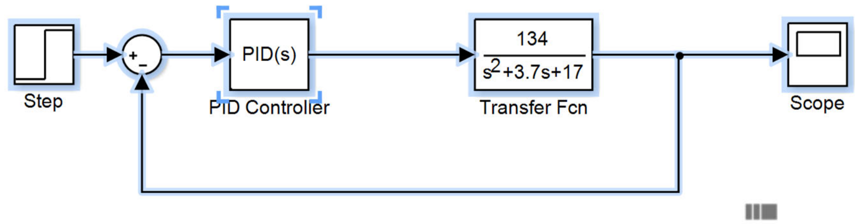


Figure 1. Model for PD/PID Control. Note that the PID block in Simulink has a built-in low-pass filter on the derivative term. This is necessary to achieve an accurate numerical simulation in the presence of a step input. Without the filter, the derivative of a step results in an impulse of infinite magnitude, which cannot be accurately reproduced by Simulink. Simulink can only produce a finite-magnitude impulse, which will not contain as much energy and will not lead to an accurate step response. The low-pass filter alleviates this problem, and as long as the low-pass filter is 5x faster than the dominant poles, it should not have any noticeable effect on the step response.

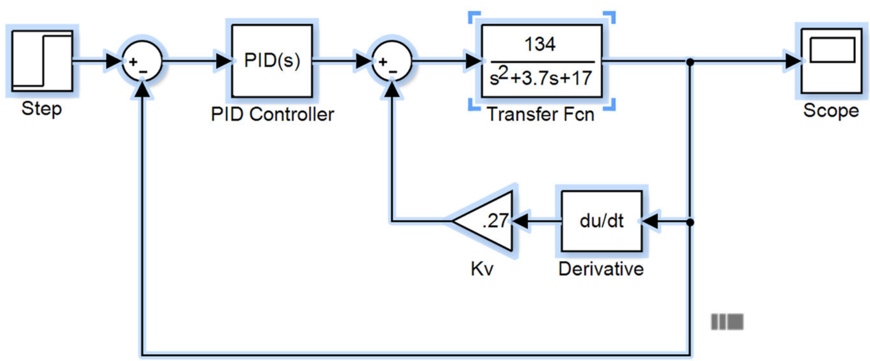


Figure 2. Model for PV/PIV Control. In this case, the D gain in the PID block should be set to zero, or the PID block could be replaced by individual P and I branches in parallel.

Comparison of closed loop transfer functions for PD/PID vs PV/PIV control:

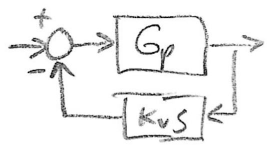
PD:

$$\frac{\Theta}{\Theta_d} = \frac{\dfrac{(K_p+K_D s)\,134}{s^2+3.7s+17}}{1+\dfrac{(K_p+K_D s)\,134}{s^2+3.7s+17}} = \frac{134K_D(s+K_P/K_D)}{s^2+(3.7+134K_D)s+(17+134K_p)}$$

$$= \frac{36.5(s+30)}{(s+20+27j)(s+20-27j)}$$

PV: Need to compute C.L.T.F. in two steps:

inner loop:



$$\Rightarrow \boxed{\frac{G_p}{1+K_v s G_p}}$$

use this as O.L.T.F. for outer loop

outer loop:

$$\frac{\Theta}{\Theta_d} = \frac{K_p\left(\dfrac{G_p}{1+K_v s G_p}\right)}{1+K_p\left(\dfrac{G_p}{1+K_v s G_p}\right)} = \frac{K_p G_p}{1+(K_p+K_v s)G_p}$$

$$= \frac{\dfrac{K_p(134)}{s^2+3.7s+17}}{1+\dfrac{(K_p+K_v s)\,134}{s^2+3.7s+17}} = \frac{K_p(134)}{s^2+(3.7+134K_v)s+(17+134K_p)}$$

$$= \frac{(8.2)(134)}{(s+20+27j)(s+20-27j)}$$

same C.L. poles as PD but no C.L. zero

PID:

$$\frac{\Theta}{\Theta_d} = \frac{\left(\frac{K_D s^2 + K_P s + K_I}{s}\right)\left(\frac{134}{s^2+3.7s+17}\right)}{1+\left(\frac{K_D s^2+K_P s+K_I}{s}\right)\left(\frac{134}{s^2+3.7s+17}\right)}$$

$$= \frac{134 K_D\left(s^2 + \frac{K_P}{K_D}s + \frac{K_I}{K_D}\right)}{s^3 + (3.7+134K_D)s^2 + (17+134K_P)s + 134K_I}$$

$$= \frac{47.4\,(s+16)(s+16)}{(s+12)(s+20+27j)(s+20-27j)}$$

PIV: inner loop is same as PV

outer loop: $$\frac{\Theta}{\Theta_d} = \frac{\left(\frac{K_P s + K_I}{s}\right)\left(\frac{G_p}{1+K_v s G_p}\right)}{1+\left(\frac{K_P s+K_I}{s}\right)\left(\frac{G_p}{1+K_v s G_p}\right)} = \frac{(K_P s + K_I)G_p}{s + (K_v s^2 + K_P s + K_I)G_p}$$

$$= \frac{\frac{(K_P s + K_I)\,134}{s^2+3.7s+17}}{\frac{s + (K_v s^2 + K_P s + K_I)\,134}{s^2+3.7s+17}}$$

$$= \frac{134 K_P\left(s + \frac{K_I}{K_P}\right)}{s^3 + (3.7+134K_v)s^2 + (17+134K_P s) + 134K_I}$$

$$= \frac{(8.2)(134)(s+9)}{(s+12)(s+20+27j)(s+20-27j)}$$

same C.L. poles as PID, but one zero @ $s=-9$ instead of two zeros @ $s=-16$

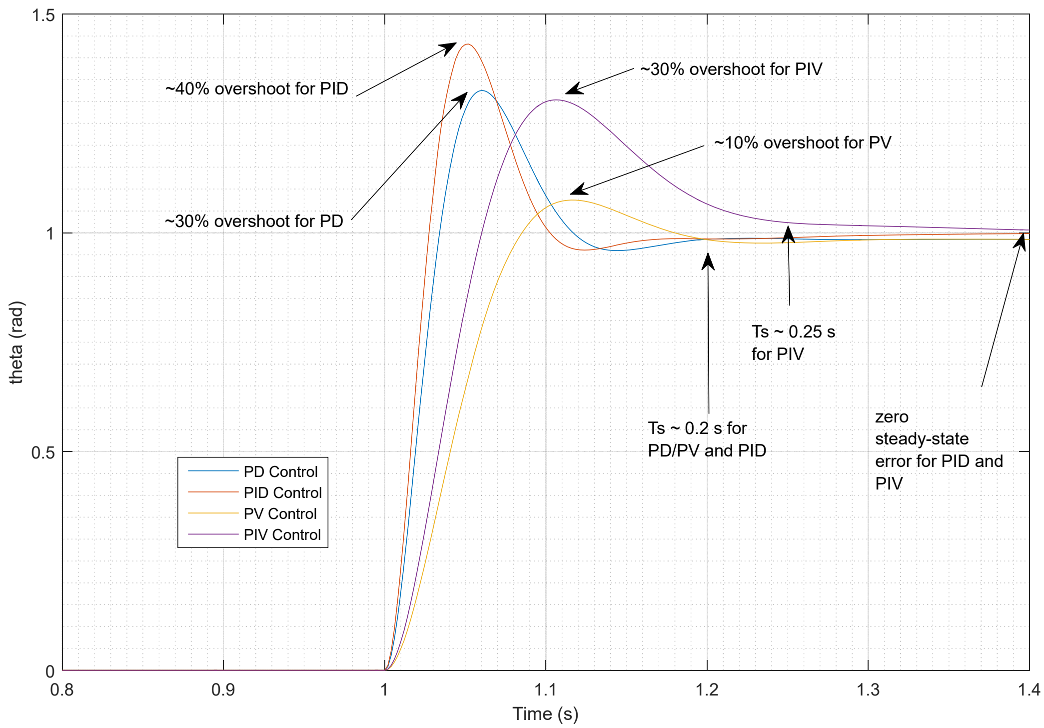


Figure 3. Step responses for PD/PID and PV/PIV controllers.

Note that although both PD and PID controllers result in the specified settling time, neither of them achieves the specified 10% overshoot. This is likely due to the proximity of the closed-loop zeros (and the 3rd closed-loop pole in the case of the PID controller). Unless they are 5x faster than the dominant poles (which is not the case in our design), they will interfere with the step response.

Notice that the PV controller (which has no closed-loop zeros) does achieve the desired settling time and overshoot. The PIV controller has one closed-loop zero at s=-9 (versus two at s=-16 for the PID controller), but that one closed-loop zero and the 3rd closed-loop pole (at s=-12) interfere with both the overshoot and settling time. We are actually fortunate with both the PID and PIV control that the 3rd C.L. pole at s=-12 is nearly cancelled by one of the nearby C.L. zeros, which is why the complex pair of poles are still mostly dominant. If the extra overshoot and/or settling time is unsatisfactory, we could try using the SISO tool in MATLAB to manually tune the gains until the desired performance is achieved, but I would say that in this case, our controller will do just fine as it is.

Another way to think about the difference between D and V control action is that the D action differentiates the desired position while the V does not. Thus the D reacts more quickly to the step, resulting in a faster rise time at the expense of more overshoot. However remember that step responses are only a tool for gain tuning. Once we are done tuning the gains, we will generally use smooth desired trajectories rather than steps, in which case the D action won't cause extra overshoot.

The PID and PIV controllers both result in zero steady-state error, while the PD and PV controllers do not.

Problem 3.5

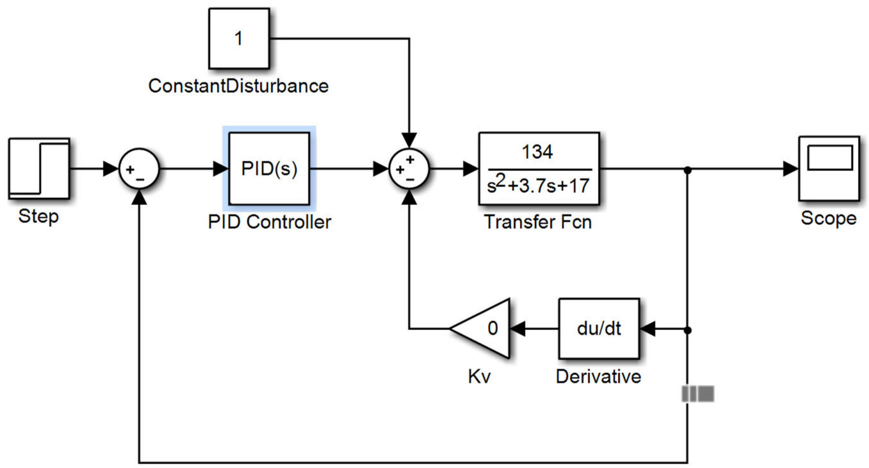


Figure 4. Constant Disturbance to the Plant.

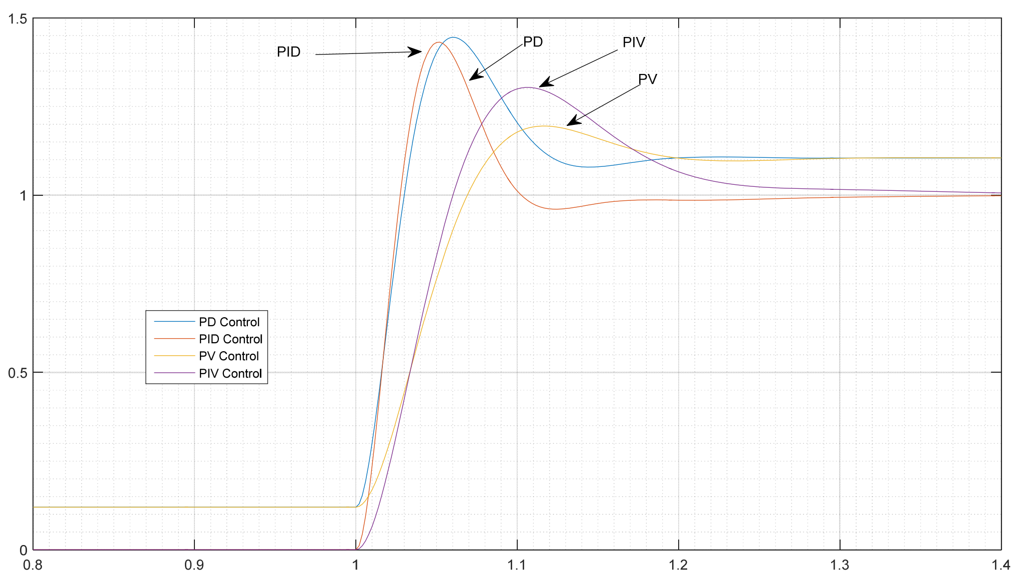


Figure 5. Step responses in presence of the disturbance.

Now the effect of the integral control is more noticeable. The PD and PV controllers both have large steady-state errors, while the PID and PIV controllers have zero steady-state error.