Chapter 7

Velocity and Acceleration

Trajectory planning results in a certain velocity of the manipulator end link and any grasped tool. This velocity comprised both time-varying positions and orientations. We now focus on how these velocities of objects and manipulators may be described.

Previously, we considered how object position and orientation may be represented, before we considered the forward and inverse kinematics for position of manipulators. Similarly, before discussing the velocities of manipulators, we will consider the linear and angular velocity, as well as acceleration, of rigid bodies.

7.1 Planar Motion of Objects

We begin by describing motion of single objects in the plane. First consider a single link rotating about origin O_0 (Figure 7.1(A)). The position $\mathbf{d}_{01} = a_1\mathbf{x}_1$ of the endpoint is

$$ {}^0\mathbf{d}_{01} = a_1\,{}^0\mathbf{x}_1 = \begin{bmatrix} a_1\mathrm{c}\theta_1 \\ a_1\mathrm{s}\theta_1 \end{bmatrix} = a_1 \begin{bmatrix} \mathrm{c}\theta_1 \\ \mathrm{s}\theta_1 \end{bmatrix} \tag{7.1} $$

Suppose the link is moving due to joint angle velocity $\dot{\theta}_1$. Then the velocity of the endpoint is found by differentiating (7.1):

$$ {}^0\dot{\mathbf{d}}_{01} = \begin{bmatrix} -a_1\dot{\theta}_1\mathrm{s}\theta_1 \\ a_1\dot{\theta}_1\mathrm{c}\theta_1 \end{bmatrix} = a_1\dot{\theta}_1 \begin{bmatrix} -\mathrm{s}\theta_1 \\ \mathrm{c}\theta_1 \end{bmatrix} = a_1\dot{\theta}_1\,{}^0\mathbf{y}_1 \tag{7.2} $$

The velocity direction is $\mathbf{y}_1$, i.e., tangential to the position vector $\mathbf{d}_{01}$ which points along the $\mathbf{x}_1$ axis. The one-link motion is just like twirling with rotational velocity ω a ball on a string of radius r, where the tangential velocity of the ball is $v = r\omega$ (Figure 7.1(B)).

A more general derivation is based on the use of rotation matrices. Rewrite (7.1) as

$$ {}^0\mathbf{d}_{01} = {}^0\mathbf{R}_1 a_1\,{}^1\mathbf{x}_1 \tag{7.3} $$

Differentiating,

$$ {}^0\dot{\mathbf{d}}_{01} = {}^0\dot{\mathbf{R}}_1 a_1\,{}^1\mathbf{x}_1 + {}^0\mathbf{R}_1 a_1\,{}^1\dot{\mathbf{x}}_1 = {}^0\dot{\mathbf{R}}_1 a_1\,{}^1\mathbf{x}_1 \tag{7.4} $$

Joint axis ${}^1\mathbf{x}_1 = \mathbf{i}$ is constant, so its time derivative is zero. The planar rotation matrix is:

$$ {}^0\mathbf{R}_1 = \mathbf{R}(\theta_1) = \begin{bmatrix} \mathrm{c}\theta_1 & -\mathrm{s}\theta_1 \\ \mathrm{s}\theta_1 & \mathrm{c}\theta_1 \end{bmatrix} \tag{7.5} $$

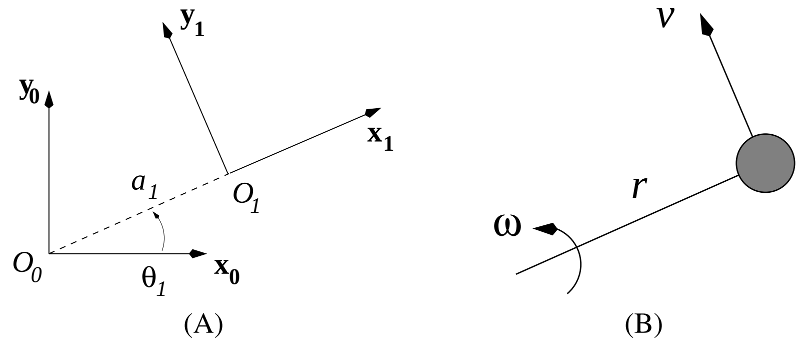

Figure 7.1: (A) A one-link manipulator. (B) An analogous twirling ball.

It's time derivative is most easily found using the chain rule:

$$\dot{\mathbf{R}}(\theta_1) = \frac{\partial \mathbf{R}(\theta_1)}{\partial \theta_1} \; \frac{d\theta_1}{dt} = \dot{\theta}_1 \begin{bmatrix} -\mathrm{s}\theta_1 & -\mathrm{c}\theta_1 \\ \mathrm{c}\theta_1 & -\mathrm{s}\theta_1 \end{bmatrix} = \dot{\theta}_1 \; \mathbf{R}(\theta_1 + \pi/2) \tag{7.6}$$

Substituting into (7.4),

$$\begin{aligned} {}^0\dot{\mathbf{d}}_{01} &= \dot{\theta}_1 \mathbf{R}(\theta_1 + \pi/2) a_1 \, {}^1\mathbf{x}_1 = \dot{\theta}_1 \mathbf{R}(\theta_1)\mathbf{R}(\pi/2) a_1 \, {}^1\mathbf{x}_1 = \dot{\theta}_1 \mathbf{R}(\theta_1) a_1 \, {}^1\mathbf{y}_1 \\ &= a_1 \dot{\theta}_1 \, {}^0\mathbf{y}_1 \end{aligned} \tag{7.7}$$

where $\mathbf{y}_1$ is obtained by rotating $\mathbf{x}_1$ by $\pi/2$.

Joint 1 constrains the motion of link 1 to be a simple rotation about O_0. Consider the more general case of an unconstrained object with an embedded frame 1 located relative to frame 0 or ground by ${}^0\mathbf{d}_{01}$ and θ_1 (Figure 7.2(A)). Suppose ${}^1\mathbf{p}_1$ is a fixed vector measured in frame 1.

$${}^0\mathbf{p}_0 = {}^0\mathbf{d}_{01} + \mathbf{R}(\theta_1) \, {}^1\mathbf{p}_1 \tag{7.8}$$

$${}^0\dot{\mathbf{p}}_0 = {}^0\dot{\mathbf{d}}_{01} + \dot{\mathbf{R}}(\theta_1) \, {}^1\mathbf{p}_1 \tag{7.9}$$

where ${}^1\dot{\mathbf{p}}_1 = \mathbf{0}$ because this vector is constant in frame 1. The interorigin velocity ${}^0\dot{\mathbf{d}}_{01}$ has to be provided independently of $\dot{\theta}_1$. That is to say, 3 numbers are required to describe the general motion of an object in a plane: 2 numbers for the linear motion of some reference point and one number to describe the rotary motion of the whole object. For the one-link case, the joint constraint causes the interorigin vector to lie along $\mathbf{x}_1$ and the interorigin velocity to depend on $\dot{\theta}_1$ via (7.7). Hence only one number is required to describe the motion of a link, the joint angle velocity.

Substituting for the time derivative of a rotation matrix (7.6),

$${}^0\dot{\mathbf{p}}_0 = \dot{\theta}_1 \; \mathbf{R}(\theta_1 + \pi/2)^1\mathbf{p}_1 + {}^0\dot{\mathbf{d}}_{01} \tag{7.10}$$

The vector $\mathbf{R}(\theta + \pi/2) \, {}^1\mathbf{p}_1$ is rotated from ${}^0\mathbf{p}_1$ by $+90^o$ and hence is perpendicular to ${}^0\mathbf{p}_1$ (Figure 7.2(B)). The velocity

$$\dot{\theta}_1 \; \mathbf{R}(\theta_1 + \pi/2)^1\mathbf{p}_1$$

is the tangential velocity due purely to rotational velocity $\dot{\theta}_1$, and sums with the velocity of displacement of the origin O_1. Its magnitude is $||\mathbf{p}_1|| \, \dot{\theta}_1$, which is again like the equation for the tangential velocity of circular motion $v = r\dot{\theta}_1$.

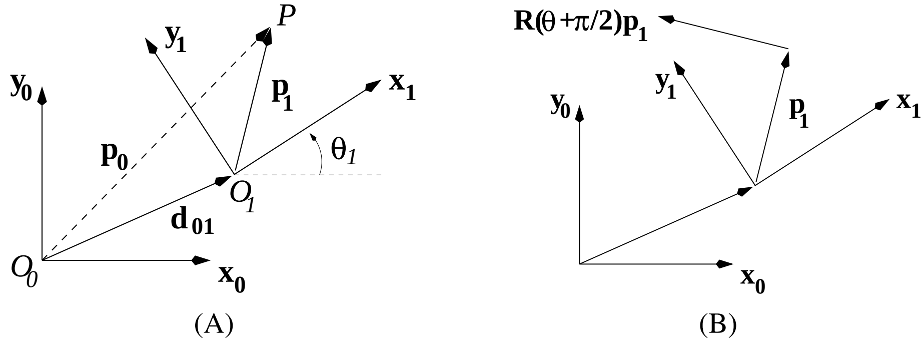

Figure 7.2: Motion of frame 1 relative to frame 0.

7.2 Spherical Motion of Objects

Reconsider the one-link manipulator in 3 dimensions, where $\mathbf{z}_0$ is now the explicit rotation axis (Figure 7.3). The direction $\mathbf{y}_1$ of the tangential velocity $a_1\dot{\theta}_1\mathbf{y}_1$ (7.7) could have been generated by crossing the rotation axis $\mathbf{z}_0$ with with the unit vector $\mathbf{x}_1$ pointing from O_0 to O_1, since $\mathbf{d}_{01} = a_1\mathbf{x}_1$: $\mathbf{y}_1 = \mathbf{z}_0 \times \mathbf{x}_1$ where $\mathbf{z}_0 = \mathbf{z}_1$. Multiplying through by $a_1\dot{\theta}_1$,

$$\begin{aligned} a_1\dot{\theta}_1\mathbf{y}_1 &= \dot{\theta}_1\mathbf{z}_0 \times a_1\mathbf{x}_1 \\ \dot{\mathbf{d}}_{01} &= \boldsymbol{\omega}_{01} \times \mathbf{d}_{01} \end{aligned} \tag{7.11}$$

where

$$\boldsymbol{\omega}_{01} = \dot{\theta}_1\mathbf{z}_0 \tag{7.12}$$

is the *angular velocity vector* of link 1 relative to the base 0. It is the product of the axis of rotation with the rotational velocity. The time derivative of the interorigin vector is the cross product of the angular velocity vector with the interorigin vector.

As before, let's repeat the process using the rotation matrix ${}^0\mathbf{R}_1 = \mathbf{R}_z(\theta_1)$ from frame 1 to 0. The result (7.4) has to be reinterpreted in 3 dimensions:

$${}^0\dot{\mathbf{d}}_{01} = \dot{\mathbf{R}}_z(\theta_1)a_1\,{}^1\mathbf{x}_1 \tag{7.13}$$

and so the question is what is $\dot{\mathbf{R}}_z(\theta_1)$. Again using the chain rule,

$$\begin{aligned} \dot{\mathbf{R}}_z(\theta_1) &= \frac{\partial \mathbf{R}_z(\theta_1)}{\partial\theta_1}\frac{d\theta_1}{dt} \\ &= \dot{\theta}_1\frac{\partial}{\partial\theta_1}\begin{bmatrix} \mathrm{c}\theta_1 & -\mathrm{s}\theta_1 & 0 \\ \mathrm{s}\theta_1 & \mathrm{c}\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \dot{\theta}_1\begin{bmatrix} -\mathrm{s}\theta_1 & -\mathrm{c}\theta_1 & 0 \\ \mathrm{c}\theta_1 & -\mathrm{s}\theta_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{7.14}$$

The result is not a rotation matrix because of the 0 in the 33 position. However, factor the result as

$$\dot{\mathbf{R}}_z(\theta_1) = \dot{\theta}_1\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathrm{c}\theta_1 & -\mathrm{s}\theta_1 & 0 \\ \mathrm{s}\theta_1 & \mathrm{c}\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

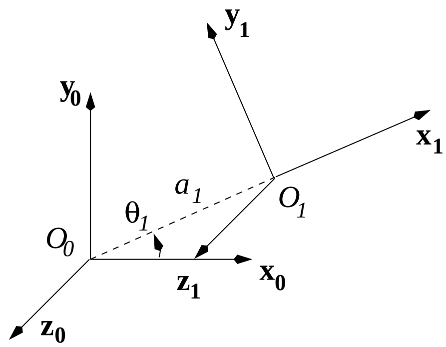

Figure 7.3: One-link rotation in 3D.

$$= \quad \dot{\theta}_1 \mathbf{S}({}^0\mathbf{z}_0)\mathbf{R}_z(\theta_1) \tag{7.15}$$

where

$$\mathbf{S}({}^0\mathbf{z}_0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.16}$$

is the skew-symmetric matrix formed from the components of ${}^0\mathbf{z}_0 = (0, 0, 1)$ (Appendix A). Substituting into (7.13),

$${}^0\dot{\mathbf{d}}_{01} = a_1\dot{\theta}_1 \mathbf{S}({}^0\mathbf{z}_0)\mathbf{R}_z(\theta_1)\ {}^1\mathbf{x}_1 = a_1\dot{\theta}_1\mathbf{S}({}^0\mathbf{z}_0)\ {}^0\mathbf{x}_1 = \mathbf{S}({}^0\boldsymbol{\omega}_0)\ {}^0\mathbf{d}_{01} \tag{7.17}$$

noindent which agrees with (7.11).

7.2.1 Rotational Motion about an Arbitrary Vector

The previous discussion was for rotation about a coordinate axis $\mathbf{z}_0$, of a vector perpendicular to the rotation axis. We generalize this result to rotational motion $\dot{\phi}$ about an arbitrary axis $\mathbf{k}$, of a vector $\mathbf{p}$ not necessarily perpendicular to the rotation axis $\mathbf{k}$. Suppose vector $\mathbf{p}$ has its tail fixed at origin O with the unit vector $\mathbf{k}$ (Figure 7.4). That is to say, the tip of $\mathbf{p}$ is undergoing circular motion about $\mathbf{k}$. Suppose $\mathbf{r}$ is the radial vector from the center of the tip motion to $\mathbf{p}$. Then

$$\mathbf{p} \quad = \quad \mathbf{k}(\mathbf{k}\cdot\mathbf{p}) + \mathbf{r} \tag{7.18}$$

$$\dot{\mathbf{p}} \quad = \quad \dot{\mathbf{r}} \tag{7.19}$$

since the projection of $\mathbf{p}$ onto $\mathbf{k}$ is fixed. The velocity of the radial vector $\dot{\mathbf{r}}$ is tangent to the circle with direction $\mathbf{k}\times\mathbf{r}$, and from the well-known relation $\dot{s} = r\dot{\phi}$ for circular motion,

$$\dot{\mathbf{r}} = \dot{\phi}\mathbf{k}\times\mathbf{r} \tag{7.20}$$

Hence

$$\dot{\mathbf{p}} = \dot{\phi}\mathbf{k}\times\mathbf{p} = \boldsymbol{\omega}\times\mathbf{p} \tag{7.21}$$

where $\boldsymbol{\omega} = \dot{\phi}\mathbf{k}$ is the angular velocity vector.

Now let's repeat the derivation with the angle-axis formulation $\mathbf{R}_k(\phi)$ of a rotation matrix. The rotation matrix ${}^0\mathbf{R}_1 = \mathbf{R}_k(\phi)$ describes the evolution of coordinate system 1 relative to coordinate system 0, by

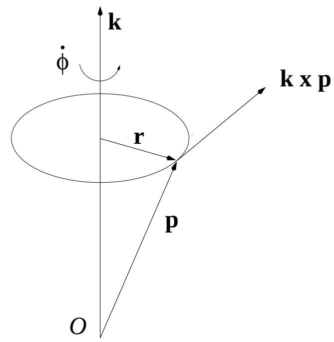

Figure 7.4: Rotation of vector $\mathbf{p}$ about axis $\mathbf{k}$ with angular rate $\dot{\phi}$.

a time-dependent rotation $\phi(t)$ about the fixed axis $\mathbf{k}$ (Figure 7.5). The case where $\mathbf{k}$ can vary as well is implicit in the derivation of the next section. This problem is a generalization of the time derivative of $\mathbf{R}_z(\theta_1)$ from (7.15).

Proceed by using the chain rule:

$$\frac{d\mathbf{R}_k(\phi)}{dt} = \frac{d\mathbf{R}_k(\phi)}{d\phi}\dot{\phi} \tag{7.22}$$

So the problem is that of finding $d\mathbf{R}_k(\phi)/d\phi$. Because the inverse of a rotation matrix is its transpose,

$$\mathbf{R}_k(\phi)\mathbf{R}_k(\phi)^T = \mathbf{I} \tag{7.23}$$

$$\frac{d\mathbf{R}_k(\phi)}{d\phi}\mathbf{R}_k(\phi)^T + \mathbf{R}_k(\phi)\frac{d\mathbf{R}_k(\phi)^T}{d\phi} = \mathbf{0} \tag{7.24}$$

where (7.24) is the derivative of (7.23) with respect to ϕ. Define

$$\mathbf{S} = \frac{d\mathbf{R}_k(\phi)}{d\phi}\mathbf{R}_k(\phi)^T \tag{7.25}$$

$$\mathbf{S}^T = \mathbf{R}_k(\phi)\frac{d\mathbf{R}_k(\phi)^T}{d\phi}$$

From the definitions above,

$$\mathbf{S} + \mathbf{S}^T = 0 \tag{7.26}$$

which is the defining condition for a skew-symmetric matrix. Reworking (7.24) using (7.26),

$$\frac{d\mathbf{R}_k(\phi)}{d\phi} = -\mathbf{S}^T\mathbf{R}_k(\phi) = \mathbf{S}\mathbf{R}_k(\phi) \tag{7.27}$$

Now we want to show that $\mathbf{S} = \mathbf{S}(\mathbf{k})$. From the derivation of the rotational matrix corresponding to an angle-axis rotation,

$$\mathbf{R}_k(\phi) = \mathbf{I}\mathrm{c}\phi + \mathbf{k}\mathbf{k}^T(1 - \mathrm{c}\phi) + \mathbf{S}(\mathbf{k})\mathrm{s}\phi \tag{7.28}$$

$$\frac{d\mathbf{R}_k(\phi)}{d\phi} = -\mathbf{I}\mathrm{s}\phi + \mathbf{k}\mathbf{k}^T\mathrm{s}\phi + \mathbf{S}(\mathbf{k})\mathrm{c}\phi \tag{7.29}$$

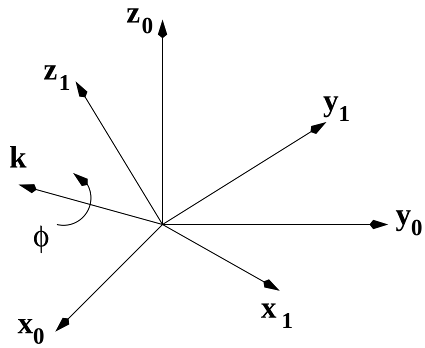

Figure 7.5: Frame 1 rotates relative to frame 0 about $\mathbf{k}$ with angular velocity $\dot{\phi}$.

Hence the evaluation of (7.25) is

$$\begin{aligned}\mathbf{S} &= \left(-\mathbf{I}\mathrm{s}\phi + \mathbf{k}\mathbf{k}^T\mathrm{s}\phi + \mathbf{S}(\mathbf{k})\mathrm{c}\phi\right)\left(\mathbf{I}\mathrm{c}\phi + \mathbf{k}\mathbf{k}^T(1-\mathrm{c}\phi) - \mathbf{S}(\mathbf{k})\mathrm{s}\phi\right) \\ &= -\mathbf{I}\mathrm{s}\phi\mathrm{c}\phi - \mathbf{k}\mathbf{k}^T\mathrm{s}\phi(1-\mathrm{c}\phi) + \mathbf{S}(\mathbf{k})\mathrm{s}\phi^2 + \mathbf{k}\mathbf{k}^T\mathrm{s}\phi\mathrm{c}\phi + \mathbf{k}\mathbf{k}^T\mathbf{k}\mathbf{k}^T\mathrm{s}\phi(1-\mathrm{c}\phi) \\ &\quad -\mathbf{k}\mathbf{k}^T\mathbf{S}(\mathbf{k})\mathrm{s}\phi^2 + \mathbf{S}(\mathbf{k})\mathrm{c}\phi^2 + \mathbf{S}(\mathbf{k})\mathbf{k}\mathbf{k}^T\mathrm{c}\phi(1-\mathrm{c}\phi) - \mathbf{S}(\mathbf{k})\mathbf{S}(\mathbf{k})\mathrm{s}\phi\mathrm{c}\phi \qquad (7.30)\end{aligned}$$

The terms with $\mathbf{S}(\mathbf{k})\mathbf{k}$ or $\mathbf{k}^T\mathbf{S}(\mathbf{k})$ are zero, because they involve taking the cross product of $\mathbf{k}$ with itself. The term involving the double cross product can be simplified by seeing the result of applying to an arbitrary vector $\mathbf{v}$:

$$\mathbf{S}(\mathbf{k})\mathbf{S}(\mathbf{k})\mathbf{v} = \mathbf{k}\times(\mathbf{k}\times\mathbf{v}) = \mathbf{k}(\mathbf{k}^T\mathbf{v}) - \mathbf{v} = (\mathbf{k}\mathbf{k}^T - \mathbf{I})\mathbf{v} \qquad (7.31)$$

Substituting into (7.30) and simplifying,

$$\begin{aligned}\mathbf{S} &= -\mathbf{I}\mathrm{s}\phi\mathrm{c}\phi - \mathbf{k}\mathbf{k}^T\mathrm{s}\phi(1-\mathrm{c}\phi) + \mathbf{S}(\mathbf{k})(\mathrm{s}\phi^2+\mathrm{c}\phi^2) + \mathbf{k}\mathbf{k}^T\mathrm{s}\phi\mathrm{c}\phi + \mathbf{k}\mathbf{k}^T\mathrm{s}\phi(1-\mathrm{c}\phi) \\ &\quad -(\mathbf{k}\mathbf{k}^T - \mathbf{I})\mathrm{s}\phi\mathrm{c}\phi \\ &= \mathbf{S}(\mathbf{k}) \qquad (7.32)\end{aligned}$$

which is the desired result. Hence

$$\frac{d\mathbf{R}_k(\phi)}{d\phi} = \mathbf{S}(\mathbf{k})\mathbf{R}_k(\phi) \qquad (7.33)$$

$$\dot{\mathbf{R}}_k(\phi) = \dot{\phi}\mathbf{S}(\mathbf{k})\mathbf{R}_k(\phi) = \mathbf{S}(\boldsymbol{\omega})\mathbf{R}_k(\phi) \qquad (7.34)$$

where $\boldsymbol{\omega} = \dot{\phi}\mathbf{k}$ is the angular velocity of coordinate system 1 relative to coordinate system 0.

Example 7.1: $\mathbf{R} = \mathbf{R}_x(\phi)$. Then

$$\mathbf{R}_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\phi & -\mathrm{s}\phi \\ 0 & \mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix}, \qquad \frac{d\mathbf{R}_x(\phi)}{d\phi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mathrm{s}\phi & -\mathrm{c}\phi \\ 0 & \mathrm{c}\phi & -\mathrm{s}\phi \end{bmatrix}$$

Consequently,

$$\mathbf{S} = \frac{d\mathbf{R}_x(\phi)}{d\phi}\mathbf{R}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mathrm{s}\phi & -\mathrm{c}\phi \\ 0 & \mathrm{c}\phi & -\mathrm{s}\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\phi & \mathrm{s}\phi \\ 0 & -\mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \mathbf{S}(\mathbf{x})$$

Hence

$$\frac{d\mathbf{R}_x(\phi)}{d\phi} = \mathbf{S}(\mathbf{x})\mathbf{R}_x(\phi)$$

and the skew-symmetric matrix is formed from the axis of rotation $\mathbf{x}$.

7.2.2 Time Derivative of an Arbitrary 3D Rotation Matrix

In the previous section, the rotational motion was aligned with the angle-axis orientation description. This will not be true in general: the rotational motion will have no relation to the rotation matrix. We repeat the process of the previous section with an arbitrary 3D rotation matrice $\mathbf{R} = \mathbf{R}(t)$ which is a function of time. Then

$$\mathbf{R}\mathbf{R}^T = \mathbf{I} \tag{7.35}$$

$$\dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = \mathbf{0} \tag{7.36}$$

$$\mathbf{S} + \mathbf{S}^T = \mathbf{0} \tag{7.37}$$

where

$$\begin{aligned} \mathbf{S} &= \dot{\mathbf{R}}\mathbf{R}^T \\ \mathbf{S}^T &= \mathbf{R}\dot{\mathbf{R}}^T \end{aligned}$$

and (7.37) shows $\mathbf{S}$ is a skew-symmetric matrix. Then from (7.36),

$$\dot{\mathbf{R}} = -\mathbf{S}^T\mathbf{R} = \mathbf{S}\mathbf{R}$$

Extract a vector $\boldsymbol{\omega}$ from the skew-symmetric matrix $\mathbf{S} = \mathbf{S}(\boldsymbol{\omega})$ (Appendix A), where $\boldsymbol{\omega}$ is again the angular velocity vector of the rotating frame with respect to the fixed frame. Then

$$\dot{\mathbf{R}} = \mathbf{S}(\boldsymbol{\omega})\mathbf{R} \tag{7.38}$$

Example 7.2: Suppose $\mathbf{R}(t) = \mathbf{R}_x(\phi(t))$. Then from (7.38) and (7.27),

$$\dot{\mathbf{R}} = \frac{\partial\mathbf{R}}{\partial\phi}\frac{d\phi}{dt} = \dot{\phi}\mathbf{S}(\mathbf{x})\mathbf{R} = \mathbf{S}(\dot{\phi}\mathbf{x})\mathbf{R} = \mathbf{S}(\boldsymbol{\omega})\mathbf{R}$$

where $\boldsymbol{\omega} = \dot{\phi}\mathbf{x}$ is the angular velocity vector parallel to the $\mathbf{x}$-axis.

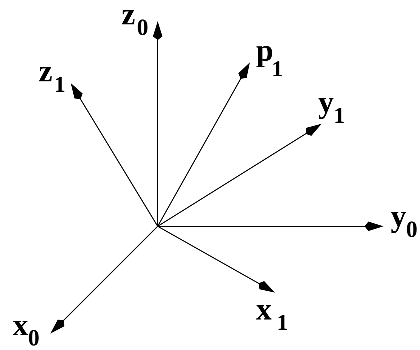

Figure 7.6: Frame 1 rotates relative to frame 0 with angular velocity $\boldsymbol{\omega}$. Vector $\mathbf{p}_1$ is fixed in frame 1.

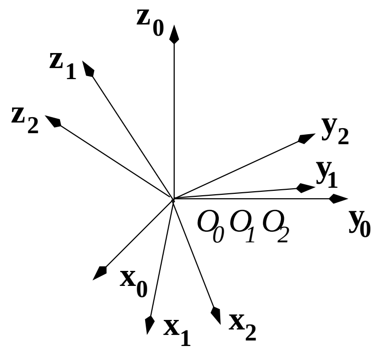

Figure 7.7: Frame 2 rotates with respect to frame 1, while frame 1 rotates with respted to frame 0. The origins stay coincident.

Consider two frames 0 and 1 with the same origin, and suppose frame 1 is rotating with respect to frame 0 and is instantaneously located by ${}^0\mathbf{R}_1(t)$ (Figure 7.6). The time derivative of a vector fixed in frame 1, ${}^1\mathbf{p}_1$, relative to frame 0 is succinctly expressed using the concept of angular velocity vector.

$$
\begin{aligned}
{}^0\mathbf{p}_1 &= {}^0\mathbf{R}_1\,{}^1\mathbf{p}_1 &\quad (7.39)\\
{}^0\dot{\mathbf{p}}_1 &= {}^0\dot{\mathbf{R}}_1\,{}^1\mathbf{p}_1 &\quad (7.40)
\end{aligned}
$$

since again ${}^1\dot{\mathbf{p}}_1 = \mathbf{0}$. Recalling (7.38),

$$
{}^0\dot{\mathbf{p}}_1 = \mathbf{S}(\boldsymbol{\omega})\,{}^0\mathbf{R}_1\,{}^1\mathbf{p}_1 = \mathbf{S}(\boldsymbol{\omega})\,{}^0\mathbf{p}_1 = \boldsymbol{\omega} \times\,{}^0\mathbf{p}_1 \qquad (7.41)
$$

Sometimes one formally writes (7.38) as

$$
\dot{\mathbf{R}} = \boldsymbol{\omega} \times \mathbf{R} \qquad (7.42)
$$

where the cross product of a vector with a matrix is interpreted when applied to another vector as (7.41).

7.2.3 Addition of Angular Velocities

Consider 3 frames whose origins are coincident, and suppose frame 2 rotates with respect to frame 1, which in turn rotates with respect to frame 0 (Figure 7.7). Differentiate the rotational relationship as follows:

$$
{}^0\mathbf{R}_2 = {}^0\mathbf{R}_1{}^1\mathbf{R}_2 \qquad (7.43)
$$

$$ {}^0\dot{\mathbf{R}}_2 = {}^0\dot{\mathbf{R}}_1{}^1\mathbf{R}_2 + {}^0\mathbf{R}_1{}^1\dot{\mathbf{R}}_2 \tag{7.44} $$

Each of the differentiated rotation matrices can be rewritten as:

$$ \begin{aligned} {}^1\dot{\mathbf{R}}_2 &= \mathbf{S}({}^1\boldsymbol{\omega}_{12}){}^1\mathbf{R}_2 \\ {}^0\dot{\mathbf{R}}_1 &= \mathbf{S}({}^0\boldsymbol{\omega}_{01}){}^0\mathbf{R}_1 \\ {}^0\dot{\mathbf{R}}_2 &= \mathbf{S}({}^0\boldsymbol{\omega}_{02}){}^0\mathbf{R}_2 \end{aligned} $$

where ${}^1\boldsymbol{\omega}_{12}$ is the angular velocity of frame 2 with respect to frame 1, expressed in frame 1, etc. Then

$$ \begin{aligned} \mathbf{S}({}^0\boldsymbol{\omega}_{02}){}^0\mathbf{R}_2 &= \mathbf{S}({}^0\boldsymbol{\omega}_{01}){}^0\mathbf{R}_1{}^1\mathbf{R}_2 + {}^0\mathbf{R}_1\mathbf{S}({}^1\boldsymbol{\omega}_{12}){}^1\mathbf{R}_2 \\ &= \mathbf{S}({}^0\boldsymbol{\omega}_{01}){}^0\mathbf{R}_2 + {}^0\mathbf{R}_1\mathbf{S}({}^1\boldsymbol{\omega}_{12})({}^0\mathbf{R}_1)^{T\,0}\mathbf{R}_1{}^1\mathbf{R}_2 \\ &= \mathbf{S}({}^0\boldsymbol{\omega}_{01}){}^0\mathbf{R}_2 + \mathbf{S}({}^0\boldsymbol{\omega}_{12}){}^0\mathbf{R}_2 \end{aligned} \tag{7.45} $$

where

$$ \mathbf{S}({}^0\boldsymbol{\omega}_{12}) = {}^0\mathbf{R}_1\mathbf{S}({}^1\boldsymbol{\omega}_{12})({}^0\mathbf{R}_1)^T \tag{7.46} $$

To prove (7.46), we apply each side to an arbitrary vector ${}^0\mathbf{b}$ and show that the results are the same. Applying the left side,

$$ \mathbf{S}({}^0\boldsymbol{\omega}_{12})\ {}^0\mathbf{b} = {}^0\boldsymbol{\omega}_{12} \times {}^0\mathbf{b} \tag{7.47} $$

Applying the right hand side,

$$ \begin{aligned} {}^0\mathbf{R}_1\mathbf{S}({}^1\boldsymbol{\omega}_{12})({}^0\mathbf{R}_1)^T\ {}^0\mathbf{b} &= {}^0\mathbf{R}_1\mathbf{S}({}^1\boldsymbol{\omega}_{12})\ {}^1\mathbf{b} \\ &= {}^0\mathbf{R}_1({}^1\boldsymbol{\omega}_{12} \times {}^1\mathbf{b}) \\ &= ({}^0\mathbf{R}_1\ {}^1\boldsymbol{\omega}_{12}) \times ({}^0\mathbf{R}_1\ {}^1\mathbf{b}) \\ &= {}^0\boldsymbol{\omega}_{12} \times {}^0\mathbf{b} \end{aligned} \tag{7.48} $$

Since (7.47) and (7.48) are the same for all $\mathbf{b}$, then (7.46) is established. From (7.45), we have the final result:

$$ \begin{aligned} \mathbf{S}({}^0\boldsymbol{\omega}_{02}) &= \mathbf{S}({}^0\boldsymbol{\omega}_{01}) + \mathbf{S}({}^0\boldsymbol{\omega}_{12}) \\ {}^0\boldsymbol{\omega}_{02} &= {}^0\boldsymbol{\omega}_{01} + {}^0\boldsymbol{\omega}_{12} \end{aligned} \tag{7.49} $$

This result holds for any number of coordinate systems:

$$ \begin{aligned} {}^0\mathbf{R}_n &= {}^0\mathbf{R}_1{}^1\mathbf{R}_2 \cdots {}^{n-1}\mathbf{R}_n \\ {}^0\boldsymbol{\omega}_{0n} &= {}^0\boldsymbol{\omega}_{01} + {}^0\boldsymbol{\omega}_{12} + \cdots + {}^0\boldsymbol{\omega}_{n-1,n} \\ &= \sum_{i=1}^{n} {}^0\boldsymbol{\omega}_{i-1,i} \end{aligned} \tag{7.50} $$

7.2.4 Euler Angle Rates

As mentioned previously, a rotation matrix is a redundant representation of orientation, and other representations such as Euler angles or quaternions are often employed. It is natural then to formulate motion in terms of time derivatives of, say, Euler angles, rather than time derivatives of rotation matrices. This section addresses the relation of Euler angle rates to the angular velocity vector.

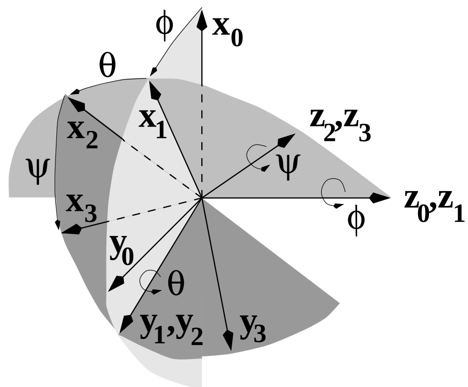

Figure 7.8: The ZYZ Euler angles.

Let's consider the ZYZ Euler angles (Figure 7.8). Then

$$\mathbf{R} = \mathbf{R}_z(\phi)\mathbf{R}_y(\theta)\mathbf{R}_z(\psi)$$

where $\mathbf{R}_z(\psi)$ transforms from frame 3 to 2,
$\mathbf{R}_y(\theta)$ transforms from frame 2 to 1, and
$\mathbf{R}_z(\phi)$ transforms from frame 1 to 0.

Then supposing that the Euler angles are changing with rates $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$ and utilizing the result about summation of angular velocities (7.50),

$$^0\boldsymbol{\omega}_{03} = \dot{\phi}\ {}^0\mathbf{z}_0 + \dot{\theta}\ {}^0\mathbf{y}_1 + \dot{\psi}\ {}^0\mathbf{z}_2 \tag{7.51}$$

where

$$^0\mathbf{y}_1 = \mathbf{R}_z(\phi)\ {}^1\mathbf{y}_1 = \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi & 0 \\ \mathrm{s}\phi & \mathrm{c}\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\mathrm{s}\phi \\ \mathrm{c}\phi \\ 0 \end{bmatrix} \tag{7.52}$$

$$^0\mathbf{z}_2 = \mathbf{R}_z(\phi)\mathbf{R}_y(\theta)\ {}^2\mathbf{z}_2 = \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi & 0 \\ \mathrm{s}\phi & \mathrm{c}\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{c}\theta & 0 & \mathrm{s}\theta \\ 0 & 1 & 0 \\ -\mathrm{s}\theta & 0 & \mathrm{c}\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{c}\phi\mathrm{s}\theta \\ \mathrm{s}\phi\mathrm{s}\theta \\ \mathrm{c}\theta \end{bmatrix} \tag{7.53}$$

Substituting into (7.51),

$$^0\boldsymbol{\omega} = \dot{\phi} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \dot{\theta} \begin{bmatrix} -\mathrm{s}\phi \\ \mathrm{c}\phi \\ 0 \end{bmatrix} + \dot{\psi} \begin{bmatrix} \mathrm{c}\phi\mathrm{s}\theta \\ \mathrm{s}\phi\mathrm{s}\theta \\ \mathrm{c}\theta \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{s}\phi & \mathrm{c}\phi\mathrm{s}\theta \\ 0 & \mathrm{c}\phi & \mathrm{s}\phi\mathrm{s}\theta \\ 1 & 0 & \mathrm{c}\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \equiv \boldsymbol{\Omega}_{zyz} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{7.54}$$

where

$$\mathbf{\Omega}_{zyz} = \begin{bmatrix} 0 & -\mathrm{s}\phi & \mathrm{c}\phi\mathrm{s}\theta \\ 0 & \mathrm{c}\phi & \mathrm{s}\phi\mathrm{s}\theta \\ 1 & 0 & \mathrm{c}\theta \end{bmatrix} \tag{7.55}$$

The complication of the relation between the angular velocity vector $\boldsymbol{\omega}$ and the Euler angle rates is due to the non-orthogonality of the Euler angle rotation axes, as reflected in matrix $\mathbf{\Omega}_{zyz}$.

The angular velocity vector is a fundamental aspect of object motion, which is explicitly represented in the dynamics of motion. If object motion is planned in terms of Euler angle rates, then we must be able to convert from the angular velocity vector to corresponding Euler angle rates. This is done by inverting (7.54):

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{\Omega}_{zyz}^{-1}\ {}^{0}\boldsymbol{\omega} \tag{7.56}$$

There is a serious problem: $\mathbf{\Omega}_{zyz}^{-1}$ may not exist. This happens when $\theta = 0$. At such configurations, there are valid angular velocities which cannot be represented by Euler angle rates. This is not a problem of physics, but of the representation. Hence it is called a *representational singularity*.

This is a really bad problem. Together with the problem of indeterminate orientation when $\theta = 0$, this makes Euler angles in practice not that useful. One could switch between different Euler angle representations to avoid this problem, and this is sometimes done, but this artifice is distasteful.

In dynamic simulation of objects or robots in motion, applied forces or torques result in accelerations, which are then integrated once to derive velocity and then again to derive position. Although angular velocity can be found by integrating angular acceleration, there is a problem: what is the integral of angular velocity?

The answer unfortunately is that the integral of $\boldsymbol{\omega}$ has no clear meaning. There is no set of Euler angles that are its integral, because Euler angles represent rotation among nonorthogonal axes. On the other hand, the components of $\boldsymbol{\omega}$ are about orthogonal axes. More generally, there is no representation of orientation which is the integral of the angular velocity vector.

Example 7.3: Consider the following two cases. Coordinate systems 0 and 1 are coincident at $t = 0$.

Case 1:

$${}^{0}\boldsymbol{\omega}_{01} = \begin{cases} \begin{bmatrix} \pi/2 \\ 0 \\ 0 \end{bmatrix} & 0 \le t < 1 \\[2ex] \begin{bmatrix} 0 \\ \pi/2 \\ 0 \end{bmatrix} & 1 \le t \le 2 \end{cases}$$

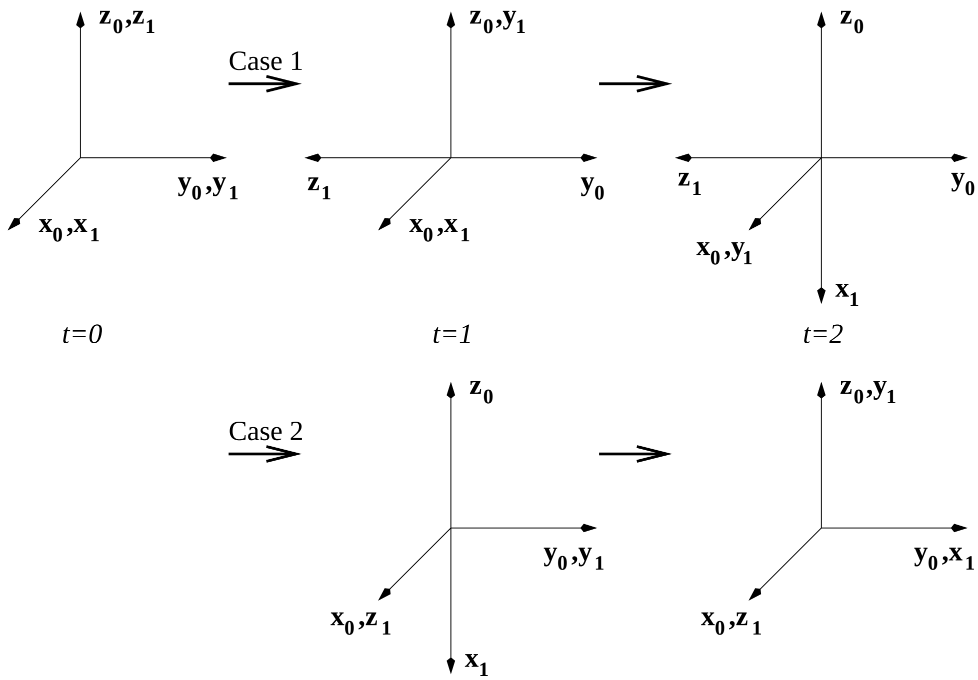

Figure 7.9: Integration of angular velocity vector corresponding to cases 1 and 2.

Case 2:

$$
{}^0\boldsymbol{\omega}_{01} = \begin{cases} \begin{bmatrix} 0 \\ \pi/2 \\ 0 \end{bmatrix} & 0 \le t < 1 \\ \\ \begin{bmatrix} \pi/2 \\ 0 \\ 0 \end{bmatrix} & 1 \le t \le 2 \end{cases}
$$

In both cases, the integral is the same.

$$
\int_0^2 {}^0\boldsymbol{\omega}_{01} = \begin{bmatrix} \pi/2 \\ \pi/2 \\ 0 \end{bmatrix}
$$

Yet the results are different (Figure 7.9). For example, in case 1 the $\mathbf{z}_1$ axis ends up as $-\mathbf{y}_0$, whereas in case 2 it ends up a $\mathbf{x}_0$.

On the other hand, the integral of Euler angle rates is meaningful, except for the representational singularity problem mentioned above. Thus one converts from angular velocity to Euler angle rate, then integrates.

7.2.5 Quaternion Rates

To get around problems with Euler angles, the use of quaternions and quaternion rates is to be recommended. They are often used in simulation and virtual reality, robotics, and control. The relationship between quaternion rates and the angular velocity vector is stated below:

$$\frac{d}{dt}\begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\mathbf{q}^T \\ -\mathbf{S}(\mathbf{q}) + q_0\mathbf{I} \end{bmatrix}\boldsymbol{\omega} \tag{7.57}$$

The derivation is complicated. In simulation, one converts from $\boldsymbol{\omega}$ to $\dot{\boldsymbol{q}}$ as above, then integrates $\dot{\boldsymbol{q}}$ to arrive at $\boldsymbol{q}$. Integration keeps $||\boldsymbol{q}||$ constant.

7.2.6 Euler Angle Accelerations

Suppose the Euler angles are also accelerating as $\ddot{\phi}$, $\ddot{\theta}$, and $\ddot{\psi}$. Differentiating (7.51) yields the angular acceleration vector corresponding to these Euler angle accelerations.

$$\begin{aligned} {}^0\dot{\boldsymbol{\omega}}_{03} &= \ddot{\phi}\,{}^0\mathbf{z}_0 + \dot{\phi}\,{}^0\dot{\mathbf{z}}_0 + \ddot{\theta}\,{}^0\mathbf{y}_1 + \dot{\theta}\,{}^0\dot{\mathbf{y}}_1 + \ddot{\psi}\,{}^0\mathbf{z}_2 + \dot{\psi}\,{}^0\dot{\mathbf{z}}_2 \\ &= \ddot{\phi}\,{}^0\mathbf{z}_0 + \ddot{\theta}\,{}^0\mathbf{y}_1 + \dot{\theta}\,{}^0\boldsymbol{\omega}_{01} \times {}^0\mathbf{y}_1 + \ddot{\psi}\,{}^0\mathbf{z}_2 + \dot{\psi}\,{}^0\boldsymbol{\omega}_{02} \times {}^0\mathbf{z}_2 \end{aligned}$$

where ${}^0\dot{\mathbf{z}}_0 = \mathbf{0}$ since it is stationary. The angular velocity of coordinate system 1 with respect to coordinate system 0 is ${}^0\boldsymbol{\omega}_{01} = \dot{\phi}\,{}^0\mathbf{z}_0$, while the angular velocity of coordinate system 2 with respect to coordinate system 1 is ${}^0\boldsymbol{\omega}_{12} = \dot{\theta}\,{}^0\mathbf{y}_1$. The angular velocity of coordinate system 2 with respect to coordinate system 0 sums these two contributions: $\boldsymbol{\omega}_{02} = \dot{\phi}\,{}^0\mathbf{z}_0 + \dot{\theta}\,{}^0\mathbf{y}_1$. Substituting,

$${}^0\dot{\boldsymbol{\omega}}_{03} = \ddot{\phi}\,{}^0\mathbf{z}_0 + \ddot{\theta}\,{}^0\mathbf{y}_1 + \dot{\theta}\dot{\phi}\,{}^0\mathbf{z}_0 \times {}^0\mathbf{y}_1 + \ddot{\psi}\,{}^0\mathbf{z}_2 + \dot{\psi}(\dot{\phi}\,{}^0\mathbf{z}_0 + \dot{\theta}\,{}^0\mathbf{y}_1) \times {}^0\mathbf{z}_2$$

where $\mathbf{z}_0 = \mathbf{z}_1$ and $\mathbf{y}_1 = \mathbf{y}_2$. Hence

$$\begin{aligned} {}^0\mathbf{z}_0 \times {}^0\mathbf{y}_1 &= {}^0\mathbf{z}_1 \times {}^0\mathbf{y}_1 \\ &= -{}^0\mathbf{x}_1 \\ {}^0\mathbf{y}_1 \times {}^0\mathbf{z}_2 &= {}^0\mathbf{y}_2 \times {}^0\mathbf{z}_2 \\ &= {}^0\mathbf{x}_2 \\ {}^0\mathbf{z}_0 \times {}^0\mathbf{z}_2 &= {}^0\mathbf{z}_1 \times (\mathrm{s}\theta\,{}^0\mathbf{x}_1 + \mathrm{c}\theta\,{}^0\mathbf{z}_1) \\ &= \mathrm{s}\theta\,{}^0\mathbf{y}_1 \end{aligned}$$

Substituting,

$${}^0\dot{\boldsymbol{\omega}}_{03} = \ddot{\phi}\,{}^0\mathbf{z}_0 + \ddot{\theta}\,{}^0\mathbf{y}_1 - \dot{\theta}\dot{\phi}\,{}^0\mathbf{x}_1 + \ddot{\psi}\,{}^0\mathbf{z}_2 + \dot{\psi}(\dot{\phi}\mathrm{s}\theta\,{}^0\mathbf{y}_1 + \dot{\theta}\,{}^0\mathbf{x}_2)$$

7.3 Spatial Motion of Objects

Next we describe the motion of objects which are translating and rotating with respect to each other at the same time.

7.3.1 Spatial Motion of Single Objects

Consider a frame 1 that is moving with respect to frame 0, and a point P (located by $\mathbf{p}_1$ relative to O_1 and $\mathbf{p}_0$ relative to O_0) that is fixed with respect to frame 1 (Figure 7.10). Then the spatial transformation is

$$ {}^0\mathbf{p}_0 = {}^0\mathbf{d}_{01} + {}^0\mathbf{R}_1\,{}^1\mathbf{p}_1 \tag{7.58} $$

and its time derivative is

$$ \begin{aligned} {}^0\dot{\mathbf{p}}_0 &= {}^0\dot{\mathbf{R}}_1\,{}^1\mathbf{p}_1 + {}^0\dot{\mathbf{d}}_{01} \\ &= \mathbf{S}({}^0\boldsymbol{\omega}_{01})\,{}^0\mathbf{R}_1\,{}^1\mathbf{p}_1 + {}^0\dot{\mathbf{d}}_{01} \\ &= {}^0\boldsymbol{\omega}_{01} \times {}^0\mathbf{R}_1\,{}^1\mathbf{p}_1 + {}^0\dot{\mathbf{d}}_{01} \\ &= {}^0\boldsymbol{\omega}_{01} \times {}^0\mathbf{p}_1 + {}^0\dot{\mathbf{d}}_{01} \end{aligned} \tag{7.59} $$

where ${}^0\boldsymbol{\omega}_{01}$ is the angular velocity of frame 1 with respect to frame 0. Like any other vector, we have to specify the frame in terms of which $\boldsymbol{\omega}_{01}$ is expressed, in this case frame 0. The motion of point P is due both to linear motion of body 1 away from body 0, and rotational motion of body 1.

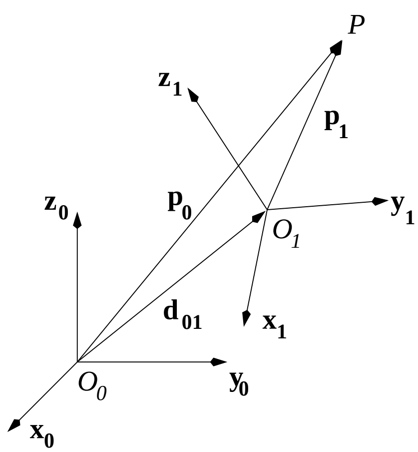

Figure 7.10: Frame 1 is moving with respect to frame 0, and point P is fixed with respect to frame 1.

7.3.2 Spatial Motion of Multiple Objects

In section 7.2.3 the angular motion of multiple objects was described in the circumstance of coincident origins. If the objects are translating as well as rotating, the relative angular motions are still the same as derived in that section.

Now let's describe the linear velocity of point P in Figure 7.11. In addition to the rotational motions, each origin is moving with respect to its neighbor with a linear velocity ${}^i\dot{\mathbf{d}}_{i-1,i}$. Then

$$ {}^0\mathbf{p}_0 = {}^0\mathbf{d}_{01} + {}^0\mathbf{R}_1{}^1\mathbf{d}_{12} + {}^0\mathbf{R}_2{}^2\mathbf{p}_2 \tag{7.60} $$

Then differentiate (7.60) to get:

$$ \begin{aligned} {}^0\dot{\mathbf{p}}_0 &= {}^0\dot{\mathbf{d}}_{01} + {}^0\dot{\mathbf{R}}_1{}^1\mathbf{d}_{12} + {}^0\mathbf{R}_1{}^1\dot{\mathbf{d}}_{12} + {}^0\dot{\mathbf{R}}_2{}^2\mathbf{p}_2 \\ &= {}^0\dot{\mathbf{d}}_{01} + {}^0\boldsymbol{\omega}_{01} \times {}^0\mathbf{d}_{12} + {}^0\mathbf{R}_1{}^1\dot{\mathbf{d}}_{12} + {}^0\boldsymbol{\omega}_{02} \times {}^0\mathbf{p}_2 \end{aligned} \tag{7.61} $$

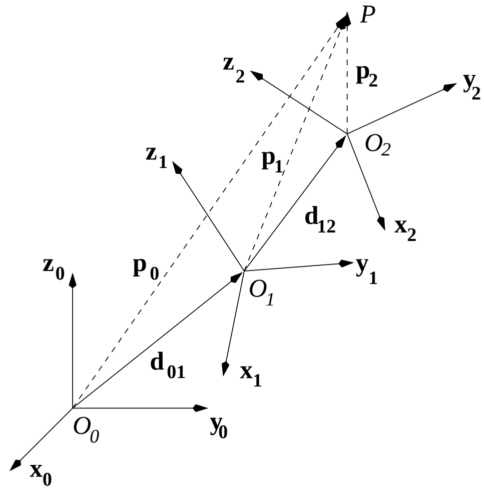

Figure 7.11: Frame 2 is also moving with respect to frame 1, and point P is fixed with respect to frame 2.

The motion of origin O_2 is due both to rotation of frame 1 about frame 0, ${}^0\boldsymbol{\omega}_{01} \times {}^0\mathbf{d}_{12}$, and linear movement of O_2 away from O_1, ${}^0\mathbf{R}_1{}^1\dot{\mathbf{d}}_{12}$.

7.4 Acceleration of Objects

Refer again to Figure 7.10. Differentiating (7.59),

$$\begin{aligned} {}^0\ddot{\mathbf{p}}_0 &= {}^0\dot{\boldsymbol{\omega}}_{01} \times {}^0\mathbf{R}_1\,{}^1\mathbf{p}_1 + {}^0\boldsymbol{\omega}_{01} \times {}^0\dot{\mathbf{R}}_1\,{}^1\mathbf{p}_1 + {}^0\ddot{\mathbf{d}}_{01} \\ &= {}^0\dot{\boldsymbol{\omega}}_{01} \times {}^0\mathbf{p}_1 + {}^0\boldsymbol{\omega}_{01} \times ({}^0\boldsymbol{\omega}_{01} \times {}^0\mathbf{p}_1) + {}^0\ddot{\mathbf{d}}_{01} \end{aligned} \quad (7.62)$$

where $\dot{\boldsymbol{\omega}}_{01}$ is the *angular acceleration* vector. The first term on the right is the *transverse acceleration* and the second term is the *centripetal acceleration*.

Example 7.4: Consider motion in the $\mathbf{x}_0, \mathbf{y}_0$ plane (Figure 7.12). Although the motion is planar, all vectors are three-dimensional. Then

$$\begin{aligned} {}^0\boldsymbol{\omega}_{01} &= \dot{\theta}_1\,{}^0\mathbf{z}_0 \\ {}^0\dot{\boldsymbol{\omega}}_{01} &= \ddot{\theta}_1\,{}^0\mathbf{z}_0 \end{aligned}$$

Hence since ${}^0\boldsymbol{\omega}_{01} \perp {}^0\mathbf{p}_1$, the centripetal acceleration is

$$\begin{aligned} {}^0\boldsymbol{\omega}_{01} \times ({}^0\boldsymbol{\omega}_{01} \times {}^0\mathbf{p}_1) &= {}^0\boldsymbol{\omega}_{01}({}^0\boldsymbol{\omega}_{01} \cdot {}^0\mathbf{p}_1) - {}^0\mathbf{p}_1({}^0\boldsymbol{\omega}_{01} \cdot {}^0\boldsymbol{\omega}_{01}) \\ &= -\dot{\theta}_1^2\,{}^0\mathbf{p}_1 \end{aligned}$$

As expected, the centripetal acceleration points inwards towards the center of rotation. Next, the tangential component of acceleration due to rotation is

$$\begin{aligned} {}^0\dot{\boldsymbol{\omega}}_{01} \times {}^0\mathbf{p}_1 &= \ddot{\theta}_1\,{}^0\mathbf{z}_0 \times {}^0\mathbf{p}_1 \\ &= \ddot{\theta}_1\,\mathbf{R}_z(\theta_1 + \pi/2){}^1\mathbf{p}_1 \end{aligned}$$

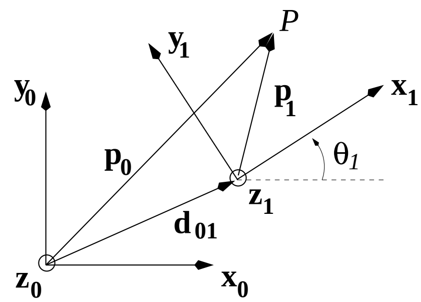

Figure 7.12: Motion in the $\mathbf{x}_0$, $\mathbf{y}_0$ plane. Axes of rotation $\mathbf{z}_0$ and $\mathbf{z}_1$ are parallel, and are indicated by circles.

Suppose $\mathbf{p}_1$ is moving relative to frame 1, i.e., ${}^1\dot{\mathbf{p}}_1 \neq \mathbf{0}$. Then the derivative of (7.58) becomes

$$\begin{aligned} {}^0\dot{\mathbf{p}}_0 &= {}^0\dot{\mathbf{R}}_1{}^1\mathbf{p}_1 + {}^0\mathbf{R}_1\,{}^1\dot{\mathbf{p}}_1 + {}^0\dot{\mathbf{d}}_{01} \\ &= {}^0\boldsymbol{\omega}_{01} \times {}^0\mathbf{R}_1{}^1\mathbf{p}_1 + {}^0\mathbf{R}_1\,{}^1\dot{\mathbf{p}}_1 + {}^0\dot{\mathbf{d}}_{01} \end{aligned} \tag{7.63}$$

The second derivative is:

$$\begin{aligned} {}^0\ddot{\mathbf{p}}_0 &= {}^0\dot{\boldsymbol{\omega}}_{01} \times {}^0\mathbf{R}_1{}^1\mathbf{p}_1 + {}^0\boldsymbol{\omega}_{01} \times {}^0\dot{\mathbf{R}}_1{}^1\mathbf{p}_1 + {}^0\boldsymbol{\omega}_{01} \times {}^0\mathbf{R}_1\,{}^1\dot{\mathbf{p}}_1 + {}^0\dot{\mathbf{R}}_1\,{}^1\dot{\mathbf{p}}_1 + {}^0\mathbf{R}_1\,{}^1\ddot{\mathbf{p}}_1 + {}^0\ddot{\mathbf{d}}_{01} \\ &= {}^0\dot{\boldsymbol{\omega}}_{01} \times {}^0\mathbf{p}_1 + {}^0\boldsymbol{\omega}_{01} \times ({}^0\boldsymbol{\omega}_{01} \times {}^0\mathbf{p}_1) + 2\,{}^0\boldsymbol{\omega}_{01} \times {}^0\mathbf{R}_1\,{}^1\dot{\mathbf{p}}_1 + {}^0\mathbf{R}_1\,{}^1\ddot{\mathbf{p}}_1 + {}^0\ddot{\mathbf{d}}_{01} \end{aligned} \tag{7.64}$$

The third term on the right is the *coriolis acceleration*, which is at right angles with the angular velocity $\boldsymbol{\omega}$ and the velocity ${}^1\dot{\mathbf{p}}_1$.