FORMULA SHEET ROBOT CONTROL

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

CHAIN RULE $\frac{d}{dx}\left[f(x)\, g(x)\right] = f'(x)\, g(x) + f(x)\, g'(x)$

PRODUCT RULE $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\, g'(x)$

QUOTIENT RULE $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

IF $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; TRANSPOSE $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$; INVERSE $A^{-1} = \frac{1}{ab-cd}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

DH PARAMETERS

a_i : FROM z_{i-1} TO z_i ALONG x_i

d_i : FROM x_{i-1} TO x_i ALONG z_{i-1}

α_i : $\measuredangle$ FROM z_{i-1} TO z_i ABOUT x_i

θ_i : $\measuredangle$ FROM x_{i-1} TO x_i ABOUT z_{i-1}

IF ROTARY, $x_i = x_{i-1}$ IN ZERO $\measuredangle$

JOINT i : CONNECTS LINK i TO LINK $i-1$

z_{i-1} : LOCATED AT JOINT i

O_i : INTERSECTION OF n_i FROM z_i

x_i : // TO n_i IN DIRECTION FROM z_{i-1} TO z_i

$${}^{i-1}T_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0R_1 = \begin{bmatrix} C\theta_1 & -S\theta_1 C\alpha_1 & S\theta_1 S\alpha_1 \\ S\theta & C\theta_1 C\alpha_1 & -C\theta_1 S\alpha_1 \\ 0 & S\alpha_1 & C\alpha_1 \end{bmatrix}$$

$${}^1T_0 = \begin{bmatrix} {}^0R_1 & -{}^0R_1^T\, {}^0d_{01} \\ 0 & 1 \end{bmatrix}$$

JACOBIANS

JACOBIAN: MATRIX REPRESENTING DERIVATIVES OF A VECTOR FUNCTION W.R.T. ITS INPUTS

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

- JACOBIAN OF f FROM R^n TO R^m IS SIZE $m \times n$

MANIPULATOR · POSITION/VELOCITY TO JOINT VELOCITY

$${}^0\dot{d}_{on} = J\dot{\theta}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \cdots & \frac{\partial x}{\partial \theta_n} \\ \frac{\partial y}{\partial \theta_1} & \cdots & \frac{\partial y}{\partial \theta_n} \\ \frac{\partial z}{\partial \theta_1} & \cdots & \frac{\partial z}{\partial \theta_n} \end{bmatrix}$$

J = MANIPULATOR JACOBIAN

- LOTS OF COMPUTATION, ONLY POSITION

VELOCITY : · VELOCITY & $\measuredangle$ VELOCITY TO JOINT VELOCITY

$$J_v = \begin{bmatrix} {}^0z_0 \times {}^0d_{on} & {}^0z_1 \times {}^0d_{1n} & \cdots & {}^0z_{n-1} \times {}^0d_{n-1,n} \\ {}^0z_0 & {}^0z_1 & \cdots & {}^0z_{n-1} \end{bmatrix}$$

- LESS COMPUTATION, PLUS IT GIVES YOU $\measuredangle$ VELOCITY

$${}^0z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad {}^0z_1 = {}^0R_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} {}^0\dot{d}_{on} \\ {}^0\omega_{on} \end{bmatrix}}_{\text{TYPICALLY } 6\times 1} = \underbrace{J_v}_{\text{TYPICALLY } 6\times n} \underbrace{\dot{\Theta}}_{n\times 1}$$

J_v = VELOCITY JACOBIAN

IF PRISMATIC... $\begin{bmatrix} {}^0z_{i-1} \\ 0 \end{bmatrix}$

YOU CAN REMOVE ZERO ENTRIES HERE

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = J\dot{\Theta} \quad \text{OR} \quad \begin{bmatrix} f_x \\ f_y \\ f_z \\ n_x \\ n_y \\ n_z \end{bmatrix} = J_v\dot{\Theta}$$

$\dot{\Theta}$ - STATE VECTOR (JOINT $\measuredangle$'S, OR d^*)

${}^0\dot{d}_{on}$ = VELOCITY END EFFECTOR

${}^0\omega_{on}$ = $\measuredangle$ VELOCITY OF END EFFECTOR

JACOBIAN'S USES

INVERSE VELOCITY:
- YOU CAN INVERT J_v TO GIVE VELOCITIES
- ONLY WORKS IF J_v IS SQUARE, AND THERE AREN'T SINGULARITIES ($\det(J_v) = 0$)
- SINGULARITIES OCCUR AT EDGE OF ROBOT'S REACH, OR OUTSIDE OF ITS CAPABILITIES
- IF YOU HAVE REDUNDANT DoF, YOU USE A PSEUDO-INVERSE TO CANCEL REDUNDANCIES

TRANSMISSION

- LETS YOU REWRITE FROM JOINT SPACE INTO MOTOR SPACE

$$\dot{\Theta} = J_t \dot{\Phi}$$

$\dot{\Phi}$ = MOTOR SPEEDS

$$J_t = \begin{bmatrix} \frac{1}{N_1} & 0 & \cdots & \\ 0 & \frac{1}{N_2} & & \\ \vdots & & \ddots & \\ & & & \frac{1}{N_n} \end{bmatrix}$$

NOT A HARD/FAST RULE — WRITE EQNS OF JOINT VELOCITIES, & HOW THEY RELATE TO MOTOR VELOCITIES. USE THAT TO BUILD J_t

COMPOUND

$$J_c = \underbrace{J}_{J \text{ OR } J_v} J_t$$

$$\dot{d}_{on} = J_c \dot{\Theta}$$

JACOBIANS + DYNAMICS

DYNAMICS

$$\tau_{JOINT} = H(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) + F(\dot{\Theta})$$

OR

$$\tau_{JOINT} = H'(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

V = CENTRIPETAL TERMS $\quad G$ = GRAVITY TERMS $\quad F$ = DAMPING/FRICTION TERMS

H = INERTIAL MATRIX

H' = INERTIAL MATRIX WITH MOTOR INERTIA

MAPPING BETWEEN SPACES

JOINT SPACE → MOTOR SPACE $\quad \tau_m = J_t^T \tau_{JOINT} = J_t^T H(\Theta) J_t \ddot{\Theta} + J_t^T V(\Theta, \dot{\Theta}) + J_t^T G(\Theta)$

MOTOR SPACE → JOINT SPACE $\quad \tau_{JOINT} = J_t^{-T} H(\Phi) J_t^{-1} \ddot{\Phi} + J_t^{-T} V(\dot{\Phi}) + J_t^{-T} G(\Phi)$

WHEN $\dot{\Theta} = J_t \dot{\Phi}$:

$$\tau_m = J_t^T \tau_{JOINT}$$

OR

$$\tau_{JOINT} = J_t^{-T} \tau_m$$

TRANSPOSES ARE SO WE DON'T HAVE A COLUMN VECTOR EQUAL TO A ROW VECTOR

NOTE: • H' & H MATRIX MULTIPLIED BY JACOBIAN SANDWICH BECAUSE IT IS MULTIPLIED BY $\ddot{\Theta}$ INSIDE + OUTSIDE MATRIX

IMPORTANT :
- H, V, G FUNCTIONS LIKE $\cos(\theta_1)$ & $\dot{\theta}_1^2$ NEED TO BE RESOLVED WITH J_t EQNS, AND THEN FOLLOWED WITH A JACOBIAN MULTIPLICATION

$$\tau = J_v^T W \quad \text{i.e.} \quad \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} = J^T \begin{bmatrix} f_x \\ \vdots \\ n_z \end{bmatrix}$$

τ = JOINT TORQUES $\qquad \dot{d}_{on} = J_v \dot{\Theta}$

W = FORCES + TORQUES

GIVEN JOINT SPACE TO MOTOR SPACE

$H(\Theta)$ = INERTIA MATRIX WRT Θ

$H'(\Theta)$ = INERTIA MATRIX W/MOTOR INERTIA INCLUDED

$H'(\Phi)$ = INERTIA MATRIX WRT Φ_m

IMPEDANCE MATCHING EXAMPLE

$$N\tau_m = (I_{ARM} + N^2 J_{MOTOR})\ddot{\theta}$$

ASSUMPTION: NEGLECTING COUPLING AND $\dot{\phi} = \phi = 0$

- ASSUME ALL OTHER TERMS GO TO ZERO

$$\rightsquigarrow \ddot{\theta} = \frac{N}{I + N^2 J}\tau_m$$

$\frac{d}{dN} \rightsquigarrow$ SET NUMERATOR EQUAL TO ZERO
WE WANT BOTTOM TO GO TO ZERO SO WE GET TORQUE AS HIGH AS POSSIBLE

$$\Rightarrow N = \sqrt{\frac{I}{J}}$$

J = MOTOR INERTIA
I = INERTIAL TERMS

EXAMPLE H'

$$H' = \begin{bmatrix} H_{11} + N_1^2 J_1 & H_{12} & \cdots & H_{1n} \\ H_{22} & H_{22} + N_2^2 J_2 & \cdots & \\ \vdots & & \ddots & \\ H_{n1} & & & H_{nn} + N_n^2 J_n \end{bmatrix}$$

J_t = TRANSMISSION JACOBIAN

$$\tau = J_c^T W$$

$$\tau_m = J_v^T W_{n\times 1}$$

W = FORCES IN WORKSPACE DIMENSIONS (TORQUES POSSIBLE TOO)

J = SOME COMPOUND JACOBIAN (CAN POTENTIALLY BE JUST A MANIPULATOR JACOBIAN)

DRIVETRAIN DYNAMICS

$$N_i = \frac{r_B}{r_A}$$

N_i = GEAR RATIO

$$\dot{\phi}_i = N\dot{\theta}_i$$

$\dot{\theta}_i$ = JOINT VELOCITY
$\dot{\phi}_i$ = MOTOR VELOCITY

$$\tau_B = N_i \tau_A$$

DC MOTOR DYNAMICS

$$V_b = K_b \dot{\phi}$$

ϕ = MOTOR SPEED

$$\tau_m = K_t i_a$$

K_b = BACK EMF

IF IN S.I. UNITS : $K_t = K_b$

$$V_a = R_a i_a + K_t \ddot{\phi}$$

V_a = ARMATURE VOLTAGE

R_a = ARMATURE RESISTANCE

i_a = ARMATURE CURRENT

$$\tau_m = \frac{K_t V_a}{R_a} - \frac{K_t}{R_a}\dot{\phi}$$

ROOT LOCUS DESIGN

$t_r = \frac{1.8}{\omega_n}$ RISE TIME

$t_s = \frac{4}{\xi \omega_n}$ SETTLING TIME

$t_p = \frac{\pi}{\omega\sqrt{1-\xi^2}}$ TIME TO PEAK

$\xi = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$

POLE LOCATIONS

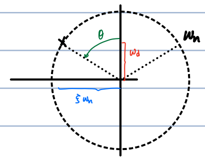
$s = -\sigma \pm j\omega$

$\%OS = 100 e^{-\pi\xi/\sqrt{1-\xi^2}}$ OVERSHOOT

only works when $0 \le \xi \le 1$

2ND ORDER SYSTEM

$\xi \omega_n = t_s$



$\xi \omega_n$

$\omega_d = \omega_n \sqrt{1-\xi^2}$

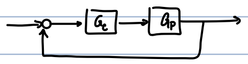
ASYMPTOTE ∡

$\theta = \frac{(2q+1)180}{p-z}$

ROOT LOCUS DESIGN

OLT.F = WHEN YOU DON'T CONSIDER FEEDBACK

↳ $OLTF = G_c G_p$



PID DESIGN

PD
1. FIND DESIRED POLE LOCATIONS (use $\%OS$, t_s ...)
2. FIND WHERE A ZERO SHOULD BE PLACED ON R.L

$$\underbrace{\Sigma\theta_z - \Sigma\theta_p = -180^\circ}_{\text{GRAPH METHOD (R.L.)}}$$ OR CHARACTERISTIC EQN

3. FIND WHERE A POLE SHOULD BE PLACED

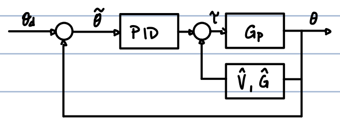
$$K_{overall} = \frac{\Pi l_p}{\Pi l_z} \quad \text{OR} \quad K = \frac{1}{\|GH\|}$$

K WILL USUALLY BE K_d SOMETHING IN A PD CONTROLLER

PI
4. SATISFY ξ CONDITION
5. AN OPEN LOOP POLE CANCELS AN OPEN LOOP ZERO

CONTROL SCHEMES

COMPENSATOR

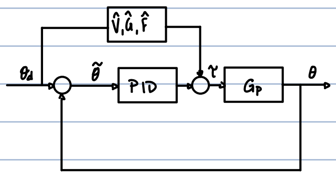


CONTROL LAW = BALANCE EQN DEFINING τ

$\hat{V}, \hat{G}$ = MODELS

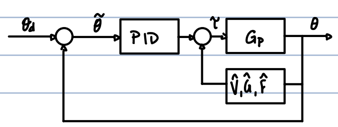
$$\tau = \hat{V}(\theta) + G(\theta) + K_p\tilde{\theta} + K_d\dot{\tilde{\theta}} + K_I\int\tilde{\theta}$$

FEEDFORWARD



$$\tau = \hat{V}(\theta_d) + G(\theta_d) + F(\dot{\theta}_d) + K_p(\tilde{\theta}) + K_d(\dot{\tilde{\theta}}) + K_I\int\tilde{\theta}$$

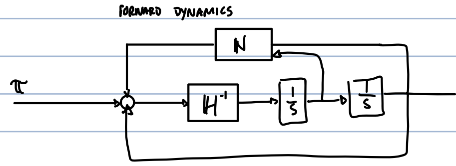
NO GRAVITY/FRICTION



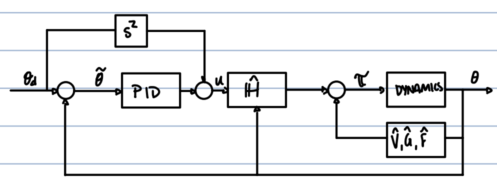
UNMODELED POLES CREATE INSTABILITY

$$\tau = K_p\tilde{\theta} + K_d\dot{\tilde{\theta}} + \underbrace{\hat{f}(\theta, \dot{\theta})}_{\text{INCLUDES GRAVITY/FRICTION/VELOCITY TERMS}}$$

FORWARD DYNAMICS



INVERSE DYNAMICS



u = CONTROL EFFORT

$$\tau = \mathbb{H}(\Theta)\,u + \mathbb{N}(\Theta, \dot{\Theta})$$

$$\mathbb{H}(\Theta)\ddot{\Theta} + \mathbb{N}(\dot{\Theta}, \Theta) = \mathbb{H}(\Theta)\,u + \mathbb{N}(\Theta, \dot{\Theta})$$

$$\Rightarrow \ddot{\Theta}_d - \ddot{\Theta} = u$$

$$\therefore \ddot{\tilde{\Theta}} = u$$

LINEAR + DECOUPLED

$$\tau = \mathbb{H}(\Theta)\ddot{\Theta} + \mathbb{N}(\Theta, \dot{\Theta})$$

$$\Theta(s) = \frac{1}{s^2}\mathbb{H}^{-1}\left[\tau - \mathbb{N}\right]$$