Chapter 2 Planar Transformations + Displacements

Pass 1:

- Headings
  - Points/Vectors
    - Affine Spaces
      - Affine Spaces don't have any vectors, they're basically where points live. (You can't add points, only subtract them)
    - Vector Spaces
      - Vectors usually expressed as other vectors
    - Orthonormal bases
      - The most useful bases (orthogonal and unit vector as base) — minimal set of vectors
    - Coordinate systems
    - Locating arbitrary points
  - Planar rotational Transformations
    - Composition of Planar Rotations
    - Planar coordinate Transforms
    - Composition of Coordinate Transformations
  - Two-link Planar manipulator kinematics
  - Homogeneous Transformations in a plane
    - Composition of Homogeneous Transformations
    - Homogeneous Coordinates and the Affine Plane
  - Operators
    - Composition of operators
    - Operation about a different frame
  - Poles of planar displacements
- There's a summary at the end of all these chapters….. Just start with that stupid thing

$\vec{v} \in \mathbb{R}^2$ — belongs to 2D space

$\vec{v} \in \mathbb{R}^3$ — belongs to 3D space

Pass 2:

·Addition of points doesn't make sense, but subtraction of points does

Basis = minimal set of vectors in a vector space

$\underbrace{\vec{x}_j \quad \vec{y}_j}$ most important orthonormal bases (unit length)

Vector coordinates are linear combinations of orthonormal base vectors

Point notation
·Italic capital letter w/subscript

O_i = fixed origin i

Vector notation
${}^{i}\vec{v}_j$ i = coordinate axes/base

Rotation matrix: ${}^{j-1}R_i = R(\theta_i)$

- Describes $\measuredangle\ \theta_j$ from axes $j-1$ to axes j
- Columns are the axes of j expressed in terms of axes $j-1$
- Inverse of rotation matrix is its transpose

Coordinate transformation:

- Relationship between two arbitrary coordinate systems by a displacement between origins + rotation between axes

Pole

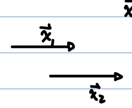
- Point that doesn't move under planar operator
- Unique, unless there's pure translation

- Operators can be related to pole by pre & post multiplying by coordinate transformation to different frame + its inverse

Pass 3 :

- We attach coordinate systems to objects + rotate them
- Unit vectors are equal if they are pointed in the same direction

$\vec{x}_1 = \vec{x}_2$



- We can describe points as $X_2 = \underbrace{O_2 + \vec{x}_2}_{\text{relative to } O_1}$

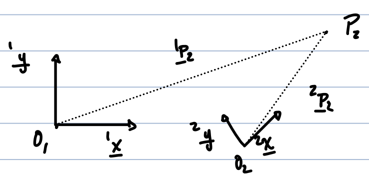
$X_j = O_j + \vec{x}_j$ or.... $X_2{}' - O_1 = \vec{x}_2$ ← vector

Arbitrary points

Point: P_2

$^2P_2 = {}^2p_{21}\,{}^2x_2 + {}^2p_{22}\,{}^2y_2$] Point P_2 vector wrt basis {2}

$^1P_2 = {}^1p_{21}\,{}^1x_1 + {}^1p_{22}\,{}^1y_1$] " wrt basis {1}



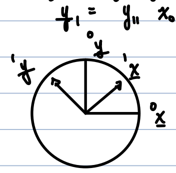
Planar Rotation Transformation

Two coordinate systems: 0 , 1
- Origins coincide
- Axes 1 rotated by θ_1 from axes 0

Axes 1 wrt axes 0

$$^0x_1 = {}^0x_{11}\,{}^0x_0 + {}^0x_{12}\,{}^0y_0$$

$$^0y_1 = {}^0y_{11}\,{}^0x_0 + {}^0y_{12}\,{}^0y_0$$



Relationships:

$$^0x_1 = {}^0x_0 \cos\theta_1 + {}^0y_0 \sin\theta_1 = \begin{bmatrix} ^0x_0 & ^0y_0 \end{bmatrix} \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix}$$

$$^0y_1 = -{}^0x_0 \sin\theta_1 + {}^0y_0 \cos\theta_1 = \begin{bmatrix} ^0x_0 & ^0y_0 \end{bmatrix} \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix} = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix}$$

Columns are coordinate axes {1} wrt axes {0}

$$\Rightarrow \begin{bmatrix} ^0x_1 & ^0y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \doteq \underset{\sim}{R}(\theta_1)$$

↑ Planar rotation matrix :)

$$\left[\,{}^{1}\underline{x}_0 \quad {}^{1}\underline{y}_1\,\right] = \underset{\sim}{R}(\theta_1)^T$$

$$ {}^{1}\underline{v} = \underline{R}(\theta_1)^T \; {}^{0}\underline{v}$$

COMPOSITION OF ROTATION MATRICIES:

$$ {}^{i-1}\underset{\sim}{R} = \underset{\sim}{R}(\theta_i)$$

$$ {}^{i}\underset{\sim}{R}_j = \underset{\sim}{R}(\theta_{ij})$$

ROTATION MATRIX FROM AXES j TO i

NOT TRUE IN 3D

$$\underset{\sim}{R}(\theta_1)\,\underset{\sim}{R}(\theta_2) = \underset{\sim}{R}(\theta_1 + \theta_2)$$

ALWAYS TRUE:

$$ {}^{i}\underline{v} = {}^{i}\underset{\sim}{R}_j \, {}^{j}\underline{v}$$

$$\left({}^{i}\underline{R}_j\right)^{-1} = {}^{i}\underset{\sim}{R}_j^{\,T} = {}^{j}\underline{R}_i$$

PLANAR COORDINATE TRANSFORMATIONS

ORIGINS

$\underline{d}_{ij}$ = VECTOR FROM O_i TO O_j

$\underline{P}_0$ IS A POINT WRT O_0, THEREFORE:

$$ {}^{0}\underline{P}_0 = {}^{0}\underline{d}_{01} + {}^{0}\underline{P}_1$$

AND $\quad {}^{0}\underline{P}_1 = {}^{0}\underset{\sim}{R}_1 \, {}^{1}\underline{P}_1$

$$\Rightarrow {}^{0}\underline{P}_0 = {}^{0}\underline{d}_{01} + {}^{0}\underset{\sim}{R}_1 \, {}^{1}\underline{P}_1$$

$$= {}^{0}\underline{d}_{01} + {}^{0}\underline{d}_{12} + {}^{0}P_2$$

TWO-LINK PLANAR MANIPULATOR KINEMATICS

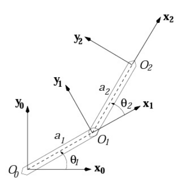


Figure 2.15: Two-link planar manipulator.

$\{0\}$ = GND

$\{1\}$ = DISTAL END LINK 1 $\quad O_0$ TO O_1

$\{2\}$ = " " " 2 $\quad O_1$ TO O_2

FORWARD KINEMATICS

- FIND ENDPOINT POSITION ${}^{0}\underline{d}_{02}$ $\quad$ (O_2 WRT O_0)

$$\rightsquigarrow {}^{0}d_{02} = {}^{0}d_{01} + {}^{0}d_{12} = a_1 \, {}^{0}\underset{\sim}{R}_1 \, {}^{1}\underline{x}_1 + a_2 \, {}^{0}R_2 \, {}^{2}\underline{x}_2$$

UNIT VECTORS

Homogeneous Transformation in a Plane

$$\underset{\sim}{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}^T \rightsquigarrow P = \begin{bmatrix} \underset{\sim}{p} \\ 1 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix}$$

↑ point P

Coordinate transform

$$\begin{bmatrix} {}^0\underset{\sim}{P}_1 \\ 1 \end{bmatrix} = \begin{bmatrix} {}^0\underset{\sim}{R}_1 \, {}^1\underset{\sim}{P}_1 + {}^0\underline{d}_{01} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^0\underset{\sim}{R}_1 & {}^0\underline{d}_{01} \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} {}^1\underset{\sim}{P}_0 \\ 1 \end{bmatrix}$$

$\underline{0}^T = [0 \quad 0]$

$$\rightsquigarrow {}^0P = {}^0\underset{\sim}{T}_1 \, {}^1P$$

${}^0\underset{\sim}{T} = 3 \times 3$ homogeneous transformation

Composition of Homogeneous Transformations

$${}^0\underset{\sim}{T}_2 = {}^0\underset{\sim}{T}_1 \, {}^1\underset{\sim}{T}_2$$

↑

transform from $\{1\}$ to $\{0\}$

$$\left({}^0\underset{\sim}{T}_1\right)^{-1} = \begin{bmatrix} {}^0\underset{\sim}{R}_1 & -{}^0\underset{\sim}{R}_1^T \, {}^0d_{01} \\ \underset{\sim}{0}^T & 1 \end{bmatrix} = {}^1\underset{\sim}{T}_0$$

Homogeneous means same coordinate space

Operators

- Coordinate transforms are static representations of relative location within a non-static system
  - Operator = Displacement = A process of moving a point wrt to a coordinate axis

Operations are in the <u>fixed frame</u>

Coordinate transforms are wrt current frame

Operators:

$\text{Trans}(\underline{v})$ ← translates point P_1

$$P_2 = \text{Trans}({}^0\underline{d}_1) \, P_1$$

$${}^0\underset{\sim}{P}_2 = {}^0\underset{\sim}{P}_1 + {}^0\underline{d}$$

↑ P_1 wrt origin

$\text{Rot}(\theta_1)$ = rotates P_1 by θ_1 to P_2

General transformation operator

$$P_2 = \begin{bmatrix} \underset{\sim}{R}(\theta_1) & {}^0\underline{d} \\ \underline{0}^T & 1 \end{bmatrix} P_1 = {}^0\underline{T}_1 P_1$$

POLES OF PLANAR DISPLACEMENTS

- THERE ARE POINTS WHERE THE POINT <u>DOESN'T</u> MOVE

$^1\underline{c}$ = POLE VECTOR $\quad$ $\underset{\sim}{R}(\theta)$ = ARBITRARY ROTATION

C = POLE POINT $\quad$ $^1\underline{d}$ = TRANSLATION

$$^1\underline{c} = {}^1\underset{\sim}{R}{}^1 c + {}^1\underline{d} \Rightarrow {}^1\underline{c} = (\underset{\sim}{I} - \underset{\sim}{R})^{-1}\, {}^1\underline{d}$$

PURE TRANSLATION HAS $\underset{\sim}{R} = \underset{\sim}{I}$

$\underset{\sim}{D}_1$