Chapter 9

Statics

To consider the motion of a body i, the net force $\mathbf{f}_i$ and the net torque $\mathbf{n}_i$ acting on the body are computed. The net force gives rise to linear acceleration described by Newton's second law, while the net torque gives rise to angular acceleration described by Euler's equation. These Newton-Euler equations are the subject of Chapter 10. When the net force and torque are zero, the body is in static equilibrium, i.e., it does not move. It will be seen that there is a reciprocal relation between statics and kinematics.

9.1 Forces and Torques

Suppose a force $\mathbf{f}_{ij}$ acts on a body i at point Q_{ij} (Figure 9.1(A)). This force may be the action of gravity at the center of gravity, or a constraint force due to contact with another body. With respect to the reference point O_{i-1}, the force $\mathbf{f}_{ij}$ also creates a torque $\mathbf{n}_{i-1,j}$ by virtue of action through the moment arm $\mathbf{q}_{i-1,j} = Q_{ij} - O_{i-1}$:

$$\mathbf{n}_{i-1,j} = \mathbf{q}_{i-1,j} \times \mathbf{f}_{ij} \tag{9.1}$$

This force-originating torque depends on the reference point. Suppose the reference point is changed to point O_i on the body, located relative to O_{i-1} by vector $\mathbf{d}_{i-1,i}$. Then the torque about O_i is:

$$\mathbf{n}_{ij} = (\mathbf{q}_{i-1,j} - \mathbf{d}_{i-1,i}) \times \mathbf{f}_{ij} \tag{9.2}$$

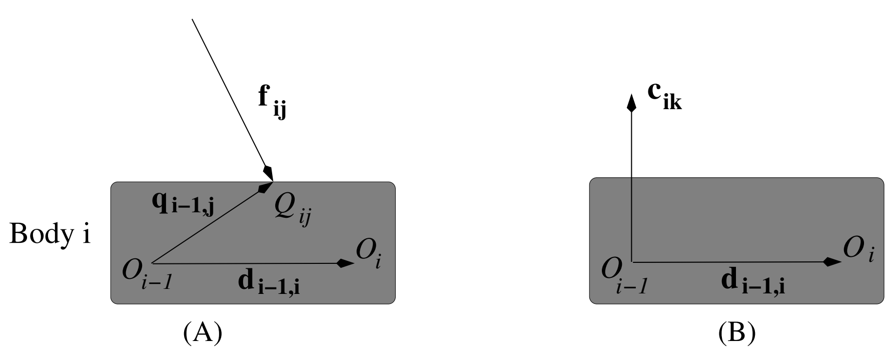


Figure 9.1: (A) Force-originating torque depends on reference point. (B) A pure torque acting on the body.

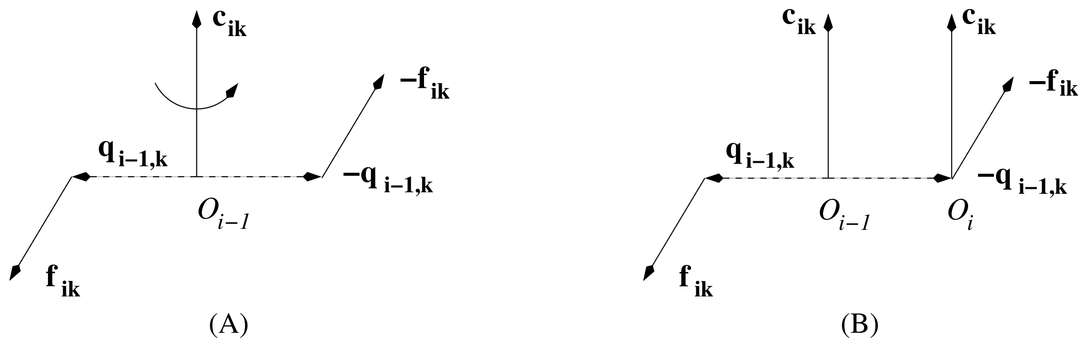


Figure 9.2: (A) A pure torque represented by a couple. (B) A change of reference point to O_i doesn't change the pure torque.

Besides force-originating torques, pure torques may arise. For example, a torque motor at a joint creates a pure torque. Suppose a pure torque $\mathbf{c}_{ik}$ acts on the body (Figure 9.1(B)). A pure torque can be represented by an equivalent *couple*. For a given reference point O_{i-1}, a couple is two oppositely directed forces $\mathbf{f}_{ik}$ and $-\mathbf{f}_{ik}$ located symmetrically about O_{i-1} by $\mathbf{q}_{i-1,k}$ and $-\mathbf{q}_{i-1,k}$ respectively (Figure 9.2(A)), such that:

$$\begin{aligned} \mathbf{c}_{ik} &= \mathbf{q}_{i-1,k} \times \mathbf{f}_{ik} + (-\mathbf{q}_{i-1,k}) \times (-\mathbf{f}_{ik}) \\ &= 2\mathbf{q}_{i-1,k} \times \mathbf{f}_{ik} \end{aligned} \tag{9.3}$$

The representation of a pure torque by a couple is not unique. There are infinitely many choices of $\mathbf{f}_{ik}$ and $\mathbf{q}_{i-1,k}$ that realize a particular $\mathbf{c}_{ik}$.

In terms of force-originating torques, a force $\mathbf{f}_{ij}$ is a *sliding vector.* Exactly the same torque is obtained if $\mathbf{f}_{ij}$ is applied to $\mathbf{q}'_{ij}$ such that $\mathbf{q}'_{ij} - \mathbf{q}_{ij}$ is parallel to $\mathbf{f}_{ij}$ (Figure 9.3(A)). From (9.1),

$$\begin{aligned} \mathbf{n}_{i-1,j} &= \mathbf{q}_{i-1,j} \times \mathbf{f}_{ij} \\ &= \mathbf{q}_{i-1,j} \times \mathbf{f}_{ij} + (\mathbf{q}'_{i-1,j} - \mathbf{q}_{i-1,j}) \times \mathbf{f}_{ij} \\ &= \mathbf{q}'_{i-1,j} \times \mathbf{f}_{ij} \end{aligned} \tag{9.4}$$

The force $\mathbf{f}_{ij}$ can slide anywhere along the line joining $\mathbf{q}_{i-1,j}$ and $\mathbf{q}'_{i-1,j}$ and still produce the same torque. However, if the force $\mathbf{f}_{ij}$ lies on a different line, a different torque is obtained; in that sense, it is also a *line-bound vector.*

Pure torque $\mathbf{c}_{ik}$ on the other hand is a free vector, which does not depend on the reference point O_{i-1}. Suppose that in Figure 9.2(A) the vector $-\mathbf{q}_{i-1,k}$ had been chosen so that it pointed to another reference point O_i, i.e., $\mathbf{q}_{i-1,k} = O_{i-1} - O_i$ (Figure 9.2(B)). Then the torque about O_i due to this couple is:

$$\begin{aligned} (\mathbf{0} \times (-\mathbf{f}_{ik})) + (\mathbf{q}_{i-1,k} + \mathbf{q}_{i-1,k}) \times \mathbf{f}_{ik} &= 2\mathbf{q}_{i-1,k} \times \mathbf{f}_{ik} \\ &= \mathbf{c}_{ik} \end{aligned} \tag{9.5}$$

which is the same as (9.3). By placing one half of the couple at O_i, a torque of zero is created by that component due to a zero moment arm.

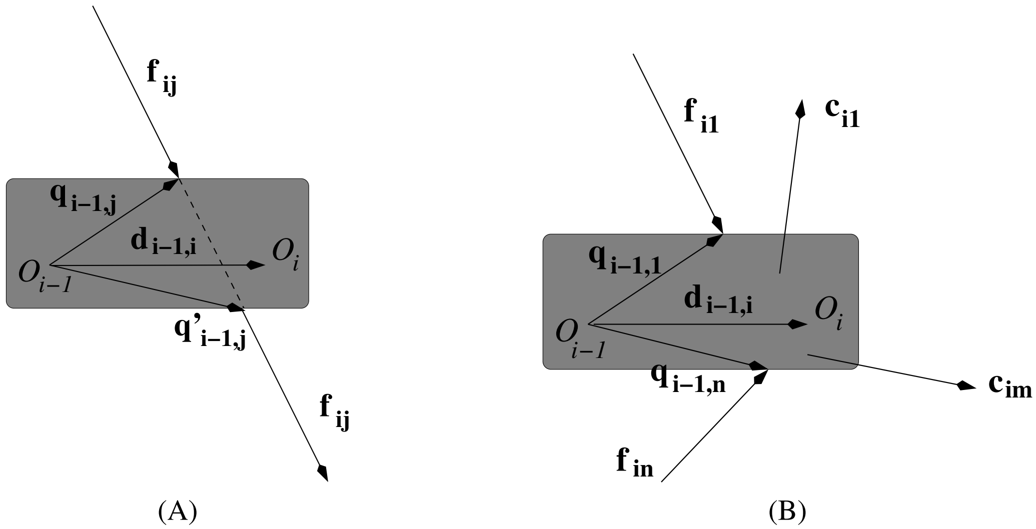


Figure 9.3: (A) Force is a sliding vector when generating torque. (B) Force and torque balance.

9.2 Force and Torque Balance

Consider a single rigid body, acted upon by n forces $\mathbf{f}_{ij}$ at points $\mathbf{q}_k$ with respect to origin O_i, and by m torques $\mathbf{c}_j$ (Figure 9.3(B)). The force balance equation yields the net force $\mathbf{f}_i$:

$$\mathbf{f}_i = \sum_{j=1}^{n} \mathbf{f}_{ij} \tag{9.6}$$

In terms of the force balance equation, the forces $\mathbf{f}_{ij}$ are *free vectors:* they are not line-bound.

The torque balance equation yields the net torque $\mathbf{n}_{i-1}$ about reference point O_{i-1}:

$$\mathbf{n}_{i-1} = \sum_{k=1}^{m} \mathbf{c}_{ik} + \sum_{j=1}^{n} \mathbf{q}_{i-1,j} \times \mathbf{f}_{ij} \tag{9.7}$$

Changing the reference point to O_i changes the net torque to $\mathbf{n}_i$ due to the change of moment arms.

$$\begin{aligned} \mathbf{n}_i &= \sum_{k=1}^{m} \mathbf{c}_{ik} + \sum_{j=1}^{n} (\mathbf{q}_{i-1,j} - \mathbf{d}_{i-1,i}) \times \mathbf{f}_{ij} \\ &= \mathbf{n}_{i-1} - \sum_{j=1}^{n} \mathbf{d}_{i-1,i} \times \mathbf{f}_{ij} \\ &= \mathbf{n}_{i-1} - \mathbf{d}_{i-1,i} \times \mathbf{f}_i \end{aligned} \tag{9.8}$$

where (9.6) has been used.

Example 9.1: Consider the system of forces acting on a 1 meter cube in Figure 9.4. The net force and torque about O_0 are:

$$\mathbf{f}_0 = \mathbf{x}_0(3-4) + \mathbf{y}_0(7-8) + \mathbf{z}_0(5-6)$$

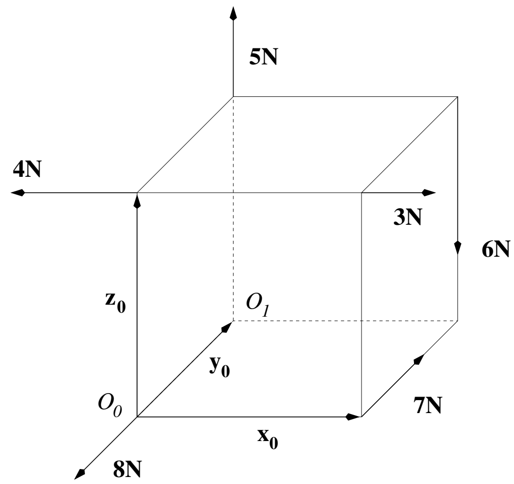


Figure 9.4: Force and torques acting on a cube.

$$\begin{aligned}
&= -(\mathbf{x}_0 + \mathbf{y}_0 + \mathbf{z}_0) \\
\mathbf{n}_0 &= \mathbf{x}_0 \times 7\mathbf{y}_0 + \mathbf{z}_0 \times (3\mathbf{x}_0 - 4\mathbf{x}_0) + \mathbf{y}_0 \times 5\mathbf{z}_0 + (\mathbf{x}_0 + \mathbf{y}_0) \times (-6\mathbf{z}_0) \\
&= 7\mathbf{z}_0 - \mathbf{y}_0 + 5\mathbf{x}_0 + 6\mathbf{y}_0 - 6\mathbf{x}_0 \\
&= -\mathbf{x}_0 + 5\mathbf{y}_0 + 7\mathbf{z}_0
\end{aligned}$$

The net torque about O_1 is found from (9.8):

$$\begin{aligned}
\mathbf{n}_1 &= (-\mathbf{x}_0 + 5\mathbf{y}_0 + 7\mathbf{z}_0) - \mathbf{y}_0 \times -(\mathbf{x}_0 + \mathbf{y}_0 + \mathbf{z}_0) \\
&= -\mathbf{x}_0 + 5\mathbf{y}_0 + 7\mathbf{z}_0 - \mathbf{z}_0 + \mathbf{x}_0 \\
&= 5\mathbf{y}_0 + 6\mathbf{z}_0
\end{aligned}$$

9.3 Center of Mass

Suppose a body i is made up of a system of N individual point masses or particles with mass m_{ik}, where $k = 1, \ldots, N$ (Figure 9.5(A)). Each particle is located by $\mathbf{q}_{0,ik} = Q_{ik} - O_0$ relative to frame 0. The particles are fixed relative to each other and move rigidly together. The total mass m_i of body i is:

$$m_i = \sum_{k=1}^{N} m_{ik} \tag{9.9}$$

The first moment of a mass particle is defined as $m_{ik}\,\mathbf{q}_{0,ik}$, also called its *mass moment.* The total mass moment $\mathbf{m}_{0i}$ of the system of particles about O_0 is:

$$\mathbf{m}_{0i} = \sum_{k=1}^{N} m_{ik}\mathbf{q}_{0,ik} \tag{9.10}$$

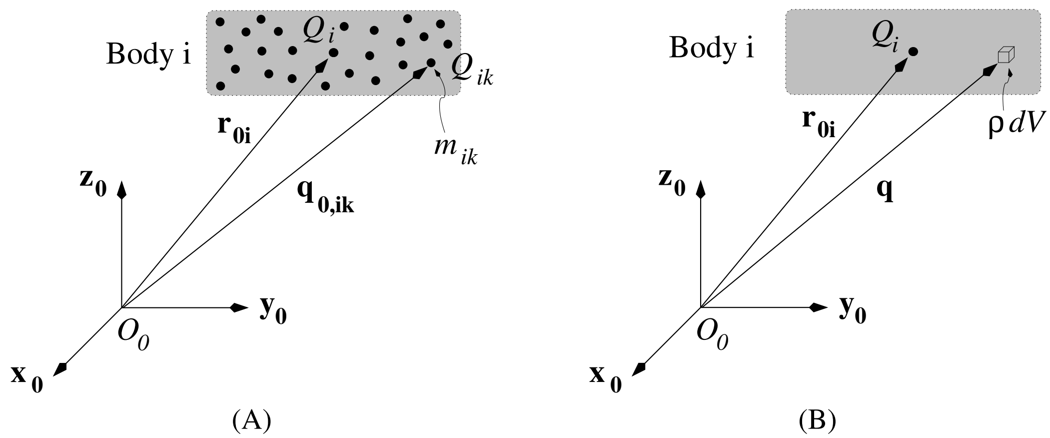


Figure 9.5: (A) The center of mass of a system of particles. (B) The center of mass for a body with continuously distributed mass.

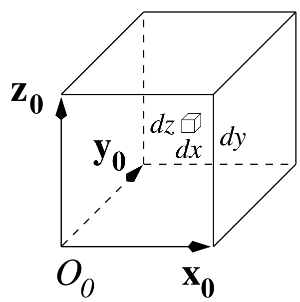


Figure 9.6: The center of mass of a uniform unit cube.

Suppose we factor the mass moment as $\mathbf{m}_{0i} = m_i \mathbf{r}_{0i}$, so that

$$\mathbf{r}_{0i} = \frac{\mathbf{m}_{0i}}{m_i} = \frac{1}{m_i} \sum_{k=1}^{N} m_{ik} \mathbf{q}_{0,ik} \tag{9.11}$$

The *center of mass* $\mathbf{r}_{0i}$ is defined as the first moment of the individual masses, normalized by the total mass. Point Q_i in body i depicts the location of the center of mass, so that $\mathbf{r}_{0i} = Q_i - O_0$.

For a body with continuously distributed mass, let ρ represent the density at a volume element dV located by $\mathbf{q}$. The mass of this element is then ρdV (Figure 9.5(B)). Then the total mass m_i is:

$$m_i = \int\int\int \rho dV \tag{9.12}$$

The center of mass $\mathbf{r}_{0i}$ is:

$$\mathbf{r}_{0i} = \frac{1}{m_i} \int\int\int \rho \mathbf{q} dV \tag{9.13}$$

The density ρ can be uniform throughout a body, or it may vary from region to region.

Example 9.2: Consider the unit cube in Figure 9.6(A). Axes 0 are embedded in a corner of the cube and are aligned with the edges; the cube is considered as body 0. The mass is uniformly distributed with density

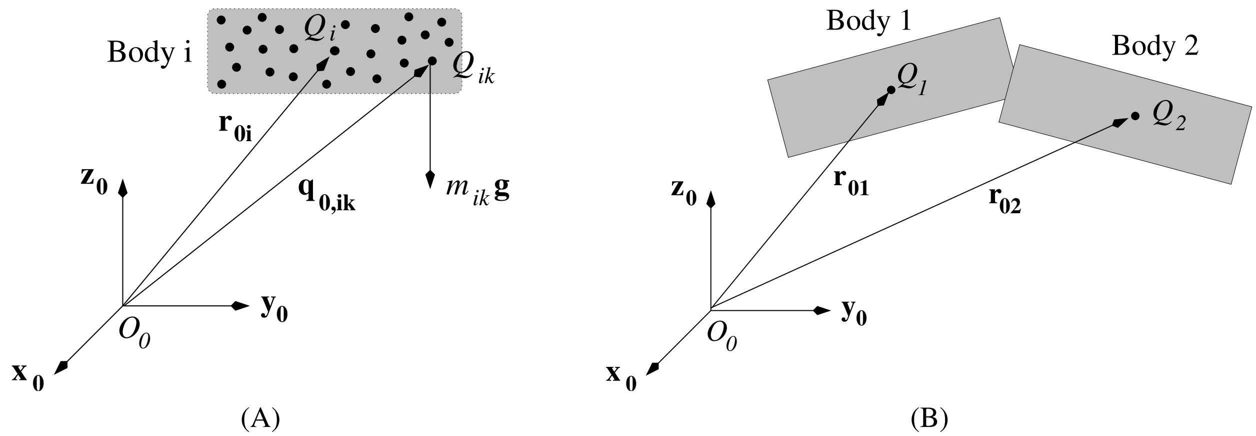


Figure 9.7: (A) Torque exerted by the weight $m_{ik}\mathbf{g}$ of a particle. (B) Composite center of mass for two objects.

$\rho = 1\,kg/m^3$. Then a mass element is $dV = dxdydz$, and the total mass is

$$m_0 = \int_0^1 \int_0^1 \int_0^1 \rho dxdydz = \int_0^1 \int_0^1 dxdy\; z]_0^1 = \int_0^1 dx\; y]_0^1 = x]_0^1 = 1\,kg \tag{9.14}$$

The center of mass $\mathbf{r}_{00} = Q_0 - O_0$ is in the center, as expected:

$$\mathbf{r}_{00} = \frac{1}{m}\int_0^1 \int_0^1 \int_0^1 \rho \begin{bmatrix} x \\ y \\ z \end{bmatrix} dxdydz \tag{9.15}$$

$$= \int_0^1 \int_0^1 \begin{bmatrix} xz \\ yz \\ z^2/2 \end{bmatrix}_0^1 dxdy = \int_0^1 \begin{bmatrix} xy \\ y^2/2 \\ y/2 \end{bmatrix}_0^1 dx = \begin{bmatrix} x^2/2 \\ x/2 \\ x/2 \end{bmatrix}_0^1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Since for all practical purposes the gravity field is constant throughout a body, the center of mass is also the *Center Of Gravity* (COG). If one could suspend a body from the center of gravity by a string, such as in a mobile, the body would be balanced (it wouldn't tip). The center of mass represents the most important point of the body, because it plays a central role in the body's dynamics.

9.3.1 Gravity Torques

For body i considered as a system of particles (Figure 9.7(A)), the weight of each particle is $m_{ik}\mathbf{g}$ where $\mathbf{g}$ is the gravity vector. The magnitude of the gravity vector is of course $||\mathbf{g}|| = 9.8m/sec^2 \equiv g$, and its direction is $\mathbf{g}/g$. For example, if $\mathbf{z}_0$ represented the upwards direction, then $\mathbf{g} = -g\mathbf{z}_0$.

Each particle exerts a torque $\mathbf{q}_{0,ik} \times m_{ik}\mathbf{g}$ about origin O_0, and the total torque is:

$$\mathbf{n}_0 = \sum_{k=1}^{n} \mathbf{q}_{0,ik} \times m_{ik}\mathbf{g} \tag{9.16}$$

Gravity is independent of the summation index, and can be pulled outside of the summation. The quantity inside the parentheses is just the mass moment $\mathbf{m}_{0i}$ (9.10).

$$\begin{aligned}\mathbf{n}_0 &= \left(\sum_{k=1}^{n} m_{ik}\mathbf{q}_{0,ik}\right) \times \mathbf{g} \\ &= \mathbf{m}_{0i} \times \mathbf{g}\end{aligned}$$

Introducing the total mass m_i as a divisor of the mass moment and a multiplier of the gravity vector, a relation is found with the center of gravity vector $\mathbf{r}_{0i}$ (9.11):

$$\begin{aligned}\mathbf{n}_0 &= \frac{\mathbf{m}_{0i}}{m_i} \times m_i\mathbf{g} \\ &= \mathbf{r}_{0i} \times m_i\mathbf{g}\end{aligned} \tag{9.17}$$

The gravity torque $\mathbf{n}_0$ is expressed as the action of the object's weight $m_i\mathbf{g}$ on the moment arm $\mathbf{r}_{0i}$. That is to say, the body's mass distribution is compactly represented by the center of gravity. Calculating the torque caused by an object's weight simply involves crossing the object's weight with the center of gravity vector.

9.3.2 Composite Center of Mass

Suppose two rigid bodies are connected, for example two links on either side of a manipulator's joint. Using the particle description again, suppose there are N particles in body 1 and M particles in body 2. The mass is m_1 for body 1 and m_2 for body 2, and the center of mass is $\mathbf{r}_{01}$ for body 1 and $\mathbf{r}_{02}$ for body 2 (Figure 9.7(B)). Considered as a composite body 3 with $N + M$ particles, the composite total mass m_3 is just the sum of all the particle masses, or the sum of the total masses of each body:

$$m_3 = \sum_{j=1}^{N} m_{1j} + \sum_{k=1}^{M} m_{2k} = m_1 + m_2 \tag{9.18}$$

The mass moment of the composite body 3 about origin O_0 is:

$$\mathbf{m}_{03} = \sum_{j=1}^{N} m_{1j}\mathbf{q}_{0,1j} + \sum_{k=1}^{M} m_{2j}\mathbf{q}_{0,2k} = \mathbf{m}_{01} + \mathbf{m}_{02} \tag{9.19}$$

Introduce mass multipliers and divisors to reveal a relation between the centers of mass:

$$\begin{aligned}\frac{\mathbf{m}_{03}}{m_3} &= \frac{m_1\mathbf{m}_{01}}{m_3 m_1} + \frac{m_2\mathbf{m}_{02}}{m_3 m_2} \\ \mathbf{r}_{03} &= \frac{m_1\mathbf{r}_{01} + m_2\mathbf{r}_{02}}{m_1 + m_2}\end{aligned} \tag{9.20}$$

where $\mathbf{r}_{03}$ is the center of mass of the composite body. It is the weighted average of the centers of mass of the individual bodies.

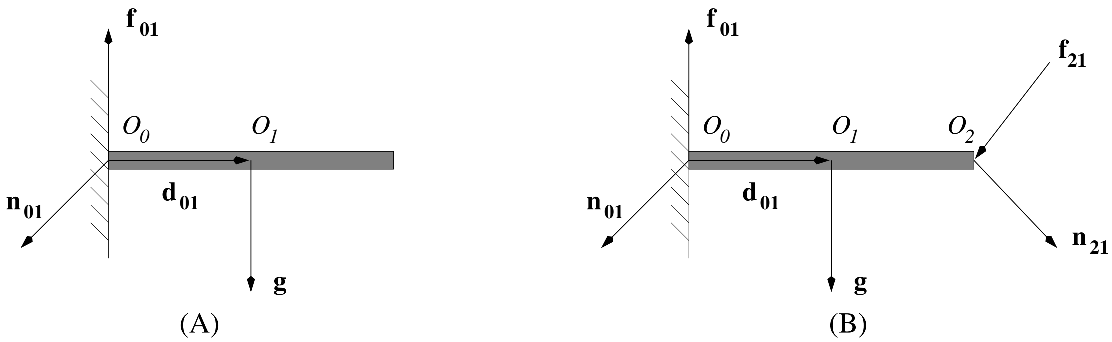


Figure 9.8: (A) Beam rigidly attached to the wall, with gravity. (B) External force $\mathbf{f}_{21}$ and torque $\mathbf{n}_{21}$ exerted on the end O_2.

9.4 Constraint Forces and Torques

Consider the horizontal thin beam of mass m_1 in Figure 9.8(A), which is rigidly attached to a wall on its left end. A gravity force of $m_1\mathbf{g}$ acts at the middle $\mathbf{d}_{01} = O_1 - O_0$, where $\mathbf{d}_{01} \perp \mathbf{g}$. Gravity would simply pull the beam down, were it not for a constraint force $\mathbf{f}_{01}$ and constraint torque $\mathbf{n}_{01}$ acting at the attachment point. Since the beam is in static equilibrium, the net force $\mathbf{f}_1$ on body 1 and the net torque $\mathbf{n}_0$ about point O_0 are:

$$\begin{aligned} \mathbf{f}_1 &= \mathbf{0} \\ &= m_1\mathbf{g} + \mathbf{f}_{01} \end{aligned} \tag{9.21}$$

$$\begin{aligned} \mathbf{n}_0 &= \mathbf{0} \\ &= \mathbf{d}_{01} \times m_1\mathbf{g} + \mathbf{n}_{01} \end{aligned} \tag{9.22}$$

Consequently the constraint force and torque are:

$$\mathbf{f}_{01} = -m_1\mathbf{g} \tag{9.23}$$

$$\mathbf{n}_{01} = -\mathbf{d}_{01} \times m_1\mathbf{g} \tag{9.24}$$

The constraint force and torque are to be considered to be applied by the wall *on* the beam. By Newton's third law, there is an equal and opposite force $\mathbf{f}_{10}$ and torque $\mathbf{n}_{10}$ exerted by the beam on the wall:

$$\mathbf{f}_{10} = -\mathbf{f}_{01} \tag{9.25}$$

$$\mathbf{n}_{10} = -\mathbf{n}_{01} \tag{9.26}$$

Example 9.3: Suppose $m_1 = 1\,kg$, $\mathbf{g} = -9.8\mathbf{y}_0\,m/s^2$, and $\mathbf{d}_{01} = \mathbf{x}_0\,m$. Then from (9.23)-(9.24),

$$\begin{aligned} \mathbf{f}_{01} &= -(1\,kg)(-9.8\mathbf{y}_0\,m/s^2) = 9.8\mathbf{y}_0\,N \\ \mathbf{n}_{01} &= -(\mathbf{x}_0\,m) \times (1\,kg)(-9.8\mathbf{y}_0\,m/s^2) = 9.8\mathbf{z}_0\,Nm \end{aligned}$$

Suppose someone grabs the end of the beam, at point O_2, and exerts a force $\mathbf{f}_{21}$ and torque $\mathbf{n}_{21}$ on the end (Figure 9.8(B)). Then the new force and torque balance equations are:

$$\begin{aligned} \mathbf{f}_1 &= \mathbf{0} \\ &= m_1\mathbf{g} + \mathbf{f}_{01} + \mathbf{f}_{21} \end{aligned} \tag{9.27}$$

$$\begin{aligned} \mathbf{n}_0 &= \mathbf{0} \\ &= \mathbf{d}_{01} \times m_1\mathbf{g} + \mathbf{d}_{02} \times \mathbf{f}_{21} + \mathbf{n}_{01} + \mathbf{n}_{21} \end{aligned} \tag{9.28}$$

Again, there are an equal and opposite force $\mathbf{f}_{12} = -\mathbf{f}_{21}$ and torque $\mathbf{n}_{12} = -\mathbf{n}_{21}$ exerted by the beam on the person. The reaction force and torque from the wall are:

$$\mathbf{f}_{01} = -m_1\mathbf{g} - \mathbf{f}_{21} \tag{9.29}$$

$$\mathbf{n}_{01} = -\mathbf{d}_{01} \times m_1\mathbf{g} - \mathbf{d}_{02} \times \mathbf{f}_{21} - \mathbf{n}_{21} \tag{9.30}$$

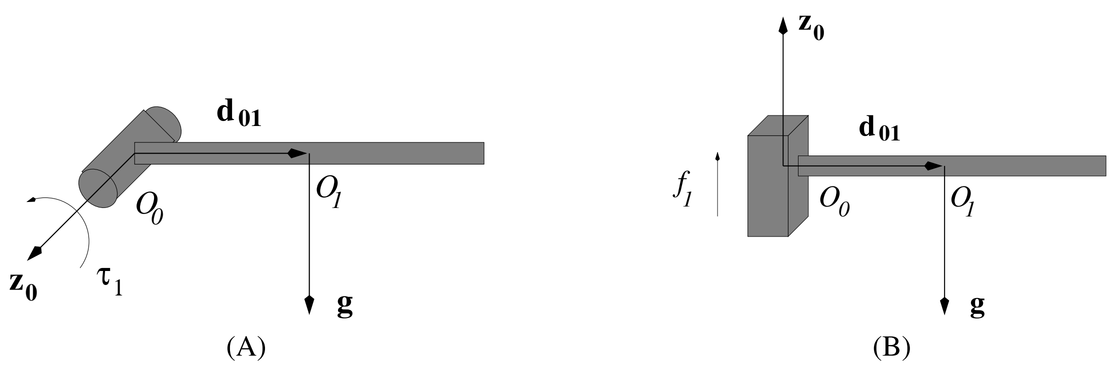


Figure 9.9: Beam attached to the wall through a rotary joint (A) or a prismatic joint (B).

9.5 Joint Torque and Joint Force

Suppose that the beam is not rigidly attached to the wall, but rather there is a rotary joint, as if the beam were a link of a robot (Figure 9.9(A)). The link is free to rotate, unless there is a joint torque τ_1 about the joint axis counteracting the reaction torque $\mathbf{n}_{01}$:

$$\tau_1 = \mathbf{z}_0 \cdot \mathbf{n}_{01} \tag{9.31}$$

Now suppose that the beam is attached to the wall through a prismatic joint (Figure 9.9(B)). The beam would slide down along the axis $\mathbf{z}_0$ of the link under the action of gravity, unless there is a counteracting force f_1 exerted at the prismatic joint. This joint force f_1 is the z-component of the constraint force $\mathbf{f}_{01}$:

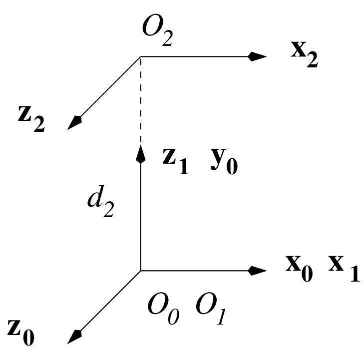
$$f_1 = \mathbf{z}_0 \cdot \mathbf{f}_{01} \tag{9.32}$$

Example 9.4: For the previous example, in the case of a rotary joint,

$$\tau_1 = \mathbf{z}_0 \cdot 9.8\mathbf{z}_0 \, Nm = 9.8 \, Nm$$

In the case of a prismatic joint, suppose $\mathbf{g}$ points in the $-\mathbf{z}_0$ direction. Then

$$f_1 = \mathbf{z}_0 \cdot 9.8\mathbf{z}_0 \, N = 9.8 \, N$$



i	a_i	d_i	α_i	θ_i
1	0	0	$-\pi/2$	$*$
2	0	$*$	$\pi/2$	0

Figure 9.10: The polar manipulator.

9.5.1 Polar Manipulator

Consider the polar manipulator in Figure 9.10. Assume that each link i's mass is m_i with COG at origin O_i, and that gravity acts as $\mathbf{g} = -g\mathbf{y}_0$. The constraint forces and torques all act at $O_0 = O_1$. The force balance equations for links 2 and 1 are respectively:

$$\mathbf{0} = \mathbf{f}_{12} + m_2\mathbf{g} \tag{9.33}$$

$$\mathbf{0} = \mathbf{f}_{01} + \mathbf{f}_{21} + m_1\mathbf{g} \tag{9.34}$$

Consequently,

$$\mathbf{f}_{12} = m_2 g \mathbf{y}_0 \tag{9.35}$$

$$\mathbf{f}_{01} = (m_1 + m_2) g \mathbf{y}_0 \tag{9.36}$$

Because $\mathbf{f}_{01}$ passes through the link 1 COG, it exerts no torque on link 1. The same for $\mathbf{f}_{12}$ and link 2. So the torque balance equations for links 2 and 1 are:

$$\mathbf{0} = \mathbf{n}_{12} + d_2\mathbf{z}_1 \times m_2\mathbf{g} \tag{9.37}$$

$$\mathbf{0} = \mathbf{n}_{01} + \mathbf{n}_{21} \tag{9.38}$$

Consequently,

$$\mathbf{n}_{12} = d_2 m_2 g \mathbf{z}_1 \times \mathbf{y}_0 \tag{9.39}$$

$$\mathbf{n}_{01} = \mathbf{n}_{12} \tag{9.40}$$

Solving for the joint force f_2,

$$\begin{aligned} f_2 &= \mathbf{z}_1 \cdot \mathbf{f}_{12} \\ &= \mathbf{z}_1 \cdot m_2 g \mathbf{y}_0 \\ &= m_2 g \cos\theta_1 \end{aligned} \tag{9.41}$$

Solving for the joint torque τ_1,

$$\begin{aligned} \tau_1 &= \mathbf{z}_0 \cdot \mathbf{n}_{01} \\ &= \mathbf{z}_0 \cdot \mathbf{n}_{12} \\ &= \mathbf{z}_0 \cdot (d_2 m_2 g \mathbf{z}_1 \times \mathbf{y}_0) \\ &= d_2 m_2 g \mathbf{z}_1 \cdot (\mathbf{y}_0 \times \mathbf{z}_0) \\ &= d_2 m_2 g \mathbf{z}_1 \cdot \mathbf{x}_0 \\ &= -d_2 m_2 g \mathrm{s}\theta_1 \end{aligned} \tag{9.42}$$

9.5.2 Two-Link Planar Manipulator

Consider the two-link planar manipulator in Figure 9.11 in a non-zero angle position. The interjoint vectors are $\mathbf{d}_{01} = a_1 \mathbf{x}_1$ and $\mathbf{d}_{12} = a_2 \mathbf{x}_2$. Suppose the center of gravity is in the middle of each link, the manipulator is at rest, and m_1 and m_2 are the masses of links 1 and 2. The force/torque balance equations for link 2 referred to origin O_1 are:

$$\mathbf{0} = m_2 \mathbf{g} + \mathbf{f}_{12} \tag{9.43}$$

$$\mathbf{0} = \frac{\mathbf{d}_{12}}{2} \times m_2 \mathbf{g} + \mathbf{n}_{12} \tag{9.44}$$

Note the use of the force $\mathbf{f}_{12}$ and torque $\mathbf{n}_{12}$ exerted *by* link 1 *on* link 2, due to the attachment at joint 2. Solving for the joint 2 constraints,

$$\mathbf{f}_{12} = -m_2 \mathbf{g} \tag{9.45}$$

$$\mathbf{n}_{12} = -\frac{\mathbf{d}_{12}}{2} \times m_2 \mathbf{g} \tag{9.46}$$

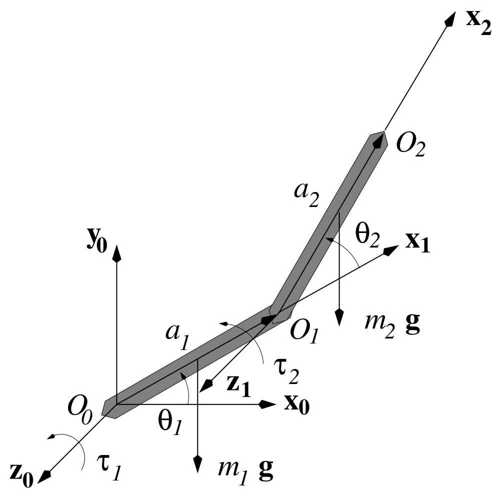


Figure 9.11: Joint torques for the two-link planar manipulator.

The force/torque balance equations for link 1 referred to origin O_0 are:

$$\mathbf{0} = m_1\mathbf{g} + \mathbf{f}_{01} + \mathbf{f}_{21} \tag{9.47}$$

$$\mathbf{0} = \frac{\mathbf{d}_{01}}{2} \times m_1\mathbf{g} + \mathbf{d}_{01} \times \mathbf{f}_{21} + \mathbf{n}_{01} + \mathbf{n}_{21} \tag{9.48}$$

Here we use the force $\mathbf{f}_{21}$ and torque $\mathbf{n}_{21}$ exerted by link 2 on link 1. Substituting the equal and opposite force and torque by link 1 on link 2, and solving for the joint 1 constraints:

$$\begin{aligned}\mathbf{f}_{01} &= \mathbf{f}_{12} - m_1\mathbf{g} \\ &= -(m_1 + m_2)\mathbf{g}\end{aligned} \tag{9.49}$$

$$\begin{aligned}\mathbf{n}_{01} &= \mathbf{n}_{12} - \frac{\mathbf{d}_{01}}{2} \times m_1\mathbf{g} + \mathbf{d}_{01} \times \mathbf{f}_{12} \\ &= -\frac{\mathbf{d}_{12}}{2} \times m_2\mathbf{g} - \frac{\mathbf{d}_{01}}{2} \times m_1\mathbf{g} - \mathbf{d}_{01} \times m_2\mathbf{g} \\ &= -\frac{\mathbf{d}_{01}}{2} \times m_1\mathbf{g} - (\mathbf{d}_{01} + \frac{\mathbf{d}_{12}}{2}) \times m_2\mathbf{g}\end{aligned} \tag{9.50}$$

These equations make intuitive sense. The weight of the combined links $(m_1 + m_2)\mathbf{g}$ has to be opposed by the joint 1 constraint force. The joint 1 constraint torque counteracts the gravity torque from the first link plus the gravity torque from the second link. The vector $\mathbf{d}_{01} + \mathbf{d}_{12}/2$ is the moment arm from O_0 to the link 2 center of gravity.

The joint torques τ_1 and τ_2 are the projections of the constraint torques $\mathbf{n}_{01}$ and $\mathbf{n}_{12}$ onto the joint axes $\mathbf{z}_0$:

$$\tau_2 = \mathbf{z}_0 \cdot \mathbf{n}_{12} \tag{9.51}$$

$$\tau_1 = \mathbf{z}_0 \cdot \mathbf{n}_{01} \tag{9.52}$$

Supposing that $\mathbf{g} = -g\mathbf{y}_0$ where $g = 9.8m/s^2$ and substituting for $\mathbf{d}_{01}$ and $\mathbf{d}_{12}$,

$$\begin{aligned}
\tau_2 &= -\mathbf{z}_0 \cdot \frac{\mathbf{d}_{12}}{2} \times m_2\mathbf{g} \\
&= \mathbf{z}_0 \cdot \frac{1}{2} m_2 g a_2 \mathbf{x}_2 \times \mathbf{y}_0 \\
&= \mathbf{z}_0 \cdot \frac{1}{2} m_2 g a_2 \mathbf{z}_0 \sin(\pi/2 - \theta_1 - \theta_2) \\
&= \frac{1}{2} m_2 g a_2 \cos(\theta_1 + \theta_2)
\end{aligned} \tag{9.53}$$

$$\begin{aligned}
\tau_1 &= -\mathbf{z}_0 \cdot \left(\frac{\mathbf{d}_{01}}{2} \times m_1\mathbf{g} + (\mathbf{d}_{01} + \frac{\mathbf{d}_{12}}{2}) \times m_2\mathbf{g} \right) \\
&= \mathbf{z}_0 \cdot \left(\frac{1}{2} m_1 g a_1 \mathbf{x}_1 \times \mathbf{y}_0 + m_2 g (a_1\mathbf{x}_1 + \frac{1}{2} a_2 \mathbf{x}_2) \times \mathbf{y}_0 \right) \\
&= \mathbf{z}_0 \cdot \left(\frac{1}{2} m_1 g a_1 \mathbf{z}_0 \sin(\pi/2 - \theta_1) + m_2 g \mathbf{z}_0 (a_1 \sin(\pi/2 - \theta_1) + \frac{1}{2} a_2 \sin(\pi/2 - \theta_1 - \theta_2)) \right) \\
&= \frac{1}{2} m_1 g a_1 \cos\theta_1 + m_2 g (a_1 \cos\theta_1 + \frac{1}{2} a_2 \cos(\theta_1 + \theta_2))
\end{aligned} \tag{9.54}$$

Again, these expressions make intuitive sense. For example, in the first term for τ_1, $(a_1/2)\cos\theta_1$ is the perpendicular component of the moment arm through which the gravity force $m_1 g$ acts. In the second term, $a_1 \cos\theta_1 + a_2/2\cos(\theta_1 + \theta_2)$ is the perpendicular component of the moment arm through which the gravity force $m_2 g$ acts.

Another viewpoint is to consider that links 1 and 2 form a composite link 3, with mass $m_3 = m_1 + m_2$ and center of gravity $\mathbf{r}_{03}$ derived from (9.20):

$$\mathbf{r}_{03} = \frac{m_1\mathbf{d}_{01}/2 + m_2(\mathbf{d}_{01} + \mathbf{d}_{12}/2)}{m_3} \tag{9.55}$$

The constraint torque at joint 1 (9.50) can then be rewritten as:

$$\mathbf{n}_{01} = -\mathbf{r}_{03} \times m_3\mathbf{g} \tag{9.56}$$

This torque is just the first moment of the composite body weight.

9.6 Spatial Manipulators

In general, the center of gravity of any link i of a manipulator will not be aligned with the interorigin vector $\mathbf{d}_{i-1,i}$, but will have an arbitrary location Q_i in the link (Figure 9.12(A)). Define $\mathbf{r}_{ji} = Q_i - O_j$. Usually the center of mass of link i is known with respect to the link i coordinate system, i.e., ${}^i\mathbf{r}_{ii}$. The other vectors are derived from it and the interorigin vectors:

$$\begin{aligned}
{}^i\mathbf{r}_{i-1,i} &= {}^i\mathbf{d}_{i-1,i} - {}^i\mathbf{r}_{ii} \\
{}^i\mathbf{r}_{0i} &= {}^i\mathbf{d}_{0i} - {}^i\mathbf{r}_{ii}
\end{aligned}$$

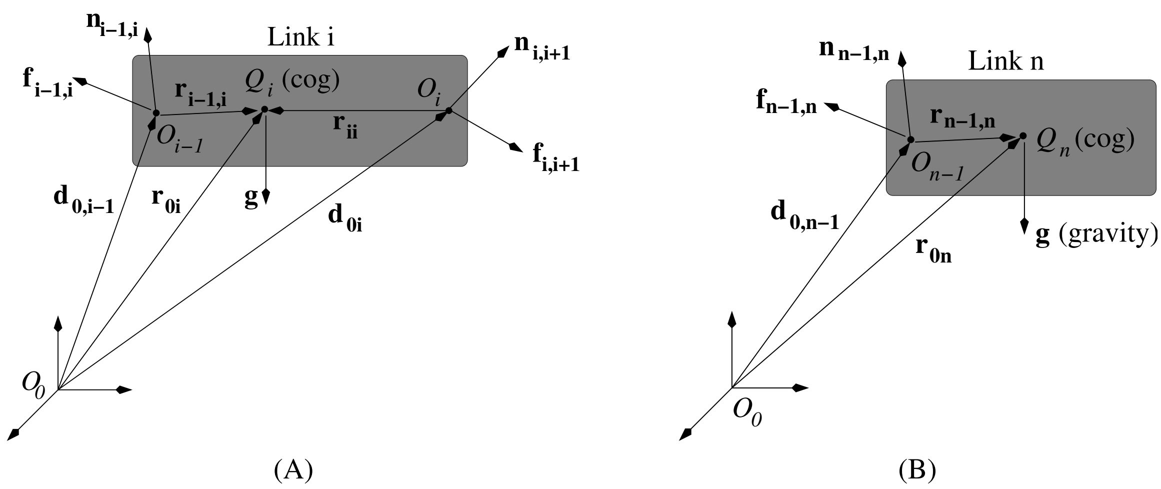


Figure 9.12: (A) Center of gravity location for an intermediate link i, and the constraint forces and torques. (B) Constraint force and torque for the last link n.

The force and torque balance equations for link i are:

$$\mathbf{0} = \mathbf{f}_{i-1,i} - \mathbf{f}_{i,i+1} + m_i\mathbf{g} \tag{9.57}$$

$$\mathbf{0} = \mathbf{n}_{i-1,i} - \mathbf{n}_{i,i+1} - \mathbf{r}_{i-1,i} \times \mathbf{f}_{i-1,i} + \mathbf{r}_{ii} \times \mathbf{f}_{i,i+1} \tag{9.58}$$

The gravity load at joint i depends not only on link i's weight, but also on the weight of all links distal to it. These distal links exert their effect through the force $\mathbf{f}_{i,i+1}$ and torque $\mathbf{n}_{i,i+1}$ of constraint at joint $i+1$. These loads are calculated recursively starting with the last link and moving to the base. For the last link, assume that the link is not touching anything. Then there are no constraint forces $\mathbf{f}_{n,n+1}$ and torques $\mathbf{n}_{n,n+1}$ (Figure 9.12(B)).

$$\mathbf{0} = \mathbf{f}_{n-1,n} + m_n\mathbf{g} \tag{9.59}$$

$$\mathbf{0} = \mathbf{n}_{n-1,n} - \mathbf{r}_{n-1,n} \times \mathbf{f}_{n-1,n} \tag{9.60}$$

Consequently, the force and torque of constraint at joint n may be calculated:

$$\mathbf{f}_{n-1,n} = -m_n\mathbf{g} \tag{9.61}$$

$$\mathbf{n}_{n-1,n} = \mathbf{r}_{n-1,n} \times \mathbf{f}_{n-1,n} \tag{9.62}$$

The force and torque of constraint at joint n are then known in terms of the force and torque balance for link $n-1$. This allows the force and torque of constraint at joint $n-1$ to be calculated. More generally, suppose that for link i the distal joint force $\mathbf{f}_{i,i+1}$ and torque $\mathbf{n}_{i,i+1}$ of constraint have previously been calculated. Then from (9.57)-(9.58),

$$
\begin{aligned}
\mathbf{f}_{i-1,i} &= \mathbf{f}_{i,i+1} - m_i\mathbf{g} && (9.63)\\
\mathbf{n}_{i-1,i} &= \mathbf{n}_{i,i+1} + \mathbf{r}_{i-1,i} \times \mathbf{f}_{i-1,i} - \mathbf{r}_{ii} \times \mathbf{f}_{i,i+1} && (9.64)
\end{aligned}
$$

Once the forces and torques of constraint have been calculated for all joints, the actuator force f_i or torque τ_i required at a particular joint i can be calculated:

$$
\begin{aligned}
f_i &= \mathbf{z}_{i-1} \cdot \mathbf{f}_{i-1,i} \qquad \text{joint } i \text{ prismaqtic} && (9.65)\\
\tau_i &= \mathbf{z}_{i-1} \cdot \mathbf{n}_{i-1,i} \qquad \text{joint } i \text{ rotary} && (9.66)
\end{aligned}
$$

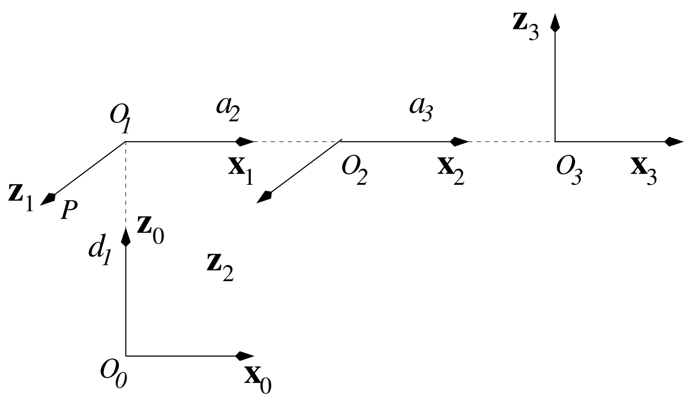


Figure 9.13: Spatial 3R arm.

9.6.1 3R Spatial Arm

As an example, the 3R spatial arm in Figure 9.13 is assumed to have centers of mass at the distal joints for simplicity ($Q_i = O_i$). This means $\mathbf{r}_{ii} = 0$ and $\mathbf{r}_{i-1,i} = \mathbf{d}_{i-1,i}$ in (9.64). The reference point for the net torque calculations for link i will be taken to be the proximal joint O_{i-1}. The force balance equations (9.64) for links 3, 2 and 1 are:

$$
\begin{aligned}
\mathbf{f}_{23} &= -m_3\mathbf{g}\\
\mathbf{f}_{12} &= \mathbf{f}_{23} - m_2\mathbf{g}\\
\mathbf{f}_{01} &= \mathbf{f}_{12} - m_1\mathbf{g}
\end{aligned}
$$

Solving for the constraint forces,

$$
\begin{aligned}
\mathbf{f}_{23} &= -m_3\mathbf{g}\\
\mathbf{f}_{12} &= -(m_2 + m_3)\mathbf{g}\\
\mathbf{f}_{01} &= -(m_1 + m_2 + m_3)\mathbf{g}
\end{aligned}
$$

These equations are the same for any 3-link manipulator. They say simply that joint 3 has to bear the weight of the 3rd link, joint 2 has to bear the weight of links 2 and 3, and joint 1 has to bear the weight of all 3 links. The torque balance equations (9.64) are:

$$\begin{aligned}
\mathbf{n}_{23} &= \mathbf{d}_{23} \times \mathbf{f}_{23} \\
&= -a_3\mathbf{x}_3 \times m_3\mathbf{g} \\
\mathbf{n}_{12} &= \mathbf{n}_{23} + \mathbf{d}_{12} \times \mathbf{f}_{12} \\
&= \mathbf{n}_{23} - a_2\mathbf{x}_2 \times (m_2 + m_3)\mathbf{g} \\
\mathbf{n}_{01} &= \mathbf{n}_{12} + \mathbf{d}_{01} \times \mathbf{f}_{01} \\
&= \mathbf{n}_{12} - d_1\mathbf{z}_0 \times (m_1 + m_2 + m_3)\mathbf{g}
\end{aligned}$$

To find the joint torques, project the torques of constraints onto their respective joint axes. Starting with τ_3,

$$\begin{aligned}
\tau_3 &= \mathbf{z}_2 \cdot \mathbf{n}_{23} \\
&= \mathbf{z}_2 \cdot (-a_3\mathbf{x}_3 \times m_3\mathbf{g}) \\
&= -a_3 m_3(\mathbf{z}_2 \times \mathbf{x}_3) \cdot \mathbf{g} \\
&= -a_3 m_3 \mathbf{z}_3 \cdot \mathbf{g}
\end{aligned}$$

since $\mathbf{z}_2 = -\mathbf{y}_3$. Next, τ_2 is

$$\begin{aligned}
\tau_2 &= \mathbf{z}_1 \cdot \mathbf{n}_{12} \\
&= \mathbf{z}_1 \cdot (\mathbf{n}_{23} - a_2\mathbf{x}_2 \times (m_2 + m_3)\mathbf{g}) \\
&= \mathbf{z}_1 \cdot \mathbf{n}_{23} - \mathbf{z}_1 \cdot a_2\mathbf{x}_2 \times (m_2 + m_3)\mathbf{g} \\
&= \tau_3 - a_2(m_2 + m_3)(\mathbf{z}_2 \times \mathbf{x}_2) \cdot \mathbf{g} \\
&= \tau_3 - a_2(m_2 + m_3)\mathbf{y}_2 \cdot \mathbf{g}
\end{aligned}$$

since $\mathbf{z}_1 = \mathbf{z}_2$. Finally, τ_1 is

$$\begin{aligned}
\tau_1 &= \mathbf{z}_0 \cdot \mathbf{n}_{01} \\
&= \mathbf{z}_0 \cdot (\mathbf{n}_{12} - d_1\mathbf{z}_0 \times (m_1 + m_2 + m_3)\mathbf{g}) \\
&= \mathbf{z}_0 \cdot \mathbf{n}_{12} \\
&= \mathbf{z}_0 \cdot (\mathbf{n}_{23} - a_2\mathbf{x}_2 \times (m_2 + m_3)\mathbf{g})
\end{aligned}$$

These equations don't simplify further without making an assumption about the gravity vector. Suppose $\mathbf{g} = -g\mathbf{y}_0$. Then

$$\begin{aligned}
\tau_3 &= -a_3 m_3 \mathbf{z}_3 \cdot (-g\mathbf{y}_0) \\
&= a_3 m_3 g(\mathbf{z}_3 \cdot \mathbf{y}_0) \\
\tau_2 &= \tau_3 - a_2(m_2 + m_3)\mathbf{y}_2 \cdot (-g\mathbf{y}_0) \\
&= \tau_3 + a_2(m_2 + m_3)g(\mathbf{y}_2 \cdot \mathbf{y}_0) \\
\tau_1 &= \mathbf{z}_0 \cdot (\mathbf{n}_{23} - a_2\mathbf{x}_2 \times (m_2 + m_3)(-g\mathbf{y}_0) \\
&= \mathbf{z}_0 \cdot (-a_3\mathbf{x}_3 \times m_3(-g\mathbf{y}_0) + \mathbf{z}_0 \cdot a_2\mathbf{x}_2 \times (m_2 + m_3)g\mathbf{y}_0 \\
&= -a_3 m_3 g(\mathbf{z}_0 \times \mathbf{y}_0) \cdot \mathbf{x}_3 - a_2(m_2 + m_3)g(\mathbf{z}_0 \times \mathbf{y}_0) \cdot \mathbf{x}_2 \\
&= a_3 m_3 g(\mathbf{x}_0 \cdot \mathbf{x}_3) + a_2(m_2 + m_3)g(\mathbf{x}_0 \cdot \mathbf{x}_2)
\end{aligned}$$

The dot products can be evaluated by mid-frame computation.

9.7 End Link Contact

When a manipulator contacts the environment, picks up an object, or accomplishes some task such as assembly of two parts, forces $\mathbf{f}_{n,n+1}$ and torques $\mathbf{n}_{i,i+1}$ are generated at the end link. These forces and torques have to be ocvercome by a robot's motors. The situation is similar to the forces and torques at the end of the beam in Figure 9.8(B), but now applied to the end link of a manipulator chain. The end link force and torque have to be measured with a six-axis force/torque sensor, and its value used in link n's force-torque balance equation:

$$\mathbf{f}_{n-1,n} = \mathbf{f}_{n,n+1} - m_n\mathbf{g} \tag{9.67}$$

$$\mathbf{n}_{n-1,n} = \mathbf{n}_{n,n+1} + \mathbf{r}_{n-1,n} \times \mathbf{f}_{n-1,n} - \mathbf{r}_{nn} \times \mathbf{f}_{n,n+1} \tag{9.68}$$

Another approach is to separately compute the actuator forces and torques for gravity and for the endpoint force and torque. The reason for doing so may be that the robot is pushing against a surface with varying forces, but the joint angles don't change and the gravity load remains the same. Suppose that the joint forces and torques have been calculated to overcome gravity, but not taking into consideration the endpoint force and torque.

The reference point for the endpoint force and torque will be considered to be the last frame origin O_n (Figure 9.14). For simplicity, suppose that all joints are rotary. We will now compute the joint torque τ_i for each joint due to the end link force and torque alone; gravity will be ignored. The corresponding constraint torque $\mathbf{n}_{i-1,i}$ referred to a link origin O_{i-1} is:

$$\mathbf{n}_{i-1,i} = \mathbf{d}_{i-1,n} \times \mathbf{f}_{n,n+1} + \mathbf{n}_{n,n+1} \tag{9.69}$$

The joint torque τ_i is the projection of the torque of constraint $\mathbf{n}_{i-1,i}$ onto the joint axis $\mathbf{z}_{i-1}$:

$$\tau_i = \mathbf{n}_{i-1,i} \cdot \mathbf{z}_{i-1}$$

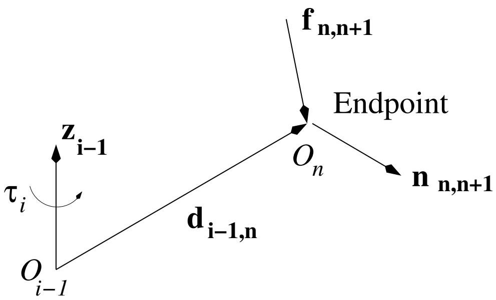


Figure 9.14: Endpoint wrench and resulting torque on joint i.

$$= (\mathbf{d}_{i-1,n} \times \mathbf{f}_{n,n+1} + \mathbf{n}_{n,n+1}) \cdot \mathbf{z}_{i-1} \tag{9.70}$$

Rearranging this equation,

$$\begin{aligned}
\tau_i &= \mathbf{z}_{i-1} \cdot (\mathbf{d}_{i-1,n} \times \mathbf{f}_{n,n+1}) + \mathbf{n}_{n,n+1} \cdot \mathbf{z}_{i-1} \\
&= (\mathbf{z}_{i-1} \times \mathbf{d}_{i-1,n}) \cdot \mathbf{f}_{n,n+1} + \mathbf{z}_{i-1} \cdot \mathbf{n}_{n,n+1} \\
&= \begin{bmatrix} (\mathbf{z}_{i-1} \times \mathbf{d}_{i-1,n})^T & \mathbf{z}_{i-1}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}_{n,n+1} \\ \mathbf{n}_{n,n+1} \end{bmatrix} \qquad (9.71)
\end{aligned}$$

Combining the results for all joints,

$$\begin{aligned}
\begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} &= \begin{bmatrix} (\mathbf{z}_0 \times \mathbf{d}_{0n})^T & \mathbf{z}_0^T \\ \vdots & \vdots \\ (\mathbf{z}_{n-1} \times \mathbf{d}_{n-1,n})^T & \mathbf{z}_{n-1}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}_{n,n+1} \\ \mathbf{n}_{n,n+1} \end{bmatrix} \\
\boldsymbol{\tau} &= \mathbf{J}_v^T \begin{bmatrix} \mathbf{f}_{n,n+1} \\ \mathbf{n}_{n,n+1} \end{bmatrix} \qquad (9.72)
\end{aligned}$$

For a prismatic joint i, appropriate substitution is made for the column of the velocity Jacobian $\mathbf{J}_v$. Correspondingly, there is a linear force f_i exerted by joint i's linear actuator.

This fundamental result relates statics and velocity kinematics, because of the velocity Jacobian $\mathbf{J}_v$. This is because the velocity Jacobian contains moment arms and rotation axes, which in one instance is used to transmit forces and torques, and in the other instance linear and angular velocities.

9.7.1 Virtual Work Derivation

The previous derivation of (9.72) was achieved straightforwardly from geometric considerations. Another derivation method employs the *principle of virtual work,* which states that the rate of doing work is the same regardless of the parameterization during a virtual motion.

Suppose that there is a virtual motion $\dot{\theta}_i$ at joint i and a corresponding virtual motion $\dot{\mathbf{d}}_{0n}$ and $\boldsymbol{\omega}_{0n}$ at the endpoint. The kinematic relation between these virtual motions is:

$$\dot{\mathbf{d}}_{0n} = \dot{\theta}_i \mathbf{z}_{i-1} \times \mathbf{d}_{i-1,n} \tag{9.73}$$

$$\boldsymbol{\omega}_{0n} = \dot{\theta}_i \mathbf{z}_{i-1} \tag{9.74}$$

The rate of doing work from the joint perspective is $\tau_i\dot{\theta}_i$, and is equal to the rate of doing work from the endpoint perspective:

$$\tau_i\dot{\theta}_i \quad = \quad \mathbf{f}_{n,n+1}\cdot\dot{\mathbf{d}}_{0n} + \mathbf{n}_{n,n+1}\cdot\boldsymbol{\omega}_{0n} \tag{9.75}$$

Substituting the kinematic relations,

$$\tau_i\dot{\theta}_i \quad = \quad \mathbf{f}_{n,n+1}\cdot\dot{\theta}_i(\mathbf{z}_{i-1}\times\mathbf{d}_{i-1,n}) + \mathbf{n}_{n,n+1}\cdot\dot{\theta}_i\mathbf{z}_{i-1} \tag{9.76}$$

Factoring out $\dot{\theta}_i$ from (9.76),

$$\begin{aligned}\tau_i &= \mathbf{f}_{n,n+1}\cdot(\mathbf{z}_{i-1}\times\mathbf{d}_{i-1,n}) + \mathbf{n}_{n,n+1}\cdot\mathbf{z}_{i-1}\\ &= \begin{bmatrix}(\mathbf{z}_{i-1}\times\mathbf{d}_{i-1,n})^T & \mathbf{z}_{i-1}^T\end{bmatrix}\begin{bmatrix}\mathbf{f}_{n,n+1}\\ \mathbf{n}_{n,n+1}\end{bmatrix}\end{aligned} \tag{9.77}$$

For all n joints of a manipulator,

$$\begin{bmatrix}\tau_1\\ \vdots\\ \tau_n\end{bmatrix} = \begin{bmatrix}(\mathbf{z}_0\times\mathbf{d}_{0n})^T & \mathbf{z}_0^T\\ \vdots & \vdots\\ (\mathbf{z}_{n-1}\times\mathbf{d}_{n-1,n})^T & \mathbf{z}_{n-1}^T\end{bmatrix}\begin{bmatrix}\mathbf{f}_{n,n+1}\\ \mathbf{n}_{n,n+1}\end{bmatrix} \tag{9.78}$$

which is the same result as before.

9.8 Wrenches

The relation of the force/torque with respect to point O_i to the force/torque with respect to point O_{i-1} can be expressed by a combined matrix-vector relation from (9.6) and (9.8):

$$\begin{aligned}\begin{bmatrix}\mathbf{f}_{i-1}\\ \mathbf{n}_{i-1}\end{bmatrix} &= \begin{bmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{S}(\mathbf{d}_{i-1,i}) & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{f}_i\\ \mathbf{n}_i\end{bmatrix}\\ \mathbf{w}_{i-1} &\equiv \mathbf{W}_{i-1,i}\,\mathbf{w}_i\end{aligned} \tag{9.79}$$

where $\mathbf{I}$ is the 3-by-3 identity matrix. The stacked vector $\mathbf{w}_i$ is termed the *wrench*. The matrix $\mathbf{W}_{i-1,i}$ is the *wrench transmission matrix,* which succinctly describes how the wrench changes with the point of reference.

The wrench is a screw coordinate. Let $\mathbf{w}_1$ be the wrench at point O_1:

$$\mathbf{w}_1 = \begin{bmatrix}\mathbf{f}\\ \mathbf{n}_1\end{bmatrix} \tag{9.80}$$

Then for a translation of origin to O_0, the new wrench $\mathbf{w}_0$ is:

$$\mathbf{w}_0 = \begin{bmatrix}\mathbf{f}\\ \mathbf{n}_1 + \mathbf{d}_{01}\times\mathbf{f}\end{bmatrix} \tag{9.81}$$

There are two special cases.

1. Pure force

$$\mathbf{w}_0 = \begin{bmatrix} \mathbf{f} \\ \mathbf{r} \times \mathbf{f} \end{bmatrix} \tag{9.82}$$

2. Pure torque (couple).

$$\mathbf{w}_0 = \begin{bmatrix} \mathbf{0} \\ \mathbf{c} \end{bmatrix} \tag{9.83}$$

For the more general case, we can formally decompose the wrench as we did for a twist:

$$\begin{aligned} \mathbf{w}_0 &= \begin{bmatrix} \mathbf{f} \\ \mathbf{n}_0 \end{bmatrix} \\ &\equiv f \begin{bmatrix} \mathbf{s} \\ \mathbf{s}_0 \end{bmatrix} \qquad \text{where } \mathbf{s} \cdot \mathbf{s} = 1 \text{ and } f = |\mathbf{f}| \\ &\equiv f \begin{bmatrix} \mathbf{s} \\ \mathbf{s}_0 - h\mathbf{s} \end{bmatrix} + f \begin{bmatrix} \mathbf{0} \\ h\mathbf{s} \end{bmatrix} \end{aligned} \tag{9.84}$$

where h is the wrench given by:

$$h = \frac{\mathbf{n}_0 \cdot \mathbf{f}}{\mathbf{f} \cdot \mathbf{f}} \tag{9.85}$$

The first term on the right in (9.84) represents a pure force, whereas the second term represents a pure torque.

This result is exactly analogous to the result for velocities. In fact, there is a reciprocity between forces and torques and linear and angular velocities.

Velocities		**Statics**	
Twist	$\begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{r} \times \boldsymbol{\omega} + h\boldsymbol{\omega} \end{bmatrix}$	Wrench	$\begin{bmatrix} \mathbf{f} \\ \mathbf{r} \times \mathbf{f} + h\mathbf{f} \end{bmatrix}$
Pitch	$h = \dfrac{\mathbf{v} \cdot \boldsymbol{\omega}}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}$	Pitch	$h = \dfrac{\mathbf{n} \cdot \mathbf{f}}{\mathbf{f} \cdot \mathbf{f}}$
Rotation ($h = 0$)	$\begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{r} \times \boldsymbol{\omega} \end{bmatrix}$	Force ($h = 0$)	$\begin{bmatrix} \mathbf{f} \\ \mathbf{r} \times \mathbf{f} \end{bmatrix}$
Translation ($h = \infty$)	$\begin{bmatrix} \mathbf{0} \\ \mathbf{v} \end{bmatrix}$	Couple ($h = \infty$)	$\begin{bmatrix} \mathbf{0} \\ \mathbf{c} \end{bmatrix}$

Note that the roles of free and sliding vectors are interchanged:

- $\boldsymbol{\omega}$ and $\mathbf{c}$ are circular concepts.

- $\mathbf{v}$ and $\mathbf{f}$ are rectilinear concepts.

The table shows that both twists and wrenches are similar, as they can be expressed as scalar multiples of a screw:

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{bmatrix} &= \omega \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_{01} \end{bmatrix} \quad \text{and} \quad & \begin{bmatrix} \mathbf{f} \\ \mathbf{n} \end{bmatrix} &= f \begin{bmatrix} \mathbf{s}_2 \\ \mathbf{s}_{02} \end{bmatrix} \\ &= \omega \$_1 & &= f \$_2 \end{aligned} \tag{9.86}$$

9.8.1 Reciprocal Screws

Suppose a body is constrained to twist about a screw $\$_{\mathbf{1}}$. A force $\mathbf{f}$ and torque $\mathbf{n}$ act on the body, resulting in a linear $\mathbf{v}$ and angular $\boldsymbol{\omega}$ velocity. The rate of doing work W is:

$$\begin{aligned} \frac{dW}{dt} &= \mathbf{f} \cdot \mathbf{v} + \boldsymbol{\omega} \cdot \mathbf{n} \\ &= f\omega(\mathbf{s}_1 \cdot \mathbf{s}_{02} + \mathbf{s}_2 \cdot \mathbf{s}_{01}) \end{aligned} \tag{9.87}$$

This is the reciprocal product of the two screws $\$_{\mathbf{1}}$ and $\$_{\mathbf{2}}$. Under equilibrium, the rate of doing work is zero:

$$\frac{dW}{dt} = 0$$

For non-zero f and ω, we have from (9.87):

$$\begin{aligned} (\mathbf{s}_1 \cdot \mathbf{s}_{02} + \mathbf{s}_2 \cdot \mathbf{s}_{01}) &= 0 \\ \$_{\mathbf{1}} \circ \$_{\mathbf{2}} &= 0 \end{aligned} \tag{9.88}$$

When the reciprocal product is zero, the two screws are said to be *reciprocal.* A wrench acting along $\$_{\mathbf{2}}$ does no work when applied to a rigid body constrained to twist about screw $\$_{\mathbf{1}}$; this doesn't depend on the magnitude $f\omega$.