Chapter 3 Spacial Transformations + Displacements

3D rotations <u>don't</u> commute like 2D transformations

3.1

- 3 vectors in any basis $\mathbb{R}^3$ space

3.2 Spatial Rotation Matricies

$$\begin{bmatrix} {}^0\underset{\sim}{x}_1 & {}^0\underset{\sim}{y}_1 & {}^0\underset{\sim}{z}_1 \end{bmatrix} = \begin{bmatrix} {}^0\underset{\sim}{x}_0 & {}^0\underset{\sim}{y}_0 & {}^0\underset{\sim}{z}_0 \end{bmatrix} \begin{bmatrix} {}^0x_{11} & {}^0y_{11} & {}^0z_{11} \\ {}^0x_{12} & {}^0y_{12} & {}^0z_{12} \\ {}^0x_{13} & {}^0y_{13} & {}^0z_{13} \end{bmatrix}$$

$${}^0\underset{\sim}{R}_1 =$$

Rotation about principal axis found in class notes

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \qquad \underset{\sim}{R}_y = \begin{bmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{bmatrix}$$

Inverse of rotation matrix

$$({}^0\underset{\sim}{R}_1)^{-1} = {}^1\underset{\sim}{R}_0 = ({}^0R_1)^T$$

$${}^0\underset{\sim}{T}_1 = \begin{bmatrix} {}^0\underset{\sim}{R}_1 & {}^0\underset{\sim}{d}_{01} \\ \underset{\sim}{0}^T & 1 \end{bmatrix}$$

inverse

$$({}^0\underset{\sim}{T}_{01})^{-1} \neq ({}^0T_{01})^T$$

$$({}^0\underset{\sim}{T}_1)^{-1} = {}^1\underset{\sim}{T}_0$$

3.5 Angle - Axis Derivation

Skew matrix

↑ does a cross-product on two vectors

$$\underset{\sim}{S}(\underline{k})\,\underline{v} = \underline{k} \times \underline{v} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

⟶ rotation about an arbitrary vector $\underline{k}$ by θ

$$\boxed{\underset{\sim}{R}_k = \underset{\sim}{I}\, C_\theta + \underset{\sim}{S}(\underline{k})\, S_\theta + \underline{k}\,\underline{k}^T (1 - C_\theta)}$$

HUGE DERIVATION

3.6 → 3.7 CH's

EULER ∡'s

· MOST COMMON, NOT THE BEST

$\underset{\sim}{R}_{zyz}(\phi, \theta, \psi)$

ZYZ
- $\phi = z_1$ EULER ∡
- $\theta = y$ EULER ∡
- $\psi = z_2$ EULER ∡

↑ DO NOT INTEGRATE (IT DOESN'T WORK, BECAUSE AXES ARE NOT ORTHOGONAL)

· ∡'s DEGENERATE, MAKING ϕ & ψ IMPOSSIBLE TO SOLVE UNIQUELY

· ⇒ USE RPY ∡'s WHEN POSSIBLE

OR

ROLL, PITCH, YAW ∡'s

$\underset{\sim}{R}_{xyz}(\phi, \theta, \psi)$ ← ROTATION ABOUT FIXED AXES

- ϕ = ROLL (x)
- θ = PITCH (y)
- ψ = YAW (z)

· USEFUL FOR DIFFERENTIAL ROTATIONS

· THESE DO COMMUTE

3.8 RODRIGUES VECTOR

· SCALED VERSION OF $\underline{k}$ VECTOR (ARBITRARY ∡ ROTATION)

· 3 COMPONENT

$$\rho = \underline{k} \tan\left(\frac{\theta}{2}\right)$$

$\|\underline{k}\| = 1$ ← CONSTRAINT ON $\underline{k}$

$$\tan\left(\frac{\theta}{2}\right) = \|\rho\|$$

WHEN $\theta = \pi$ RODRIGUEZ IS NOT USEABLE

3.9 EULER PARAMETERS

· MAKING RODRIGUES VECTOR 4 PARAMETERS (GETS AROUND π PROBLEM)

$$\left[\cos\left(\frac{\theta}{2}\right),\ k_1 \sin\left(\frac{\theta}{2}\right),\ k_2 \sin\left(\frac{\theta}{2}\right),\ k_3 \sin\left(\frac{\theta}{2}\right)\right]$$

CONSTRAINT EQN

$$1 = \cos^2\left(\frac{\theta}{2}\right) + \underline{k}^2 \sin^2\left(\frac{\theta}{2}\right)$$

3.10 QUATERNIONS

· HAMILTON, PERFECT FOR ROTATION MATRICIES

$$\underline{q} = q_0 + \underline{q} \qquad q_0 = \cos\left(\frac{\theta}{2}\right) \qquad \underline{q} = \underline{k} \sin\left(\frac{\theta}{2}\right)$$

$\underline{q}$ = 4x1 QUATERNION VECTOR

SCALAR

VECTOR

q = VECTOR $\quad \underline{q}$ = FULL QUATERNION $\rightsquigarrow \underline{q} = \begin{bmatrix} q_0 \\ q \end{bmatrix}$

$\underline{q}$ = QUATERNION

$$\underbrace{\underset{\sim}{q}\cdot\underset{\sim}{q} = q_0^2 + \underline{q}\cdot\underline{q} = 1}_{\text{CONSTANT FOR QUATERNION ROTATIONS}} \longleftarrow \text{UNIT QUATERNIONS}$$

- IF $q_0 = 0 \rightsquigarrow \underset{\sim}{q} = \underline{q}$ IS A VECTOR
- IF $\underline{q} = 0 \rightsquigarrow \underset{\sim}{q} = q_0$ IS A SCALAR

EXAMPLE

LET $\underset{\sim}{q}_x$ IS QUATERNION WHERE: $k = \underline{x}_0 \qquad \theta = \frac{\pi}{2}$

$$\underset{\sim}{q}_x = \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\,\underline{x}_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\underline{x}_0$$

$\underset{\sim}{q}_z$ HAS $k = \underline{z}_0 \quad \theta = \frac{\pi}{2}$

$$\underset{\sim}{q}_z = \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\,\underline{z}_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\underline{z}_0$$

COMPOSITION OF QUATERNIONS

- WRITTEN AS A QUATERNIONS JUXTAPOSED

$$\underset{\sim}{p}\,\underset{\sim}{q} = (p_0 + \underline{p})(q_0 + \underline{q})$$

$$= p_0 q_0 + p_0\underline{q} + q_0\underline{p} + \underline{p}\,\underline{q}$$

$$= p_0 q_0 + p_0\underline{q} + q_0\underline{p} + \underline{p}\times\underline{q} - \underline{p}\cdot\underline{q}$$

$\underline{p}\,\underline{q} = \underline{p}\times\underline{q} - \underline{p}\cdot\underline{q}$ ⟵ DEFINITION OF TWO JUXTAPOSED VECTORS

$$\Rightarrow \underset{\sim}{p}\,\underset{\sim}{q} = (p_0 q_0 - \underline{p}\cdot\underline{q}) + (p_0\underline{q} + q_0\underline{p} + \underline{p}\times\underline{q})$$

QUATERNION COMPOSITIONS

$\underset{\sim}{v}$ = ZERO SCALAR PART QUATERNION

$$\rightsquigarrow \underset{\sim}{v} = 0 + \underline{v}$$

$$\Rightarrow \underset{\sim}{v}\,\underset{\sim}{q}^* = (0 + \underline{v})(q_0 - \underline{q})$$

$$= \underline{v}\cdot\underline{q} + (q_0\underline{v} - \underline{v}\times\underline{q})$$

$$\underset{\sim}{q}\,\underset{\sim}{v}\,\underset{\sim}{q}^* = (q_0 + \underline{q})(\underline{v}\cdot\underline{q} + (q_0\underline{v} - \underline{v}\times\underline{q}))$$

$$= (q_0(\underline{v}\cdot\underline{q}) - \underline{q}\cdot(q_0\underline{v} - \underline{v}\times\underline{q})) + (q_0(q_0\underline{v} - \underline{v}\times\underline{q}) + (\underline{v}\cdot\underline{q})\underline{q} + \underline{q}\times(q_0\underline{v} - \underline{v}\times\underline{q}))$$

$$= q_0(\underline{v}\cdot\underline{q}) - q_0(\underline{q}\cdot\underline{v}) + q_0^2\underline{v} - q_0\underline{v}\times\underline{q} + (\underline{v}\cdot\underline{q})\underline{q} + q_0\underline{q}\times\underline{v} - \underline{q}\times(\underline{v}\times\underline{q})$$

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