Chapter 3

Spatial Transformations and Displacements

Spatial transformations have the same form as the 2D case, except that vector and matrix dimensions are increased by 1. The extension of linear transformations and displacements from 2D to 3D is straightforward, but there are significant new issues and difficulties in 3D rotations as compared to 2D rotations. The difficulties hinge on the mathematical fact that 3D rotations do not commute, whereas 3D translations do. The new issues concern alternative ways of parameterizing a rotation matrix, including the angle-axis, Rodrigues vector, Euler parameter, and quaternion representations. Finally, it is shown that there is an invariant line corresponding to an arbitrary spatial displacement, called the screw of a spatial displacement with an associated pitch.

3.1 Spatial Affine and Vector Spaces

Points P_i are now spatial points, and vectors $\mathbf{v} = P_i - P_j$ are now the displacements between spatial points. The vector space is now three-dimensional, so we write $\mathbf{v} \in \mathcal{R}^3$. Hence there are three vectors in any basis for $\mathcal{R}^3$. Again, the most useful are the orthonormal bases, designated as $\mathbf{x}_j, \mathbf{y}_j, \mathbf{z}_j$. There are three coordinates for any vector expressed with respect to a particular basis j (Figure 3.1(A)):

$$ {}^j\mathbf{v} = {}^jv_1\,{}^j\mathbf{x}_j + {}^jv_2\,{}^j\mathbf{y}_j + {}^jv_3\,{}^j\mathbf{z}_j = \begin{bmatrix} {}^jv_1 \\ {}^jv_2 \\ {}^jv_3 \end{bmatrix} \tag{3.1} $$

where the coordinates of any orthonormal basis with respect to itself are:

$$ {}^j\mathbf{x}_j = \mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad {}^j\mathbf{y}_j = \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad {}^j\mathbf{z}_j = \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{3.2} $$

and

$$ \begin{bmatrix} {}^j\mathbf{x}_j & {}^j\mathbf{y}_j & {}^j\mathbf{z}_j \end{bmatrix} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{bmatrix} = \mathbf{I}, \tag{3.3} $$

the 3-by-3 identity matrix. Hence the coordinates of ${}^j\mathbf{v}$ in (3.1) emerge from the mathematical manipulations:

$$ {}^j\mathbf{v} = {}^jv_1\mathbf{i} + {}^jv_2\mathbf{j} + {}^jv_3\mathbf{k} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{bmatrix} \begin{bmatrix} {}^jv_1 \\ {}^jv_2 \\ {}^jv_3 \end{bmatrix} = \mathbf{I} \begin{bmatrix} {}^jv_1 \\ {}^jv_2 \\ {}^jv_3 \end{bmatrix} = \begin{bmatrix} {}^jv_1 \\ {}^jv_2 \\ {}^jv_3 \end{bmatrix} \tag{3.4} $$

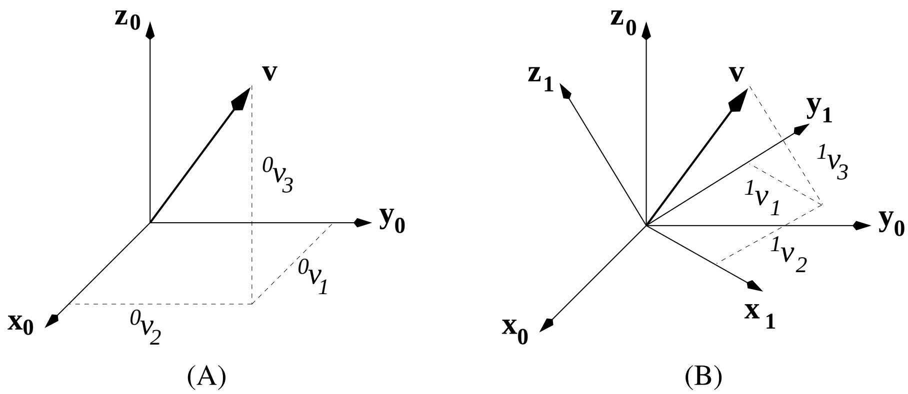

Figure 3.1: Representation of $\mathbf{v}$ with respect to (A) coordinate axes 0 and (B) coordinate axes 1.

Figure 3.1 shows the coordinates of $\mathbf{v}$ with respect to orthonormal bases 0 and 1.

The combination of an origin O_j with a set of axes $\mathbf{x}_j, \mathbf{y}_j, \mathbf{z}_j$ is called *coordinate system j* or *frame j*. The axes follow the right-hand rule: $\mathbf{x}_j \times \mathbf{y}_j = \mathbf{z}_j$. The *base coordinates* are the axes which serve as the reference. For reasons which will be described later, the base coordinates are typically numbered as 0, and the numbering increases from there. Because we work so often with the base coordinates 0, by convention we will sometimes drop the superscript for vectors expressed with respect to coordinate axes 0. At other times, if no superscript is indicated, this means we don't care what coordinate axes a vector is represented with respect to. This happens when we wish to make formal arguments about the relations between vectors, which are true for any coordinate axes (such as the vector cross product, discussed later).

3.2 Spatial Rotational Transformations

Let's consider how to relate two different coordinate axes, say axes 1 relative to axes 0 (Figure 3.2(A)):

$$\begin{aligned} {}^0\mathbf{x}_1 &= {}^0x_{11}\,{}^0\mathbf{x}_0 + {}^0x_{12}\,{}^0\mathbf{y}_0 + {}^0x_{13}\,{}^0\mathbf{z}_0 \\ {}^0\mathbf{y}_1 &= {}^0y_{11}\,{}^0\mathbf{x}_0 + {}^0y_{12}\,{}^0\mathbf{y}_0 + {}^0y_{13}\,{}^0\mathbf{z}_0 \\ {}^0\mathbf{z}_1 &= {}^0z_{11}\,{}^0\mathbf{x}_0 + {}^0z_{12}\,{}^0\mathbf{y}_0 + {}^0z_{13}\,{}^0\mathbf{z}_0 \end{aligned} \tag{3.5}$$

That is to say, the coordinates of axis ${}^0\mathbf{x}_1$ are $[{}^0x_{11}\ {}^0x_{12}\ {}^0x_{13}]^T$ with respect to axes 0, etc. Rewrite (3.5) as

$$\begin{aligned} \begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 & {}^0\mathbf{z}_1 \end{bmatrix} &= \begin{bmatrix} {}^0\mathbf{x}_0 & {}^0\mathbf{y}_0 & {}^0\mathbf{z}_0 \end{bmatrix} \begin{bmatrix} {}^0x_{11} & {}^0y_{11} & {}^0z_{11} \\ {}^0x_{12} & {}^0y_{12} & {}^0z_{12} \\ {}^0x_{13} & {}^0y_{13} & {}^0z_{13} \end{bmatrix} \\ &= \begin{bmatrix} {}^0x_{11} & {}^0y_{11} & {}^0z_{11} \\ {}^0x_{12} & {}^0y_{12} & {}^0z_{12} \\ {}^0x_{13} & {}^0y_{13} & {}^0z_{13} \end{bmatrix} \qquad (3.6) \\ &\doteq {}^0\mathbf{R}_1 \end{aligned}$$

where ${}^0\mathbf{R}_1$ is the *3D rotation matrix* from coordinate axes 1 to 0. As for the planar case, the columns of the rotation matrix are the axes 1 in terms of axes 0. We call this the *coordinate axis form* of a rotation matrix.

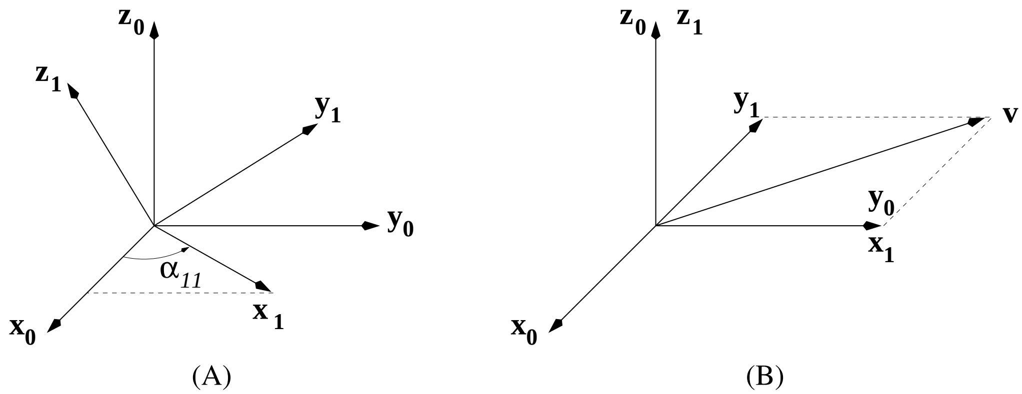

Figure 3.2: (A) Relation of axes 1 to axes 0. α_{11} is the direction cosine between $\mathbf{x}_0$ and $\mathbf{x}_1$. (B) Example where $\mathbf{x}_1 = \mathbf{y}_0$, $\mathbf{y}_1 = -\mathbf{x}_0$, and $\mathbf{z}_1 = \mathbf{z}_0$.

- ${}^i\mathbf{R}_j$ will denote the rotation matrix from coordinate axes j to i.

Because the axes make up the columns of ${}^0\mathbf{R}_1$, it is clear that rotation matrices are orthogonal: their columns are mutually perpendicular and have unit length.

Example 3.1: In Figure 3.2(B), axes 1 have a simple relation to axes 0. Therefore the rotation matrix can simply be written by inspection:

$$ {}^0\mathbf{R}_1 = \begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 & {}^0\mathbf{z}_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Suppose a particular vector $\mathbf{v}$ is expressed with respect to coordinate axes 1 (Figure 3.1):

$$ {}^1\mathbf{v} = {}^1v_1\,{}^1\mathbf{x}_1 + {}^1v_2\,{}^1\mathbf{y}_1 + {}^1v_3\,{}^1\mathbf{z}_1 = \begin{bmatrix} {}^1v_1 \\ {}^1v_2 \\ {}^1v_3 \end{bmatrix} \tag{3.7} $$

What is the relation between ${}^1\mathbf{v}$ and ${}^0\mathbf{v}$? Refer (3.7) to axes 0, rearrange, and substitute (3.7):

$$ \begin{aligned} {}^0\mathbf{v} &= {}^1v_1\,{}^0\mathbf{x}_1 + {}^1v_2\,{}^0\mathbf{y}_1 + {}^1v_3\,{}^0\mathbf{z}_1 \\ &= \begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 & {}^0\mathbf{z}_1 \end{bmatrix} \begin{bmatrix} {}^1v_1 \\ {}^1v_2 \\ {}^1v_3 \end{bmatrix} \\ &= {}^0\mathbf{R}_1\,{}^1\mathbf{v} \end{aligned} \tag{3.8} $$

The spatial rotational transformation ${}^0\mathbf{R}_1$ converts a representation of a vector ${}^1\mathbf{v}$ in axes 1 to a representation ${}^0\mathbf{v} = {}^0\mathbf{R}_1\,{}^1\mathbf{v}$ in axes 0.

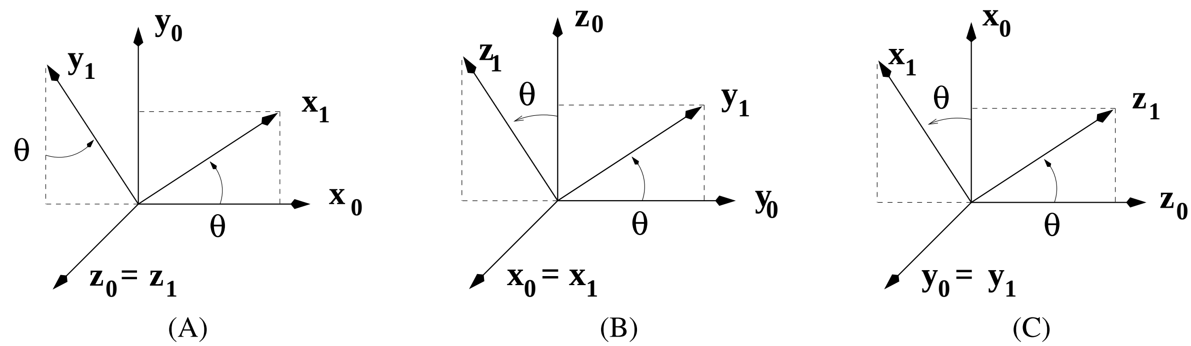

Figure 3.3: Rotation by θ about the (A) $\mathbf{z}$, (B) $\mathbf{x}$, and (C) $\mathbf{y}$ axes.

Example 3.2: Let ${}^1\mathbf{v} = [1\ 1\ 0]^T$ in Figure 3.2(B). Then

$$ {}^0\mathbf{v} = {}^0\mathbf{R}_1\, {}^1\mathbf{v} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} $$

3.2.1 Rotation about Principal Axes

Three elemental rotation matrices result from rotation about the $\mathbf{x}$, $\mathbf{y}$, or $\mathbf{z}$ axes. Rotation about the $\mathbf{z}$ axis is shown in Figure 3.3(A), where $\mathbf{z}_1 = \mathbf{z}_0$ and axis $\mathbf{x}_1$ makes an angle θ with respect to $\mathbf{x}_0$. Then

$$ \begin{aligned} \mathbf{x}_1 &= \mathbf{x}_0 \cos\theta + \mathbf{y}_0 \sin\theta \\ \mathbf{y}_1 &= -\mathbf{x}_0 \sin\theta + \mathbf{y}_0 \cos\theta \\ \mathbf{z}_1 &= \mathbf{z}_0 \end{aligned} \tag{3.9} $$

and therefore from (3.13)

$$ {}^0\mathbf{R}_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \doteq \mathbf{R}_z(\theta) \tag{3.10} $$

This has the same form as a planar rotation matrix, with an extra row and column. In fact, a planar rotation can be considered as a 3D rotation about the $\mathbf{z}$ axis. In the planar case, the rotation axis was implicit, but here it is explicit. For elemental rotation about the $\mathbf{z}$ axis, the rotation matrix will henceforth be designated as $\mathbf{R}_z(\theta)$.

Similar expressions derive for $\mathbf{R}_x(\theta)$ and $\mathbf{R}_y(\theta)$, (Figures 3.3(B)-(C)) and are left as an exercise.

$$ \mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} \tag{3.11} $$

$$ \mathbf{R}_y(\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \tag{3.12} $$

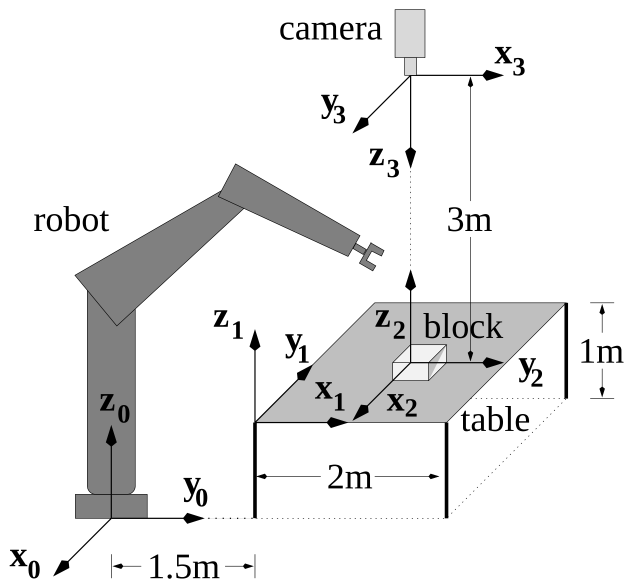

Figure 3.4: Relative locations of a robot, table, block, and camera.

Example 3.3: Frame 0 is located at the base of a robot (Figure 3.4). Frame 1 is located at the corner of a 2m-square table, which is 1m off the ground and displaced by 1.5m along $\mathbf{y}_0$. It is rotated by $\pi/2$ about the $\mathbf{z}$ axis relative to frame 0. Hence

$$^0\mathbf{R}_1 = \mathbf{R}_z(\pi/2) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is the same result as the first example. Frame 2 is at the corner of a block, situated in the middle of the table, and rotated by $-\pi/2$ about the $\mathbf{z}$ axis relative to frame 1. Hence

$$^1\mathbf{R}_2 = \mathbf{R}_z(-\pi/2) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Frame 3 locates a camera, which sits 3m directly above the block; its orientation relative to frame 2 cannot be described by a single elemental rotation. However, the rotation matrix $^2\mathbf{R}_3$ can be simply written by inspection using the coordinate axis form (3.7), since $\mathbf{x}_3 = \mathbf{y}_2$, $\mathbf{y}_3 = \mathbf{x}_2$, and $\mathbf{z}_3 = -\mathbf{z}_2$:

$$^2\mathbf{R}_3 = \begin{bmatrix} {}^2\mathbf{x}_3 & {}^2\mathbf{y}_3 & {}^2\mathbf{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Similarly, $^0\mathbf{R}_1$ and $^0\mathbf{R}_2$ could be written by inspection without solving (3.10).

3.2.2 Direction Cosine Form

The rotation matrix was shown in (3.7) to have as its columns the coordinate axes 1 expressed with respect to axes 0, called the coordinate axis form. Another view about what a rotation matrix is, is obtained by taking the dot product of each equation in (3.5) with each of ${}^0\mathbf{x}_0, {}^0\mathbf{y}_0, {}^0\mathbf{z}_0$. For example,

$${}^0\mathbf{x}_1 \cdot {}^0\mathbf{x}_0 = {}^0\mathbf{x}_1{}^T \; {}^0\mathbf{x}_0 = {}^0x_{11}$$

since ${}^0\mathbf{x}_0 \cdot {}^0\mathbf{y}_0 = 0$ and ${}^0\mathbf{x}_0 \cdot {}^0\mathbf{z}_0 = 0$ because of orthogonality. The result is that

$${}^0\mathbf{R}_1 \doteq \begin{bmatrix} {}^0x_{11} & {}^0y_{11} & {}^0z_{11} \\ {}^0x_{12} & {}^0y_{12} & {}^0z_{12} \\ {}^0x_{13} & {}^0y_{13} & {}^0z_{13} \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{x}_1^T\,{}^0\mathbf{x}_0 & {}^0\mathbf{y}_1^T\,{}^0\mathbf{x}_0 & {}^0\mathbf{z}_1^T\,{}^0\mathbf{x}_0 \\ {}^0\mathbf{x}_1^T\,{}^0\mathbf{y}_0 & {}^0\mathbf{y}_1^T\,{}^0\mathbf{y}_0 & {}^0\mathbf{z}_1^T\,{}^0\mathbf{y}_0 \\ {}^0\mathbf{x}_1^T\,{}^0\mathbf{z}_0 & {}^0\mathbf{y}_1^T\,{}^0\mathbf{z}_0 & {}^0\mathbf{z}_1^T\,{}^0\mathbf{z}_0 \end{bmatrix} \tag{3.13}$$

Suppose α_{11} is the angle from $\mathbf{x}_0$ to $\mathbf{x}_1$ (Figure 3.2). Then

$${}^0\mathbf{x}_1 \cdot {}^0\mathbf{x}_0 = ||{}^0\mathbf{x}_1||\;||{}^0\mathbf{x}_0|| \cos\alpha_{11} = \cos\alpha_{11} = {}^0x_{11}$$

since the axes all have unit length. Hence ${}^0x_{11}$ is the *direction cosine* between the two $\mathbf{x}$ axes. For example, $\alpha_{11} = \theta_1$ in Figure 3.3.

Similarly, every other dot product and associated coordinate in (3.13) represents a direction cosine between corresponding axes. Therefore (3.13) is the direction cosine form of a rotation matrix, and represents another important view of what a rotation matrix is. The relation to the coordinate axes form is that the direction cosines are the components of coordinate axes 1. The two forms represent the same rotation matrix, it's the interpretation that differs.

3.2.3 Inverse of a Rotation Matrix

Similar to (3.5), express axes 0 relative to axes 1:

$$\begin{aligned} {}^1\mathbf{x}_0 &= {}^1x_{01}\,{}^1\mathbf{x}_1 + {}^1x_{02}\,{}^1\mathbf{y}_1 + {}^1x_{03}\,{}^1\mathbf{z}_1 \\ {}^1\mathbf{y}_0 &= {}^1y_{01}\,{}^1\mathbf{x}_1 + {}^1y_{02}\,{}^1\mathbf{y}_1 + {}^1y_{03}\,{}^1\mathbf{z}_1 \\ {}^1\mathbf{z}_0 &= {}^1z_{01}\,{}^1\mathbf{x}_1 + {}^1z_{02}\,{}^1\mathbf{y}_1 + {}^1z_{03}\,{}^1\mathbf{z}_1 \end{aligned} \tag{3.14}$$

and write the direction cosine form of the rotational transformation ${}^1\mathbf{R}_0$ similar to (3.13)::

$${}^1\mathbf{R}_0 \doteq \begin{bmatrix} {}^1x_{01} & {}^1y_{01} & {}^1z_{01} \\ {}^1x_{02} & {}^1y_{02} & {}^1z_{02} \\ {}^1x_{03} & {}^1y_{03} & {}^1z_{03} \end{bmatrix} = \begin{bmatrix} {}^1\mathbf{x}_0^T\,{}^1\mathbf{x}_1 & {}^1\mathbf{y}_0^T\,{}^1\mathbf{x}_1 & {}^1\mathbf{z}_0^T\,{}^1\mathbf{x}_1 \\ {}^1\mathbf{x}_0^T\,{}^1\mathbf{y}_1 & {}^1\mathbf{y}_0^T\,{}^1\mathbf{y}_1 & {}^1\mathbf{z}_0^T\,{}^1\mathbf{y}_1 \\ {}^1\mathbf{x}_0^T\,{}^1\mathbf{z}_1 & {}^1\mathbf{y}_0^T\,{}^1\mathbf{z}_1 & {}^1\mathbf{z}_0^T\,{}^1\mathbf{z}_1 \end{bmatrix} \tag{3.15}$$

Compare the $1,1$ elements of (3.15) and (3.13). Since the value of the dot product does not depend on the basis, then ${}^1\mathbf{x}_0^T\,{}^1\mathbf{x}_1 = {}^0\mathbf{x}_1^T\,{}^0\mathbf{x}_0$. Similar comparisons with the other elements shows that

$${}^1\mathbf{R}_0 = {}^0\mathbf{R}_1^T \tag{3.16}$$

Apply ${}^1\mathbf{R}_0$, which converts from a representation in axes 0 to axes 1, to (3.8):

$${}^1\mathbf{v} = {}^1\mathbf{R}_0\,{}^0\mathbf{v} = {}^1\mathbf{R}_0\,{}^0\mathbf{R}_1\,{}^1\mathbf{v} \tag{3.17}$$

This can only be true for an arbitrary vector $\mathbf{v}$ if

$${}^1\mathbf{R}_0\,{}^0\mathbf{R}_1 = \mathbf{I}, \tag{3.18}$$

the 3-by-3 identity matrix. In view of (3.16), therefore

$${}^1\mathbf{R}_0 = {}^0\mathbf{R}_1{}^{-1} = {}^0\mathbf{R}_1^T \tag{3.19}$$

This is an important result: *the inverse of a rotation matrix is its transpose.*

3.2.4 Composition of Spatial Rotations

Spatial rotations compose as in the 2D case, with appropriate change of dimensions. Consider three sets of coordinate axes 0, 1, and 2 that have the same origin, and suppose $\mathbf{p}$ locates a point relative to the common origin. Then

$$
\begin{aligned}
{}^1\mathbf{p} &= {}^1\mathbf{R}_2\,{}^2\mathbf{p} \\
{}^0\mathbf{p} &= {}^0\mathbf{R}_1\,{}^1\mathbf{p} = {}^0\mathbf{R}_1\,{}^1\mathbf{R}_2\,{}^2\mathbf{p} = {}^0\mathbf{R}_2\,{}^2\mathbf{p}
\end{aligned}
$$

where ${}^0\mathbf{R}_2 = {}^0\mathbf{R}_1\,{}^1\mathbf{R}_2$.

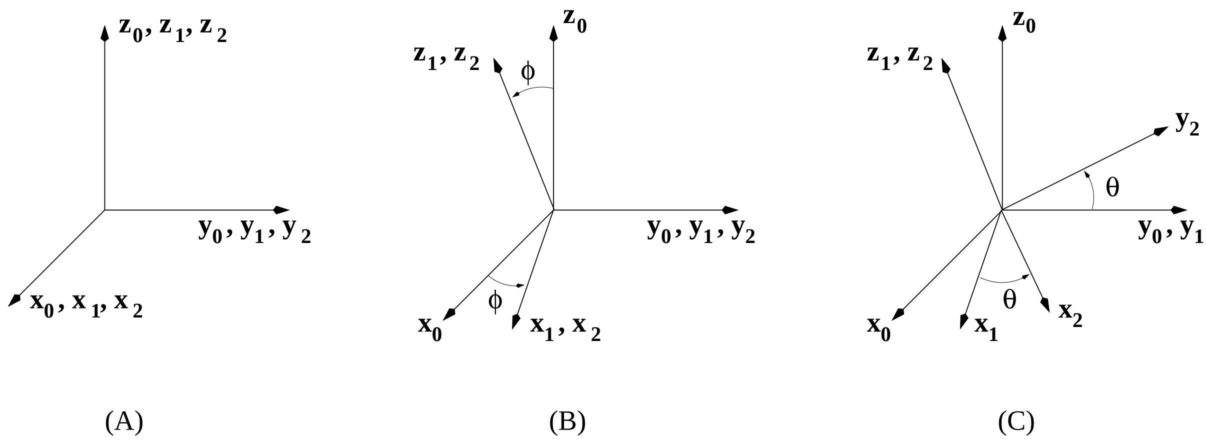

Figure 3.5: (A) Initial overlapping axes 1, 2, and 3. (B) A rotation of ϕ of axes 1 and 2 about the $\mathbf{y}_0$ axis. (C) A rotation of θ of axes 2 about the $\mathbf{z}_1$ axis.

As an example, consider the composition of the elementary rotations $\mathbf{R} = \mathbf{R}_y(\phi)\mathbf{R}_z(\theta)$. As coordinate transformations, we may reinterpret these rotations from left to right.

1. Assume initially all three coordinate frames 0, 1, and 2 coincide (Figure 3.5(A)).
2. Rotate frames 1 and 2 about $\mathbf{y}_0$ by ϕ (Figure 3.5(B)).
3. Rotate frame 2 about $\mathbf{z}_1$ by θ (Figure 3.5(C)).

In each case, the frame relative to which rotation occurs is the *current frame.*

- *Note that 3D rotations do not commute.* Therefore

$$\mathbf{R}_y(\phi)\mathbf{R}_z(\theta) \neq \mathbf{R}_z(\theta)\mathbf{R}_y(\phi)$$

This is unlike the 2D case, where the implied rotation axis is always the same (e.g., the $\mathbf{z}$-axis for the $\mathbf{x}, \mathbf{y}$ plane).

3.3 Spatial Coordinate Transformations

Spatial coordinate transformations are like the 2D case, after appropriate change of dimensions (Figure 3.6):

$$ {}^0\mathbf{p}_0 = {}^0\mathbf{d}_{01} + {}^0\mathbf{R}_1\,{}^1\mathbf{p}_1 \tag{3.20}$$

where all vectors have 3 components and ${}^0\mathbf{R}_1$ is a spatial rotation matrix. The composition of spatial transformations also has the same form as for planar transformations:

$$\begin{aligned} {}^0\mathbf{p}_0 &= {}^0\mathbf{d}_{01} + {}^0\mathbf{R}_1\,{}^1\mathbf{p}_1 \\ &= {}^0\mathbf{d}_{01} + {}^0\mathbf{R}_1\,{}^1\mathbf{d}_{12} + {}^0\mathbf{R}_1\,{}^1\mathbf{R}_2\,{}^2\mathbf{p}_2 \\ &= {}^0\mathbf{d}_{01} + {}^0\mathbf{d}_{12} + {}^0\mathbf{p}_2 \end{aligned} \tag{3.21}$$

The homogeneous transformation from frame 1 to frame 0

$${}^0\mathbf{T}_1 = \begin{bmatrix} {}^0\mathbf{R}_1 & {}^0\mathbf{d}_{01} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{3.22}$$

is comprised of a spatial rotation ${}^0\mathbf{R}_1$ and spatial displacement ${}^0\mathbf{d}_{01}$, and $\mathbf{0}^T = [0\ 0\ 0]$. Again, a homogeneous coordinate is indicated by an upper case italic letter, e.g.,

$$P = \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} \tag{3.23}$$

Then

$${}^0P = {}^0\mathbf{T}_1\,{}^1P \tag{3.24}$$

Let ${}^0\mathbf{T}_1$ represent the homogeneous transformation from frame 1 to frame 0 and ${}^1\mathbf{T}_2$ from frame 2 to frame 1. Then the homogeneous transformation from frame 2 to 0 is ${}^0\mathbf{T}_2 = {}^0\mathbf{T}_1\,{}^1\mathbf{T}_2$. Applied to homogeneous coordinates,

$${}^0P = {}^0\mathbf{T}_2\,{}^2P = {}^0\mathbf{T}_1\,{}^1P \tag{3.25}$$

where ${}^1P = {}^1\mathbf{T}_2\,{}^2P$.

Given a coordinate transformation from 1 to 0, we can reverse the transformation to go from 0 to 1. Given ${}^0\mathbf{p}_0 = {}^0\mathbf{R}_1\,{}^1\mathbf{p}_1 + {}^0\mathbf{d}_{01}$, by rearrangement we find:

$${}^1\mathbf{p}_1 = {}^0\mathbf{R}_1{}^T({}^0\mathbf{p}_0 - {}^0\mathbf{d}_{01}) \tag{3.26}$$

where ${}^0\mathbf{R}_1{}^T = {}^0\mathbf{R}_1{}^{-1}$. The inverse coordinate transformation is again easier to express using homogeneous transformations.

$${}^1P = {}^0\mathbf{T}_1{}^{-1}\,{}^0P \tag{3.27}$$

$${}^0\mathbf{T}_1{}^{-1} = \begin{bmatrix} {}^0\mathbf{R}_1{}^T & -{}^0\mathbf{R}_1{}^T\,{}^0\mathbf{d}_{01} \\ \mathbf{0}^T & 1 \end{bmatrix} = {}^1\mathbf{T}_0 \tag{3.28}$$

The inverse of the composition of transformations may be simply written:

$${}^2P = ({}^0\mathbf{T}_1\,{}^1\mathbf{T}_2)^{-1}\,{}^0P = {}^1\mathbf{T}_2{}^{-1}\,{}^0\mathbf{T}_1{}^{-1}\,{}^0P = {}^2\mathbf{T}_0\,{}^0P \tag{3.29}$$

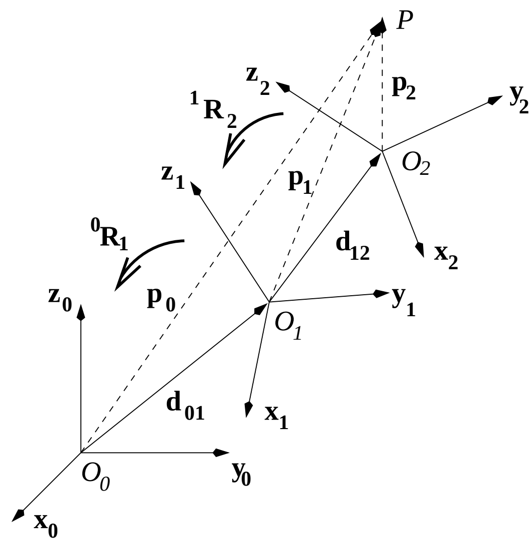

Figure 3.6: Spatial coordinate transformations between frames 0, 1, and 2.

The following elementary homogeneous transformations express pure translation and pure rotation.

$$\mathbf{Trans}(\mathbf{d}) = \begin{bmatrix} \mathbf{I} & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{3.30}$$

$$\mathbf{Rot}(\mathbf{R}) = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{3.31}$$

The columns of ${}^0\mathbf{T}_1$ can be simply interpreted as the axes and displacement of frame 1 expressed in frame 0:

$${}^0\mathbf{T}_1 = \begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 & {}^0\mathbf{z}_1 & {}^0\mathbf{d}_{01} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.32}$$

Therefore ${}^0\mathbf{T}_1$ is often itself called frame 1.

Example 3.4: Returning to the example of Figure 3.4, the homogeneous transformations between neighboring frames are:

$${}^2\mathbf{T}_3 = \begin{bmatrix} {}^2\mathbf{x}_3 & {}^2\mathbf{y}_3 & {}^2\mathbf{z}_3 & {}^2\mathbf{d}_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1\mathbf{T}_2 = \begin{bmatrix} {}^1\mathbf{x}_2 & {}^1\mathbf{y}_2 & {}^1\mathbf{z}_2 & {}^1\mathbf{d}_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
{}^0\mathbf{T}_1 = \begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 & {}^0\mathbf{z}_1 & {}^0\mathbf{d}_{01} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The relationships of frames 2 and 3 to frame 0 are:

$$
{}^0\mathbf{T}_2 = \begin{bmatrix} {}^0\mathbf{x}_2 & {}^0\mathbf{y}_2 & {}^0\mathbf{z}_2 & {}^0\mathbf{d}_{02} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0\mathbf{T}_3 = \begin{bmatrix} {}^0\mathbf{x}_3 & {}^0\mathbf{y}_3 & {}^0\mathbf{z}_3 & {}^0\mathbf{d}_{03} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 2.5 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

In this case, the relative coordinate axes and origin displacements can be found from inspection, rather than multiplying out homogeneous transformations.

3.3.1 Spatial Operators

Spatial translation, rotation, and transformation operators work exactly the same as planar operators.

3.4 Representations of Orientation

Although a rotation matrix $\mathbf{R} = \{r_{ij}\}$ is a unique representation of orientation, with 9 numbers it is a redundant representation, because elements are related. Consider the columns $\mathbf{v}_i$ of $\mathbf{R}$:

$$
\mathbf{v}_i = \begin{bmatrix} r_{1i} \\ r_{2i} \\ r_{3i} \end{bmatrix} \qquad \text{for } i = 1, 2, 3 \tag{3.33}
$$

Then because the rows (and columns) of a rotation matrix are orthogonal,

$$
\mathbf{v}_i^T \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
$$

This comprises 6 equations, hence 9-6=3 elements are independent.

Euler angles and the *Rodrigues vector* are two examples of 3-number orientation representations. Yet 3-number representations have distinctive drawbacks, and we will examine the 4-number representation called *quaternions,* which have become a standard way of orientation representation. Quaternions are based on the representation of any rotation matrix as an equivalent rotation about an associated vector.

3.5 Angle-Axis Formula

We begin by deriving the rotation matrix that results from rotation about an arbitrary vector. Previously, we examined elementary rotations about coordinate axes. It is possible to rotate about any arbitrary axis $\mathbf{k}$ (a

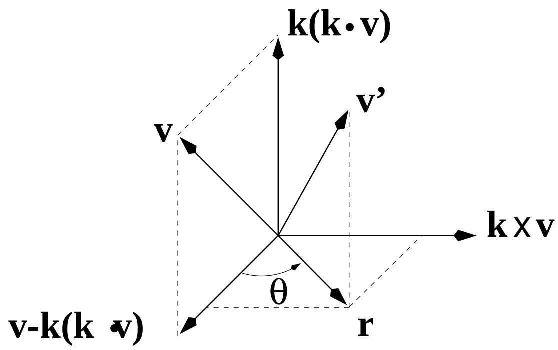

Figure 3.7: Vector $\mathbf{v}$ is rotated by θ about $\mathbf{k}$ to yield $\mathbf{v}'$.

unit vector) by θ. An operator derivation is simplest. We want to rotate a vector $\mathbf{v}$ about $\mathbf{k}$ to yield $\mathbf{v}'$, where θ is measured as the angle between projections onto the plane whose normal is $\mathbf{k}$ (Figure 3.7). To proceed, form a right-handed coordinate system as follows.

$\mathbf{k}(\mathbf{k}\cdot\mathbf{v})$:	component of $\mathbf{v}$ $\parallel$ to $\mathbf{k}$.
$\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})$:	component of $\mathbf{v}$ $\perp$ to $\mathbf{k}$.
$\mathbf{k}\times\mathbf{v}$:	component $\perp$ to the other two.

The orthogonality of these 3 vectors may be simply verified. Note that $\mathbf{k}\times\mathbf{v} = \mathbf{k}\times(\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v}))$. Hence the 2nd and 3rd vectors are equal in length:

$$||\mathbf{k}\times\mathbf{v}|| = ||\mathbf{k}\times(\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v}))|| = ||\mathbf{k}||\,||\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})||\sin\frac{\pi}{2} = ||\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})|| \tag{3.34}$$

Next, rotate $\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})$ by θ about $\mathbf{k}$ to give a vector $\mathbf{r}$ in the $\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})$ and $\mathbf{k}\times\mathbf{v}$ plane. If we knew $\mathbf{r}$, then we would have:

$$\mathbf{v}' = \mathbf{r}+\mathbf{k}(\mathbf{k}\cdot\mathbf{v}) \tag{3.35}$$

Note that $||\mathbf{r}|| = ||\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})|| = ||\mathbf{k}\times\mathbf{v}||$, since a rotational operator does not change a vector's length. Since $\mathbf{r}$ lies in the plane spanned by vectors $\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})$ and $\mathbf{k}\times\mathbf{v}$ plane, we can write $\mathbf{r}$ as a linear sum of these two vectors:

$$\mathbf{r} = (\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v}))\,\mathrm{c}\theta + (\mathbf{k}\times\mathbf{v})\,\mathrm{s}\theta \tag{3.36}$$

Substituting (3.36) into (3.35),

$$\begin{aligned}\mathbf{v}' &= (\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v}))\cos\theta + (\mathbf{k}\times\mathbf{v})\,\mathrm{s}\theta + \mathbf{k}(\mathbf{k}\cdot\mathbf{v}) \\ &= \mathbf{v}\cos\theta + \mathbf{k}\times\mathbf{v}\mathrm{s}\theta + \mathbf{k}(\mathbf{k}\cdot\mathbf{v})(1-\cos\theta)\end{aligned} \tag{3.37}$$

3.5.1 Equivalent Matrix Representation

To rewrite the result (3.37) as a matrix/vector operation, we will employ the *outer product* of two vectors:

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{v}) = (\mathbf{k}\mathbf{k}^T)\mathbf{v} \tag{3.38}$$

where the outer product $\mathbf{k}\mathbf{k}^T$ is a 3-by-3 matrix. It is distinguished from the *inner product* of two vectors, $\mathbf{k} \cdot \mathbf{v} = \mathbf{k}^T\mathbf{v}$, which is a scalar. We will also employ the matrix representation $\mathbf{S}(\mathbf{k})\mathbf{v}$ of the cross product $\mathbf{k} \times \mathbf{v}$:

$$\mathbf{k} \times \mathbf{v} = \begin{vmatrix} k_1 & k_2 & k_3 \\ v_1 & v_2 & v_3 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix} = \begin{bmatrix} k_2 v_3 - k_3 v_2 \\ k_3 v_1 - k_1 v_3 \\ k_1 v_2 - k_2 v_1 \end{bmatrix} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{S}(\mathbf{k})\mathbf{v} \qquad (3.39)$$

where the skew-symmetric matrix $\mathbf{S}(\mathbf{k})$ represents the cross product by $\mathbf{k}$:

$$\mathbf{S}(\mathbf{k}) = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \qquad (3.40)$$

Substituting (3.38) and (3.39) into (3.37),

$$\begin{aligned} \mathbf{v}' &= (\mathbf{I}\mathrm{c}\theta)\mathbf{v} + \mathbf{S}(\mathbf{k})\mathbf{v}\mathrm{s}\theta + (\mathbf{k}\mathbf{k}^T)\mathbf{v}(1 - \mathrm{c}\theta) \\ &= \Big(\mathbf{I}\mathrm{c}\theta + \mathbf{S}(\mathbf{k})\mathrm{s}\theta + \mathbf{k}\mathbf{k}^T(1 - \mathrm{c}\theta)\Big)\,\mathbf{v} \\ &= \mathbf{R}_k(\theta)\mathbf{v} \end{aligned}$$

where $\mathbf{I}$ is the 3-by-3 identity matrix. The equivalent matrix operation $\mathbf{R}_k(\theta)$ is:

$$\mathbf{R}_k(\theta) = \mathbf{I}\mathrm{c}\theta + \mathbf{S}(\mathbf{k})\mathrm{s}\theta + \mathbf{k}\mathbf{k}^T(1 - \mathrm{c}\theta) \qquad (3.41)$$

$$= \begin{bmatrix} k_1^2\mathrm{v}\theta + \mathrm{c}\theta & k_1k_2\mathrm{v}\theta - k_3\mathrm{s}\theta & k_1k_3\mathrm{v}\theta + k_2\mathrm{s}\theta \\ k_1k_2\mathrm{v}\theta + k_3\mathrm{s}\theta & k_2^2\mathrm{v}\theta + \mathrm{c}\theta & k_2k_3\mathrm{v}\theta - k_1\mathrm{s}\theta \\ k_1k_3\mathrm{v}\theta - k_2\mathrm{s}\theta & k_2k_3\mathrm{v}\theta + k_1\mathrm{s}\theta & k_3^2\mathrm{v}\theta + \mathrm{c}\theta \end{bmatrix} \qquad (3.42)$$

The versine of θ is abbreviated $\mathrm{v}\theta = 1 - \mathrm{c}\theta$.

3.5.2 Equivalent Angle-Axis Representation from a Matrix

Any rotation matrix can be represented as some rotation θ about some axis $\mathbf{k}$. We show this by construction, given the previous expression (3.42) for $\mathbf{R}_k(\theta)$. Let $\mathbf{R} = \{r_{ij}\}$ be an arbitrary 3D rotation matrix with elements r_{ij}. Equate $\mathbf{R} = \mathbf{R}_k(\theta)$ to solve for θ and $\mathbf{k}$.

First, solve for θ by taking the trace of $\mathbf{R}$:

$$\begin{aligned} \mathrm{Tr}(\mathbf{R}) &= r_{11} + r_{22} + r_{33} \\ &= k_1^2\mathrm{v}\theta + \mathrm{c}\theta + k_2^2\mathrm{v}\theta + \mathrm{c}\theta + k_3^2\mathrm{v}\theta + \mathrm{c}\theta \\ &= (k_1^2 + k_2^2 + k_3^2)\mathrm{v}\theta + 3\mathrm{c}\theta \\ &= 1 + 2\mathrm{c}\theta \qquad (3.43) \end{aligned}$$

$$\mathrm{c}\theta = \frac{\mathrm{Tr}(\mathbf{R}) - 1}{2} \qquad (3.44)$$

since we are requiring $\mathbf{k}$ to be a unit vector. An independent estimate of sθ is obtained by differencing diagonally opposite elements of $\mathbf{R}$:

$$\begin{aligned} r_{32} - r_{23} &= k_2k_3\mathrm{v}\theta + k_1\mathrm{s}\theta - (k_2k_3\mathrm{v}\theta - k_1\mathrm{s}\theta) \\ &= 2k_1\mathrm{s}\theta \\ r_{13} - r_{31} &= k_1k_3\mathrm{v}\theta + k_2\mathrm{s}\theta - (k_1k_3\mathrm{v}\theta - k_2\mathrm{s}\theta) \\ &= 2k_2\mathrm{s}\theta \\ r_{21} - r_{12} &= k_1k_2\mathrm{v}\theta + k_3\mathrm{s}\theta - (k_1k_2\mathrm{v}\theta - k_3\mathrm{s}\theta) \\ &= 2k_3\mathrm{s}\theta \end{aligned} \tag{3.45}$$

Hence

$$(r_{32} - r_{23})^2 + (r_{13} - r_{31})^2 + (r_{21} - r_{12})^2 = 4(k_1^2 + k_2^2 + k_3^2)\mathrm{s}\theta^2$$

$$\mathrm{s}\theta = \pm\frac{1}{2}\sqrt{(r_{32} - r_{23})^2 + (r_{13} - r_{31})^2 + (r_{21} - r_{12})^2} \tag{3.46}$$

We may now use the 4-quadrant arctangent function `atan2`, using independent estimates of sθ (3.46) and cθ (3.44). Two solutions exist for θ, corresponding to the two solutions for sθ in (3.46). We can now solve for $\mathbf{k}$ from (3.45):

$$\mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \frac{1}{2\mathrm{s}\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \tag{3.47}$$

Note that $\mathbf{R}_k(\theta) = \mathbf{R}_{-k}(-\theta)$. The rotation matrix $\mathbf{R}_{-k}(-\theta)$ would have resulted if we had chosen the other of the two solutions for sθ in (3.46). Also note that $\mathbf{R}_k(\theta)^T = \mathbf{R}_k(-\theta)$. This is like the result of rotation in a plane. Actually, $\mathbf{k}$ can be considered as a normal to a plane, just like $\mathbf{z}$ is normal to the x, y plane. By restricting the rotation to the general plane identified by $\mathbf{k}$, then rotating forward in the plane is simply reversed by rotating backwards in the same plane.

Example 3.5: Consider the following rotation matrix, where the relative orientation of frame 1 to frame 0 is shown in Figure 3.8.

$${}^0\mathbf{R}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then from (3.44) and (3.46), using the positive value for sθ,

$$\mathrm{c}\theta = -1/2, \quad \mathrm{s}\theta = \sqrt{3}/2, \quad \theta = \mathrm{atan2}(\sqrt{3}/2, -1/2) = 120^o$$

Hence

$$\mathbf{k} = \frac{1}{2\frac{\sqrt{3}}{2}}\begin{bmatrix} 1-0 \\ 1-0 \\ 1-0 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

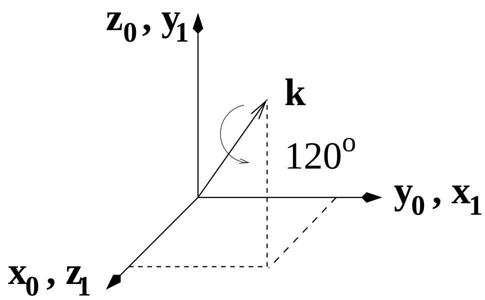

Figure 3.8: Equivalent angle-axis rotation.

Rotating by 120^o about axis $\mathbf{k}$ gives axes 1 starting from axes 0 (Figure 3.8). If one thinks of the axes forming a tetrahedron, then the rotation axis bisects the spatial right angle of the tetrahedron. The rotation of axes then just moves one side of the tetrahedron to the other.

Why didn't we apply the arccos in (3.44) or the arcsin in (3.46) to estimate θ? Besides the issue of uniqueness, which the 4-quadrant arctangent gives, there is an issue of poor numerical robustness. The derivative of $\cos\theta$ is of course $-\sin\theta$, and when $\theta = 0$ or π the slope of the cosine curve is 0. This means that changes in angle have a small effect on the value of the cosine. If there is error in the knowledge of robot position, which is quite likely since it is often derived from sensor data, then the angle between these vectors will be derived quite inaccurately. Similarly, for $\sin^{-1}$ the angle is inaccurately derived near $\theta = \pm\pi/2$. The derivative of $\tan\theta$ is $\sec^2\theta$, and the slope of the tangent curve is never less than 1. Thus the arctangent has good numerical sensitivity everywhere.

3.6 Euler Angles

Euler angles are the most common 3-parameter representation of rotation, although not the best way. A set of Euler angles are any nonredundant set of 3 successive rotations about (current) principle axes. There are 12 different Euler angle systems, depending on the axes chosen. One of the most common sets is the ZYZ Euler angles (ϕ, θ, ψ):

$$\begin{aligned}
\mathbf{R}_{zyz}(\phi,\theta,\psi) &= \mathbf{R}_z(\phi)\mathbf{R}_y(\theta)\mathbf{R}_z(\psi) \qquad (3.48)\\
&= \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi & 0 \\ \mathrm{s}\phi & \mathrm{c}\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{c}\theta & 0 & \mathrm{s}\theta \\ 0 & 1 & 0 \\ -\mathrm{s}\theta & 0 & \mathrm{c}\theta \end{bmatrix} \begin{bmatrix} \mathrm{c}\psi & -\mathrm{s}\psi & 0 \\ \mathrm{s}\psi & \mathrm{c}\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}\\
&= \begin{bmatrix} \mathrm{c}\phi\mathrm{c}\theta\mathrm{c}\psi - \mathrm{s}\phi\mathrm{s}\psi & -\mathrm{c}\phi\mathrm{c}\theta\mathrm{s}\psi - \mathrm{s}\phi\mathrm{c}\psi & \mathrm{c}\phi\mathrm{s}\theta \\ \mathrm{s}\phi\mathrm{c}\theta\mathrm{c}\psi + \mathrm{c}\phi\mathrm{s}\psi & -\mathrm{s}\phi\mathrm{c}\theta\mathrm{s}\psi + \mathrm{c}\phi\mathrm{c}\psi & \mathrm{s}\phi\mathrm{s}\theta \\ -\mathrm{s}\theta\mathrm{c}\psi & \mathrm{s}\theta\mathrm{s}\psi & \mathrm{c}\theta \end{bmatrix} \qquad (3.49)
\end{aligned}$$

where Figure 3.9 shows left-to-right evaluation of (3.48) to derive the final coordinate system 3 beginning from frame 0. Again, the Euler angle transformation (3.48) is considered a coordinate transformation, hence the left-to-right evaluation.

Conversely, given an arbitrary matrix $\mathbf{R} = \{r_{ij}\}$, we can extract the ZYZ Euler angles. To find ϕ, use the 13 and 23 elements of (3.49):

$$r_{13} = \mathrm{c}\phi\mathrm{s}\theta$$

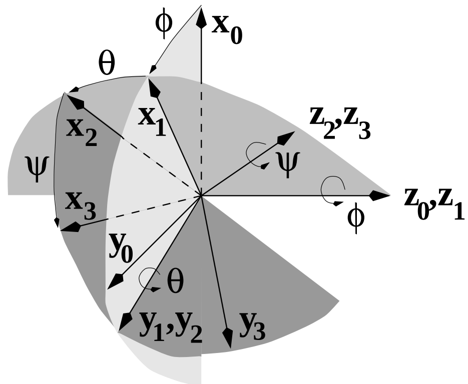

Figure 3.9: ZYZ Euler angles ϕ, θ, ψ.

$$r_{23} \quad = \quad \mathrm{s}\phi\mathrm{s}\theta$$

Hence we get 2 solutions for ϕ:

$$\begin{aligned} \tan\phi &= \frac{r_{23}}{r_{13}} \\ \phi &= \tan^{-1}\left(\frac{r_{23}}{r_{13}}\right) \end{aligned} \tag{3.50}$$

As will be seen below, given one of these two solutions of ϕ, the other Euler angles will be determined uniquely. This means that there are two possible Euler angle sets corresponding to a given rotation matrix, but these sets are related.

When $r_{13} = r_{23} = 0$, then $\theta = 0$ or π and the solution is ill-defined. Then

$$\mathbf{R}_{zyz}(\phi, 0, \psi) = \mathbf{R}_z(\phi)\,\mathbf{I}\,\mathbf{R}_z(\psi) = \mathbf{R}_z(\phi + \psi) \tag{3.51}$$

and we cannot resolve ϕ and ψ uniquely. This is a *degeneracy,* as ϕ and ψ correspond to the same rotation. We can only determine the sum:

$$\phi + \psi = \text{ATAN2}(r_{21}, r_{11}) \tag{3.52}$$

One convention is to choose $\phi = 0$. This degeneracy of the Euler angles is one of the reasons that the use of Euler angles is currently not preferred.

To find θ given ϕ, we have

$$r_{33} = \mathrm{c}\theta$$

To find $\mathrm{s}\theta$ independently, we use r_{13} and r_{23}, but not directly:

$$\begin{aligned} r_{13}\mathrm{c}\phi + r_{23}\mathrm{s}\phi &= (\mathrm{c}\phi\mathrm{s}\theta)\mathrm{c}\phi + (\mathrm{s}\phi\mathrm{s}\theta)\mathrm{s}\phi \\ &= (\mathrm{c}^2\phi + \mathrm{s}^2\phi)\mathrm{s}\theta \\ &= \mathrm{s}\theta \end{aligned}$$

Hence

$$\theta = \text{ATAN2}(r_{13}\mathrm{c}\phi + r_{23}\mathrm{s}\phi, r_{33})$$

To find ψ given ϕ and θ, note that

$$\begin{aligned}
r_{21}\mathrm{c}\phi - r_{11}\mathrm{s}\phi &= (\mathrm{s}\phi\mathrm{c}\theta\mathrm{c}\psi + \mathrm{c}\phi\mathrm{s}\psi)\mathrm{c}\phi - (\mathrm{c}\phi\mathrm{c}\theta\mathrm{c}\psi - \mathrm{s}\phi\mathrm{s}\psi)\mathrm{s}\phi \\
&= \mathrm{c}^2\phi\mathrm{s}\psi + \mathrm{s}^2\phi\mathrm{s}\psi \\
&= \mathrm{s}\psi \\
r_{22}\mathrm{c}\phi - r_{12}\mathrm{s}\phi &= (-\mathrm{s}\phi\mathrm{c}\theta\mathrm{s}\psi + \mathrm{c}\phi\mathrm{c}\psi)\mathrm{c}\phi - (-\mathrm{c}\phi\mathrm{c}\theta\mathrm{s}\psi - \mathrm{s}\phi\mathrm{c}\psi)\mathrm{s}\phi \\
&= \mathrm{c}^2\phi\mathrm{c}\psi + \mathrm{s}^2\phi\mathrm{c}\psi \\
&= \mathrm{c}\psi
\end{aligned}$$

Hence

$$\psi = \text{ATAN2}(r_{21}\mathrm{c}\phi - r_{11}\mathrm{s}\phi, r_{22}\mathrm{c}\phi - r_{12}\mathrm{s}\phi)$$

Example 3.6: In the previous example, we can interpret $\mathbf{R}$ as the application of the ZYZ Euler angles $0, \pi/2, \pi/2$, since $\mathbf{R}_z(0) = \mathbf{I}$:

$$\mathbf{R} = \mathbf{R}_z(0)\mathbf{R}_y(\pi/2)\mathbf{R}_z(\pi/2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then the first Euler angle ϕ is:

$$\phi = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

where we have taken the positive solution 0 rather than π. Then

$$\begin{aligned}
\theta &= \text{ATAN2}(1, 0) = \pi/2 \\
\psi &= \text{ATAN2}(1, 0) = \pi/2
\end{aligned}$$

Thus we have extracted the correct Euler angles.

3.7 Roll, Pitch, Yaw Angles

The roll, pitch, yaw (RPY) angles (Figure 3.10) are successive rotations about the fixed XYZ axes, in that order:

$$\mathbf{R}_{zyx}(\phi, \theta, \psi) = \mathbf{R}_z(\phi)\mathbf{R}_y(\theta)\mathbf{R}_x(\psi) \tag{3.53}$$

where (3.53) represents an operator. Note that the order is right-to-left because of the rotation about the fixed axes. By contrast, Euler angles are successive rotations about the current axes. We would get the same result as RPY with the ZYX Euler angles.

In a process similar to the ZYZ Euler angles, we can extract the RPY angles from any rotation matrix $\mathbf{R}$. RPY angles are particularly useful for differential rotations, which commute, because these rotations are about orthogonal axes.

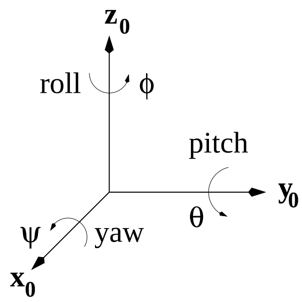

Figure 3.10: The roll, pitch, yaw angles ϕ, θ, ψ.

3.8 Rodrigues Vector

Euler angles or RPY angles are difficult to visualize in combinations. They also suffer from degeneracy, and (as will be seen later) they have a complicated relation to the angular velocity vector. We now consider several representations based on the angle-axis form of a rotation matrix.

The angle/axis $\mathbf{k}, \theta$ is itself a representation, but it is redundant, with 4 parameters. There is a constraint equation: $||\mathbf{k}|| = 1$. The *Rodrigues vector* is a scaled version of $\mathbf{k}$, which has only 3 components:

$$\boldsymbol{\rho} = \mathbf{k} \tan\left(\frac{\theta}{2}\right) \tag{3.54}$$

Given $\boldsymbol{\rho}$, we can then extract the angle and the axis:

$$\tan\left(\frac{\theta}{2}\right) = ||\boldsymbol{\rho}|| \tag{3.55}$$

$$\mathbf{k} = \frac{\boldsymbol{\rho}}{||\boldsymbol{\rho}||} \tag{3.56}$$

A problem is when $\theta = \pi$, which often makes the Rodriguez parameters not useable.

3.9 Euler Parameters

It appears that any 3-parameter representation of orientation has some problem, and hence redundant 4 parameter representations have been proposed. To get around the problem with π of the Rodrigues vector, we can separate $\tan(\theta/2)$ into $\sin(\theta/2)$ and $\cos(\theta/2)$ to yield 4 parameters.

$$[\cos\left(\frac{\theta}{2}\right),\ k_1 \sin\left(\frac{\theta}{2}\right),\ k_2 \sin\left(\frac{\theta}{2}\right),\ k_3 \sin\left(\frac{\theta}{2}\right)] \tag{3.57}$$

Generally, these 4 Euler parameters are preferred for their robustness, despite the extra parameters plus constraint equation:

$$1 = \cos^2\left(\frac{\theta}{2}\right) + \mathbf{k}^2 \sin^2\left(\frac{\theta}{2}\right)$$

3.10 Quaternions

Proposed by Hamilton, these are a substitute for rotation matrices. The Euler parameters make up the components of a quaternion. We write:

$$\boldsymbol{q} = q_0 + \mathbf{q} \qquad \text{where } q_0 = \cos\left(\frac{\theta}{2}\right), \quad \mathbf{q} = \mathbf{k}\sin\left(\frac{\theta}{2}\right) \tag{3.58}$$

The notation $\boldsymbol{q}$ denotes a 4-by-1 quaternion vector, where q_0 is a scalar part and $\mathbf{q}$ is a vector part.

- A quaternion will be denoted by a slanted bold-font lower-case letter, such as $\boldsymbol{q}$.

The sum in (3.58) is purely a formal way to designate these components; we don't actually add the scalar part to the vector part. Thus the Euler parameters would be written $[q_0\ \mathbf{q}^T]$. There is one constraint equation for quaternions that represent rotations:

$$\boldsymbol{q} \cdot \boldsymbol{q} = q_0^2 + \mathbf{q} \cdot \mathbf{q} = 1 \tag{3.59}$$

Such $\boldsymbol{q}$ are called *unit quaternions.* Scalars and vectors are special cases of quaternions:

- If $q_0 = 0$, then $\boldsymbol{q} = \mathbf{q}$ is a vector, i.e., a vector is a quaternion with a zero scalar part.
- If $\mathbf{q} = 0$, then $\boldsymbol{q} = q_0$ is a scalar, i.e., a scalar is a quaternion with a zero vector part.

Example 3.7: Let $\boldsymbol{q}_x$ be the quaternion with $\mathbf{k} = \mathbf{x}_0$ and $\theta = \pi/2$. Then

$$\boldsymbol{q}_x = \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\mathbf{x}_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathbf{x}_0$$

Let $\boldsymbol{q}_z$ be the quaternion with $\mathbf{k} = \mathbf{z}_0$ and $\theta = \pi/2$. Then

$$\boldsymbol{q}_z = \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\mathbf{z}_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathbf{z}_0$$

Given two quaternions $\boldsymbol{p} = p_0 + \mathbf{p}$ and $\boldsymbol{q}$, the composition of two quaternions is written as the two quaternions juxtaposed, and is defined as follows:

$$\begin{aligned} \boldsymbol{pq} &= (p_0 + \mathbf{p})(q_0 + \mathbf{q}) \\ &= p_0 q_0 + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{pq} \\ &= p_0 q_0 + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q} - \mathbf{p} \cdot \mathbf{q} \end{aligned}$$

where we define two vectors $\mathbf{p}$ and $\mathbf{q}$ juxtaposed as:

$$\mathbf{pq} = \mathbf{p} \times \mathbf{q} - \mathbf{p} \cdot \mathbf{q} \tag{3.60}$$

The resulting quaternion is rewritten to group the scalar and vector parts together:

$$\boldsymbol{pq} = (p_0 q_0 - \mathbf{p} \cdot \mathbf{q}) + (p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q}) \tag{3.61}$$

Example 3.8: The composition $\boldsymbol{q}_x\boldsymbol{q}_z$ is:

$$\begin{aligned}\boldsymbol{q}_x\boldsymbol{q}_z &= \left(\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} - (\frac{\sqrt{2}}{2}\mathbf{x}_0)\cdot(\frac{\sqrt{2}}{2}\mathbf{z}_0)\right) + \left(\frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2}\mathbf{z}_0) + \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2}\mathbf{x}_0) + (\frac{\sqrt{2}}{2}\mathbf{x}_0)\times(\frac{\sqrt{2}}{2}\mathbf{z}_0)\right) \\ &= \frac{1}{2} + \left(\frac{1}{2}\mathbf{z}_0 + \frac{1}{2}\mathbf{x}_0 + \frac{1}{2}(-\mathbf{y}_0)\right) \\ &= \frac{1}{2} + \frac{1}{2}\left(\mathbf{z}_0 + \mathbf{x}_0 - \mathbf{y}_0\right)\end{aligned}$$

The inverse of $\boldsymbol{q}$ is denoted as $\boldsymbol{q}^*$, where

$$\boldsymbol{q}^* = q_0 - \mathbf{q} \tag{3.62}$$

Then

$$\begin{aligned}\boldsymbol{q}\boldsymbol{q}^* &= (q_0q_0 - \mathbf{q}\cdot(-\mathbf{q})) + (q_0(-\mathbf{q}) + q_0\mathbf{q} + \mathbf{q}\times(-\mathbf{q})) \\ &= q_0^2 + \mathbf{q}\cdot\mathbf{q} \\ &= 1\end{aligned}$$

since we only deal with unit quaternions (3.59). We can therefore define the dot product of two quaternions as the equivalent composition:

$$\boldsymbol{q}\cdot\boldsymbol{q} = \boldsymbol{q}\boldsymbol{q}^*$$

3.10.1 Representing rotations by quaternion compositions

Consider a vector $\mathbf{v}$ (i.e., a quaternion $\boldsymbol{v}$ with a zero scalar part $v_0 = 0$) composed with a quaternion $\boldsymbol{q}^*$ as follows:

$$\begin{aligned}\mathbf{v}\boldsymbol{q}^* &= (0 + \mathbf{v})(q_0 - \mathbf{q}) \\ &= \mathbf{v}\cdot\mathbf{q} + (q_0\mathbf{v} - \mathbf{v}\times\mathbf{q})\end{aligned}$$

Now compose this result on the left with $\boldsymbol{q}$:

$$\begin{aligned}\boldsymbol{q}\mathbf{v}\boldsymbol{q}^* &= (q_0 + \mathbf{q})(\mathbf{v}\cdot\mathbf{q} + (q_0\mathbf{v} - \mathbf{v}\times\mathbf{q})) \\ &= (q_0(\mathbf{v}\cdot\mathbf{q}) - \mathbf{q}\cdot(q_0\mathbf{v} - \mathbf{v}\times\mathbf{q})) + (q_0(q_0\mathbf{v} - \mathbf{v}\times\mathbf{q}) + (\mathbf{v}\cdot\mathbf{q})\mathbf{q} + \mathbf{q}\times(q_0\mathbf{v} - \mathbf{v}\times\mathbf{q})) \\ &= q_0(\mathbf{v}\cdot\mathbf{q}) - q_0(\mathbf{q}\cdot\mathbf{v}) + q_0^2\mathbf{v} - q_0\mathbf{v}\times\mathbf{q} + (\mathbf{v}\cdot\mathbf{q})\mathbf{q} + q_0\mathbf{q}\times\mathbf{v} - \mathbf{q}\times(\mathbf{v}\times\mathbf{q}) \\ &= q_0^2\mathbf{v} - q_0\mathbf{v}\times\mathbf{q} + (\mathbf{v}\cdot\mathbf{q})\mathbf{q} + q_0\mathbf{q}\times\mathbf{v} - \mathbf{v}(\mathbf{q}\cdot\mathbf{q}) + \mathbf{q}(\mathbf{q}\cdot\mathbf{v}) \\ &= (q_0^2 - \mathbf{q}\cdot\mathbf{q})\mathbf{v} + 2q_0\mathbf{q}\times\mathbf{v} + 2\mathbf{q}(\mathbf{q}\cdot\mathbf{v})\end{aligned} \tag{3.63}$$

It will now be shown that this result is equivalent to a rotation of $\mathbf{v}$ by the quaternion $\boldsymbol{q}$. Substitute for $q_0 = \cos(\theta/2)$ and $\mathbf{q} = \mathbf{k}\sin(\theta/2)$:

$$\begin{aligned}\boldsymbol{q}\mathbf{v}\boldsymbol{q}^* &= \left(\cos^2\left(\frac{\theta}{2}\right) - \mathbf{k}^2\sin^2\left(\frac{\theta}{2}\right)\right)\mathbf{v} + 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathbf{k}\times\mathbf{v} + 2\sin^2\left(\frac{\theta}{2}\right)\mathbf{k}(\mathbf{k}\cdot\mathbf{v}) \\ &= \mathbf{v}\cos\theta + \mathbf{k}\times\mathbf{v}\sin\theta + \mathbf{k}(\mathbf{k}\cdot\mathbf{v})(1-\cos\theta) \end{aligned} \tag{3.64}$$

This is the same as a rotation matrix expressed in angle-axis form $\mathbf{R}_k(\theta)\mathbf{v}$ in (3.37).

Example 3.9: Suppose we wish to rotate the vector $\mathbf{x}_0$ by the quaternion $\boldsymbol{q}_z$. Then

$$\begin{aligned}\boldsymbol{q}_z\mathbf{x}_0\boldsymbol{q}_z^* &= \left((\frac{\sqrt{2}}{2})^2 - \frac{\sqrt{2}}{2}\mathbf{z}_0\cdot\frac{\sqrt{2}}{2}\mathbf{z}_0\right)\mathbf{x}_0 + 2\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\mathbf{z}_0\times\mathbf{x}_0\right) + 2\frac{\sqrt{2}}{2}\mathbf{z}_0\left(\frac{\sqrt{2}}{2}\mathbf{z}_0\cdot\mathbf{x}_0\right) \\ &= \mathbf{0} + \mathbf{y}_0 + \mathbf{0} \\ &= \mathbf{y}_0\end{aligned}$$

One can verify the same result by $\mathbf{R}_z(\pi/2)\mathbf{x}_0$.

Quaternions can be composed just like rotation matrices.

Example 3.10: Consider the quaternion composition $\boldsymbol{q}_x\boldsymbol{q}_z$. Then from the previous example and re-grouping,

$$\boldsymbol{q}_x\boldsymbol{q}_z\mathbf{x}_0(\boldsymbol{q}_x\boldsymbol{q}_z)^* = \boldsymbol{q}_x\boldsymbol{q}_z\mathbf{x}_0\boldsymbol{q}_z^*\boldsymbol{q}_x^* = \boldsymbol{q}_x(\boldsymbol{q}_z\mathbf{x}_0\boldsymbol{q}_z^*)\boldsymbol{q}_x^* = \boldsymbol{q}_x\mathbf{y}_0\boldsymbol{q}_x^* = 2\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\mathbf{x}_0\times\mathbf{y}_0\right) = \mathbf{z}_0$$

Alternatively, we may compose $\boldsymbol{q}_x\boldsymbol{q}_z$ directly, as in a previous example, then apply to $\mathbf{x}_0$:

$$\begin{aligned}(\boldsymbol{q}_x\boldsymbol{q}_z)\mathbf{x}_0(\boldsymbol{q}_x\boldsymbol{q}_z)^* &= \left((\frac{1}{2})^2 - \frac{1}{2}(\mathbf{z}_0+\mathbf{x}_0-\mathbf{y}_0)\cdot\frac{1}{2}(\mathbf{z}_0+\mathbf{x}_0-\mathbf{y}_0)\right)\mathbf{x}_0 + 2(\frac{1}{2})\left(\frac{1}{2}(\mathbf{z}_0+\mathbf{x}_0-\mathbf{y}_0)\times\mathbf{x}_0\right) \\ &\quad +2(\frac{1}{2})(\mathbf{z}_0+\mathbf{x}_0-\mathbf{y}_0)\left(\frac{1}{2}(\mathbf{z}_0+\mathbf{x}_0-\mathbf{y}_0)\cdot\mathbf{x}_0\right) \\ &= -\frac{1}{2}\mathbf{x}_0 + \frac{1}{2}(\mathbf{y}_0+\mathbf{z}_0) + \frac{1}{2}(\mathbf{z}_0+\mathbf{x}_0-\mathbf{y}_0) \\ &= \mathbf{z}_0\end{aligned}$$

One can verify the same result from $\mathbf{R}_x(\pi/2)\mathbf{R}_z(\pi/2)\mathbf{x}_0$.

3.10.2 Converting from a quaternion to a rotation matrix

We can find the equivalent rotation matrix $\mathbf{R} = \{r_{ij}\}$ directly from the components $\boldsymbol{q} = q_0 + \mathbf{q}$ of a quaternion. Rewrite (3.63) to extract the rotation matrix:

$$\mathbf{R} = (q_0^2 - \mathbf{q}\cdot\mathbf{q})\mathbf{I} + 2q_0\mathbf{S}(\mathbf{q}) + 2\mathbf{q}\mathbf{q}^T \tag{3.65}$$

From (3.59), we can substitute for the left coefficient:

$$\begin{aligned} q_0^2 - \mathbf{q}\cdot\mathbf{q} &= (1 - \mathbf{q}\cdot\mathbf{q}) - \mathbf{q}\cdot\mathbf{q} \\ &= 1 - 2\mathbf{q}\cdot\mathbf{q} \\ &= 1 - 2(q_1^2 + q_2^2 + q_3^2) \end{aligned}$$

where $\mathbf{q} = [q_1\ q_2\ q_3]^T$. For the r_{11} element,

$$\begin{aligned} r_{11} &= 1 - 2(q_1^2 + q_2^2 + q_3^2) + 2q_1^2 \\ &= 1 - 2q_2^2 - 2q_3^2 \end{aligned}$$

For the r_{12} element,

$$r_{12} = -2q_0q_3 + 2q_1q_2$$

In a similar manner, we can find all elements r_{ij} of $\mathbf{R}$ to yield:

$$\mathbf{R} = \begin{bmatrix} 1 - 2q_2^2 - 2q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & 1 - 2q_1^2 - 2q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & 1 - 2q_1^2 - 2q_2^2 \end{bmatrix} \tag{3.66}$$

3.10.3 Converting from a rotation matrix to a quaternion

Given a rotation matrix with elements $\mathbf{R} = \{r_{ij}\}$, we can find the equivalent quaternion $\boldsymbol{q} = q_0 + \mathbf{q}$. From (3.66),

$$\begin{aligned} r_{11} + r_{22} + r_{33} &= (1 - 2q_2^2 - 2q_3^2) + (1 - 2q_1^2 - 2q_3^2) + (1 - 2q_1^2 - 2q_2^2) \\ &= 3 - 4(q_1^2 + q_2^2 + q_3^2) \\ &= 3 - 4(1 - q_0^2) \\ &= 4q_0^2 - 1 \end{aligned}$$

Hence

$$q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

We take the positive value for the square root, so as to choose

$$0 \le q_0 = \cos\left(\frac{\theta}{2}\right) \le 1 \quad \Longleftrightarrow \quad -\frac{\pi}{2} \le \frac{\theta}{2} \le \frac{\pi}{2} \quad \Longleftrightarrow \quad -\pi \le \theta \le \pi$$

Next, solve for q_1, q_2, q_3 from

$$\begin{aligned} r_{32} - r_{23} &= 4q_0q_1 \\ r_{13} - r_{31} &= 4q_0q_2 \\ r_{21} - r_{12} &= 4q_0q_3 \end{aligned} \tag{3.67}$$

This procedure is similar to extracting $\mathbf{k}$ and θ from $\mathbf{R} = \mathbf{R}_k(\theta)$ in Section 3.5.2.

There are two problems.

1. If $\theta \approx 0$, then $\mathbf{q} = \mathbf{k}\sin(\theta/2)$ is poorly defined.
2. If $|\theta| \approx \pi$, then $\mathbf{q}$ is well defined, but $q_0 = 0$ and you can't divide by it in (3.68).

We sketch Salamin's [6] procedure for handling the case $|\theta| \approx \pi$. Similar to above, it may be shown that:

$$\begin{aligned} q_0^2 &= \frac{1}{4}(1 + r_{11} + r_{22} + r_{33}) \\ q_1^2 &= \frac{1}{4}(1 + r_{11} - r_{22} - r_{33}) \\ q_2^2 &= \frac{1}{4}(1 - r_{11} + r_{22} - r_{33}) \\ q_3^2 &= \frac{1}{4}(1 - r_{11} - r_{22} + r_{33}) \end{aligned}$$

Also

$$\begin{aligned} q_0q_1 &= \frac{1}{4}(r_{32} - r_{23}) \\ q_0q_2 &= \frac{1}{4}(r_{13} - r_{31}) \\ q_0q_3 &= \frac{1}{4}(r_{21} - r_{12}) \\ q_1q_2 &= \frac{1}{4}(r_{12} + r_{21}) \\ q_1q_3 &= \frac{1}{4}(r_{13} + r_{31}) \\ q_2q_3 &= \frac{1}{4}(r_{23} + r_{32}) \end{aligned}$$

Choose the largest of the q_i^2 (at least one of them must be significantly different from 0). Then solve from the q_iq_j products for the other q_j's. This will successfully handle the case $|\theta| \approx \pi$.

3.10.4 Normalization

With many successive rotations, computational errors may build up that cause quaternions to become of non-unit length. The correction is easy:

$$\boldsymbol{q}' = \frac{\boldsymbol{q}}{\sqrt{\boldsymbol{q} \cdot \boldsymbol{q}}}$$

By contrast, normalization of rotation matrices that become nonorthogonal is much more involved. We need to find the $\mathbf{R}'$ such that:

$$f = \sum_{i,j}(r'_{ij} - r_{ij})^2 \text{ is a minimum subject to } \mathbf{R}'(\mathbf{R}')^T = \mathbf{I}.$$

The answer is $\mathbf{R}' = \mathbf{R}(\mathbf{R}^T\mathbf{R})^{-\frac{1}{2}}$ [6].

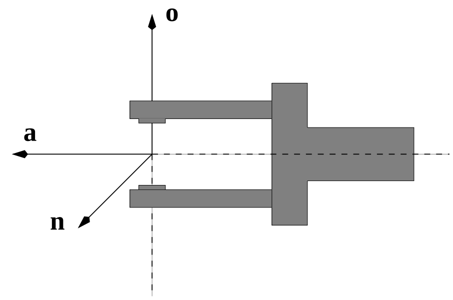

Figure 3.11: Approach $\mathbf{a}$, orientation $\mathbf{o}$, and normal $\mathbf{n}$ axes.

3.10.5 Computational efficiency of rotation matrices versus quaternions

The following comparisons are from [6].

Operation	Matrices	Quaternions
Storage (numbers)	9	4
Transformation	9M 6A	15M 15A
Composition	27M 18A	16M 12A
Normalization	Complicated	8M 3A 1sqrt

where M is the number of multiples and A the number of adds.

Funda et al. [3] suggest just storing two columns of $\mathbf{R}$. Let

$$\mathbf{R} = \begin{bmatrix} \mathbf{n} & \mathbf{o} & \mathbf{a} \end{bmatrix}$$

where the endpoint coordinate system is defined by vectors (Figure 3.11):

- $\mathbf{a}$ is the approach vector, directed towards the line of approach.
- $\mathbf{o}$ is the orientation vector, directed between the fingers of a vice gripper.
- $\mathbf{n}$ is the normal vector, forming a right-hand coordinate system with the other two: $\mathbf{n} = \mathbf{o} \times \mathbf{a}$.

Then store $\mathbf{o}$ and $\mathbf{a}$, and compute $\mathbf{n} = \mathbf{o} \times \mathbf{a}$. The rationale is that memory fetches are often more expensive than computation. This increases the transformation costs by 6M 3A. Also normalization is different; refer to Table III of Funda et al. [3].

Funda et al. [3] conclude that the use of matrices versus quaternions is a toss-up. In terms of representing orientation, however, quaternions these days are overwhelmingly preferred over the other alternatives (Euler angles and Rodriguez parameters) in graphics, control, and robotics because of handling rotation by π.

3.11 The Screw of a Spatial Displacement

In Chapter 2 it was seen that the pole is an invariant point for a planar displacement: a given planar displacement can be viewed as a pure rotation about the pole. For a spatial rotation $\mathbf{R}_k(\theta)$ described in terms of an angle θ and axis $\mathbf{k}$, points on the axis of rotation are invariant (do not move) since:

$$\mathbf{R}_k(\theta)\mathbf{k} = \mathbf{k} \tag{3.68}$$

Are there invariant points for spatial displacements? The answer is no, but there is an invariant line called the screw of the spatial displacement. Points on the screw displace to other points on the screw by a fixed offset.

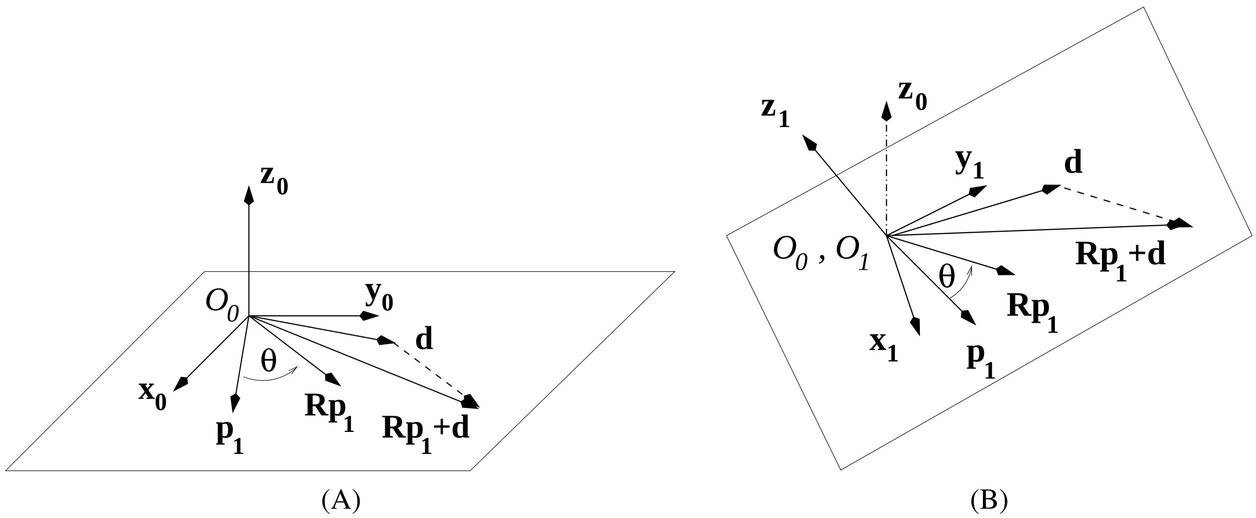

Figure 3.12: Planar rotation embedded in 3D. Different planes (A) and (B) through the origin O_0 give rise to different planar rotations.

3.11.1 Planar Displacements in 3D

To develop this important viewpoint, first let us generalize the notion of poles to arbitrary planar displacements. The planar displacements of Chapter 2 can be viewed as embedded in 3D-space, confined to the plane defined by the normal $\mathbf{z}_0$ and the origin O_0 lying in the $\mathbf{x}_0, \mathbf{y}_0$ plane (Figure 3.12(A)). The normal $\mathbf{z}_0$ is always the axis of rotation, so rotational displacements have the form $\mathbf{R} = \mathbf{R}_{z_0}(\theta)$; let us call it plane $\mathbf{z}_0$. The translational displacement $\mathbf{d}$ lies in the $\mathbf{x}_0, \mathbf{y}_0$ plane; i.e., it is perpendicular to the rotation axis $\mathbf{z}_0$. If vector $\mathbf{p}_1$ is in the plane, then its displacement $\mathbf{p}_2 = \mathbf{R}\mathbf{p}_1 + \mathbf{d}$ also lies in the plane.

Plane $\mathbf{z}_0$ is only one of infinitely many planes in 3D. Consider another plane with normal $\mathbf{z}_1$ and coordinate origin $O_1 = O_0$ (Figure 3.12(B)). What makes a spatial displacement planar is that points in plane $\mathbf{z}_1$ are mapped to other points in plane $\mathbf{z}_1$. Let $\mathbf{R}, \mathbf{d}$ represent a planar displacement for plane $\mathbf{z}_1$. Since $\mathbf{z}_1$ is always the axis of rotation, then $\mathbf{R}$ has the form $\mathbf{R}_{z_1}(\theta)$.

Next we prove that $\mathbf{d}$ is perpendicular to $\mathbf{z}_1$ as the other defining condition for a planar displacement in 3D. Let $\mathbf{p}_1$ be a vector in this plane; hence $\mathbf{z}_1^T \mathbf{p}_1 = 0$. Its displacement yields another vector $\mathbf{p}_2$:

$$\mathbf{p}_2 = \mathbf{R}\mathbf{p}_1 + \mathbf{d} \tag{3.69}$$

Since $\mathbf{p}_2$ is supposed to lie in the $\mathbf{z}_1$ plane, then $\mathbf{z}_1^T \mathbf{p}_2 = 0$. Project the displacement (3.69) onto the $\mathbf{z}_1$ axis:

$$\mathbf{z}_1^T \mathbf{p}_2 = \mathbf{z}_1^T \mathbf{R}\mathbf{p}_1 + \mathbf{z}_1^T \mathbf{d} \tag{3.70}$$

The left side is zero, while $\mathbf{z}_1^T \mathbf{R} = \mathbf{z}_1^T \mathbf{R}_{z_1}(\theta) = \mathbf{z}_1^T$. Then

$$\begin{aligned} 0 &= \mathbf{z}_1^T \mathbf{p}_1 + \mathbf{z}_1^T \mathbf{d} \\ &= \mathbf{z}_1^T \mathbf{d} \end{aligned} \tag{3.71}$$

Thus $\mathbf{d}$ is perpendicular to the rotation axis $\mathbf{z}_1$, as claimed.

The pole $\mathbf{c}$ of the displacement is defined from (3.69):

$$\mathbf{c} = \mathbf{R}\mathbf{c} + \mathbf{d} \tag{3.72}$$

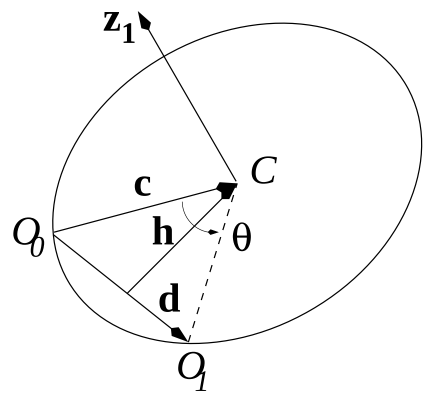

Figure 3.13: Pole of a spatially embedded planar displacement.

Because $\mathbf{I} - \mathbf{R}$ is not invertible, the first solution procedure in Chapter 2.7 cannot be used. Instead, we follow the second solution procedure. In the plane with normal $\mathbf{z}_1$, suppose the pole is located at point C (Figure 3.13). Draw a circle centered at the pole C that intersects the origin O_0; hence $\mathbf{c} = C - O_0$. Under the displacement $\mathbf{R}, \mathbf{d}$, the origin is displaced to O_1, located by $\mathbf{d} = O_1 - O_0$. The resultant point O_1 is on the circle, such that the angle subtended between C and O_1 is θ. The perpendicular bisector $\mathbf{h}$ of θ, drawn as a vector from the middle of $\mathbf{d}$ to C, is:

$$\mathbf{h} = \mathbf{z}_1 \times \frac{\mathbf{d}}{||\mathbf{d}||} \cot(\theta/2)\frac{||\mathbf{d}||}{2} = \frac{1}{2\tan(\theta/2)}\mathbf{z}_1 \times \mathbf{d} \tag{3.73}$$

The pole is found by simple vector addition:

$$\mathbf{c} = \frac{1}{2}\mathbf{d} + \mathbf{h} = \frac{1}{2}\mathbf{d} + \frac{1}{2\tan(\theta/2)}\mathbf{z}_1 \times \mathbf{d} \tag{3.74}$$

This result is similar to that of Chapter 2.7. The cross product with $\mathbf{z}_1$ has been used to generate the vector $\mathbf{h}$ perpendicular to $\mathbf{d}$, rather than a 90-degree rotation of $\mathbf{d}$.

Example 3.11: Suppose $\mathbf{z}_1 = \mathbf{x}_0$, $\theta = \pi/2$, and $\mathbf{d} = \mathbf{y}_0$. Then the pole is

$$\mathbf{C} = \frac{1}{2}\mathbf{y}_0 + \frac{1}{2\tan(90/2)}\mathbf{x}_0 \times \mathbf{y}_0 = \frac{1}{2}\mathbf{y}_0 + \frac{1}{2}\mathbf{z}_0$$

3.11.2 The Screw Location

For general spatial displacements, there is no pole. Instead, as mentioned earlier there is a line in space called the *screw* that has the invariance that points on the line are mapped to other points on the line by a fixed offset. Describe $\mathbf{R}$ by the angle-axis form $\mathbf{R}_k(\theta)$, and divide the translation into two components:

$$\mathbf{d} = (\mathbf{d} - \mathbf{k}(\mathbf{k}^T\mathbf{d})) + \mathbf{k}(\mathbf{k}^T\mathbf{d}) \tag{3.75}$$

where the first term on the right is perpendicular to $\mathbf{k}$ and the second is parallel to $\mathbf{k}$. The displacement comprised of rotation $\mathbf{R}$ and translation $\mathbf{d} - \mathbf{k}(\mathbf{k}^T\mathbf{d})$ is therefore a planar displacement with pole $\mathbf{c}$ adapted from (3.74):

$$\begin{aligned} \mathbf{c} &= \frac{1}{2}(\mathbf{d} - \mathbf{k}(\mathbf{k}^T\mathbf{d})) + \frac{1}{2\tan(\theta/2)}\mathbf{k} \times (\mathbf{d} - \mathbf{k}(\mathbf{k}^T\mathbf{d})) \\ &= \frac{1}{2}(\mathbf{d} - \mathbf{k}(\mathbf{k}^T\mathbf{d})) + \frac{1}{2\tan(\theta/2)}\mathbf{k} \times \mathbf{d} \end{aligned} \tag{3.76}$$

The residual $\mathbf{k}(\mathbf{k}^T\mathbf{d})$ is an extra translation tacked onto the planar displacement.

This pole locates a line with direction $\mathbf{k}$; this line turns out to be the screw of the original displacement $\mathbf{R}, \mathbf{d}$. Any point on the line can be represented by the linear combination:

$$\mathbf{p}_1 = \mathbf{c} + t_1\mathbf{k} \tag{3.77}$$

for some constant t_1 (Figure 3.14(A)). Apply the displacement $\mathbf{R}_k(\theta), \mathbf{d}$ to $\mathbf{p}_1$ to yield a vector $\mathbf{p}_2$:

$$\begin{aligned} \mathbf{p}_2 &= \mathbf{R}_k(\theta)(\mathbf{c} + t_1\mathbf{k}) + \mathbf{d} \\ &= \mathbf{R}_k(\theta)\mathbf{c} + t_1\mathbf{R}_k(\theta)\mathbf{k} + (\mathbf{d} - \mathbf{k}(\mathbf{k}^T\mathbf{d})) + \mathbf{k}(\mathbf{k}^T\mathbf{d}) \\ &= \mathbf{R}_k(\theta)\mathbf{c} + (\mathbf{d} - \mathbf{k}(\mathbf{k}^T\mathbf{d})) + t_1\mathbf{k} + \mathbf{k}(\mathbf{k}^T\mathbf{d}) \end{aligned} \tag{3.78}$$

The first two terms on the right side represent a planar transformation of the pole $\mathbf{c}$ from (3.76), whose result is $\mathbf{c}$. Rearranging and substituting (3.77),

$$\mathbf{p}_2 = \mathbf{c} + t_1\mathbf{k} + (\mathbf{k}^T\mathbf{d})\mathbf{k} \tag{3.79}$$

$$= \mathbf{p}_1 + (\mathbf{k}^T\mathbf{d})\mathbf{k} \tag{3.80}$$

Hence $\mathbf{p}_2$ is also on the line, and is displaced from $\mathbf{p}_1$ by the fixed amount $\mathbf{k}^T\mathbf{d}$ (Figure 3.14(A)). This demonstrates that this line is the screw for this general spatial displacement.

3.11.3 Rodrigues' Equation

Why is this line in space called a screw? Consider the spatial displacement of an arbitrary vector $\mathbf{p}_1$, reworked in terms of the screw as follows (Figure 3.14):

$$\begin{aligned} \mathbf{p}_2 &= \mathbf{R}_k(\theta)\mathbf{p}_1 + \mathbf{d} \\ &= \mathbf{R}_k(\theta)(\mathbf{p}_1 - \mathbf{c} + \mathbf{c}) + (\mathbf{d} - \mathbf{k}(\mathbf{k}^T\mathbf{d}) + \mathbf{k}(\mathbf{k}^T\mathbf{d}) \\ &= \mathbf{R}_k(\theta)(\mathbf{p}_1 - \mathbf{c}) + \mathbf{R}_k(\theta)\mathbf{c} + (\mathbf{d} - \mathbf{k}(\mathbf{k}^T\mathbf{d})) + \mathbf{k}(\mathbf{k}^T\mathbf{d}) \end{aligned}$$

The second and third terms on the right represent the planar displacement of the pole, which is again $\mathbf{c}$. Hence

$$\mathbf{p}_2 = \mathbf{R}_k(\theta)(\mathbf{p}_1 - \mathbf{c}) + \mathbf{c} + \mathbf{k}(\mathbf{k}^T\mathbf{d}) \tag{3.81}$$

The equivalent displacement is that the point represented by $\mathbf{p}_1$ has been rotated about the screw by angle θ and translated along the screw by $\mathbf{k}^T\mathbf{d}$. Figure 3.14(B) demonstrates the displacement. Point P_1 is located

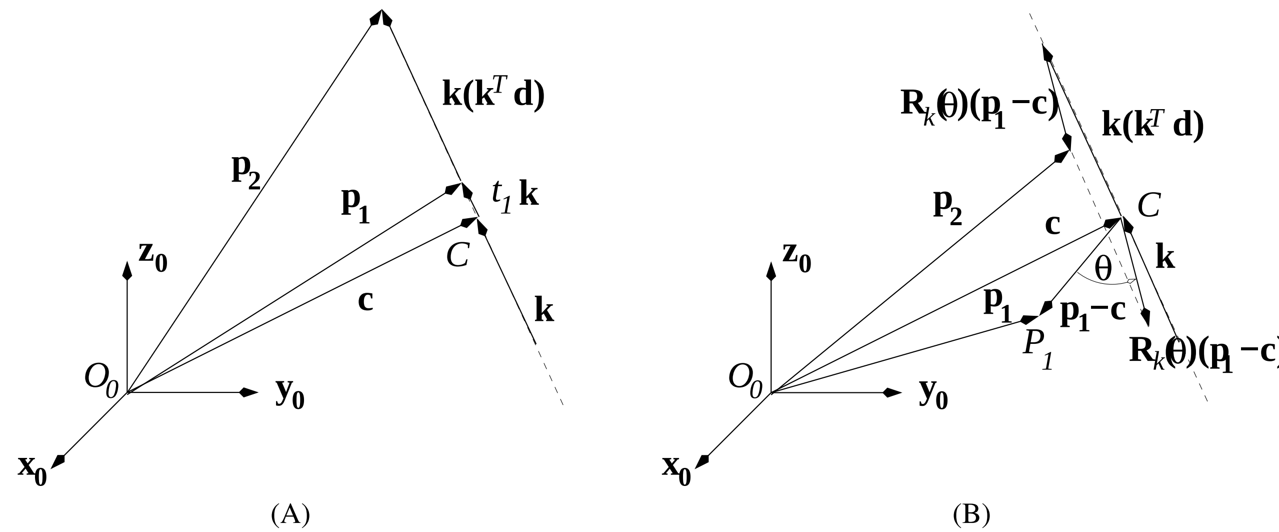

Figure 3.14: (A) Invariant mapping of a point on a screw. (B) Spatial displacement viewed as rotation about a screw by θ and translation along the screw by $\mathbf{k}^T\mathbf{d}$.

relative to O_0 by $\mathbf{p}_1$ and relative to C, the location of the pole, by $\mathbf{p}_1 - \mathbf{p}$. Relative to C and rotation axis $\mathbf{k}$, point P_1 is rotated by θ to another point represented by the vector $\mathbf{R}_k(\theta)(\mathbf{p}_1 - \mathbf{c})$. This point is translated up $\mathbf{k}$ by the amount $\mathbf{k}^T\mathbf{d}$. The final location $\mathbf{p}_2$ also includes the offset of C from O_0.

P_1 has been rotated by θ around the screw and translated by $\mathbf{k}^T\mathbf{d}$ along the screw. The ratio

$$p = \frac{\mathbf{k}^T\mathbf{d}}{\theta} \tag{3.82}$$

is called the *pitch* of the screw, analogous to a real screw as depicted in Figure 3.15. There is a nut on a screw, and a thin rod attached to the nut. As the nut rotates on the screw, the end of the rod both rotates to the right and moves up. The ratio of translation to rotation is determined by the number of threads per unit distance; e.g., 10 threads per cm means that the pitch is 1 mm/ 2π radians. Thus any spatial displacement is neatly described by the associated screw and pitch.

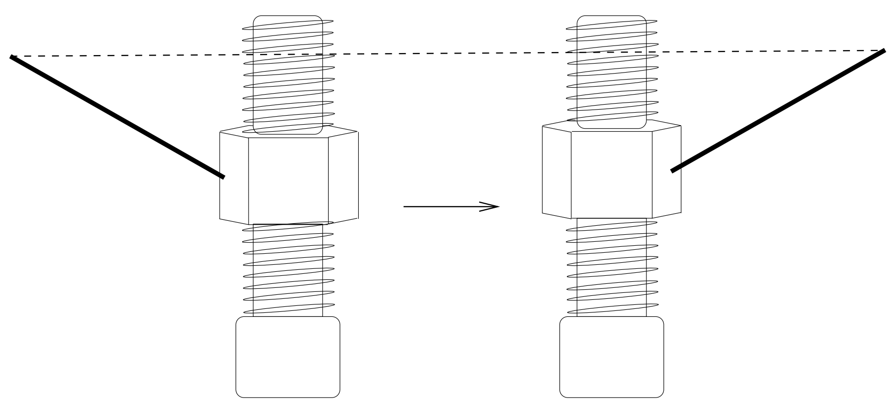

Figure 3.15: The pitch of a screw determines the ratio of translation to rotation.

The limiting cases are pure rotations and pure translations. If the pitch is zero, then the spatial displacement is a pure rotation. This corresponds to a planar displacement and its pole. If the pitch is infinite, then the spatial displacement is a pure translation. The screw has no definite location in space, and corresponds to a free vector. For more information on screws, consult [1, 4, 5, 7].

Substituting (3.82) into (3.81) yields Rodrigues' equation:

$$\mathbf{p}_2 = \mathbf{R}_k(\theta)(\mathbf{p}_1 - \mathbf{c}) + \mathbf{c} + p\theta\mathbf{k} \tag{3.83}$$

which describes a displacement in terms of the screw parameters $\mathbf{k}$, $\mathbf{c}$, θ and p. Rearranging yields an equivalent operator form:

$$\begin{bmatrix} \mathbf{p}_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_k(\theta) & (\mathbf{I} - \mathbf{R}_k(\theta))\mathbf{c} + p\theta\mathbf{k} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ 1 \end{bmatrix}$$

where the equivalent operator $\mathbf{D}$ is

$$\mathbf{D} = \begin{bmatrix} \mathbf{R}_k(\theta) & (\mathbf{I} - \mathbf{R}_k(\theta))\mathbf{c} + p\theta\mathbf{k} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{3.84}$$

This has the form of a homogeneous transformation, where the interorigin vector $\mathbf{d}$ is identified as:

$$\mathbf{d} = (\mathbf{I} - \mathbf{R}_k(\theta))\mathbf{c} + p\theta\mathbf{k} \tag{3.85}$$

3.12 Matrix Exponential Parameterization

3.12.1 Exponential Coordinates for Rotation

Analogous to the power series expansion of the exponential e^a with scalar a, the matrix exponential for a matrix exponent $\mathbf{A}$ is defined as:

$$\begin{aligned} e^{\mathbf{A}} &= \sum_{i=1}^{\infty} \frac{1}{i!}\mathbf{A}^i \\ &= \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \frac{1}{4!}\mathbf{A}^4 + \frac{1}{5!}\mathbf{A}^5 + \frac{1}{6!}\mathbf{A}^6 + \frac{1}{7!}\mathbf{A}^7 + \cdots \end{aligned} \tag{3.86}$$

Consider $\mathbf{A} = \theta\mathbf{S}(\mathbf{k})$ as the 3-by-3 matrix derived from the angle-axis formula. Then

$$e^{\theta\mathbf{S}(\mathbf{k})} = \mathbf{I} + \theta\mathbf{S}(\mathbf{k}) + \frac{\theta^2}{2!}\mathbf{S}(\mathbf{k})^2 + \frac{\theta^3}{3!}\mathbf{S}(\mathbf{k})^3 + \frac{\theta^4}{4!}\mathbf{S}(\mathbf{k})^4 + \frac{\theta^5}{5!}\mathbf{S}(\mathbf{k})^5 + \frac{\theta^6}{6!}\mathbf{S}(\mathbf{k})^6 + \frac{\theta^7}{7!}\mathbf{S}(\mathbf{k})^7 + \cdots$$

This infinite series can be simplified by the relation $\mathbf{S}(\mathbf{k})^3 = -\mathbf{S}(\mathbf{k})$, proven by simplifying the equivalent triple cross product. Applying to a dummy vector $\mathbf{p}$ and noting that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$,

$$
\begin{aligned}
\mathbf{S}(\mathbf{k})^3\mathbf{p} &= \mathbf{k}\times(\mathbf{k}\times(\mathbf{k}\times\mathbf{p})) \\
&= \mathbf{k}(\mathbf{k}\cdot(\mathbf{k}\times\mathbf{p})) - (\mathbf{k}\times\mathbf{p})(\mathbf{k}\cdot\mathbf{k}) \\
&= \mathbf{k}(0) - (\mathbf{k}\times\mathbf{p})(1) = -\mathbf{S}(\mathbf{k})\mathbf{p}
\end{aligned}
$$

This simplifies the odd power terms in the series; for example,

$$
\begin{aligned}
\mathbf{S}(\mathbf{k})^5 &= \mathbf{S}(\mathbf{k})^3\mathbf{S}(\mathbf{k})^2 \\
&= -\mathbf{S}(\mathbf{k})\mathbf{S}(\mathbf{k})^2 \\
&= -\mathbf{S}(\mathbf{k})^3 \\
&= \mathbf{S}(\mathbf{k})
\end{aligned}
$$

Similarly, $\mathbf{S}(\mathbf{k})^7 = -\mathbf{S}(\mathbf{k})$, $\mathbf{S}(\mathbf{k})^9 = \mathbf{S}(\mathbf{k})$, etc., showing that the minus sign alternates. For the even powers,

$$
\begin{aligned}
\mathbf{S}(\mathbf{k})^4 &= \mathbf{S}(\mathbf{k})^3\mathbf{S}(\mathbf{k}) \\
&= -\mathbf{S}(\mathbf{k})\mathbf{S}(\mathbf{k}) \\
&= -\mathbf{S}(\mathbf{k})^2 \\
\mathbf{S}(\mathbf{k})^6 &= \mathbf{S}(\mathbf{k})^5\mathbf{S}(\mathbf{k}) \\
&= \mathbf{S}(\mathbf{k})\mathbf{S}(\mathbf{k}) \\
&= \mathbf{S}(\mathbf{k})^2
\end{aligned}
$$

and the minus sign is also alternating. Substituting both results into the series expansion,

$$
e^{\theta\mathbf{S}(\mathbf{k})} = \mathbf{I} + \theta\mathbf{S}(\mathbf{k}) + \frac{\theta^2}{2!}\mathbf{S}(\mathbf{k})^2 - \frac{\theta^3}{3!}\mathbf{S}(\mathbf{k}) - \frac{\theta^4}{4!}\mathbf{S}(\mathbf{k})^2 + \frac{\theta^5}{5!}\mathbf{S}(\mathbf{k}) + \frac{\theta^6}{6!}\mathbf{S}(\mathbf{k})^2 - \frac{\theta^7}{7!}\mathbf{S}(\mathbf{k}) + \cdot \quad (3.87)
$$

Collecting similar terms,

$$
e^{\theta\mathbf{S}(\mathbf{k})} = \mathbf{I} + \mathbf{S}(\mathbf{k})(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots) + \mathbf{S}(\mathbf{k})^2(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots)
$$

where the two series in parentheses are related to the series expansions for sine and cosine:

$$
\begin{aligned}
\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \\
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots
\end{aligned}
$$

Substituting,

$$e^{\theta \mathbf{S}(\mathbf{k})} = \mathbf{I} + \mathbf{S}(\mathbf{k})\mathrm{s}\theta + \mathbf{S}(\mathbf{k})^2(1 - \mathrm{c}\theta) \tag{3.88}$$

This is seen to be equivent to the angle-axis matrix formula (3.42) by substituting that $\mathbf{S}(\mathbf{k})^2 = \mathbf{k}\mathbf{k}^T - \mathbf{I}$:

$$\begin{aligned} e^{\theta \mathbf{S}(\mathbf{k})} &= \mathbf{I} + \mathbf{S}(\mathbf{k})\mathrm{s}\theta + (\mathbf{k}\mathbf{k}^T - \mathbf{I})(1 - \mathrm{c}\theta) \\ &= \mathbf{I}\mathrm{c}\theta + \mathbf{S}(\mathbf{k})\mathrm{s}\theta + \mathbf{k}\mathbf{k}^T(1 - \mathrm{c}\theta) \end{aligned}$$

3.12.2 Exponential Coordinates for Spatial Transformation

This section shows how the matrix exponential can be used to represent a spatial transformation. The generalization of the skew-symmetric matrix $\mathbf{S}(\mathbf{k})$ is called the twist $\boldsymbol{\xi}$ [5, 7]:

$$\boldsymbol{\xi} = \begin{bmatrix} \mathbf{S}(\mathbf{k}) & \mathbf{c} \times \mathbf{k} \\ \mathbf{0}^T & 0 \end{bmatrix} \tag{3.89}$$

which is a 4-by-4 matrix. The matrix exponential in terms of $\theta\boldsymbol{\xi}$ is:

$$e^{\theta \boldsymbol{\xi}} = \mathbf{I}_4 + \theta\boldsymbol{\xi} + \frac{\theta^2}{2!}\boldsymbol{\xi}^2 + \frac{\theta^3}{3!}\boldsymbol{\xi}^3 + \frac{\theta^4}{4!}\boldsymbol{\xi}^4 + \frac{\theta^5}{5!}\boldsymbol{\xi}^5 + \frac{\theta^6}{6!}\boldsymbol{\xi}^6 + \frac{\theta^7}{7!}\boldsymbol{\xi}^7 + \cdots \tag{3.90}$$

where $\mathbf{I}_4$ is the 4-by-4 identity matrix, to distinguish it from the 3-by-3 identity matrix $\mathbf{I}$. As before, higher powers of $\boldsymbol{\xi}$ simplify. For $\boldsymbol{\xi}^2$,

$$\begin{aligned} \boldsymbol{\xi}\boldsymbol{\xi} &= \begin{bmatrix} \mathbf{S}(\mathbf{k}) & \mathbf{c} \times \mathbf{k} \\ \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{S}(\mathbf{k}) & \mathbf{c} \times \mathbf{k} \\ \mathbf{0}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{S}(\mathbf{k})^2 & \mathbf{S}(\mathbf{k})(\mathbf{c} \times \mathbf{k}) \\ \mathbf{0}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{S}(\mathbf{k})^2 & \mathbf{k} \times (\mathbf{c} \times \mathbf{k}) \\ \mathbf{0}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{S}(\mathbf{k})^2 & \mathbf{c} \\ \mathbf{0}^T & 0 \end{bmatrix} \end{aligned}$$

where the double cross product $\mathbf{k} \times (\mathbf{c} \times \mathbf{k}) = \mathbf{c}$ because $\mathbf{c} \perp \mathbf{k}$ by construction. For $\boldsymbol{\xi}^3$,

$$\begin{aligned}
\boldsymbol{\xi}^3 &= \boldsymbol{\xi}\boldsymbol{\xi}^2 \\
&= \begin{bmatrix} \mathbf{S}(\mathbf{k}) & \mathbf{c}\times\mathbf{k} \\ \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{S}(\mathbf{k})^2 & \mathbf{c} \\ \mathbf{0}^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{S}(\mathbf{k})^3 & \mathbf{S}(\mathbf{k})\mathbf{c} \\ \mathbf{0}^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{S}(\mathbf{k})^3 & -\mathbf{c}\times\mathbf{k} \\ \mathbf{0}^T & 0 \end{bmatrix}
\end{aligned}$$

where the cross product $\mathbf{k}\times\mathbf{c}$ has been reversed to be analogous to the form for $\boldsymbol{\xi}$. The next two terms $\boldsymbol{\xi}^4$ and $\boldsymbol{\xi}^5$ continue the trend but alternate signs in the vector position from $\boldsymbol{\xi}^2$ and $\boldsymbol{\xi}^3$ respectively::

$$\begin{aligned}
\boldsymbol{\xi}^4 &= \boldsymbol{\xi}\boldsymbol{\xi}^3 \\
&= \begin{bmatrix} \mathbf{S}(\mathbf{k}) & \mathbf{c}\times\mathbf{k} \\ \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{S}(\mathbf{k})^3 & -\mathbf{c}\times\mathbf{k} \\ \mathbf{0}^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{S}(\mathbf{k})^4 & -\mathbf{c} \\ \mathbf{0}^T & 0 \end{bmatrix} \\
\boldsymbol{\xi}^5 &= \boldsymbol{\xi}\boldsymbol{\xi}^4 \\
&= \begin{bmatrix} \mathbf{S}(\mathbf{k}) & \mathbf{c}\times\mathbf{k} \\ \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{S}(\mathbf{k})^4 & -\mathbf{c} \\ \mathbf{0}^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{S}(\mathbf{k})^5 & \mathbf{c}\times\mathbf{k} \\ \mathbf{0}^T & 0 \end{bmatrix}
\end{aligned}$$

The alternating sign trend continues indefinitely. Looking at the upper left 3-by-3 matrices in the series (3.90), the result $e^{\theta\mathbf{S}(\mathbf{k})}$ is the same as (3.87). For the upper right vector in the series, separate the even powers of $\boldsymbol{\xi}$ from the odd powers, and use the series relations for sine and cosine as before:

$$\mathbf{c}(1+\frac{\theta^2}{2!}-\frac{\theta^4}{4!}+\frac{\theta^6}{6!}+\cdots) = (1-\mathrm{c}\theta)\mathbf{c}$$

$$\mathbf{c}\times\mathbf{k}(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots) = \mathrm{s}\theta(\mathbf{c}\times\mathbf{k})$$

Substituting these results,

$$e^{\theta\boldsymbol{\xi}} = \begin{bmatrix} e^{\theta\mathbf{S}(\mathbf{k})} & (1-\mathrm{c}\theta)\mathbf{c} + \mathrm{s}\theta(\mathbf{c}\times\mathbf{k}) \\ \mathbf{0}^t & 1 \end{bmatrix} \tag{3.91}$$

From the angle-axis formula (3.37) with $\mathbf{c}$ substituted for $\mathbf{v}$ and $\mathbf{R}_k(\theta)\mathbf{c}$ for $\mathbf{v}'$,

$$\begin{aligned} (1-\mathrm{c}\theta)\mathbf{c} + \mathrm{s}\theta(\mathbf{c}\times\mathbf{k}) &= \mathbf{c} - \mathbf{R}_k(\theta)\mathbf{c} \\ &= \mathbf{d} \end{aligned}$$

where (3.72) has been substituted. Substituting for the interorigin vector $\mathbf{d}$ from (3.85), a form equivalent to Rodrigues' equation (3.84) is obtained:

$$e^{\theta\boldsymbol{\xi}} = \begin{bmatrix} e^{\theta\mathbf{k}} & (\mathbf{I} - e^{\theta\mathbf{k}})\mathbf{c} + p\theta\mathbf{k} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{3.92}$$

Bibliography

[1] Angeles, J., *Fundamentals of Robotic Mechanical Systems*. New York: Springer, 1997.

[2] Funda, J., and Paul, R.P., "A computational analyis of screw transformations in robotics," *IEEE Trans. Automation and Robotics*, vol. 6, pp. 357-367, 1990.

[3] Funda, J., Taylor, R.H., and Paul, R.P., "On homogeneous transforms, quaternions, and computational efficiency," *IEEE Trans. Automation and Robotics*, vol. 6, pp. 382-388, 1990.

[4] McCarthy, J.M., *Introduction to Theoretical Kinematics*. Cambridge, MA: The MIT Press, 1990.

[5] Murray, R.M., Li, Z., and Sastry, S.S., *A Mathematical Introduction to Robotic Manipulation*. Boca Raton, FL: CRC Press, 1994.

[6] Salamin, E., "Application of quaternions to computation with rotations," Internal Working Paper, Stanford AI Lab, 1979.

[7] Waldron, K., and Schmiedeler, J., "Kinematics," *Springer Handbook of Robotics*, edited by B. Siciliano and O. Khatib. Springer, pp. 9-33, 2008.