Chapter 8

Velocity Kinematics

8.1 Joint-Constrained Motion

The effect of joint constraints is to make the interorigin velocity ${}^0\dot{\mathbf{d}}_{i-1,i}$ dependent on joint angle velocities $\dot{\theta}_j$ for rotary joints $j \le i$ and also on the joint translational velocity $\dot{d}_i$ for a prismatic joint i.

8.1.1 Motion at a Single Joint

Consider first that all joints are stationary except joint i. For a rotary joint i (Figure 8.1(A)), the angular velocity ${}^0\boldsymbol{\omega}_{i-1,i}$ of link i relative to link $i-1$ is:

$$ {}^0\boldsymbol{\omega}_{i-1,i} \;=\; \dot{\theta}_i\,{}^0\mathbf{z}_{i-1} \qquad \text{for a rotary joint } j \tag{8.1} $$

That is to say, this is the angular velocity imparted to link i due solely to the rotation of joint i, irrespective of any other joints. If joint i is prismatic (Figure 8.1(B)), it imparts no rotation to link i, hence

$$ {}^0\boldsymbol{\omega}_{i-1,i} = \mathbf{0} \qquad \text{for a prismatic joint } j \tag{8.2} $$

Linear motion is more complicated. The interorigin vector ${}^0\mathbf{d}_{i-1,i}$ according to the DH parameters is:

$$ {}^0\mathbf{d}_{i-1,i} = d_i\,{}^0\mathbf{z}_{i-1} + a_i\,{}^0\mathbf{x}_i \tag{8.3} $$

If joint i is rotary, then since only joint i is moving the axis $\mathbf{z}_{i-1}$ is stationary but $\mathbf{x}_i$ is rotating about $\mathbf{z}_{i-1}$ with joint angle velocity $\dot{\theta}_i$. Then the linear velocity ${}^0\dot{\mathbf{d}}_{i-1,i}$ of origin O_i relative to O_{i-1} is

$$ {}^0\dot{\mathbf{d}}_{i-1,i} \;=\; \dot{\theta}_i\,{}^0\mathbf{z}_{i-1} \times a_i\,{}^0\mathbf{x}_i \qquad \text{(joint } i \text{ rotary, only joint } i \text{ is moving)} \tag{8.4} $$

Even though $\mathbf{z}_{i-1}$ is stationary, it is convenient to write formally that

$$ \begin{aligned} {}^0\dot{\mathbf{d}}_{i-1,i} &= \dot{\theta}_i\,{}^0\mathbf{z}_{i-1} \times (d_i\,{}^0\mathbf{z}_{i-1} + a_i\,{}^0\mathbf{x}_i) \\ &= {}^0\boldsymbol{\omega}_{i-1,i} \times {}^0\mathbf{d}_{i-1,i} \qquad \text{(joint } i \text{ rotary, only joint } i \text{ is moving)} \end{aligned} \tag{8.5} $$

where (8.1) and (8.3 have been substituted and the term with $\mathbf{z}_{i-1}$ crossed with itself is zero and doesn't change the linear velocity.

If joint i is prismatic, motion is due only to the prismatic variable d_i, and so $\boldsymbol{\omega}_{i-1,i} = \mathbf{0}$. Differentiating (8.3) yields

$$ {}^0\dot{\mathbf{d}}_{i-1,i} \;=\; \dot{d}_i\,{}^0\mathbf{z}_{i-1} \qquad \text{(joint } i \text{ prismatic, only joint } i \text{ is moving)} \tag{8.6} $$

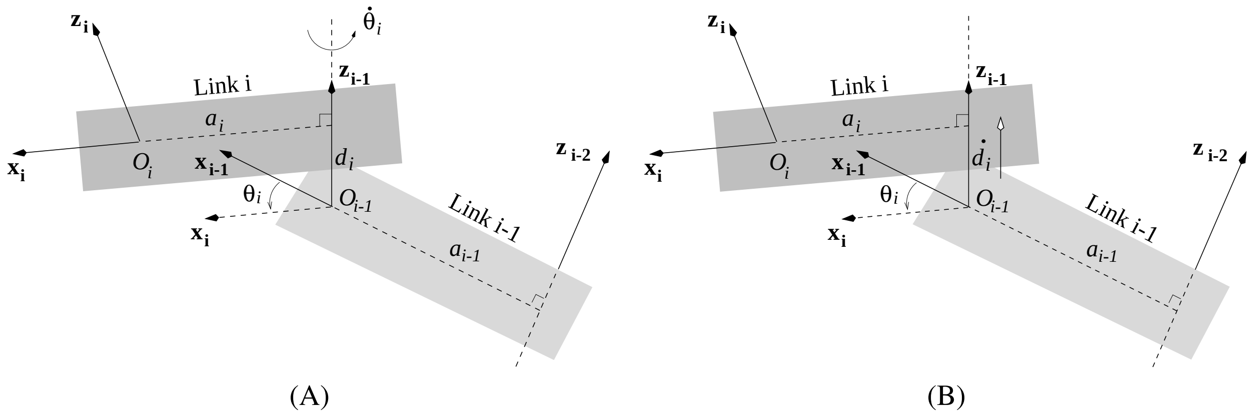


Figure 8.1: (A) Rotary-joint motion of link i relative to link $i-1$. (B) Prismatic-joint motion of link i relative to link $i-1$.

8.1.2 Motion at All Joints

Now consider that all joints 1 to i are moving. The relative angular velocity ${}^0\boldsymbol{\omega}_{i-1,i}$ stays the same according to (8.1) or (8.2). The total angular velocity ${}^0\boldsymbol{\omega}_{0i}$ of link i is from Chapter 7 the sum of the angular velocity contributions due to all joints from 1 to i:

$$\begin{aligned} {}^0\boldsymbol{\omega}_{0i} &= \sum_{j=1}^{i} {}^0\boldsymbol{\omega}_{j-1,j} \\ &= \sum_{j=1}^{i} \begin{cases} \dot{\theta}_j \, {}^0\mathbf{z}_{j-1} & \text{for a rotary joint } j \\ \mathbf{0} & \text{for a prismatic joint } j \end{cases} \end{aligned} \tag{8.7}$$

It will also be useful to express the angular velocity recursively:

$$\begin{aligned} {}^0\boldsymbol{\omega}_{0i} &= {}^0\boldsymbol{\omega}_{0,i-1} + {}^0\boldsymbol{\omega}_{i-1,i} \\ &= {}^0\boldsymbol{\omega}_{0,i-1} + \begin{cases} \dot{\theta}_i \, {}^0\mathbf{z}_{i-1} & \text{for a rotary joint } i \\ \mathbf{0} & \text{for a prismatic joint } i \end{cases} \end{aligned} \tag{8.8}$$

The interorigin linear velocity is again obtained by differentiating (8.3). If joint i is rotary, then axis $\mathbf{x}_i$ is spinning with angular velocity $\boldsymbol{\omega}_{0i}$ while axis $\mathbf{z}_{i-1}$ is spinning with angular velocity $\boldsymbol{\omega}_{0,i-1}$. Hence

$$ {}^0\dot{\mathbf{d}}_{i-1,i} = {}^0\boldsymbol{\omega}_{0,i-1} \times d_i \, {}^0\mathbf{z}_{i-1} + {}^0\boldsymbol{\omega}_{0i} \times a_i \, {}^0\mathbf{x}_i \qquad \text{(joint } i \text{ rotary, all joints moving)} $$

Again, it is convenient to rework this expression into a different form. From the recursive definition (8.8) applied to a rotary joint i,

$$\begin{aligned} {}^0\dot{\mathbf{d}}_{i-1,i} &= ({}^0\boldsymbol{\omega}_{0,i-1} + \dot{\theta}_i \, {}^0\mathbf{z}_{i-1}) \times d_i \, {}^0\mathbf{z}_{i-1} + {}^0\boldsymbol{\omega}_{0i} \times a_i \, {}^0\mathbf{x}_i \\ &= {}^0\boldsymbol{\omega}_{0i} \times {}^0\mathbf{d}_{i-1,i} \qquad \text{(joint } i \text{ rotary, all joints moving)} \end{aligned} \tag{8.9}$$

where the interorigin vector (8.3) has also been substituted. We can say that $\mathbf{z}_{i-1}$ is also spinning with angular velocity $\boldsymbol{\omega}_{0i}$ because the last rotational motion is about itself and has no effect on it.

If joint i is prismatic, then joint i does not impart any additional angular velocity to link i. Hence $\boldsymbol{\omega}_{0i} = \boldsymbol{\omega}_{0,i-1}$. Not only is link i rotating with angular velocity ${}^0\boldsymbol{\omega}_{0,i-1}$ but it is translating with velocity $\dot{d}_i$ along $\mathbf{z}_{i-1}$. Differentiating (8.3),

$$\begin{aligned} {}^0\dot{\mathbf{d}}_{i-1,i} &= \dot{d}_i\,{}^0\mathbf{z}_{i-1} + {}^0\boldsymbol{\omega}_{0,i-1} \times d_i\,{}^0\mathbf{z}_{i-1} + {}^0\boldsymbol{\omega}_{0i} \times a_i\,{}^0\mathbf{x}_i \\ &= \dot{d}_i\,{}^0\mathbf{z}_{i-1} + {}^0\boldsymbol{\omega}_{0i} \times {}^0\mathbf{d}_{i-1,i} \qquad \text{(joint } i \text{ prismatic, all joints moving)} \end{aligned} \tag{8.10}$$

8.1.3 Manipulator Velocity Kinematics

Now consider a manipulator with n links. The angular velocity of the last link is obtained from (8.7):

$$\begin{aligned} {}^0\boldsymbol{\omega}_{0n} &= \sum_{i=1}^{n} {}^0\boldsymbol{\omega}_{i-1,i} \\ &= \sum_{i=1}^{n} \begin{cases} \dot{\theta}_i\,{}^0\mathbf{z}_{i-1} & \text{for a rotary joint } i \\ \mathbf{0} & \text{for a prismatic joint } i \end{cases} \end{aligned} \tag{8.11}$$

The position of the last link origin is:

$${}^0\mathbf{d}_{0n} = \sum_{i=1}^{n} {}^0\mathbf{d}_{i-1,i}$$

Its linear velocity is

$$\begin{aligned} {}^0\dot{\mathbf{d}}_{0n} &= \sum_{i=1}^{n} {}^0\dot{\mathbf{d}}_{i-1,i} \\ &= \sum_{i=1}^{n} \begin{cases} {}^0\boldsymbol{\omega}_{0i} \times {}^0\mathbf{d}_{i-1,i} & \text{for a rotary joint } i \\ {}^0\boldsymbol{\omega}_{0i} \times {}^0\mathbf{d}_{i-1,i} + \dot{d}_i\,{}^0\mathbf{z}_{i-1} & \text{for a prismatic joint } i \end{cases} \end{aligned} \tag{8.12}$$

where (8.9) and (8.10) have been substituted.

8.2 The Velocity Jacobian $\mathbf{J}_v$

An alternative formulation for the endpoint linear velocity ${}^0\dot{\mathbf{d}}_{0n}$ will be useful, where the contributions of individual joint motions are separated out. In (8.12), a particular joint angle velocity $\dot{\theta}_j$ will appear as a part of all ${}^0\dot{\mathbf{d}}_{i-1,i}$ when $i \geq j$, but we want a reformulation where the effect of $\dot{\theta}_j$ on the endpoint velocity is only indicated in one term. This could be done by subsituting for ${}^0\boldsymbol{\omega}_{0,i} = \sum_{j=1}^{i} {}^0\boldsymbol{\omega}_{j-1,j}$ and collecting all terms with $\dot{\theta}_j$, but a more intuitive derivation is possible.

In Figure 8.2 the vector $\mathbf{d}_{i-1,n} = O_n - O_{i-1}$ from coordinate origin O_{i-1} to the end frame origin O_n is formed for each joint i. If joint i is rotary, the effect of the rotation of joint i on linear velocity of the endpoint is

$$\dot{\theta}_i\,{}^0\mathbf{z}_{i-1} \times {}^0\mathbf{d}_{i-1,n}$$

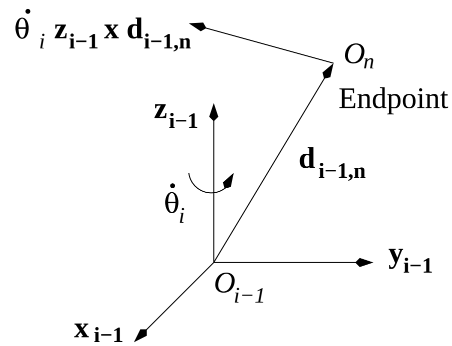


Figure 8.2: Linear velocity at endpoint due to rotation at joint i.

If joint i is prismatic, the effect is $\dot{d}_i\ {}^0\mathbf{z}_{i-1}$ as before. The total linear velocity of the endpoint sums all of these individual joint contributions:

$$
{}^0\dot{\mathbf{d}}_{0n} = \sum_{i=1}^{n} \begin{cases} \dot{\theta}_i\ {}^0\mathbf{z}_{i-1} \times {}^0\mathbf{d}_{i-1,n} & \text{for a rotary joint } i \\ \dot{d}_i\ {}^0\mathbf{z}_{i-1} & \text{for a prismatic joint } i \end{cases} \tag{8.13}
$$

Initially assume that all joints are rotary. Then from (8.11) and (8.13),

$$
\begin{aligned}
{}^0\boldsymbol{\omega}_{0n} &= \sum_{i=1}^{n} \dot{\theta}_i\, {}^0\mathbf{z}_{i-1} \\
{}^0\dot{\mathbf{d}}_{0n} &= \sum_{i=1}^{n} \dot{\theta}_i\, {}^0\mathbf{z}_{i-1} \times {}^0\mathbf{d}_{i-1,n}
\end{aligned}
$$

Rearranging into matrix/vector form,

$$
\begin{aligned}
\begin{bmatrix} {}^0\dot{\mathbf{d}}_{0n} \\ {}^0\boldsymbol{\omega}_{0n} \end{bmatrix} &= \sum_{i=1}^{n} \dot{\theta}_i \begin{bmatrix} {}^0\mathbf{z}_{i-1} \times {}^0\mathbf{d}_{i-1,n} \\ {}^0\mathbf{z}_{i-1} \end{bmatrix} \\
&= \begin{bmatrix} {}^0\mathbf{z}_0 \times {}^0\mathbf{d}_{0n} & {}^0\mathbf{z}_1 \times {}^0\mathbf{d}_{1n} & \cdots & {}^0\mathbf{z}_{n-1} \times {}^0\mathbf{d}_{n-1,n} \\ {}^0\mathbf{z}_0 & {}^0\mathbf{z}_1 & \cdots & {}^0\mathbf{z}_{n-1} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} \\
&\equiv \mathbf{J}_v \dot{\boldsymbol{\theta}}
\end{aligned} \tag{8.14}
$$

where $\dot{\boldsymbol{\theta}} = (\dot{\theta}_1, \ldots, \dot{\theta}_n)$ is the vector of joint angle velocities and $\mathbf{J}_v$ is the *velocity Jacobian:*

$$
\mathbf{J}_v = \begin{bmatrix} {}^0\mathbf{z}_0 \times {}^0\mathbf{d}_{0n} & {}^0\mathbf{z}_1 \times {}^0\mathbf{d}_{1n} & \cdots & {}^0\mathbf{z}_{n-1} \times {}^0\mathbf{d}_{n-1,n} \\ {}^0\mathbf{z}_0 & {}^0\mathbf{z}_1 & \cdots & {}^0\mathbf{z}_{n-1} \end{bmatrix} \tag{8.15}
$$

For a prismatic joint i, column i of $\mathbf{J}_v$ becomes:

$$
\begin{bmatrix} {}^0\mathbf{z}_{i-1} \\ \mathbf{0} \end{bmatrix} \tag{8.16}
$$

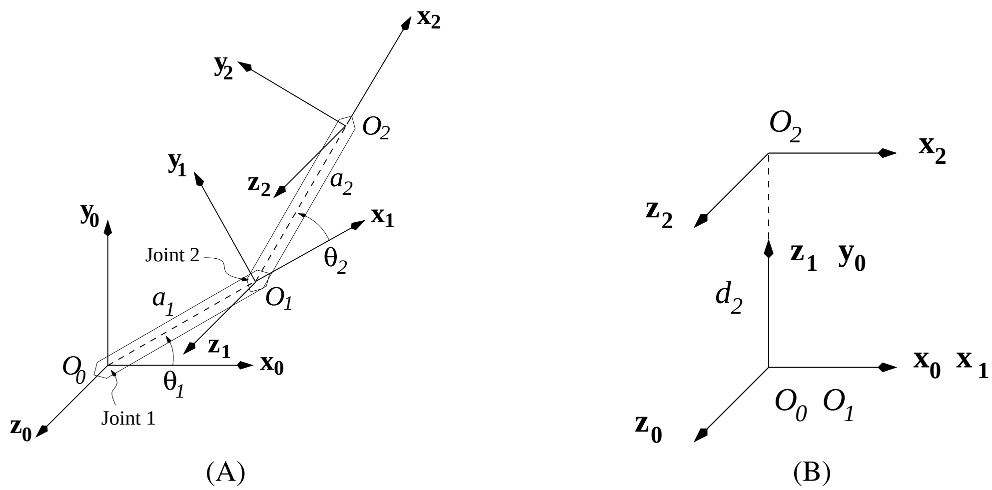


Figure 8.3: (A) The two-link rotary planar manipulator. (B) The polar robot.

and $\dot{\theta}_i$ is replaced by $\dot{d}_i$.

8.2.1 Planar Two-Link Manipulator

As the first example, the velocity Jacobian for the two-link rotary planar manipulator (Figure 8.3(A)) has the form:

$$\mathbf{J}_v = \begin{bmatrix} {}^0\mathbf{z}_0 \times {}^0\mathbf{d}_{02} & {}^0\mathbf{z}_1 \times {}^0\mathbf{d}_{12} \\ {}^0\mathbf{z}_0 & {}^0\mathbf{z}_1 \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{z}_0 \times (a_1\,{}^0\mathbf{x}_1 + a_2\,{}^0\mathbf{x}_2) & {}^0\mathbf{z}_1 \times a_2\,{}^0\mathbf{x}_2 \\ {}^0\mathbf{z}_0 & {}^0\mathbf{z}_1 \end{bmatrix} \tag{8.17}$$

The cross products are simplified by noting that $\mathbf{z}_0 = \mathbf{z}_1 = \mathbf{z}_2$.

$$\mathbf{J}_v = \begin{bmatrix} a_1\,{}^0\mathbf{y}_1 + a_2\,{}^0\mathbf{y}_2 & a_2\,{}^0\mathbf{y}_2 \\ {}^0\mathbf{z}_0 & {}^0\mathbf{z}_1 \end{bmatrix} \tag{8.18}$$

8.2.2 Polar Robot

The polar robot has the second joint prismatic (Figure 8.3(B)), and therefore the second column of the velocity Jacobian has the form (8.16).

$$\mathbf{J}_v = \begin{bmatrix} {}^0\mathbf{z}_0 \times {}^0\mathbf{d}_{02} & {}^0\mathbf{z}_1 \\ {}^0\mathbf{z}_0 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{z}_0 \times d_2\,{}^0\mathbf{z}_1 & {}^0\mathbf{z}_1 \\ {}^0\mathbf{z}_0 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -d_2\,{}^0\mathbf{x}_1 & {}^0\mathbf{z}_1 \\ {}^0\mathbf{z}_0 & \mathbf{0} \end{bmatrix} \tag{8.19}$$

8.2.3 The Elbow Robot

Consider the elbow robot where the last link coordinate origin O_6 is located at the wrist O_5 (Figure 8.4(A)). Because of the spherical wrist, $\mathbf{d}_{46} = \mathbf{d}_{56} = \mathbf{0}$ and $\mathbf{d}_{36} = d_4\mathbf{z}_3$. The particularized form of the velocity Jacobian (8.14) therefore has zeros in the upper right, greatly simplifying the linear velocity computation.

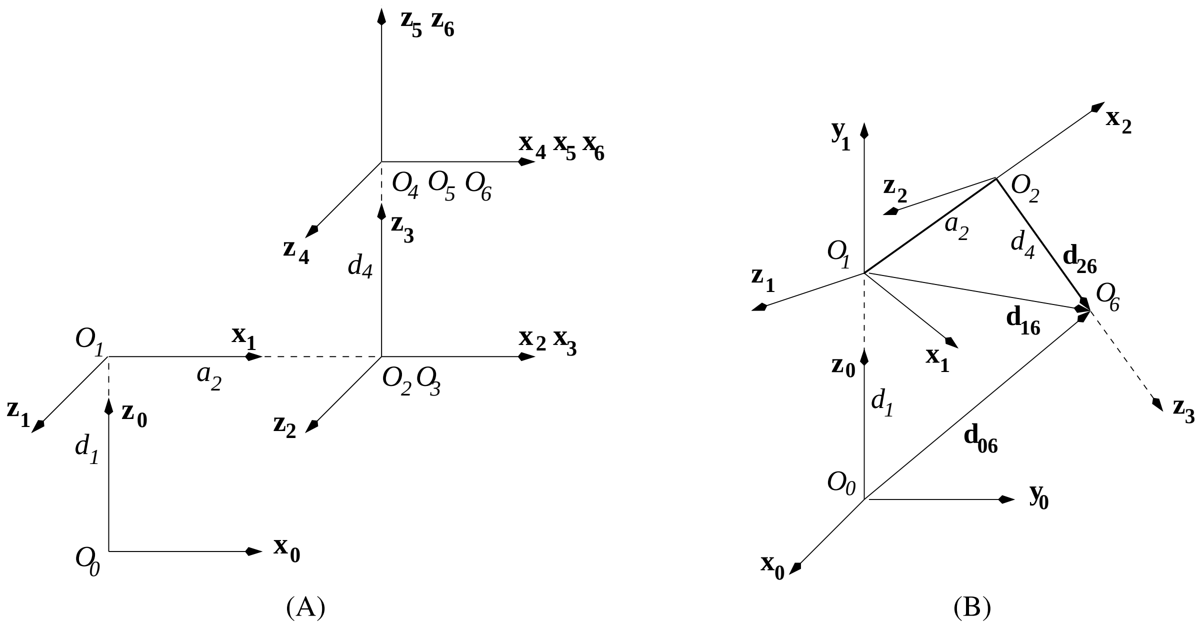


Figure 8.4: (A) Zero position of the elbow robot. (B) Diagram to derive the velocity Jacobian.

$$\mathbf{J}_v = \begin{bmatrix} {}^0\mathbf{z}_0 \times {}^0\mathbf{d}_{06} & {}^0\mathbf{z}_1 \times {}^0\mathbf{d}_{16} & {}^0\mathbf{z}_2 \times {}^0\mathbf{d}_{26} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ {}^0\mathbf{z}_0 & {}^0\mathbf{z}_1 & {}^0\mathbf{z}_2 & {}^0\mathbf{z}_3 & {}^0\mathbf{z}_4 & {}^0\mathbf{z}_5 \end{bmatrix} \tag{8.20}$$

$$= \begin{bmatrix} {}^0\mathbf{z}_0 \times (d_1\,{}^0\mathbf{z}_0 + a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3) & {}^0\mathbf{z}_1 \times (a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3) & {}^0\mathbf{z}_2 \times d_4\,{}^0\mathbf{z}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ {}^0\mathbf{z}_0 & {}^0\mathbf{z}_1 & {}^0\mathbf{z}_2 & {}^0\mathbf{z}_3 & {}^0\mathbf{z}_4 & {}^0\mathbf{z}_5 \end{bmatrix}$$

The second and third cross products are simplified by noting that $\mathbf{z}_1 = \mathbf{z}_2$.

$$\begin{aligned} {}^0\mathbf{z}_2 \times d_4\,{}^0\mathbf{z}_3 &= -d_4\,{}^0\mathbf{x}_3 \\ {}^0\mathbf{z}_1 \times (a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3) &= a_2\,{}^0\mathbf{y}_2 - d_4\,{}^0\mathbf{x}_3 \end{aligned}$$

The second and third joints form a planar two-link manipulator, and therefore the form of columns two and three of the velocity Jacobian is analogous to (8.18) with the difference due to the variation in DH parameters. The first cross product is simplified by noting that $\mathbf{z}_0 = \mathbf{y}_1$.

$${}^0\mathbf{z}_0 \times (d_1\,{}^0\mathbf{z}_0 + a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3) = {}^0\mathbf{y}_1 \times (a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3)$$

The cross product is further simplified by performing a mid-frame evaluation of $\mathbf{y}_1$ and $\mathbf{z}_3$ in terms of axes 2.

$$\begin{aligned} \mathbf{y}_1 &= \mathrm{s}\theta_2\mathbf{x}_2 + \mathrm{c}\theta_2\mathbf{y}_2 \\ \mathbf{z}_3 &= -\mathrm{s}\theta_3\mathbf{x}_2 + \mathrm{c}\theta_3\mathbf{y}_2 \end{aligned}$$

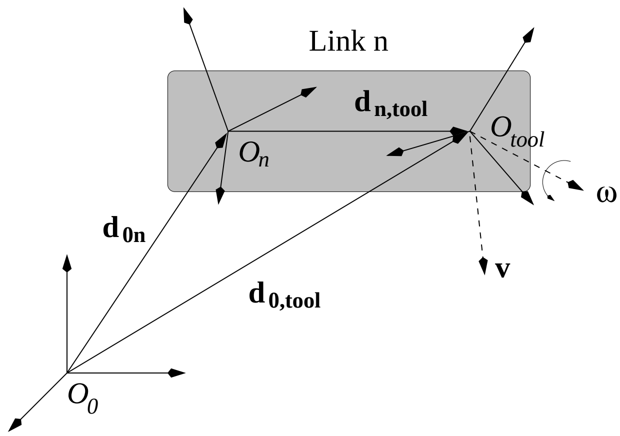


Figure 8.5: Relating frame n velocity to tool frame velocity.

Substituting,

$$\begin{aligned}
{}^0\mathbf{y}_1 \times (a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3) &= (\mathrm{s}\theta_2\,{}^0\mathbf{x}_2 + \mathrm{c}\theta_2\,{}^0\mathbf{y}_2) \times (a_2\,{}^0\mathbf{x}_2 + d_4(-\mathrm{s}\theta_3\,{}^0\mathbf{x}_2 + \mathrm{c}\theta_3\,{}^0\mathbf{y}_2)) \\
&= d_4\mathrm{s}\theta_2\mathrm{c}\theta_3\,{}^0\mathbf{z}_2 + (d_4\mathrm{c}\theta_2\mathrm{s}\theta_3 - a_2\mathrm{c}\theta_2)\,{}^0\mathbf{z}_2 \\
&= (d_4\sin(\theta_2+\theta_3) - a_2\mathrm{c}\theta_2)\,{}^0\mathbf{z}_2
\end{aligned}$$

The result could be determined by inspection by noting that the angle between $\mathbf{y}_1$ and $\mathbf{z}_3$ is $\theta_1 + \theta_2$. The simplified velocity Jacobian is:

$$\mathbf{J}_v = \begin{bmatrix} (d_4\sin(\theta_2+\theta_3) - a_2\mathrm{c}\theta_2)\,{}^0\mathbf{z}_2 & a_2\,{}^0\mathbf{y}_2 - d_4\,{}^0\mathbf{x}_3 & -d_4\,{}^0\mathbf{x}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ {}^0\mathbf{z}_0 & {}^0\mathbf{z}_1 & {}^0\mathbf{z}_2 & {}^0\mathbf{z}_3 & {}^0\mathbf{z}_4 & {}^0\mathbf{z}_5 \end{bmatrix} \tag{8.21}$$

8.2.4 Tool Frame Considerations

If there are extra frames in the last link beyond frame n of an n-link manipulator, for example to locate a tool frame, then one may either modify the development above so that the summations extend to the last frame, or one may refer velocities of the tool frame to frame n. The latter is often the best.

Suppose the tool transform ${}^n\mathbf{T}_{tool}$ is given, and that the linear and angular velocities are specified for the tool frame (Figure 8.5). Then

$$\dot{\mathbf{d}}_{0n} = \dot{\mathbf{d}}_{0,tool} - \dot{\mathbf{d}}_{n,tool} = \dot{\mathbf{d}}_{0,tool} - \boldsymbol{\omega}_{0n} \times \mathbf{d}_{n,tool} \tag{8.22}$$

The angular velocity doesn't change with this referral.

In the case of the elbow robot or other spherical-wrist arm, referring velocities to frame 6 located at the wrist rather than the tool frame results in computational savings. This is because the last three joint angles act through a zero moment arm, and hence do not affect wrist linear velocity.

8.3 Manipulator Jacobian J

Often a trajectory is specified in terms of endpoint position and orientation expressed via ZYZ Euler angles, say. Let

$$\begin{aligned} {}^0\mathbf{d}_{0n} &\equiv (x_1, x_2, x_3) = \text{endpoint position} \\ (\phi, \theta, \psi) &\equiv (x_4, x_5, x_6) = \text{endpoint orientation} \end{aligned}$$

Then the vector describing position plus orientation is:

$$\mathbf{x} = (x_1, \ldots, x_6)$$

The forward kinematics for position and orientation can be alternatively specified as:

$$\begin{aligned} x_i &= f_i(\boldsymbol{\theta}) \\ \mathbf{x} &= \mathbf{f}(\boldsymbol{\theta}) \end{aligned}$$

for a vector function $\mathbf{f} = (f_1, \ldots, f_6)$ extracted from ${}^0\mathbf{T}_n = {}^0\mathbf{T}_1 \cdots {}^{n-1}\mathbf{T}_n$. This extraction would be straightforward for the position components, but would be a very complicated function indeed for the Euler angles. We don't actually do this extraction, but are representing this operation for formal development. Next,

$$\dot{x}_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial \theta_j} \dot{\theta}_j \tag{8.23}$$

$$\begin{aligned} \dot{\mathbf{x}} &= \frac{\partial \mathbf{f}}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} \\ &\equiv \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1} & \cdots & \frac{\partial f_1}{\partial \theta_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_6}{\partial \theta_1} & \cdots & \frac{\partial f_6}{\partial \theta_n} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} \\ &\equiv \mathbf{J} \dot{\boldsymbol{\theta}} \end{aligned} \tag{8.24}$$

where the *manipulator Jacobian* $\mathbf{J}$ has elements

$$J_{ij} = \frac{\partial f_i}{\partial \theta_j} \tag{8.25}$$

This manipulator Jacobian $\mathbf{J}$ represents the differential relationship between two (nonlinearly) related sets of variables, $\mathbf{x}$ and $\boldsymbol{\theta}$.

Older textbooks in robotics used to advocate computation of this manipulator Jacobian $\mathbf{J}$ by direct evaluation. This extraordinarily tedious procedure is however unnecessary, and we now never compute $\mathbf{J}$ directly. Instead, it can be derived simply from the velocity Jacobian. The key step is to relate $\dot{\mathbf{x}}$ to the linear and angular velocities:

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{d}}_{0n} \\ \boldsymbol{\omega}_{0n} \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \Omega_{zyz} \end{bmatrix} \dot{\mathbf{x}} \\ &\equiv \mathbf{C} \dot{\mathbf{x}} \end{aligned} \tag{8.26}$$

where the matrix

$$\mathbf{C} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \Omega_{zyz} \end{bmatrix} \tag{8.27}$$

converts between the two end link velocity representations. Equating (8.20), (8.24), and (8.26),

$$\mathbf{J}_v = \mathbf{CJ} \tag{8.28}$$

$$\mathbf{J} = \mathbf{C}^{-1}\mathbf{J}_v \tag{8.29}$$

This method of deriving $\mathbf{J}$ is vastly superior. On the other hand, there is a problem again with the invertibility of $\mathbf{C}$ due to Ω_{zyz}. In the literature, one often sees that people have mixed up $\mathbf{J}_v$ and $\mathbf{J}$ and are unaware of this invertibility problem.

8.4 Manipulator Accelerations

Differentiating (8.14), the linear acceleration $\ddot{\mathbf{d}}_{0n}$ and angular acceleration $\dot{\boldsymbol{\omega}}_{0n}$ of the hand can be related to the joint angle accelerations (for an all-rotary manipulator):

$$\begin{bmatrix} {}^0\ddot{\mathbf{d}}_{0n} \\ {}^0\dot{\boldsymbol{\omega}}_{0n} \end{bmatrix} = \mathbf{J}_v\ddot{\boldsymbol{\theta}} + \dot{\mathbf{J}}_v\dot{\boldsymbol{\theta}} \tag{8.30}$$

While this acceleration relationship is conceptually straightforward, the form of $\dot{\mathbf{J}}_v$ is complicated.

To compute the forward kinematic accelerations, the best procedure is to express the velocity relations (8.11) and (8.12) recursively and differentiate those relations. Then

$$ {}^0\boldsymbol{\omega}_{0i} = {}^0\boldsymbol{\omega}_{0,i-1} + \begin{cases} \dot{\theta}_i\, {}^0\mathbf{z}_{i-1} & \text{for a rotary joint } i \\ \mathbf{0} & \text{for a prismatic joint } i \end{cases} \tag{8.31}$$

$$ {}^0\dot{\mathbf{d}}_{0i} = {}^0\dot{\mathbf{d}}_{0,i-1} + \begin{cases} {}^0\boldsymbol{\omega}_{0i} \times {}^0\mathbf{d}_{i-1,i} & \text{for a rotary joint } i \\ {}^0\boldsymbol{\omega}_{0i} \times {}^0\mathbf{d}_{i-1,i} + \dot{d}_i\, {}^0\mathbf{z}_{i-1} & \text{for a prismatic joint } i \end{cases} \tag{8.32}$$

Differentiating,

$$ {}^0\dot{\boldsymbol{\omega}}_{0i} = {}^0\dot{\boldsymbol{\omega}}_{0,i-1} + \begin{cases} \ddot{\theta}_i\, {}^0\mathbf{z}_{i-1} + \dot{\theta}_i\, {}^0\boldsymbol{\omega}_{0,i-1} \times {}^0\mathbf{z}_{i-1} & \text{for a rotary joint } i \\ \mathbf{0} & \text{for a prismatic joint } i \end{cases} \tag{8.33}$$

$$ {}^0\ddot{\mathbf{d}}_{0i} = {}^0\ddot{\mathbf{d}}_{0,i-1} + \begin{cases} {}^0\dot{\boldsymbol{\omega}}_{0i} \times {}^0\mathbf{d}_{i-1,i} + {}^0\boldsymbol{\omega}_{0i} \times ({}^0\boldsymbol{\omega}_{0i} \times {}^0\mathbf{d}_{i-1,i}) & \text{for a rotary joint } i \\ {}^0\dot{\boldsymbol{\omega}}_{0i} \times {}^0\mathbf{d}_{i-1,i} + {}^0\boldsymbol{\omega}_{0i} \times ({}^0\boldsymbol{\omega}_{0i} \times {}^0\mathbf{d}_{i-1,i}) & \\ +\ddot{d}_i\, {}^0\mathbf{z}_{i-1} + 2\dot{d}_i\, {}^0\boldsymbol{\omega}_{0,i-1} \times {}^0\mathbf{z}_{i-1} & \text{for a prismatic joint } i \end{cases} \tag{8.34}$$

Suppose that there is a fixed frame -1 in the environment with respect to which the base frame 0 is referred. If the base is stationary, then the initial conditions for these recursive relations are $\dot{\mathbf{d}}_{-1,0} = 0$, $\boldsymbol{\omega}_{-1,0} = 0$, $\ddot{\mathbf{d}}_{-1,0} = 0$, and $\dot{\boldsymbol{\omega}}_{-1,0}$. If the base is moving, such as for a mobile manipulator, then these initial conditions must be provided to start the recursion.

8.5 Inverse Kinematic Velocities

The problem of inverse kinematic velocities is to find the joint angle rates, given the end link velocities ${}^0\dot{\mathbf{d}}_{0n}$ and ${}^0\boldsymbol{\omega}_{0n}$. We presume to have previously solved for the inverse kinematic positions, which is necessary because velocities are calculated at an instantaneous configuration. The general solution procedure we will adopt is intelligent projection of the velocity equations (8.13) and (8.12), that is to say, to find linear or angular velocity directions that are due only to one joint.

8.5.1 Planar Manipulators

For the two-link planar manipulator and the polar arm (Figure 8.6), the velocity ${}^0\dot{\mathbf{d}}_{02}$ of each endpoint is only controllable in terms of the x, y components; the z component must be zero because the arm cannot move out of the plane. Because there are only two joints, the angular velocity ${}^0\boldsymbol{\omega}_{02}$ of the end link cannot be independently controlled, and so is ignored. Therefore a velocity specification for the endpoint is only in terms of the x, y components of ${}^0\dot{\mathbf{d}}_{02}$.

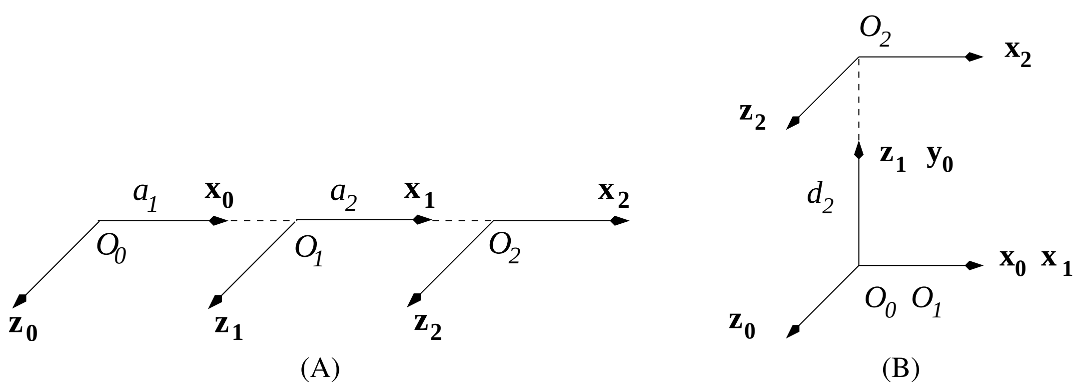


Figure 8.6: (A) The two-link planar manipulator and (B) the polar manipulator in zero-angle positions.

Two-Link Planar Manipulator

From (8.13), the endpoint velocity is

$$
\begin{aligned}
{}^0\dot{\mathbf{d}}_{02} &= \dot{\theta}_1\, {}^0\mathbf{z}_0 \times {}^0\mathbf{d}_{02} + \dot{\theta}_2\, {}^0\mathbf{z}_1 \times {}^0\mathbf{d}_{12} && (8.35)\\
&= \dot{\theta}_1\, {}^0\mathbf{z}_0 \times (a_1\, {}^0\mathbf{x}_1 + a_2\, {}^0\mathbf{x}_2) + \dot{\theta}_2\, {}^0\mathbf{z}_1 \times a_2\, {}^0\mathbf{x}_2 && (8.36)
\end{aligned}
$$

Unlike the result in equation (8.18), it is best not to carry out the cross products before figuring out which axis to project the velocity onto. This is because the relation $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ can be applied to yield a simpler procedure. We will adopt this procedure for all of the subsequent examples.

Noting that the $\mathbf{z}_i$ axes are all equal, the terms on the right involve a cross product between a $\mathbf{z}_i$ axis and either axis $\mathbf{x}_1$ or $\mathbf{x}_2$. By taking the dot product of this equation with $\mathbf{x}_2$, joint rate $\dot{\theta}_1$ is isolated since $\mathbf{x}_2 \cdot (\mathbf{z}_i \times \mathbf{x}_2) = 0$.

$$
{}^0\dot{\mathbf{d}}_{02} \cdot {}^0\mathbf{x}_2 = \dot{\theta}_1({}^0\mathbf{z}_0 \times a_1\, {}^0\mathbf{x}_1) \cdot {}^0\mathbf{x}_2
$$

$$
\begin{aligned}
&= a_1\dot\theta_1\,{}^0\mathbf{z}_0 \cdot ({}^0\mathbf{x}_1 \times {}^0\mathbf{x}_2) \\
&= a_1\dot\theta_1\,{}^0\mathbf{z}_2 \cdot {}^0\mathbf{z}_2 \sin\theta_2 \\
&= a_1\dot\theta_1 \sin\theta_2
\end{aligned}
$$

Therefore

$$\dot\theta_1 = \frac{{}^0\dot{\mathbf{d}}_{02} \cdot {}^0\mathbf{x}_2}{a_1 \sin\theta_2} \tag{8.37}$$

Projecting (8.36) onto the $\mathbf{x}_1$ axis, the remaining joint rate can be found.

$$
\begin{aligned}
{}^0\dot{\mathbf{d}}_{02} \cdot {}^0\mathbf{x}_1 &= \dot\theta_1({}^0\mathbf{z}_0 \times a_2\,{}^0\mathbf{x}_2) \cdot {}^0\mathbf{x}_1 + \dot\theta_2({}^0\mathbf{z}_1 \times a_2\,{}^0\mathbf{x}_2) \cdot {}^0\mathbf{x}_1 \\
&= a_2(\dot\theta_1 + \dot\theta_2)\,{}^0\mathbf{z}_0 \cdot ({}^0\mathbf{x}_2 \times {}^0\mathbf{x}_1) \\
&= -a_2(\dot\theta_1 + \dot\theta_2)\sin\theta_2
\end{aligned}
$$

Therefore

$$\dot\theta_2 = -\dot\theta_1 - \frac{{}^0\dot{\mathbf{d}}_{02} \cdot {}^0\mathbf{x}_1}{a_2 \sin\theta_2} \tag{8.38}$$

When $\sin\theta_2 = 0$, which happens when the arm is straight or folded back on itself, no solution exists. There can be no component of velocity in the direction of the $\mathbf{x}_1$ or $\mathbf{x}_2$ axes, which is called a motion singularity. This particular singularity is called an *elbow singularity,* since it depends on the elbow joint angle. Since the two-link planar manipulator is a substructure of more complicated arms such as the elbow manipulator, the elbow singularity carries over to these other arms as well (e.g., Figure 8.9).

Polar Manipulator

The motions of the polar manipulator are more decoupled than for the two-link planar manipulator, so the solution for the joint rates is easier. From (8.13), the endpoint velocity is:

$$
\begin{aligned}
{}^0\dot{\mathbf{d}}_{02} &= \dot\theta_1\,{}^0\mathbf{z}_0 \times {}^0\mathbf{d}_{02} + \dot d_2\,{}^0\mathbf{z}_1 \\
&= \dot\theta_1\,{}^0\mathbf{z}_0 \times d_2\,{}^0\mathbf{z}_1 + \dot d_2\,{}^0\mathbf{z}_1
\end{aligned}
$$

Project onto $\mathbf{x}_1$ and $\mathbf{z}_1$ to find the joint rates:

$$
\begin{aligned}
{}^0\dot{\mathbf{d}}_{02} \cdot {}^0\mathbf{x}_1 &= (\dot\theta_1\,{}^0\mathbf{z}_0 \times d_2\,{}^0\mathbf{z}_1) \cdot {}^0\mathbf{x}_1 \\
&= -d_2\dot\theta_1\,{}^0\mathbf{x}_1 \cdot {}^0\mathbf{x}_1 \\
\dot\theta_1 &= -\frac{{}^0\dot{\mathbf{d}}_{02} \cdot {}^0\mathbf{x}_1}{d_2} \qquad (8.39) \\
{}^0\dot{\mathbf{d}}_{02} \cdot \mathbf{z}_1 &= \dot d_2 \qquad (8.40)
\end{aligned}
$$

Although there isn't apparently a singularity for the polar arm, the linear axis does have a practical limit of travel.

8.5.2 Elbow Robot

Inverse Jacobian Method

For a six degree-of-freedom manipulator, one can simply solve for $\dot{\boldsymbol{\theta}}$ by matrix inversion of (8.14):

$$\dot{\boldsymbol{\theta}} = \mathbf{J}_v^{-1} \begin{bmatrix} {}^0\dot{\mathbf{d}}_{06} \\ {}^0\boldsymbol{\omega}_{06} \end{bmatrix} \tag{8.41}$$

The assumption of course is that $\mathbf{J}_v$ is non-singular. As we will see later, this assumption is often violated at *manipulator singularities.* How to handle these singularities is a significant issue.

While apparently a straight-forward procedure, matrix inversion is the worst way for solving for the joint angle velocities. Matrix inversion has computational complexity $O(n^3)$. Some computational savings result from using Gaussian elimination instead of matrix inversion, since we are not ordinarily interested in the inverse Jacobian $\mathbf{J}_v^{-1}$ per se.

Block Matrix Method

It is better to take advantage of special geometries, such as spherical wrists, to partition the Jacobian. The computational savings are great. Consider our previous example of the 6R elbow manipulator. As in the procedure for inverse kinematic positions, the procedure starts by referring velocities to the wrist. The velocity Jacobian for the elbow robot was developed in (8.20). Because the last frame 6 is at the spherical wrist, the upper right 3-by-3 portion of $\mathbf{J}_v$ is the zero matrix. By representing $\mathbf{J}_v$ in terms of 3-by-3 block matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{0}$, the inverse kinematic velocities can be found more efficiently by explicit block matrix manipulation.

$$\mathbf{J}_v \equiv \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{bmatrix} \tag{8.42}$$

$$\begin{bmatrix} {}^0\dot{\mathbf{d}}_{06} \\ {}^0\boldsymbol{\omega}_{06} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \\ \begin{bmatrix} \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix} \end{bmatrix} \tag{8.43}$$

where

$$\mathbf{A} = \begin{bmatrix} {}^0\mathbf{z}_0 \times {}^0\mathbf{d}_{06} & {}^0\mathbf{z}_1 \times {}^0\mathbf{d}_{16} & {}^0\mathbf{z}_2 \times {}^0\mathbf{d}_{26} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} {}^0\mathbf{z}_0 & {}^0\mathbf{z}_1 & {}^0\mathbf{z}_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} {}^0\mathbf{z}_3 & {}^0\mathbf{z}_4 & {}^0\mathbf{z}_5 \end{bmatrix} \tag{8.44}$$

Consequently, we solve straightforwardly for the first 3 joint angle rates:

$${}^0\dot{\mathbf{d}}_{06} = \mathbf{A} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{8.45}$$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \mathbf{A}^{-1} \left({}^0\dot{\mathbf{d}}_{06}\right) \tag{8.46}$$

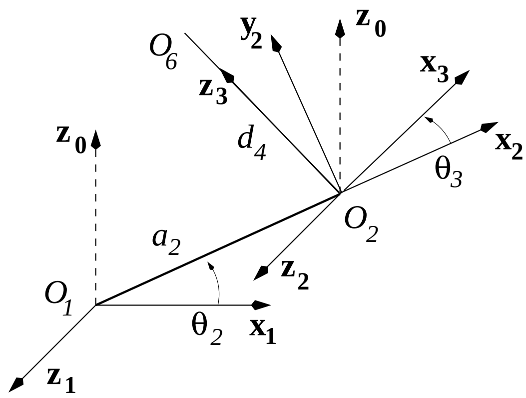


Figure 8.7: Axis relationships for the first three joints of the elbow robot.

A 3-by-3 matrix inversion is much more efficient than a 6-by-6 matrix inversion. Next,

$$\begin{bmatrix} \dot\theta_4 \\ \dot\theta_5 \\ \dot\theta_6 \end{bmatrix} = \mathbf{C}^{-1}\left({}^0\boldsymbol{\omega}_{06} - \mathbf{B}\begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{bmatrix}\right) \tag{8.47}$$

Again, the issue of the invertibility of $\mathbf{A}$ and $\mathbf{C}$ arises and is dealt with later.

Wrist Partitioned Method

A second more efficient way is not to invert at all, but to perform explicit manipulations to extract the joint angle rates [1]. From (8.20), the linear velocity depends only on the first three joint rates:

$${}^0\dot{\mathbf{d}}_{06} = \dot\theta_1\,{}^0\mathbf{z}_0 \times (a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3) + \dot\theta_2\,{}^0\mathbf{z}_1 \times (a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3) + \dot\theta_3\,{}^0\mathbf{z}_2 \times d_4\,{}^0\mathbf{z}_3 \tag{8.48}$$

Only $\dot\theta_1$ causes motion in the $\mathbf{z}_1 = \mathbf{z}_2$ direction, so take the dot product with ${}^0\mathbf{z}_1$:

$$\begin{aligned}
{}^0\dot{\mathbf{d}}_{06}\cdot{}^0\mathbf{z}_1 &= \dot\theta_1\,{}^0\mathbf{z}_0 \times (a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3)\cdot{}^0\mathbf{z}_1 \\
&= \dot\theta_1({}^0\mathbf{z}_1 \times {}^0\mathbf{z}_0)\cdot(a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3) \\
&= -\dot\theta_1\,{}^0\mathbf{x}_1\cdot(a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3) \\
&= -\dot\theta_1(a_2 \mathrm{c}\theta_2 + d_4\,{}^0\mathbf{x}_1\cdot{}^0\mathbf{z}_3)
\end{aligned}$$

The angle between $\mathbf{x}_1$ and $\mathbf{z}_3$ is $\pi/2 + \theta_2 + \theta_3$, and so

$$\begin{aligned}
{}^0\dot{\mathbf{d}}_{06}\cdot{}^0\mathbf{z}_1 &= -\dot\theta_1(a_2\mathrm{c}\theta_2 + d_4\cos(\pi/2 + \theta_2 + \theta_3)) \\
&= -\dot\theta_1(a_2\mathrm{c}\theta_2 - d_4\sin(\theta_2 + \theta_3)) \\
\dot\theta_1 &= \frac{{}^0\dot{\mathbf{d}}_{06}\cdot{}^0\mathbf{z}_1}{-a_2\mathrm{c}\theta_2 + d_4\sin(\theta_2+\theta_3)}
\end{aligned}$$

The next two joints form a planar two-link manipulator, and the solution procedure is similar to (8.37) and (8.38). Since we now know $\dot{\theta}_1$, substract that component from $\dot{\mathbf{d}}_{06}$ in (8.48) to derive a residual velocity $\mathbf{v}_1$ dependent only on the joint rates 2 and 3.

$$\begin{aligned} {}^0\mathbf{v}_1 &= {}^0\dot{\mathbf{d}}_{06} - \dot{\theta}_1\,{}^0\mathbf{z}_0 \times (a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3) && (8.49)\\ &= \dot{\theta}_2\,{}^0\mathbf{z}_1 \times (a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3) + \dot{\theta}_3\,{}^0\mathbf{z}_2 \times d_4\,{}^0\mathbf{z}_3 && (8.50)\end{aligned}$$

Next, take the dot product with $\mathbf{z}_3$ to isolate $\dot{\theta}_2$ while noting that $\mathbf{z}_1 = \mathbf{z}_2$.

$$\begin{aligned} {}^0\mathbf{v}_1 \cdot {}^0\mathbf{z}_3 &= \dot{\theta}_2\,{}^0\mathbf{z}_1 \times (a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3) \cdot {}^0\mathbf{z}_3\\ &= a_2\dot{\theta}_2({}^0\mathbf{z}_3 \times {}^0\mathbf{z}_2) \cdot {}^0\mathbf{x}_2\\ &= a_2\dot{\theta}_2\,{}^0\mathbf{x}_3 \cdot {}^0\mathbf{x}_2\\ &= a_2\mathrm{c}\theta_3\dot{\theta}_2\end{aligned}$$

Therefore

$$\dot{\theta}_2 = \frac{{}^0\mathbf{v}_1 \cdot {}^0\mathbf{z}_3}{a_2\mathrm{c}\theta_3} \tag{8.51}$$

Redefine a residual velocity vector $\mathbf{v}_2$ which is now only a function of joint rate 3.

$$\begin{aligned} {}^0\mathbf{v}_2 &= {}^0\mathbf{v}_1 - \dot{\theta}_2\,{}^0\mathbf{z}_1 \times (a_2\,{}^0\mathbf{x}_2 + d_4\,{}^0\mathbf{z}_3) && (8.52)\\ &= \dot{\theta}_3\,{}^0\mathbf{z}_2 \times d_4\,{}^0\mathbf{z}_3 && (8.53)\end{aligned}$$

Joint rate $\dot{\theta}_3$ is now found by projecting (8.48) onto $\mathbf{x}_2$.

$$\begin{aligned} {}^0\mathbf{v}_2 \cdot {}^0\mathbf{x}_2 &= \dot{\theta}_3\,{}^0\mathbf{z}_2 \times d_4\,{}^0\mathbf{z}_3 \cdot {}^0\mathbf{x}_2\\ &= -d_4\dot{\theta}_3\,{}^0\mathbf{x}_3 \cdot {}^0\mathbf{x}_2\\ &= -d_4\mathrm{c}\theta_3\dot{\theta}_3\end{aligned}$$

Therefore

$$\dot{\theta}_3 = -\frac{{}^0\mathbf{v}_2 \cdot {}^0\mathbf{x}_2}{d_4\mathrm{c}\theta_3} \tag{8.54}$$

The third step, which is the analog to the inverse kinematic position method, is to find the hand angular velocity ${}^0\boldsymbol{\omega}_{36}$ relative to the forearm:

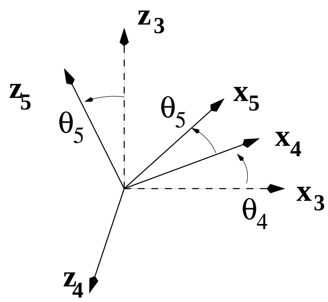


Figure 8.8: Axis relationships for the last 3 joints of elbow robot.

$$
\begin{aligned}
{}^0\boldsymbol{\omega}_{36} &\equiv \dot{\theta}_4\, {}^0\mathbf{z}_3 + \dot{\theta}_5\, {}^0\mathbf{z}_4 + \dot{\theta}_6\, {}^0\mathbf{z}_5 && (8.55)\\
&= {}^0\boldsymbol{\omega}_{06} - \sum_{i=1}^{3} \dot{\theta}_i\, {}^0\mathbf{z}_{i-1} && (8.56)
\end{aligned}
$$

By taking the dot product with ${}^0\mathbf{z}_4$, we readily find $\dot{\theta}_5$:

$$
{}^0\boldsymbol{\omega}_{36} \cdot {}^0\mathbf{z}_4 = \dot{\theta}_5 \qquad (8.57)
$$

Proceed by taking dot products with both ${}^0\mathbf{z}_3$ and ${}^0\mathbf{z}_5$:

$$
\begin{aligned}
{}^0\boldsymbol{\omega}_{36} \cdot {}^0\mathbf{z}_3 &= \dot{\theta}_4 + \dot{\theta}_6\, {}^0\mathbf{z}_3 \cdot {}^0\mathbf{z}_5\\
{}^0\boldsymbol{\omega}_{36} \cdot {}^0\mathbf{z}_5 &= \dot{\theta}_4\, {}^0\mathbf{z}_3 \cdot {}^0\mathbf{z}_5 + \dot{\theta}_6 && (8.58)
\end{aligned}
$$

where ${}^0\mathbf{z}_3 \cdot {}^0\mathbf{z}_5 = \cos\theta_5$ (Figure 8.8). Solve by Gaussian elimination:

$$
\begin{aligned}
\boldsymbol{\omega}_{36} \cdot {}^0\mathbf{z}_3 - (\boldsymbol{\omega}_{36} \cdot {}^0\mathbf{z}_5)\cos\theta_5 &= \dot{\theta}_4 - \dot{\theta}_4 \cos^2\theta_5\\
\dot{\theta}_4 &= \frac{\boldsymbol{\omega}_{36} \cdot {}^0\mathbf{z}_3 - (\boldsymbol{\omega}_{36} \cdot {}^0\mathbf{z}_5)\cos\theta_5}{\sin^2\theta_5} && (8.59)
\end{aligned}
$$

Consequently, from (8.58),

$$
\dot{\theta}_6 = \boldsymbol{\omega}_{36} \cdot {}^0\mathbf{z}_5 - \dot{\theta}_4 \cos\theta_5 \qquad (8.60)
$$

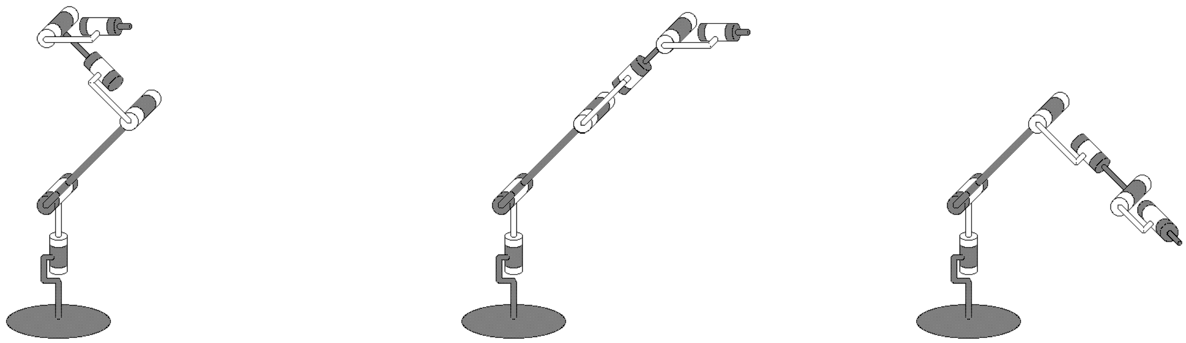
8.5.3 Singularities

For any manipulator, there are a number of joint angle configurations for which the Jacobian matrix becomes singular. For our example elbow manipulator, there are 3 singular configurations (Figure 8.9).

Shoulder singularity: This happens when the wrist point lies along $\mathbf{z}_0$.

$$
a_2 \cos\theta_2 = d_4 \sin(\theta_2 + \theta_3) \qquad (8.61)
$$

Linear motion of the wrist normal to the upper arm/forearm plane is not possible.



(A) (B) (C)

Figure 8.9: (A) Shoulder, (B) elbow, and (C) wrist singularities for the elbow robot.

Elbow singularity: When the forearm is fully extended, i.e.,

$$\cos\theta_3 = 0 \tag{8.62}$$

it is coincident with the upper arm. Then radial motion in the direction of the forearm/upper arm direction is not possible. It is unavoidable that every manipulator has a maximum reach beyond which it cannot extend.

Wrist singularity: For the roll-pitch-roll wrist, when the hand is straight out the $\mathbf{z}_3$ and $\mathbf{z}_5$ rotation axes coincide.

$$\sin\theta_5 = 0 \tag{8.63}$$

Angular motion is not possible normal to $\mathbf{z}_4 \times \mathbf{z}_5$.

The wrist singularity is a problem, because the hand straight out is a desirable working configuration for the robot. A roll-pitch-yaw wrist avoids this problem, at the expense of a more difficult mechanical design problem because of colliding actuators.

There are many ways to get around problems of singularities in trajectory planning. The most generally useful approach is probably damped least squares.

8.6 Inverse Kinematic Accelerations

Differentiating (8.14), the linear acceleration ${}^0\dot{\mathbf{d}}_{0n}$ and angular acceleration ${}^0\dot{\boldsymbol{\omega}}_{0n}$ of the hand can be related to the joint angle accelerations (for an all-rotary manipulator):

$$\begin{bmatrix} {}^0\ddot{\mathbf{d}}_{0n} \\ {}^0\dot{\boldsymbol{\omega}}_{0n} \end{bmatrix} = \mathbf{J}_v\ddot{\boldsymbol{\theta}} + \dot{\mathbf{J}}_v\dot{\boldsymbol{\theta}} \tag{8.64}$$

Assuming that $\mathbf{J}_v$ is a square matrix and non-singular, then the corresponding joint angle accelerations can be found by inversion:

$$\ddot{\boldsymbol{\theta}} = \mathbf{J}_v^{-1}\left(\begin{bmatrix} {}^0\ddot{\mathbf{d}}_{0n} \\ {}^0\dot{\boldsymbol{\omega}}_{0n} \end{bmatrix} - \dot{\mathbf{J}}_v\dot{\boldsymbol{\theta}}\right) \tag{8.65}$$

While these acceleration relationships are conceptually straightforward, the form of $\dot{\mathbf{J}}_v$ is complicated and the inversion of $\mathbf{J}_v$ is inefficient for simple robots such as the 6R arm.

Another approach modifies the formulation so that exactly the same mathematics as that for the inverse kinematic velocities of the elbow robot can be applied. The term $\dot{\mathbf{J}}_v\dot{\boldsymbol{\theta}}$ can be viewed as an acceleration bias vector that is subtracted from the original acceleration requirements:

$$\begin{bmatrix} \mathbf{a} \\ \boldsymbol{\alpha} \end{bmatrix} \equiv \begin{bmatrix} {}^0\ddot{\mathbf{d}}_{06} \\ {}^0\dot{\boldsymbol{\omega}}_{06} \end{bmatrix} - \dot{\mathbf{J}}_v\dot{\boldsymbol{\theta}} = \mathbf{J}_v\ddot{\boldsymbol{\theta}} \tag{8.66}$$

The acceleration bias vector represents the velocity dependent terms. Therefore the modified linear $\mathbf{a}$ and angular $\boldsymbol{\alpha}$ accelerations represent the robot as if it were accelerating from zero velocity. The form of this equation is exactly the same as (8.14), save that $\ddot{\boldsymbol{\theta}}$ replaces $\dot{\boldsymbol{\theta}}$. Therefore the solution equations from the last section can be applied intact.

It remains to evaluate the bias vector $\dot{\mathbf{J}}_v\dot{\boldsymbol{\theta}}$. Differentiating (8.12) and (8.11) for an all-rotary manipulator,

$$ {}^0\ddot{\mathbf{d}}_{06} = \sum_{i=1}^{3}\left({}^0\dot{\boldsymbol{\omega}}_{0i} \times {}^0\mathbf{d}_{i-1,i} + {}^0\boldsymbol{\omega}_{0i} \times ({}^0\boldsymbol{\omega}_{0i} \times {}^0\mathbf{d}_{i-1,i})\right) \tag{8.67}$$

$$ {}^0\dot{\boldsymbol{\omega}}_{06} = \sum_{i=1}^{6}\left({}^0\mathbf{z}_{i-1}\ddot{\theta}_i + {}^0\boldsymbol{\omega}_{0,i-1} \times {}^0\mathbf{z}_{i-1}\dot{\theta}_i\right) \tag{8.68}$$

The term

$$\sum_{i=1}^{6} {}^0\boldsymbol{\omega}_{0,i-1} \times {}^0\mathbf{z}_{i-1}\dot{\theta}_i$$

represents the last three components of the acceleration bias vector. To find the first three components, relation (8.67) needs more work. The angular acceleration $\dot{\boldsymbol{\omega}}_{0i}$ for link i is:

$$ {}^0\dot{\boldsymbol{\omega}}_{0i} = \sum_{j=1}^{i}\left({}^0\mathbf{z}_{j-1}\ddot{\theta}_j + {}^0\boldsymbol{\omega}_{0,j-1} \times {}^0\mathbf{z}_{j-1}\dot{\theta}_j\right)$$

Substituting into (8.67),

$$\begin{aligned} {}^0\ddot{\mathbf{d}}_{06} &= \sum_{i=1}^{3}\sum_{j=1}^{i}\left({}^0\mathbf{z}_{j-1}\ddot{\theta}_j + {}^0\boldsymbol{\omega}_{0,j-1} \times {}^0\mathbf{z}_{j-1}\dot{\theta}_j\right) \times {}^0\mathbf{d}_{i-1,i} + \sum_{i=1}^{3} {}^0\boldsymbol{\omega}_{0i} \times ({}^0\boldsymbol{\omega}_{0i} \times {}^0\mathbf{d}_{i-1,i}) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{i} {}^0\mathbf{z}_{j-1}\ddot{\theta}_j \times {}^0\mathbf{d}_{i-1,i} + \sum_{i=1}^{3}\sum_{j=1}^{i}\left({}^0\boldsymbol{\omega}_{0,j-1} \times {}^0\mathbf{z}_{j-1}\dot{\theta}_j\right) \times {}^0\mathbf{d}_{i-1,i} \\ &\quad + \sum_{i=1}^{3} {}^0\boldsymbol{\omega}_{0i} \times ({}^0\boldsymbol{\omega}_{0i} \times {}^0\mathbf{d}_{i-1,i}) \end{aligned} \tag{8.69}$$

The first three components of the acceleration bias vector are therefore:

$$\sum_{i=1}^{3}\sum_{j=1}^{i}\left({}^0\boldsymbol{\omega}_{0,j-1} \times {}^0\mathbf{z}_{j-1}\dot{\theta}_j\right) \times {}^0\mathbf{d}_{i-1,i} + \sum_{i=1}^{3} {}^0\boldsymbol{\omega}_{0i} \times ({}^0\boldsymbol{\omega}_{0i} \times {}^0\mathbf{d}_{i-1,i}) \tag{8.70}$$

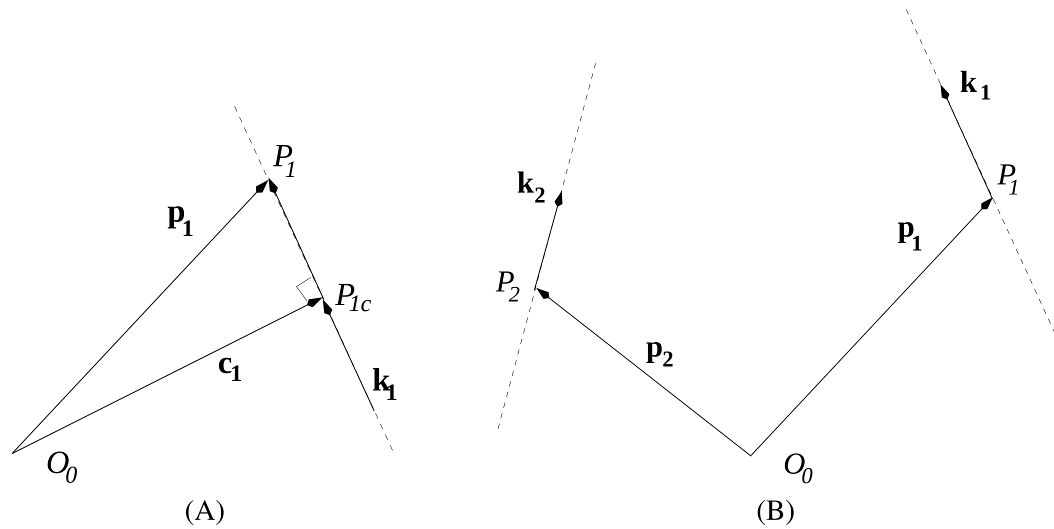


Figure 8.10: (A) Locating a line. (B) A second line in relation to the first.

8.7 Screw Theory

Chapter 3 introduced the finite screw transformation as an alternative representation of spatial displacement. Actually, other than insight the finite screw transformation does not offer any advantage over a coordinate transformation approach. The main use of screw transformations is for infinitessimal displacements or velocities, also called instantaneous kinematics.

8.7.1 Plücker Coordinates

As seen in Chapter 3, a line-bound vector can be represented by a point on the line $\mathbf{p}_1 = P_1 - O_0$, plus a unit normal $\mathbf{k}_1$ (Figure 8.10). The vector $\mathbf{c}_1 = \mathbf{P}_{1c} - O_o$ represents that point on the line closest to the origin. It was determined that 4 numbers are required to locate a line, given constraint relations $||\mathbf{k}_1|| = 1$ and $\mathbf{c}_1 \times \mathbf{k}_1 = 0$. A way of representing this line is the *Plücker coordinates*:

$$\begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_{01} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{p}_1 \times \mathbf{s}_1 \end{bmatrix} \tag{8.71}$$

where

$$\begin{aligned} \mathbf{s}_1 \cdot \mathbf{s}_1 &= 1 \\ \mathbf{s}_1 \cdot (\mathbf{p}_1 \times \mathbf{s}_1) &= 0 \end{aligned}$$

or $\mathbf{s}_1 \cdot \mathbf{s}_{01} = 0$ in terms of the Plücker coordinates. The identification is clearly $\mathbf{s}_1 = \mathbf{k}_1$. Since $\mathbf{p}_1 \times \mathbf{k}_1 = \mathbf{c}_1 \times \mathbf{k}_1$, then $||\mathbf{p}_1 \times \mathbf{k}_1|| = ||\mathbf{c}_1||$, the distance from the origin to the line.

1. Line intersecting the origin. This occurs when $\mathbf{s}_{01} = \mathbf{p}_1 \times \mathbf{k}_1$ is zero, i.e., $\mathbf{c}_1 = \mathbf{0}$.

2. Line at infinity. Consider the normalized Plücker coordinates

$$\begin{bmatrix} \mathbf{s}_1/||\mathbf{s}_{01}|| \\ \mathbf{s}_{01}/||\mathbf{s}_{01}|| \end{bmatrix} \tag{8.72}$$

As the line goes to infinity, the direction vector $\mathbf{s}_1/||\mathbf{s}_{01}||$ becomes $\mathbf{0}$. Therefore

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{s}_{01} \end{bmatrix} \tag{8.73}$$

is the Plücker coordinates of a line at infinity [3].

Consider a second line located by $\mathbf{p}_2$ and $\mathbf{k}_2$ (Figure 8.10(B)). The moment of the second line about point P_1 is defined as:

$$(\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{k}_2 \tag{8.74}$$

The moment of line 2 with respect to line 1 is defined as:

$$\mathbf{k}_1 \cdot (\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{k}_2 = \mathbf{s}_1 \cdot \mathbf{s}_{02} + \mathbf{s}_2 \cdot \mathbf{s}_{01} \tag{8.75}$$

which is called the *reciprocal product* of two lines, represented by the special operator "$\circ$":

$$\begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_{01} \end{bmatrix} \circ \begin{bmatrix} \mathbf{s}_2 \\ \mathbf{s}_{02} \end{bmatrix} \equiv \mathbf{s}_1 \cdot \mathbf{s}_{02} + \mathbf{s}_2 \cdot \mathbf{s}_{01} \tag{8.76}$$

If α is the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$, and d is the shortest distance between the lines, then the moment between the two lines is $d \sin \alpha$. Two lines intersect if the reciprocal product is zero.

8.7.2 Screw Coordinates

Plücker coordinates can represent joint motion: the axis and axis moment are reversed over the velocity Jacobian. For a single link with a rotary joint,

$$\begin{bmatrix} \boldsymbol{\omega}_{01} \\ \dot{\mathbf{d}}_{01} \end{bmatrix} = \dot{\theta}_1 \begin{bmatrix} \mathbf{z}_0 \\ \mathbf{z}_0 \times \mathbf{d}_{01} \end{bmatrix} = \dot{\theta}_1 \begin{bmatrix} \mathbf{z}_0 \\ \mathbf{r}_1 \times \mathbf{z}_0 \end{bmatrix} \equiv \dot{\theta}_1 \, \$, \quad \text{where } \$ = \begin{bmatrix} \mathbf{z}_0 \\ \mathbf{r}_1 \times \mathbf{z}_0 \end{bmatrix} \tag{8.77}$$

where $\mathbf{r}_1 = \mathbf{d}_{10}$ to show the relation to Plücker coordinates. Thus the reference point is considered the end point. For a prismatic joint,

$$\begin{bmatrix} \boldsymbol{\omega}_{01} \\ \dot{\mathbf{d}}_{01} \end{bmatrix} = \dot{d}_1 \begin{bmatrix} \mathbf{0} \\ \mathbf{z}_0 \end{bmatrix} \equiv \dot{d}_1 \, \$, \quad \text{where } \$ = \begin{bmatrix} \mathbf{0} \\ \mathbf{z}_0 \end{bmatrix} \tag{8.78}$$

Next consider a two-link manipulator with axes $\mathbf{z}_0$ and $\mathbf{z}_1$ located by $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to O_2. The motion of the reference point initially located at O_o is:

$$\left[\begin{array}{c} \boldsymbol{\omega}_{02} \\ \dot{\mathbf{d}}_{02} \end{array}\right] = \dot{\theta}_1 \left[\begin{array}{c} \mathbf{z}_0 \\ \mathbf{r}_1 \times \mathbf{z}_0 \end{array}\right] + \dot{\theta}_2 \left[\begin{array}{c} \mathbf{z}_1 \\ \mathbf{r}_2 \times \mathbf{z}_1 \end{array}\right]$$

Let us rewrite this formally as:

$$\begin{aligned} \dot{\theta} \left[\begin{array}{c} \mathbf{s} \\ \mathbf{s}_0 \end{array}\right] &= \dot{\theta}_1 \left[\begin{array}{c} \mathbf{z}_0 \\ \mathbf{r}_1 \times \mathbf{z}_0 \end{array}\right] + \dot{\theta}_2 \left[\begin{array}{c} \mathbf{z}_1 \\ \mathbf{r}_2 \times \mathbf{z}_1 \end{array}\right] \\ \dot{\theta}\,\$ &\equiv \dot{\theta}_1 \$_1 + \dot{\theta}_2 \$_2 \end{aligned} \tag{8.79}$$

where $\mathbf{s} \cdot \mathbf{s} = 1$, but $\mathbf{s} \cdot \mathbf{s}_0 \neq 0$ in general. Thus the resulting motion is not a pure rotation. Factor the left side of (8.79) formally as follows:

$$\dot{\theta} \left[\begin{array}{c} \mathbf{s} \\ \mathbf{s}_0 \end{array}\right] = \dot{\theta} \left[\begin{array}{c} \mathbf{s} \\ \mathbf{s}_0 - h\mathbf{s} \end{array}\right] + \dot{\theta} \left[\begin{array}{c} \mathbf{0} \\ h\mathbf{s} \end{array}\right] \tag{8.80}$$

where

$$h = \frac{\mathbf{s} \cdot \mathbf{s}_0}{\mathbf{s} \cdot \mathbf{s}} \tag{8.81}$$

Note that $\mathbf{s} \cdot (\mathbf{s}_0 - h\mathbf{s}) = 0$; hence the first term on the right in (8.80) represents a pure rotation. The second term represents a pure translation. The equivalent motion to that of the two joints is that of a nut about a screw:

h	is the pitch.
$	is the screw.
$\dot{\theta}$	is the *twist* amplitude about the screw.

A pure rotation has zero pitch ($\mathbf{s} \cdot \mathbf{s}_0 = 0$) while a pure translation has infinite pitch ($\mathbf{s} \cdot \mathbf{s} = 0$). The location of s is given by:

$$\mathbf{r} \times \mathbf{s} = \mathbf{s}_0 - h\mathbf{s}$$

For an arbitrary number n of serially connected links, where we consider prismatic as well as rotary joints and hence use the variable q to represent either joint variable,

$$\dot{q} \left[\begin{array}{c} \mathbf{s} \\ \mathbf{s}_0 \end{array}\right] = \sum_{i=1}^{n} \dot{q}_i \left[\begin{array}{c} \mathbf{s}_i \\ \mathbf{s}_{0i} \end{array}\right] \tag{8.82}$$

$$\dot{q}\,\$ = \sum_{i=1}^{n} \dot{q}_i\,\$_i \tag{8.83}$$

8.7.3 The Screw Jacobian

The *screw Jacobian* $\mathbf{J}_s$ has as its columns the joint screws.

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\omega}_{0n} \\ \dot{\mathbf{d}}_{0n} \end{bmatrix} &= \begin{bmatrix} \$_1 & \cdots & \$_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} & (8.84) \\ &\equiv \mathbf{J}_s \dot{\mathbf{q}} & (8.85) \end{aligned}$$

The relation to the velocity Jacobian is:

$$\mathbf{J}_s = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{J}_v \qquad (8.86)$$

where the first and last three rows have been interchanged. The columns of $\mathbf{J}_v$ are not screws because the order is wrong.

8.7.4 Analyzing Singularities via Screw Theory

At a singularity, a certain motion is not possible. Along this direction, a wrench can do no work (see Chapter 9). Thus there is a reciprocal screw $\$_r$, representing the wrench, to the set of motions:

$$\begin{aligned} 0 &= \$_r \circ \sum_{i=1}^{6} \dot{\theta}_i \$_i \\ &= \sum_{i=1}^{6} \dot{\theta}_i (\$_r \circ \$_i) & (8.87) \end{aligned}$$

Since (8.87) holds for general $\dot{\theta}_i$'s, there is a unique screw reciprocal to each individual joint screw:

$$\$_r \circ \$_i = 0 \qquad i = 1, \ldots, 6 \qquad (8.88)$$

This reciprocal screw $\$_r$ is unique, because it has 5 independent components, and there are five independent equations (8.88) because of the singularity [2, 4].

Consider the special case where the reciprocal screw $\$_r$ is a Plücker coordinate. The individual joint screws $\$_i$ are already Plücker coordinates. What is the geometrical relation between the lines in space that they represent? Write each screw in terms of its line coordinates:

$$\$_r = \begin{bmatrix} \mathbf{s}_r \\ \mathbf{r}_r \times \mathbf{s}_r \end{bmatrix} \qquad \$_i = \begin{bmatrix} \mathbf{s}_i \\ \mathbf{r}_i \times \mathbf{s}_i \end{bmatrix} \qquad (8.89)$$

where the $\mathbf{r}$ vectors locate the axes w.r.t. a reference point. Expand the reciprocal product in terms of these components:

$$0 = \begin{bmatrix} \mathbf{s}_r \\ \mathbf{r}_r \times \mathbf{s}_r \end{bmatrix} \circ \begin{bmatrix} \mathbf{s}_i \\ \mathbf{r}_i \times \mathbf{s}_i \end{bmatrix}$$

$$
\begin{aligned}
&= \mathbf{s}_r \cdot (\mathbf{r}_i \times \mathbf{s}_i) + \mathbf{s}_i \cdot (\mathbf{r}_r \times \mathbf{s}_r) \\
&= (\mathbf{s}_i \times \mathbf{s}_r) \cdot \mathbf{r}_i + (\mathbf{s}_r \times \mathbf{s}_i) \cdot \mathbf{r}_r \\
&= (\mathbf{s}_i \times \mathbf{s}_r) \cdot (\mathbf{r}_i - \mathbf{r}_r) \qquad (8.90)
\end{aligned}
$$

There are three conditions under which (8.90) is zero.

1. $\mathbf{s}_i \times \mathbf{s}_r = 0$. Then $\mathbf{s}_r \parallel \mathbf{s}_i$.
2. $\mathbf{r}_i = \mathbf{r}_r$. Then $\mathbf{s}_r$ intersects $\mathbf{s}_i$.
3. $(\mathbf{s}_i \times \mathbf{s}_r) \perp (\mathbf{r}_i - \mathbf{r}_r)$. Then $\mathbf{s}_r$, $\mathbf{s}_i$ and $\mathbf{r}_i - \mathbf{r}_r$ are coplanar. Since the axes $\mathbf{s}_r$ and $\mathbf{s}_i$ lie in the same plane, then they either intersect or they are parallel. Thus this case is not distinct from the previous two.

Hence the conclusion is that two screws that are Plücker coordinates are reciprocal if either

- they are parallel, or
- they intersect.

Singularities can now be analyzed by screw theory, by finding configurations for which there exists a reciprocal screw to each joint screw: $\$_r \circ \$_i = 0$ for each i. We should particularly look for reciprocal screws which are Plücker coordinates, because the condition of being parallel to or intersecting the joint screws is easier to visualize geometrically. Failing the reciprocal screw being a Plücker coordinate, we have to find the reciprocal screw algebraically rather than geometrically by solving (8.88).

8.7.5 Singularities of the Elbow Manipulator

As an example, let us analyze the singularities of the elbow manipulator. Take the reference point as the wrist.

(1) *Shoulder singularity.* Linear motion normal to the $\mathbf{x}_2, \mathbf{z}_3$ plane is impossible. This is represented by a reciprocal screw $\$_{r_1}$ parallel to $\$_2$:

$$
\$_{r_1} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{0} \end{bmatrix} \qquad (8.91)
$$

Note that $\$_{r_1}$ is a Plücker coordinate. It intersects screws $\$_1$, $\$_4$, $\$_5$, and $\$_6$ which also intersect the wrist, and is parallel to screws $\$_2$ and $\$_3$. Hence $\$_r$ is reciprocal to each joint screw.

(2) *Elbow singularity.* Linear motion along the $\mathbf{x}_2 = \mathbf{z}_3$ axes is not possible, represented by the reciprocal screw $\$_{r_2}$:

$$
\$_{r_2} = \begin{bmatrix} \mathbf{z}_3 \\ \mathbf{0} \end{bmatrix} \qquad (8.92)
$$

In this case, $\$_{r_2}$ is a Plücker coordinate, and intersects every joint screw.

(3) *Wrist singularity.* It is much harder to construct the reciprocal screw because there is no simple Cartesian statement of the impossible motion. Proximal joints also produce rotary as well as linear motion.

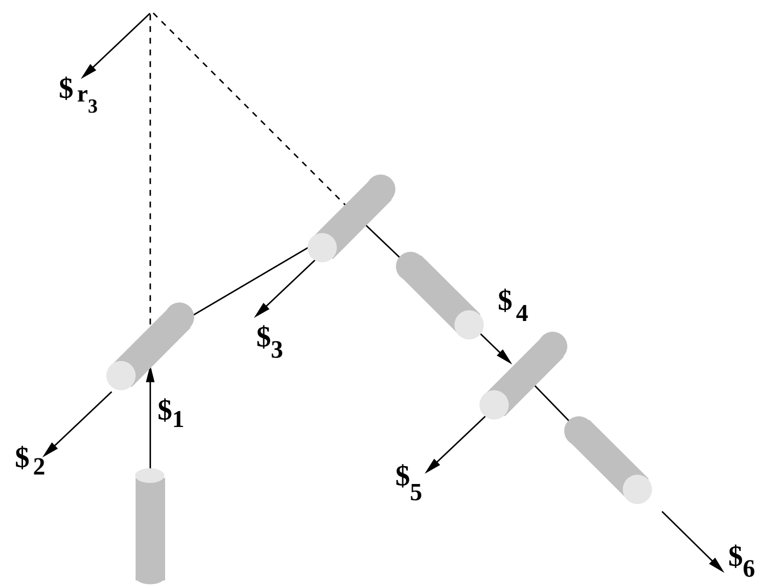


Figure 8.11: Reciprocal screw $\$_{r3}$ for the wrist singularity when axis $\mathbf{z}_4$ is parallel to $\mathbf{z}_1$.

Although $\mathbf{z}_3 \times \mathbf{z}_4$ rotary motions are not possible due to the wrist joints alone, some component of this motion is provided by rotation about $\mathbf{z}_0$, $\mathbf{z}_1$, and $\mathbf{z}_2$.

In the special case $\mathbf{z}_4 \parallel \mathbf{z}_1$, then $\$_{r3}$ can be simply constructed:

$$\$_{r3} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{r} \times \mathbf{z}_1 \end{bmatrix} \tag{8.93}$$

where $\mathbf{r}$ is a vector from the wrist to the point of intersection of axis $\mathbf{z}_3$ with axis $\mathbf{z}_0$. $\$_{r3}$ intersects joint screws $\$_1$, $\$_4$, and $\$_6$, and is parallel to the rest (Figure 8.11).

In general, the reciprocal screw for the wrist singularity is not a Plücker coordinate and must be found algebraically. The screw Jacobian reverses the role of the velocity Jacobian (8.21).

$$\mathbf{J}_s = \begin{bmatrix} \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 & \mathbf{z}_4 & \mathbf{z}_3 \\ (-a_2\cos\theta_2 + d_4\sin(\theta_2+\theta_3))\mathbf{z}_1 & a_2\mathbf{y}_2 - d_4\mathbf{x}_3 & -d_4\mathbf{x}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{8.94}$$

where the last joint axis $\mathbf{z}_5 = \mathbf{z}_3$. Start by the reciprocal product of $\$_{r3}$ with the last two joint screws:

$$\begin{aligned} 0 &= \$_{r3} \circ \$_5 \\ &= \begin{bmatrix} \mathbf{s}_r \\ \mathbf{s}_{0r} \end{bmatrix} \circ \begin{bmatrix} \mathbf{z}_3 \\ \mathbf{0} \end{bmatrix} \\ &= \mathbf{s}_{0r} \cdot \mathbf{z}_3 \end{aligned} \tag{8.95}$$

and

$$0 = \$_{r3} \circ \$_4$$

$$
\begin{aligned}
&= \begin{bmatrix} \mathbf{s}_r \\ \mathbf{s}_{0r} \end{bmatrix} \circ \begin{bmatrix} \mathbf{z}_4 \\ \mathbf{0} \end{bmatrix} \\
&= \mathbf{s}_{0r} \cdot \mathbf{z}_4
\end{aligned} \tag{8.96}
$$

Consequently $\mathbf{s}_{0r}$ is parallel to $\mathbf{z}_3 \times \mathbf{z}_4 = \mathbf{x}_4$, with unknown length c:

$$\mathbf{s}_{0r} = c\mathbf{x}_4 \tag{8.97}$$

Next, take the reciprocal product of $\$_{r_3}$ with $\$_3$:

$$
\begin{aligned}
0 &= \begin{bmatrix} \mathbf{s}_r \\ c\mathbf{x}_4 \end{bmatrix} \circ \begin{bmatrix} \mathbf{z}_2 \\ -d_4\mathbf{x}_3 \end{bmatrix} \\
&= \mathbf{s}_r \cdot (-d_4\mathbf{x}_3) + c\mathbf{z}_2 \cdot \mathbf{x}_4
\end{aligned} \tag{8.98}
$$

Continue with the reciprocal product of $\$_{r_3}$ with $\$_2$:

$$
\begin{aligned}
0 &= \begin{bmatrix} \mathbf{s}_r \\ c\mathbf{x}_4 \end{bmatrix} \circ \begin{bmatrix} \mathbf{z}_1 \\ a_2\mathbf{y}_2 - d_4\mathbf{x}_3 \end{bmatrix} \\
&= \$_{r_3} \circ \$_3 + \begin{bmatrix} \mathbf{s}_r \\ c\mathbf{x}_4 \end{bmatrix} \circ \begin{bmatrix} \mathbf{0} \\ a_2\mathbf{y}_2 \end{bmatrix} \\
&= \mathbf{s}_r \cdot a_2\mathbf{y}_2
\end{aligned} \tag{8.99}
$$

Lastly, take the reciprocal product of $\$_{r_3}$ with $\$_1$:

$$
\begin{aligned}
0 &= \begin{bmatrix} \mathbf{s}_r \\ c\mathbf{x}_4 \end{bmatrix} \circ \begin{bmatrix} \mathbf{z}_0 \\ (-a_2\cos\theta_2 + d_4\sin(\theta_2+\theta_3))\mathbf{z}_1 \end{bmatrix} \\
&= \mathbf{s}_r \cdot (-a_2\cos\theta_2 + d_4\sin(\theta_2+\theta_3))\mathbf{z}_1 + \mathbf{z}_0 \cdot c\mathbf{x}_4
\end{aligned} \tag{8.100}
$$

Combine (8.98), (8.99), and (8.100) into a matrix equation:

$$
c \begin{bmatrix} \mathbf{x}_4 \cdot \mathbf{z}_2 \\ 0 \\ \mathbf{x}_4 \cdot \mathbf{z}_0 \end{bmatrix} = \begin{bmatrix} d_4\mathbf{x}_3^T \\ a_2\mathbf{y}_2^T \\ (a_2\cos\theta_2 - d_4\sin(\theta_2+\theta_3))\mathbf{z}_1^T \end{bmatrix} \mathbf{s}_r \tag{8.101}
$$

There are 4 unknowns in (8.101), plus the constraint relation $\mathbf{s}_r \cdot \mathbf{s}_r = 1$. One way to solve this equation is to set $c = 1$, solve for $\mathbf{s}_r$ by matrix inversion, then normalize the result. This procedure works if the true $c \neq 0$ (assumed to be so because the special case of $\mathbf{z}_1 \parallel \mathbf{z}_4$ was treated earlier), and if the matrix is invertible (true as long as we are not at the elbow singularity where $\mathbf{y}_2 \parallel \mathbf{x}_3$). Finally, solve for c from (8.98).

Bibliography

[1] Hollerbach, J. M., and Sahar, G., "Wrist-partitioned inverse kinematic accelerations and manipulator dynamics," *Intl. J. Robotics Research*, vol. 2, no. 4, pp. 61-76, 1983.

[2] Lipkin, H., and Duffy, J., "Analysis of industrial robots via the theory of screws," in *Proc. 12th Intl. Symp. Industrial Robots*, Paris, pp. 359-370, June 9-11, 1982.

[3] Mason, M.T., *Mechanics of Robotic Manipulation*. Cambridge, Mass.: MIT Press, 2001.

[4] Sugimoto, K., and Duffy, J., "Special configurations of industrial robots," in *Proc. 11th Intl. Symp. Industrial Robots*, Tokyo, pp. 309-316, Oct. 7-9, 1981.

[5] Wampler, II, C.W., "Manipulator inverse kinematic solutions based on vector formulations and damped least-squares methods," *IEEE Trans. Systems, Man, Cybern.*, vol. SMC-16, pp. 93-101, 1986.