Stability and Robustness of PID Feedback Control for Robot Manipulators of Sensory Capability

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A PID local feedback scheme for the position control of robot manipulators is proposed. In addition to the ordinary use of joint-angle position and velocity feedbacks at each joint actuator, the feedback of integration of positional reference errors is introduced so as to reduce the steady state offset. In spite of the existence of strong couplings between joints, nonlinear terms of Coriolis and centrifugal forces, and bias terms due to friction or the change of payload, the stability and robustness of the proposed scheme is proved by the Lyapunov method of stability analysis incorporating a singular perturbation technique. The PID feedback scheme is then modified to suit the case when the target point, into which the end effector of the manipulator is maneuvered, is described in terms of the cartesian coordinates or any other task-oriented coordinates. The feedback signal for this case is calculated based upon the data of sensory signals taken from both inside and outside of the manipulator, and hence the localization of feedback becomes impossible. Nevertheless, the asympototic stability of such a PID sensory feedback scheme is also proved under some condition on the Jacobian matrix.

Introduction

The control of existing industrial manipulators with many degrees of freedom consists of many actuator servomechanisms, each of which controls independently a corresponding joint with a single degree of freedom, though there are dynamic couplings among joints. On the other hand, a good number of papers propose control algorithms by taking into consideration the dynamics of manipulators and, in particular, the strong coupling among joints. Most of them could not be easily implemented in the manipulator system, since the algorithm was apt to be complex due to the involved expression of dynamics of the manipulator. However, it was shown recently [Takegaki and Arimoto, 1981] that for the purpose of so-called "point to point" control the torque control based on the feedback of the state variable of joint angles and their derivatives is effective and can be easily implemented. This is due to the global stability, which is assured even if the feedback at each joint does not refer to any other joint angles. That is, the independent local feedback at each joint assures the global stability for most mechanical manipulators with many degrees of freedom.

In this paper we propose a PID independent local feedback scheme as a natural extension of the control algorithm proposed previously. The feedback of integration of the reference error is introduced in order to reduce the steady offset, which may arise from the unknown term of disturbances such as friction torques and evaluation errors of physical constants, for example, inertia moments and centers of gravity. Our principal concern is to prove the global asymptotic stability of the system under the condition of a PID local feedback control scheme. It should be noted that the stability problem is nontrivial since there are some strong couplings in the inertia matrix and coupled nonlinear terms of Coriolis and centrifugal forces, and moreover the inertia matrix varies depending on the manipulator configuration.

The proposed PID feedback control scheme is easily implemented by using tachogenerators attached to motors for "differential" feedback, potentiometers or photoencoders for "proportional" feedback, and analog circuits or microcomputers for "integration" feedback. With other sensors, such as position or line sensors, or vision sensors (which will be available in the near future), it is rather convenient to describe the task in terms of the cartesian coordinates or other adequate task-oriented coordinates. It is then possible to adopt a modified PID feedback control scheme in which the feedback signal is calculated on the basis of sensor signals taken from both inside and outside of the manipulator. In this case, the localization of PID feedback necessarily becomes unavailable.

Throughout this paper, a standard model of robot manipulators with three degrees of freedom is conveniently employed for explaining the problem and idea explicity and, in particular, for gaining an insight into the actual effectiveness of stabilization by the proposed PID feedback control scheme. Detailed discussions on the asymptotic stability of the local feedback and its robustness as a control method are given in sections 2 and 3. The proof of central results is based upon Lyapunov's direct method of stability analysis, which in some cases is incorporated into a singular perturbation technique. In section 4, the method of proof is modified and extended to cover the stabilization problem for the case where a task-oriented coordinates system is used.

1 Stability of Idealized Motion for a Standard Manipulator Model

First we consider a standard model of typical manipulators existing in industries, which is a serial link manipulator as shown in figure 1. The first link is a base which can be rotated about a fixed vertical axis (z axis). The second link is connected to the base link by joint 1, which can revolve. In other words, the actuator of joint 1 can make link 2 bend in a plane which contains the z axis and is perpendicular to the joint axis of rotation. The third link is connected to the second one at joint 2, which can revolve, too. Joint angles θ_i, $i = 1, 2, 3$, are taken as indicated in figure 1.

It is assumed in this paper that the model is a series of rigid bodies. Then, it is not difficult to evaluate the kinetic energy, which is written as [Paul, 1981; Vukobratovic, 1982]

$$T = \tfrac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}} R(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}, \tag{1}$$

where $\dot{\theta}_i$ denotes $d\theta_i/dt$ and

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}, \quad R(\boldsymbol{\theta}) = \begin{bmatrix} r_{11}(\boldsymbol{\theta}) & 0 & 0 \\ 0 & r_{22}(\boldsymbol{\theta}) & r_{23}(\boldsymbol{\theta}) \\ 0 & r_{23}(\boldsymbol{\theta}) & r_{33}(\boldsymbol{\theta}) \end{bmatrix},$$

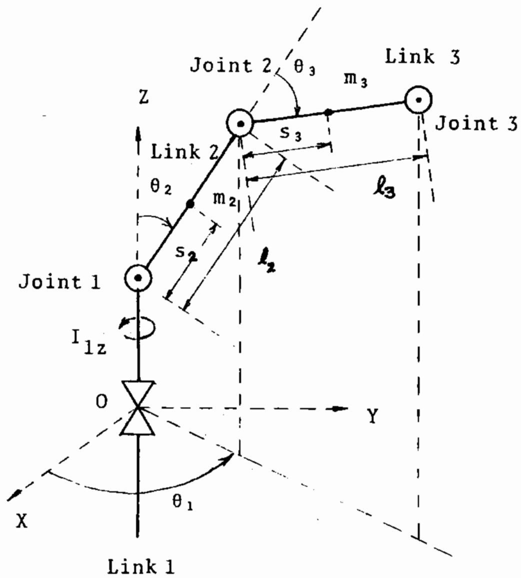

Figure 1
Manipulator with three degrees of freedom.

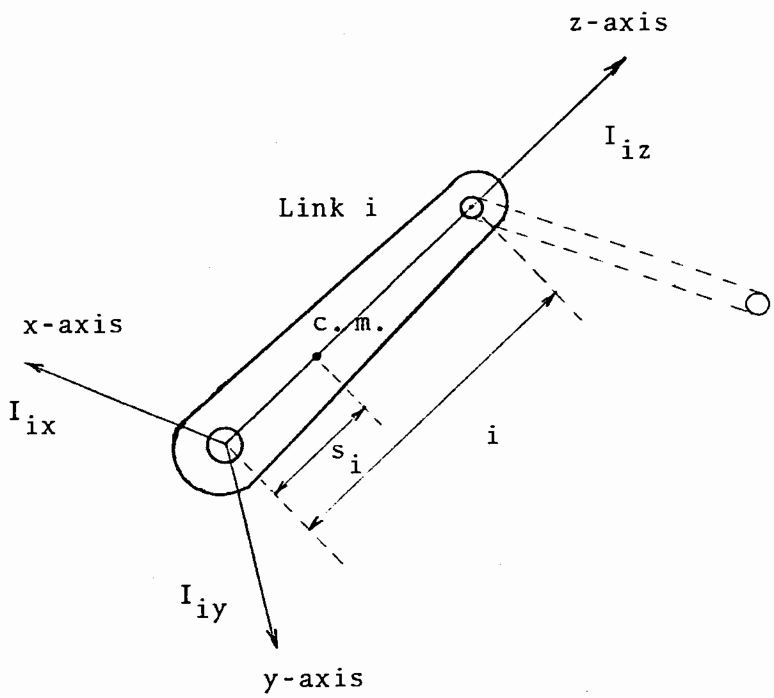

Figure 2
Physical constants for link i; c.m. denotes the center of mass.

$$r_{11}(\boldsymbol{\theta}) = I_{1z} + m_2(s_2 \sin\theta_2)^2 + m_3(l_2 \sin\theta_2 + s_3 \sin(\theta_2 + \theta_3))^2$$
$$+ I_{2y}(\sin\theta_2)^2 + I_{3y}(\sin(\theta_2 + \theta_3))^2 + I_{2z}(\cos\theta_2)^2 + I_{3z}(\cos(\theta_2 + \theta_3))^2,$$
$$r_{22}(\boldsymbol{\theta}) = m_2 s_2^2 + m_3 l_2^2 + I_{2x} + m_3 s_3^2 + I_{3x} + 2m_3 l_2 s_3 \cos\theta_3,$$
$$r_{23}(\boldsymbol{\theta}) = m_3 s_3^2 + I_{3x} + m_3 l_2 s_3 \cos\theta_3,$$
$$r_{33}(\boldsymbol{\theta}) = m_3 s_3^2 + I_{3x}.$$

Note that $I_{i\xi}$ denotes the inertia moment of link i around the ξ axis and m_i, l_i, and s_i denote the mass of link i, the length between centers of joints $i-1$ and i, and the distance from the center of joint $i-1$ to the center of mass for link i, respectively, as indicated in figures 1 and 2. It is also important to note that the inertia matrix $R(\boldsymbol{\theta})$ is positive definite for any value of $\boldsymbol{\theta}$. The potential energy is given by

$$P(\boldsymbol{\theta}) = (m_2 s_2 + m_3 s_2) g \cos\theta_2 + m_3 s_3 g \cos(\theta_2 + \theta_3). \tag{2}$$

First we ignore the term of frictions which may arize at each actuated joint. Then, by introducing a Lagrangian $L = T - P$, we obtain the equation of motion for the model, which is described by the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}}\right) - \frac{\partial L}{\partial \boldsymbol{\theta}} = \mathbf{u}, \tag{3}$$

where $\mathbf{u}$ denotes the vector of generalized force and

$$\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}} = \begin{bmatrix} \partial L/\partial \dot{\theta}_1 \\ \partial L/\partial \dot{\theta}_2 \\ \partial L/\partial \dot{\theta}_3 \end{bmatrix}, \quad \frac{\partial L}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \partial L/\partial \theta_1 \\ \partial L/\partial \theta_2 \\ \partial L/\partial \theta_3 \end{bmatrix}.$$

Since the potential energy P is independent of $\dot{\boldsymbol{\theta}}$, the Lagrange equation (3) can be rewritten as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}\right) - \frac{\partial T}{\partial \boldsymbol{\theta}} + \frac{\partial P}{\partial \boldsymbol{\theta}} = \mathbf{u}. \tag{4}$$

If symbol $\mathbf{p}$ is introduced to express the momentum vector as defined by $\mathbf{p} = R(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$, then equation (4) yields a state equation

$$\begin{aligned} \dot{\mathbf{p}} &= \frac{\partial T}{\partial \boldsymbol{\theta}} - \frac{\partial P}{\partial \boldsymbol{\theta}} + \mathbf{u}, \\ \dot{\boldsymbol{\theta}} &= R^{-1}(\boldsymbol{\theta})\mathbf{p}. \end{aligned} \tag{5}$$

It should be noted that the kinetic energy is described in terms of $\mathbf{p}$ as follows:

$$T = \tfrac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}} R(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \tfrac{1}{2}\mathbf{p}^{\mathrm{T}} R^{-1}(\boldsymbol{\theta})\mathbf{p}. \tag{6}$$

Throughout this paper we treat only the case of so-called point-to-point position control. Hence, we suppose that a desired posture of the manipulator is given, which is described by the position vector $\boldsymbol{\theta}_0 = (\theta_{01}, \theta_{02}, \theta_{03})^{\mathrm{T}}$. The difference between the given target position $\boldsymbol{\theta}_0$ and the present value $\boldsymbol{\theta}(t)$ of the joint-angle coordinates $\boldsymbol{\theta}$ is denoted $\mathbf{e}(t) = \boldsymbol{\theta}(t) - \boldsymbol{\theta}_0$ and called the reference error. For the time being, we assume, second, that not only the first actuator at the base link but also each actuator at corresponding joints is capable of producing any torque at any instant. In other words, it is assumed that the generalized force $\mathbf{u}$ can be regarded as the control input $\mathbf{u}(t)$, which is freely chosen. We assume, third, that the gravity term in equation (5) can be compensated by a feedforward method, which mathematically turns out to be the substitution of

$$\mathbf{u} = \partial P/\partial \boldsymbol{\theta} + \mathbf{v}, \tag{7}$$

into equation (5). This yields

$$\begin{aligned} \dot{\mathbf{p}} &= \frac{\partial T}{\partial \boldsymbol{\theta}} + \mathbf{v}, \\ \dot{\boldsymbol{\theta}} &= R^{-1}(\boldsymbol{\theta})\mathbf{p}. \end{aligned} \tag{8}$$

Here, it should be noted that, though the on-line computation of $\partial P/\partial \boldsymbol{\theta}$ is feasible in the present status of microcomputer technique, it gives rise to some amount of computation errors due to the inaccurate evaluation of physical constants such as l_i, s_i, m_i, and $I_{i\xi}$ or due to the change of payload. Within this section, these kinds of errors are also excluded from equation (8).

Now we introduce an ordinary feedback control scheme, which is described in terms of the following expression:

$$v_i = -a_i\dot{\theta}_i - b_i(\theta_i - \theta_{0i}), \quad i = 1, 2, 3. \tag{9}$$

Then, substitution of (9) into (8) results in the following equation of the closed-loop system:

$$\begin{aligned} \dot{\mathbf{p}} &= \frac{\partial T}{\partial \boldsymbol{\theta}} - AR^{-1}(\boldsymbol{\theta})\mathbf{p} - B(\boldsymbol{\theta} - \boldsymbol{\theta}_0), \\ \dot{\boldsymbol{\theta}} &= R^{-1}(\boldsymbol{\theta})\mathbf{p}, \end{aligned} \tag{10}$$

where A and B are constant diagonal matrices such that

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}.$$

Now it is possible to prove the asymptotic stability of the equilibrium point ($\mathbf{p} = 0$, $\boldsymbol{\theta} = \boldsymbol{\theta}_0$) with the aid of Lyapunov's direct method of stability analysis. To do this, set

$$W = \tfrac{1}{2}\mathbf{p}^{\mathrm{T}}R^{-1}(\boldsymbol{\theta})\mathbf{p} + \tfrac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}}B(\boldsymbol{\theta} - \boldsymbol{\theta}_0) \tag{11}$$

and differentiate this with respect to t. Then, along the solution trajectory of equation (10) we have

$$
\begin{aligned}
\dot{W} &= \mathbf{p}^{\mathrm{T}} R^{-1}(\boldsymbol{\theta})\dot{\mathbf{p}} + (\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}} B\dot{\boldsymbol{\theta}} + \tfrac{1}{2}\mathbf{p}^{\mathrm{T}}\left[\frac{d}{dt}R^{-1}\right]\mathbf{p} \\
&= -\mathbf{p}^{\mathrm{T}} R^{-1} A R^{-1}\mathbf{p} + \mathbf{p}^{\mathrm{T}} R^{-1}\frac{\partial T}{\partial \boldsymbol{\theta}} + \tfrac{1}{2}\mathbf{p}^{\mathrm{T}}\left[\frac{d}{dt}R^{-1}\right]\mathbf{p}.
\end{aligned} \tag{12}
$$

Since $\partial R^{-1}/\partial\theta_i = -R^{-1}(\partial R/\partial\theta_i)R^{-1}$, it is easy to see that

$$
\begin{aligned}
\mathbf{p}^{\mathrm{T}}\left(\frac{d}{dt}R^{-1}\right)\mathbf{p} &= \sum_{i=1}^{3}\mathbf{p}^{\mathrm{T}}\dot{\theta}_i\left(\frac{\partial}{\partial\theta_i}R^{-1}\right)\mathbf{p} \\
&= -\sum_{i=1}^{3}\dot{\theta}_i\mathbf{p}^{\mathrm{T}}R^{-1}\frac{\partial R}{\partial\theta_i}R^{-1}\mathbf{p} \\
&= -\sum_{i=1}^{3}\dot{\theta}_i\dot{\boldsymbol{\theta}}^{\mathrm{T}}\frac{\partial R}{\partial\theta_i}\dot{\boldsymbol{\theta}} \\
&= -\mathbf{p}^{\mathrm{T}}R^{-1}\frac{\partial}{\partial\boldsymbol{\theta}}(\dot{\boldsymbol{\theta}}^{\mathrm{T}}R\dot{\boldsymbol{\theta}}) \\
&= -\mathbf{p}^{T}R^{-1}\frac{\partial T}{\partial\boldsymbol{\theta}}.
\end{aligned} \tag{13}
$$

Substituting this into equation (12), we obtain

$$\dot{W} = -\mathbf{p}^{\mathrm{T}}R^{-1}AR^{-1}\mathbf{p}. \tag{14}$$

If all a_i are chosen to be positive, then it follows from this equation that

$$\dot{W} \leqslant 0. \tag{15}$$

Furthermore, since $R^{-1}(\boldsymbol{\theta})$ is periodic in $\boldsymbol{\theta}$ and positive definite for all $\boldsymbol{\theta}$ due to the positive definiteness of $R(\boldsymbol{\theta})$, it follows from the definition of W described in equation (11) that

$$W > 0 \quad \text{for all} \quad (\mathbf{p}, \boldsymbol{\theta}) \quad \text{except} \quad (\mathbf{0}, \boldsymbol{\theta}_0), \tag{16}$$

provided all b_i are chosen to be positive.

At this stage, it is necessary to recall the well-known theorem of LaSalle [LaSalle, 1960; Yoshizawa, 1966]:

If there exists such a function $W(\mathbf{x})$ defined in a certain domain Ω of the state space of $\mathbf{x} = (\mathbf{p}, \boldsymbol{\theta})$ containing the equilibrium point $\mathbf{x}_0 = (\mathbf{0}, \boldsymbol{\theta}_0)$, then for any initial condition $\mathbf{x}(0) = (\mathbf{p}(0), \boldsymbol{\theta}(0))$ in a neighborhood of $\mathbf{x}_0$ the solution trajectory $(\mathbf{p}(t), \boldsymbol{\theta}(t))$ of equation (10) approaches asymptotically to the maximal invariant set M contained in the set

$$E = \{\mathbf{x} = (\mathbf{p}, \boldsymbol{\theta}) \in \Omega : \dot{W} = 0\}. \tag{17}$$

In our case, according to equation (14), $\dot{W} = 0$ means $\mathbf{p} = \mathbf{0}$. Therefore, it holds along any solution trajectory in E that

$$\dot{\mathbf{p}} = -B(\boldsymbol{\theta} - \boldsymbol{\theta}_0).$$

This in turn implies that M is composed of the single point $\mathbf{x}_0 = (\mathbf{p} = \mathbf{0}, \boldsymbol{\theta} = \boldsymbol{\theta}_0)$. Thus, owing to the theorem of LaSalle, we have proved the asymptotic stability of the equilibrium point $\mathbf{x}_0 = (\mathbf{0}, \boldsymbol{\theta}_0)$. That is, the solution trajectory $\mathbf{x}(t) = (\mathbf{p}(t), \boldsymbol{\theta}(t))$ of the closed-loop system (10) approaches asymptotically the equilibrium state $\mathbf{x}_0 = (\mathbf{0}, \boldsymbol{\theta}_0)$.

2 Stability of PID Feedback Scheme

We begin with this section by discussing an additional term $\mathbf{f}(\dot{\boldsymbol{\theta}}, \boldsymbol{\theta})$, which should be added to equation (8) in the following way:

$$\begin{aligned} \dot{\mathbf{p}} &= \frac{\partial T}{\partial \boldsymbol{\theta}} - \mathbf{f}(\dot{\boldsymbol{\theta}}, \boldsymbol{\theta}) + \mathbf{v}, \\ \dot{\boldsymbol{\theta}} &= R^{-1}(\boldsymbol{\theta})\mathbf{p}. \end{aligned} \tag{18}$$

The term $\mathbf{f}$ is unknown in practical situations, since it may come from frictions at each joint and changes or evaluation errors of m_i or physical constants l_i, s_i, m_i, and $I_{i\xi}$, respectively. Without loss of generality, it is possible to rewrite $\mathbf{f}$ as

$$\mathbf{f}(\dot{\boldsymbol{\theta}}, \boldsymbol{\theta}) = \mathbf{d} + \mathbf{g}(\mathbf{p}, \boldsymbol{\theta} - \boldsymbol{\theta}_0), \tag{19}$$

where $\mathbf{d}$ is a bias term defined as

$$\mathbf{d} = \mathbf{f}(\mathbf{0}, \boldsymbol{\theta}_0), \tag{20}$$

which may be unknown too. With the aid of this expression, it may be reasonable to assume that

$$|\mathbf{g}(\mathbf{p}, \boldsymbol{\theta} - \boldsymbol{\theta}_0)| \leqslant k_1 |\boldsymbol{\theta} - \boldsymbol{\theta}_0| + k_2 |\mathbf{p}| \tag{21}$$

in a certain domain defined by

$$\Omega = \{\mathbf{x} = (\mathbf{p}, \boldsymbol{\theta}) : |\mathbf{p}| < \beta_1, |\boldsymbol{\theta} - \boldsymbol{\theta}_0| < \pi\}. \tag{22}$$

Then equation (18) is rewritten as

$$\begin{aligned} \dot{\mathbf{p}} &= \frac{\partial T}{\partial \boldsymbol{\theta}} - \mathbf{g}(\mathbf{p}, \boldsymbol{\theta} - \boldsymbol{\theta}_0) - \mathbf{d} + \mathbf{v}, \\ \dot{\boldsymbol{\theta}} &= R^{-1}(\boldsymbol{\theta})\mathbf{p}. \end{aligned} \tag{23}$$

For this system, the ordinary control scheme employed in the previous section does not assure the asymptotic stability of the equilibrium point owing to the existence of bias term $\mathbf{d}$. To overcome this, we propose a PID feedback control scheme as defined by the form

$$v_i = -a_i \dot{\theta}_i - b_i(\theta_i - \theta_{0i}) - c_i \int_0^t \{\theta_i(\tau) - \theta_{0i}\}\, d\tau \tag{24}$$

for $i = 1, 2, 3$. Substituting this into equation (23) and introducing new vectors

$$\boldsymbol{\eta}(t) = \int_0^t (\boldsymbol{\theta}(\tau) - \boldsymbol{\theta}_0)\, d\tau, \tag{25}$$

$$\boldsymbol{\xi}(t) = \boldsymbol{\eta}(t) - C^{-1}\mathbf{d}, \tag{26}$$

we obtain the equation of motion for the closed-loop system as follows:

$$\begin{aligned}
\dot{\mathbf{p}} &= \frac{\partial T}{\partial \boldsymbol{\theta}} - \mathbf{g}(\mathbf{p}, \boldsymbol{\theta} - \boldsymbol{\theta}_0) - AR^{-1}\mathbf{p} - B(\boldsymbol{\theta} - \boldsymbol{\theta}_0) - C\boldsymbol{\xi}, \\
\dot{\boldsymbol{\theta}} &= R^{-1}\mathbf{p}, \\
\dot{\boldsymbol{\xi}} &= \boldsymbol{\theta} - \boldsymbol{\theta}_0,
\end{aligned} \tag{27}$$

where $A = \operatorname{diag}(a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, and $C = \operatorname{diag}(c_1, c_2, c_3)$.

Before entering into the discussion on the stability for this system, it is further necessary to introduce the state vector $\mathbf{x} = (\mathbf{p}, \boldsymbol{\theta}, \boldsymbol{\xi})$ and define Ω as a domain in the state space such that

$$\Omega = \{\mathbf{x} = (\mathbf{p}, \boldsymbol{\theta}, \boldsymbol{\xi}) : |\boldsymbol{\theta} - \boldsymbol{\theta}_0| < \pi, |\mathbf{p}| < \beta_1, |\boldsymbol{\xi}| < \beta_2\}, \tag{28}$$

where β_1 and β_2 are appropriate positive constants so that inequality (21) is valid. Now we propose a Lyapunov function of the form

$$W(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^{\mathrm{T}} \begin{bmatrix} R^{-1}(\boldsymbol{\theta}) & \alpha I & 0 \\ \alpha I & B & C \\ 0 & C & \alpha C \end{bmatrix} \mathbf{x}. \tag{29}$$

Here, α is a positive constant that will be determined later, I denotes the 3×3 identity matrix, and B and C are constant diagonal matrices already introduced in equation (27). Let us differentiate W with respect to time and evaluate it along the solution trajectory of equation (27). Then, after a careful manipulation of equations, we finally arrive at the result

$$\begin{aligned}
\dot{W} = {} & \alpha(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}} \frac{\partial T}{\partial \boldsymbol{\theta}} - \mathbf{p}^{\mathrm{T}}(R^{-1}AR^{-1} - \alpha R^{-1})\mathbf{p} \\
& - (\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}}(\alpha B - C)(\boldsymbol{\theta} - \boldsymbol{\theta}_0) \\
& - \alpha(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}} AR^{-1}\mathbf{p} \\
& - \mathbf{p}^{\mathrm{T}} R^{-1}\mathbf{g} - \alpha(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}}\mathbf{g}.
\end{aligned} \tag{30}$$

Next we note that

$$-\alpha(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}} AR^{-1}\mathbf{p} \leqslant \tfrac{1}{2}\alpha(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}} A(\boldsymbol{\theta} - \boldsymbol{\theta}_0) + \frac{\alpha}{2}\mathbf{p}^{\mathrm{T}} R^{-1}AR^{-1}\mathbf{p}$$

and

$$\alpha(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}} \frac{\partial T}{\partial \boldsymbol{\theta}} = \frac{\alpha}{2}\mathbf{p}^{\mathrm{T}} \left[\sum_{i=1}^{3} (\theta_i - \theta_{0i}) \frac{\partial R^{-1}}{\partial \theta_i} \right] \mathbf{p}$$

$$= -\frac{\alpha}{2}\mathbf{p}^{\mathrm{T}}R^{-1}\left[\sum_{i=1}^{3}(\theta_i - \theta_{0i})\frac{\partial R}{\partial \theta_i}\right]R^{-1}\mathbf{p}.$$

Substituting these into equation (30), we obtain

$$\begin{aligned}\dot{W} \leqslant & -(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}}[\alpha B - C - \tfrac{1}{2}\alpha A](\boldsymbol{\theta} - \boldsymbol{\theta}_0) \\ & -\mathbf{p}^{\mathrm{T}}R^{-1}\left[A - \tfrac{1}{2}\alpha A - \alpha R + \frac{\alpha}{2}\sum_{i=1}^{3}(\theta_i - \theta_{0i})\frac{\partial R}{\partial \theta_i}\right]R^{-1}\mathbf{p} \\ & -\mathbf{p}^{\mathrm{T}}R^{-1}\mathbf{g} - \alpha(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}}\mathbf{g}.\end{aligned} \tag{31}$$

In order to evaluate the right side of this equation, we choose A, B, and C as follows:

$$C > 0, \quad A > R_0, \quad B > A + 2\alpha^{-1}C, \tag{32}$$

where R_0 is a positive definite constant matrix such that $R_0 > R(\boldsymbol{\theta})$ for all $\boldsymbol{\theta}$. Then, it follows that

$$\alpha B - C - \tfrac{1}{2}\alpha A > (\alpha/2)A + C > 0. \tag{33}$$

Moreover, it is possible to choose α so small that

$$A - \frac{\alpha}{2}A - \alpha R + \frac{\alpha}{2}\sum_{i=1}^{3}(\theta_i - \theta_{0i})\frac{\partial R}{\partial \theta_i} > \tfrac{1}{2}A \tag{34}$$

for all $\mathbf{x} = (\mathbf{p}, \boldsymbol{\theta}, \boldsymbol{\xi}) \in \Omega$. Substitution of equations (33) and (34) into equation (31) yields

$$\begin{aligned}\dot{W} \leqslant & -(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}}\left(\frac{\alpha}{2}A + C\right)(\boldsymbol{\theta} - \boldsymbol{\theta}_0) - \tfrac{1}{2}\mathbf{p}^{\mathrm{T}}R^{-1}AR^{-1}\mathbf{p} \\ & - \mathbf{p}^{\mathrm{T}}R^{-1}\mathbf{g} - \alpha(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}}\mathbf{g}.\end{aligned} \tag{35}$$

In addition to this, it is easy to see that the matrix proposed in equation (29) becomes positive definite with this choice of A, B, C, and α. Furthermore, the assumption on $\mathbf{g}$ described in equation (21) means the existence of constants $k_3 \sim k_6$ such that

$$\begin{aligned}|\mathbf{p}^{\mathrm{T}}R^{-1}\mathbf{g}| + \alpha|(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}}\mathbf{g}| \leqslant & \; k_3 p^{\mathrm{T}}R^{-1}\mathbf{p} \\ & + \alpha(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}}(k_4 R^{-1} + k_5 I + k_6 R)(\boldsymbol{\theta} - \boldsymbol{\theta}_0).\end{aligned} \tag{36}$$

Hence, it is possible to choose matrices A and C so that

$$\dot{W} \leqslant -\tfrac{1}{4}\mathbf{p}^{\mathrm{T}}R^{-1}AR^{-1}\mathbf{p} - \tfrac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}}C(\boldsymbol{\theta} - \boldsymbol{\theta}_0). \tag{37}$$

Thus it is concluded that for certain constant diagonal matrices A, B, and C

$$\begin{aligned}& W > 0 \text{ for all } \mathbf{x} = (\mathbf{p}, \boldsymbol{\theta}, \boldsymbol{\xi}) \text{ except } \mathbf{x}_0 = (\mathbf{0}, \boldsymbol{\theta}_0, \mathbf{0}), \\ & \dot{W} \leqslant -\tfrac{1}{4}\mathbf{p}^{\mathrm{T}}R^{-1}AR^{-1}\mathbf{p} - \tfrac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}}C(\boldsymbol{\theta} - \boldsymbol{\theta}_0) \leqslant 0.\end{aligned} \tag{38}$$

Finally we observe that $\dot{W} = 0$ implies $\mathbf{p} = \mathbf{0}$ and $\boldsymbol{\theta} - \boldsymbol{\theta}_0$ and hence, under the condition $\dot{W} = 0$, the right side of equation (27) vanishes if and only if $\boldsymbol{\xi} = \mathbf{0}$. This fact shows

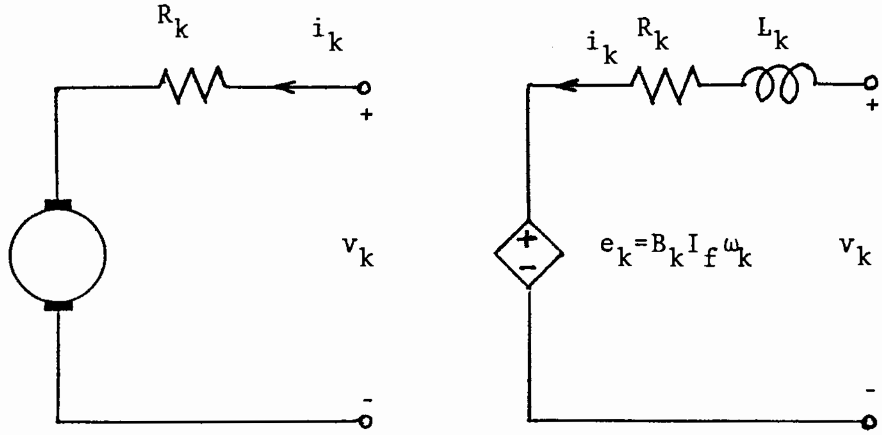

Figure 3
Equivalent armature circuit of armature-controlled dc motor acting on link k.

that the maximal invariant set satisfying $\dot{W} = 0$ consists of the single point $\mathbf{x} = (\mathbf{0}, \boldsymbol{\theta}_0, \mathbf{0})$. Hence, due to the theorem of LaSalle, it holds that

$$\dot{\boldsymbol{\theta}}(t) \to \mathbf{0} \quad \text{and} \quad \boldsymbol{\theta}(t) \to \boldsymbol{\theta}_0 \quad \text{as } t \to \infty$$

as far as the initial state $\mathbf{x}(0) = (\mathbf{p}(0), \boldsymbol{\theta}(0), \boldsymbol{\xi}(0))$ is given in some domain Ω_0 ($\subset \Omega$).

Thus we have proved the asymptotic stability for the PID local feedback control scheme under the presence of friction and some other disturbances.

3 More Detailed Discussion of Robustness and Stability of PID Feedback Control

It was assumed that the input torque $\mathbf{u}$ in equation (5) is freely realized at any instant. In a torque motor or a current-controlled dc servomotor, the input torque is simply proportional to current, which can be regulated by fast feedback with the aid of power electronics in conjunction with fast digital processing of sensory signals. Therefore, there is no need to discuss further the actuator dynamics, that is, the discussion given previously is valid for a manipulator system in which current-controlled motors are used as actuators at joints. However, for a voltage-controlled dc servomotor, it is necessary to modify the previous discussion since the input torque is realized indirectly through the electromechanical coupling in the motor.

Consider the armature circuit of a dc servomotor that may act on the $(k-1)$th joint, as shown in figure 3. With the use of physical variables and constants indicated in figure 3, it follows that

$$L_k \frac{d}{dt} i_k + R_k i_k + B_k I_{fk} \omega_k = v_k, \tag{39}$$

where ω_k, I_{fk}, and B_k denote the angular velocity, field current, and electromechanical constant for motor k, respectively. The last term of the left side in equation (39) is a voltage drop due to the inverse electromotive force. Since in the ordinary use of armature-controlled dc motors the field current I_f is maintained at a constant value, we set a constant K_k to

$$B_k I_{fk} = K_k,$$

which is called the torque constant for motor k. Then, the torque generated by the motor is

$$T_k = K_k i_k. \tag{40}$$

If a gear with reduction ratio r_k is attached at motor k, then we have $\dot{\boldsymbol{\theta}} = r_k \omega_k$, which permits equation (39) to be written

$$\frac{L_k}{R_k}\frac{d}{dt} i_k + i_k = -\frac{K_k}{r_k R_k}\dot{\theta}_k + \frac{1}{R_k} v_k. \tag{41}$$

Now we are in a position to incorporate these servo loops in the manipulator dynamics. First we deals with a specialized case where inductance L_k is negligibley small and thereby can be ignored in equation (41). In that case, the input torque u_k is expressed by

$$\begin{aligned} u_k = T_k = K_k i_k &= -\frac{K_k^2}{r_k R_k}\dot{\theta}_k + \frac{K_k}{R_k} v_k \\ &= \zeta_k \theta_k + \gamma_k v_k, \end{aligned} \tag{42}$$

where

$$\zeta_k = K_k^2 / r_k R_k, \quad \gamma_k = K_k / R_k.$$

Incorporating equation (42) into equation (5) and taking into account the disturbance due to friction etc., we have

$$\begin{aligned} \dot{\mathbf{p}} &= \frac{\partial T}{\partial \boldsymbol{\theta}} - \frac{\partial P}{\partial \boldsymbol{\theta}} - \mathbf{f} - G R^{-1}\mathbf{p} + \Gamma \mathbf{p}, \\ \dot{\boldsymbol{\theta}} &= R^{-1}\mathbf{p}, \end{aligned} \tag{43}$$

where

$$G = \operatorname{diag}(\zeta_1, \zeta_2, \zeta_3), \quad \Gamma = \operatorname{diag}(\gamma_1, \gamma_2, \gamma_3).$$

For this case, it is also possible to apply a PID local feedback scheme in conjunction with a feedforward that compensates the gravity term. That is, we set

$$\begin{aligned} \mathbf{v} = \Gamma^{-1}\frac{\partial P}{\partial \boldsymbol{\theta}} &- \Gamma^{-1} A \dot{\boldsymbol{\theta}} - \Gamma^{-1} B(\boldsymbol{\theta} - \boldsymbol{\theta}_0) \\ &- \Gamma^{-1} C_0^t (\boldsymbol{\theta}(\tau) - \boldsymbol{\theta}_0)\, d\tau, \end{aligned} \tag{44}$$

which, by using the same argument as presented in previous section, gives rise to the following state equation:

$$\begin{aligned}
\dot{\mathbf{p}} &= \frac{\partial T}{\partial \boldsymbol{\theta}} - \mathbf{g}(\mathbf{p}, \boldsymbol{\theta} - \boldsymbol{\theta}_0) - GR^{-1}\mathbf{p} - AR^{-1}\mathbf{p} \\
&\quad - B(\boldsymbol{\theta} - \boldsymbol{\theta}_0) - C\boldsymbol{\xi}, \\
\dot{\boldsymbol{\theta}} &= R^{-1}\mathbf{p}, \\
\dot{\boldsymbol{\xi}} &= \boldsymbol{\theta} - \boldsymbol{\theta}_0.
\end{aligned} \tag{45}$$

Comparing this with equation (27), we observe that addition of the term $-GR^{-1}\mathbf{p}$ to the right side of the first equation in (45) is the only difference between equations (27) and (45). In view of the diagonal and positive definite property of matrix G and the arbitrariness of choice for matrix A, we arrive at the same conclusion as that reached previously.

Next we treat the case that time constants

$$\varepsilon_k = L_k/R_k, \quad i = 1, 2, 3, \tag{46}$$

are relatively small. In this case it is necessary to incorporate the vector $\boldsymbol{\sigma} = (i_1, i_2, i_3)$ in the state vector. It results in

$$\begin{aligned}
\dot{\mathbf{p}} &= \frac{\partial T}{\partial \boldsymbol{\theta}} - \frac{\partial P}{\partial \boldsymbol{\theta}} - \mathbf{f} - K\boldsymbol{\sigma}, \\
\dot{\boldsymbol{\theta}} &= R^{-1}\mathbf{p}, \\
\varepsilon\dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma} &= -K^{-1}GR^{-1}\mathbf{p} + K^{-1}\Gamma\mathbf{v},
\end{aligned} \tag{47}$$

where

$$K = \mathrm{diag}(K_1, K_2, K_3), \quad \varepsilon = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3).$$

This system of equations is of a well-known form called a singular perturbed system (for example, see [Kokotovic, O'Malley, and Sannuti, 1976; Levin and Levinson, 1954]) if each ε_k is sufficiently small. When $\varepsilon = 0$ in (47), equation (43) follows by substituting the third equation of (47) into the first one. On the other hand, it has already been shown that the system described by equation (43) can be stabilized by the control scheme given by equation (44). Hence, in accord with the stability theorem for a singular perturbation system, it is possible to prove the stability of the same PID control scheme as in equation (44) for the system of equation (47). Otherwise an extension of the method of proof given in the appendix to the case of multiple joints could be applicable to this system. However, a precise discussion will be presented elsewhere.

The time constant for typical small-scaled permanent-magnet dc motors ranges from 0.001 to 0.0001 sec. Therefore, it is reasonable to assume that time constants ε_2 and ε_3 for motors at joints 1 and 2, respectively, are sufficiently small. However, for the basement link, it may be necessary to use a more powerful motor with time constant ε_1, which may be still small, ranging from 0.01 to 0.001 sec, but not sufficiently small. In addition, it is unnecessary for the basement link to use the integration feedback because the dynamics of this link are not affected by the gravity and, moreover, the inertia matrix R is independent of θ_1. To steer clear of involvement in such a conflict among joints, we conveniently exclude the dynamic equation of motion for the basement link from the overall equation (5) and analyze them separately. Since T and P in equation (4) are independent of θ_1, the equation of motion for the first link is described by

$$\dot{p}_1 = K_1\sigma_1, \quad \dot{\theta}_1 = r_{11}^{-1}p_1. \tag{48}$$

Stabilization for the system (48) is established by means of an ordinary feedback

$$v_1 = -\gamma_1^{-1}a_1\dot{\theta}_1 - \gamma_1^{-1}b_1(\theta_1 - \theta_{01}), \tag{49}$$

where the input voltage v_1 is related to the equation of armature circuit:

$$\varepsilon_1\dot{\sigma}_1 + \sigma_1 = -K_1^{-1}\zeta_1 r_{11}^{-1}p_1 + K_1^{-1}\gamma_1 v_1. \tag{50}$$

The proof for the asymptotic stability of the desired state $(p_1, \theta_1, \sigma_1) = (0, \theta_{01}, 0)$ will be given in the appendix under appropriate assumptions on r_{11}. The remaining part of the dynamics for links 2 and 3 can be regarded as a singular perturbation system with forcing terms $(\partial r_{11}/\partial\theta_i)\dot{\theta}_1^2$, $i = 2, 3$. Note that in most practical situations these forcing terms are small relative to other principal terms at the whole stage of motion and vanish rapidly with increases in time.

So far we have treated the standard model of robot manipulators with three degrees of freedom as shown in figure 1. We remark, however, that the present discussion is applicable to most kinds of serial-link manipulators with many degrees of freedom as long as the inertia matrix R is positive definite for any point $\boldsymbol{\theta} = (\theta_1, \theta_1, \ldots, \theta_n)$ in the joint-angle coordinates.

If it is necessary to take the shaft flexibility into account, the overall dynamics of motion must be described in a more complex form. However, if it is possible to regard the dynamics of torsional motion for the shaft as a perturbation to the principal equation of joint dynamics, then the stabilization is analyzed by means of a similar singular perturbation technique. This problem will be discussed elsewhere.

4 Stability in the Space of Task-Oriented Coordinates

Instead of the posture control discussed previously, we consider the control problem of maneuvering the manipulator so that only its end effector could reach a given target in space, without consideration of intermediate positions of other joints. In that case, the target point is described usually in terms of outer coordinates of a reference frame fixed in space which are convenient for description of the task.

Let $\mathbf{y} = (y_1, \ldots, y_m)$ be such a task-oriented outer coordinates system (for example, cartesian coordinates) and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ be an inner coordinates system (for example, joint-angle coordinates). Suppose that a transformation from $\boldsymbol{\theta}$ to $\mathbf{y}$ is given by

$$y_i = F_i(\theta_1, \ldots, \theta_n), \quad i = 1, \ldots, m. \tag{51}$$

Throughout this paper, we assume that $n \geqslant m$. The Jacobian matrix is defined as

$$J = \left(\frac{\partial F_i}{\partial \theta_j}\right). \tag{52}$$

Then, obviously it holds that

$$\dot{\mathbf{y}} = J\dot{\boldsymbol{\theta}}. \tag{53}$$

We now propose a modified PID feedback control scheme, defined as

$$\mathbf{v} = -A\dot{\boldsymbol{\theta}} - J^{\mathrm{T}}[B(\mathbf{y} - \mathbf{y}_0) + C\int_0^t (\mathbf{y}(\tau) - \mathbf{y}_0)\,d\tau, \tag{54}$$

where $\mathbf{y}_0$ is a given target point, A is an $n \times n$ constant diagonal matrix, and B and C are $m \times m$ constant diagonal matrices. The first term of the right side in equation (54) means a local velocity feedback, but the second and third terms cannot be localized due to involvement of the multiplication of the Jacobian matrix, regardless of the diagonal property of matrices B and C. A feedback scheme similar to that of equation (54) was first proposed, except the term of integration feedback, in [Takegai and Arimoto, 1981].

After compensating for the gravity term, we apply $\mathbf{y}$ to the system (8). This means the substitution of (54) into (8), by which we obtain the closed-loop system

$$\begin{aligned}
\dot{\mathbf{p}} &= \frac{\partial T}{\partial \boldsymbol{\theta}} - AR^{-1}\mathbf{p} - J^{\mathrm{T}}[B(\mathbf{y} - \mathbf{y}_0) + C\boldsymbol{\xi}],\\
\dot{\mathbf{y}} &= JR^{-1}\mathbf{p},\\
\dot{\boldsymbol{\xi}} &= \mathbf{y} - \mathbf{y}_0.
\end{aligned} \tag{55}$$

Note that the term $\partial T/\partial \boldsymbol{\theta}$ can not be replaced with any function of $\mathbf{y}$ if $n > m$. Despite this, let us introduce a Lyapunov function

$$W = \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\begin{bmatrix} R^{-1} & \alpha J^{\mathrm{T}}(JJ^{\mathrm{T}})^{-1} & 0 \\ \alpha(JJ^{\mathrm{T}})^{-1}J & B & C \\ 0 & C & \alpha C \end{bmatrix}\mathbf{x}, \tag{56}$$

where

$$\mathbf{x} = (\mathbf{p}, \mathbf{y} - \mathbf{y}_0, \boldsymbol{\xi})^{\mathrm{T}}. \tag{57}$$

In what follows, we assume that JJ^{T} is nonsingular while maneuvering the manipulator. Then, by applying the same argument as in section 2, we obtain

$$\begin{aligned}
\dot{W} = &-\mathbf{p}^{\mathrm{T}}R^{-1}(A - \alpha R)R^{-1}\mathbf{p} - \mathbf{y}^{\mathrm{T}}(\alpha B - C)\mathbf{y}\\
&-\alpha\mathbf{y}^{\mathrm{T}}(JJ^{\mathrm{T}})^{-1}JAR^{-1}\mathbf{p} - \mathbf{y}^{\mathrm{T}}(JJ^{\mathrm{T}})^{-1}J\frac{\partial T}{\partial \boldsymbol{\theta}}\\
&+\alpha\mathbf{p}^{\mathrm{T}}\left[\frac{d}{dt}J^{\mathrm{T}}(JJ^{\mathrm{T}})^{-1}\right]\mathbf{y}.
\end{aligned} \tag{58}$$

Bearing in mind that $\partial T/\partial \boldsymbol{\theta}$ is of the same order as $\mathbf{p}^{\mathrm{T}}R^{-1}\mathbf{p}$, JJ^{T} is positive definite by the assumption, and each entry of the matrix $dJ^{\mathrm{T}}(JJ^{\mathrm{T}})^{-1}/dt$ can be represented by a linear combination of components of vector $\mathbf{p}$ whose coefficients are sinusoidal functions of $\boldsymbol{\theta}$, we can rewrite equation (58) by taking α small and diagonal elements of matrices A and B large, so that

$$\dot{W} \leqslant -\gamma_1 \mathbf{p}^{\mathrm{T}} R^{-1} A R^{-1} \mathbf{p} - \gamma_2 \mathbf{y}^{\mathrm{T}} B \mathbf{y}$$

with appropriate constants γ_1 and γ_2 in a certain domain

$$\Omega = \{(\mathbf{p}, \mathbf{y}, \boldsymbol{\xi}) : |\mathbf{p}| < \beta_1, |\mathbf{y} - \mathbf{y}_0| < \beta_2, |\boldsymbol{\xi}| < \beta_3\}.$$

Hence, in accordance with the theorem of LaSalle, the asymptotic stability of the target point has been established.

For the case that additional terms of frictions and bias disturbances exist in the equation of motion, it is possible to apply an argument similar to that presented in section 3.

Appendix

Without loss of generality, we consider the problem concerning the asymptotic stability of the equilibrium point $\mathbf{x}_0 = (p = 0, \theta = 0, \sigma = 0)$ for the system

$$\begin{aligned} \dot{p} &= -(\zeta + a) r^{-1} p - b\theta - K\varepsilon\dot{\sigma}, \\ \dot{\theta} &= r^{-1} p, \\ \varepsilon\dot{\sigma} &= -\sigma - K^{-1}(\zeta + a) r^{-1} p - K^{-1} b\theta, \end{aligned} \tag{A.1}$$

where a, b, and c are constants to be chosen arbitrarily, K and ζ are fixed positive constants, ε is sufficiently small, and r is a periodic function of joint-angles satisfying

$$r = r_0(1 + \phi(\boldsymbol{\theta})), \quad |\phi(\boldsymbol{\theta})| < \delta. \tag{A.2}$$

Actually δ is relatively small since the inertia moment I_{1z} of the basement link is large in comparison with other constants appearing in r_{11}.

Here, we regard $\phi(\boldsymbol{\theta})$ as a given time-varying function since $\phi(\boldsymbol{\theta})$ is independent of θ_1. Set a Lyapunov function

$$W = \tfrac{1}{2}[r^{-1}p^2 + (b + \beta(\zeta + a))\theta^2 + \alpha\varepsilon^2\sigma^2] + \beta p\theta + \varepsilon K\sigma(\beta\theta + r^{-1}p), \tag{A.3}$$

where constants α and β will be determined later. By differentiating this with respect to t, we obtain

$$\begin{aligned} \dot{W} = &-p^2[r^{-2}(\zeta + a) - \beta r^{-1}] - \theta^2[\beta b] - \varepsilon\sigma^2[\alpha - K^2 r^{-1}] \\ &+ \varepsilon\sigma p[-\alpha(\zeta + a)K^{-1}r^{-1} + K\beta r^{-1}] - \varepsilon\sigma\theta[\alpha b K^{-1}] \\ &+ \tfrac{1}{2}p^2\left(\frac{d}{dt}r^{-1}\right) + \varepsilon\sigma p K\left(\frac{d}{dt}r^{-1}\right). \end{aligned} \tag{A.4}$$

Since

$$\begin{aligned} \varepsilon\sigma p &\leqslant \frac{\sqrt{\varepsilon}}{2}(p^2 + \varepsilon\sigma^2), \\ \varepsilon\sigma\theta &\leqslant \frac{\sqrt{\varepsilon}}{2}(\theta^2 + \varepsilon\sigma^2), \end{aligned} \tag{A.5}$$

equation (A.4) can be rewritten as

$$\begin{aligned}\dot W \leqslant{} & -p^2\left[r^{-2}(\zeta + a) - \beta r^{-1} + O(\sqrt{\varepsilon}) + \frac{r^{-2}}{2}\dot r\right] \\ & - \theta^2\left[\beta b - \frac{\sqrt{\varepsilon}}{2}\alpha b K^{-1} + O(\sqrt{\varepsilon})\right] - \varepsilon\sigma^2[\alpha - K^2 r^{-1} + O(\sqrt{\varepsilon})].\end{aligned} \tag{A.6}$$

If we choosen α and β as

$$\alpha = 4K^2/r_0(1-\delta), \quad \beta = \sqrt{\varepsilon}\alpha K^{-1}, \tag{A.7}$$

then we have

$$\dot W \leqslant -p^2\left[r^{-2}(\zeta + a) + O(\sqrt{\varepsilon}) + \frac{r^{-2}}{2}\dot r\right] - \theta^2\left[\frac{\beta b}{2} + O(\sqrt{\varepsilon})\right] - \varepsilon\sigma^2[\tfrac{3}{4}\alpha + O(\sqrt{\varepsilon})]. \tag{A.8}$$

On the other hand, it holds that, with the choice of α and β as in (A.7),

$$\begin{aligned}\varepsilon K\sigma(\beta\theta + r^{-1}p) &= r^{-1}\left(\frac{p}{\sqrt{2}}\sqrt{2}\varepsilon K\sigma\right) + \beta K(\varepsilon\sigma\theta) \\ &\leqslant \tfrac{1}{4}r^{-1}p^2 + K^2 r^{-1}\varepsilon^2\sigma^2 + (\beta K/2)(\varepsilon^2\sigma^2 + \theta^2) \\ &\leqslant \tfrac{1}{4}r^{-1}p^2 + \left(\frac{1}{4} + \frac{\sqrt{\varepsilon}}{2}\right)\alpha\varepsilon^2\sigma^2 + (\alpha/2)\sqrt{\varepsilon}\theta^2\end{aligned}$$

and

$$\beta p\theta \leqslant (\sqrt{\varepsilon}\alpha K^{-1}/2)(p^2 + \theta^2).$$

Substituting these into equation (A.3), we have

$$W \geqslant \tfrac{1}{4}[r^{-1}p^2 + (2b + O(\sqrt{\varepsilon}))^2 + (\alpha + O(\sqrt{\varepsilon}))\varepsilon^2\sigma^2], \tag{A.9}$$

which shows that W is a positive definite function. Moreover, if a is chosen so that

$$a + \zeta > \max_t |\dot r|, \tag{A.10}$$

then $\dot W$ becomes a negative definite function. Thus, we conclude that the equilibrium point $\mathbf{x} = (0,0,0)$ is asymptotically stable if condition (A.10) is satisfied while maneuvering the manipulator.

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