Solutions to Midterm

1. a) $$J = \begin{bmatrix} \frac{dx}{dd_1} & \frac{dx}{d\theta_2} \\ \frac{dy}{dd_1} & \frac{dy}{d\theta_2} \end{bmatrix} = \begin{bmatrix} 0 & -a_2 \sin\theta_2 \\ 1 & a_2 \cos\theta_2 \end{bmatrix}$$

b) $$\begin{bmatrix} f_1 \\ \tau_2 \end{bmatrix} = \underbrace{\begin{bmatrix} m_1+m_2 & a_2 m_2 c\theta_2 \\ a_2 m_2 c\theta_2 & a_2^2 m_2 \end{bmatrix}}_{H(\Theta)} \begin{bmatrix} \ddot{d}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -a_2 m_2 s\theta_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{d}_1^2 \\ \dot{\theta}_1^2 \end{bmatrix}}_{V(\Theta,\dot{\Theta})} + \underbrace{\begin{bmatrix} (m_1+m_2)g \\ a_2 m_2 g\, c\theta_2 \end{bmatrix}}_{G(\Theta)}$$

inertial torques Centripetal torque gravity torques

c) $$\begin{bmatrix} \dot{d}_1 \\ \dot{\theta}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{N_1} & 0 \\ 0 & \frac{1}{N_2} \end{bmatrix}}_{J_t} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

J_t (transmission Jacobian)

d) $$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{d}_1 \\ \dot{\theta}_2 \end{bmatrix} = \underbrace{J J_t}_{J_c} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

J_c compound Jacobian

$$J_c = \begin{bmatrix} 0 & -\frac{a_2 \sin\theta_2}{N_2} \\ \frac{1}{N_1} & \frac{a_2 \cos\theta_2}{N_2} \end{bmatrix}$$

$$\tau_m = J_c^T W$$

motor torques wrench $W = \begin{bmatrix} F_x \\ F_y \end{bmatrix}$

e) $$\begin{bmatrix} \tau_{m1} \\ \tau_{m2} \end{bmatrix} = J_t^T \begin{bmatrix} f_1 \\ \tau_2 \end{bmatrix}$$

$$= J_t^T H(\Theta) J_t \ddot{\Phi} + J_t^T V(\Theta, \dot{\Theta}) + J_t^T G(\Theta)$$

$$= \underbrace{\begin{bmatrix} \frac{m_1 + m_2}{N_1^2} & \frac{a_2 m_2 c\left(\frac{\phi_2}{N_2}\right)}{N_1 N_2} \\ \frac{a_2 m_2 c\left(\frac{\phi_2}{N_2}\right)}{N_1 N_2} & \frac{a_2^2 m_2}{N_2^2} \end{bmatrix}}_{H(\Phi) = \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix}} \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & \frac{-a_2 m_2 s\left(\frac{\phi_2}{N_2}\right)}{N_1 N_2^2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1^2 \\ \dot{\phi}_2^2 \end{bmatrix}}_{V(\Phi, \dot{\Phi})}$$

$$+ \underbrace{\begin{bmatrix} \frac{(m_1 + m_2) g}{N_1} \\ \frac{a_2 m_2 g\, c\left(\frac{\phi_2}{N_2}\right)}{N_2} \end{bmatrix}}_{G(\Phi)}$$

f) $$\begin{bmatrix} \tau_{m1} \\ \tau_{m2} \end{bmatrix} = H(\Phi)\ddot{\Phi} + V(\Phi, \dot{\Phi}) + G(\Phi) + \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} b_1 \dot{\phi}_1 \\ b_2 \dot{\phi}_2 \end{bmatrix}}_{F(\dot{\Phi})}$$

$$= \underbrace{\begin{bmatrix} H_{11} + J_1 & H_{12} \\ H_{12} & H_{22} + J_2 \end{bmatrix}}_{H'(\Phi)} \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{bmatrix} + V(\Phi, \dot{\Phi}) + G(\Phi) + F(\dot{\Phi})$$

2. a) neglecting inertial coupling and starting from rest:

$$\tau_{m1} = \left(\frac{m_1 + m_2}{N_1^2} + J_1 \right) \ddot{\phi}_1$$

impedance matching: $\frac{m_1+m_2}{N_1^2} = J_1$

$$N_1 = \sqrt{\frac{m_1+m_2}{J_1}} = \sqrt{\frac{128}{2}} = \sqrt{64} = 8$$

$$\boxed{N_1 = 8}$$

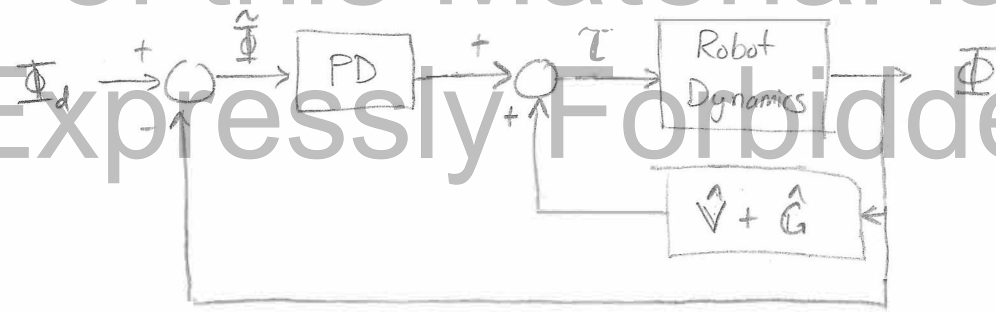
$$\tau_{m2} = \left(\frac{a_2^2 m_2}{N_2^2} + J_2 \right) \ddot{\phi}_2$$

impedance matching: $\frac{a_2^2 m_2}{N_2^2} = J_2$

$$N_2 = \sqrt{\frac{a_2^2 m_2}{J_2}} = \sqrt{\frac{2^2 \cdot 50}{2}} = \sqrt{\frac{200}{2}} = \sqrt{100} = 10$$

$$\boxed{N_2 = 10}$$

b) PD Control plus Feedback Comp



Control Law: $\tau = K_p \tilde{\Phi} + K_D \dot{\tilde{\Phi}} + \hat{V}(\Phi, \dot{\Phi}) + \hat{G}(\Phi)$

c) Cancelling V & G terms, and neglecting inertial coupling:

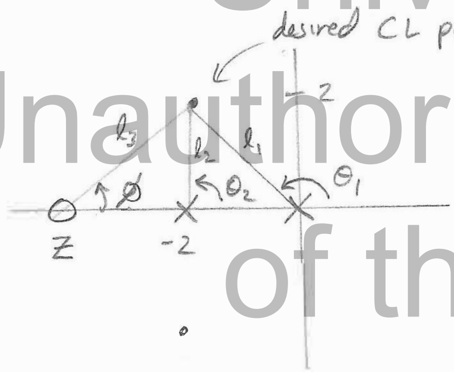
$$\tau_{m_1} = \left(\frac{m_1+m_2}{N_1^2} + J_1\right)\ddot{\phi}_1 + b_1\dot{\phi}_1 = 4\ddot{\phi}_1 + 8\dot{\phi}_1$$

$$\frac{\phi_1(s)}{\tau_{m_1}(s)} = \frac{1}{4s^2+8s}$$

$$\tau_{m_2} = \left(\frac{a_2^2 m_2}{N_2^2} + J_2\right)\ddot{\phi}_2 + b_2\dot{\phi}_2 = 4\ddot{\phi}_2 + 8\dot{\phi}_2$$

$$\frac{\phi_2(s)}{\tau_{m_2}(s)} = \frac{1}{4s^2+8s}$$

d) Placed closed loop poles at $s = -2 \pm 2j$



$$O.L.T.F. = \frac{(K_P + K_D s)}{4s^2+8s} = \frac{\overbrace{\frac{K_D}{4}}^{\text{root locus gain}}\overbrace{\left(s+\frac{K_P}{K_D}\right)}^{\text{OL zero}}}{\underbrace{s(s+2)}_{\text{OL poles}}}$$

Angle Condition:

$$\phi - \theta_1 - \theta_2 = \pm 180°$$

$$\phi = \pm 180° + 135° + 90° = 45°$$

so zero is at $s = -4 \Rightarrow \frac{K_P}{K_D} = 4$

Magnitude Condition:

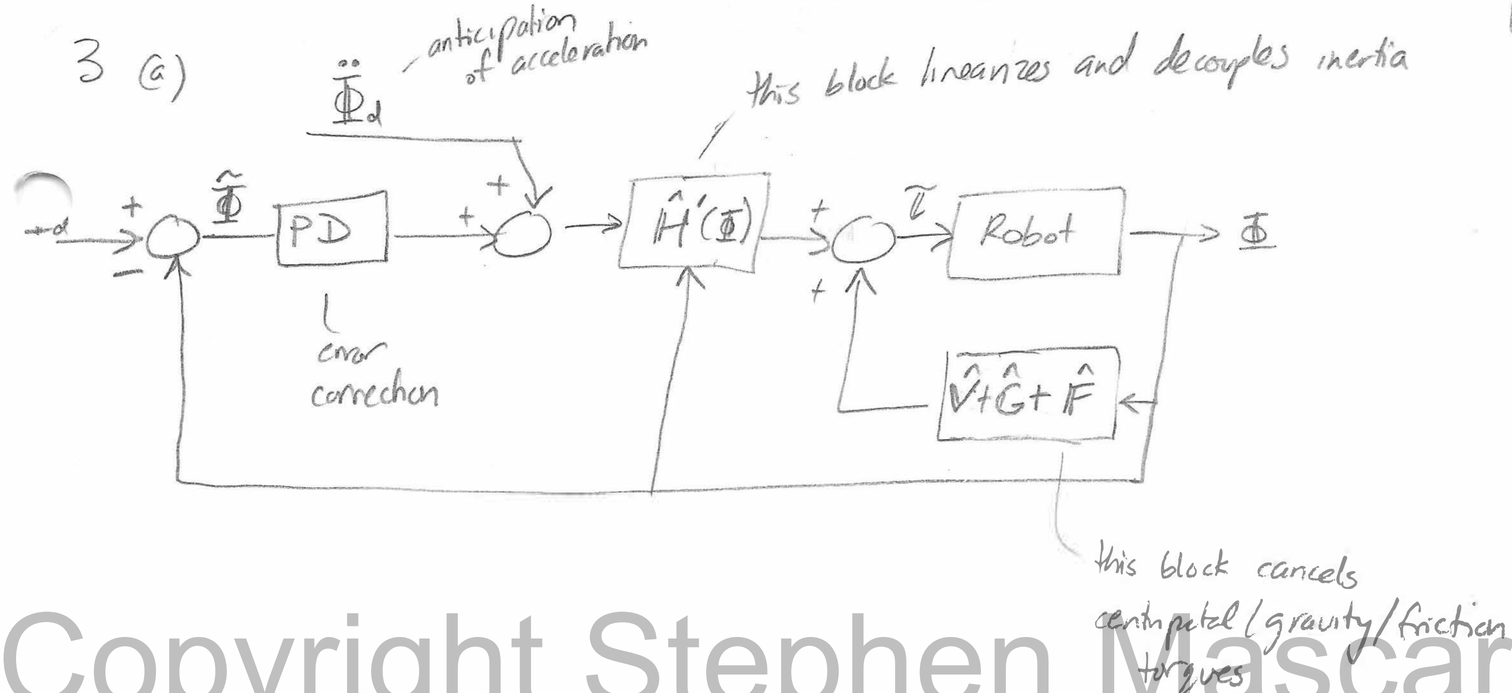
$$K = \frac{l_1 l_2}{l_3} = l_2 = 2 \quad \Rightarrow \quad \frac{K_D}{4} = 2$$

$$\boxed{K_D = 8}$$

$$K_P = 4K_D = 4 \cdot 8 = 32$$

$$\boxed{K_P = 32}$$

3 (a)



(b) $\tau = \hat{H}'(\Phi)\left[K_p \tilde{\Phi} + K_D \dot{\tilde{\Phi}} + \ddot{\Phi}_d\right] + \hat{V}(\Phi, \dot{\Phi}) + \hat{G}(\Phi) + \hat{F}(\dot{\Phi})$

where $\hat{H}, \hat{V}, \hat{G}, \hat{F}$ are best estimates of dynamic parameters

(c) We should not expect to use same PD gains as in Problem 2. Since inertia matrix multiplies PD gains by ~ factor of 4, we should expect to reduce our PD gains by a factor of 4 to get comparable performance. Or another way to look at it is that the effective OLTF has now been changed to $\frac{1}{s^2}$ instead of $\frac{1}{4s^2+8s}$, so we would need different PD gains to put CL poles at same spot.