Chapter 2 Planar Transformations + Displacements

Pass 1:

- Headings
  - Points/Vectors
    - Affine Spaces
      - Affine Spaces don't have any vectors, they're basically where points live. (You can't add points, only subtract them)
    - Vector Spaces
      - Vectors usually expressed as other vectors
    - Orthonormal bases
      - The most useful bases (minimal set of vectors) (orthogonal and unit vector as base)
    - Coordinate systems
    - Locating arbitrary points
  - Planar rotational Transformations
    - Composition of Planar Rotations
    - Planar coordinate Transforms
    - Composition of Coordinate Transformations
  - Two-link Planar manipulator kinematics
  - Homogeneous Transformations in a plane
    - Composition of Homogeneous Transformations
    - Homogeneous Coordinates and the Affine Plane
  - Operators
    - Composition of operators
    - Operation about a different frame
  - Poles of planar displacements
- There's a summary at the end of all these chapters….. Just start with that stupid thing

$\vec{v} \in \mathbb{R}^2$ ← belongs to 2D space

$\vec{v} \in \mathbb{R}^3$ ← belongs to 3D space

Pass 2:

· Addition of points doesn't make sense, but subtraction of points does

Basis = minimal set of vectors in a vector space

$\underbrace{\vec{x}_j \quad \vec{y}_j}_{\text{most important orthonormal bases (unit length)}}$

Vector coordinates are linear combinations of orthonormal base vectors

Point notation
· Italic capital letter w/subscript

O_i = fixed origin i

Vector notation
$^{i}\vec{v}_j$ i = coordinate axes/base

Rotation matrix: $^{j-1}R_i = R(\theta_i)$

- Describes ∡ θ_j from axes $j-1$ to axes j
- Columns are the axes of j expressed in terms of axes $j-1$
- Inverse of rotation matrix is its transpose

Coordinate transformation:

- Relationship between two arbitrary coordinate systems by a displacement between origins + rotation between axes

Pole

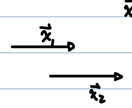
- Point that doesn't move under planar operator
- Unique, unless there's pure translation

- OPERATORS CAN BE RELATED TO POLE BY PRE & POST MULTIPLYING BY COORDINATE TRANSFORMATION TO DIFFERENT FRAME + ITS INVERSE

PASS 3 :

- WE ATTACH COORDINATE SYSTEMS TO OBJECTS + ROTATE THEM
- UNIT VECTORS ARE EQUAL IF THEY ARE POINTED IN THE SAME DIRECTION

$\vec{x}_1 = \vec{x}_2$



- WE CAN DESCRIBE POINTS AS $X_2 = \underbrace{O_2 + \vec{x}_2}_{\text{RELATIVE TO } O_1}$

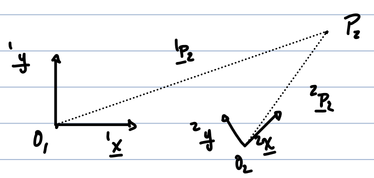
$X_j = O_j + \vec{x}_j$ OR.... $X_2' - O_1 = \vec{x}_2$ ← VECTOR

ARBITRARY POINTS

POINT: P_2

$^2P_2 = {}^2P_{21}\,{}^2\underline{x}_2 + {}^2P_{22}\,{}^2\underline{y}_2$] POINT P_2 VECTOR WRT BASIS {2}

$^1P_2 = {}^1P_{21}\,{}^1\underline{x}_1 + {}^1P_{22}\,{}^1\underline{y}_1$] " WRT BASIS {1}



PLANAR ROTATION TRANSFORMATION

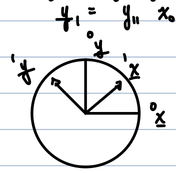
TWO COORDINATE SYSTEMS: 0 , 1

- ORIGINS COINCIDE
- AXES 1 ROTATED BY θ_1 FROM AXES 0

AXES 1 WRT AXES 0

$$^0\underline{x}_1 = {}^0x_{11}\,{}^0x_0 + {}^0x_{12}\,{}^0y_0$$

$$^0\underline{y}_1 = {}^0y_{11}\,{}^0x_0 + {}^0y_{12}\,{}^0y_0$$



RELATIONSHIPS:

$$^0\underline{x}_1 = {}^0\underline{x}_0\cos\theta_1 + {}^0\underline{y}_0\sin\theta_1 = \begin{bmatrix} {}^0\underline{x}_0 & {}^0\underline{y}_0 \end{bmatrix}\begin{bmatrix}\cos\theta_1\\ \sin\theta_1\end{bmatrix} = \begin{bmatrix}\cos\theta_1\\ \sin\theta_1\end{bmatrix}$$

$$^0\underline{y}_1 = -{}^0\underline{x}_0\sin\theta_1 + {}^0\underline{y}_0\cos\theta_1 = \begin{bmatrix} {}^0\underline{x}_0 & {}^0\underline{y}_0 \end{bmatrix}\begin{bmatrix}-\sin\theta_1\\ \cos\theta_1\end{bmatrix} = \begin{bmatrix}-\sin\theta_1\\ \cos\theta_1\end{bmatrix}$$

COLUMNS ARE COORDINATE AXES {1} WRT AXES {0}

$$\Rightarrow \begin{bmatrix} {}^0\underline{x}_1 & {}^0\underline{y}_1 \end{bmatrix} = \begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{bmatrix} \doteq \underline{R}(\theta_1)$$

↑ PLANAR ROTATION MATRIX ☺

$\begin{bmatrix} {}^1\underline{x}_0 & {}^1\underline{y}_1 \end{bmatrix} = \underset{\sim}{R}(\theta_1)^T$

$${}^1\underline{v} = \underline{R}(\theta_1)^T \, {}^0\underline{v}$$

COMPOSITION OF ROTATION MATRICIES:

$${}^{i-1}\underset{\sim}{R} = \underset{\sim}{R}(\theta_i)$$

$${}^i\underset{\sim}{R}_j = \underset{\sim}{R}(\theta_{ij})$$

ROTATION MATRIX FROM AXES j TO i

NOT TRUE IN 3D

$$R(\underset{\sim}{\theta_1})\,R(\underset{\sim}{\theta_2}) = \underset{\sim}{R}(\theta_1 + \theta_2)$$

ALWAYS TRUE:

$${}^i\underline{v} = {}^i\underset{\sim}{R}_j \, {}^j\underline{v}$$

$$({}^i\underline{R}_j)^{-1} = {}^i\underset{\sim}{R}_j{}^T = {}^j\underline{R}_i$$

PLANAR COORDINATE TRANSFORMATIONS

ORIGINS

$\underline{d}_{ij}$ = VECTOR FROM O_i TO O_j

$\underline{P}_0$ IS A POINT WRT O_0, THEREFORE:

$${}^0\underline{P}_0 = {}^0\underline{d}_{01} + {}^0\underline{P}_1$$

AND $\quad {}^0\underline{P}_1 = {}^0\underset{\sim}{R}_1 \, {}^1\underline{P}_1$

$$\Rightarrow {}^0\underline{P}_0 = {}^0\underline{d}_{01} + {}^0\underline{R}_1 \, {}^1\underline{P}_1$$

$$= {}^0\underline{d}_{01} + {}^0\underline{d}_{12} + {}^0P_2$$

TWO-LINK PLANAR MANIPULATOR KINEMATICS

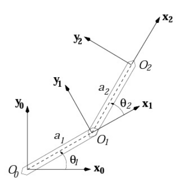


Figure 2.15: Two-link planar manipulator.

$\{0\}$ = GND

$\{1\}$ = DISTAL END LINK 1 $\quad O_0$ TO O_1

$\{2\}$ = " " " 2 $\quad O_1$ TO O_2

FORWARD KINEMATICS

- FIND ENDPOINT POSITION ${}^0\underline{d}_{02}$ $\quad$ (O_2 WRT O_0)

$$\rightsquigarrow {}^0d_{02} = {}^0d_{01} + {}^0d_{12} = a_1 \, {}^0\underset{\sim}{R}_1 \, {}^1\underline{x}_1 + a_2 \, {}^0R_2 \, {}^2\underline{x}_2$$

UNIT VECTORS

Homogeneous Transformation in a Plane

$$\underline{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}^T \rightsquigarrow P = \begin{bmatrix} \underline{p} \\ 1 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix}$$

↑ POINT P

COORDINATE TRANSFORM

$$\begin{bmatrix} {}^0\underline{p}_1 \\ 1 \end{bmatrix} = \begin{bmatrix} {}^0\underline{R}_1 {}^1\underline{p}_1 + {}^0\underline{d}_{01} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^0\underline{R}_1 & {}^0\underline{d}_{01} \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} {}^1\underline{p}_0 \\ 1 \end{bmatrix}$$

$\underline{0}^T = [0 \quad 0]$

${}^0\underset{\sim}{T}_1$ = 3x3 HOMOGENEOUS TRANSFORMATION

$$\rightsquigarrow {}^0P = {}^0\underset{\sim}{T}_1 \, {}^1P$$

Composition of Homogeneous Transformations

$${}^0\underset{\sim}{T}_2 = {}^0\underset{\sim}{T}_1 \, {}^1\underset{\sim}{T}_2$$

↑

TRANSFORM FROM $\{1\}$ TO $\{0\}$

$$\left({}^0\underset{\sim}{T}_1\right)^{-1} = \begin{bmatrix} {}^0\underline{R}_1 & -{}^0\underset{\sim}{R}_1^T \, {}^0d_{01} \\ \underset{\sim}{0}^T & 1 \end{bmatrix} = {}^1\underset{\sim}{T}_0$$

HOMOGENEOUS MEANS SAME COORDINATE SPACE

Operators

- COORDINATE TRANSFORMS ARE STATIC REPRESENTATIONS OF RELATIVE LOCATION WITHIN A NON-STATIC SYSTEM
  - OPERATOR = DISPLACEMENT = A PROCESS OF MOVING A POINT WRT TO A COORDINATE AXIS

OPERATIONS ARE IN THE FIXED FRAME

COORDINATE TRANSFORMS ARE WRT CURRENT FRAME

OPERATORS:

$\text{Trans}(\underline{v})$ ← TRANSLATES POINT P_1

$$P_2 = \text{Trans}({}^0\underline{d}_1)\, P_1$$

$${}^0\underline{p}_2 = {}^0\underline{p}_1 + {}^0\underline{d}$$

↑ P_1 WRT ORIGIN

$\text{Rot}(\theta_1)$ = ROTATES P_1 BY θ_1 TO P_2

GENERAL TRANSFORMATION OPERATOR

$$P_2 = \begin{bmatrix} \underline{R}(\theta_1) & {}^0\underline{d} \\ \underline{0}^T & 1 \end{bmatrix} P_1 = {}^0\underline{T}_1 P_1$$

POLES OF PLANAR DISPLACEMENTS

· THERE ARE POINTS WHERE THE POINT DOESN'T MOVE

$^{1}\underline{c}$ = POLE VECTOR $\quad\quad$ $\underset{\sim}{R}(\theta)$ = ARBITRARY ROTATION

C = POLE POINT $\quad\quad$ $^{1}\underline{d}$ = TRANSLATION

$$^{1}\underline{c} = {}^{1}\underset{\sim}{R}\,{}^{1}c + {}^{1}\underline{d} \Rightarrow {}^{1}\underline{c} = (\underset{\sim}{I} - \underset{\sim}{R})^{-1}\,{}^{1}\underline{d}$$

PURE TRANSLATION HAS $\underset{\sim}{R} = \underset{\sim}{I}$

$\underset{\sim}{D}_1$