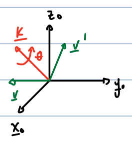
Lecture 10

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \qquad R_y(\theta) = \begin{bmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{bmatrix} \qquad R_z(\theta) = \begin{bmatrix} C_\theta & -S_\theta & 0 \\ S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- 1's on diagonal (zeros on that row/column)
- C_θ always on diagonal
- S_θ on upper right for R_x R_y
  - Bottom left for $R_y(\theta)$

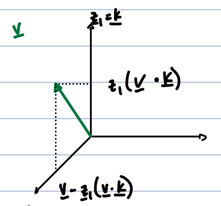
$$R_y\left(\tfrac{\pi}{2}\right) R_z\left(\tfrac{\pi}{2}\right) \neq R_z\left(\tfrac{\pi}{2}\right) R_y\left(\tfrac{\pi}{2}\right)$$

Rotation about arbitrary vector



Rotate $\underline{v}$ about $\underline{k}$ to get $\underline{v}'$
- Construct new coordinate system w/ $\underline{k}$ as axis $\underline{z}_1$

v is a dummy vector for derivation



We want that axis to act like an x-axis

$$\rightsquigarrow \quad \underline{z}_1(\underline{z}_1 \cdot \underline{v}) \cdot (\underline{v} - \underline{z}_1(\underline{z}_1 \cdot \underline{v})) = (\underline{z}_1 \cdot \underline{v})(\underline{z}_1 \cdot \underline{v}) - (\underline{z}_1 \cdot \underline{v})(\underline{z}_1 \cdot \underline{v})(\underline{z}_1 \cdot \underline{v})$$

$= 0$

$\therefore$ orthogonal

$$\rightsquigarrow \quad \underline{x}_1 = \frac{v - z_1(\underline{z}_1 \cdot \underline{v})}{\| \underline{v} - z_1(\underline{z}_1 \cdot \underline{v}) \|}$$

Let $r = \| v - z_1(z_1 \cdot \underline{v}) \|$

$$r\underline{x}_1 = \underline{v} - \underline{z}_1(\underline{z}_1 \cdot \underline{v})$$

$$\underline{y}_1 = \underline{z}_1 \times \underline{x}_1$$

$$\rightsquigarrow \quad {}^0R_1 = \begin{bmatrix} {}^0\underline{x}_1 & {}^0\underline{y}_1 & {}^0\underline{z}_1 \end{bmatrix}$$

$${}^1\underline{v} = {}^0R_1^T \, {}^0\underline{v}$$
$${}^1\underline{v} = {}^1R_0 \, {}^0\underline{v}$$

$$R_z(\theta)\, r\,{}^1\underline{x}_1 = \begin{bmatrix} C_\theta & -S_\theta & 0 \\ S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} rC_\theta \\ rS_\theta \\ 0 \end{bmatrix} = rC_\theta\,{}^1\underline{x}_1 + rS_\theta + {}^1\underline{y}_1$$

$$\longrightarrow {}^1\underline{v}' = {}^1\underline{z}_1({}^1\underline{z}_1 \cdot {}^1\underline{v}) + rC_\theta\,{}^1\underline{x}_1 + rS_\theta\,{}^1\underline{y}_1$$

$$= {}^1\underline{z}_1({}^1\underline{z}_1 \cdot {}^1\underline{v}) + C_\theta({}^1\underline{v} - {}^1\underline{z}_1({}^1\underline{z}_1 \cdot {}^1\underline{v})) + \underbrace{rS_\theta({}^1\underline{z}_1 \times {}^1\underline{x}_1)}_{\parallel}$$

$$S_\theta({}^1\underline{z}_1 \times ({}^1\underline{v} - \cancel{{}^1\underline{z}_1({}^1\underline{z}_1 \cdot {}^1\underline{v})})) \rightarrow 0$$

$$= {}^1\underline{z}_1({}^1\underline{z}_1 \cdot {}^1\underline{v}) + C_\theta({}^1\underline{v} - {}^1\underline{z}({}^1\underline{z}_1 \cdot {}^1\underline{v})) + S_\theta({}^1\underline{z}_1 \times {}^1\underline{v})$$

$${}^1\underline{v}_1 = \underbrace{(1 - C_\theta)}_{\text{VERSINE} = V_\theta}\,{}^1\underline{z}_1({}^1\underline{z} \cdot {}^1\underline{v}) + C_\theta\,{}^1\underline{v} + S_\theta\,{}^1\underline{z}_1 \times {}^1\underline{v}$$

OLD GEOMETRY CRAP HE INSISTS ON USING ̈n

$$= V_\theta\,{}^1\underline{z}_1({}^1\underline{z} \cdot {}^1\underline{v}) + C_\theta\,{}^1\underline{v} + S_\theta\,{}^1\underline{z}_1 \times {}^1\underline{v}$$

$$= C_\theta\,\underset{\sim}{I}\,{}^1\underline{v} + V_\theta({}^1\underline{z}_1({}^1\underline{z}_1)^T)\,{}^1\underline{v}$$

$$=$$

$${}^1\underline{z}_1({}^1\underline{z}_1{}^T\,{}^1\underline{v}) = \underbrace{({}^1\underline{z}_1\,{}^1\underline{z}_1{}^T)}_{3\times3 \text{ CALLED "OUTER PRODUCT"}}\,{}^1\underline{v}$$

$$\text{IF } {}^1\underline{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow {}^1\underline{z}_1({}^1\underline{z}_1)^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

CROSS PRODUCT REVIEW

$\underline{a} \times \underline{b}$

1. FIND det

$$\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \underline{i} & \underline{j} & \underline{k} \end{bmatrix} = a_1b_2\underline{k} + a_2b_3\underline{i} + a_3b_1\underline{j} - a_2b_2\underline{i} - a_1b_3\underline{j} - b_1a_2\underline{k}$$

2. VECTOR FROM $\underline{a} \times \underline{b}$ $= \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$

WE CAN DO THIS W/A MATRIX (SKEW-SYMMETRIC MATRIX)

$$S(\underline{a})(b) \quad \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

↑ SKEW SYMMETRIC

SKEW-SYMMETRIC $A^T = -A$

$S =$ MATRIX OF $\underline{a} \times \underline{b}$

RETURNING TO EARLIER...

$$\underset{\sim}{R}_k(\theta) = C_\theta \underset{\sim}{I} + V_\theta \, {}^1\underline{z}_1 \, {}^1\underline{z}_1^T + S_\theta \, S({}^1\underline{z}_1)$$

$$= C_\theta \underset{\sim}{I} + V_\theta \, {}^0\underline{k} \, {}^0\underline{k}^T + S_\theta \, S({}^0\underline{k})$$

$${}^0\underset{\sim}{R}_1 = \underset{\sim}{R}_k(\theta)$$

BASICALLY ANY ROTATION CAN BE REPRESENTED ABOUT AN ARBITRARY AXIS

i.e. $$\underset{\sim}{R}_x(\theta) = C_\theta \underset{\sim}{I} + V_\theta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + S_\theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_\theta & 0 & 0 \\ 0 & C_\theta & 0 \\ 0 & 0 & C_\theta \end{bmatrix} + \begin{bmatrix} V_\theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -S_\theta \\ 0 & S_\theta & 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_\theta + (1 - C_\theta) & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix}$$

:)

$\underline{a} \times \underline{b}$ $$\underset{\sim}{S}(\underline{a})(\underline{b}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ a_2 & a_1 & 0 \end{bmatrix} \underline{b}$$

$$\underset{\sim}{S}(\underline{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ a_2 & a_1 & 0 \end{bmatrix}$$