Chapter 2

Planar Transformations and Displacements

Kinematics is concerned with the properties of the motion of points. These points are on objects in the environment or on a robot manipulator. Two features that make kinematics difficult are (1) rotations do not commute, and (2) the mapping between joint angles and manipulator endpoint position is nonlinear. To begin, one needs to establish a formal method for describing points and their displacements; this is done by defining coordinate systems and frames. A basic distinction is between points and vectors, treated next. Another distinction is between coordinate transformations and displacements: the former describe the relationship between fixed points, the latter describe the motion of a point.

In this chapter, we will develop the concepts of coordinate transformations and displacements in the plane; subsequent chapters will treat the 3D case. The basic concepts are most easily introduced for the plane. First we consider rotations alone, then rotations plus translations. A displacement will be distinguished from a coordinate transformation. Finally, we will discuss the pole of a planar displacement.

2.1 Points and Vectors

Points and vectors are not the same, although they are intimately related. A *point* is a geometrical object, i.e., some definite location in the environment or on a manipulator, that exists independently of how we describe it.

- Points belong to an *affine space*, and are operated on by *affine transformations* such as translation and rotation.
- Points are denoted by upper-case italic letters with subscripts, e.g., O_j, P_j, etc. The subscript j just identifies a particular instance.

A *vector* represents a displacement between two points; e.g., given points P_1 and P_2, the vector $\mathbf{v} = P_2 - P_1$ defines the distance and direction from P_1 to P_2. Figure 2.1(A) represents a table top or plane, where distinctive points P_1 and P_2 have been identified. The vector $\mathbf{v}$ lies on the plane of the table.

- Vectors belong to a *vector space,* and are described by coordinates relative to particular coordinate axes.
- Vectors are denoted by lower-case bold letters with subscripts and superscripts, e.g., ${}^i\mathbf{v}_j$. The superscript i denotes the particular coordinate axes i with respect to which the coordinates of $\mathbf{v}_j$ are expressed.

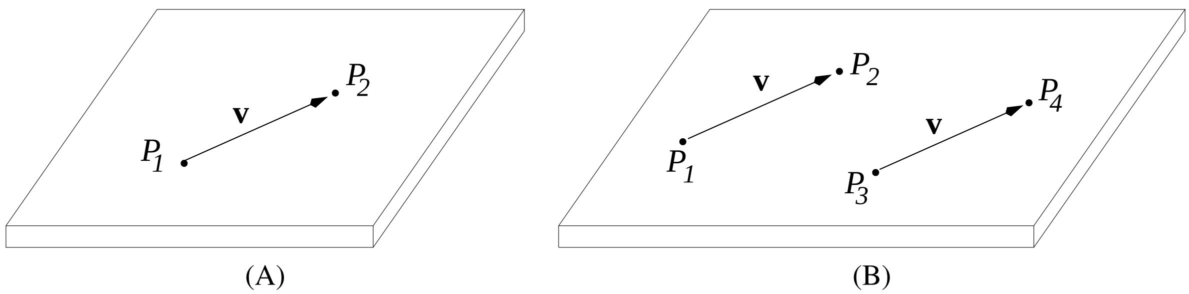

Figure 2.1: (A) Points P_1 and P_2 on a table top and the displacement vector $\mathbf{v}$ between them. (B) The same vector $\mathbf{v}$ describes the displacement from different points P_3 to P_4.

Next we provide more detail on affine spaces and vector spaces.

2.1.1 Affine Spaces

As mentioned earlier, points are geometric objects that are elements of an affine space A. Points are not vectors, because there is no preferred origin. For example, addition of points makes no sense. In Figure 2.2(A) a different result would be obtained in adding points P_1 and P_2 if expressed relative to different origins O_0 and O_1. However, subtraction of points makes sense, and gives rise to an associated vector space $\mathcal{V}$. For example, from Figure 2.1(A):

$$\mathbf{v} = P_2 - P_1 \tag{2.1}$$

where vector $\mathbf{v} \in \mathcal{V}$ is the displacement from P_1 to P_2. We may also state that P_2 is obtained from P_1 by adding a displacement $\mathbf{v}$:

$$P_2 = P_1 + \mathbf{v} \tag{2.2}$$

The formal definition of an affine space can now be stated: it consists of a set of points, an associated vector space, and the subtraction and addition operators above.

There are infinitely many point pairs whose relative displacement is the same vector $\mathbf{v}$. In Figure 2.1(B), the displacement from P_3 to P_4 is the same as from P_1 to P_2, i.e., the vector $\mathbf{v}$. Another way of thinking about a vector is that it encodes a length or distance and a direction.

Example 2.1: Suppose you are giving a person directions from an originating point, symbolically represented as P_1, to a destination point P_2. An example might be "go 100 meters to the northeast." If a person were starting out at a different originating point P_3, giving the same directions would cause the person to end up at a different point P_4. But in both cases, the same distance and direction were specified.

Thus a vector is not a geometrical object by itself, i.e., it is not tied to any point. Only when added to a preexisting point is there an association with a unique point. More formally, the length of a vector $\mathbf{v}$ is denoted as $||\mathbf{v}||$, the Euclidean norm of $\mathbf{v}$ (discussed later). The unit vector $\mathbf{u} = \mathbf{v}/||\mathbf{v}||$ represents the direction.

Often we will specify points in terms of displacements from a fixed point O_i called *origin i.* For example, in Figure 2.2(B) point P_1 is displaced from the origin O_0 by $\mathbf{v}_1$, and P_2 is displaced from P_1 by $\mathbf{v}_2$. Then P_2 is displaced from O_0 by $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. More generally, $(P + \mathbf{v}_1) + \mathbf{v}_2 = P + (\mathbf{v}_1 + \mathbf{v}_2)$, and $P + \mathbf{v} = P$ if and only if (iff) $\mathbf{v} = \mathbf{0}$.

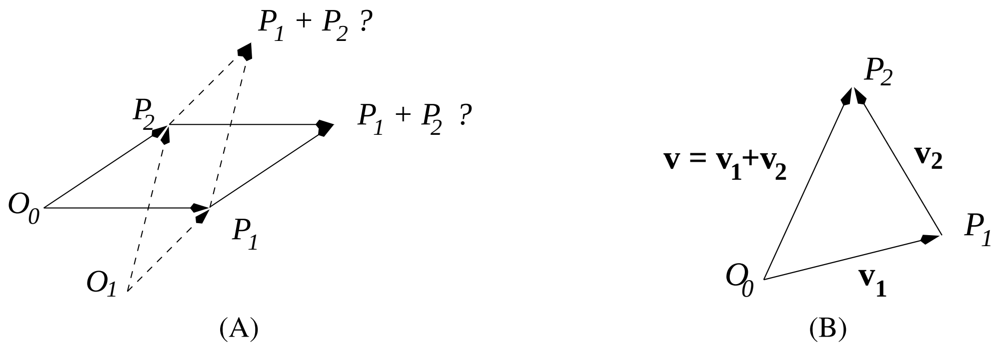

Figure 2.2: (A) Adding points is not defined. (B) Displacement of points P_1 and P_2 relative to O_0.

2.1.2 Vector Spaces

A vector space is composed of vectors, whose linear combinations are also vectors. Vectors in a two-dimensional (2D) space, i.e., the plane, have two components, while vectors in a 3D space have three components. We will write respectively that $\mathbf{v} \in \mathcal{R}^2$ or $\mathbf{v} \in \mathcal{R}^3$ depending on the context. Given vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in an n-dimensional space $\mathcal{R}^n$, then $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ is also a vector in $\mathcal{R}^n$, where a_1 and a_2 are arbitrary scalars.

- Scalars are repesented by subscripted lower-case italic letters, e.g., a_j, b_j, etc.

Figure 2.2(B) shows the familiar triangle which results from the addition of two vectors: $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, and $a_1 = a_2 = 1$.

Vectors are usually expressed in terms of other vectors, which form a *basis* for the vector space. A basis is a set of vectors that spans the vector space, that is to say, any vector can be expressed as a linear combination of the basis vectors. There are as many vectors in a basis as the dimension of the space. For the plane, i.e., the vector space $\mathcal{R}^2$, any two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ that are not parallel can form a basis. Thus any vector $\mathbf{v}$ in the plane can be represented as:

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \tag{2.3}$$

for some unique scalars a_1 and a_2.

2.1.3 Orthonormal Bases

Of all possible bases, the most useful ones are *orthonormal bases*, i.e., bases in which the vectors are orthogonal and are unit vectors. Orthogonal vectors in a plane are at right angles; the defining condition is that their dot product is zero. We will denote the two vectors in an orthonormal basis j as $\mathbf{x}_j$ and $\mathbf{y}_j$, where the right-hand rule determines that the angle from $\mathbf{x}_j$ to $\mathbf{y}_j$ is $+\pi/2$ radians. A vector $\mathbf{v}$ in terms of the orthonormal basis j will then be written as:

$$\mathbf{v} = {}^j v_1\ \mathbf{x}_j + {}^j v_2\ \mathbf{y}_j \tag{2.4}$$

The unique scalars ${}^j v_1$ and ${}^j v_2$ are the *coordinates* of $\mathbf{v}$ with respect to orthonormal basis j.

The superscript j on the scalar multipliers indicates which orthonormal basis is the reference. For a different orthonormal basis k, the same vector will have different coordinates:

$$\mathbf{v} = {}^k v_1\ \mathbf{x}_k + {}^k v_2\ \mathbf{y}_k \tag{2.5}$$

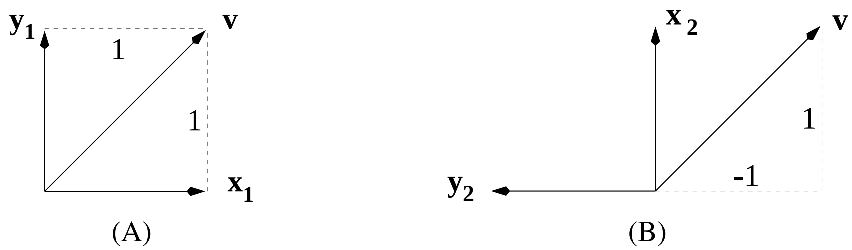

Figure 2.3: (A) The vector $\mathbf{v}$ expressed in basis 1. (B) The same vector expressed in basis 2.

The basis k coordinates ${}^k v_1$ and ${}^k v_2$ will be different from the basis j coordinates because the basis vectors are different. A key point is that the vector $\mathbf{v}$ is the same in both cases, only its description in terms of a particular basis differs.

Example 2.2: In Figure 2.3(A), a particular vector $\mathbf{v}$ has been constructed such that $\mathbf{v} = \mathbf{x}_1 + \mathbf{y}_1$ with respect to orthonormal basis 1. Hence its coordinates are ${}^1 v_1 = 1$ and ${}^1 v_2 = 1$. In Figure 2.3(B), there is a different orthonormal basis 2 that is rotated 90 degrees from basis 1. Then $\mathbf{v} = \mathbf{x}_2 - \mathbf{y}_2$, and its associated coordinates are ${}^2 v_1 = 1$ and ${}^2 v_2 = -1$.

Just like any other vectors, the basis j vectors can be represented in terms of some other basis, say k. Then

$$\begin{aligned} \mathbf{x}_j &= {}^k x_{j1}\, \mathbf{x}_k + {}^k x_{j2}\, \mathbf{y}_k \\ \mathbf{y}_j &= {}^k y_{j1}\, \mathbf{x}_k + {}^k y_{j2}\, \mathbf{y}_k \end{aligned} \tag{2.6}$$

where the double subscripts $j1$ and $j2$ refer to the first and second elements of the coordinates of the respective basis j vectors. Thus ${}^k x_{j1}$ and ${}^k x_{j2}$ are the coordinates of $\mathbf{x}_j$, and ${}^k y_{j1}$ and ${}^k y_{j2}$ are the coordinates of $\mathbf{y}_j$, with respect to basis k.

Example 2.3: For the simple case of relative rotation by 90^o as in Figure 2.3, we can see by inspection that $\mathbf{x}_2 = \mathbf{y}_1$ and $\mathbf{y}_2 = -\mathbf{x}_1$. Hence the coordinates of $\mathbf{x}_2$ are ${}^1 x_{21} = 0$ and ${}^1 x_{22} = 1$, and the coordinates of $\mathbf{y}_2$ are ${}^1 y_{21} = -1$ and ${}^1 y_{22} = 0$, with respect to basis 1.

What are the coordinates of the basis vectors with respect to their own basis? Clearly

$$\begin{aligned} \mathbf{x}_j &= 1 \cdot \mathbf{x}_j + 0 \cdot \mathbf{y}_j \\ \mathbf{y}_j &= 0 \cdot \mathbf{x}_j + 1 \cdot \mathbf{y}_j \end{aligned} \tag{2.7}$$

Hence ${}^j x_{j1} = 1$ and ${}^j x_{j2} = 0$ are the coordinates of $\mathbf{x}_j$, and ${}^j y_{j1} = 0$ and ${}^j y_{j2} = 1$ are the coordinates of $\mathbf{y}_j$, with respect to basis j.

When actually evaluating any vector equation, for example (2.4), all vectors have to be expressed with respect to the same basis. We did not state what basis the vectors $\mathbf{x}_j$ and $\mathbf{y}_j$ in (2.4) were being referred to. We explicitly indicate the reference basis, say k, with respect to which all vectors are being expressed by a left superscript on the vectors:

$$ {}^k\mathbf{v} = {}^j v_1\, {}^k\mathbf{x}_j + {}^j v_2\, {}^k\mathbf{y}_j \tag{2.8}$$

Note that the j coordinates did not change, only the expression of the three vectors.

The simplest case to evaluate is when $j = k$. For this case, the coordinates of a vector will be written as a column vector. First, consider the coordinates of basis vectors with respect to themselves, rewritten from (2.7):

$$^j\mathbf{x}_j = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{i}, \qquad ^j\mathbf{y}_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{j} \tag{2.9}$$

This familiar result states that the coordinates of the $\mathbf{x}$ and $\mathbf{y}$ axes are $[1\ \ 0]^T$ and $[0\ \ 1]^T$ respectively, also conveniently abbreviated as $\mathbf{i}$ and $\mathbf{j}$. This is true of any set of basis vectors expressed with respect to themselves. Thus for a different basis k, then $^k\mathbf{x}_k = \mathbf{i}$ and $^k\mathbf{y}_k = \mathbf{j}$. Note that when a set of orthonormal basis vectors is expressed in terms of a different basis, for example $^k\mathbf{x}_j$ and $^k\mathbf{y}_j$, they are no longer $\mathbf{i}$ and $\mathbf{j}$.

Next, the coordinates of an arbitrary vector expressed in basis j will be written as:

$$^j\mathbf{v} = {}^jv_1\,{}^j\mathbf{x}_j + {}^jv_2\,{}^j\mathbf{y}_j = \begin{bmatrix} ^j\mathbf{x}_j & ^j\mathbf{y}_j \end{bmatrix} \begin{bmatrix} ^jv_1 \\ ^jv_2 \end{bmatrix} = \begin{bmatrix} ^jv_1 \\ ^jv_2 \end{bmatrix} \tag{2.10}$$

where $[^j\mathbf{x}_j\ \ ^j\mathbf{y}_j] = [\mathbf{i}\ \ \mathbf{j}] = \mathbf{I}$ is the 2-by-2 identity matrix. Thus the vector $^j\mathbf{v}$ referred to basis j is represented as a 2-by-1 column vector of its coordinates. (Higher-dimensional vectors will also be represented as column vectors.)

Example 2.4: From Figure 2.3(A), we would write that

$$^1\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad ^2\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We can also write that $^1\mathbf{v} = [1\ 1]^T$ and $^2\mathbf{v} = [1\ \ -1]^T$, where $\mathbf{v}^T$ means the transpose of vector $\mathbf{v}$. In this context, the transpose operator converts a 1-by-2 row vector into a 2-by-1 column vector.

We are now in a position to define the length of a vector and the dot product. The dot product of two vectors $^j\mathbf{v}$ and $^j\mathbf{w}$ is:

$$^j\mathbf{v} \cdot {}^j\mathbf{w} = (^j\mathbf{v})^T\ \ ^j\mathbf{w} = \begin{bmatrix} ^jv_1 & ^jv_2 \end{bmatrix} \begin{bmatrix} ^jw_1 \\ ^jw_2 \end{bmatrix} = {}^jv_1\,{}^jw_1 + {}^jv_2\,{}^jw_2 \tag{2.11}$$

This definition of the dot product generalizes to vector spaces of any dimensions. Although the dot product is evaluated with respect to a particular basis, the value of the dot product is the same irrespective of the basis. This is because the dot product represents the projection of $\mathbf{w}$ onto $\mathbf{v}$, and this projection always has the same value. Two vectors $\mathbf{v}$ and $\mathbf{w}$ are *orthogonal* if their dot product is zero: $\mathbf{v} \cdot \mathbf{w} = 0$.

The length $||^j\mathbf{v}||$ of a vector is defined in terms of the dot product:

$$||^j\mathbf{v}|| = \sqrt{^j\mathbf{v} \cdot {}^j\mathbf{v}} = \sqrt{(^jv_1)^2 + (^jv_2)^2} \tag{2.12}$$

The length of a vector does not depend on the basis; it's always the same. Hence $||^j\mathbf{v}|| = ||^k\mathbf{v}||$ for different bases j and k. A *unit vector* has a length of 1: $||^j\mathbf{u}|| = 1$. Any vector can be converted to a unit vector by dividing by its length: $\mathbf{u} = \mathbf{v}/||\mathbf{v}||$.

Example 2.5: For the vector $\mathbf{v}$ of Figure 2.3, we have that $||^1\mathbf{v}|| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $||^2\mathbf{v}|| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. The length is the same.

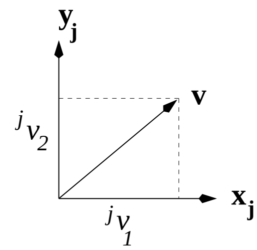

Figure 2.4: The projections of a vector $\mathbf{v}$ onto axes j are its coordinates $^jv_1$ and $^jv_2$.

An *orthonormal* basis $\mathbf{x}_j, \mathbf{y}_j$ consists of orthogonal ($\mathbf{x}_j \cdot \mathbf{y}_j = 0$) and unit ($||\mathbf{x}_j|| = ||\mathbf{y}_j|| = 1$) vectors.

Let's reexamine the meaning of the coordinates of a vector, in view of the dot product and its definition as a projection (Figure 2.4). Take the dot product of (2.8) first with $^k\mathbf{x}_j$, and then with $^k\mathbf{y}_j$:

$$^k\mathbf{v} \cdot \,^k\mathbf{x}_j = {}^jv_1 \,^k\mathbf{x}_j \cdot \,^k\mathbf{x}_j + {}^jv_2 \,^k\mathbf{y}_j \cdot \,^k\mathbf{x}_j = {}^jv_1 \tag{2.13}$$

since $^k\mathbf{x}_j \cdot \,^k\mathbf{x}_j = 1$ and $^k\mathbf{y}_j \cdot \,^k\mathbf{x}_j = 0$. Hence the coordinate $^jv_1$ represents the projection of $\mathbf{v}$ onto $\mathbf{x}_j$, and is independent of the basis k with respect to which the vectors in (2.8) are expressed. Similarly,

$$^k\mathbf{v} \cdot \,^k\mathbf{y}_j = {}^jv_1 \,^k\mathbf{x}_j \cdot \,^k\mathbf{y}_j + {}^jv_2 \,^k\mathbf{y}_j \cdot \,^k\mathbf{y}_j = {}^jv_2 \tag{2.14}$$

since $^k\mathbf{x}_j \cdot \,^k\mathbf{y}_j = 0$ and $^k\mathbf{y}_j \cdot \,^k\mathbf{y}_j = 1$. Hence the coordinate $^jv_2$ is the projection of $\mathbf{v}$ onto $\mathbf{y}_j$.

2.1.4 Coordinate Systems

The combination of an origin O_j with a set of orthonormal basis vectors $\mathbf{x}_j, \mathbf{y}_j$ is called *coordinate system j* or *frame j*. The basis vectors $\mathbf{x}_j, \mathbf{y}_j$ are called the *axes* of coordinate system j. The axes are pictured as displacements from the origin in Figure 2.5(A), to implied points $X_j = O_j + \mathbf{x}_j$ and $Y_j = O_j + \mathbf{y}_j$. Thus a coordinate system really consists of 3 points O_j, X_j, Y_j, and the axes are deduced as $\mathbf{x}_j = X_j - O_j$ and $\mathbf{y}_j = Y_j - O_j$. While these implied points are to be understood, we will usually draw a coordinate system with just the origin and axes indicated (Figure 2.5(B)).

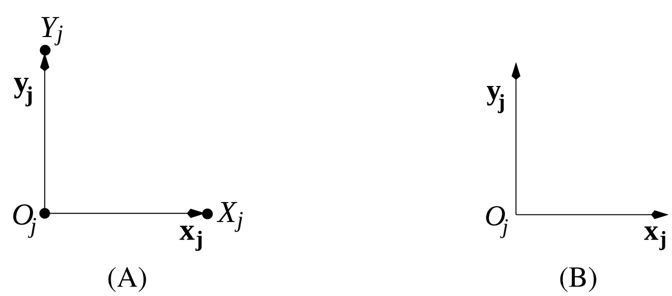

Figure 2.5: (A) A coordinate system j with implied points $X_j = O_j + \mathbf{x}_j$ and $Y_j = O_j + \mathbf{y}_j$. (B) A coordinate system is usually depicted without the implied points.

To describe where objects in the environment are, we will attach coordinate systems to them and relate the various coordinate systems. We will also be attaching coordinate systems to each link of a manipulator, in order to deduce joint angles and to relate the position of a manipulator relative to objects.

How coordinate systems are defined and attached to objects is a matter of convenience. For example, consider the two trapezoids in arbitrary placement in Figure 2.6(A), where the origins have been placed in corresponding corners and the axes on corresponding sides. In this general case, the origins O_1 and O_2 are different, as are the axes $\mathbf{x}_1, \mathbf{y}_1$ and $\mathbf{x}_2, \mathbf{y}_2$. The position and orientation of each trapezoid is considered to be synonymous with the position and orientation of its coordinate system.

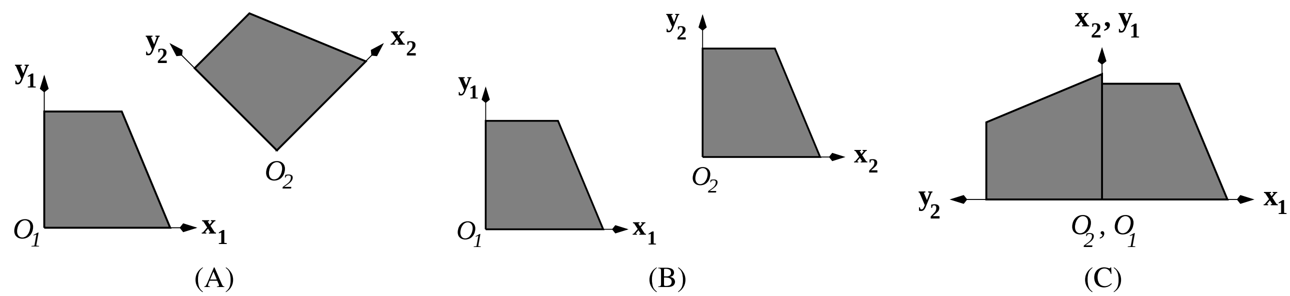

Figure 2.6: (A) Two trapezoids with attached coordinate systems in general position relative to each other. (B) The two trapezoids are oriented the same. (C) The two trapezoids are in the same position and abutted.

In Figure 2.6(B), trapezoid 2 has been rotated such that $\mathbf{x}_1 = \mathbf{x}_2$ and $\mathbf{y}_1 = \mathbf{y}_2$. When the axes are aligned like this, we say that the two trapezoids are in the same orientation; however, they are not in the same position becasue their origins differ. Furthermore, when we say something like $\mathbf{x}_1 = \mathbf{x}_2$, we are not saying that these vectors are pointing to the same point. What we are saying is that the two displacements are the same. In terms of the implied points X_1 and X_2 (not shown, but in the manner of Figure 2.5(A)), we are stating that $X_1 - O_1 = X_2 - O_2$, which are the definitions of $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively.

In Figure 2.6(C), trapezoid 2's origin has been made coincident with trapezoid 1's origin, and trapezoid 2 has been rotated so that $\mathbf{x}_2 = \mathbf{y}_1$. Hence we have abutted the two trapezoids. Their position is now the same because $O_1 = O_2$. Their orientations differ by 90^o, which is the angle from $\mathbf{x}_1$ to $\mathbf{x}_2$.

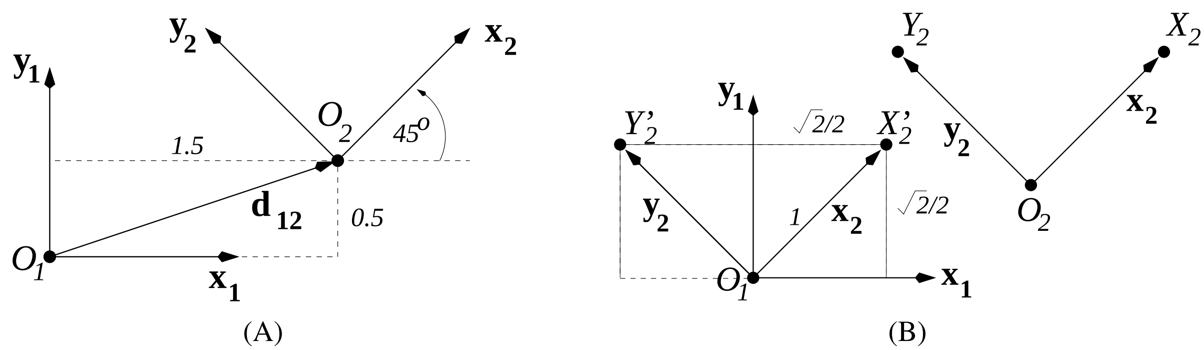

Figure 2.7: (A) Trapezoid 2 is displaced by $(1.5, 0.5)$ from trapezoid 1, and rotated by 45^o. (B) Axes 2 are displaced to origin O_1.

Example 2.6: Suppose in Figure 2.6(A) that axes 2 are rotated by 45^o with repect to axes 1, and that the origin O_2 is displaced from origin O_1 by ${}^1\mathbf{d}_{12} = O_2 - O_1 = 1.5\ {}^1\mathbf{x}_1 + 0.5\ {}^1\mathbf{y}_1 = [1.5\ 0.5]^T$ (see Figure 2.7(A)). The same linear displacement ${}^1\mathbf{d}_{12}$ also applies to Figure 2.6(B). What are the coordinates of $\mathbf{x}_2, \mathbf{y}_2$ with respect to axes 1 in the three figures?

1. The coordinate description for Figure 2.6(C) is the same as for Figure 2.3.

2. For Figure 2.6(B), ${}^1\mathbf{x}_2 = [1\ 0]^T$ and ${}^1\mathbf{y}_2 = [0\ 1]^T$ since axes 2 are parallel to axes 1. Note that the displacement ${}^1\mathbf{d}_{12}$ between origins makes no difference, because we are comparing displacements, not points.

3. The same is true for Figure 2.6(A), where

$$\begin{aligned} {}^1\mathbf{x}_2 &= \sqrt{2}/2\ {}^1\mathbf{x}_1 + \sqrt{2}/2\ {}^1\mathbf{y}_1 &= \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \\ {}^1\mathbf{y}_2 &= -\sqrt{2}/2\ {}^1\mathbf{x}_1 + \sqrt{2}/2\ {}^1\mathbf{y}_1 &= \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \end{aligned} \tag{2.15}$$

   We are not describing where the points represented by $O_2 + \mathbf{x}_2$ and $O_2 + \mathbf{y}_2$ are located relative to O_1. We are describing the displacements $\mathbf{x}_2$ and $\mathbf{y}_2$ relative to the displacements $\mathbf{x}_1$ and $\mathbf{y}_1$. Another way to view this problem is to define points X_2' and Y_2' such that the displacements of each point relative to O_1 are (Figure 2.7(B)):

$$X_2' - O_1 = \mathbf{x}_2, \qquad Y_2' - O_1 = \mathbf{y}_2$$

   Again, we can do this because there are infinitely many pairs of points whose displacement is the same vector. From the construction of Figure 2.7(B) where the various vectors in question all start from O_1, it is clear that ${}^1\mathbf{x}_2 = \sqrt{2}/2\ {}^1\mathbf{x}_1 + \sqrt{2}/2\ {}^1\mathbf{y}_1$ and ${}^1\mathbf{y}_2 = -\sqrt{2}/2\ {}^1\mathbf{x}_1 + \sqrt{2}/2\ {}^1\mathbf{y}_1$. Hence when comparing vectors in coordinate systems with different origins, we can just pretend the origins coincide in the derivation.

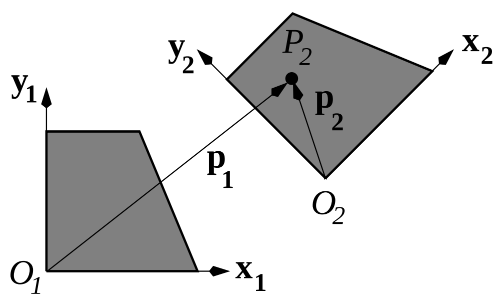

Figure 2.8: An arbitrary point P_2 on trapezoid 2 is located relative to O_2 by $\mathbf{p}_2$ and relative to O_1 by $\mathbf{p}_1$.

2.1.5 Locating Arbitrary Points

Suppose that there is an arbitrary point P_2 on trapezoid 2 that we wish to locate (Figure 2.8). The vector $\mathbf{p}_2 = P_2 - O_2$ locates point P_2 relative to origin O_2, while the vector $\mathbf{p}_1 = P_2 - O_1$ locates point P_2 relative to origin O_1. We can relate either vector to bases 1 or 2:

$${}^2\mathbf{p}_2 = {}^2p_{21}{}^2\mathbf{x}_2 + {}^2p_{22}{}^2\mathbf{y}_2 \tag{2.16}$$

$${}^1\mathbf{p}_2 = {}^1p_{21}{}^1\mathbf{x}_1 + {}^1p_{22}{}^1\mathbf{y}_1 \tag{2.17}$$

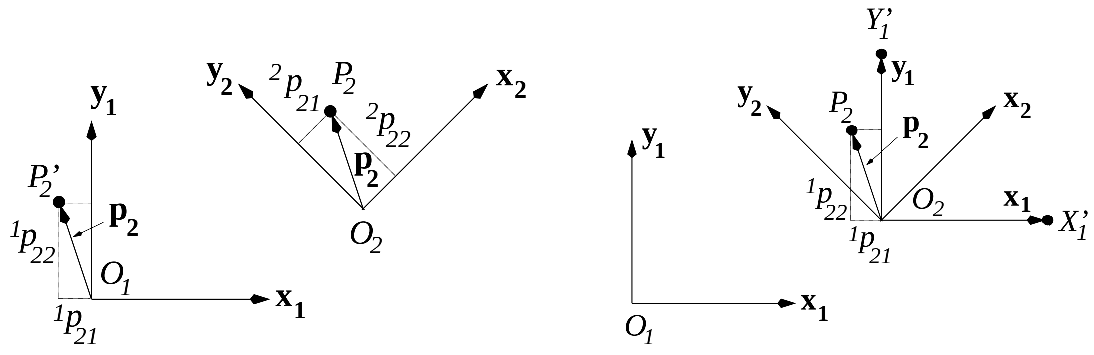

Figure 2.9: (A) The coordinates of $\mathbf{p}_2$ relative to axes 1 and 2. (B) Axes $\mathbf{x}_2, \mathbf{y}_2$ are referred to origin 2 to find the coordinates of ${}^1\mathbf{p}_2$.

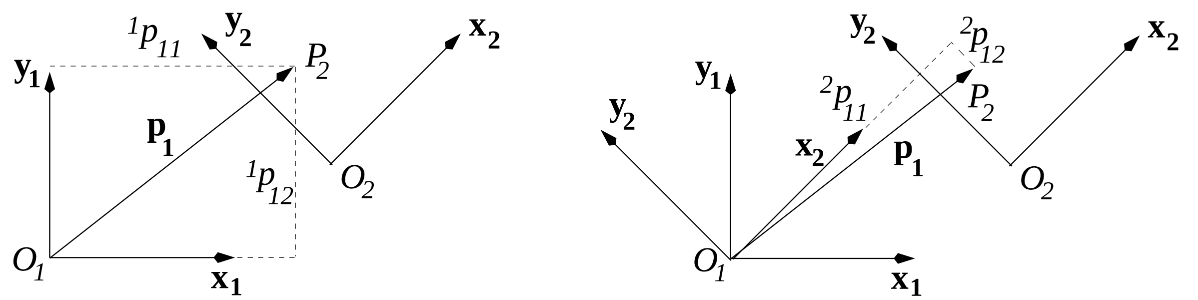

Figure 2.10: The coordinates of ${}^1\mathbf{p}_1$ (A) and ${}^2\mathbf{p}_1$ (B).

$$
\begin{aligned}
{}^2\mathbf{p}_1 &= {}^2p_{11}{}^2\mathbf{x}_2 + {}^2p_{12}{}^2\mathbf{y}_2 && (2.18) \\
{}^1\mathbf{p}_1 &= {}^1p_{11}{}^1\mathbf{x}_1 + {}^1p_{12}{}^1\mathbf{y}_1 && (2.19)
\end{aligned}
$$

The coordinates of each vector are in general different from each other. Figure 2.9(A) illustrates the coordinates for vector $\mathbf{p}_2$. Relative to basis 2, the coordinates ${}^2\mathbf{p}_2 = [{}^2p_{21} \;\; {}^2p_{22}]^T$ are simply the projections of $\mathbf{p}_2$ onto the axes $\mathbf{x}_2, \mathbf{y}_2$ as indicated. Relative to basis 1, again we imagine that a point P_2' has been created such that $\mathbf{p}_2 = P_2' - O_1$ describes the same displacement. Then the coordinates ${}^1\mathbf{p}_2 = [{}^1p_{21} \;\; {}^1p_{22}]^T$ are obtained by projection onto the axes $\mathbf{x}_1, \mathbf{y}_1$ as before.

A different construction for the coordinates of ${}^1\mathbf{p}_2$ is to imagine points X_1' and Y_1' created such that $\mathbf{x}_1 = X_1' - O_2$ and $\mathbf{y}_1 = Y_1' - O_2$ (Figure 2.9(B)). This time we are dealing with the original vector $\mathbf{p}_2$, but copies of $\mathbf{x}_1, \mathbf{y}_1$ referred to origin O_2. The coordinates ${}^1\mathbf{p}_2 = [{}^1p_{21} \;\; {}^1p_{22}]^T$ are obtained by projection onto the $\mathbf{x}_1, \mathbf{y}_1$ axes referred to origin O_2.

With regard to vector $\mathbf{p}_1$, the coordinates ${}^1\mathbf{p}_1 = [{}^1p_{11} \;\; {}^1p_{12}]^T$ are obtained by projecting onto the $\mathbf{x}_1, \mathbf{y}_1$ axes in Figure 2.10(A). For the coordinates ${}^2\mathbf{p}_1 = [{}^2p_{11} \;\; {}^2p_{12}]^T$, we have chosen in Figure 2.10(B) to refer axes 2 to origin 1 before finding the coordinates by projection. The process of actually evaluating the coordinates of $\mathbf{p}_2$ and $\mathbf{p}_1$ with respect to different axes is the subject of subsequent sections.

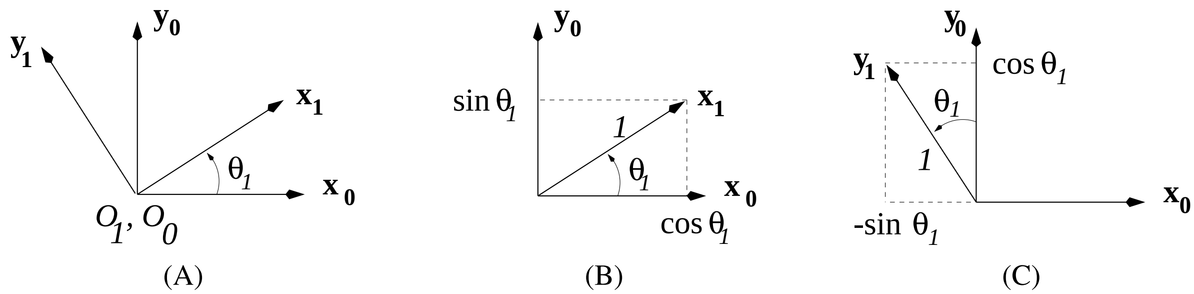

Figure 2.11: (A) Coordinate origins O_0 and O_1 are coincident, while axes 1 are rotated by θ_1 from axes 0. (B) Derivation of ${}^0\mathbf{x}_1$ and (C) ${}^0\mathbf{y}_1$.

2.2 Planar Rotational Transformations

Consider two coordinate systems 0 and 1, whose origins coincide and where axes 1 are rotated by θ_1 from axes 0 (Figure 2.11(A)). The general expression of axes 1 relative to axes 0 is obtained from (2.6):

$$\begin{aligned} {}^0\mathbf{x}_1 &= {}^0x_{11}{}^0\mathbf{x}_0 + {}^0x_{12}{}^0\mathbf{y}_0 \\ {}^0\mathbf{y}_1 &= {}^0y_{11}{}^0\mathbf{x}_0 + {}^0y_{12}{}^0\mathbf{y}_0 \end{aligned} \tag{2.20}$$

What is the relationship of the coordinates ${}^0x_{11}, {}^0x_{12}$ and ${}^0y_{11}, {}^0y_{12}$ to the rotation angle θ_1? Since these coordinates represent the projection of axes 1 onto axes 0, from Figure 2.11(B)-(C) we have that

$$\begin{aligned} {}^0x_{11} &= \cos\theta_1 & {}^0y_{11} &= -\sin\theta_1 \\ {}^0x_{12} &= \sin\theta_1 & {}^0y_{12} &= \cos\theta_1 \end{aligned} \tag{2.21}$$

Substituting,

$$\begin{aligned} {}^0\mathbf{x}_1 &= {}^0\mathbf{x}_0\cos\theta_1 + {}^0\mathbf{y}_0\sin\theta_1 &= \begin{bmatrix} {}^0\mathbf{x}_0 & {}^0\mathbf{y}_0 \end{bmatrix}\begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} &= \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} \\ {}^0\mathbf{y}_1 &= -{}^0\mathbf{x}_0\sin\theta_1 + {}^0\mathbf{y}_0\cos\theta_1 &= \begin{bmatrix} {}^0\mathbf{x}_0 & {}^0\mathbf{y}_0 \end{bmatrix}\begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix} &= \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix} \end{aligned} \tag{2.22}$$

where again $[\,{}^0\mathbf{x}_0 \;\; {}^0\mathbf{y}_0] = \mathbf{I}$, the 2-by-2 identity matrix, and ${}^0\mathbf{x}_1$ and ${}^0\mathbf{y}_1$ just equal their components relative to coordinate axes 0. Combining the axes equations (2.22) into one matrix equation,

$$\begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \doteq \mathbf{R}(\theta_1) \tag{2.23}$$

where $\mathbf{R}(\theta_1)$ is the planar *rotation matrix* corresponding to angle θ_1 from coordinate axes 1 to coordinate axes 0. While this rotation matrix in terms of θ_1 may be very familiar, it is important to recognize that its columns are the coordinate axes 1 referred to axes 0. We are able to perform a rotation because we know how axes 1 look relative to axes 0.

Example 2.7: Suppose that $\theta_1 = \pi/4$, or 45^o, between axes 0 and 1. Refer to Figure 2.7(B) for a similar relationship between axes 1 and 2. Then

$$\mathbf{R}(\pi/4) = \begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 \end{bmatrix} \tag{2.24}$$

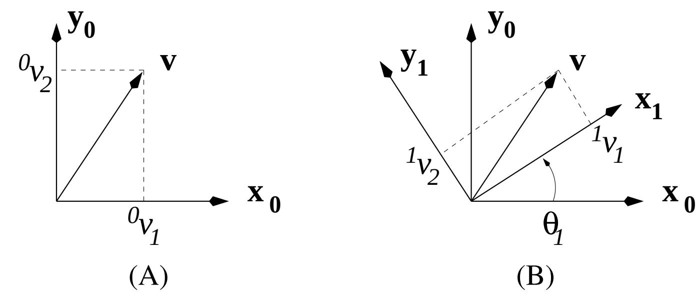

Figure 2.12: (A) Representation of a vector $\mathbf{v}$ in the plane. (B) Representation of $\mathbf{v}$ with respect to coordinate axes 1, which are at an angle θ_1 with respect to coordinate axes 0.

Compare this result with (2.15).

Conversely, we may express coordinate axes 0 relative to coordinate axes 1. The result is:

$$\begin{bmatrix} {}^1\mathbf{x}_0 & {}^1\mathbf{y}_0 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} = \mathbf{R}(\theta_1)^T \tag{2.25}$$

The derivation is left as an exercise. Note that the rotation matrix (2.25) from axes 0 to axes 1 is the transpose of the rotation matrix (2.23) from axes 1 to 0, which is true in general.

Next consider a vector $\mathbf{v}$, with components ${}^0v_1$ and ${}^0v_2$ relative to axes 0, and components ${}^1v_1$ and ${}^1v_2$ relative to axes 1 (Figure 2.12). Then

$${}^0\mathbf{v} = {}^1v_1\,{}^0\mathbf{x}_1 + {}^1v_2\,{}^0\mathbf{y}_1 = \begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 \end{bmatrix} \begin{bmatrix} {}^1v_1 \\ {}^1v_2 \end{bmatrix} \tag{2.26}$$

From (2.23) and the definition of the components of ${}^1\mathbf{v}$,

$${}^0\mathbf{v} = \mathbf{R}(\theta_1)\,{}^1\mathbf{v} \tag{2.27}$$

The rotation matrix $\mathbf{R}(\theta_1)$ converts ${}^1\mathbf{v}$ into ${}^0\mathbf{v}$.

Example 2.8: Let us return to the problem of determining ${}^1\mathbf{p}_2$ in Figure 2.9(B), when $\theta_2 = \pi/4$ and ${}^2\mathbf{p}_2 = [{}^2p_{21}\ {}^2p_{22}]^T = [1/4\ \ 3/4]^T$. Then

$${}^1\mathbf{p}_2 = \mathbf{R}(\theta_2)\,{}^2\mathbf{p}_2 = \mathbf{R}(\pi/4)\begin{bmatrix} {}^2p_{21} \\ {}^2p_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/4 \\ \sqrt{2}/2 \end{bmatrix}$$

A similar development utilizing (2.25) will show that

$${}^1\mathbf{v} = \mathbf{R}(\theta_1)^T\,{}^0\mathbf{v} \tag{2.28}$$

This result can be gotten immediately by inverting $\mathbf{R}(\theta_1)$ in (2.27). For this and all rotation matrices, the inverse is the transpose:

$$\mathbf{R}(\theta_1)^{-1} = \mathbf{R}(\theta_1)^T \tag{2.29}$$

In the planar case, another way of looking at the inverse or transpose of a rotation matrix is to note that

$$\mathbf{R}(\theta_1)^T = \mathbf{R}(-\theta_1) \tag{2.30}$$

If we rotate forward by θ_1, then the inverse operation is just to rotate backwards by $-\theta_1$. This works, because in the plane the rotation axis is implicitly always the same.

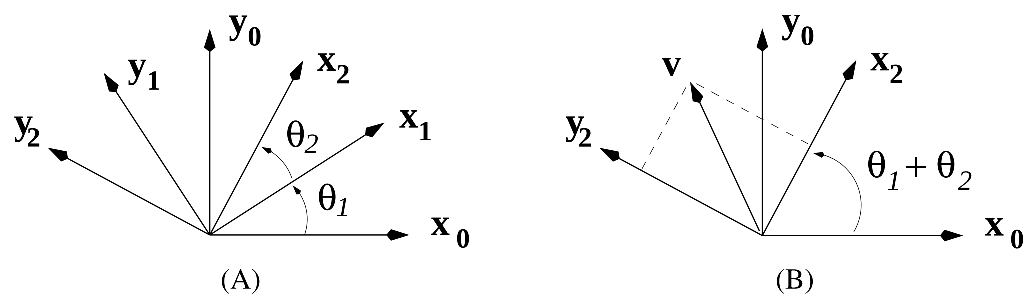

Figure 2.13: (A) Composition of rotations θ_1 and θ_2. (B). Arbitrary vector $\mathbf{v}$ represented in frame 2.

2.2.1 Composition of Planar Rotations

Suppose that another set of coordinate axes $\mathbf{x}_2$, $\mathbf{y}_2$ are introduced, that make an angle θ_2 with respect to axes 1 (Figure 2.13(A)). Then the rotation matrix relating axes 2 to axes 1 has the same form as (2.23):

$$\left[\begin{array}{cc} {}^1\mathbf{x}_2 & {}^1\mathbf{y}_2 \end{array}\right] = \left[\begin{array}{cc} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{array}\right] = \mathbf{R}(\theta_2) \tag{2.31}$$

Consequently, the rotation matrix relating axes 2 to axes 1 is obtained by using (2.27):

$$\left[\begin{array}{cc} {}^0\mathbf{x}_2 & {}^0\mathbf{y}_2 \end{array}\right] = \left[\begin{array}{cc} \mathbf{R}(\theta_1)\,{}^1\mathbf{x}_2 & \mathbf{R}(\theta_1)\,{}^1\mathbf{y}_2 \end{array}\right] = \mathbf{R}(\theta_1)\left[\begin{array}{cc} {}^1\mathbf{x}_2 & {}^1\mathbf{y}_2 \end{array}\right] = \mathbf{R}(\theta_1)\mathbf{R}(\theta_2) = \mathbf{R}(\theta_1+\theta_2) \tag{2.32}$$

This result is clear, because $\theta_1 + \theta_2$ is the angle between coordinate axes 0 and 2. The composition of rotations in the plane, in this example of $\mathbf{R}(\theta_1)$ with $\mathbf{R}(\theta_2)$, is very simple, as the result is the rotation matrix which sums the angles, e.g., $\mathbf{R}(\theta_1 + \theta_2)$. In general, for axes j related to axes i with $i < j$ and a number of intermediate axes $i+1, \ldots, j-1$,

$$\left[\begin{array}{cc} {}^i\mathbf{x}_j & {}^i\mathbf{y}_j \end{array}\right] = \mathbf{R}(\theta_{i+1} + \cdots + \theta_j) = \mathbf{R}(\theta_{ij}) \qquad \text{where} \quad \theta_{ij} = \theta_{i+1} + \cdots + \theta_j \tag{2.33}$$

Similar to the previous development, an arbitrary vector $\mathbf{v}$ expressed with respect to coordinate axes 2, i.e., ${}^2\mathbf{v}$ (Figure 2.13(B)), can be related to ${}^1\mathbf{v}$, and then to ${}^0\mathbf{v}$, by:

$${}^0\mathbf{v} = \mathbf{R}(\theta_1)^1\mathbf{v} = \mathbf{R}(\theta_1)\mathbf{R}(\theta_2)^2\mathbf{v} = \mathbf{R}(\theta_1+\theta_2)^2\mathbf{v} \tag{2.34}$$

In keeping with the spirit of later chapters, we introduce an additional notation for rotation matrices derived from (2.33):

- ${}^i\mathbf{R}_j = \mathbf{R}(\theta_{ij})$ is the rotation matrix from axes j to axes i.

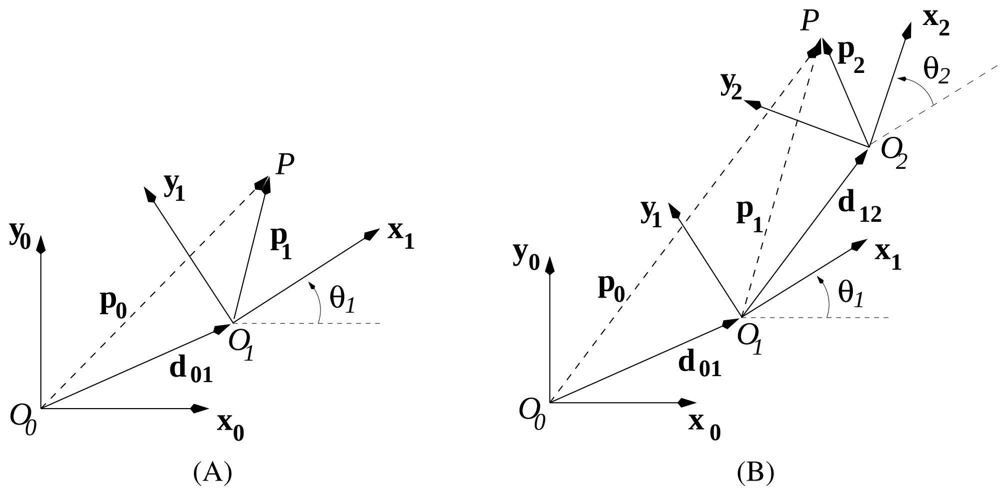

Figure 2.14: Coordinate transformation between (A) frame 1 and frame 0, and (B) also including frame 2.

The atomic relation is ${}^{i-1}\mathbf{R}_i = \mathbf{R}(\theta_i)$. More generally,

$$\begin{aligned} {}^{i}\mathbf{R}_j &= \mathbf{R}(\theta_{i+1})\mathbf{R}(\theta_{i+2})\cdots\mathbf{R}(\theta_j) \\ &= {}^{i}\mathbf{R}_{i+1}\,{}^{i+1}\mathbf{R}_{i+2}\cdots{}^{j-1}\,\mathbf{R}_j \end{aligned} \tag{2.35}$$

For 3D rotations, we cannot express a rotation matrix as a function of a single parameter, such as θ_{ij} for ${}^{i}\mathbf{R}_j = \mathbf{R}(\theta_{ij})$ for the planar case. Instead, we will see that 3 parameters are required to represent 3D rotations. Nevertheless, ${}^{i}\mathbf{R}_j$ will have the same meaning, as the rotation from axes j to axes i, whether for 2D or 3D.

The reason for the notation ${}^{i}\mathbf{R}_j$ is apparent from (2.33):

$${}^{i}\mathbf{R}_j = \left[\begin{array}{cc} {}^{i}\mathbf{x}_j & {}^{i}\mathbf{y}_j \end{array}\right] \tag{2.36}$$

Thus ${}^{i}\mathbf{R}_j$ has the same form as its column constituents. Another view is to reexpress (2.34) as

$${}^{0}\mathbf{v} = {}^{0}\mathbf{R}_1\,{}^{1}\mathbf{v} = {}^{0}\mathbf{R}_2\,{}^{2}\mathbf{v} \tag{2.37}$$

The right subscript 1 in ${}^{0}\mathbf{R}_1$ matches the left superscript 1 in ${}^{1}\mathbf{v}$, while the left superscript 0 in ${}^{0}\mathbf{R}_1$ matches the left superscript 0 in ${}^{0}\mathbf{v}$. More generally,

$${}^{i}\mathbf{v} = {}^{i}\mathbf{R}_j\,{}^{j}\mathbf{v} \tag{2.38}$$

Finally, the inverse relation (2.29) is

$$\left({}^{i}\mathbf{R}_j\right)^{-1} = {}^{i}\mathbf{R}_j^T = {}^{j}\mathbf{R}_i \tag{2.39}$$

Exactly the same relation will hold for 3D rotations.

2.3 Planar Coordinate Transformations

For the rotational planar transformations considered so far, it was assumed that the origins corresponding to different coordinate axes were the same. In general, the origins O_i corresponding to coordinate axes i will be located in different places, for example the two trapezoids in Figure 2.6(A). Let us relabel the coordinate systems as 0 and 1; in general, the environmentally fixed reference frame will be labeled as frame 0. Figure 2.14(A) shows origin O_1 displaced from origin O_0 by the vector $\mathbf{d}_{01}$.

- $\mathbf{d}_{ij}$ is defined as the vector from O_i to O_j.

The vector $\mathbf{p}_1$ identifies some point P relative to O_1, while $\mathbf{p}_0$ identifies P relative to O_0. Clearly

$$ {}^0\mathbf{p}_0 = {}^0\mathbf{d}_{01} + {}^0\mathbf{p}_1 \tag{2.40}$$

Often a point P will only be known relative to the frame of the body in which it is located. For example, point P_2 in Figure 2.6(A) could be at the opposite corner of trapezoid 2, and its location $\mathbf{p}_2 = P_2 - O_2$ would be known relative to axes 2. In the present example, we assume we know the coordinates ${}^1\mathbf{p}_1 = [\,{}^1p_{11} \;\; {}^1p_{12}]$. From the previous section, the rotation matrix ${}^0\mathbf{R}_1 = \mathbf{R}(\theta_1)$ applied to ${}^1\mathbf{p}_1$ yields ${}^0\mathbf{p}_1$ (see equation (2.27)):

$$ {}^0\mathbf{p}_1 = {}^0\mathbf{R}_1\, {}^1\mathbf{p}_1 \tag{2.41}$$

We assume that θ_1 and ${}^0\mathbf{d}_{01}$ are given, i.e., we know where frame 1 is relative to frame 0. To find ${}^0\mathbf{p}_0$ given ${}^1\mathbf{p}_1$, substitute (2.41) into (2.40):

$$ {}^0\mathbf{p}_0 = {}^0\mathbf{d}_{01} + {}^0\mathbf{R}_1\, {}^1\mathbf{p}_1 \tag{2.42}$$

Equation (2.42) is said to represent a *coordinate transformation:* a point originally represented in frame 1 is transformed to its representation in frame 0. Note the order in (2.42): first ${}^1\mathbf{p}_1$ is rotated by ${}^0\mathbf{R}_1$ so that it is expressed relative to axes 0, i.e., ${}^0\mathbf{p}_1$, then it is added to the displacement between origins ${}^0\mathbf{d}_{01}$, which is also expressed relative to axes 0.

Example 2.9: Let us return to the problem of determining ${}^1\mathbf{p}_1$ in Figure 2.10(A) for $\theta_2 = \pi/4$, ${}^2\mathbf{p}_2 = [1/4 \;\; 3/4]^T$, and ${}^1\mathbf{d}_{12} = [3/2 \;\; 1/2]^T$. Then

$$ {}^1\mathbf{p}_1 = {}^1\mathbf{d}_{12} + \mathbf{R}(\theta_2)\, {}^2\mathbf{p}_2 = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -\sqrt{2}/4 \\ \sqrt{2}/2 \end{bmatrix} $$

Given a coordinate transformation from 1 to 0, we can reverse the transformation to go from 0 to 1. Given (2.42), by rearrangement we find:

$$ {}^1\mathbf{p}_1 = {}^0\mathbf{R}_1^T ({}^0\mathbf{p}_0 - {}^0\mathbf{d}_{01}) \tag{2.43}$$

where again ${}^0\mathbf{R}_1^T = {}^1\mathbf{R}_0 = \mathbf{R}(-\theta_1)$.

Example 2.10: For the problem of determining ${}^2\mathbf{p}_1$ in Figure 2.10(B) for $\theta_2 = \pi/4$, ${}^2\mathbf{p}_2 = [1/4 \;\; 3/4]^T$, and ${}^1\mathbf{d}_{12} = [3/2 \;\; 1/2]^T$,

$$ \begin{aligned} {}^2\mathbf{p}_1 &= {}^2\mathbf{d}_{12} + {}^2\mathbf{p}_2 = \mathbf{R}(\theta_2)\, {}^1\mathbf{d}_{12} + {}^2\mathbf{p}_2 \\ &= \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -\sqrt{2}/2 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} \end{aligned} $$

2.3.1 Composition of Coordinate Transformations

Next introduce frame 2, which is displaced from frame 1 by $\mathbf{d}_{12}$ and whose axes are at an angle θ_2 with respect to axes 1 (Figure 2.14(B)). Suppose point P is located relative to O_2 by $\mathbf{p}_2$, relative to O_1 by $\mathbf{p}_1$, and relative to O_0 by $\mathbf{p}_0$. Then

$$\mathbf{p}_1 = \mathbf{d}_{12} + \mathbf{p}_2 \tag{2.44}$$
$$\mathbf{p}_0 = \mathbf{d}_{01} + \mathbf{p}_1 \tag{2.45}$$

Again, suppose $\mathbf{p}_2$ is known relative to frame 2, i.e., as ${}^2\mathbf{p}_2$. The location of P relative to frame 1 is:

$${}^1\mathbf{p}_1 = {}^1\mathbf{d}_{12} + {}^1\mathbf{R}_2\,{}^2\mathbf{p}_2 \tag{2.46}$$

Utilizing (2.46), the location of P relative to frame 0 is:

$$\begin{aligned}
{}^0\mathbf{p}_0 &= {}^0\mathbf{d}_{01} + {}^0\mathbf{R}_1\,{}^1\mathbf{p}_1 \\
&= {}^0\mathbf{d}_{01} + {}^0\mathbf{R}_1({}^1\mathbf{d}_{12} + {}^1\mathbf{R}_2\,{}^2\mathbf{p}_2) \\
&= {}^0\mathbf{d}_{01} + {}^0\mathbf{R}_1{}^1\mathbf{d}_{12} + {}^0\mathbf{R}_2\,{}^2\mathbf{p}_2 \\
&= {}^0\mathbf{d}_{01} + {}^0\mathbf{d}_{12} + {}^0\mathbf{p}_2
\end{aligned} \tag{2.47}$$

Equation (2.47) demonstrates the composition of two coordinate transformations. Note again the composition of rotations: ${}^0\mathbf{R}_1\,{}^1\mathbf{R}_2 = {}^0\mathbf{R}_2$.

2.4 Two-Link Planar Manipulator Kinematics

As an example of the application of coordinate transformations, we will solve for the forward kinematics of a planar two-link manipulator (Figure 2.15).

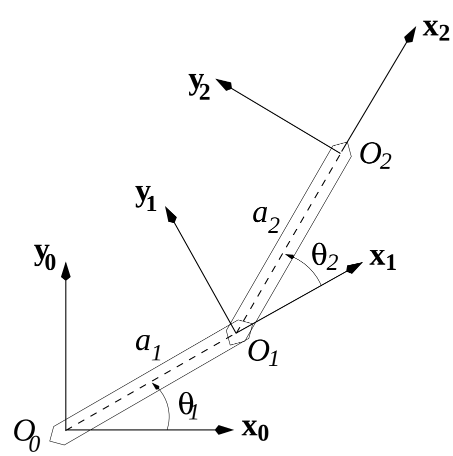

Figure 2.15: Two-link planar manipulator.

- Frame 0, the stationary or ground reference, is placed at the proximal end of link 1, such that origin O_0 is located at joint 1.

- Frame 1 is embedded in link 1 at the distal end, such that origin O_1 is the location of joint 2. Axis $\mathbf{x}_1$ lies along the long axis of link 1, which connects O_0 to O_1. Frame 1 is the intrinsic coordinate system of link 1.

- Frame 2 is embedded in link 2 in the distal end, where O_2 is considered the manipulator *endpoint*. The endpoint is the working point of the manpulator, which we wish to interact with the environment. Axis $\mathbf{x}_2$ is along the long axis of link 2, which connects O_1 to O_2. Frame 2 is the intrinsic coordinate system of link 2.

- The length of link 1 is a_1, and that of link 2 is a_2.

- Joint angle θ_1 is the angle from $\mathbf{x}_0$ to $\mathbf{x}_1$, while joint angle θ_2 is the angle from $\mathbf{x}_1$ to $\mathbf{x}_2$.

This method of setting up coordinate systems in the links is consistent with the Denavit-Hartenberg convention, discussed in a later chapter. Note that the coordinate system for link i is placed in link i's distal end. Other conventions, not employed in this course, place it in the proximal end.

The problem of *forward kinematics* is to find the endpoint position ${}^0\mathbf{d}_{02}$, i.e., point O_2 relative to point O_0, given the joint angles θ_1 and θ_2. From Figure 2.15,

$$\begin{aligned}
{}^0\mathbf{d}_{02} &= {}^0\mathbf{d}_{01} + {}^0\mathbf{d}_{12} \\
&= a_1 {}^0\mathbf{x}_1 + a_2 {}^0\mathbf{x}_2 \\
&= a_1 \, {}^0\mathbf{R}_1 \, {}^1\mathbf{x}_1 + a_2 \, {}^0\mathbf{R}_2 \, {}^2\mathbf{x}_2 \\
&= a_1 \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} + a_2 \begin{bmatrix} \cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) \end{bmatrix}
\end{aligned} \tag{2.48}$$

Ordinarily, we wouldn't go through all these transformations for the planar case, because this relation can be directly found from inspection. Let the coordinates of the endpoint be (x, y) in the simplified diagram, Figure 2.16. Then

$$x = a_1 \mathrm{c}\theta_1 + a_2 \mathrm{c}(\theta_1 + \theta_2) \tag{2.49}$$

$$y = a_1 \mathrm{s}\theta_1 + a_2 \mathrm{s}(\theta_1 + \theta_2) \tag{2.50}$$

where throughout this course,

- the shorthand notation $\mathrm{c}\theta$ is used for $\cos\theta$ and $\mathrm{s}\theta$ for $\sin\theta$.

2.5 Homogeneous Transformations in a Plane

A coordinate transformation such as (2.42) separately represents rotations and translations. These transformations can be combined into a single matrix operation by increasing the dimension of a planar vector to 3, where the third dimension is 1. Let $\mathbf{p} = [p_1 \; p_2]^T$ be a 2-D column vector with its xy components indicated. Then the *homogeneous coordinate* created from $\mathbf{p}$ is indicated by the upper case italic letter P:

$$P = \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix} \tag{2.51}$$

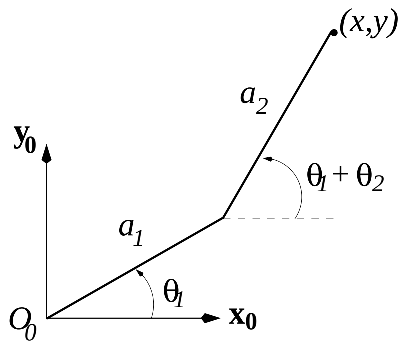

Figure 2.16: Endpoint position (x, y) of two-link planar manipulator found by inspection.

A coordinate transformation can now be represented as:

$$\begin{bmatrix} {}^0\mathbf{p}_0 \\ 1 \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{R}_1\,{}^1\mathbf{p}_1 + {}^0\mathbf{d}_{01} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{R}_1 & {}^0\mathbf{d}_{01} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} {}^1\mathbf{p}_1 \\ 1 \end{bmatrix} \tag{2.52}$$

or employing the homogeneous coordinate notation,

$$ {}^0P = {}^0\mathbf{T}_1\,{}^1P \tag{2.53}$$

where $\mathbf{0}^T = [0\ \ 0]$ and ${}^0\mathbf{T}_1$ is the 3-by-3 *homogeneous transformation,* from frame 1 to frame 0, for the plane.

Note that a superscript j can be employed for the homogeneous coordinate description of a point ${}^jP$, to indicate the axes with respect to which the coordinates are expressed:

$$ {}^jP = \begin{bmatrix} {}^j\mathbf{p} \\ 1 \end{bmatrix} \tag{2.54}$$

2.5.1 Composition of Homogeneous Transformations

Coordinate transformations can easily be chained between pairs of coordinate systems. Let ${}^0\mathbf{T}_1$ represent the homogeneous transformation from frame 1 to frame 0 and ${}^1\mathbf{T}_2$ from frame 2 to frame 1. Then the homogeneous transformation from frame 2 to 0 is

$$ {}^0\mathbf{T}_2 = {}^0\mathbf{T}_1{}^1\mathbf{T}_2 \tag{2.55}$$

In this notation, note how the subscript of ${}^0\mathbf{T}_1$ lines up with the superscript of ${}^1\mathbf{T}_2$. Applied to homogeneous coordinates,

$$ {}^0P = {}^0\mathbf{T}_2\,{}^2P = {}^0\mathbf{T}_1\,{}^1P \tag{2.56}$$

where ${}^1P = {}^1\mathbf{T}_2\ {}^2P$. Note how much more compact the homogeneous transformation notation is than (2.47).

The inverse coordinate transformation is again easier to express using homogeneous transformations.

$$ {}^1P \;=\; ({}^0\mathbf{T}_1)^{-1}\,{}^0P \tag{2.57}$$

$$ ({}^0\mathbf{T}_1)^{-1} \;=\; \begin{bmatrix} {}^0\mathbf{R}_1^T & -\,{}^0\mathbf{R}_1^T\,{}^0\mathbf{d}_{01} \\ \mathbf{0}^T & 1 \end{bmatrix} = {}^1\mathbf{T}_0 \tag{2.58}$$

where (2.58) is derived from (2.43). With our notation, $({}^0\mathbf{T}_1)^{-1} = {}^1\mathbf{T}_0$: the inverse of the transformation from frame 1 to 0 is the transformation from frame 0 to 1. The inverse of the composition of transformations may be simply written:

$$ {}^2P = ({}^0\mathbf{T}_1\,{}^1\mathbf{T}_2)^{-1}\,{}^0P = ({}^1\mathbf{T}_2)^{-1}({}^0\mathbf{T}_1)^{-1}\,{}^0P = {}^2\mathbf{T}_1\,{}^1\mathbf{T}_0\,{}^0P = {}^2\mathbf{T}_0\,{}^0P \tag{2.59} $$

Example 2.11: Consider the following relative location of frames 0, 1, and 2 (Figure 2.17):

- Frame 1 is displaced from frame 0 by a rotation of 30^o and a translation of $(1,1)$. Then ${}^0\mathbf{R}_1 = \mathbf{R}(30^o)$ and

$$ \mathbf{R}(30^o) \;=\; \begin{bmatrix} \cos 30^o & -\sin 30^o \\ \sin 30^o & \cos 30^o \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}, \qquad {}^0\mathbf{d}_{01} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, $$

$$ {}^0\mathbf{T}_1 \;=\; \begin{bmatrix} \mathbf{R}(30^o) & {}^0\mathbf{d}_{01} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 1 \\ 1/2 & \sqrt{3}/2 & 1 \\ 0 & 0 & 1 \end{bmatrix} $$

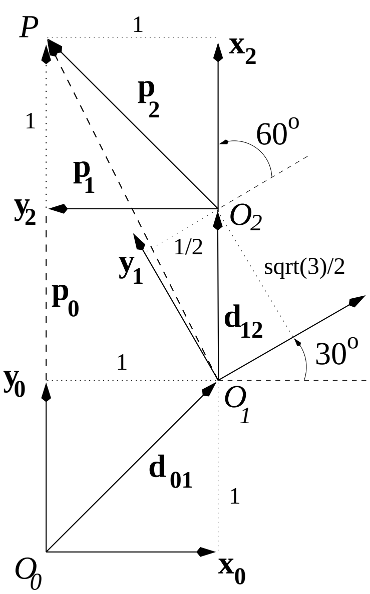

Figure 2.17: Relative locations of frames 0, 1, and 2, and a point ${}^2\mathbf{p}_2 = (1,1)$.

- Frame 2 is displaced from frame 1 by a rotation of 60^o and a translation of $(1/2, \sqrt{3}/2)$. Then ${}^1\mathbf{R}_2 = \mathbf{R}(60^o)$ and

$$ \mathbf{R}(60^o) \;=\; \begin{bmatrix} \cos 60^o & -\sin 60^o \\ \sin 60^o & \cos 60^o \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \qquad {}^1\mathbf{d}_{12} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}, $$

$$
{}^1\mathbf{T}_2 = \begin{bmatrix} \mathbf{R}(60^o) & {}^1\mathbf{d}_{12} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 1/2 \\ \sqrt{3}/2 & 1/2 & \sqrt{3}/2 \\ 0 & 0 & 1 \end{bmatrix}
$$

Then

$$
{}^0\mathbf{d}_{12} = {}^0\mathbf{R}(30^o)\, {}^1\mathbf{d}_{12} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

- Given a vector ${}^2\mathbf{p} = (1, 1)$, then

$$
{}^1\mathbf{p}_2 = \mathbf{R}(60^o)\, {}^2\mathbf{p}_2 = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1-\sqrt{3})/2 \\ (1+\sqrt{3})/2 \end{bmatrix}
$$

$$
{}^1\mathbf{p}_1 = {}^1\mathbf{d}_{12} + {}^1\mathbf{p}_2 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} + \begin{bmatrix} (1-\sqrt{3})/2 \\ (1+\sqrt{3})/2 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{3}/2 \\ 1/2+\sqrt{3} \end{bmatrix}
$$

Also,

$$
{}^0\mathbf{p}_1 = \mathbf{R}(30^o)\, {}^1\mathbf{p}_1 = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1-\sqrt{3}/2 \\ 1/2+\sqrt{3} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}
$$

$$
{}^0\mathbf{p}_0 = {}^0\mathbf{d}_{01} + {}^0\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}
$$

2.5.2 Homogeneous Coordinates and the Affine Plane

In graphics, homogeneous coordinates and transformations are used extensively to represent scaling and shear, as well as translation and rotation [1]. In addition, homogeneous coordinates are a way of concretely denoting a point by a set of coordinates, which is distinct from denoting a vector by a set of coordinates. We purposely used the same notation for a homogeneous coordinate as was introduced for a point, namely italic upper case letters such as P_i, because we now consider that a homogeneous coordinate P_i represents the coordinates of a point.

- The set of points in $\mathcal{R}^3$ whose last coordinate is 1 is referred to as the *standard affine plane* in $\mathcal{R}^3$. It is the subspace of points that lies in the plane $z = 1$ (Figure 2.18).

A vector is then considered to be a 3-coordinate vector whose last component is 0 and that lies in the affine plane. In Figure 2.18, point P_i has coordinates $[P_{i1}\ P_{i2}\ 1]^T$, while the vector $\mathbf{v}$ from P_1 to P_2 has coordinates

$$
\boldsymbol{V} = P_2 - P_1 = \begin{bmatrix} P_{21} \\ P_{22} \\ 1 \end{bmatrix} - \begin{bmatrix} P_{11} \\ P_{12} \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} \tag{2.60}
$$

Even though a 3-tuple expresses both points and vectors, the third component as 1 or 0 distinguishes them.

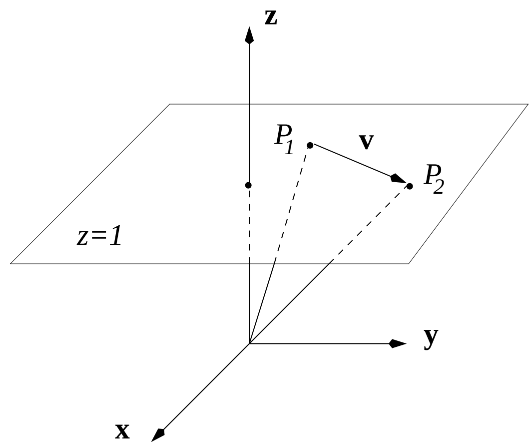

Figure 2.18: Points P_1 and P_2 on the affine plane $z = 1$, represented as homogeneous coordinates, and the displacement $\mathbf{v}$ that lies on the plane.

- The notation to distinguish the 3-by-1 vector $\boldsymbol{V} = [\mathbf{v} \quad 0]^T$ from the 2-by-1 vector $\mathbf{v}$ is upper case slanted bold font.

Because a homogeneous transformation can be considered to operate on points, they are also referred to as *affine transformations.* Two of these affine transformations are special.

- A *translational affine transformation* **Trans** is the special homogeneous transformation with the form:

$$\mathbf{Trans}(\mathbf{v}) = \begin{bmatrix} 1 & 0 & v_1 \\ 0 & 1 & v_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{v} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{2.61}$$

  There is no rotation, reflected by the identity transformation $\mathbf{I} = \mathbf{R}(0)$, and there is a displacement $\mathbf{v}$ between origins.

- A *rotational affine transformation* **Rot** implements a pure rotation without translation:

$$\mathbf{Rot}(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}(\theta_1) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{2.62}$$

Thus a general homogeneous transformation is the composition of a translational with a rotational affine transformation:

$${}^0\mathbf{T}_1 = \mathbf{Trans}({}^0\mathbf{d}_{01})\mathbf{Rot}(\theta_1) = \begin{bmatrix} \mathbf{I} & {}^0\mathbf{d}_{01} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}(\theta_1) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}(\theta_1) & {}^0\mathbf{d}_{01} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{2.63}$$

Note the order: first a translation, then a rotation. The reverse composition $\mathbf{Rot}(\theta_1)\mathbf{Trans}({}^0\mathbf{d}_{01})$ would not give the correct result.

For ${}^0\mathbf{T}_1$, the rotational part $\mathbf{R}(\theta_1)$ contains as its two columns ${}^0\mathbf{x}_1$ and ${}^0\mathbf{y}_1$; refer to (2.23). Thus

$${}^0\mathbf{T}_1 = \begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 & {}^0\mathbf{d}_{01} \\ 0 & 0 & 1 \end{bmatrix} \tag{2.64}$$

Hence the homogeneous transformation contains the axes of frame 1 relative to frame 0, plus the displacement ${}^0\mathbf{d}_{01}$ of origin 1 from origin 0. Because ${}^0\mathbf{T}_1$ contains all the information about the coordinate system 1 relative to coordinate system 0, it itself is often called *frame 1.*

2.6 Operators

A coordinate transformation is a static description of the relative location of two coordinate systems, perhaps representing two different objects. An *operator* or *displacement* is an active description of the process of moving a point relative to a fixed coordinate system, such as moving an object someplace. The same mathematics for coordinate transformations can also be interpreted as operators. The difference from the coordinate transformation viewpoint is:

- operations or displacements are with respect to the *fixed frame;*
- coordinate transformations are with respect to the *current frame.*

Another viewpoint is that displacing a point *forward* is the same as moving the coordinate system *backward.*

The *translational operator* in Figure 2.19(A) translates a point P_1, represented relative to the origin O_0 by $\mathbf{p}_1$, by the displacement $\mathbf{d}$ to the point P_2, represented relative to the origin by $\mathbf{p}_2$:

$$P_2 = \mathbf{Trans}({}^0\mathbf{d}_1)P_1 \qquad \text{or} \qquad {}^0\mathbf{p}_2 = {}^0\mathbf{p}_1 + {}^0\mathbf{d} \tag{2.65}$$

A homogeneous transformation and a vector equation have both been presented to express the results of the translational operator. The homogeneous transformation emphasizes that point P_1 is being moved to point P_2. All vectors in the vector equation are represented in the same coordinate frame 0, which demonstrates that the displacement is with respect to a fixed frame.

The coordinate transformation interpretation of the translation operator assumes coordinate frames 0 and 1 are initially coincident. Then move coordinate frame 0 *backwards* by $-{}^0\mathbf{d}$ (Figure 2.19(B)) to describe the location of point P_2.

The *rotational operator* in Figure 2.20(A) rotates point P_1 to P_2 by the angle θ_1. Then the new vector ${}^0\mathbf{p}_2$ resulting after the displacement by θ_1 *in the same coordinate frame* is:

$$P_2 = \mathbf{Rot}(\theta_1)P_1 \qquad \text{or} \qquad {}^0\mathbf{p}_2 = \mathbf{R}(\theta_1)\,{}^0\mathbf{p}_1 \tag{2.66}$$

Again, the homogeneous transformation representation of a rotational operator demonstrates that point P_1 is being rotated about point O_0 to point P_2. The vector equation again shows that all vectors are expressed with respect to the fixed frame.

The coordinate transformation interpretation assumes coordinate frames 0 and 1 are initially coincident (Figure 2.20(B)). Rotate frame 0 backwards by $\mathbf{R}(-\theta_1)$ to describe the location of point P_2.

The general *transformational operator* first rotates a point, then translates it. The homogeneous transformation and vector forms are:

$$P_2 = \begin{bmatrix} \mathbf{R}(\theta_1) & {}^0\mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix} P_1 = {}^0\mathbf{T}_1 P_1 \qquad \text{or} \qquad {}^0\mathbf{p}_2 = \mathbf{R}(\theta_1)\,{}^0\mathbf{p}_1 +{}^0\mathbf{d} \tag{2.67}$$

In Figure 2.21(A), point P_1 is rotated to become the point $O_0 + \mathbf{R}(\theta_1)\,{}^0\mathbf{p}_1$, then this intermediate point is translated by ${}^0\mathbf{d}$ to become P_2. Again, all vectors in the vector form are expressed in the same, or fixed, frame 0. We have adopted the coordinate transformation form ${}^0\mathbf{T}_1$ to represent the transformational

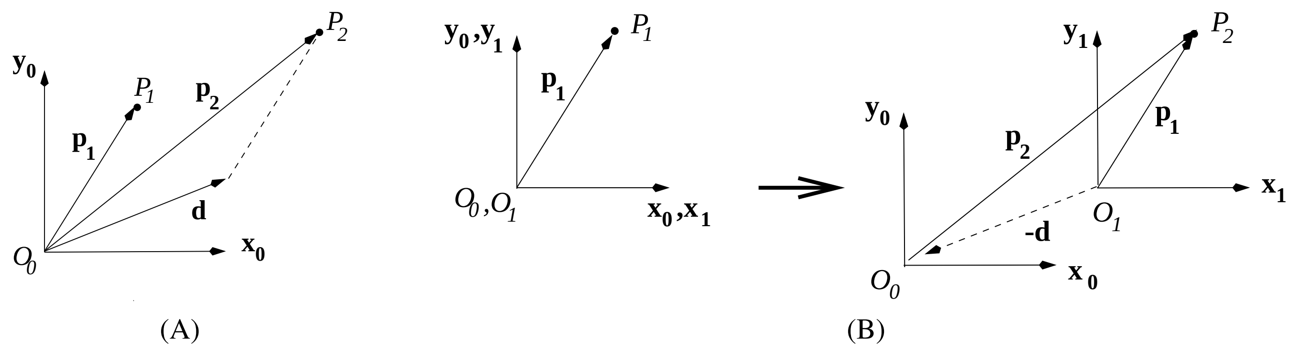

Figure 2.19: (A) The translation operator moves point P_1 to point P_2 via the displacement $\mathbf{d}$. (B) The equivalent coordinate transformation moves coordinate system 0 backwards by $-\mathbf{d}$.

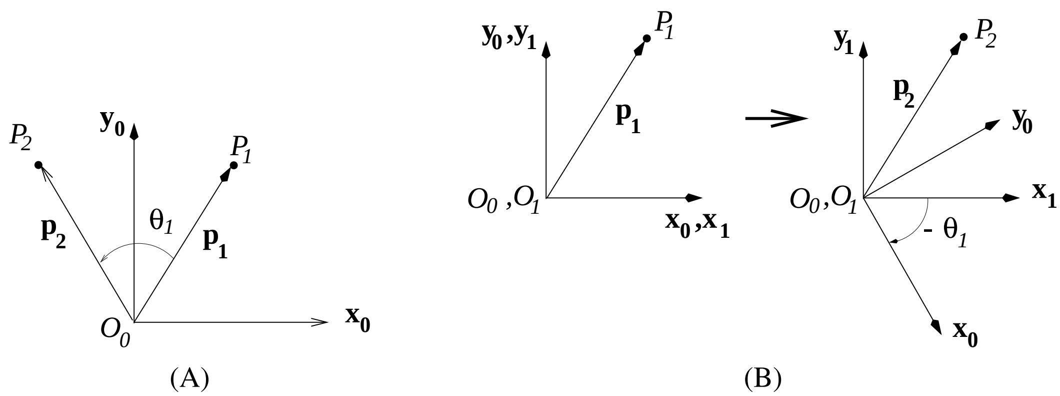

Figure 2.20: (A) The rotation operator moves point P_1 to point P_2 by rotating around O_0 by the angle θ_1. (B) The equivalent coordinate transformation rotates coordinate system 0 backwards by $-\theta_1$.

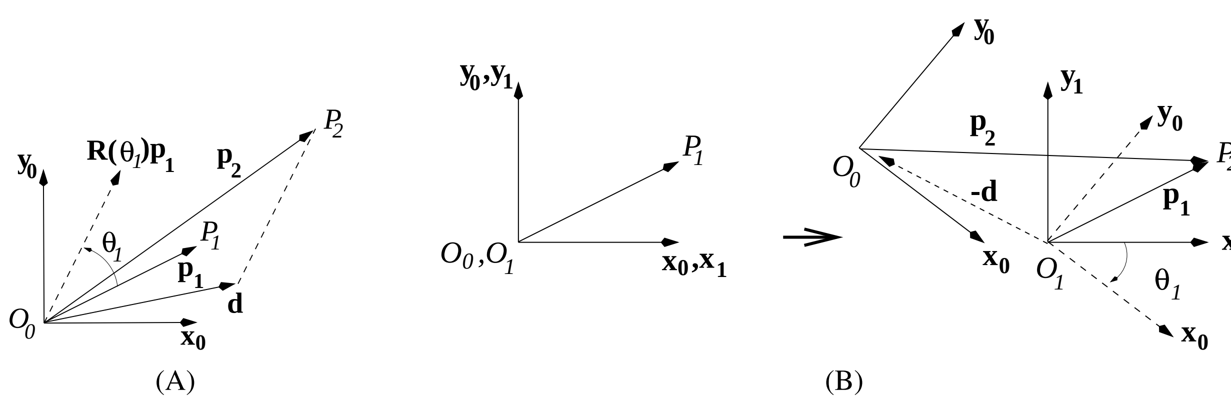

Figure 2.21: (A) The transformational operator moves point P_1 to point P_2 by rotating $\mathbf{p}_1$ to $\mathbf{R}(\theta_1)\mathbf{p}_1$, then translating by $\mathbf{d}$. (B) The equivalent coordinate transformation rotates frame 0 backwards by $-\theta_1$, then translates frame 0 backwards by $-\mathbf{d}$ relative to frame 0's current or intermediate location.

operator for convenience, and in the future we will be explicit about whether this form is being interpreted as an operator or as a coordinate transformation. The same holds for $\mathbf{Rot}$ and $\mathbf{Trans}$.

The coordinate transformation interpretation, starting from overlapping frames 0 and 1, involves rotating frame 0 *backwards* by $-\theta_1$, then translating *backwards* by $-\mathbf{d}$ relative to the *current frame,* i.e., the intermediate location of frame 0 indicated by dashed lines.

2.6.1 Composition of Operators

As another viewpoint to explain the order of translations and rotations in an operator, recast the transformational operator as a homogeneous transformation in terms of a pure translation and a pure rotation:

$$^0\mathbf{T}_1 = \mathbf{Trans}(^0\mathbf{d})\mathbf{Rot}(\theta_1) \tag{2.68}$$

That is to say, we have a composition of two elementary operators or coordinate transformations. The difference between operators and coordinate transformations can be viewed as follows:

- A coordinate transformation interpretation involves a *left-to-right evaluation of a composition about the current frame.*
- An operator interpretation involves a *right-to-left evaluation of a composition about the fixed frame.*

As an example to illustrate these two different interpretations, consider the homogeneous transformation

$$^0\mathbf{T}_1 = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 1 \\ 1/2 & \sqrt{3}/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.69}$$

1. Viewed as a coordinate transform, first $^0\mathbf{T}_1$ translates the origin of frame 1 to $(1, 1/2)$, then rotates the axes by 30^o.
2. Viewed as an operator, first $^0\mathbf{T}_1$ rotates frame 2 by 30^o, then translates it by $(1, 1/2)$, all still relative to frame 0.

There doesn't seem to be such a big difference between these two interpretations for this simple example, but the composition of two general operators illustrates the difference more clearly. Consider the composition of the homogeneous transformations $^0\mathbf{T}_2 = {}^0\mathbf{T}_1\,{}^1\mathbf{T}_2$, where $^0\mathbf{T}_1$ is the same as above and

$$^1\mathbf{T}_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 3/4 \\ -1/\sqrt{2} & 1/\sqrt{2} & 3/4 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.70}$$

Decompose these transformations in terms of the elementary transformations **Trans** and **Rot**:

$$^0\mathbf{T}_1 = \mathbf{Trans}(1, 1/2)\,\mathbf{Rot}(30^o) \tag{2.71}$$

$$^1\mathbf{T}_2 = \mathbf{Trans}(3/4, 3/4)\,\mathbf{Rot}(-45^o) \tag{2.72}$$

$$^0\mathbf{T}_2 = \mathbf{Trans}(1, 1/2)\,\mathbf{Rot}(30^o)\mathbf{Trans}(3/4, 3/4)\,\mathbf{Rot}(-45^o) \tag{2.73}$$

where we have written $^0\mathbf{d}_{01} = (1, 1/2)$ and $^1\mathbf{d}_{12} = (3/4, 3/4)$. Consider that coordinate systems 0, 1, and 2 are all initially overlapping. The interpretation of $^0\mathbf{T}_2$ as a coordinate transformation versus as an operator differs as follows.

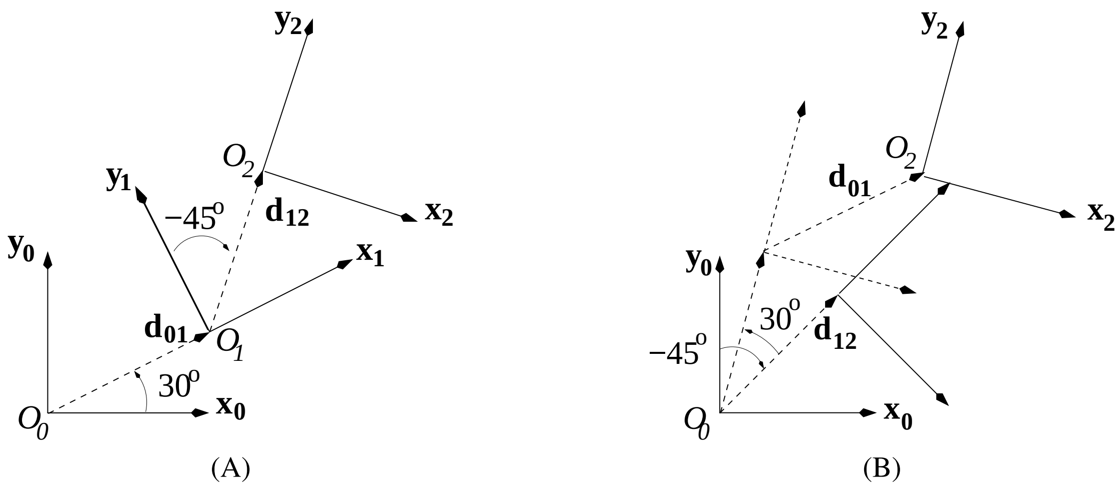

Figure 2.22: The composition ${}^0\mathbf{T}_2 = {}^0\mathbf{T}_1\ {}^1\mathbf{T}_2$ viewed as (A) a coordinate transformation (left to right evaluation about the current frame), or (B) as an operator (right to left evaluation about the fixed frame).

1. Viewed as a coordinate transform, the transformations in ${}^0\mathbf{T}_2$ are interpreted from left to right about the current frame. First ${}^0\mathbf{T}_1$ translates the origin of frames 1 and 2 to $(1, 1/2)$, then rotates the axes by 30^o. Then ${}^1\mathbf{T}_2$ translates frame 2 relative to frame 1 by $(3/4, 3/4)$, then rotates frame 2 by -45^o relative to frame 1 (Figure 2.22(A)).

2. Viewed as an operator, the transformations in ${}^0\mathbf{T}_2$ are interpreted from right to left about the fixed frame. First ${}^1\mathbf{T}_2$ rotates frames 1 and 2 by -45^o, then translates frames 1 and 2 by $(3/4, 3/4)$, all relative to frame 0. Then ${}^0\mathbf{T}_1$ rotates frame 2 by 30^o, then translates it by $(1, 1/2)$, all still relative to frame 0.

The result of ${}^0\mathbf{T}_2$ viewed as an operator or a coordinate transformation is the same, but the process of reaching the final placement of frame 2 is different.

What is the relation to the interpretation of the previous section? Figure 2.22(B) shows these operations from an observer fixed in frame 0. If the observer had been fixed in frame 2 as in the previous Figures 2.19-2.21, then frame 0 would have appeared to be undergoing the negative of the displacements just discussed.

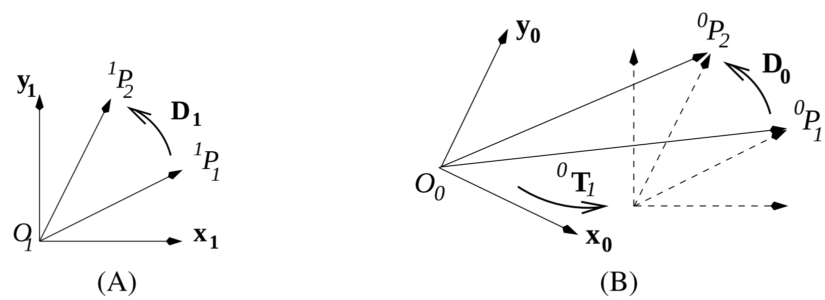

Figure 2.23: (A) Operator $\mathbf{D}_1$ moves ${}^1P_1$ to ${}^1P_2$ relative to frame 1. (B) The corresponding operator $\mathbf{D}_0$ in frame 0 moves ${}^0P_1$ to ${}^0P_2$.

2.6.2 Operation about a Different Frame

Let homogeneous transformation $\mathbf{D}_1$ operate on homogeneous coordinate ${}^1P_1$ in frame 1 to yield homogeneous coordinate ${}^1P_2$ (Figure 2.23(A)):

$$ {}^1P_2 = \mathbf{D}_1 \, {}^1P_1 \tag{2.74} $$

Consider another reference frame 0, located by homogeneous transformation ${}^0\mathbf{T}_1$ relative to frame 1 (Figure 2.23(B)). The corresponding homogeneous coordinates in frame 0 are:

$$ {}^0P_1 = {}^0\mathbf{T}_1 \, {}^1P_1 \tag{2.75} $$

$$ {}^0P_2 = {}^0\mathbf{T}_1 \, {}^1P_2 \tag{2.76} $$

Then

$$ {}^0P_2 = {}^0\mathbf{T}_1 \, \mathbf{D}_1 \, {}^1P_1 = {}^0\mathbf{T}_1 \, \mathbf{D}_1 \, {}^0\mathbf{T}_1^{-1} \, {}^0P_1 \tag{2.77} $$

Hence the operator

$$ \mathbf{D}_0 = {}^0\mathbf{T}_1 \, \mathbf{D}_1 \, {}^0\mathbf{T}_1^{-1} \tag{2.78} $$

moves ${}^0P_1$ to ${}^0P_2$ in the same way that $\mathbf{D}_1$ moves ${}^1P_1$ to ${}^1P_2$. Thus $\mathbf{D}_0$ is the same operator as $\mathbf{D}_1$, but in frame 0 rather than in frame 1.

2.7 Poles of Planar Displacements

For an arbitrary planar operator, there is one point called the *pole* that doesn't move [2]. Consider an arbitrary operator represented by rotation $\mathbf{R}(\theta)$ and translation ${}^1\mathbf{d}$, relative to frame 1. Figure 2.24(A) shows the effect of this operator on a frame 2 that is initially coincident with frame 1. The pole $\mathbf{C}$, represented by the vector ${}^1\mathbf{c}$, is that point which is invariant under this operator:

$$ {}^1\mathbf{c} = \mathbf{R} \, {}^1\mathbf{c} + {}^1\mathbf{d} \quad \Rightarrow \quad {}^1\mathbf{c} = (\mathbf{I} - \mathbf{R})^{-1} \, {}^1\mathbf{d} \tag{2.79} $$

There is one exception for which there is no solution: for a pure translation $\mathbf{R} = \mathbf{I}$. Let $\mathbf{D}_1$ be the operator, represented by the homogeneous transformation with elements $\mathbf{R}(\theta)$ and $\mathbf{d}$. Then

$$ {}^1C = \mathbf{D}_1 \, {}^1C \tag{2.80} $$

where ${}^1C$ represents the homogeneous coordinates fashioned from ${}^1\mathbf{c}$.

Suppose we change reference frames to frame 0 by the homogeneous transformation ${}^0\mathbf{T}_1$. What is the pole in frame 0? The pole identified in frame 1 is expressed in frame 0 as:

$$ {}^0C = {}^0\mathbf{T}_1 \, {}^1C \tag{2.81} $$

The corresponding operator $\mathbf{D}_0$ in frame 0 is given by (2.78), and its effect on ${}^0C$ is:

$$ \mathbf{D}_0 \, {}^0C = ({}^0\mathbf{T}_1 \, \mathbf{D}_1 \, {}^0\mathbf{T}_1^{-1})({}^0\mathbf{T}_1 \, {}^1C) = {}^0\mathbf{T}_1 \, \mathbf{D}_1 \, {}^1C = {}^0\mathbf{T}_1 \, {}^1C = {}^0C \tag{2.82} $$

Hence ${}^0C$ is the pole of $\mathbf{D}_0$.

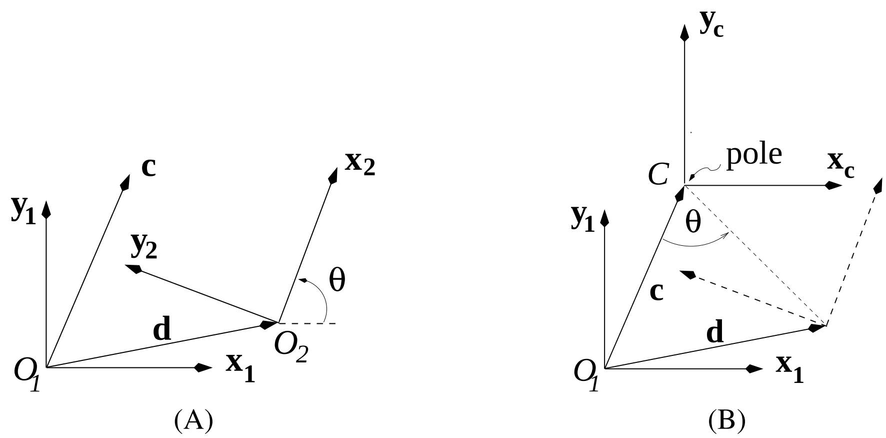

Figure 2.24: (A) The effect of operator $\mathbf{D}_1$ on frame 2, initially coincident with frame 1. (B) A pure rotation about frame c by θ captures the operator $\mathbf{D}_1$ about frame 1.

Suppose frame c is located at the pole (Figure 2.24(B)), with the same axes orientation as frame 1. Then ${}^0\mathbf{T}_1\ {}^1C = \mathbf{0}$, i.e., the pole is at the origin of frame 0. From this relation, it is easy to show that

$$ {}^0\mathbf{T}_1 = \begin{bmatrix} \mathbf{I} & -\,{}^0\mathbf{c} \\ \mathbf{0}^T & 1 \end{bmatrix} = \mathbf{Trans}(-\,{}^0\mathbf{c}) \tag{2.83} $$

where ${}^0\mathbf{c} = {}^1\mathbf{c}$ since axes 0 and 1 are parallel. Also ${}^0\mathbf{d} = {}^1\mathbf{d}$. Hence

$$
\begin{aligned}
\mathbf{D}_0 &= {}^0\mathbf{T}_1\, \mathbf{D}_1\, {}^0\mathbf{T}_1^{\,-1} \\
&= \begin{bmatrix} \mathbf{I} & -\,{}^0\mathbf{c} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & {}^0\mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & {}^0\mathbf{c} \\ \mathbf{0}^T & 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{I} & -\,{}^0\mathbf{c} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{R}\,{}^0\mathbf{c} + {}^0\mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{R} & \mathbf{R}\,{}^0\mathbf{c} + {}^0\mathbf{d} - {}^0\mathbf{c} \\ \mathbf{0}^T & 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}
\end{aligned}
$$

since ${}^0\mathbf{c} = \mathbf{R}\,{}^0\mathbf{c} + {}^0\mathbf{d}$ from (2.79). Hence $\mathbf{D}_0$ is a pure rotation about the pole C. Figure 2.24(B) shows that frame 2 is derived from frame 1 with a pure rotation relative to frame 0 at the pole. We reiterate this important result.

Any displacement in the plane is equivalent to a pure rotation about the pole.

This *center of rotation* is a useful concept in compliance control.

Example 2.12: Consider the operator (2.70). Its pole is found from (2.79):

$$\begin{aligned}
\mathbf{c} &= (\mathbf{I}-\mathbf{R})^{-1}\,\mathbf{d} \\
&= \begin{bmatrix} 1-\sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 1-\sqrt{2}/2 \end{bmatrix}^{-1} \begin{bmatrix} 3/4 \\ 3/4 \end{bmatrix} \\
&= \frac{1}{2-\sqrt{2}} \begin{bmatrix} 1-\sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & 1-\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 3/4 \\ 3/4 \end{bmatrix} \\
&= \frac{3}{4}\frac{1}{2-\sqrt{2}} \begin{bmatrix} 1 \\ 1-\sqrt{2} \end{bmatrix}
\end{aligned}$$

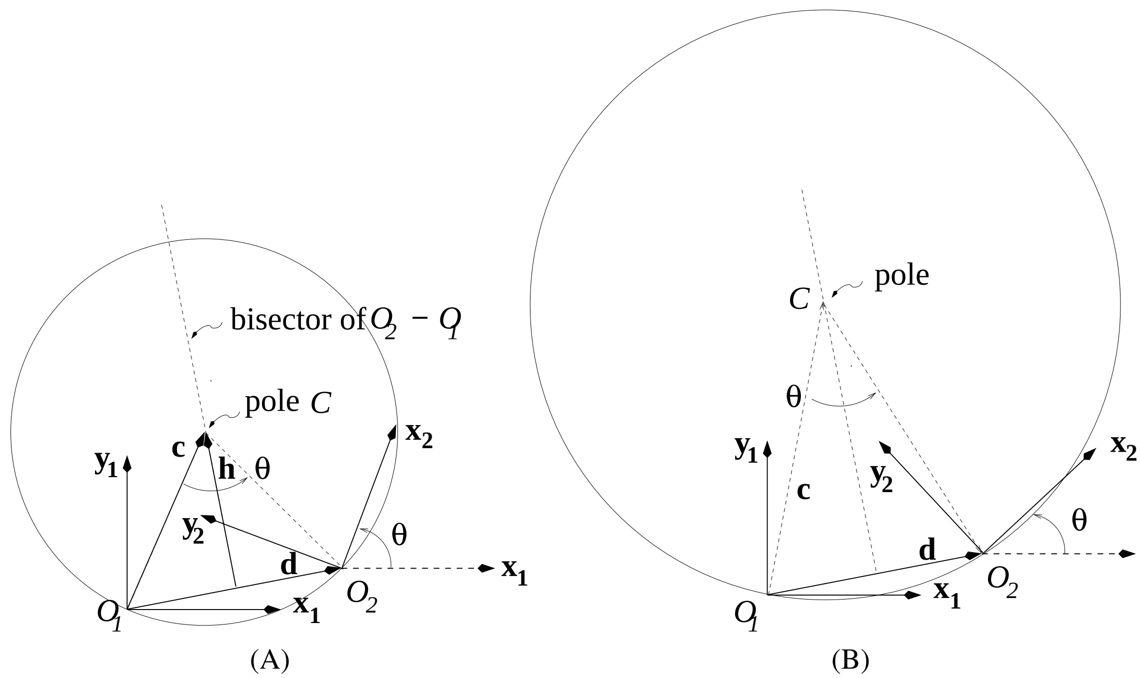

Figure 2.25: (A) A circle constructed so that its center is the pole. (B) A different circle results from a different rotation angle θ.

A more intuitive viewpoint is provided by Figure 2.25(A), which shows some construction lines added to Figure 2.24(A). Because the displacement referred to the pole is a pure rotation, origins O_1 and O_2 have to lie on the arc of a circle, and the pole C is the center of the circle. To find where the center is, the pole has to lie on the perpendicular bisector of the segment $\mathbf{d} = O_2 - O_1$. Form an isosceles triangle by drawing lines $\overline{O_1C}$ and $\overline{O_2O_1}$ and including the line $\overline{O_1O_2}$. The location of the pole on the bisector is that point where the angle between lines $\overline{O_1C}$ and $\overline{O_2C}$ is the same as the rotation angle θ between axes $\mathbf{x}_0$ and $\mathbf{x}_1$. Let $\mathbf{h}$ be the vector along the bisector from the midpoint of $\mathbf{d}$ to C, given by:

$$\mathbf{h} = \mathbf{R}(\pi/2)\frac{\mathbf{d}}{||\mathbf{d}||}\cot(\theta/2)\frac{||\mathbf{d}||}{2} = \frac{1}{2\tan(\theta/2)}\mathbf{R}(\pi/2)\mathbf{d} \tag{2.84}$$

Then the location of C, given by vector $\mathbf{c} = C - O_1$, is

$$\mathbf{c} = \frac{\mathbf{d}}{2} + \mathbf{h} = \frac{\mathbf{d}}{2} + \frac{1}{2\tan(\theta/2)}\mathbf{R}(\pi/2)\mathbf{d} \tag{2.85}$$

By moving the pole up and down on the bisector, different rotation angles are obtained. Figure 2.25(B) shows another example.

2.8 Summary

1. **Points** are distinct geometrical objects that exist independent of any coordinate description. They belong to an **affine space**, whose only meaningful operation is the difference between points.

2. **Vectors** represent a displacement between two points, and are not themselves geometric objects. Vectors belong to a **vector space,** defined as one in which the linear combination of two vectors is still a vector.

3. A **basis** is a minimal set of vectors that spans a vector space, i.e., any vector can be expressed as a linear combination of the basis vectors. Of the infinitely many possible bases, the most important are the orthonormal bases $\mathbf{x}_j, \mathbf{y}_j$, whose vectors are orthogonal and of unit length.

4. A **coordinate system** or **frame** is an origin O_j plus an orthonormal basis $\mathbf{x}_j, \mathbf{y}_j$. The orthonormal basis vectors are referred to as the **axes** of the coordinate system.

5. The **coordinates** of a vector ${}^j\mathbf{v}$ with respect to axes j are the coefficients on the linear combinations of the orthonormal basis vectors j in which ${}^j\mathbf{v}$ is expressed. They are formed into a 2-by-1 column vector.

6. A **rotation matrix** ${}^{j-1}\mathbf{R}_j = \mathbf{R}(\theta_j)$ describing the angle θ_j from axes $j-1$ to axes j has as its columns the axes j expressed in terms of axes $j-1$. The inverse of a rotation matrix is its transpose.

7. A **coordinate transformation** represents the relation between two arbitrary coordinate systems by a displacement between the origins and a rotation between the axes.

8. The **homogeneous coordinates** of a point P relative to origin O_j are formed as the 3-by-1 column vector $[\mathbf{p} \;\; 1]^T$, by adding a 3rd coordinate 1 to the vector $\mathbf{p} = P - O_j$. The homogeneous coordinates of a point can be interpreted as lying *on* the **affine plane** $z = 1$ in three-dimensional space. A vector represented by homogeneous coordinates is then interpreted as lying *in* the affine plane with 3rd coordinate 0.

9. A **homogeneous transformation** represents a coordinate transformation as a 3-by-3 matrix, applied to the homogeneous coordinates of a point. It is a compact way to combine rotations and translations into a single matrix-vector operation. The special transformations $\mathbf{Rot}$ and $\mathbf{Trans}$ perform rotation and translation, respectively, and can be combined to define an arbitrary homogeneous transformation.

10. An **operator** actively moves points with respect to the fixed frame. By contrast, a coordinate transformation describes the instantaneous relationship between two coordinate systems. However, the same mathematics represents operators and coordinate transformations, but interpreted differently. Consider a composition of several homogeneous transformations.

    - A coordinate transformation is interpreted as a left-to-right evaluation of the composition about the *current* frame.
    - An operator is interpreted as a right-to-left evaluation about the *fixed* frame.

    Another interpretation is that moving a point forwards is the same as moving the reference coordinate system backwards. This viewpoint corresponds to an observer in the last versus the first frame, respectively.

11. The **pole** of a planar displacement is that point which does not move under an arbitrary planar operator. It is unique, unless the planar operator is a pure translation. An operator can be related to a different frame by pre- and post-multiplying by the coordinate transformation to the different frame and its inverse respectively. When related to the pole, the equivalent operator is a pure rotation.

2.9 Notation Synopsis

1. **Points** are denoted by upper-case italic letters with subscripts, e.g., O_j, P_j, etc. The subscripts j just identify different instances.

    (a) O_j is the origin of coordinate system j.

2. **Scalars** are repesented by subscripted lower-case italic letters, e.g., a_j, b_j, etc.

3. **Vectors** are denoted by lower-case bold letters with subscripts and superscripts, e.g., ${}^i\mathbf{v}_j$. The superscript i denotes the particular coordinate axes i with respect to which the coordinates of $\mathbf{v}_j$ are expressed.

    (a) The **axes** for coordinate system j are $\mathbf{x}_j, \mathbf{y}_j$.

    (b) The **coordinates** of vector $\mathbf{v}$ with respect to axes j are

    $$ {}^j\mathbf{v} = {}^jv_1\, {}^j\mathbf{x}_j + {}^jv_2\, {}^j\mathbf{y}_j = \begin{bmatrix} {}^jv_1 \\ {}^jv_2 \end{bmatrix} $$

    For a subscripted vector such as $\mathbf{v}_k$, the two coordinates are indicated by double subscripts: ${}^jv_{k1}, {}^jv_{k2}$.

    (c) The coordinates of axes j with respect to themselves are

    $$ {}^j\mathbf{x}_j = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{i}, \qquad {}^j\mathbf{y}_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{j} $$

    (d) All vectors are considered as **column vectors**.

    (e) The **transpose** of the column vector $\mathbf{v}$ with respect to basis j is the row vector:

    $$ \mathbf{v}^T = \begin{bmatrix} {}^jv_1 & {}^jv_2 \end{bmatrix} $$

    (f) The **dot product** of two vectors ${}^j\mathbf{v} = [{}^jv_1 \;\; {}^jv_2]^T$ and ${}^j\mathbf{w} = [{}^jw_1 \;\; {}^jw_2]^T$ is

    $$ {}^j\mathbf{v} \cdot {}^j\mathbf{w} = {}^j\mathbf{v}^T\, {}^j\mathbf{w} = {}^jv_1\, {}^jw_1 + {}^jv_2\, {}^jw_2 $$

    (g) The **length** or **Euclidean norm** of a vector is

    $$ ||{}^j\mathbf{v}|| = \sqrt{{}^j\mathbf{v} \cdot {}^j\mathbf{v}} = \sqrt{({}^jv_1)^2 + ({}^jv_2)^2} $$

    (h) $\mathbf{d}_{ij} = O_j - O_i$ is the displacement from origin O_i to origin O_j.

4. **Matrices** are represented by upper-case bold letters, e.g., $\mathbf{C}$, $\mathbf{I}$, $\mathbf{R}$, etc., which may have superscripts or subscripts.

    (a) $\mathbf{R}(\theta)$ is the **rotation matrix** relative to rotation angle θ.

    (b) ${}^{j-1}\mathbf{R}_j = \mathbf{R}(\theta_j)$ is the more general notation for rotation matrix describing the orientation of axes j to axes $j-1$. Any two arbitrary axes i and j are related by ${}^i\mathbf{R}_j$.

    (c) $\mathbf{I}$ is the 2-by-2 **identity matrix**.

(d) $\mathbf{Rot}(\theta)$ is the **rotational homogeneous transformation**

$$\mathbf{Rot}(\theta) = \begin{bmatrix} \mathbf{R}(\theta) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

where $\mathbf{0} = [0 \ \ 0]^T$.

(e) $\mathbf{Trans}(\mathbf{d})$ is the **translational homogeneous transformation**

$$\mathbf{Trans}(\mathbf{d}) = \begin{bmatrix} \mathbf{I} & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

(f) The **homogeneous transformation** from coordinate system j to coordinate system i is ${}^i\mathbf{T}_j$. It has the form

$${}^i\mathbf{T}_j = \mathbf{Trans}({}^i\mathbf{d}_{ij})\mathbf{Rot}(\theta_{ij}) = \begin{bmatrix} \mathbf{R}(\theta_{ij}) & {}^i\mathbf{d}_{ij} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

where $\theta_{ij} = \theta_{i+1} + \cdots + \theta_j$.

5. The homogeneous coordinate representation of a point is

$${}^jP_i = \begin{bmatrix} {}^j\mathbf{p}_i \\ 1 \end{bmatrix}$$

where the superscript j indicates the reference coordinate system for the affine plane $z = 1$.

(a) The homogeneous coordinate representation of a vector ${}^j\mathbf{p}_i$ corresponding to the displacement from origin ${}^jO_j$ to point ${}^jP_i$ is

$${}^j\boldsymbol{P}_i = {}^jP_j - {}^jO_j = \begin{bmatrix} {}^j\mathbf{p}_1 \\ 0 \end{bmatrix}$$

and is represented as an upper-case letter in a slanted bold font.

Bibliography

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