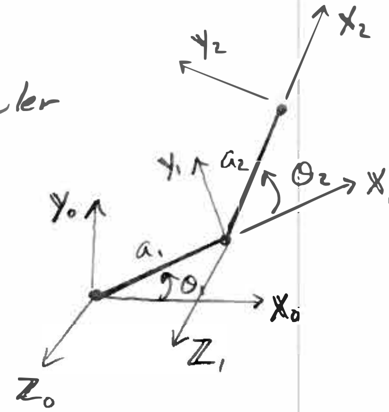
PS # 2 Solutions

Derive dynamics using Newton-Euler



Forward accelerations

<u>$i=1$</u>

$$\omega_{01} = \dot{\theta}_1 Z_0$$

$$\dot{\omega}_{01} = \ddot{\theta}_1 Z_0$$

$$\ddot{d}_{01} = \dot{\omega}_{01} \times d_{01} + \omega_{01} \times (\omega_{01} \times d_{01})$$

$$= \ddot{\theta}_1 Z_0 \times a_1 X_1 + \dot{\theta}_1 Z_0 \times (\dot{\theta}_1 Z_0 \times a_1 X_1)$$

$$= a_1 \ddot{\theta}_1 Y_1 - a_1 \dot{\theta}_1^2 X_1$$

assume $r_{01} = r_{01} X_1$

$$\ddot{r}_{01} = \dot{\omega}_{01} \times r_{01} + \omega_{01} \times (\omega_{01} \times r_{01})$$

$$= r_{01} \ddot{\theta}_1 Y_1 - r_{01} \dot{\theta}_1^2 X_1$$

<u>$i=2$</u>

$$\omega_{02} = \dot{\theta}_1 Z_0 + \dot{\theta}_2 Z_1 = (\dot{\theta}_1 + \dot{\theta}_2) Z_0$$

$$\dot{\omega}_{02} = \ddot{\theta}_1 Z_0 + \ddot{\theta}_2 Z_1 + \dot{\theta}_2 \underbrace{\omega_{01} \times Z_1}_{0}$$

$$= (\ddot{\theta}_1 + \ddot{\theta}_2) Z_0$$

assume $r_{12} = r_{12} X_2$

$$\ddot{r}_{02} = \ddot{d}_{01} + \dot{\omega}_{02} \times r_{12} + \omega_{02} \times (\omega_{02} \times r_{12})$$

$$= \ddot{d}_{01} + (\ddot{\theta}_1 + \ddot{\theta}_2) Z_0 \times r_{12} X_2 + (\dot{\theta}_1 + \dot{\theta}_2) Z_0 \times [(\dot{\theta}_1 + \dot{\theta}_2) Z_0 \times r_{12} X_2$$

$$= a_1 \ddot{\theta}_1 Y_1 - a_1 \dot{\theta}_1^2 X_1 + r_{12} (\ddot{\theta}_1 + \ddot{\theta}_2) Y_2 - r_{12} (\dot{\theta}_1 + \dot{\theta}_2)^2 X_2$$

Recursive Forces/Torques

$\underline{i=2}$

$$f_2 = m_2 \ddot{r}_{02}$$

$$n_2 = I_2 \dot{\omega}_{02} + \omega_{02} \times I_2 \omega_{02}$$

$$f_{12} = f_2 - m_2 g = m_2(\ddot{r}_{02} + g y_0) \qquad g = -g y_0$$

$$n_{12} = n_2 + r_{12} \times f_{12}$$

$$\tau_2 = Z_1 \cdot n_{12} = Z_0 \cdot (n_2 + r_{12} x_2 \times f_{12})$$

$$= Z_0 \cdot n_2 + r_{12} (Z_0 \times x_2) \cdot f_{12}$$

$$= \underbrace{Z_0 \cdot n_2}_{A} + \underbrace{r_{12} y_2 \cdot f_{12}}_{B}$$

(A) $$Z_0 \cdot n_2 = Z_0 \cdot I_2 (\ddot{\theta}_1 + \ddot{\theta}_2) Z_0 + Z_0 \cdot [(\dot{\theta}_1 + \dot{\theta}_2) Z_0 \times I_2 (\dot{\theta}_1 + \dot{\theta}_2) Z_0$$

$$= (\ddot{\theta}_1 + \ddot{\theta}_2) I_{2,33} + \underbrace{(Z_0 \times Z_0)}_{0} \cdot I_2 Z_0 (\dot{\theta}_1 + \dot{\theta}_2$$

(B) $$r_{12} y_2 \cdot f_{12} = r_{12} y_2 \cdot m_2 (\ddot{r}_{02} + g y_0)$$

$$= r_{12} y_2 \cdot m_2 (a_1 \ddot{\theta}_1 y_1 - a_1 \dot{\theta}_1^2 x_1 + r_{12} (\ddot{\theta}_1 + \ddot{\theta}_2) y_2 - r_{12} (\dot{\theta}_1 + \dot{\theta}_2)^2 x_2 + g y$$

$$= r_{12} m_2 [a_1 \ddot{\theta}_1 c\theta_2 + a_1 \dot{\theta}_1^2 s\theta_2 + r_{12} (\ddot{\theta}_1 + \ddot{\theta}_2) + 0 + g c(\theta_1 + \theta_2$$

$$\tau_2 = (\ddot{\theta}_1 + \ddot{\theta}_2) \underbrace{[I_{2,33} + m_2 r_{12}^2]}_{I_2} + a_1 r_{12} m_2 c_2 \ddot{\theta}_1 + a_1 r_{12} m_2 s_2 \dot{\theta}_1^2 + r_{12} m_2 g c_{12}$$

(moment of inertia of link 2 about joint 2 by parallel axis theorem)

$$\tau_2 = \underbrace{(I_2 + a_1 r_{12} m_2 c_2)}_{H_{21}} \ddot{\theta}_1 + \underbrace{I_2}_{H_{22}} \ddot{\theta}_2 + \underbrace{a_1 r_{12} m_2 s_2}_{h} \dot{\theta}_1^2 + \underbrace{r_{12} m_2 g c_{12}}_{G_2}$$

$\underline{i=1}$

$$f_1 = m_1 \ddot{r}_{01}$$

$$n_1 = I_1 \dot{\omega}_{01} + \omega_{01} \times I_1 \omega_{01}$$

$$f_{01} = f_1 + f_{12} - m_1 g$$

$$n_{01} = n_1 + n_{12} + r_{01} \times f_{01} - r_{11} \times f_{12}$$

$$\tau_1 = z_0 \cdot n_{01} = \underset{A}{z_0 \cdot n_1} + \underset{B}{z_0 \cdot n_{12}} + \underset{C}{(z_0 \times r_{01}) \cdot f_{01}} - \underset{D}{(z_0 \times r_{11}) \cdot f_{12}}$$

A $$z_0 \cdot n_1 = z_0 \cdot I_1 \ddot{\theta}_1 z_0 + z_0 \cdot (\dot{\theta}_1 z_0 \times I_1 \dot{\theta}_1 z_0)$$

$$= I_{1,33} \ddot{\theta}_1 + \underbrace{z_0 \times z_0}_{0} \cdot I_1 z_0 \dot{\theta}_1^2$$

B $$z_0 \cdot n_{12} = \tau_2$$

C $$(z_0 \times r_{01}) \cdot f_{01} = (z_0 \times r_{01} x_1) \cdot f_{01} = r_{01} y_1 \cdot f_{01}$$

$$= r_{01} y_1 \cdot [m_1 \ddot{r}_{01} + m_2 \ddot{r}_{02} + (m_1 + m_2) g y_0]$$

$$= r_{01} y_1 \cdot [m_1 (r_{01} \ddot{\theta}_1 y_1 - r_{01} \dot{\theta}_1^2 x_1) + m_2 (a_1 \ddot{\theta}_1 y_1 - a_1 \dot{\theta}_1^2 x_1$$
$$+ r_{12} (\ddot{\theta}_1 + \ddot{\theta}_2) y_2 - r_{12} (\dot{\theta}_1 + \dot{\theta}_2)^2 x_2) + (m_1 + m_2) g y_0]$$

$$= m_1 r_{01}^2 \ddot{\theta}_1 + 0 + a_1 r_{01} m_2 \ddot{\theta}_1 + 0 + r_{01} r_{12} (\ddot{\theta}_1 + \ddot{\theta}_2) c\theta_2$$
$$- r_{01} r_{12} (\dot{\theta}_1 + \dot{\theta}_2)^2 s\theta_2 + r_{01} (m_1 + m_2) g c\theta_1$$

D $$-(z_0 \times r_{11}) \cdot f_{12} = z_0 \times (a_1 - r_{01}) x_1 \cdot f_{12} = (a_1 - r_{01}) y_1 \cdot f_{12}$$

$$= (a_1 - r_{01}) y_1 \cdot m_2 (\ddot{r}_{02} + g y_0)$$

$$= m_2 (a_1 - r_{01}) y_1 \cdot [a_1 \ddot{\theta}_1 y_1 - a_1 \dot{\theta}_1^2 x_1 + r_{12} (\ddot{\theta}_1 + \ddot{\theta}_2) y_2 - r_{12} (\dot{\theta}_1 + \dot{\theta}_2)^2 x_2 + g$$

$$= m_2 (a_1 - r_{01}) [a_1 \ddot{\theta}_1 + 0 + r_{12} (\ddot{\theta}_1 + \ddot{\theta}_2) c\theta_2 - r_{12} (\dot{\theta}_1 + \dot{\theta}_2)^2 s\theta_2 + g c\theta$$

$$\tau_1 = I_{1,33}\ddot{\theta}_1 + \tau_2 + m_1 r_{01}^2\ddot{\theta}_1 + a_1 \cancel{r_{01}} m_2\ddot{\theta}_1 + \cancel{r_{01} r_{12}(\ddot{\theta}_1+\ddot{\theta}_2)} c\theta_2$$

$$- \cancel{r_{01} r_{12} (\dot{\theta}_1+\dot{\theta}_2)^2} s\theta_2 + r_{01}(m_1+\cancel{m_2}) g c\theta_1$$

$$+ m_2(a_1 - \cancel{r_{01}}) a_1 \ddot{\theta}_1 + m_2(a_1-\cancel{r_{01}}) r_{12}(\ddot{\theta}_1+\ddot{\theta}_2) c\theta_2$$

$$- m_2 (a_1-\cancel{r_{01}}) r_{12}(\dot{\theta}_1+\dot{\theta}_2)^2 s\theta_2 + m_2(a_1-\cancel{r_{01}}) g c\theta_1$$

$$= \underbrace{(I_{1,33} + m_1 r_{01}^2)}_{I_1}\ddot{\theta}_1 + \tau_2 + m_2 a_1^2\ddot{\theta} + m_2 a_1 r_{12}(\ddot{\theta}_1+\ddot{\theta}_2)c\theta_2$$

$$- m_2 a_1 r_{12}(\dot{\theta}_1+\dot{\theta}_2)^2 s\theta_2 + (r_{01}m_1 + a_1 m_2) g c\theta$$

moment of inertia of link 1 about joint 1 by parallel axis theorem

$$= \underbrace{(I_1 + I_2 + m_2 a_1^2 + 2a_1 r_{12} m_2 c_2)}_{H_{11}}\ddot{\theta}_1 + \underbrace{(I_2 + a_1 r_{12} m_2 c_2)}_{H_{12} = H_{21}}\ddot{\theta}_2$$

$$+ \underbrace{a_1 r_{12} m_2 s\theta_2}_{h}(\cancel{\dot{\theta}_1^2} - \cancel{\dot{\theta}_1^2} - 2\dot{\theta}_1\dot{\theta}_2 - \dot{\theta}_2^2)$$

$$+ \underbrace{(r_{01}m_1 + a_1 m_2) g c_1 + r_{12} m_2 g c_{12}}_{G_1}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}H_{11} & H_{12}\\ H_{12} & H_{22}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \underbrace{\begin{bmatrix}0 & -2h & -h\\ h & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1^2\\ \dot{\theta}_1\dot{\theta}_2\\ \dot{\theta}_2^2\end{bmatrix}}_{} + \begin{bmatrix}G_1\\ G_2\end{bmatrix}$$

$$\boldsymbol{\tau} = \mathbb{H}(\Theta)\ \ddot{\Theta} + \mathbb{V}(\Theta,\dot{\Theta}) + G(\Theta)$$