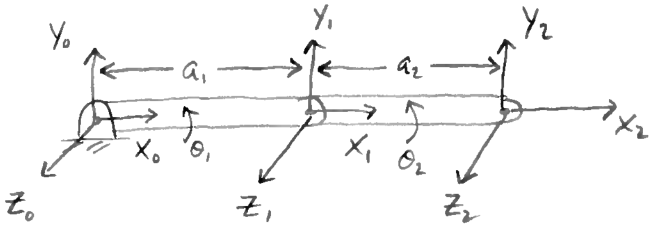
Problem Set #1 Solutions

1.



i	a_i	d_i	α_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

2.

$$
{}^0\mathbb{T}_1 = \left[\begin{array}{ccc|c} c\theta_1 & -s\theta_1 & 0 & a_1 c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1 s\theta_1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
\qquad
{}^1\mathbb{T}_2 = \left[\begin{array}{ccc|c} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

$$
{}^0\mathbb{T}_2 = \left[\begin{array}{ccc|c} C_1C_2 - S_1S_2 & -C_1S_2 - S_1C_2 & 0 & a_2C_1C_2 - a_2S_1S_2 + a_1C_1 \\ S_1C_2 + C_1S_2 & -S_1S_2 + C_1C_2 & 0 & a_2S_1C_2 + a_2C_1S_2 + a_1S_1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

$$
{}^0\mathbb{T}_2 = \left[\begin{array}{ccc|c} C_{12} & -S_{12} & 0 & a_1C_1 + a_2C_{12} \\ S_{12} & C_{12} & 0 & a_1S_1 + a_2S_{12} \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

where

$C_{12} = \cos(\theta_1+\theta_2)$

$S_{12} = \sin(\theta_1+\theta_2)$

$$
{}^0\mathbb{R}_2 = \begin{bmatrix} C_{12} & -S_{12} & 0 \\ S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\qquad
{}^0d_{02} = \begin{bmatrix} a_1C_1 + a_2C_{12} \\ a_1S_1 + a_2S_{12} \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}
$$

3.

$$ {}^0\dot{d}_{02} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} $$

$$ = \underbrace{\begin{bmatrix} -a_1 S_1 - a_2 S_{12} & -a_2 S_{12} \\ a_1 C_1 + a_2 C_{12} & a_2 C_{12} \end{bmatrix}}_{\substack{J \\ \text{manipulator Jacobian}}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} $$

Singularities occur when $|J| = 0$

$$ \begin{vmatrix} -a_1 S_1 - a_2 S_{12} & -a_2 S_{12} \\ a_1 C_1 + a_2 C_{12} & a_2 C_{12} \end{vmatrix} = \cancel{a_2} C_{12}(-a_1 S_1 - a_2 S_{12}) + \cancel{a_2} S_{12}(a_1 C_1 + a_2 C_{12}) = 0 $$

$$ \cancel{a_1}(C_1 S_{12} - S_1 C_{12}) + a_2(\cancel{S_{12} C_{12}} - \cancel{S_{12} C_{12}}) = 0 $$

$$ C_1 S_{12} - S_1 C_{12} = 0 $$

$$ \sin(\cancel{\theta_1} + \theta_2 - \cancel{\theta_1}) = 0 $$

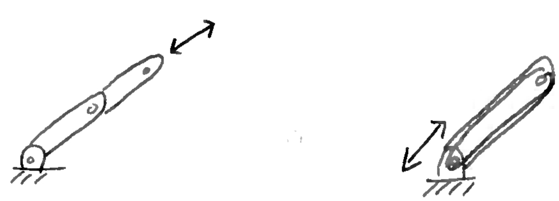
$$ \sin \theta_2 = 0 $$

$$ \theta_2 = 0, \pi $$

Trig Identities

$\sin A \cos B \pm \sin B \cos A = \sin(A \pm B)$

$\cos A \cos B \pm \sin A \sin B = \cos(A \mp B)$



velocities are limited: end-effector can't move in direction shown

4.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} {}^0Z_0 \times {}^0d_{02} & {}^0Z_1 \times {}^0d_{12} \\ {}^0Z_0 & {}^0Z_1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_1C_1 + a_2C_{12} \\ a_1S_1 + a_2S_{12} \\ 0 \end{bmatrix} & {}^0R_1\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2C_2 \\ a_2S_2 \\ 0 \end{bmatrix} \right) \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & {}^0R_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} -a_1S_1 + a_2S_{12} & {}^0R_1 \begin{bmatrix} -a_2S_2 \\ a_2C_2 \\ 0 \end{bmatrix} \\ a_1C_1 + a_2C_{12} & \\ 0 & \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$${}^0R_1 \begin{bmatrix} -a_2S_2 \\ a_2C_2 \\ 0 \end{bmatrix} = \begin{bmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -a_2S_2 \\ a_2C_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_2C_1S_2 - a_2S_1C_2 \\ -a_2S_1S_2 + a_2C_1C_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -a_2S_{12} \\ a_2C_{12} \\ 0 \end{bmatrix}$$

(using Trig. Identities

Just need rows 1, 2, and 6 for planar motion

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega_z \end{bmatrix} = \underbrace{\begin{bmatrix} -a_1S_1 - a_2S_{12} & -a_2S_{12} \\ a_1C_1 + a_2C_{12} & a_2C_{12} \\ 1 & 1 \end{bmatrix}}_{J_v} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

velocity Jacobian

1st 2 rows are same as J

Since J_v is not square, we can't invert it. Since robot is only 2-DOF, can't independently specify 3 end-effector velocities.
We can only specify $\dot{x}$ and $\dot{y}$ and use J^{-1} to solve for $\dot{\theta}_1$ and $\dot{\theta}_2$

5. Principle of virtual work (or power in = power out)

$$\tau_1 \dot{\theta}_1 + \tau_2 \dot{\theta}_2 = F_x \dot{x} + F_y \dot{y}$$

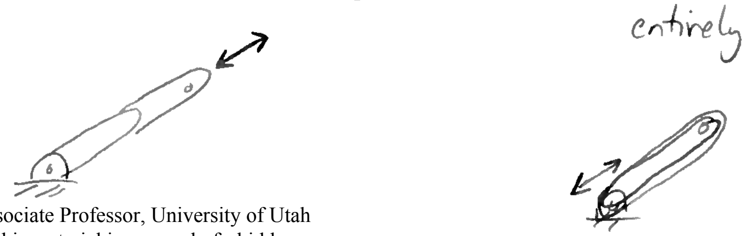
$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}^T \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \end{bmatrix}^T \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}^T \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \end{bmatrix}^T J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}^T = \begin{bmatrix} F_x \\ F_y \end{bmatrix}^T J$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = J^T \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

If robot is at singularity, force exerted on the end-effector in direction shown will result in zero joint torque. (Force born entirely by structure)



6. $$\tau_1 \dot{\theta}_1 + \tau_2 \dot{\theta}_2 = F_x \dot{x} + F_y \dot{y} + \tau_z \omega_z$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}^T \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ \tau_z \end{bmatrix}^T \begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega_z \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ \tau_z \end{bmatrix}^T J_v \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = J_v^T \begin{bmatrix} F_x \\ F_y \\ \tau_z \end{bmatrix}$$

Since J_v is not square, we can't invert J_v^T.

In this case, this means that given a set of joint torques, there is not a unique solution for the end-effector wrench.

i.e. there are multiple combinations of end-effector forces/torque that result in the same joint torques.