Chapter 10

Dynamics

This chapter describes how joint torques depend on the link motions of a robot manipulator. The basis for deriving the joint torques is the Newton-Euler equations for rigid-body motion, combined with the joint constraint forces and torques of the last chapter. Euler's equation will be derived, including the concepts of inertia and angular momentum.

10.1 Momentum

10.1.1 Linear Momentum

Suppose the velocity of the center of mass is $\dot{\mathbf{r}}_{0i}$ (Figure 10.1(A)). Then the *linear momentum* $\mathbf{p}_i$ of body i is defined as the time derivative of the mass moment:

$$\mathbf{p}_i = \frac{d(m_i \mathbf{r}_{0i})}{dt} = m_i \dot{\mathbf{r}}_{0i} \tag{10.1}$$

10.1.2 Angular Momentum

Consider again that a body is made up of a system of N particles (Figure 10.1(B)). The *angular momentum* $\mathbf{l}_{0,ik}$ for particle ik about point O_0 is defined as the moment of the linear momentum vector:

$$\mathbf{l}_{0,ik} = \mathbf{q}_{0,ik} \times m_{ik} \dot{\mathbf{q}}_{0,ik} \tag{10.2}$$

Suppose origin O_0 is fixed with respect to the body (even though it doesn't look like it in the Figure). Then each vector $\mathbf{q}_{0,ik}$ is a fixed distance from O_0 and is spinning with the angular velocity $\boldsymbol{\omega}_{0i}$ of the body. Express the velocity $\dot{\mathbf{q}}_{0,ik}$ of particle ik in terms of the angular velocity vector $\boldsymbol{\omega}_{0i}$ of the body:

$$\dot{\mathbf{q}}_{0,ik} = \boldsymbol{\omega}_{0i} \times \mathbf{q}_{0,ik} \tag{10.3}$$

Substituting,

$$\mathbf{l}_{0,ik} = \mathbf{q}_{0,ik} \times m_{ik}(\boldsymbol{\omega}_{0i} \times \mathbf{q}_{0,ik})$$

The total angular momentum $\mathbf{l}_{0i}$ for the N particles is:

$$\mathbf{l}_{0i} = \sum_{k=1}^{N} \mathbf{l}_{0,ik} = \sum_{k=1}^{N} \mathbf{q}_{0,ik} \times m_{ik}(\boldsymbol{\omega}_{0i} \times \mathbf{q}_{0,ik}) \tag{10.4}$$

The subscript $0i$ emphasizes that angular momentum of body i depends on the reference point O_0.

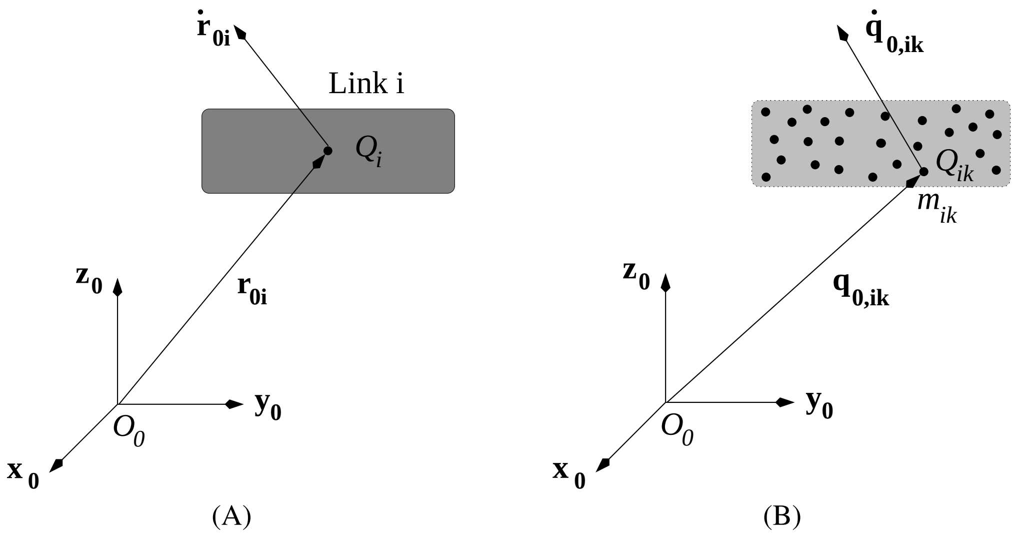

Figure 10.1: (A) The linear momentum is defined in terms of a body's center of gravity $\mathbf{r}_{0i}$. (B). The angular momentum of a body comprised of a system of particles ik.

10.2 Inertia Matrix

From the vector relation $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}^T\mathbf{c}) - \mathbf{c}(\mathbf{a}^T\mathbf{b})$,

$$\begin{aligned} \mathbf{l}_{0i} &= \sum_{k=1}^{N} \left[m_{ik}\boldsymbol{\omega}_{0i}(\mathbf{q}_{0,ik}^T\mathbf{q}_{0,ik}) - m_{ik}\mathbf{q}_{0,ik}(\boldsymbol{\omega}_{0i}^T\mathbf{q}_{0,ik}) \right] \\ &= \sum_{k=1}^{N} \left[m_{ik}||\mathbf{q}_{0,ik}||^2\boldsymbol{\omega}_{0i} - m_{ik}\mathbf{q}_{0,ik}\mathbf{q}_{0,ik}^T\boldsymbol{\omega}_{0i} \right] \end{aligned} \tag{10.5}$$

Let $\mathbf{1}$ represent the 3-by-3 identity matrix. Factor out $\boldsymbol{\omega}_{0i}$:

$$\begin{aligned} \mathbf{l}_{0i} &= \left(\sum_{k=1}^{N} \left(m_{ik}||\mathbf{q}_{0,ik}||^2\mathbf{1} - m_{ik}\mathbf{q}_{0,ik}\mathbf{q}_{0,ik}^T \right) \right) \boldsymbol{\omega}_{0i} \\ &= \mathbf{I}_0\boldsymbol{\omega}_{0i} \end{aligned} \tag{10.6}$$

where the *inertia matrix* $\mathbf{I}_{0i}$ is defined as:

$$\mathbf{I}_{0i} = \sum_{k=1}^{N} \left(m_{ik}||\mathbf{q}_{0,ik}||^2\mathbf{1} - m_{ik}\mathbf{q}_{0,ik}\mathbf{q}_{0,ik}^T \right) \tag{10.7}$$

Again, subscript 0 emphasizes that the inertia of body i depends on the reference point O_0.

For a body with continuous mass distribution as in Figure 10.2(A), the integral form of the inertia matrix is:

$$\mathbf{I}_{0i} = \int\int\int \rho \left(||\mathbf{q}||^2\mathbf{1} - \mathbf{q}\mathbf{q}^T \right) dV \tag{10.8}$$

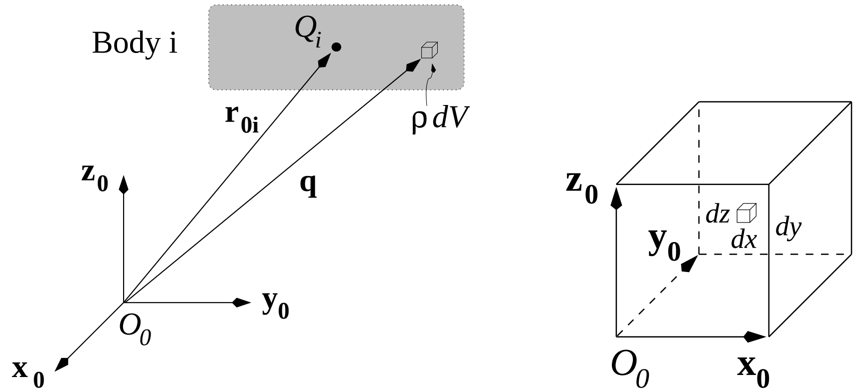

Figure 10.2: (A) The inertia for a body with continuously distributed mass. (B) The inertia of a unit cube.

Let the coordinates of $\mathbf{q} = (x, y, z)$ measure the position of a mass element relative to O_0. Then

$$\begin{aligned}\mathbf{I}_{0i} &= \int\int\int \rho \begin{bmatrix} x^2+y^2+z^2 & 0 & 0 \\ 0 & x^2+y^2+z^2 & 0 \\ 0 & 0 & x^2+y^2+z^2 \end{bmatrix} dxdydz - \\ &\int\int\int \rho \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} dxdydz \\ &= \int\int\int \rho \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dxdydz \qquad (10.9)\end{aligned}$$

__Example 10.1:__ We derive the inertia $\mathbf{I}_0$ of the uniform unit cube, considered as body 0, of Figure 10.2(B) with respect to the center of mass $Q_0 = (1/2, 1/2, 1/2)$. When the reference point is the center of mass, we omit the double subscript and just refer to the inertia $\mathbf{I}_i$ of body i. Then with $\rho = 1$,

$$\begin{aligned}\mathbf{I}_0 &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dxdydz \\ &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} dxdy \begin{bmatrix} y^2z+z^3/3 & -xyz & -xz^2/2 \\ -xyz & x^2z+z^3/3 & -yz^2/2 \\ -xz^2/2 & -yz^2/2 & x^2z+y^2z \end{bmatrix}_{z=-1/2}^{z=1/2} \\ &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \begin{bmatrix} y^2+1/12 & -xy & 0 \\ -xy & x^2+1/12 & 0 \\ 0 & 0 & x^2+y^2 \end{bmatrix} dxdy\end{aligned}$$

Note that the terms contaning even powers of z become zero. Continuing,

$$\begin{aligned}
\mathbf{I}_0 &= \int_{-1/2}^{1/2} dx \left[\begin{array}{ccc} y^3/3+y/12 & -xy^2/2 & 0 \\ -xy^2/2 & x^2y+y/12 & 0 \\ 0 & 0 & x^2y+y^3/3 \end{array}\right]_{y=-1/2}^{y=1/2} \\
&= \int_{-1/2}^{1/2} \left[\begin{array}{ccc} 1/6 & 0 & 0 \\ 0 & x^2+1/12 & 0 \\ 0 & 0 & x^2+1/12 \end{array}\right] dx \\
&= \int_{-1/2}^{1/2} \left[\begin{array}{ccc} x/6 & 0 & 0 \\ 0 & x^3/3+x/12 & 0 \\ 0 & 0 & x^3/3+x/12 \end{array}\right]_{x=-1/2}^{x=1/2} \\
&= \left[\begin{array}{ccc} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} \end{array}\right]
\end{aligned} \tag{10.10}$$

10.2.1 Origin Dependence: Parallel Axis Theorem

Suppose that the reference point has changed to O_1, as in Figure 10.3(A). Then $\mathbf{q} = \mathbf{d}_{01} + \mathbf{q}'$ and

$$\begin{aligned}
\mathbf{I}_{0i} &= \int\int\int \rho \left((\mathbf{d}_{01}+\mathbf{q}')^T(\mathbf{d}_{01}+\mathbf{q}')\mathbf{1} - (\mathbf{d}_{01}+\mathbf{q}')(\mathbf{d}_{01}+\mathbf{q}')^T \right) dV \\
&= \int\int\int \rho \left(||\mathbf{q}'||^2\mathbf{1} - \mathbf{q}'\mathbf{q}'^T \right) dV + \int\int\int \rho \left(||\mathbf{d}_{01}||^2\mathbf{1} - \mathbf{d}_{01}\mathbf{d}_{01}^T \right) dV + \int\int\int \rho \left(2\mathbf{d}_{01}^T\mathbf{q}' \right) \mathbf{1} dV \\
&\quad - \int\int\int \rho \left(\mathbf{q}'\mathbf{d}_{01}^T + \mathbf{d}_{01}\mathbf{q}'^T \right) dV
\end{aligned} \tag{10.11}$$

To simply this expression further, the critical assumption is made that $O_1 = Q_i$ is the center of mass, i.e., $\mathbf{d}_{01} = \mathbf{r}_{0i}$, so that the mass moment is zero.

$$\mathbf{r}_{1i} = \mathbf{r}_{0i} - \mathbf{d}_{01} = \mathbf{0} = \frac{1}{m_i}\int\int\int \rho\mathbf{q}' dV \tag{10.12}$$

Then the third and fourth terms on the right are zero, because after a little factorization they include the mass moment about the center of mass (10.12) which is zero. The first term on the right is the inertia $\mathbf{I}_i$ about the center of mass, while the second term factors to reveal the mass m_i. Hence

$$\mathbf{I}_{0i} = \mathbf{I}_i + m_i \left(||\mathbf{r}_{0i}||^2\mathbf{1} - \mathbf{r}_{0i}\mathbf{r}_{0i}^T \right) \tag{10.13}$$

This is the *parallel axis theorem.* If the inertia about the center of mass $\mathbf{I}_i$ is known, the inertia $\mathbf{I}_{0i}$ relative to any other point O_0 can be calculated.

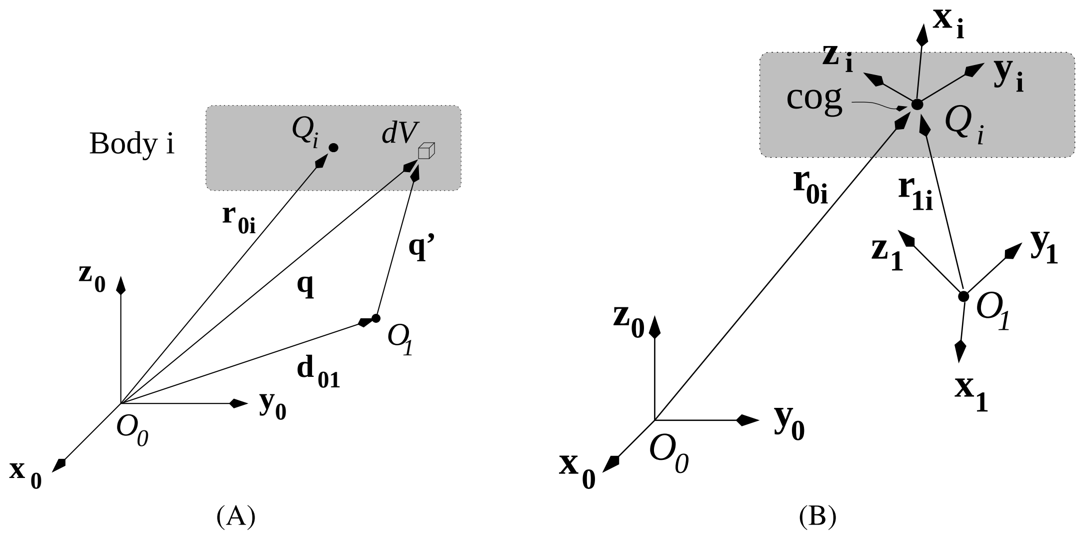

Figure 10.3: (A) Inertia referred to a different origin. (B) Inertia expressed in terms of different axes.

Example 10.2: Previously we developed the inertia matrix $\mathbf{I}_0$ for the center of mass of the cube in Figure 10.2(B). The center of mass location is $\mathbf{r}_{00} = O_1 - O_0 = 1/2(1,1,1)$. The inertia matrix $\mathbf{I}_{00}$ relative to origin O_0 is:

$$\begin{aligned}
\mathbf{I}_{00} &= \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/6 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix} \qquad (10.14)
\end{aligned}$$

10.2.2 Orientation Dependence

So far we have not been explicit as to the coordinate system axes in terms of which inertia is represented. Suppose that inertia ${}^0\mathbf{I}_{0i}$ is expressed in terms of coordinate system axes 0. How is inertia expressed in terms of different coordinate axes, say as ${}^1\mathbf{I}_{0i}$ in terms of axes 1 in Figure 10.3B? Note that the reference origin O_0 is not changing, only the axes orientation. Express the angular momentum (10.6) in terms of axes 0 orientation:

$${}^0\mathbf{l}_{0i} = {}^0\mathbf{I}_{0i}\,{}^0\boldsymbol{\omega}_{0i} \qquad (10.15)$$

When all quantities are expressed in terms of axes 1, this equation is reworked as:

$$\begin{aligned}
{}^1\mathbf{l}_{0i} &= {}^1\mathbf{I}_{0i}\,{}^1\boldsymbol{\omega}_{0i} \\
&= {}^1\mathbf{I}_{0i}\,({}^1\mathbf{R}_0\,{}^1\mathbf{R}_0^T)\,{}^1\boldsymbol{\omega}_{0i} \\
&= ({}^1\mathbf{I}_{0i}\,{}^1\mathbf{R}_0)\,({}^1\mathbf{R}_0^T\,{}^1\boldsymbol{\omega}_{0i})
\end{aligned}$$

$$= \ ({}^1\mathbf{I}_{0i}\,{}^1\mathbf{R}_0)\,{}^0\boldsymbol{\omega}_{0i} \tag{10.16}$$

We can also obtain ${}^1\mathbf{l}_{0i}$ by rotating (10.15) using ${}^1\mathbf{R}_0$:

$$\begin{aligned} {}^1\mathbf{l}_{0i} &= {}^1\mathbf{R}_0({}^0\mathbf{I}_{0i}\,{}^0\boldsymbol{\omega}_{0i}) \\ &= ({}^1\mathbf{R}_0{}^0\mathbf{I}_{0i})\,{}^0\boldsymbol{\omega}_{0i} \end{aligned} \tag{10.17}$$

Since the two expression (10.16) and (10.17) have to be equal, then

$$\begin{aligned} {}^1\mathbf{I}_{0i}\,{}^1\mathbf{R}_0 &= {}^1\mathbf{R}_0{}^0\mathbf{I}_{0i} \\ {}^1\mathbf{I}_{0i} &= {}^1\mathbf{R}_0{}^0\mathbf{I}_{0i}\,{}^1\mathbf{R}_0^T \end{aligned} \tag{10.18}$$

Equation (10.18) shows how the inertia matrix depends on the axes orientation.

10.2.3 Principal Axes and Inertias

The definition of the inertia matrix (10.8) shows that $\mathbf{I}_{0i}$ is symmetric: the identity matrix $\mathbf{1}$ is symmetric, and so is the outer product $\mathbf{q}\mathbf{q}^T$. Thus there are 6 independent numbers comprising $\mathbf{I}_{0i}$. This holds true regardless of the coordinate system axes in terms of which inertia is expressed.

It is possible to find a set of axes that diagonalizes any inertia matrix. That is to say, given ${}^0\mathbf{I}_{0i}$, it is possible to find a rotation matrix ${}^1\mathbf{R}_0$ such that

$${}^1\mathbf{I}_{0i} = \begin{bmatrix} I_{0i1} & 0 & 0 \\ 0 & I_{0i2} & 0 \\ 0 & 0 & I_{0i3} \end{bmatrix} \tag{10.19}$$

where the diagonal elements are positive. The axes defined by ${}^0\mathbf{R}_1 = [{}^0\mathbf{x}_1\,{}^0\mathbf{y}_1\,{}^0\mathbf{z}_1]$ are referred to as the *principal axes* of inertia, and the diagonal elements $I_{0i1}, I_{0i2}, I_{0i3}$ are the *principal inertias.* The principal axes and inertias are a different set of 6 numbers to represent inertia, since 3 numbers are required to describe the orientation of the principal axes.

The principal axes and inertias are the eigenvectors and eigenvalues of the inertia matrix (Appendix A). If the eigenvalues are distinct, the eigenvectors are unique. In case the inertia matrix is symmetric, one or more eigenvalues will be repeated and the eigenvectors are not unique.

Example 10.3: For the cube of Example 10.3, the inertia matrix about O_0 is

$$\mathbf{I}_{00} = \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix}$$

Eigenvalues of 11/12, 11/12, and 1/6 are found by solving the cubic equation:

$$0 = \det(\mathbf{I}_{00} - \lambda\mathbf{1})$$

An eigenvector $\mathbf{v}_1$ corresponding to the distinct eigenvalue 1/6 is found by solving the linear equation

$$(\mathbf{I}_{00} - 1/6\mathbf{1})\mathbf{v}_1 = 0$$

The unit-vector solution is

$$\mathbf{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

which is a diagonal vector from the origin to the far corner $(1, 1, 1)$. This makes sense because this diagonal line is an axis of symmetry. Let's call it the $\mathbf{z}_1$ principal axis. The other two eigenvalues are repeated, and so their eigenvectors are not unique. The mass distribution of the cube about O_0 is symmetric about the diagonal line, and any two orthogonal vectors which are also orthogonal to $\mathbf{v}_1$ can serve as eigenvectors. Substituting the eigenvalue 11/12, solution of the equation

$$(\mathbf{I}_{00} - 11/12\mathbf{1})\mathbf{v} = 0$$

where $\mathbf{v} = (v_1, v_2, v_3)$ yields only one independent equation:

$$v_1 + v_2 + v_3 = 0$$

One possible choice is to set $v_3 = 0$ and $v_1 = -1$, and call it the $\mathbf{x}_1$ axis:

$$\mathbf{x}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

The third axis is found from the cross product relation:

$$\mathbf{y}_1 = \mathbf{z}_1 \times \mathbf{x}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Another example is the inertia of a uniform sphere about its center, where all three eigenvalues are the same. Any set of coordinate axes can be placed at the center and serve as principal axes, and so the choice is arbitrary.

The inertia matrix is positive definite, and therefore the principal inertias are positive (Appendix A). Positive definiteness can be derived most easily for the particle-based description of an object, as in Figure 10.1(B). Suppose the body comprised of N particles is rotating about point O_0 with angular velocity $\boldsymbol{\omega}_{0i}$. The kinetic energy of particle ik is $1/2\, m_{ik}\dot{\mathbf{q}}_{0,ik} \cdot \dot{\mathbf{q}}_{0,ik}$, and the total (rotational) kinetic energy K is

$$K = \sum_{k=1}^{N} \frac{1}{2} m_{ik}(\dot{\mathbf{q}}_{0,ik} \cdot \dot{\mathbf{q}}_{0,ik}) \tag{10.20}$$

Clearly $K > 0$ if the body is moving because of the squared term $||\dot{\mathbf{q}}_{0,ik}||^2$. Since $\dot{\mathbf{q}}_{0,ik} = \boldsymbol{\omega}_{0i} \times \mathbf{q}_{0,ik}$, then the angular velocity $\boldsymbol{\omega}_{0i} \neq \mathbf{0}$ for a moving body. Substituting,

$$\begin{aligned} K &= \sum_{k=1}^{N} \frac{1}{2} m_{ik}(\boldsymbol{\omega}_{0i} \times \mathbf{q}_{0,ik}) \cdot (\boldsymbol{\omega}_{0i} \times \mathbf{q}_{0,ik}) \\ &= \sum_{k=1}^{N} \frac{1}{2} m_{ik}\boldsymbol{\omega}_{0i} \cdot (\mathbf{q}_{0,ik} \times (\boldsymbol{\omega}_{0i} \times \mathbf{q}_{0,ik}) \end{aligned} \tag{10.21}$$

where we have employed the vector relation $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Reworking,

$$\begin{aligned} K &= \frac{1}{2}\boldsymbol{\omega}_{0i} \cdot \sum_{k=1}^{N} m_{ik}\mathbf{q}_{0,ik} \times (\boldsymbol{\omega}_{0i} \times \mathbf{q}_{0,ik}) \\ &= \frac{1}{2}\boldsymbol{\omega}_{0i} \cdot \mathbf{l}_{0i} \\ &= \frac{1}{2}\boldsymbol{\omega}_{0i} \cdot \mathbf{I}_{0i}\boldsymbol{\omega}_{0i} \end{aligned} \tag{10.22}$$

where (10.4) and (10.6) have been used. Since the kinetic energy $K > 0$ if $\boldsymbol{\omega}_{0i} \neq 0$, this shows that $\mathbf{I}_{0i}$ is positive definite.

Let us refer inertia $\mathbf{I}_i$ to the center of mass O_i as in Figure 10.3(B), and set up the principal axes $\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i$ of $\mathbf{I}_i$ to complete frame i. Just as the center of mass location Q_i is a fixed point in the body, the principal axes and inertias at the center of mass are fixed attributes of the body. This point and these axes form the most important description of the mass distribution in the body, which may be used for control or to help with recognition.

10.3 Newton-Euler Equations

Newton's second law states that a net force $\mathbf{f}_i$ applied to body i changes the linear momentum $\mathbf{p}_i$ according to:

$$\mathbf{f}_i = \dot{\mathbf{p}}_i = m_i\ddot{\mathbf{r}}_{0i} \tag{10.23}$$

Substituting the expression for linear momentum (10.1) shows the familiar relation between net force and acceleration of the center of mass $\ddot{\mathbf{r}}_{0i}$. This is called Newton's equation for short. The net force comprises all external forces acting on the body, including gravity and constraint forces at joints if the body is a link of a manipulator.

Suppose that inertia $\mathbf{I}_i$ is referred to the center of mass Q_i as in Figure 10.3(B). Suppose coordinate axes i are fixed in the body; for example, they could represent the principal axes or the DH axes at one of the joints of a link. Then ${}^i\mathbf{I}_i$ is a constant matrix. Suppose frame 0 is an *inertial frame,* i.e., a non-accelerating frame such as a stationary frame fixed to the ground. Then ${}^0\mathbf{I}_i$ will vary as the body moves. The angular momentum ${}^0\mathbf{l}_i$ with respect to the center of mass can be expressed as:

$$\begin{aligned} {}^0\mathbf{l}_i &= {}^0\mathbf{I}_i\,{}^0\boldsymbol{\omega}_{0i} \\ &= {}^0\mathbf{R}_i({}^i\mathbf{I}_i\,{}^i\boldsymbol{\omega}_{0i}) \end{aligned} \tag{10.24}$$

where ${}^0\mathbf{R}_i$ represents the rotational transformation from axes i to axes 0. Just as Newton's equation relates net force to the change of linear momentum, Euler's equation relates net torque ${}^0\mathbf{n}_i$ at the center of mass to the change of angular momentum ${}^0\mathbf{l}_i$:

$${}^0\mathbf{n}_i = \frac{d\,{}^0\mathbf{l}_i}{dt} = \frac{d\,{}^0\mathbf{R}_i({}^i\mathbf{I}_i\,{}^i\boldsymbol{\omega}_{0i})}{dt} \tag{10.25}$$

As for inertia, when the reference point is the center of mass, we drop the double subscript on net torque. Expand using the chain rule for differentiation:

$${}^0\mathbf{n}_i = {}^0\dot{\mathbf{R}}_i\,{}^i\mathbf{I}_i\,{}^i\boldsymbol{\omega}_{0i} + {}^0\mathbf{R}_i({}^i\mathbf{I}_i\,{}^i\dot{\boldsymbol{\omega}}_{0i}) \tag{10.26}$$

because ${}^{i}\mathbf{I}_i$ is constant. Using the relation that ${}^{0}\dot{\mathbf{R}}_i = {}^{0}\boldsymbol{\omega}_{0i} \times {}^{0}\mathbf{R}_i$,

$$
\begin{aligned}
{}^{0}\mathbf{n}_i &= {}^{0}\boldsymbol{\omega}_{0i} \times {}^{0}\mathbf{R}_i({}^{i}\mathbf{I}_i\,{}^{i}\boldsymbol{\omega}_{0i}) + {}^{0}\mathbf{R}_i({}^{i}\mathbf{I}_i\,{}^{i}\dot{\boldsymbol{\omega}}_{0i}) \\
&= {}^{0}\boldsymbol{\omega}_{0i} \times {}^{0}\mathbf{I}_i\,{}^{0}\boldsymbol{\omega}_{0i} + {}^{0}\mathbf{I}_i\,{}^{0}\dot{\boldsymbol{\omega}}_{0i} \qquad (10.27)
\end{aligned}
$$

This is Euler's equation. It is more complicated than Newton's equation. First, there are the *gyroscopic torques* $\boldsymbol{\omega}_{0i} \times \mathbf{I}\boldsymbol{\omega}_{0i}$. Second, the angular acceleration $\dot{\boldsymbol{\omega}}_{0i}$ and the net torque $\mathbf{n}$ are not generally in the same direction because of the multiplication of angular acceleration by inertia $\mathbf{I}$. The net torque includes all pure torques and forces acting through moment arms on a body, such as forces and torques of constraint at joints, actuator torques, and external forces and torques from the environment.

The main developments of this chapter so far are summarized in Table 10.1. The left superscript is omitted, but these equations are valid regardless of the coordinate axes orientation. The main requirement is that the reference frame is an inertial frame, and that inertia, angular momentum, and net torque are referred to the center of mass. In the remainder of the chapter, the Newton-Euler equations will serve as the basis for deriving the dynamics of a manipulator.

Mass:	m_i	Inertia:	$\mathbf{I}_i$
Linear momentum:	$\mathbf{p}_i = m_i\dot{\mathbf{r}}_{0i}$	Angular momentum:	$\mathbf{l}_i = \mathbf{I}_i\boldsymbol{\omega}_{0i}$
Net force:	$\mathbf{f}_i$	Net torque:	$\mathbf{n}_i$
Newton's equation:	$\mathbf{f}_i = \dot{\mathbf{p}}_i = m_i\ddot{\mathbf{r}}_{0i}$	Euler's equation:	$\mathbf{n}_i = \dot{\mathbf{l}}_i = \mathbf{I}_i\dot{\boldsymbol{\omega}}_{0i} + \boldsymbol{\omega}_{0i} \times \mathbf{I}_i\boldsymbol{\omega}_{0i}$

Table 10.1: The Newton-Euler equations. Inertia, angular momentum, and net torque are referred to the center of mass $\mathbf{r}_{0i}$.

10.4 One-Link Robot Dynamics

Let's apply the Newton-Euler equations to describe the dynamics of a one-link robot (Figure 10.4(A)). Utilize the free-body diagram of the link (Figure 10.4(B)) to derive the net force $\mathbf{f}_1$ and the net torque $\mathbf{n}_1$ about the center of mass:

$$
\begin{aligned}
\mathbf{f}_1 &= \mathbf{f}_{01} && (10.28)\\
&= m_1\ddot{\mathbf{r}}_{01} && (10.29)\\
\mathbf{n}_1 &= \mathbf{n}_{01} - \mathbf{r}_{01} \times \mathbf{f}_{01} && (10.30)\\
&= \mathbf{I}_1\dot{\boldsymbol{\omega}}_{01} + \boldsymbol{\omega}_{01} \times \mathbf{I}_1\boldsymbol{\omega}_{01} && (10.31)\\
\tau_1 &= \mathbf{z}_0 \cdot \mathbf{n}_{01} && (10.32)
\end{aligned}
$$

where $\mathbf{f}_{01}$ and $\mathbf{n}_{01}$ are the force and torque of constraint of the base acting on link 1. In the previous chapter on statics, the net force and torque were zero. Now they are non-zero, and the link will accelerate as described by the Newton-Euler equations. Suppose the center of mass is at the middle of the link, $\mathbf{r}_{01} = a_1/2\mathbf{x}_1$. Then rearranging (10.28) and (10.30),

$$
\mathbf{f}_{01} = \mathbf{f}_1 \qquad (10.33)
$$

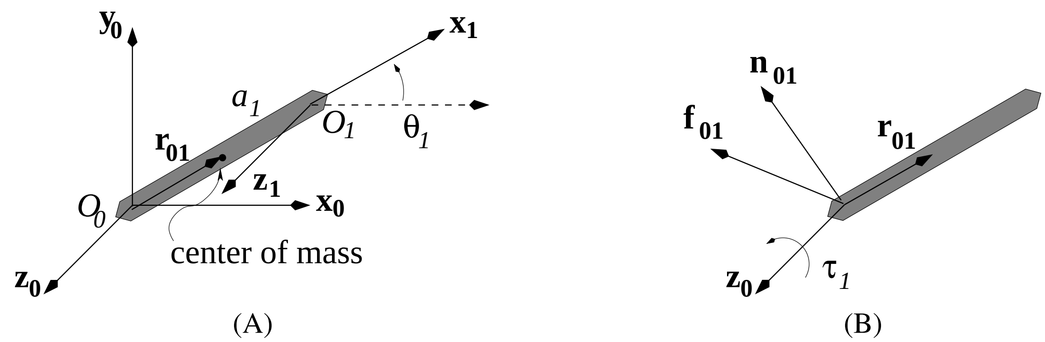

Figure 10.4: (A) A one-link robot with $\mathbf{r}_{01}$ as the center of mass location. (B) Free-body diagram of the link.

$$\begin{aligned} \mathbf{n}_{01} &= \mathbf{n}_1 + \frac{a_1}{2}\mathbf{x}_1 \times \mathbf{f}_{01} \\ &= \mathbf{n}_1 + \frac{a_1}{2}\mathbf{x}_1 \times \mathbf{f}_1 \end{aligned} \tag{10.34}$$

The joint torque (10.32) then simplifies to

$$\begin{aligned} \tau_1 &= \mathbf{z}_0 \cdot (\mathbf{n}_1 + \frac{a_1}{2}\mathbf{x}_1 \times \mathbf{f}_1) \\ &= \mathbf{z}_0 \cdot \mathbf{n}_1 + \frac{a_1}{2}\mathbf{z}_0 \cdot (\mathbf{x}_1 \times \mathbf{f}_1) \\ &= \mathbf{z}_0 \cdot \mathbf{n}_1 + \frac{a_1}{2}\mathbf{y}_1 \cdot \mathbf{f}_1 \end{aligned} \tag{10.35}$$

It remains to evaluate the Newton-Euler equations for the net force $\mathbf{f}_1$ (10.29) and the net torque $\mathbf{n}_1$ (10.31). The angular velocity and acceleration of link one are:

$$\boldsymbol{\omega}_{01} = \mathbf{z}_0\dot{\theta}_1 \tag{10.36}$$

$$\dot{\boldsymbol{\omega}}_{01} = \mathbf{z}_0\ddot{\theta}_1 \tag{10.37}$$

The term involving Euler's equation in (10.35) then becomes

$$\begin{aligned} \mathbf{z}_0 \cdot \mathbf{n}_1 &= \mathbf{z}_0 \cdot (\mathbf{I}_1\dot{\boldsymbol{\omega}}_{01} + \boldsymbol{\omega}_{01} \times \mathbf{I}_1\boldsymbol{\omega}_{01}) \\ &= \mathbf{z}_0 \cdot (\mathbf{I}_1\mathbf{z}_0\ddot{\theta}_1 + \mathbf{z}_0\dot{\theta}_1 \times \mathbf{I}_1\mathbf{z}_0\dot{\theta}_1) \\ &= \mathbf{z}_0 \cdot \mathbf{I}_1\mathbf{z}_0\ddot{\theta}_1 \end{aligned} \tag{10.38}$$

Suppose the inertia matrix is evaluated in frame 0, with elements ${}^0\mathbf{I}_1 = \{{}^0I_{1,ij}\}$. Then

$${}^0\mathbf{z}_0 \cdot {}^0\mathbf{n}_1 = {}^0I_{1,33}\ddot{\theta}_1 \tag{10.39}$$

The linear acceleration of the center of mass is found by successive differentiation of $\mathbf{r}_{01}$:

$$\mathbf{r}_{01} = \frac{a_1}{2}\mathbf{x}_1$$

$$\begin{aligned}
\dot{\mathbf{r}}_{01} &= \frac{a_1}{2}\boldsymbol{\omega}_{01} \times \mathbf{x}_1 \\
&= \frac{a_1}{2}\mathbf{y}_1\dot{\theta}_1 \\
\ddot{\mathbf{r}}_{01} &= \frac{a_1}{2}\mathbf{y}_1\ddot{\theta}_1 + \frac{a_1}{2}\boldsymbol{\omega}_{01} \times \mathbf{y}_1\dot{\theta}_1 \\
&= \frac{a_1}{2}\mathbf{y}_1\ddot{\theta}_1 - \frac{a_1}{2}\mathbf{x}_1\dot{\theta}_1^2 \qquad (10.40)
\end{aligned}$$

Then the term involving Newton's equation in (10.35) is

$$\begin{aligned}
\mathbf{y}_1 \cdot \mathbf{f}_1 &= \mathbf{y}_1 \cdot m_1\ddot{\mathbf{r}}_{01} \\
&= \mathbf{y}_1 \cdot m_1\left(\frac{a_1}{2}\mathbf{y}_1\ddot{\theta}_1 - \frac{a_1}{2}\mathbf{x}_1\dot{\theta}_1^2\right) \\
&= \frac{a_1}{2}m_1\ddot{\theta}_1 \qquad (10.41)
\end{aligned}$$

Substituting both results into (10.35),

$$\begin{aligned}
\tau_1 &= {}^0I_{1,33}\ddot{\theta}_1 + \frac{a_1^2}{4}m_1\ddot{\theta}_1 \\
&= ({}^0I_{1,33} + \frac{a_1^2}{4}m_1)\ddot{\theta}_1 \qquad (10.42) \\
&= {}^0I_{0,33}\ddot{\theta}_1 \qquad (10.43)
\end{aligned}$$

where ${}^0I_{0,33} ={}^0 I_{1,33} + m_1a_1^2/4$ is the inertia about O_0 as may be verified from the parallel axis theorem. This equation should be familiar from freshman physics.

10.5 Two-Link Planar Manipulator Dynamics

Consider again the two-link planar manipulator of the previous chapter (Figure 10.5). The force/torque balance equations for link 2 referred to origin O_1 are now not zero. The force balance equations yield net forces $\mathbf{f}_1, \mathbf{f}_2$ which result in linear acceleration according to Newton's equation:

$$\begin{aligned}
\mathbf{f}_2 &= \mathbf{f}_{12} \qquad (10.44) \\
&= m_2\ddot{\mathbf{r}}_{02} \qquad (10.45) \\
\mathbf{f}_1 &= \mathbf{f}_{01} - \mathbf{f}_{12} \qquad (10.46) \\
&= m_1\ddot{\mathbf{r}}_{01} \qquad (10.47)
\end{aligned}$$

where $\mathbf{r}_{0i}$ is the center of mass location of link i relative to O_0 and again $\mathbf{f}_{21} = -\mathbf{f}_{12}$. We are assuming that the center of mass of each link is located halfway along each interorigin vector:

$$\begin{aligned}
\mathbf{r}_{01} &= \frac{a_1}{2}\mathbf{x}_1 \qquad (10.48) \\
\mathbf{r}_{02} &= a_1\mathbf{x}_1 + \frac{a_2}{2}\mathbf{x}_2 \qquad (10.49)
\end{aligned}$$

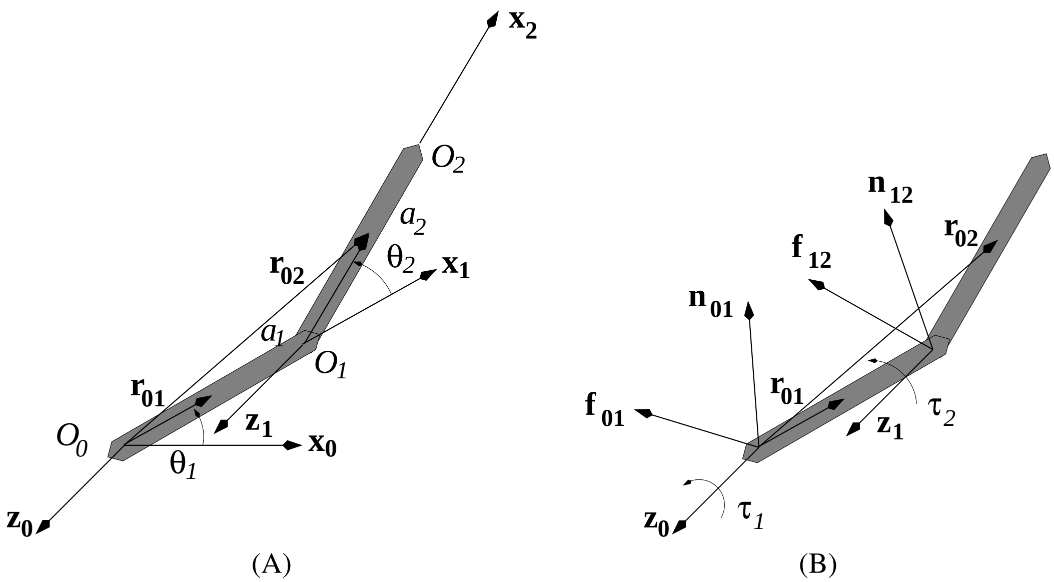

Figure 10.5: (A) Kinematic diagram of two-link planar manipulator. (B) Free-body diagram.

The torque balance equations are now referred to the center of mass for each link, and result in net torques $\mathbf{n}_1, \mathbf{n}_2$ which cause angular acceleration according to Euler's equation:

$$\mathbf{n}_2 = -\frac{a_2}{2}\mathbf{x}_2 \times \mathbf{f}_{12} + \mathbf{n}_{12} \tag{10.50}$$

$$= \mathbf{I}_2\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_{02} \times \mathbf{I}_2\boldsymbol{\omega}_{02} \tag{10.51}$$

$$\mathbf{n}_1 = -\frac{a_1}{2}\mathbf{x}_1 \times \mathbf{f}_{01} - \frac{a_1}{2}\mathbf{x}_1 \times \mathbf{f}_{12} + \mathbf{n}_{01} - \mathbf{n}_{12} \tag{10.52}$$

$$= \mathbf{I}_1\dot{\boldsymbol{\omega}}_{01} + \boldsymbol{\omega}_{01} \times \mathbf{I}_1\boldsymbol{\omega}_{01} \tag{10.53}$$

The minus signs arise either because of the direction of the moment arm from the center of mass to the point of force application or the sign of the constraint force. The joint torques are:

$$\tau_2 = \mathbf{z}_0 \cdot \mathbf{n}_{12} \tag{10.54}$$

$$\tau_1 = \mathbf{z}_0 \cdot \mathbf{n}_{01} \tag{10.55}$$

The angular velocities and accelerations are:

$$\boldsymbol{\omega}_{01} = \dot{\theta}_1\mathbf{z}_0 \tag{10.56}$$

$$\dot{\boldsymbol{\omega}}_{01} = \ddot{\theta}_1\mathbf{z}_0 \tag{10.57}$$

$$\boldsymbol{\omega}_{02} = \dot{\theta}_1\mathbf{z}_0 + \dot{\theta}_2\mathbf{z}_1$$

$$= (\dot{\theta}_1 + \dot{\theta}_2)\mathbf{z}_0 \tag{10.58}$$

$$\dot{\boldsymbol{\omega}}_{02} = (\ddot{\theta}_1 + \ddot{\theta}_2)\mathbf{z}_0 \tag{10.59}$$

The accelerations of the centers of mass are found from differentiating (10.48)-(10.49):

$$\begin{aligned}
\dot{\mathbf{r}}_{01} &= \frac{a_1}{2}\boldsymbol{\omega}_{01} \times \mathbf{x}_1 \\
&= \frac{a_1}{2}\dot{\theta}_1 \mathbf{z}_0 \times \mathbf{x}_1 \\
&= \frac{a_1}{2}\dot{\theta}_1 \mathbf{y}_1 && (10.60) \\
\ddot{\mathbf{r}}_{01} &= \frac{a_1}{2}\ddot{\theta}_1 \mathbf{y}_1 + \frac{a_1}{2}\dot{\theta}_1 \boldsymbol{\omega}_{01} \times \mathbf{y}_1 \\
&= \frac{a_1}{2}\ddot{\theta}_1 \mathbf{y}_1 + \frac{a_1}{2}\dot{\theta}_1 (\dot{\theta}_1 \mathbf{z}_0) \times \mathbf{y}_1 \\
&= \frac{a_1}{2}\ddot{\theta}_1 \mathbf{y}_1 - \frac{a_1}{2}\dot{\theta}_1^2 \mathbf{x}_1 && (10.61) \\
\dot{\mathbf{r}}_{02} &= a_1 \boldsymbol{\omega}_{01} \times \mathbf{x}_1 + \frac{a_2}{2}\boldsymbol{\omega}_{02} \times \mathbf{x}_2 \\
&= a_1 \dot{\theta}_1 \mathbf{z}_0 \times \mathbf{x}_1 + \frac{a_2}{2}(\dot{\theta}_1 + \dot{\theta}_2)\mathbf{z}_0 \times \mathbf{x}_2 \\
&= a_1 \dot{\theta}_1 \mathbf{y}_1 + \frac{a_2}{2}(\dot{\theta}_1 + \dot{\theta}_2)\mathbf{y}_2 && (10.62) \\
\ddot{\mathbf{r}}_{02} &= a_1 \ddot{\theta}_1 \mathbf{y}_1 + a_1 \dot{\theta}_1 \boldsymbol{\omega}_{01} \times \mathbf{y}_1 + \frac{a_2}{2}(\ddot{\theta}_1 + \ddot{\theta}_2)\mathbf{y}_2 + \frac{a_2}{2}(\dot{\theta}_1 + \dot{\theta}_2)\boldsymbol{\omega}_{02} \times \mathbf{y}_2 \\
&= a_1 \ddot{\theta}_1 \mathbf{y}_1 + a_1 \dot{\theta}_1 (\dot{\theta}_1 \mathbf{z}_0) \times \mathbf{y}_1 + \frac{a_2}{2}(\ddot{\theta}_1 + \ddot{\theta}_2)\mathbf{y}_2 + \frac{a_2}{2}(\dot{\theta}_1 + \dot{\theta}_2)((\dot{\theta}_1 + \dot{\theta}_2)\mathbf{z}_0) \times \mathbf{y}_2 \\
&= a_1 \ddot{\theta}_1 \mathbf{y}_1 - a_1 \dot{\theta}_1^2 \mathbf{x}_1 + \frac{a_2}{2}(\ddot{\theta}_1 + \ddot{\theta}_2)\mathbf{y}_2 - \frac{a_2}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2 \mathbf{x}_2 && (10.63)
\end{aligned}$$

where the relations $\mathbf{z}_0 = \mathbf{z}_1 = \mathbf{z}_2$ have been used. The two Newton's equations (10.47) and (10.45) are therefore

$$\mathbf{f}_1 = m_1 \left(\frac{a_1}{2}\ddot{\theta}_1 \mathbf{y}_1 - \frac{a_1}{2}\dot{\theta}_1^2 \mathbf{x}_1 \right) \tag{10.64}$$

$$\mathbf{f}_2 = m_2 \left(a_1 \ddot{\theta}_1 \mathbf{y}_1 - a_1 \dot{\theta}_1^2 \mathbf{x}_1 + \frac{a_2}{2}(\ddot{\theta}_1 + \ddot{\theta}_2)\mathbf{y}_2 - \frac{a_2}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2 \mathbf{x}_2 \right) \tag{10.65}$$

The two Euler equations (10.53) and (10.51) are:

$$\mathbf{n}_1 = \mathbf{I}_1 \ddot{\theta}_1 \mathbf{z}_0 + \dot{\theta}_1 \mathbf{z}_0 \times \mathbf{I}_1 \dot{\theta}_1 \mathbf{z}_0 \tag{10.66}$$

$$\mathbf{n}_2 = \mathbf{I}_2 (\ddot{\theta}_1 + \ddot{\theta}_2)\mathbf{z}_0 + (\dot{\theta}_1 + \dot{\theta}_2)\mathbf{z}_0 \times \mathbf{I}_2 (\dot{\theta}_1 + \dot{\theta}_2)\mathbf{z}_0 \tag{10.67}$$

10.5.1 Joint 2 Torques

Now let's evaluate the torque at joint 2. Combining (10.44), (10.50), and (10.54),

$$\tau_2 = \mathbf{z}_0 \cdot (\mathbf{n}_2 + \frac{a_2}{2}\mathbf{x}_2 \times \mathbf{f}_{12})$$

$$= \quad \mathbf{z}_0 \cdot \mathbf{n}_2 + \frac{a_2}{2}\mathbf{y}_2 \cdot \mathbf{f}_2 \tag{10.68}$$

Suppose both inertia matrices are represented in frame 0. The projection of Euler's equation (10.67) onto ${}^0\mathbf{z}_0$ is:

$$\begin{aligned} {}^0\mathbf{z}_0 \cdot {}^0\mathbf{n}_2 &= {}^0\mathbf{z}_0 \cdot \left({}^0\mathbf{I}_2(\ddot{\theta}_1 + \ddot{\theta}_2)^0\mathbf{z}_0 + (\dot{\theta}_1 + \dot{\theta}_2)\, {}^0\mathbf{z}_0 \times {}^0\mathbf{I}_2(\dot{\theta}_1 + \dot{\theta}_2)\, {}^0\mathbf{z}_0 \right) \\ &= {}^0I_{2,33}(\ddot{\theta}_1 + \ddot{\theta}_2) \end{aligned} \tag{10.69}$$

which is analogous to the one-link robot result (10.43). The projection of Newton's equation (10.65) onto ${}^0\mathbf{y}_2$ is:

$$\begin{aligned} {}^0\mathbf{y}_2 \cdot {}^0\mathbf{f}_2 &= {}^0\mathbf{y}_2 \cdot m_2 \left(a_1\ddot{\theta}_1\, {}^0\mathbf{y}_1 - a_1\dot{\theta}_1^2\, {}^0\mathbf{x}_1 + \frac{a_2}{2}(\ddot{\theta}_1 + \ddot{\theta}_2)\, {}^0\mathbf{y}_2 - \frac{a_2}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2\, {}^0\mathbf{x}_2 \right) \\ &= m_2 a_1 \ddot{\theta}_1 \cos\theta_2 + m_2 a_1 \dot{\theta}_1^2 \sin\theta_2 + m_2 \frac{a_2}{2}(\ddot{\theta}_1 + \ddot{\theta}_2) \end{aligned} \tag{10.70}$$

Substituting both results into (10.68),

$$\begin{aligned} \tau_2 &= {}^0I_{2,33}(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 a_1 \frac{a_2}{2}\cos\theta_2 \ddot{\theta}_1 + m_2 a_1 \frac{a_2}{2}\sin\theta_2 \dot{\theta}_1^2 + m_2 \frac{a_2^2}{4}(\ddot{\theta}_1 + \ddot{\theta}_2) \\ &= {}^0I'_{2,33}(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 a_1 \frac{a_2}{2}\cos\theta_2 \ddot{\theta}_1 + m_2 a_1 \frac{a_2}{2}\sin\theta_2 \dot{\theta}_1^2 \end{aligned} \tag{10.71}$$

where ${}^0I'_{2,33} = {}^0I_{2,33} + m_2 a_2^2/4$ is the inertia of link 2 referred to O_1 according to the parallel axis theorem.

10.5.2 Joint 1 Torques

Now let's evaluate the torque at joint 1. Combining (10.46), (10.52), and (10.55),

$$\begin{aligned} \tau_1 &= \mathbf{z}_0 \cdot \left(\mathbf{n}_1 + \mathbf{n}_{12} + \frac{a_1}{2}\mathbf{x}_1 \times \mathbf{f}_{01} + \frac{a_1}{2}\mathbf{x}_1 \times \mathbf{f}_{12} \right) \\ &= \mathbf{z}_0 \cdot \mathbf{n}_1 + \mathbf{z}_0 \cdot \mathbf{n}_{12} + \frac{a_1}{2}\mathbf{y}_1 \cdot (\mathbf{f}_{01} + \mathbf{f}_{12}) \\ &= \mathbf{z}_0 \cdot \mathbf{n}_1 + \tau_2 + \frac{a_1}{2}\mathbf{y}_1 \cdot \mathbf{f}_1 + a_1 \mathbf{y}_1 \cdot \mathbf{f}_2 \end{aligned} \tag{10.72}$$

The projection of Euler's equation (10.53) onto ${}^0\mathbf{z}_0$ is:

$$\begin{aligned} {}^0\mathbf{z}_0 \cdot {}^0\mathbf{n}_1 &= {}^0\mathbf{z}_0 \cdot \left({}^0\mathbf{I}_1 \ddot{\theta}_1\, {}^0\mathbf{z}_0 + \dot{\theta}_1\, {}^0\mathbf{z}_0 \times {}^0\mathbf{I}_1 \dot{\theta}_1\, {}^0\mathbf{z}_0 \right) \\ &= {}^0I_{1,33}\ddot{\theta}_1 \end{aligned} \tag{10.73}$$

The projection of Newton's equation (10.64) onto ${}^0\mathbf{y}_1$ is:

$$\begin{aligned} {}^0\mathbf{y}_1 \cdot {}^0\mathbf{f}_1 &= {}^0\mathbf{y}_1 \cdot m_1 \left(\frac{a_1}{2}\ddot{\theta}_1 \mathbf{y}_1 - \frac{a_1}{2}\dot{\theta}_1^2 \mathbf{x}_1 \right) \\ &= m_1 \frac{a_1}{2}\ddot{\theta}_1 \end{aligned} \tag{10.74}$$

The projection of Newton's equation (10.65) onto ${}^0\mathbf{y}_1$ is:

$$\begin{aligned}
{}^0\mathbf{y}_1 \cdot {}^0\mathbf{f}_2 &= {}^0\mathbf{y}_1 \cdot m_2 \left(a_1\ddot{\theta}_1\mathbf{y}_1 - a_1\dot{\theta}_1^2\mathbf{x}_1 + \frac{a_2}{2}(\ddot{\theta}_1 + \ddot{\theta}_2)\mathbf{y}_2 - \frac{a_2}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2\mathbf{x}_2 \right) \\
&= m_2 a_1\ddot{\theta}_1 + m_2\frac{a_2}{2}(\ddot{\theta}_1 + \ddot{\theta}_2)\cos\theta_2 - m_2\frac{a_2}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2\sin\theta_2
\end{aligned} \tag{10.75}$$

Substitute these three results into (10.72):

$$\begin{aligned}
\tau_1 &= {}^0I_{1,33}\ddot{\theta}_1 + \tau_2 + m_1\frac{a_1^2}{4}\ddot{\theta}_1 + m_2a_1^2\ddot{\theta}_1 + m_2a_1\frac{a_2}{2}(\ddot{\theta}_1 + \ddot{\theta}_2)\cos\theta_2 - \\
&\quad m_2a_1\frac{a_2}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2\sin\theta_2 \\
&= \left({}^0I'_{1,33} + m_2a_1^2 + m_2a_1\frac{a_2}{2}\cos\theta_2 \right)\ddot{\theta}_1 + m_2a_1\frac{a_2}{2}\ddot{\theta}_2\cos\theta_2 + \tau_2 - \\
&\quad m_2a_1\frac{a_2}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2\sin\theta_2
\end{aligned} \tag{10.76}$$

where ${}^0I'_{1,33} = {}^0I_{1,33} + m_1a_1^2/4$ is the inertia of link 1 about O_0 according to the parallel axis theorem.

This is a recursive relationship involving τ_2. Substitute for τ_2 from (10.71):

$$\begin{aligned}
\tau_1 &= \left({}^0I'_{1,33} + m_2a_1^2 + m_2a_1\frac{a_2}{2}\cos\theta_2 \right)\ddot{\theta}_1 + m_2a_1\frac{a_2}{2}\ddot{\theta}_2\cos\theta_2 + \\
&\quad {}^0I'_{2,33}(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2a_1\frac{a_2}{2}\cos\theta_2\ddot{\theta}_1 + m_2a_1\frac{a_2}{2}\sin\theta_2\dot{\theta}_1^2 - \\
&\quad m_2a_1\frac{a_2}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2\sin\theta_2 \\
&= \left({}^0I'_{1,33} + {}^0I'_{2,33} + m_2a_1(a_1 + a_2\cos\theta_2) \right)\ddot{\theta}_1 + \left({}^0I'_{2,33} + m_2a_1\frac{a_2}{2}\cos\theta_2 \right)\ddot{\theta}_2 + m_2a_1\frac{a_2}{2}\sin\theta_2\dot{\theta}_1^2 - \\
&\quad m_2a_1\frac{a_2}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2\sin\theta_2 \\
&= \left({}^0I'_{1,33} + {}^0I'_{2,33} + m_2a_1(a_1 + a_2\cos\theta_2) \right)\ddot{\theta}_1 + \left({}^0I'_{2,33} + m_2a_1\frac{a_2}{2}\cos\theta_2 \right)\ddot{\theta}_2 - \\
&\quad m_2a_1\frac{a_2}{2}\sin\theta_2\dot{\theta}_2^2 - m_2a_1a_2\sin\theta_2\dot{\theta}_1\dot{\theta}_2
\end{aligned} \tag{10.77}$$

This and the expression for τ_2 (10.71) are rather complex. Looking at (10.77), there are several different types of terms. There are *inertial torques* which are proportional to joint acceleration in the first line. The joint 1 torque depends on the acceleration of joint 2 as well as the acceleration of joint 1. There are *centripetal torques* in the second line proportional to the square of the joint 2 velocity and *coriolis torques* proportional to the product of joint 1 and 2 velocities.

10.5.3 Closed Form

Let's stack the equations for τ_1 and τ_2 to form a matrix-vector equation

$$\begin{aligned}
\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} &= \begin{bmatrix} {}^0I'_{1,33} + {}^0I'_{2,33} + m_2a_1(a_1 + a_2\cos\theta_2) & {}^0I'_{2,33} + \frac{1}{2}m_2a_1a_2\cos\theta_2 \\ {}^0I'_{2,33} + \frac{1}{2}m_2a_1a_2\cos\theta_2 & {}^0I'_{2,33} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \\
& \begin{bmatrix} 0 & -m_2a_1a_2\sin\theta_2 & \frac{1}{2}m_2a_1a_2\sin\theta_2 \\ \frac{1}{2}m_2a_1a_2\sin\theta_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2^2 \end{bmatrix} \\
\boldsymbol{\tau} &= \mathbf{H}\ddot{\boldsymbol{\theta}} + \mathbf{C} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2^2 \end{bmatrix} \qquad (10.78)
\end{aligned}$$

where $\boldsymbol{\tau}$ is the vector of joint torques, $\mathbf{H}$ is the inertia matrix of the robot in joint space, and $\mathbf{C}$ is a position-dependent matrix for the velocity product terms. This closed-form dynamics expression generalizes to all robots.

10.6 General Manipulator Dynamics

When the motion of a manipulator is prescribed and the driving forces or torques are to be found, the computation is referred to as the *inverse dynamics* computation. Inverse dynamics are typically solved to control a robot. When the driving forces or torques are prescribed and the resultant motion is to be found, the computation is referred to as the *forward dynamics* computation. When simulating a manipulator, one typically solves for the forward dynamics. As for the kinematics, the terms *inverse* and *forward* have to do with which variable is more intrinsic; in this case, actuator force and torque are more intrinsic than joint angles, because the joint angles result from the application of forces and torques. In forward dynamics, force and torque are the input and joint motion is the output, which represents the causal order. In inverse dynamics, joint motion is the input and joint forces and torques are the output, which is the inverse of the causal order.

We'll now apply the Newton-Euler equations to describe the inverse dynamics of a general robot arm.

10.6.1 Last Link Dynamics

Let us begin by describing the dynamics of the last link (Figure 10.6). This is just like the dynamics of the one-link manipulator or the last link of the two-link planar manipulator, but in a generalized setting. Suppose that the last link n is not touching anything.

m_n	is the mass of link n.
$\mathbf{f}_{n-1,n}$	is the constraint force acting at joint n's center O_{n-1}.
$\mathbf{n}_{n-1,n}$	is the constraint torque acting at joint n's center O_{n-1}.
$\mathbf{r}_{n-1,n}$	is the vector from joint n center O_{n-1} to the center of gravity Q_n.
$\mathbf{d}_{0,n-1}$	is a vector from O_0 to O_{n-1}.

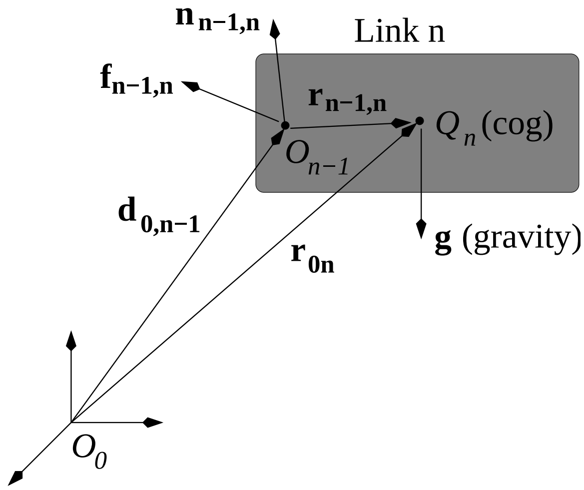

Figure 10.6: Dynamics of the last link.

The subscript $n-1, n$ on the constraint force means that $\mathbf{f}_{n-1,n}$ is the force exerted *on* link n *by* link $n-1$. By Newton's third law of action/reaction, then $\mathbf{f}_{n,n-1} = -\mathbf{f}_{n-1,n}$ is the force exerted on link $n-1$ by link n. Similarly for the constraint torque.

- If joint n is a rotary joint, then the component of constraint torque along the joint axis is the joint torque τ_n supplied by the rotary actuator:

$$\tau_n = \mathbf{z}_{n-1} \cdot \mathbf{n}_{n-1,n} \tag{10.79}$$

  This joint torque is a scalar, because we are presuming single degree-of-freedom joints. The components of constraint torque $\mathbf{n}_{n-1,n}$ orthogonal to the joint axis are passed through the structure to the more proximal joints. All three components of the constraint force $\mathbf{f}_{n-1,n}$ are passed through the structure.

- If joint n is a prismatic joint, then the component of constraint force along the joint axis is the joint force f_n supplied by the linear actuator:

$$f_n = \mathbf{z}_{n-1} \cdot \mathbf{f}_{n-1,n} \tag{10.80}$$

  Again, the components of constraint force $\mathbf{f}_{n-1,n}$ orthogonal to the joint axis are passed through the structure to the more proximal joints. All three components of the constraint torque $\mathbf{n}_{n-1,n}$ are passed through the structure.

The net force $\mathbf{f}_n$ and net torque $\mathbf{n}_n$ acting at link n's center of gravity can then be written:

$$\begin{aligned} \mathbf{f}_n &= \mathbf{f}_{n-1,n} \\ &= m_n \ddot{\mathbf{r}}_{0n} \end{aligned} \tag{10.81}$$

$$\begin{aligned} \mathbf{n}_n &= \mathbf{n}_{n-1,n} - \mathbf{r}_{n-1,n} \times \mathbf{f}_{n-1,n} \\ &= \mathbf{I}_n \dot{\boldsymbol{\omega}}_{0n} + \boldsymbol{\omega}_{0n} \times \mathbf{I}_n \boldsymbol{\omega}_{0n} \end{aligned} \tag{10.82}$$

where the Newton-Euler equations have been substituted for the net force and torque. Here

$\ddot{\mathbf{r}}_{0n}$ is the acceleration of link n's center of gravity,

$\mathbf{I}_n$ is the inertia of link n about its center of gravity.

$\boldsymbol{\omega}_{0n}$ is link n's angular velocity, and

$\dot{\boldsymbol{\omega}}_{0n}$ is link n's angular acceleration.

In the torque balance equation, note the subscript on the constraint force $\mathbf{f}_{n-1,n}$: since we are writing the torque balance for link n, we want to know the force acting on link n by link $n-1$, not vice versa. Also, the minus sign is due to the moment arm vector $\mathbf{r}_{n-1,n}$ pointing from the joint to the center of gravity, rather than vice versa.

As a result of motion planning, we usually know the joint variable positions, velocities, and accelerations. Consequently, we may find the joint n position $\mathbf{d}_{0,n-1}$, velocity $\dot{\mathbf{d}}_{0,n-1}$, and acceleration $\ddot{\mathbf{d}}_{0,n-1}$, and the link n angular velocity $\boldsymbol{\omega}_{0n}$ and angular acceleration $\dot{\boldsymbol{\omega}}_{0n}$ by solving the forward kinematics. The acceleration $\ddot{\mathbf{r}}_{0n}$ of the center of gravity is then found from:

$$
\begin{aligned}
\mathbf{r}_{0n} &= \mathbf{d}_{0,n-1} + \mathbf{r}_{n-1,n} \\
\dot{\mathbf{r}}_{0n} &= \dot{\mathbf{d}}_{0,n-1} + \boldsymbol{\omega}_{0n} \times \mathbf{r}_{n-1,n} + \begin{cases} \mathbf{0} & \text{joint } n \text{ rotary} \\ \dot{d}_n \mathbf{z}_{n-1} & \text{joint } n \text{ prismatic} \end{cases} \\
\ddot{\mathbf{r}}_{0n} &= \ddot{\mathbf{d}}_{0,n-1} + \dot{\boldsymbol{\omega}}_{0n} \times \mathbf{r}_{n-1,n} + \boldsymbol{\omega}_{0n} \times (\boldsymbol{\omega}_{0n} \times \mathbf{r}_{n-1,n}) \\
&\quad + \begin{cases} \mathbf{0} & \text{joint } n \text{ rotary} \\ 2\boldsymbol{\omega}_{0,n-1} \times \dot{d}_n \mathbf{z}_{n-1} + \ddot{d}_n \mathbf{z}_{n-1} & \text{joint } n \text{ prismatic} \end{cases}
\end{aligned}
$$

10.6.2 Intermediate Link Dynamics

An intermediate link has joint attachments at both ends, which give rise to 2 sets of constraint forces and torques (Figure 10.7). The net force $\mathbf{f}_i$ and net torque $\mathbf{n}_i$ acting at link i's center of gravity can then be written:

$$
\begin{aligned}
\mathbf{f}_i &= \mathbf{f}_{i-1,i} - \mathbf{f}_{i,i+1} \qquad (10.83) \\
&= m_i \ddot{\mathbf{r}}_{0i}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{n}_i &= \mathbf{n}_{i-1,i} - \mathbf{n}_{i,i+1} - \mathbf{r}_{i-1,i} \times \mathbf{f}_{i-1,i} + \mathbf{r}_{ii} \times \mathbf{f}_{i,i+1} \qquad (10.84) \\
&= \mathbf{I}_i \dot{\boldsymbol{\omega}}_{0i} + \boldsymbol{\omega}_{0i} \times \mathbf{I}_i \boldsymbol{\omega}_{0i}
\end{aligned}
$$

where the Newton-Euler equations have again been substituted for the net force and torque. The meanings of the symbols is similar to that for link n, save for the change in link number. Also,

$\mathbf{r}_{ii}$ is the vector from O_i to Q_i.

In the net force equation, note the minus sign on $\mathbf{f}_{i,i+1}$. We are interested in the force that link $i+1$ exerts on link i, which is $\mathbf{f}_{i+1,i} = -\mathbf{f}_{i,i+1}$. We prefer to employ the $\mathbf{f}_{i,i+1}$ term for consistency in considering all manipulator links. Similarly, in the torque balance equation there appears the term $-\mathbf{n}_{i,i+1}$. The torque due to the joint $i+1$ constraint force is actually $(-\mathbf{r}_{ii}) \times (-\mathbf{f}_{i,i+1})$, but the two minus signs cancel out.

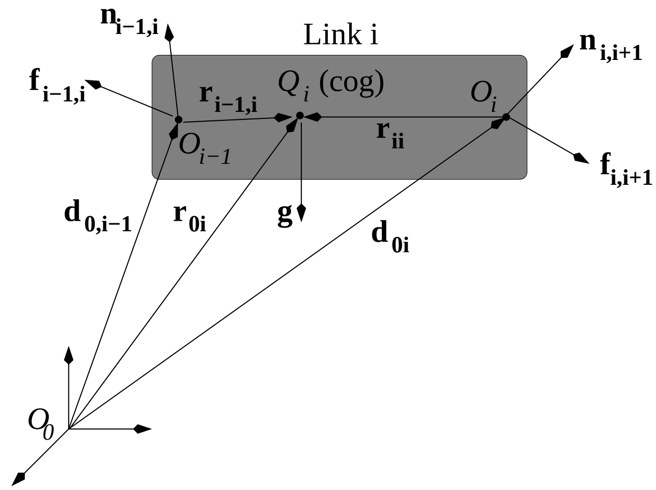

Figure 10.7: Dynamics of an intermediate link i.

If the kinematics and inertia matrix are known, then the actuator force f_i or torque τ_i at joint i may be found similar as before:

$$\tau_i = \mathbf{z}_{i-1} \cdot \mathbf{n}_{i-1,i} \qquad \text{for a rotary joint} \tag{10.85}$$

$$f_i = \mathbf{z}_{i-1} \cdot \mathbf{f}_{i-1,i} \qquad \text{for a prismatic joint} \tag{10.86}$$

The linear acceleration $\ddot{\mathbf{r}}_{0i}$ of link i's center of mass is similar to (10.83).

10.6.3 Coordinate System Considerations

All vectors and matrices can be expressed with regard to a convenient coordinate system orientation, such as the base coordinates 0. Usually, we know the location of the center of mass as vectors ${}^i\mathbf{r}_{i-1,i}$ and ${}^i\mathbf{r}_{ii}$ that are constant in link i coordinates; appropriate rotational transformations would have to be performed to find them in terms of link 0 coordinates as ${}^0\mathbf{r}_{i-1,i}$ and ${}^0\mathbf{r}_{ii}$. Similarly, the inertia matrix ${}^i\mathbf{I}_i$ is constant in link i coordinates, and would have to be transformed according to (10.18) to find ${}^0\mathbf{I}_i$.

This procedure is reasonably efficient and transparent. A little more efficiency can be obtained by carrying out the dynamics computations for each link in terms of its own coordinate system, although the efficiency gains are relatively minor. The Newton-Euler equations for link i are computed as:

$${}^i\mathbf{f}_i = m_i \, {}^i\ddot{\mathbf{r}}_{0i} \tag{10.87}$$

$${}^i\mathbf{n}_i = {}^i\mathbf{I}_i \, {}^i\dot{\boldsymbol{\omega}}_{0i} + {}^i\boldsymbol{\omega}_{0i} \times {}^i\mathbf{I}_i \, {}^i\boldsymbol{\omega}_{0i} \tag{10.88}$$

This procedure avoids transforming the inertia matrix ${}^i\mathbf{I}_i$. The force and torque balance equations (10.83)-(10.84) become:

$${}^i\mathbf{f}_{i-1,i} = {}^i\mathbf{f}_i + {}^i\mathbf{f}_{i,i+1} \tag{10.89}$$

$${}^i\mathbf{n}_{i-1,i} = {}^i\mathbf{n}_i + {}^{i+1}\mathbf{n}_{i,i+1} + {}^i\mathbf{r}_{i-1,i} \times {}^i\mathbf{f}_{i-1,i} - {}^i\mathbf{r}_{ii} \times {}^i\mathbf{f}_{i,i+1} \tag{10.90}$$

This procedure avoids transforming ${}^i\mathbf{r}_{i-1,i}$ and ${}^i\mathbf{r}_{ii}$. The joint force or torque is then found from:

$$\tau_i = {}^{i-1}\mathbf{z}_{i-1} \cdot {}^{i-1}\mathbf{n}_{i-1,i} \qquad \text{for a rotary joint} \tag{10.91}$$

$$f_i = {}^{i-1}\mathbf{z}_{i-1} \cdot {}^{i-1}\mathbf{f}_{i-1,i} \qquad \text{for a prismatic joint} \tag{10.92}$$

This procedure amounts to just copying the last component of the dot product. The constraint force and torque have to be transformed to the $i-1$ frame.

10.6.4 Recursive Newton-Euler Dynamics

With respect to inverse dynamics, the driving forces and torques to realize a desired joint motion can be readily found through an efficient recursive procedure. At the last link n, since there is only one set of constraint force and torque due to attachment to link $n-1$, the solution for the net force $\mathbf{f}_n$ and torque $\mathbf{n}_n$ by the Newton-Euler equations leads to the solution for the constraint force $\mathbf{f}_{n-1,n}$ and torque $\mathbf{n}_{n-1,n}$ in (10.81)-(10.82). This constraint force and torque then becomes an input to the link $n-1$ dynamics, which have the form (10.83)-(10.84). After solving for the net force $\mathbf{f}_{n-1}$ and net torque $\mathbf{n}_{n-1}$ from the Newton-Euler equations, then the proximal joint constraint force $\mathbf{f}_{n-2,n-1}$ and constraint torque $\mathbf{n}_{n-2,n-1}$ may be found. This procedure is worked recursively all the way to the base. This *recursive Newton-Euler dynamics computation* is concretely summarized as a series of steps.

Assume that the joint motions have been derived from a trajectory planner.

1. Express coordinate axes $\mathbf{x}_i$ and $\mathbf{z}_i$ and center of gravity locations $\mathbf{r}_{i-1,i}$ and $\mathbf{r}_{ii}$ in a convenient coordinate frame, say frame 0, for all links. Also solve for the inertia matrices $\mathbf{I}_i$ in this same coordinate frame.

2. Set the link counter to $i = 1$. Assuming that the base is at rest, then at the beginning of the recursion for link 1,

$$\boldsymbol{\omega}_{01} = \begin{cases} \dot{\theta}_1 \mathbf{z}_0 & \text{joint 1 rotary} \\ \mathbf{0} & \text{joint 1 prismatic} \end{cases} \tag{10.93}$$

$$\dot{\boldsymbol{\omega}}_{01} = \begin{cases} \ddot{\theta}_i \mathbf{z}_0 & \text{joint 1 rotary} \\ \mathbf{0} & \text{joint 1 prismatic} \end{cases} \tag{10.94}$$

   Derive the acceleration of the link 1 origin O_1 from:

$$\ddot{\mathbf{d}}_{01} = \begin{cases} \dot{\boldsymbol{\omega}}_{01} \times \mathbf{d}_{01} + \boldsymbol{\omega}_{01} \times (\boldsymbol{\omega}_{01} \times \mathbf{d}_{01}) & \text{joint 1 rotary} \\ \ddot{d}_1 \mathbf{z}_0 & \text{joint 1 prismatic} \end{cases} \tag{10.95}$$

   where $\mathbf{d}_{01} = d_1\mathbf{z}_0 + a_1\mathbf{x}_1$ and it is assumed that the base is stationary. If the base is moving, say with respect to coordinate system -1, then provide $\ddot{\mathbf{d}}_{-1,0}$, $\boldsymbol{\omega}_{-1,0}$, and $\dot{\boldsymbol{\omega}}_{-1,0}$ as initial conditions and use the more general relationships in step 3. Solve for the acceleration of link 1's center of mass:

$$\ddot{\mathbf{r}}_{01} = \begin{cases} \dot{\boldsymbol{\omega}}_{01} \times \mathbf{r}_{01} + \boldsymbol{\omega}_{01} \times (\boldsymbol{\omega}_{01} \times \mathbf{r}_{01}) & \text{joint 1 rotary} \\ \ddot{d}_1 \mathbf{z}_0 & \text{joint 1 prismatic} \end{cases} \tag{10.96}$$

3. Set the link counter to $i = i + 1$. If $i > n$, go to the next step. Otherwise, solve for the angular velocity and acceleration of link i:

$$\boldsymbol{\omega}_{0i} = \boldsymbol{\omega}_{0,i-1} + \begin{cases} \dot{\theta}_i \mathbf{z}_{i-1} & \text{joint } i \text{ rotary} \\ \mathbf{0} & \text{joint } i \text{ prismatic} \end{cases} \tag{10.97}$$

$$\dot{\boldsymbol{\omega}}_{0i} = \dot{\boldsymbol{\omega}}_{0,i-1} + \begin{cases} \ddot{\theta}_i \mathbf{z}_{i-1} + \dot{\theta}_i \boldsymbol{\omega}_{0,i-1} \times \mathbf{z}_{i-1} & \text{joint } i \text{ rotary} \\ \mathbf{0} & \text{joint } i \text{ prismatic} \end{cases} \tag{10.98}$$

Solve for the joint acceleration $\ddot{\mathbf{d}}_{0i}$:

$$\mathbf{d}_{0i} = \mathbf{d}_{0,i-1} + \mathbf{d}_{i-1,i} \tag{10.99}$$

$$\begin{aligned} \ddot{\mathbf{d}}_{0i} &= \ddot{\mathbf{d}}_{0,i-1} + \dot{\boldsymbol{\omega}}_{0i} \times \mathbf{d}_{i-1,i} + \boldsymbol{\omega}_{0i} \times (\boldsymbol{\omega}_{0i} \times \mathbf{d}_{i-1,i}) \\ &\quad + \begin{cases} \mathbf{0} & \text{joint } i \text{ rotary} \\ 2\dot{d}_i \boldsymbol{\omega}_{0,i-1} \times \mathbf{z}_{i-1} + \ddot{d}_i \mathbf{z}_{i-1} & \text{joint } i \text{ prismatic} \end{cases} \end{aligned} \tag{10.100}$$

where $\mathbf{d}_{i-1,i} = O_i - O_{i-1} = d_i \mathbf{z}_{i-1} + a_i \mathbf{x}_i$. Find the acceleration of link i's center of mass according to (10.83):

$$\begin{aligned} \ddot{\mathbf{r}}_{0i} &= \ddot{\mathbf{d}}_{0,i-1} + \dot{\boldsymbol{\omega}}_{0i} \times \mathbf{r}_{i-1,i} + \boldsymbol{\omega}_{0i} \times (\boldsymbol{\omega}_{0i} \times \mathbf{r}_{i-1,i}) \\ &\quad + \begin{cases} \mathbf{0} & \text{joint } i \text{ rotary} \\ 2\dot{d}_i \boldsymbol{\omega}_{0,i-1} \times \mathbf{z}_{i-1} + \ddot{d}_i \mathbf{z}_{i-1} & \text{joint } i \text{ prismatic} \end{cases} \end{aligned} \tag{10.101}$$

Repeat this step.

4. Set link counter $i = n$. Solve the Newton-Euler dynamics at the last link to find the net force and torque acting on the last link using (10.81)-(10.82):

$$\mathbf{f}_n = m_n \ddot{\mathbf{r}}_{0n} \tag{10.102}$$

$$\mathbf{n}_n = \mathbf{I}_n \dot{\boldsymbol{\omega}}_{0n} + \boldsymbol{\omega}_{0n} \times \mathbf{I}_n \boldsymbol{\omega}_{0n} \tag{10.103}$$

Then solve for the constraint force and torque at joint n:

$$\mathbf{f}_{n-1,n} = \mathbf{f}_n \tag{10.104}$$

$$\mathbf{n}_{n-1,n} = \mathbf{n}_n + \mathbf{r}_{n-1,n} \times \mathbf{f}_{n-1,n} \tag{10.105}$$

Find the actuator force f_n or torque τ_n according to (10.80) or (10.79):

$$\tau_n = \mathbf{z}_{n-1} \cdot \mathbf{n}_{n-1,n} \qquad \text{if joint } n \text{ is rotary} \tag{10.106}$$

$$f_n = \mathbf{z}_{n-1} \cdot \mathbf{f}_{n-1,n} \qquad \text{if joint } n \text{ is prismatic} \tag{10.107}$$

5. Set the link counter to $i = i - 1$. If $i = 0$, we are at the base and have finished the procedure. Otherwise, solve the Newton-Euler equations for the net force $\mathbf{f}_i$ and net torque $\mathbf{n}_i$:

$$\mathbf{f}_i = m_i \ddot{\mathbf{r}}_{0i} \tag{10.108}$$

$$\mathbf{n}_i = \mathbf{I}_i \dot{\boldsymbol{\omega}}_{0i} + \boldsymbol{\omega}_{0i} \times \mathbf{I}_i \boldsymbol{\omega}_{0i} \tag{10.109}$$

Then find the constraint force $\mathbf{f}_{i-1,i}$ and constraint torque $\mathbf{n}_{i-1,i}$ from (10.83)-(10.84).

$$\mathbf{f}_{i-1,i} = \mathbf{f}_i + \mathbf{f}_{i,i+1} \tag{10.110}$$

$$\mathbf{n}_{i-1,i} = \mathbf{n}_i + \mathbf{n}_{i,i+1} + \mathbf{r}_{i-1,i} \times \mathbf{f}_{i-1,i} - \mathbf{r}_{ii} \times \mathbf{f}_{i,i+1} \tag{10.111}$$

Find the actuator force f_n or torque τ_n according to (10.86) or (10.85).

$$\tau_i = \mathbf{z}_{i-1} \cdot \mathbf{n}_{i-1,i} \quad \text{for a rotary joint} \tag{10.112}$$

$$f_i = \mathbf{z}_{i-1} \cdot \mathbf{f}_{i-1,i} \quad \text{for a prismatic joint} \tag{10.113}$$

Repeat this step.

This procedure is very efficient, as the computional complexity is linear with the number of joints n [1].

10.6.5 Closed-Form Equations

As for the two-link case, the manipulator dynamics can be computed in a closed rather than a recursive form. Similar to (10.78), the expression for rotary manipulators is:

$$\boldsymbol{\tau} = \mathbf{H}\ddot{\boldsymbol{\theta}} + \mathbf{C} \circ \dot{\boldsymbol{\theta}}\dot{\boldsymbol{\theta}}^T \tag{10.114}$$

where we define the tensor operation $\circ$ to represent the velocity product terms:

$$\mathbf{v} \equiv \mathbf{C} \circ \dot{\boldsymbol{\theta}}\dot{\boldsymbol{\theta}}^T \quad \text{where} \quad v_k = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ijk}\dot{\theta}_i\dot{\theta}_j \tag{10.115}$$

The closed-form solution allows the manipulator dynamics to be represented in compact notation, although the actual elements of $\mathbf{H}$ and $\mathbf{C}$ are extremely complex. This compact notation is typically used in making formal arguments about robot control or in dynamic simulation [3]. In the past, researchers developed the symbolic expressions for particular manipulators in laborious fashion. Often we are only interested in the numerical values of the elements, which can be derived using the recursive Newton-Euler equations. For example H_{11} can be derived by setting $\dot{\boldsymbol{\theta}} = \mathbf{0}$ and $\ddot{\boldsymbol{\theta}} = \mathbf{e}_1 = (1, 0, \ldots, 0)$, then solving for τ_1.

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