In class we learned that you can put the inverse dynamic solution into a closed form:

$$\tau = H(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta)$$

where **H** is the inertia matrix, **V** is a matrix of centripetal and coriolis terms, and **G** is a matrix of body or gravitational torques. For the 2-DOF robot example in class, this matrix equation took the form of:

$$\begin{bmatrix}\tau_1\\\tau_2\end{bmatrix} = \begin{bmatrix}H_{11} & H_{12}\\H_{21} & H_{22}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\\ddot{\theta}_2\end{bmatrix} + \begin{bmatrix}0 & -2h & -h\\h & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1^2\\\dot{\theta}_1\dot{\theta}_2\\\dot{\theta}_2^2\end{bmatrix} + \begin{bmatrix}G_1\\G_2\end{bmatrix}$$

Use the recursive Newton-Euler algorithm to derive the algebraic solutions for the terms in the above equations.

Hint: You can follow the derivations in Ch. 10 of Hollerbach's notes, EXCEPT for the part where he assumes the centers of mass of the links are exactly halfway along each link, and you'll need to include gravitational effects.

10.3 NEWTON - EULER

FRAME 0 = INERTIAL FRAME

$\underline{f}_i = \dot{\underline{p}}_i = m_i \ddot{\underline{r}}_{0i}$

FORCE MOMENTUM ACCELERATION

$\underline{I}_i =$ C.O.G.

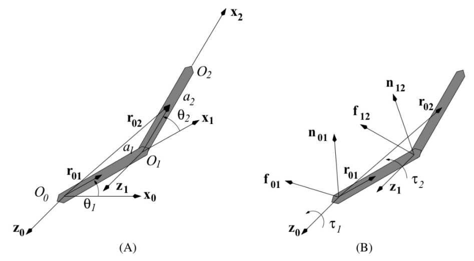


Figure 10.5: (A) Kinematic diagram of two-link planar manipulator. (B) Free-body diagram.

$\underline{r}_{01}$ & $\underline{r}_{02}$ ARE NOT NECESSARILY IN EXACT CENTER OF ROBOT

RECURSIVE NEWTON-EULER

$\underline{f}_2 = \underline{f}_{12} + m_2 \underline{g}$

$\underline{f}_1 = \underline{f}_{01} - \underline{f}_{12} + m_1 \underline{g}$

ASSUME

$\dot{r}_{01} = \dot{r}_{12} = 0$ (RIGID)

WHERE: $\underline{r}_{01} = r_{01}\ \underline{x}_1$

$\underline{r}_{02} = r_{12}\ \underline{x}_2 + a_1\underline{x}_1$

$\underline{g} = -\underline{y}_0 g$

WHERE $\underline{\omega}_{01} = \dot{\theta}_1 \underline{z}_0$

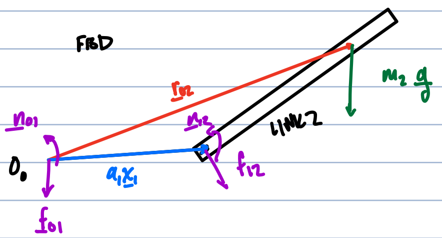
$\dot{\underline{\omega}}_{01} = \ddot{\theta}_1 \underline{z}_0$

$\underline{\omega}_{02} = \dot{\theta}_1 \underline{z}_0 + \dot{\theta}_2 \underline{z}_0 =$

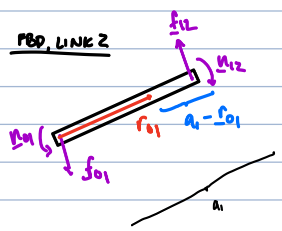
$\dot{\underline{\omega}}_{02} = \ddot{\theta}_1 \underline{z}_0 + \ddot{\theta}_2 \underline{z}_0$

$= (\ddot{\theta}_1 + \ddot{\theta}_2)\underline{z}_0$

FBD



FBD, LINK 2



C.O.G. DERIVATIVES

LINK 1

$\dot{\underline{r}}_{01} = \frac{d}{dt}(r_{01}\underline{x}_1) = r_{01}\underline{\omega}_{01} \times \underline{x}_1$

$= r_{01}\dot{\theta}_1 \underline{z}_0 \times \underline{x}_1$ $\quad \underline{z}_0 = \underline{z}_1$

$\therefore \underline{z}_1 \times \underline{x}_1 = \underline{y}_1$

$= r_{01}\dot{\theta}_1 \underline{y}_1$

$\ddot{\underline{r}}_{01} = \frac{d}{dt}(r_{01}\dot{\theta}_1\underline{y}_1) = r_{01}\ddot{\theta}_1\underline{y}_1 + r_{01}\dot{\theta}_1\underline{\omega}_{01} \times \underline{y}_1$

$= r_{01}\ddot{\theta}_1\underline{y}_1 + r_{01}\dot{\theta}_1^2 \underline{z}_0 \times \underline{y}_1$

$= r_{01}\ddot{\theta}_1 \underline{y}_1 - r_{01}\dot{\theta}_1^2\underline{x}_1$

LINK 2

$\dot{\underline{r}}_{02} = \frac{d}{dt}(a_1\underline{x}_1 + r_{12}\underline{x}_2)$

$\dot{\underline{r}}_{02} = \dot{\underline{r}}_{01} + \underline{\omega}_{02} \times r_{12}\underline{x}_2$

$= r_{01}\dot{\theta}_1\underline{y}_1 + (\dot{\theta}_1 + \dot{\theta}_2) r_{12}\underline{z}_0 \times \underline{x}_2$ $\quad \underline{z}_0 = \underline{z}_1 = \underline{z}_2$

$= r_{01}\dot{\theta}_1\underline{y}_1 + (\dot{\theta}_1 + \dot{\theta}_2) r_{12}\underline{y}_2$ $\quad \therefore \underline{z}_2 \times \underline{x}_2 = \underline{y}_2$

$\ddot{\underline{r}}_{02} = \frac{d}{dt}(\dot{\underline{r}}_{02}) = \ddot{\underline{r}}_{01} + \frac{d}{dt}((\dot{\theta}_1 + \dot{\theta}_2) r_{12}\underline{y}_2)$

$\underline{\omega}_{02} = (\dot{\theta}_1 + \dot{\theta}_2)$

$= \ddot{\underline{r}}_{01} + (\ddot{\theta}_1 + \ddot{\theta}_2) r_{12}\underline{y}_2 + (\dot{\theta}_1 + \dot{\theta}_2) r_{12}\underline{\omega}_{02} \times \underline{y}_2$

$= r_{01}\ddot{\theta}_1\underline{y}_1 - r_{01}\dot{\theta}_1^2\underline{x}_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) r_{12}\underline{y}_2 + (\dot{\theta}_1 + \dot{\theta}_2)^2 r_{12}\underline{z}_0 \times \underline{y}_2$

$= r_{01}\ddot{\theta}_1\underline{y}_1 - r_{01}\dot{\theta}_1^2\underline{x}_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) r_{12}\underline{y}_2 - (\dot{\theta}_1 + \dot{\theta}_2)^2 r_{12}\underline{x}_2$

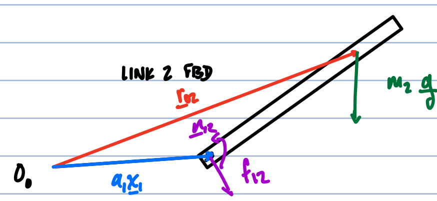
FORCE BALANCE

$\underline{f}_2 = m_2\ddot{\underline{r}}_{02} = \underline{f}_{12} + m_2\underline{g}$ $\qquad \underline{g} = -g\underline{y}_0$

$\hookrightarrow m_2(r_{01}\ddot{\theta}_1\underline{y}_1 - r_{01}\dot{\theta}_1^2\underline{x}_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) r_{12}\underline{y}_2 - (\dot{\theta}_1 + \dot{\theta}_2)^2 r_{12}\underline{x}_2) = \underline{f}_{12} + m_2\underline{g} = \underline{f}_{12} - m_2 g\underline{y}_0$

$\therefore \underline{f}_{12} = m_2 g\underline{y}_0 + m_2(r_{01}\ddot{\theta}_1\underline{y}_1 - r_{01}\dot{\theta}_1^2\underline{x}_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) r_{12}\underline{y}_2 - (\dot{\theta}_1 + \dot{\theta}_2)^2 r_{12}\underline{x}_2)$

LINK 2 FBD



$\underline{f}_1 = m_1\ddot{\underline{r}}_{01} = \underline{f}_{01} - \underline{f}_{12} + m_1\underline{g}$ $\qquad r_{12} \times m_2 r_{01}\dot{\theta}_1^2$

$\hookrightarrow m_1(r_{01}\ddot{\theta}_1\underline{y}_1 - r_{01}\dot{\theta}_1^2\underline{x}_1) = \underline{f}_{01} - \underline{f}_{12} + m_1\underline{g}$

$\therefore \underline{f}_{01} = \underline{f}_{12} + m_1 g\underline{y}_0 + m_1 r_{01}\ddot{\theta}_1\underline{y}_1 - m_1 r_{01}\dot{\theta}_1^2\underline{x}_1$

MOMENT BALANCE

$$\underline{n}_2 = I_2 \dot{\underline{\omega}}_{02} + \underline{\omega}_{02} \times I_2 \underline{\omega}_{02}$$

$$\underline{n}_2 = \underline{n}_{12} - (r_{12}\underline{x}_2)\times \underline{f}_{12}$$

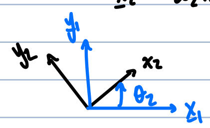
$$\therefore \underline{n}_{12} = I_2\dot{\underline{\omega}}_{02} + \underline{\omega}_{02}\times I_2\underline{\omega}_{02} + r_{12}\underline{x}_2\times\underline{f}_{12}$$

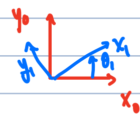
$$\therefore \underline{n}_{01} = I_1\dot{\underline{\omega}}_{01} + \underline{\omega}_{01}\times I_1\underline{\omega}_{01} + r_{01}\underline{x}_1\times\underline{f}_{01} + (a_1 - r_{01})\underline{x}_1\times\underline{f}_{12} + \underline{n}_{12}$$

$$= I_2\dot{\underline{\omega}}_{02} + \underline{\omega}_{02}\times I_2\underline{\omega}_{02} + (r_{12}\underline{x}_2)\times\left[\left(m_2\left(r_{01}\ddot\theta_1\underline{y}_1 - r_{01}\dot\theta_1^2\underline{x}_1 + (\ddot\theta_1+\ddot\theta_2)r_{12}\underline{y}_2 - (\dot\theta_1+\dot\theta_2)^2 r_{12}\underline{x}_2\right) + m_2 g\underline{y}_0\right)\right]$$

$$= I_2(\ddot\theta_1+\ddot\theta_2)\underline{z}_0 + (\dot\theta_1+\dot\theta_2)\underline{z}_0\times I_2(\dot\theta_1+\dot\theta_2)\underline{z}_0 + r_{12}\underline{x}_2\times\left[m_2 r_{01}(\ddot\theta_1\underline{y}_1 - \dot\theta_1^2\underline{x}_1) + m_2 r_{12}(\ddot\theta_1+\ddot\theta_2)\underline{y}_2 - r_{12}m_2(\dot\theta_1+\dot\theta_2)^2\underline{x}_2 + m_2 g\underline{y}_0\right]$$

$\underline{x}_2 = c\theta_2 \underline{x}_1 + s\theta_2\underline{y}_1$ $\qquad$ $\underline{y}_0 = c\theta_1\underline{y}_1 + s\theta_1\underline{x}_1$





$$= r_{12}\underline{x}_2\times\left[m_2 r_{01}(\ddot\theta_1\underline{y}_1 - \dot\theta_1^2\underline{x}_1) + m_2 r_{12}(\ddot\theta_1+\ddot\theta_2)\underline{y}_2 - r_{12}m_2(\dot\theta_1+\dot\theta_2)^2\underline{x}_2 + m_2 g\underline{y}_0\right]$$

$$r_{12}(c\theta_2\underline{x}_1 + \cancel{s\theta_2\underline{y}_1}^{0})\times m_2 r_{01}\ddot\theta_1\underline{y}_1 = r_{12}r_{01}m_2\ddot\theta_1 c\theta_2\underline{z}_1$$

$$r_{12}(\cancel{c\theta_2\underline{x}_1}^{0} + s\theta_2\underline{y}_1)\times m_2 r_{01}(-\dot\theta_1^2\underline{x}_1) = r_{12}r_{01}m_2 s\theta_2\dot\theta_1^2\underline{z}_1$$

$$r_{12}\underline{x}_2\times m_2 r_{12}(\ddot\theta_1+\ddot\theta_2)\underline{y}_2 = m_2 r_{12}^2(\ddot\theta_1+\ddot\theta_2)\underline{z}_2$$

$$r_{12}\underline{x}_2\times(-r_{12}m_2(\dot\theta_1+\dot\theta_2)^2\underline{x}_2) = 0$$

$$r_{12}(c\theta_2\underline{x}_1 + s\theta_2\underline{y}_1)\times(c\theta_1\underline{y}_1 + s\theta_1\underline{x}_1)m_2 g = r_{12}m_2 g(c\theta_2 c\theta_1\underline{z}_1 + 0 + 0 - s\theta_2 s\theta_1\underline{z}_1)$$

$\underline{x}_1\times\underline{y}_1 = \underline{z}_1$
$\underline{x}_1\times\underline{x}_1 = 0$
$\underline{y}_1\times\underline{y}_1 = 0$
$\underline{y}_1\times\underline{x}_1 = -\underline{z}_1$

$$= r_{12}m_2 g(c\theta_2 c\theta_1 - s\theta_2 s\theta_1)\underline{z}_1$$

$$\underline{n}_{12} = I_2(\ddot\theta_1+\ddot\theta_2)\underline{z}_0 + (\dot\theta_1+\dot\theta_2)\underline{z}_0\times I_2(\dot\theta_1+\dot\theta_2)\underline{z}_0 + r_{12}r_{01}m_2\ddot\theta_1 c\theta_2\underline{z}_1 + r_{12}r_{01}m_2 s\theta_2\dot\theta_1^2\underline{z}_1 + m_2 r_{12}^2(\ddot\theta_1+\ddot\theta_2)\underline{z}_2 + r_{12}m_2 g(c\theta_2 c\theta_1 - s\theta_2 s\theta_1)\underline{z}_1$$

$$\tau_2 = \underline{z}_0\cdot\underline{n}_{12} = \underline{z}_1\cdot\underline{n}_{12}$$

$$= \underbrace{\underline{z}_1\cdot\left[I_2(\ddot\theta_1+\ddot\theta_2)\underline{z}_0\right.}_{\underline{z}_0\cdot\underline{z}_0\, I_2(\ddot\theta_1+\ddot\theta_2) \,=\, I_{2,33}(\ddot\theta_1+\ddot\theta_2)} + \underbrace{\left.(\dot\theta_1+\dot\theta_2)\underline{z}_0\times \cancel{I_2(\dot\theta_1+\dot\theta_2)\underline{z}_0}^{0}\right]}_{\underline{a}\cdot(\underline{a}\times\underline{b}) = 0} + \underline{z}_1\cdot\left[r_{12}r_{01}m_2\ddot\theta_1 c\theta_2\underline{z}_1 + r_{12}r_{01}m_2 s\theta_2\dot\theta_1^2\underline{z}_1 + m_2 r_{12}^2(\ddot\theta_1+\ddot\theta_2)\underline{z}_2 + r_{12}m_2 g(c\theta_2 c\theta_1 - s\theta_2 s\theta_1)\underline{z}_1\right]$$

$$= I_{2,33}(\ddot\theta_1+\ddot\theta_2) + \left[r_{12}r_{01}m_2\ddot\theta_1 c\theta_2 + r_{12}r_{01}m_2 s\theta_2\dot\theta_1^2 + m_2 r_{12}^2(\ddot\theta_1+\ddot\theta_2) + r_{12}m_2 g(c\theta_2 c\theta_1 - s\theta_2 s\theta_1)\right]$$

$$\tau_2 = (I_{2,33} + m_2 r_{12}^2)(\ddot\theta_1+\ddot\theta_2) + r_{01}r_{12}m_2 c\theta_2(\ddot\theta_1) + r_{01}r_{12}m_2 s\theta_2(\dot\theta_1^2) + \underbrace{r_{12}m_2 g(c\theta_2 c\theta_1 - s\theta_2 s\theta_1)}_{G \text{ TERMS}}$$

$$\tau_2 = r_{01}r_{12}m_2(\dot\theta_1^2)$$

FROM ABOVE...

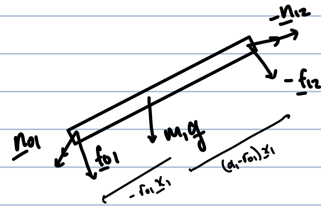
$$\underline{n}_{01} = I_1\dot{\underline{\omega}}_{01} + \underline{\omega}_{01}\times I_1\underline{\omega}_{01} + r_{01}\underline{x}_1\times\underline{f}_{01} + (a_1 - r_{01})\underline{x}_1\times\underline{f}_{12} + \underline{n}_{12}$$

WHERE: $\underline{f}_{01} = \underline{f}_{12} + m_1 g\underline{y}_0 + r_{01}\ddot\theta_1\underline{y}_1 - r_{01}\dot\theta_1^2\underline{x}_1$

SINCE WE'LL BE MULTIPLYING BY $\underline{z}_0$ TO GET $\tau_1 = \underline{z}_0 \cdot \underline{n}_{01}$, WE CAN MAKE SOME SIMPLIFICATIONS :

$$\underline{n}_{12} = \tau_2$$

$$I_1 = I_{1,33} \Rightarrow \begin{cases} I_1 \dot{\underline{\omega}}_{01} = (\ddot{\theta}_1) I_{1,33} \underline{z}_0 \\ \underline{\omega}_{01} \times I_1 \underline{\omega}_{01} = 0 \end{cases}$$



$$\underline{n}_1 = I_1 \dot{\underline{\omega}}_{01} + \underline{\omega}_{01} \times I_1 \underline{\omega}_{01}$$

$$\underline{n}_1 = -r_{01} \underline{x}_1 \times \underline{f}_{01} + (a_1 - r_{01}) \underline{x}_1 \times (-\underline{f}_{12}) + \underline{n}_{01} - \underline{n}_{12}$$

$$\therefore \underline{n}_{01} = I_1 \dot{\underline{\omega}}_{01} + \underline{\omega}_{01} \times I_1 \underline{\omega}_{01} + r_{01} \underline{x}_1 \times \underline{f}_{01} + (a_1 - r_{01}) \underline{x}_1 \times \underline{f}_{12} + \underline{n}_{12}$$

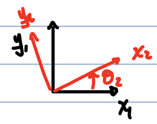
(SAME RULES AS I_2)

$$\therefore \tau_1 = \underline{z}_0 \cdot \underline{n}_{01} = \underline{z}_0 \cdot \left[I_{1,33} (\ddot{\theta}_1) \underline{z}_0 + r_{01} \underline{x}_1 \times \underbrace{\left(\cancel{\underline{f}_{12}}^{\,0} + m_1 g \underline{y}_0 + m_1 r_{01} \ddot{\theta}_1 \underline{y}_1 - m_1 r_{01} \dot{\theta}_1^2 \underline{x}_1 \right)}_{= \underline{f}_{01}} + (a_1 - \cancel{r_{01}}^{\,0}) \underline{x}_1 \times \underline{f}_{12} \right] + \tau_2$$

$$r_{01} \underline{x}_1 \times \underline{f}_{12} - r_{01} \underline{x}_1 \times \underline{f}_{12} = 0$$

$$\tau_1 = \tau_2 + I_{2,33}(\ddot{\theta}_1) + \underline{z}_0 \cdot \left[r_{01} \underline{x}_1 \times (m_1 g \underline{y}_0 + r_{01} \ddot{\theta}_1 \underline{y}_1 - r_{01} \dot{\theta}_1^2 \underline{x}_1) + a_1 \underline{x}_1 \times \left(m_2 g \underline{y}_0 + m_2 \left(m_1 r_{01} \ddot{\theta}_1 \underline{y}_1 - m_1 r_{01} \dot{\theta}_1^2 \underline{x}_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) r_{12} \underline{y}_2 - (\dot{\theta}_1 + \dot{\theta}_2)^2 r_{12} \underline{x}_2 \right) \right) \right]$$

FROM EARLIER: $\underline{x}_2 = C\theta_2 \underline{x}_1 + S\theta_2 \underline{y}_1$ $\underline{y}_0 = C\theta_1 \underline{y}_1 + S\theta_1 \underline{x}_1$



$$r_{01} \underline{x}_1 \times (m_1 g \underline{y}_0 + r_{01} \ddot{\theta}_1 \underline{y}_1 - r_{01} \dot{\theta}_1^2 \underline{x}_1)$$

$$r_{01} \underline{x}_1 \times m_1 g (C\theta_1 \underline{y}_1 + S\theta_1 \underline{x}_1) = r_{01} m_1 g C\theta_1 \underline{z}_1$$

$$r_{01} \underline{x}_1 \times r_{01} \ddot{\theta}_1 \underline{y}_1 = r_{01}^2 \ddot{\theta}_1 \underline{z}_1$$

$$\underline{x}_1 = C\theta_2 \underline{x}_2 - S\theta_2 \underline{y}_2$$

$$r_{01} \underline{x}_1 \times (-r_{01} \dot{\theta}_1^2) \underline{x}_1 = 0$$

$$\therefore = (r_{01} m_1 g C\theta_1 + r_{01}^2 \ddot{\theta}_1) \underline{z}_1$$

$$a_1 \underline{x}_1 \times \left(m_2 g \underline{y}_0 + m_2 \left(m_1 r_{01} \ddot{\theta}_1 \underline{y}_1 - m_1 r_{01} \dot{\theta}_1^2 \underline{x}_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) r_{12} \underline{y}_2 - (\dot{\theta}_1 + \dot{\theta}_2)^2 r_{12} \underline{x}_2 \right) \right)$$

$$a_1 \underline{x}_1 \times m_2 g (C\theta_1 \underline{y}_1 + S\theta_1 \underline{x}_1) = a_1 m_2 C\theta_1 \underline{z}_1$$

$$a_1 \underline{x}_1 \times m_2 r_{01} \ddot{\theta}_1 \underline{y}_1 = a_1 r_{01} m_1 m_2 \ddot{\theta}_1 \underline{z}_1$$

$$a_1 \underline{x}_1 \times -(m_2 r_{01} \dot{\theta}_1^2 \underline{x}_1) = 0$$

$$a_1 (C\theta_2 \underline{x}_2 - S\theta_2 \underline{y}_2) \times m_2 r_{12} (\ddot{\theta}_1 + \ddot{\theta}_2) \underline{y}_2 = a_1 m_1 r_{12} C\theta_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \underline{z}_2$$

$$\underline{z}_2 = \underline{z}_1$$

$$a_1 (C\theta_2 \underline{x}_2 - S\theta_2 \underline{y}_2) \times m_2 (-r_{12} (\dot{\theta}_1 + \dot{\theta}_2)^2) \underline{x}_2 = -a_1 r_{12} m_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 S\theta_2 \underline{z}_2$$

$$\rightsquigarrow \left(a_1 m_2 C\theta_1 + a_1 r_{01} m_1 m_2 \ddot{\theta}_1 + a_1 m_1 r_{12} C\theta_2 (\ddot{\theta}_1 + \ddot{\theta}_2) - a_1 r_{12} (\dot{\theta}_1 + \dot{\theta}_2)^2 S\theta_2 \right) \underline{z}_1$$

$$\tau_1 = \tau_2 + I_{2,33}(\ddot{\theta}_1) + \underline{z}_0 \cdot \left[(r_{01} m_1 g C\theta_1 + r_{01}^2 \ddot{\theta}_1) \underline{z}_1 + \left(a_1 m_2 C\theta_1 + a_1 r_{01} m_1 m_2 (\ddot{\theta}_1) + a_1 m_1 r_{12} C\theta_2 (\ddot{\theta}_1 + \ddot{\theta}_2) - a_1 r_{12} (\dot{\theta}_1 + \dot{\theta}_2)^2 S\theta_2 \right) \underline{z}_1 \right]$$

$$= I_{2,33}(\ddot{\theta}_1) + \tau_2 + r_{01}^2 (\ddot{\theta}_1) + r_{01} m_1 g C\theta_1 + a_1 m_2 C\theta_1 + a_1 r_{01} m_1 m_2 (\ddot{\theta}_1) + a_1 m_1 r_{12} C\theta_2 (\ddot{\theta}_1 + \ddot{\theta}_2) - m_2 a_1 r_{12} S\theta_2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$\tau_2 = (I_{2,33} + m_2 r_{12}^2)(\ddot{\theta}_1 + \ddot{\theta}_2) + r_{01} r_{12} m_2 C\theta_2 (\ddot{\theta}_1) + r_{01} r_{12} m_2 S\theta_2 (\dot{\theta}_1^2) + r_{12} m_2 g (C\theta_2 C\theta_1 - S\theta_2 S\theta_1)$$

$$\therefore \tau_1 = (I_{2,33} + r_{01}^2 + a_1 m_1 r_{01} m_2 + r_{01} r_{12} m_2 C\theta_2) \ddot{\theta}_1 + (a_1 m_1 r_{12} C\theta_2 + I_{2,33} + m_2 r_{12}^2)(\ddot{\theta}_1 + \ddot{\theta}_2) + (-m_2 a_1 r_{12} S\theta_2)(\dot{\theta}_1 + \dot{\theta}_2)^2 + (r_{01} r_{12} m_2 S\theta_2)(\dot{\theta}_1^2)$$

$$+ (r_{01} m_1 g + a_1 m_2)(C\theta_1) + r_{12} m_2 g \underline{(C\theta_2 C\theta_1 - S\theta_2 S\theta_1)}$$

$$C_{12} \overset{\shortparallel}{=} c(\theta_1+\theta_2)$$

SUMMING $\dot\theta_1^2$, $\dot\theta_1 + \dot\theta_2$, $\dot\theta_2^2$ TERMS

$$-a_1 r_{12} S\theta_2 \left(\dot\theta_1 + \dot\theta_2\right)^2 = -a_1 r_{12} m_2 S\theta_2 \left(\dot\theta_1^2 + 2\dot\theta_1\dot\theta_2 + \dot\theta_2^2\right)$$

$$r_{01} r_{12} m_2 S\theta_2 \dot\theta_1^2 - a_1 r_{12} m_2 S\theta_2 \dot\theta_1^2 - 2a_1 r_{12} m_2 S\theta_2 - a_1 r_{12} m_2 S\theta_2 \dot\theta_2^2$$

$$\rightsquigarrow \begin{bmatrix} (r_{01}-a_1) r_{12} m_2 S\theta_2 & -2a_1 r_{12} m_2 S\theta_2 & -a_1 r_{12} m_2 S\theta_2 \\ r_{01} r_{12} m_2 S\theta_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot\theta_1^2 \\ \dot\theta_1\dot\theta_2 \\ \dot\theta_2^2 \end{bmatrix}$$

$$\mathbb{V} = \begin{bmatrix} (r_{01}-a_1) r_{12} m_2 S\theta_2 & -2a_1 r_{12} m_2 S\theta_2 & -a_1 r_{12} m_2 S\theta_2 \\ r_{01} r_{12} m_2 S\theta_2 & 0 & 0 \end{bmatrix}$$

SUMMING GRAVITATIONAL TERMS :

$$\mathbb{G} = \begin{bmatrix} r_{01} m_1 g + a_1 m_2 & r_{12} m_2 g \\ 0 & r_{12} m_2 g \end{bmatrix} \begin{bmatrix} C\theta_1 \\ c(\theta_1+\theta_2) \end{bmatrix}$$

$$\Rightarrow \mathbb{H} = \begin{bmatrix} \left(I_{2,33} + r_{01}^2 + a_1 r_{01} m_1 m_2 + r_{01} r_{12} m_2 C\theta_2\right) + \left(a_1 m_1 r_{12} C\theta_2 + I_{2,33} + m_2 r_{12}^2\right) & \left(a_1 m_1 r_{12} C\theta_2 + I_{2,33} + m_2 r_{12}^2\right) \\ \left(I_{2,33} + m_2 r_{12}^2\right) & \left(I_{2,33} + m_2 r_{12}^2\right) \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2h & -h \\ h & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot\theta_1^2 \\ \dot\theta_1\dot\theta_2 \\ \dot\theta_2^2 \end{bmatrix} \sim$$

FULL DYNAMICS

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \mathbb{H} \begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} + \mathbb{V} \begin{bmatrix} \dot\theta_1^2 \\ \dot\theta_1\dot\theta_2 \\ \dot\theta_2^2 \end{bmatrix} + \mathbb{G}$$

FIND $\mathbb{H}$ $\mathbb{V}$ $\mathbb{G}$ ABOVE