1. (24 pts) Consider the homogeneous transformations

$$^0\mathbf{T}_1 = \mathbf{Trans}([1\ \ -1]^T)\ \mathbf{Rot}(\pi) \qquad ^1\mathbf{T}_2 = \mathbf{Trans}([-1\ \ -2]^T)\ \mathbf{Rot}(-\pi/2)$$

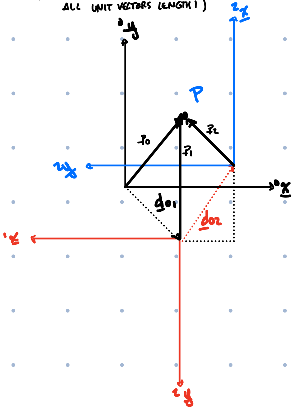
(a) (8pts) Draw a diagram to scale showing the relative locations of frames 0, 1, and 2.

(b) (4pts) Suppose point P is fixed in coordinate system 2, and is located by $\mathbf{p}_2 = P - O_2$. Suppose $^2\mathbf{p}_2 = [1\ 1]^T$; draw $\mathbf{p}_2$ in your diagram. Find $^1\mathbf{p}_2$ and $^0\mathbf{p}_2$.

(c) (6pts) Suppose $\mathbf{p}_1 = P - O_1$. Draw $\mathbf{p}_1$, and find $^2\mathbf{p}_1$, $^1\mathbf{p}_1$, and $^0\mathbf{p}_1$.

(d) (6pts) Suppose $\mathbf{p}_0 = P - O_0$. Draw $\mathbf{p}_0$, and find $^2\mathbf{p}_0$, $^1\mathbf{p}_0$, and $^0\mathbf{p}_0$.

A) (NOT DRAWN TO SCALE, ALL UNIT VECTORS LENGTH 1)



$\underline{d}_{01} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$\underline{d}_{12} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

B) $\underline{p}_2 = P - O_2$

$^1\underline{p}_2 =$

$R(\pi) = \begin{bmatrix} C\theta & -S\theta \\ S\theta & C\theta \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$^1\underset{\sim}{T}_0 = \begin{bmatrix} \underset{\sim}{R}(\pi) & {}^0\underline{d}_{01} \\ \underline{0}^T & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$^1\underset{\sim}{R}_2 = \underset{\sim}{R}\left(\frac{-\pi}{2}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$^1\underset{\sim}{T}_2 = \begin{bmatrix} ^1\underset{\sim}{R}_2 & ^1\underline{d}_{12} \\ \underline{0}^T & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

B) $\underline{p}_2 = P - O_2 = {}^2\underline{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$^0\underset{\sim}{T}_2 = {}^0\underset{\sim}{T}_1\ {}^1\underset{\sim}{T}_2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} =$

$^1\underline{p}_2 = {}^1\underset{\sim}{R}_2\ {}^2\underline{p}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$^0\underline{p}_2 = {}^0\underset{\sim}{R}_2\ {}^2\underline{p}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{^1\underline{p}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad ^0\underline{p}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

C) $\underline{p}_1 = P - O_1 \qquad {}^2\underline{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$^1\underline{p}_1 = {}^1\underline{d}_{12} + {}^1\underset{\sim}{R}_2\,{}^2\underline{p}_2$

$= \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & +1 \\ -2 & -1 \end{bmatrix}$

$\boxed{^1\underline{p}_1 = \begin{bmatrix} 0 \\ -3 \end{bmatrix}}$

$^1\underset{\sim}{R}_2^T = {}^2\underset{\sim}{R}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$^0\underline{p}_1 = {}^0\underset{\sim}{R}_1\ {}^1\underline{p}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$^2\underline{p}_1 = {}^2\underset{\sim}{R}_1\ {}^1\underline{p}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$\boxed{^0\underline{p}_1 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \qquad ^2\underline{p}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}}$

D)

$$ {}^{0}\underline{p}_0 = {}^{0}\underline{d}_{02} + {}^{0}\underset{\sim}{R}_2 \, {}^{2}\underline{p}_2 \qquad \Rightarrow \; {}^{0}\underline{d}_{02} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} $$

$$ = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + {}^{0}\underset{\sim}{R}_1 \, {}^{1}\underset{\sim}{R}_2 \, {}^{2}\underline{p}_2 $$

$$ = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} $$

$$ \boxed{ {}^{0}\underline{p}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} } $$

$$ {}^{0}\underset{\sim}{R}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} $$

$$ {}^{1}\underset{\sim}{R}_0 = {}^{0}\underset{\sim}{R}_1^{\,T} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} $$

$$ {}^{2}\underset{\sim}{R}_0 = {}^{0}\underset{\sim}{R}_2^{\,T} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} $$

$$ {}^{1}\underline{p}_0 = {}^{1}\underset{\sim}{R}_0 \, {}^{0}\underline{p}_2 $$

$$ = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} $$

$$ \boxed{ {}^{1}\underline{p}_0 = \begin{bmatrix} -1 \\ -2 \end{bmatrix} } $$

$$ {}^{2}\underline{p}_0 = {}^{2}\underset{\sim}{R}_0 \, {}^{0}\underline{p}_0 $$

$$ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} $$

$$ \boxed{ {}^{2}\underline{p}_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} } $$

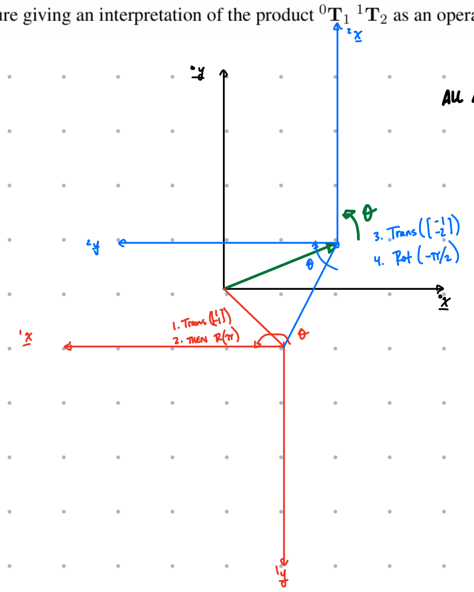
2. (20pts) Again consider the homogeneous transformations

$$^0\mathbf{T}_1 = \mathbf{Trans}([1\ \ -1]^T)\ \mathbf{Rot}(\pi) \qquad {}^1\mathbf{T}_2 = \mathbf{Trans}([-1\ \ -2]^T)\ \mathbf{Rot}(-\pi/2)$$

(a) (10pts) Show a figure giving an interpretation of the product $^0\mathbf{T}_1\ {}^1\mathbf{T}_2$ as a coordinate transformation.

(b) (10pts) Show a figure giving an interpretation of the product $^0\mathbf{T}_1\ {}^1\mathbf{T}_2$ as an operator.

A)



ALL AXES ARE UNIT VECTORS (NOT SCALE)

EXTRINSIC = ROTATE

CURRENT FRAME

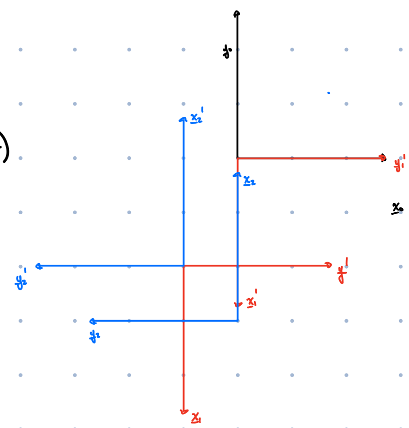
TRANSFORM = FIXED (TRANSLATION → ROTATE)

OPERATOR = NOT FIXED

FIXED FRAME

B) EQUIVALENT OPERATORS

1. Rot(-π/2)



NOT DRAWN TO SCALE (UNIT VECTORS)

1. Rot(-π/2)
2. Trans([-2 -1])
3. Rot(π)
4. Trans([1 -1])

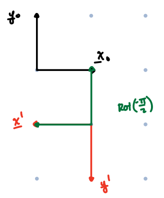
3. (10pts) Again consider the transformation ${}^0\mathbf{T}_1$ from above:

$$\underline{{}^0\mathbf{T}_1 = \mathbf{Trans}([1\ \ -1]^T)\ \mathbf{Rot}(\pi)}$$

but now consider that

$$\underline{\mathbf{D}_1 = \mathbf{Trans}([-1\ \ -2]^T)\ \mathbf{Rot}(-\pi/2)}$$

is an operator in frame 1. Find the equivalent operator $\mathbf{D}_0$ in frame 0.



$$\underset{\sim}{R}(\pi) = \begin{bmatrix} C\theta & -S\theta \\ S\theta & C\theta \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$${}^0\underset{\sim}{T}_1 = \begin{bmatrix} \underset{\sim}{R}(\pi) & {}^0\underset{\sim}{d}_{01} \\ \underset{\sim}{0}^T & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det\left({}^0\underset{\sim}{T}_1\right) = -1(-1+0) - 0(0) + 1(0)$$
$$= 1$$

$$\rightsquigarrow \left({}^0\underset{\sim}{T}_1\right)^{-1} = \frac{1}{1} \begin{bmatrix} \begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \end{bmatrix}^T$$

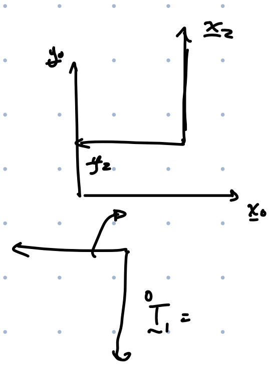
$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}^T$$

$$\left({}^0\underset{\sim}{T}_1\right)^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{D}_1 = \text{Trans}\left(\begin{bmatrix} -1 \\ -2 \end{bmatrix}\right) \text{Rot}\left(\frac{-\pi}{2}\right)$$

$$\underset{\sim}{R}\left(\frac{-\pi}{2}\right) = \begin{bmatrix} C\theta & -S\theta \\ S\theta & C\theta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\underset{\sim}{D}_1 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$



$${}^0\underset{\sim}{T}_1 =$$

$$\underset{\sim}{D}_0 = {}^0\underset{\sim}{T}_1\,\underset{\sim}{D}_1\left({}^0\underset{\sim}{T}_1\right)^{-1}$$

$$= \underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{=}\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0, & -1, & 1+1 \\ 1, & 0, & 2-1 \\ 0, & 0, & 1 \end{bmatrix}$$

$$\underset{\sim}{D}_0 = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1+2 \\ -1 & 0 & 1+1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\underset{\sim}{D}_0 = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$$

4. (42pts) This question concerns the concatenation of two polynomial trajectories $x_1(t)$ and $x_2(t)$.

- Polynomial $x_1(t)$ is valid in the time range $0 \leq t \leq t_1$. At $t = 0$, the position and velocity are zero.
- Polynomial $x_2(t)$ is valid in the time range $t_1 \leq t \leq t_2$. At $t = t_2$, the position is $x2$, and the velocity is zero.
- At $t = t_1$, the two polynomials have the same position, velocity and acceleration.

(a) (8pts) Identify the constraints at time 0, t_1 and t_2.

(b) (6pts) What are the minimal degrees of the two polynomials? Write their equations, and first and second derivatives.

PART B

(c) (5pts) Substitute the unary constraints into the equations above. Identify which lead immediately to coefficient solutions.

(d) (3pts) Now substitute the above polynomial equations into the binary constraints.

(e) (16pts) Find the coefficients of polynomial 2.

(f) (4pts) Find the coefficients of polynomial 1.

1 of 2

2 CONSTRAINTS

A) $x_1(0) = 0$ $\quad x_1(t_1) = x_2(t_1)$ $\quad x_2(t_2) = x_2$

$\dot{x}_1(0) = 0$ $\quad \dot{x}_1(t_1) = \dot{x}_2(t_1)$ $\quad \dot{x}_2(t_2) = 0$

$\ddot{x}_1(t_1) = \ddot{x}_2(t_1)$

B) 7 CONSTRAINTS $\quad n$ = DEGREE OF POLYNOMIAL

$\Rightarrow$ x_1 IS $n=4$ $\quad x_2$ IS $n=2$

WHAT WE'LL CHOOSE $\longrightarrow$ x_1 IS $n=3$ $\quad x_2$ IS $n=2$

SYSTEM OF EQNS

$$\begin{cases} x_1(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 & \dot{x}_1(t) = a_1 + 2a_2 t + 3a_3 t^2 & \ddot{x}_1(t) = 2a_2 + 6a_3 t \\ x_2(t) = b_0 + b_1(t - t_1) + b_2(t - t_1)^2 & \dot{x}_2(t) = b_1 + 2b_2(t - t_1) & \ddot{x}_2(t) = 2b_2 \end{cases}$$

t_1 = CONST

$\Delta t = t - t_1$

c) UNARY CONSTRAINT @ $x_1(0)$, $\dot{x}_1(0)$ (WE HAVE A VALUE)

$x_1(0) = 0 = a_0 + a_1 t + a_2 t^2 + a_3 t^3 = a_0$

$\dot{x}_1(0) = 0 = a_1 + 2a_2 t + 3a_3 t^2 = a_1$

$x_2(t_2) = \underline{X2 = b_0 + b_1(t_2 - t_1) + b_2(t_2 - t_1)^2}$ ①

$\dot{x}_2(t_2) = 0 = b_1 + 2b_2(t_2 - t_1) \rightsquigarrow \underline{b_1 = 2b_2(t_2 - t_1)}$ ②

$\Rightarrow \boxed{a_0 = 0 \quad a_1 = 0} \Rightarrow$

$x_1(t) = a_2 t^2 + a_3 t^3$

$\dot{x}_1(t) = 2a_2 t + 3a_3 t^2$

$\ddot{x}_1(t) = 2a_2 t + 6a_3 t$

D) BINARY CONSTRAINTS (NO IDEA WHAT VALUE IS)

$\Delta t = (t - t_1)$

$x_1(t_1) = x_2(t_1) \rightsquigarrow$ $\quad a_2 t_1^2 + a_3 t_1^3 = b_0 + b_1 \cancel{\Delta t}^0 + b_2 \cancel{\Delta t}^0$

$\dot{x}_1(t_1) = \dot{x}_2(t_1) \rightsquigarrow$ $\quad 2a_2 t_1 + 3a_3 t_1^2 = b_1 + 2b_2 \cancel{\Delta t}^0$

$\ddot{x}_1(t_1) = \ddot{x}_2(t_1) \rightsquigarrow$ $\quad 2a_2 + 6a_3 t_1 = 2b_2$

$\rightsquigarrow$

③ $a_2 t_1^2 + a_3 t_1^3 = b_0$

④ $2a_2 t_1 + 3a_3 t_1^2 = b_1$

⑤ $2a_2 + 6a_3 t_1 = 2b_2$

× -3 and $-3a_2 t_1^2 - 3a_3 t_1^2 = -3b_0$

⑥ $b_1 t_1 - 3b_0 = -a_2 t_1^2$

6 EQNS 6 UNKWN

E) IN TERMS OF b_2...

F)

$$\boxed{\begin{array}{ll} \text{FROM ②:} & b_1 = 2b_2(t_2 - t_1) \\ ① & b_0 = X2 - b_1(t_2 - t_1) + b_2(t_2 - t_1) \\ ⑥ & a_2 = \dfrac{3b_0 - b_1 t_1}{t_1^2} \\ ③ & a_3 = \dfrac{b_0 - a_2 t_1^2}{t_1^3} \\ & a_1 = 0 \\ & a_0 = 0 \end{array}}$$