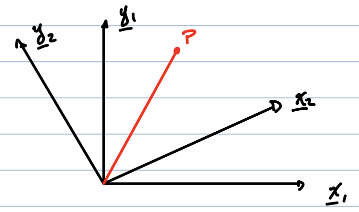
Lecture 4 Rotation Matrices

Rotation Matrices

Two coincident coordinate systems



$\{1\}$, $\{2\}$ ← coordinate systems

$$\left.\begin{aligned}\underline{p} &= P - O_1\\ &= P - O_2\end{aligned}\right\}\ \text{can be expressed either way}$$

Suppose we know ${}^2\underline{p}$

· How do we write ${}^1\underline{p}$?

$$\underline{p} = {}^1p_1\,\underline{x}_1 + {}^1p_2\,\underline{y}_1$$
$$= {}^2p_1\,\underline{x}_2 + {}^2p_2\,\underline{y}_2$$

$${}^2\underline{p} = {}^2p_1\,\underline{x}_1 + {}^2p_2\,\underline{y}_1$$
$$= \begin{bmatrix} {}^2p_1 \\ {}^2p_2 \end{bmatrix} \leftarrow \text{coordinates of } \{2\}$$

What is ${}^1\underline{p}$?

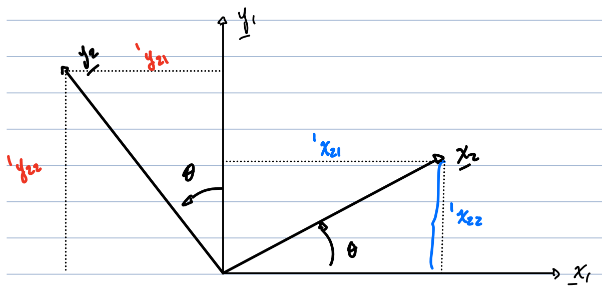
$${}^1\underline{p} = {}^2p_1\,{}^1\underline{x}_2 + {}^2p_2\,{}^1\underline{x}_2$$

$$= \underbrace{\begin{bmatrix} {}^1\underline{x}_2 & {}^1\underline{y}_2 \end{bmatrix}}_{2\times2 \text{ matrix}} \begin{bmatrix} {}^2p_1 \\ {}^2\underline{p}_1 \end{bmatrix}$$

Block form

$$= {}^1\underset{\sim}{R}_2\,{}^2\underline{p}$$

How axes $\{2\}$ are seen in another coordinate system



If we know θ, we know how to express $\underset{\sim}{R}$

$${}^1\underline{x}_2 = {}^1x_{21}\,{}^1\underline{x}_1 + {}^1x_{22}\,{}^1\underline{y}_1$$

↑ $\underline{x}_2$ in terms of $\{1\}$

$${}^1\underline{y}_2 = {}^1y_{21}\,{}^1\underline{x}_1 + {}^1y_{22}\,{}^1\underline{y}_1$$

$${}^1\underline{x}_2 = \begin{bmatrix} {}^1x_{21} \\ {}^1x_{22} \end{bmatrix} = \begin{bmatrix} C_\theta \\ S_\theta \end{bmatrix}$$

$${}^1\underline{y}_2 = \begin{bmatrix} {}^1y_{21} \\ {}^1y_{22} \end{bmatrix} = \begin{bmatrix} -S_\theta \\ C_\theta \end{bmatrix}$$

$$\Rightarrow {}^1R_2 = \left[{}^1x_2 \;\; {}^1y_2\right] = \begin{bmatrix} {}^1x_{21} & {}^1y_{21} \\ {}^1x_{22} & {}^1y_{22} \end{bmatrix} = \begin{bmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{bmatrix}$$

EXAMPLE $\quad R(\theta) = R(60^\circ) = R\left(\frac{\pi}{3}\right)$

$$R\left(\frac{\pi}{3}\right) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

SUPPOSE $\quad {}^2p = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

FIND $\quad {}^1p = {}^1R_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} 1 - \frac{\sqrt{3}}{2} \\ \sqrt{3} + \frac{3}{2} \end{bmatrix}}_{2\times 1}$

$$\boxed{{}^1p = {}^1R_2 \, {}^2p}$$

NOTATION

$${}^1R_2 = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \{r_{ij}\}$$

ROTATION MATRIX

WHAT IS θ?

$$\left.\begin{array}{l} C_\theta = r_{11} \\ S_\theta = r_{21} \end{array}\right\} \Rightarrow \theta = \tan^{-1}(r_{21}, r_{11})$$

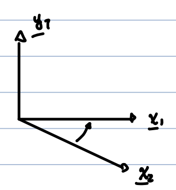
IF WE DIVIDE THESE, WE LOSE THE SIGNS

4 QUADRANT ARCTANGENT FUNCTION

EXAMPLE:

$${}^1R_2 = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

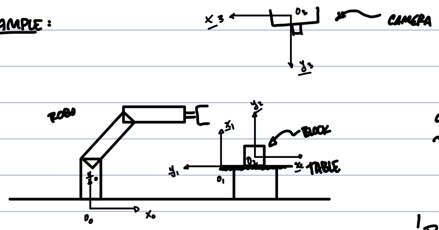
WHAT IS θ?



$$\left.\begin{array}{l} S_\theta = \frac{-1}{2} \\ C_\theta = \frac{\sqrt{3}}{2} \end{array}\right. \Rightarrow \theta = \frac{-\pi}{6}$$

4 QUADRANT ARCTANGENT FUNCTION

EXAMPLE :



$${}^0\underset{\sim}{R}_1 = \left[{}^0\underline{x}_1 \quad {}^0\underline{y}_1\right] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \underset{\sim}{R}\left(\frac{\pi}{2}\right)$$

THESE ARE ALL UNIT VECTORS

=> MAGNITUDES = 1

$${}^1\underset{\sim}{R}_2 = \left[{}^1\underline{x}_2 \quad {}^1\underline{y}_2\right] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \underset{\sim}{R}\left(\frac{-\pi}{2}\right)$$

$${}^2\underset{\sim}{R}_3 = \left[{}^2\underline{x}_3 \quad {}^2\underline{y}_3\right] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \underset{\sim}{R}(\pi)$$

$${}^0\underset{\sim}{R}_3 = \left[{}^0\underline{x}_3 \quad {}^0\underline{y}_3\right] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \underset{\sim}{R}(\pi)$$

TRANSFORMING COORDINATES

$${}^1\underline{p} = {}^1\underset{\sim}{R}_2 \, {}^2\underline{p}$$

HENCE $\quad {}^2\underline{p} = \left({}^1\underset{\sim}{R}_2\right)^{-1} {}^1\underline{p}$

$${}^2\underline{p} = {}^1p_1 \, \underline{x}_1 + {}^1p_2 \, {}^2\underline{y}$$

$$= \left[{}^2\underline{x}_1 \quad {}^2\underline{y}_1\right] \begin{bmatrix} {}^1p_1 \\ {}^1p_2 \end{bmatrix}$$

INVERSES

$$= {}^2\underset{\sim}{R}_1 \, {}^1\underline{p}$$

ALWAYS TRUE

$${}^1\underset{\sim}{R}_2 = \left({}^2\underset{\sim}{R}_1\right)^{-1}$$

$$\left({}^1\underset{\sim}{R}_2\right)^{-1} = {}^2\underset{\sim}{R}_1$$

$${}^1\underset{\sim}{R}_2 = \left[{}^1\underline{x}_2 \quad {}^1\underline{y}_2\right]$$

${}^1\underset{\sim}{R}_2^{\,T} =$ COLUMN VECTORS INTO ROW VECTORS $= \begin{bmatrix} ({}^1\underline{x}_2)^T \\ ({}^1\underline{y}_2)^T \end{bmatrix}$

$${}^1\underset{\sim}{R}_2^{\,T} \, {}^1\underset{\sim}{R}_2 = \begin{bmatrix} {}^1\underline{x}_2^{\,T} \\ {}^1\underline{y}_2^{\,T} \end{bmatrix} \left[{}^1\underline{x}_2 \quad {}^1\underline{y}_2\right] = \begin{bmatrix} {}^1\underline{x}_2^{\,T}\,{}^1\underline{x}_2 & {}^1\underline{x}_2^{\,T}\,{}^1\underline{y}_2 \\ {}^1\underline{y}_2^{\,T}\,{}^1\underline{x}_2 & {}^1\underline{y}_2^{\,T}\,{}^1\underline{y}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underset{\sim}{I}_{2\times 2}$$

$$\left.\begin{array}{l} \Rightarrow \underbrace{R^{-1} = R^T} \\ \text{HOW TO } \underline{\text{INVERT}} \\ \text{ROTATION MATRIX} \end{array}\right\} \text{ \underline{ALWAYS} TRUE IN ALL DIMENSIONS}$$

${}^1\underline{p} \cdot {}^1\underline{p} \quad$ VS $\quad {}^1\underline{p} \cdot {}^2\underline{p}$

$${}^1\underline{p}^{\,T} \cdot {}^1\underline{p} = \left({}^1\underset{\sim}{R}_2 \, {}^2\underline{p}\right)^T \left({}^1\underset{\sim}{R}_2 \, {}^2\underline{p}\right)$$

$$= {}^2\underline{p}^{\,T} \, {}^1\underset{\sim}{R}_2^{\,T} \, {}^1\underset{\sim}{R}_2 \, {}^2p$$

$$= 1$$