Name:

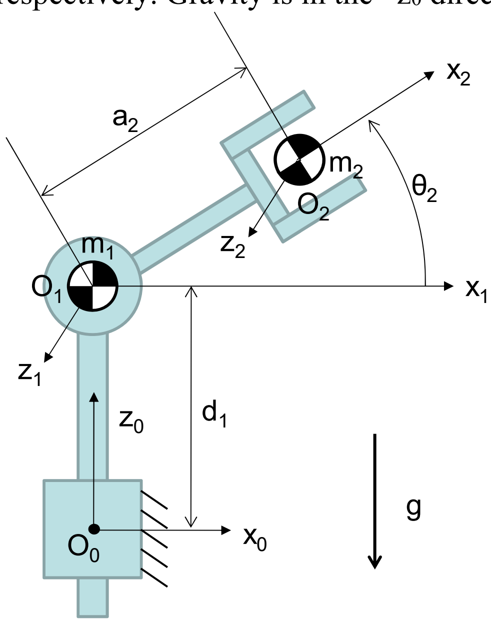
uNID:

ME EN 5230/6230, CS 6330, ECE 6651
Intro to Robot Control
Practice Midterm Exam
Spring 2023
80 minutes

- **Closed Book**
- **1 sheets of notes allowed plus Newton/Euler reference sheet**

Problem 1: Dynamics (Total 40 Points)

The figure below shows a planar 2-DOF robot with a prismatic joint and a rotary joint. Assume that the rotational inertia I_2 is negligible and the COGs of m_1 and m_2 are at O_1 and O_2 respectively. Gravity is in the $-z_0$ direction.



The forward kinematics are given by:

$$x = a_2 \cos(\theta_2)$$
$$y = d_1 + a_2 \sin(\theta_2)$$

The manipulator dynamics are given by:

$$f_1 = (m_1 + m_2)\ddot{d}_1 + a_2 m_2 \ddot{\theta}_2 \cos\theta_2 - a_2 m_2 \dot{\theta}_2^2 \sin\theta_2 + (m_1 + m_2)g$$

$$\tau_2 = a_2 m_2 \ddot{d}_1 \cos\theta_2 + a_2^2 m_2 \ddot{\theta}_2 + a_2 m_2 g \cos\theta_2$$

SLIGHT ERROR, DID THIS LATE AT NIGHT

(a) (5 pts) Use the forward kinematics to find the manipulator Jacobian for this robot.

$$J = \begin{bmatrix} \frac{\partial x}{\partial d_1} & \frac{\partial x}{\partial \theta_2} \\ \frac{\partial y}{\partial d_1} & \frac{\partial y}{\partial \theta_2} \end{bmatrix} \rightsquigarrow \frac{\partial x}{\partial d_1} = 0 \quad \frac{\partial x}{\partial \theta_2} = -\dot{\theta}_2 a_2 S_2 \quad \frac{\partial y}{\partial \theta_1} = 1 \quad \frac{\partial y}{\partial \theta_2} = \dot{\theta}_2 a_2 C_2 \rightsquigarrow J = \begin{bmatrix} 0 & -a_2 S_2 \\ 1 & a_2 C_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{d}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

FINE ◡̈

(b) (5 pts) Put the dynamics in matrix form and explain what type of force/torque each term represents.

$$\begin{bmatrix} f_1 \\ \tau_2 \end{bmatrix} = \underbrace{\begin{bmatrix} m_1+m_2 & a_2 m_2 C_2 \\ a_2 m_2 C_2 & a_2^2 m_2 \end{bmatrix}}_{\text{INERTIA}} \begin{bmatrix} \ddot{d}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & -a_2 m_2 S_2 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{VELOCITY TERMS}} \begin{bmatrix} \dot{d}_1^2 \\ \dot{d}_1 \dot{\theta}_2 \\ \dot{\theta}_2^2 \end{bmatrix} + \underbrace{\begin{bmatrix} (m_1+m_2)g \\ a_2 m_2 g \end{bmatrix}}_{\text{GRAVITY}} \begin{bmatrix} 1 \\ C\theta_2 \end{bmatrix}$$

(c) (5 pts) Suppose the joint axes are connected to the motor shafts through transmission ratios $N_1 = \phi_1/d_1$ and $N_2 = \phi_2/\theta_2$. Find the transmission Jacobian that relates the joint variables d_1 and θ_2 to the motor angles ϕ_1 and ϕ_2.

$$N_1 = \frac{\dot{\phi}_1}{\dot{d}_1} \qquad N_2 = \frac{\dot{\phi}_2}{\dot{\theta}_2} \qquad d_1 = \frac{\dot{\phi}_1}{N_1}$$

$$\rightsquigarrow \begin{cases} \dot{d}_1 = \frac{1}{N_1}\dot{\phi}_1 \\ \dot{\theta}_2 = \frac{1}{N_2}\dot{\phi}_2 \end{cases} \quad \begin{bmatrix} \dot{d}_1 \\ \dot{\theta}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{N_1} & 0 \\ 0 & \frac{1}{N_2} \end{bmatrix}}_{J_t} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

$$\dot{\Theta} = J_t \dot{\Phi}$$

(d) (5 pts) Derive the compound Jacobian for this robot that relates end-effector velocities to motor velocities. Show how this same Jacobian relates end-effector wrench to motor torques.

$$J_c = J J_t = \begin{bmatrix} 0 & -a_2 S_2 \\ 1 & a_2 C_2 \end{bmatrix} \begin{bmatrix} \frac{1}{N_1} & 0 \\ 0 & \frac{1}{N_2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-a_2 S_2}{N_2} \\ \frac{1}{N_1} & \frac{a_2 C_2}{N_2} \end{bmatrix} \qquad \dot{d}_{02} = J_c \dot{\Theta}$$

$$J_c^T = \begin{bmatrix} 0 & 1 \\ \frac{-a_2 S_2}{N_2} & \frac{a_2 C_2}{N_2} \end{bmatrix}$$

$$\tau = J^T W = \begin{bmatrix} f_d \\ \tau_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-a_2 S_2}{N_2} & \frac{a_2 C_2}{N_2} \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

(e) (15 pts) Transform your manipulator dynamics to motor space.

$$\underset{n\times 1}{\tau_m} = \underset{n\times n}{J_t}\ \underset{n\times 1}{\tau_{JOINT}}$$

$$\leadsto \tau_m = J_t^{-T}\tau_{JOINT}$$

$$\begin{bmatrix} f_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} m_1+m_2 & a_2m_2C_2 \\ a_2m_2C_2 & a_2^2m_2 \end{bmatrix}\begin{bmatrix} \ddot{d}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -a_2m_2S_2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{d}_1^2 \\ \dot{d}_1\dot{\theta}_2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} (m_1+m_2)g \\ a_2m_2g \end{bmatrix}\begin{bmatrix} 1 \\ c\theta_2 \end{bmatrix}$$

WE REMOVE $\dot{d}_1\dot{\theta}_2$ BECAUSE IT DOESN'T CONTRIBUTE TO PROBLEM + MESSES UP OUR MATRIX MULTIPLICATION

$$\tau_{JOINT} = \mathbb{H}\ddot{\Theta} + \mathbb{V}(\dot{\Theta}) + \mathbb{G}(\Theta)$$

$$\tau_m = \underbrace{J_t^T\,\mathbb{H}(\Theta)\,J_t}_{=}\,\ddot{\Theta} + \underbrace{J_t^T\,\mathbb{V}(\dot{\Theta})}\ + J_t^T\,\mathbb{G}(\Theta)$$

$$\underbrace{\begin{bmatrix} \frac{1}{N_1} & 0 \\ 0 & \frac{1}{N_2} \end{bmatrix}\begin{bmatrix} m_1+m_2 & a_2m_2C_2 \\ a_2m_2C_2 & a_2^2m_2 \end{bmatrix}}_{=}\begin{bmatrix} \frac{1}{N_1} & 0 \\ 0 & \frac{1}{N_2} \end{bmatrix}$$

$$C_2 = \cos\theta_2 = \cos\left(\tfrac{1}{N_2}\phi_2\right)$$

$$\begin{bmatrix} \frac{m_1+m_2}{N_1} & \frac{a_2m_2C_2}{N_2} \\ \frac{a_2m_2C_2}{N_2} & \frac{a_2^2m_2}{N_2} \end{bmatrix}\begin{bmatrix} \frac{1}{N_1} & 0 \\ 0 & \frac{1}{N_2} \end{bmatrix} = \begin{bmatrix} \frac{m_1+m_2}{N_1^2} & \frac{a_2m_2C_2}{N_1N_2} \\ \frac{a_2m_2C_2}{N_1N_2} & \frac{a_2^2m_2}{N_2^2} \end{bmatrix} = \begin{bmatrix} \frac{m_1+m_2}{N_1^2} & \frac{a_2m_2\cos\left(\frac{1}{N_2}\phi_2\right)}{N_1N_2} \\ \frac{a_2m_2\cos\left(\frac{1}{N_2}\phi_2\right)}{N_1N_2} & \frac{a_2^2m_2}{N_2^2} \end{bmatrix} = \mathbb{H}(\Phi)$$

$$\begin{bmatrix} 0 & -a_2m_2S_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{d}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix}$$

$$-\dot{d}_1^2a_2m_2\sin(\theta_2)$$

WHERE: $\dot{d}_1 = \frac{1}{N_1}\dot{\phi}_1$

$\dot{\theta}_2 = \frac{1}{N_2}\dot{\phi}_2$

$$\leadsto -\dot{d}_1^2 = -\frac{1}{N_1^2}\dot{\phi}_1^2$$

$$\sin(\theta_2) = \sin\left(\tfrac{1}{N_2}\phi_2\right)$$

$$\therefore -\dot{d}_1^2a_2m_2\sin(\theta_2) = \frac{-1}{N_1^2}\dot{\phi}_1^2\,a_2m_2\sin\left(\tfrac{1}{N_2}\phi_2\right)$$

$$\leadsto \mathbb{V}(\dot{\Theta}) = \underbrace{\begin{bmatrix} \frac{-1}{N_1^2}a_2m_2\sin\left(\frac{1}{N_2}\phi_2\right) & 0 \\ 0 & 0 \end{bmatrix}}\begin{bmatrix} \dot{\phi}_1^2 \\ \dot{\phi}_2^2 \end{bmatrix}$$

$$J^T\,\mathbb{V}(\dot{\Theta}) = \begin{bmatrix} \frac{1}{N_1} & 0 \\ 0 & \frac{1}{N_2} \end{bmatrix}\begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \frac{-1}{N_1^3}a_2m_2S\left(\frac{\phi_2}{N_2}\right) & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \mathbb{V}(\dot{\Phi}) = \begin{bmatrix} \frac{-1}{N_1^3}a_2m_2S\left(\frac{\phi_2}{N_2}\right) & 0 \\ 0 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} (m_1+m_2)g \\ a_2m_2g \end{bmatrix}\begin{bmatrix} 1 \\ c\theta_2 \end{bmatrix}$$

$$c\theta_2 = c\left(\frac{\phi_2}{N_2}\right)$$

$$\leadsto J_t\,\mathbb{G}(\Theta) = \underset{2\times 2}{\begin{bmatrix} \frac{1}{N_1} & 0 \\ 0 & \frac{1}{N_2} \end{bmatrix}}\ \underset{2\times 1}{\begin{bmatrix} (m_1+m_2)g \\ a_2m_2g\,c\left(\frac{\phi_2}{N_2}\right) \end{bmatrix}} = \begin{bmatrix} \frac{m_1+m_2g}{N_1} \\ \frac{a_2m_2\,c\left(\frac{\phi_2}{N_2}\right)}{N_2} \end{bmatrix}$$

(f) (5 pts) Suppose the motor shafts have rotational inertias J_1 and J_2 and viscous damping constants b_1 and b_2. Show how these augment the dynamic equations in motor space.

$\rightsquigarrow$ $\mathbb{H}(\Phi) \rightsquigarrow \mathbb{H}'(\Phi)$

WHERE $$\mathbb{H}'(\Phi) = \begin{bmatrix} \frac{m_1+m_2}{N_1^2} & \frac{a_2 m_2 \cos\left(\frac{1}{N_2}\phi_2\right)}{N_1 N_2} \\ \frac{a_2 m_2 \cos\left(\frac{1}{N_2}\phi_2\right)}{N_1 N_2} & \frac{a_2^2 m_2}{N_2^2} \end{bmatrix} + \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

b_1

$\mathbb{V}(\Phi) \rightsquigarrow \mathbb{V}'(\Phi)$ OR $\mathbb{F} = \mathbb{V}(\Phi)$

$$\mathbb{V}'(\dot{\Phi}) = \begin{bmatrix} -\frac{1}{N_1^3} a_2 m_2 S\left(\frac{\dot{\phi}_2}{N_2}\right) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1^2 \\ \dot{\phi}_2^2 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

Problem 2: Decentralized Control (Total 40 Points)

Suppose you decide to design a simple decentralized PD control system for the robot. To do this, you decide to neglect inertial coupling and use a simple feedback compensator to cancel out all the centripetal and gravity forces/torques. Use the following parameter values:

$$m_1=78$$
$$m_2=50$$
$$a_2=2$$
$$J_1=J_2=2$$
$$b_1=b_2=8$$

(a) (10 pts) Use impedance matching to find the optimal gear ratios N_1 and N_2 to maximize the ability to accelerate each joint from rest. Use these values in the remainder of this problem.

$m_1 = 78$

ASSUME $\dot{\phi}_1 = \phi_1 = \dot{\phi}_2 = \phi_2 = 0$ & IGNORE INERTIAL COUPLING

$\rightsquigarrow$ ONLY INERTIA TERMS REMAIN

$$\begin{bmatrix} \tau_{m1} \\ \tau_{m2} \end{bmatrix} = \begin{bmatrix} \frac{m_1+m_2}{N_1^2} + J_1 & \frac{a_2 m_2 \cos\left(\frac{1}{N_2}\phi_2\right)}{N_1 N_2} \\ \frac{a_2 m_2 \cos\left(\frac{1}{N_2}\phi_2\right)}{N_1 N_2} & \frac{a_2^2 m_2}{N_2^2} + J_2 \end{bmatrix} \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{bmatrix}$$

$$\tau_{m_1} = \ddot{\phi}_1\left(\frac{m_1+m_2}{N_1^2} + J_1\right)$$

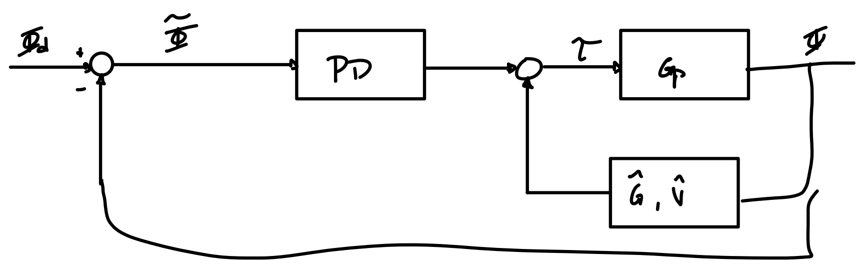
$$\frac{\partial}{\partial N}\left(\ddot{\phi}_1 = \frac{N_1^2 - J_1 N_1^2}{m_1+m_2}\right)$$

$=$

$\rightsquigarrow$ $N_1 = 8$

$N_2 = 10$

(b) (10 pts) Sketch a block diagram of your control system including the PD control and feedback compensator. Also write your control law in terms of the dynamic parameters.



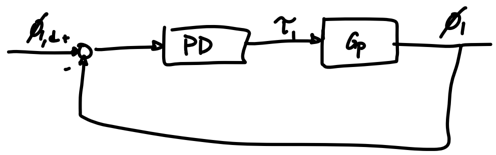
$\tau = K_p \tilde{\Phi} + K_d \tilde{\Phi} s + \hat{G}(\Phi) + \hat{V}(\Phi s)$

(c) (5 pts) Assuming that you have now cancelled out centripetal and gravity forces/torques (and neglecting inertial coupling), show that you get the same open-loop transfer function for both joints:

$$\frac{\phi_1(s)}{\tau_{m1}(s)} = \frac{1}{4s^2 + 8s} \qquad and \qquad \frac{\phi_2(s)}{\tau_{m2}(s)} = \frac{1}{4s^2 + 8s}$$

IF $\hat{G}(\Phi) = G(\Phi)$ & $\hat{V}(\dot{\Phi}) = \hat{V}(\dot{\Phi})$

$\Rightarrow$



$\phi = G_p \tau_1 \rightsquigarrow G_p =$

$K_p = 8$

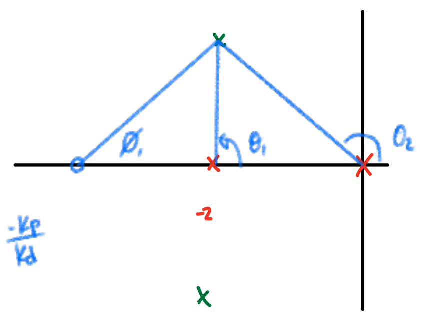
$K_d = 10$

(d) (15 pts) Since both joints happen to have the same transfer function, we can use the same PD gains for both. Sketch the root locus and use the angle and magnitude conditions to find the PD gains necessary to place the closed loop poles at $s = -2 \pm 2j$.

$s = -2 \pm 2j$

O.L.T.F $\rightsquigarrow \dfrac{K_p + K_d s}{4s^2 + 8s} \rightsquigarrow \dfrac{K_d\left(s + \frac{K_p}{K_d}\right)}{4s(s+2)} = \underbrace{\dfrac{K_d}{4}}_{\text{GAIN}} \dfrac{\left(s + \frac{K_p}{K_d}\right)}{s(s+2)}$

$4s(s+2)$



DESIRED C.L.TF

$\theta_1 = 90$

$\theta_2 = 135$

$\Rightarrow \phi_3 = 45$

$\Rightarrow \boxed{\frac{-K_p}{K_d} = -4}$

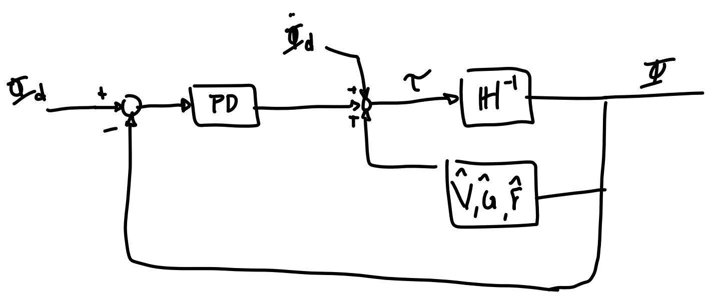
$\frac{l_1 l_2}{l_3} = l_2 \sim 2$

$\frac{K_d}{4} =$

Problem 3: Centralized Control (Total 20 Points)

Design an Inverse Dynamics Controller for the robot.

(a) (10 pts) Draw a complete block diagram of the control system.



(b) (5 pts) Write the control law in terms of the dynamic parameters.

$$\tau = K_p \tilde{\Phi} + K_d(\dot{\tilde{\Phi}}) + \hat{V}(\dot{\Phi}) + G(\Phi)$$

(c) (5 pts) Why is this controller superior to the one in Problem 2? Would you use the same PD gains as you designed in Problem 2? Explain.

- IT IS LINEARIZED TO THE PD CONTROLLER, EXCEPT FOR INERTIAL TERMS
- NO GAINS WILL BE DIFFERENT AS PD CONTROLLER IS CONTROLLING A MORE LINEAR SYSTEM