Lyapunov Stability

Consider a non-linear autonomous system

$$\dot{\mathbf{x}} = f(\mathbf{x}) \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ states}$$

if $f(0) = \underline{0}$

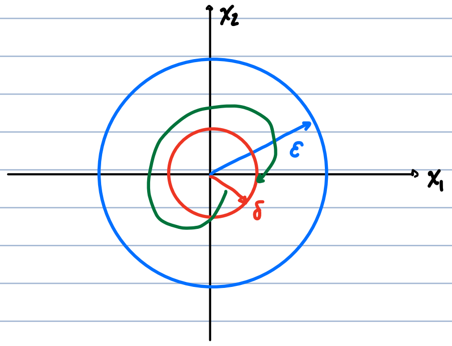
then the origin $\mathbf{x} = 0$ is said to be an equilibrium

IFF = if & only if

Lyapunov Stability

- The equilibrium $\mathbf{x} = 0$ is stable IFF for arbitrary $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\text{If } \|\mathbf{x}(t_0)\| < \delta \quad \text{then } \|\mathbf{x}(t)\| < \varepsilon \text{ for all } t > t_0$$



$\|\mathbf{x}\|$: norm of $\mathbf{x}$ & scalar distance of origin

- The equilibrium is asymptotically stable IFF there exists a δ such that $\|\mathbf{x}(t_0)\| < \delta$ then

$$\|\mathbf{x}(t)\| \to 0 \quad \text{as } t \to \infty$$

Definition: A scalar function $V(\mathbf{x})$ is said to be positive definite IFF $V(\mathbf{x}) > 0$ for all $\mathbf{x}$ other than $\mathbf{x} = 0$, where $V(\underline{0}) = 0$

Ex:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}, \qquad P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow V(\mathbf{x}) = 2x_1^2 + x_2^2$$

← For this to be positive definite, all values other than $\mathbf{x} = 0$ is positive

it is positive definite

Ex:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}, \qquad P = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$V(x) = 2x_1^2$$

x_2 can be anything, & $V(\mathbf{x})$ can be 0

∴ positive semi-definite

LYAPUNOV STABILITY (A.K.A. LYAPUNOV'S DIRECT METHOD)

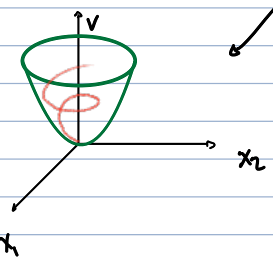
THE EQUILIBRIUM $X(t) = 0$ IS STABLE IF THERE EXISTS A SCALAR FUNCTION $V(X)$ WHICH IS CONTINUOUSLY DIFFERENTIABLE (1^{st} ORDER PDE'S CONTINUOUS) SUCH THAT:

1. $V(X)$ IS POSITIVE DEFINITE
2. $\frac{dV}{dt}$ IS NEGATIVE SEMI-DEFINITE

THEN IT IS STABLE

IF 3. $\frac{dV}{dt}$ IS NEGATIVE DEFINITE, THEN IT IS ASYMPTOTICALLY STABLE

THINK OF V AS AN ENERGY-LIKE FUNCTION OF STATES. IF V IS POSITIVE, BUT ALWAYS DECREASING, SYSTEM IS STABLE



- IF V = ENERGY, IT FUNNELS ENERGY TO ZERO

EX $I\ddot{\theta} + b\dot{\theta} + k\theta = 0$ SPRING MASS DAMPER

PUT IN S.S. FORM (STATE SPACE)
- WE NEED 2 STATES (2^{ND} ORDER)

$$x_1 = \theta \qquad x_2 = \dot{\theta} \quad \longrightarrow \quad X = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

STATE EQUATIONS

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{I}(-bx_2 - kx_1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{I} & -\frac{b}{I} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

IS ORIGIN $(\theta = 0, \dot{\theta} = 0)$ STABLE BY LYAPUNOV?

$$V(X) = \frac{1}{2} X^T \begin{bmatrix} k & 0 \\ 0 & I \end{bmatrix} X$$

THAT HE CHOSE (WE'D THINK UP AN EQUATION THAT MADE THIS WORK)

$$= \underbrace{\frac{1}{2} k x_1^2}_{\text{P.E. OF SYSTEM}} + \underbrace{\frac{1}{2} I x_2^2}_{\text{K.E. OF SYSTEM}}$$

$V(X)$ IS POSITIVE DEFINITE ✓

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = kx_1\dot{x}_1 + Ix_2\dot{x}_2$$

$x_1 = x_2$

$x_2 = \frac{-bx_2 - kx_1}{I}$

PLUG IN STATE EQNS

$$= kx_1x_2 + I_2 x_2\left(\frac{-bx_2 - kx_1}{I}\right) = -bx_2^2$$

ONLY NEGATIVE SEMI-DEFINITE

$\Rightarrow$ OUR SYSTEM IS STABLE BUT NOT ASYMPTOTICALLY STABLE

Corollary of La Salle's Invariant Set Theorem

If
1. $V(x)$ is pos. def.
2. $\frac{dV}{dt}$ is neg. semi-def

$R = \{x \mid \dot{V}(x) = 0\}$

SET OF ALL TRAJECTORIES WHERE $\dot{V}(x) = 0$

contains no trajectory other than $x = 0$

$\therefore$ Equilibrium is asymptotically stable

For earlier ex.

$\dot{V} = 0$ IMPLIES $x_2 = 0$

If we plug $x_2 = 0$ into state equations...

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0\\ -\frac{kx_1}{I_1}\end{bmatrix}$$

If $\dot{x}_1 \neq 0 \Rightarrow \dot{x}_2$ will not be zero

$\Rightarrow$ x_2 may be momentarily equal to zero w/non-zero x_1.

If $x_1 \neq 0$ too,
x_2 will not stay equal to 0

The only way for $\dot{V}(x) = 0$, x_1 & x_2 must be zero.

$\Rightarrow$ Our origin is asymptotically stable by La Salle's Corollary

NONLINEAR EXAMPLE

(1.) $I\ddot{\theta} + b\dot{\theta} + \underline{K\sin\theta} = \tau$ FROM SINGLE DOF ROBOT BEFORE LINEARIZATION

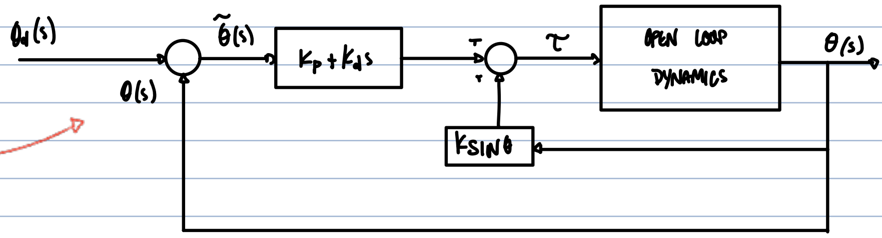
NON-LINE GRAVITY

OPEN LOOP DYNAMICS

WE'LL DEFINE A NEW CONTROL LAW:

(2.) $\tau = (K_p + K_d s)(\theta_d(s) - \theta(s)) + K\sin\theta$

PD CONTROL w/ GRAVITY COMPENSATION



WILL THIS CONTROLLER BE STABLE?

· USE LYAPUNOV TO PROVE STABILITY (WE WANT IT TO BE NON-LINEAR)

WE SKIPPED MATRIX MULTIPLICATION (IT IS IN TEXTBOOK)

CONSIDER LYAPUNOV CANDIDATE $V(\tilde{\theta}, \dot{\theta}) = \underbrace{\frac{1}{2} K_p \tilde{\theta}^2}_{\text{ARTIFICIAL POTENTIAL ERROR}} + \underbrace{\frac{1}{2} I \dot{\theta}^2}_{\text{REAL KINETIC ENERGY}}$

$X = \begin{bmatrix} \tilde{\theta} \\ \dot{\theta} \end{bmatrix}$

ERROR IS ONE OF OUR STATES
· AS LONG AS $\tilde{\theta} \to 0$, THEN WE CAN HAVE θ BE NON-ZERO
· IF $\tilde{\theta} = 0$ & $\dot{\theta} = 0$, WE'RE STABLE ◡̈

1. THIS IS POSITIVE DEFINITE ✓

2. IS $\frac{dV}{dt}$ NEGATIVE?

IS CLOSED LOOP SYS STABLE?

∴ TAKE CONTROL LAW + PLUG INTO CLOSED LOOP DYNAMICS

(1.) = (2.)

$\therefore I\ddot{\theta} + b\dot{\theta} + \cancel{K\sin\theta} = K_p\tilde{\theta} + K_D\dot{\tilde{\theta}} + \cancel{K\sin\theta}$

ASSUME $\dot{\theta}_d = 0 \Rightarrow \dot{\tilde{\theta}} = -\dot{\theta}$
ALLOWS US TO COMBINE $\dot{\tilde{\theta}}$ & $\dot{\theta}$

$\ddot{\theta} = \frac{1}{I}\left(-(K_D + b)\dot{\theta} + K_p\tilde{\theta})\right)$

$\frac{dV}{dt} = \frac{\partial V}{\partial \tilde{\theta}} + \frac{\partial V}{\partial \dot{\theta}}$

$= K_p \tilde{\theta}\dot{\tilde{\theta}} + I\dot{\theta}\ddot{\theta}$

$\dot{\tilde{\theta}} = -\dot{\theta}$

PLUG IN $\ddot{\theta}$ FORMULA

$= -K_p\tilde{\theta}\dot{\theta} + \dot{\theta}\left[-(K_D + b)\dot{\theta} + K_p\tilde{\theta}\right]$

$$= -(K_d + b)\dot{\theta}^2$$ ← NEGATIVE SEMI-DEFINITE

IF $\dot{V} = 0$, THEN $\dot{\theta} = 0$ $\Rightarrow$ $\ddot{\theta} = \frac{K_p \tilde{\theta}}{I}$

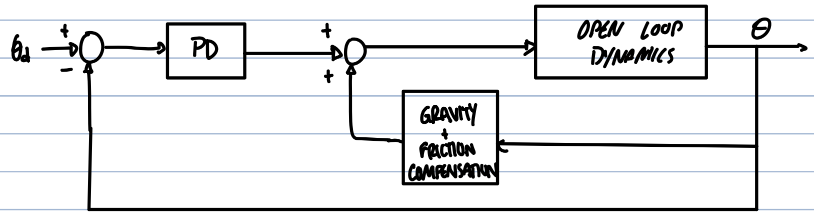
UNLESS $\tilde{\theta}$ IS ALSO 0, THEN SYSTEM WILL ACCELERATE AND $\dot{V}$ IS MOMENTARILY ZERO

$\Rightarrow$ THE SYSTEM IS ASYMPTOTICALLY STABLE BY LA SALLE'S COROLLARY

- THE ORIGIN IS $\tilde{\theta}, \dot{\theta}$ ($\theta_{error} = 0$, ZERO VELOCITY)

$\Rightarrow$ WITH A PD CONTROLLER, YOU CAN CONTROL FOR A DESIRED θ

CAN WE ALSO COMPENSATE FOR COULOMB FRICTION?



$\rightsquigarrow$ OPEN LOOP DYNAMICS IN THE FORM:

$$I\ddot{\theta} + \underbrace{f(\theta, \dot{\theta})}_{\text{COMBINES NON-LINEAR FRICTION + GRAVITY}} = \tau$$

$$\rightsquigarrow \tau = K_p\tilde{\theta} + K_d\dot{\tilde{\theta}} + \underbrace{\hat{f}(\theta, \tilde{\theta})}_{\text{MODELLED FRICTION + GRAVITY}}$$

IF OUR MODEL ISN'T PERFECT, WE CAN'T PROVE ASYMPTOTIC STABILITY