Chapter 4

Forward Kinematics for Position

Forward kinematics is the problem of locating the position and orientation of the end link, given the joint angles. This computation is straightforward for serial-link manipulators, but difficult for parallel-link manipulators. In this Chapter, we consider only serial-link manipulators. We begin by discussion the types of robot joints, and how they are typically arranged in complete robot systems. We then discuss how to set up coordinate systems and transformations between them, through the use of the so-called Denavit-Hartenberg representation.

4.1 Types of Robot Joints

The joints connecting links and allowing relative motion between them are classified either as low-order pairs or high-order pairs.

4.1.1 Low-Order Pairs

Rotary joint: The joint angle θ about the joint rotation axis $\mathbf{z}$ describes the joint displacement.

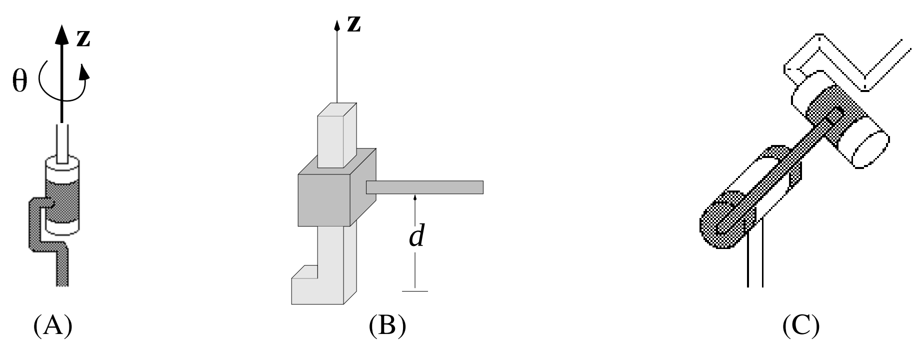
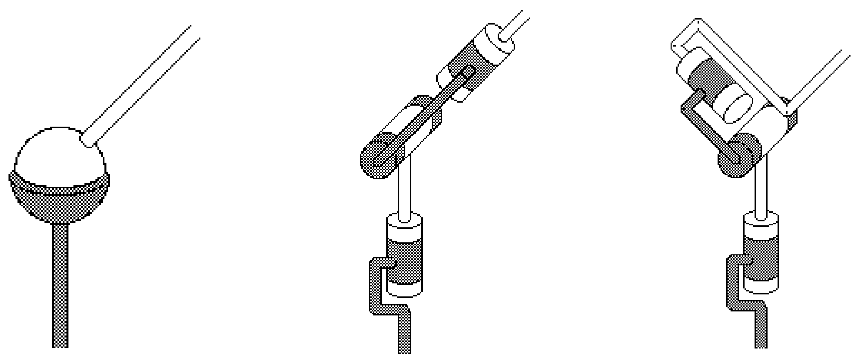


Figure 4.1: (A) Rotary joint. (B) Prismatic joint. (C) Hooke joint.

Prismatic or sliding joint: The linear displacement d along the joint axis $\mathbf{z}$ describes the displacement (Figure 4.1(B)).

Hooke joint: This is a two degree-of-freedom rotary joint described by two rotation angles, e.g., yaw and pitch (Figure 4.1(C)). Thus a Hooke joint is represented by two consecute rotary joints with intersecting rotation axes.

Spherical joint: Three joint angles about three rotation axes are required to describe this displacement. A spherical joint can be represented by three consecutive rotary joints with intersecting rotation axes. There are many ways to configure 3 such rotary joints. The human shoulder and wrist are spherical joints (Figure 4.2(A)). Most robot wrists are spherical joints with one of two arrangements:



(A) (B) (C)

Figure 4.2: (A) Spherical joint. (B) Roll-pitch-roll and (C) roll-pitch-yaw equivalents.

- *roll-pitch-roll wrist.* In their zero postions, the first and third rotation axes are parallel and the second axis is perpendicular to them. Hence the first and third axes may be considered to generate roll motions, while the second generates a pitch motion (Figure 4.2(B)). A disadvantage is that the wrist singularity is located in the pointing direction. An advantage is that motors can be separated into the forearm, wrist, and hand.
- *roll-pitch-yaw wrist.* In their zero positions, the rotation axes are at right angles and may be considered to be described by roll-pitch-yaw angles (Figure 4.2(C)). A disadvantage of this wrist is that motors would seem to have to be coincident. An advantage is that the wrist singularity (a degeneracy of alignment of two axes) is at right angles to the pointing direction.

Planar pair: A planar pair is any combination of 3 rotary and prismatic joints that generate motions of the plane; e.g., x-y translations and planar rotation (Figure 4.3(B)). Two common planar pair combinations are:

- *RRR planar pair.* This is a planar pair constructed of three consecutive rotary joints, whose axes are all parallel (Figure 4.3). A planar 3-link manipulator is the equivalent result.
- *RPR planar pair.* A rotary joint is followed by a prismatic joint, which is followed by a rotary joint. All three axes are parallel.

4.1.2 High-Order Pairs

Higher-order pairs have complex joint axes or movements, such as a curved joint axis, gears, and cams. We will not treat such joints in this class.

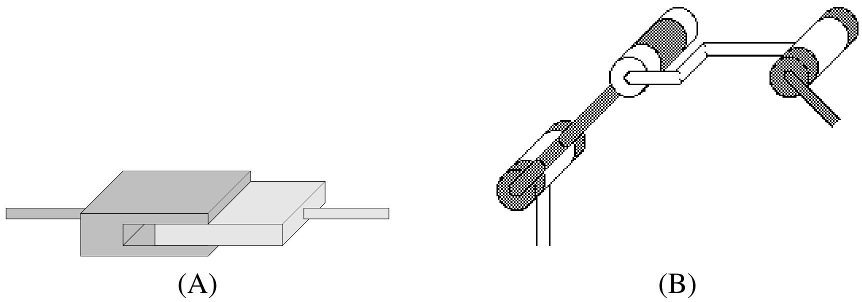


Figure 4.3: (A) Planar pair. (B) Three-rotary joint equivalent.

4.1.3 Common Robot Joint Arrangements

Most robots can be thought of as being composed of two parts: a three degree-of-freedom (DOF) regional structure which positions the wrist, and a 3-DOF wrist structure which controls hand orientation.

Spherical wrist robot: This is the most common 6 degree-of-freedom (DOF) robot. An example is the PUMA 560 robot. The regional structure is a Hooke shoulder joint followed by a rotary elbow joint. The wrist structure is typically roll-pitch-roll (Figure 4.4(A)).

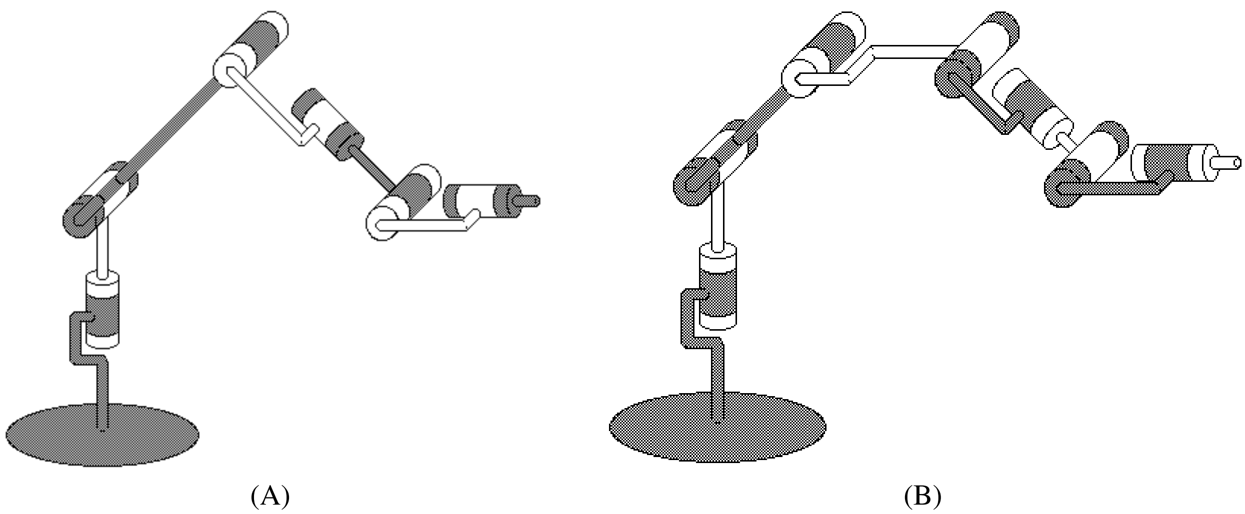
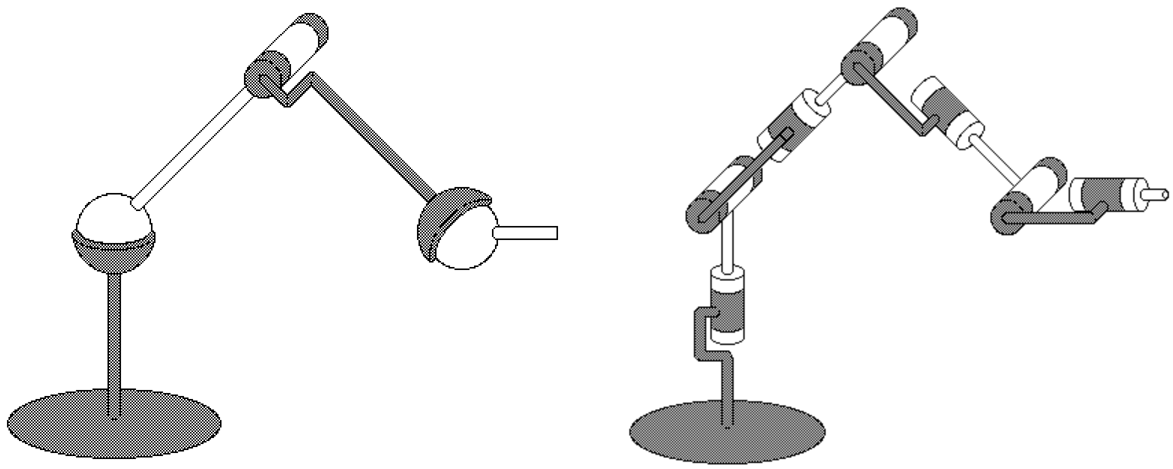


Figure 4.4: (A) The elbow robot with 6 degrees of freedom. (B) The planar pair robot with six degrees of freedom.

Planar pair robot: Another common 6-DOF kinematic arrangement is to incorporate a planar pair as joints 2-4. The wrist joints 5-6 vary, and Figure 4.4(B) shows one possibility.

Cartesian robot: The first three degrees of freedom are a regional structure comprised of prismatic joints with mutually perpendicular axes. The wrist is typically roll-pitch-roll. An example is the IBM RS-1 robot. Kinematics of Cartesian robots are particularly simple, because the wrist position is simply controlled by setting the x, y, z coordinates of the prismatic joints.

Anthropomorphic robot: Not counting shoulder shrug, the human arm has 7 DOFs: a spherical shoulder joint, a rotary elbow joint, and a spherical wrist joint (Figure 4.5(A)). Commercial robots that have a similar structure include the Sarcos Dextrous Arm, which is then said to be anthropomorphic. One realization of this structure is a roll-pitch-roll shoulder joint, a rotary elbow joint, and a roll-pitch-roll wrist (Figure 4.5(B)).



(A) (B)

Figure 4.5: (A) Human arm kinematics. (B) Equivalent 7-DOF anthropomorphic robot arm.

SCARA robot: This 4-DOF robot is primarily meant for stacking operations (essentially planar motion with height control). A commercial example is the Adept-One robot. Two rotary joints form a horizontal two-link planar manipulator. A prismatic joint whose axis is parallel to the first two follows. Finally, a rotary joint follows whose axis is coincident with that of the prismatic joint.

4.2 Setting Up Coordinate Systems in the Links

We need to set up a coordinate system in each link, in order to define transformations between links and ultimately the location of the end link relative to the base of the robot. There are a variety of ways of defining such coordinate systems. We could, for example, set up coordinate systems in each link arbitrarily, then define a transformation ${}^{i-1}\mathbf{T}_i$ between links i and $i-1$. However, the method of defining coordinate systems should take into account constraints imposed by the joint type.

In robotics we typically deal only with rotary and prismatic joints. The six numbers (three position, three orientation) for arbitrary coordinate system placement in a link are too many, because neighboring links don't undergo arbitrary relative motion to each other. Prismatic or rotary joints allow only certain motions between links, and hence fewer than 6 numbers are required to locate a link's coordinate system. The exact numbers are derived below.

4.2.1 Rotation Axes as Line Vectors

The rotation axis of a rotary joint is a line in space: this line has a definite direction $\mathbf{k}$ and location $\mathbf{c}$ (the unique normal to $\mathbf{k}$ from the origin) relative to a fixed frame (Figure 4.6(A)). The reference point on this

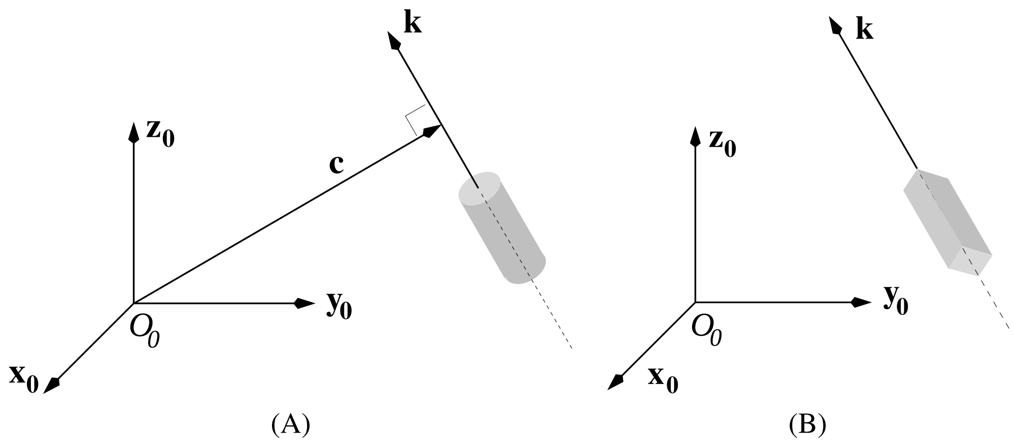


Figure 4.6: (A) A rotary joint axis is a line vector, with direction $\mathbf{k}$ and displacement $\mathbf{c}$ from O_0. (B) A prismatic joint axis is a free vector with direction $\mathbf{k}$, whose location relative to O_0 is arbitrary.

line is arbitrary; hence a rotary axis is an example of a *sliding vector*. We need 4 numbers to locate a line in space.

- The vectors $\mathbf{k}$ and $\mathbf{c}$ contain together 6 numbers; however,
- There is one constraint because $\mathbf{k}$ is a unit vector ($||\mathbf{k}|| = 1$), and
- There is another constraint because $\mathbf{c}$ is normal to $\mathbf{k}$ ($\mathbf{c} \cdot \mathbf{k} = 0$).

Thus 6 numbers minus 2 constraints means that there are exactly 4 independent numbers required to locate a rotation axis.

4.2.2 Translation Axes as Free Vectors

The translation axis for a sliding or prismatic joint only consists of a direction $\mathbf{k}$ (Figure 4.6(B)). The position in space of this line does not matter, because all we care about is the direction of motion. The translation axis is said to be a *free vector*: it does not matter where it is located. Hence there are exactly 2 numbers required to locate the unit vector $\mathbf{k}$, and hence 2 numbers to locate a translation axis.

4.3 Denavit-Hartenberg Coordinates

The standard way for setting up coordinate systems in links is the Denavit-Hartenberg (DH) representation. The key observation in the DH representation is that there is a unique normal between any two lines in space, with two exceptions:

1. Parallel lines: the common normal is not unique, as there are infinitely many common normals.
2. Intersecting lines: the common normal has zero length, and hence is not well defined.

We will return to these exceptions in a moment. For the general case, suppose $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$ are joint axes, represented as lines in space. Then

$$\mathbf{n}_i = \frac{\mathbf{z}_{i-1} \times \mathbf{z}_i}{||\mathbf{z}_{i-1} \times \mathbf{z}_i||} \tag{4.1}$$

is a unit vector which is normal to both lines in space.

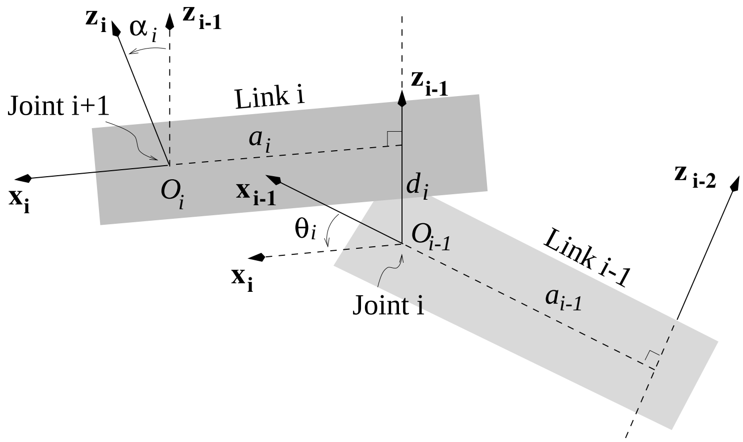


Figure 4.7: The Denavit-Hartenberg link labeling.

4.3.1 Recursive DH Definition in Middle of Chain

The base or ground is considered to be link 0. Before considering how to set up the DH parameters at either end of the manipulator, let us define them recursively starting someplace in the middle of the chain of links i. Consider that the rotation axes $\mathbf{z}_i$ are lines in space. First, we set up an origin and coordinate axes in each link (Figure 4.7).

- Joint i connects link $i-1$ to link i.
- Axis $\mathbf{z}_{i-1}$ is located at joint i. Thus link i rotates or translates about or along this axis relative to link $i-1$.
- The intersection of the common normal $\mathbf{n}_i$ from (4.1) with $\mathbf{z}_i$ defines the origin O_i of link i. Depending on $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$, the vector $\mathbf{n}_i$ either points from $\mathbf{z}_{i-1}$ to $\mathbf{z}_i$ or vice versa (Figure 4.8). We take $\mathbf{x}_i$ to be parallel to $\mathbf{n}_i$, but pointing from $\mathbf{z}_{i-1}$ to $\mathbf{z}_i$.

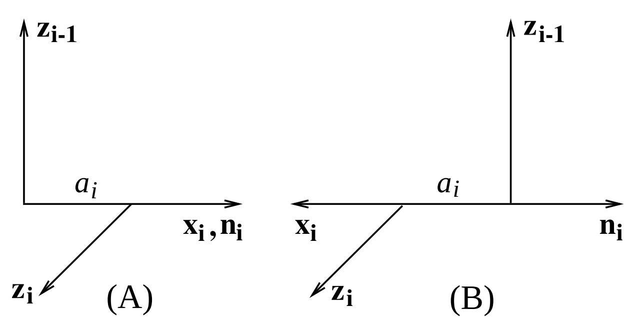


Figure 4.8: Comparison between $\mathbf{n}_i$ and $\mathbf{x}_i$ when (A) they are equal or (B) oppositely directed.

- $\mathbf{y}_i = \mathbf{z}_i \times \mathbf{x}_i$ completes the coordinate system.

Some points should be emphasized about these coordinate systems. The coordinate system for link i is located at the *distal* end of link i. The coordinate system at joint i is the $i-1$ coordinate system.

Next, four parameters (the so-called DH parameters) are defined that relate neighboring coordinate systems resulting from above.

- α_i is the skew angle from $\mathbf{z}_{i-1}$ to $\mathbf{z}_i$, measured about $\mathbf{x}_i$. Note that

$$\cos \alpha_i = \mathbf{z}_{i-1} \cdot \mathbf{z}_i$$

- d_i is the distance from $\mathbf{x}_{i-1}$ to $\mathbf{x}_i$, as measured along $\mathbf{z}_{i-1}$. Note that d_i can be negative.
- a_i is the distance from $\mathbf{z}_{i-1}$ to $\mathbf{z}_i$ measured along $\mathbf{x}_i$. Note that a_i is always positive because of the definition of $\mathbf{x}_i$.
- θ_i is the angle from $\mathbf{x}_{i-1}$ to $\mathbf{x}_i$, measured about $\mathbf{z}_{i-1}$.

For a rotary joint, θ_i varies and is called the joint angle. For a prismatic joint, d_i varies and is called the joint displacement.

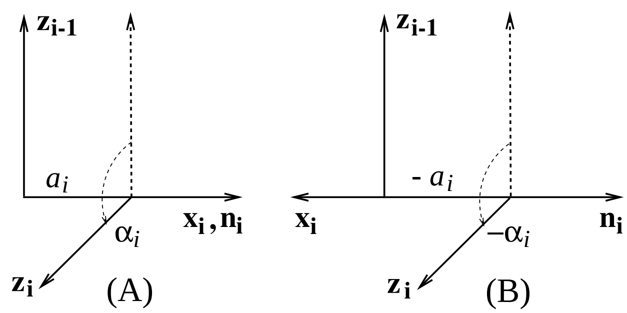


Figure 4.9: The coordinate axis $\mathbf{x}_i$ can be chosen to point from $\mathbf{z}_{i-1}$ to $\mathbf{z}_i$ (A), as in the strict DH definition, or vice versa (B). The DH parameters in (B) are the negative of those in (A).

For convenience, one can relax the strict DH definition for the direction of $\mathbf{x}_i$ (Figure 4.9(A)), so that it points from $\mathbf{z}_i$ to $\mathbf{z}_{i-1}$ (Figure 4.9(B)), and still obtain a valid parameterization. The resulting DH parameter $-a_i$ in Figure 4.9(B) is the negative of that in Figure 4.9(A); its definition as the (directed) distance from $\mathbf{z}_{i-1}$ to $\mathbf{z}_i$ along $\mathbf{x}_i$ still holds. In addition, the skew angle $-\alpha_i$ in Figure 4.9(B) is the negative of that in Figure 4.9(A); its definition as the (directed) angle from $\mathbf{z}_{i-1}$ to $\mathbf{z}_i$ also still holds. The reason that one might want to flip the direction of $\mathbf{x}_i$ is to set a more convenient reference configuration of the manipulator when the joint angles are all zero.

4.3.2 Parallel or Intersecting Neighboring Axes

When neighboring axes intersect, $a_i = 0$ and the common normal has zero length; hence it is not well defined. As for the non-intersecting case above, the axis $\mathbf{x}_i$ is made parallel to the common normal $\mathbf{n}_i$ (4.1). There are two choices relative to $\mathbf{n}_i$, similar in spirit to Figure 4.9:

- $\mathbf{x}_i = \mathbf{n}_i$ (Figure 4.10(A)). This is similar to the choice made in Figure 4.9(A).
- $\mathbf{x}_i = -\mathbf{n}_i$ (Figure 4.10(B)). This is similar to the choice made in Figure 4.9(B), with the skew angle $-\alpha_i$ the negative of that in Figure 4.10(A).

Another important special case is when $\mathbf{z}_{i-1}$ is parallel to $\mathbf{z}_i$, because the common normal is not uniquely defined. There are two common arbitary choices.

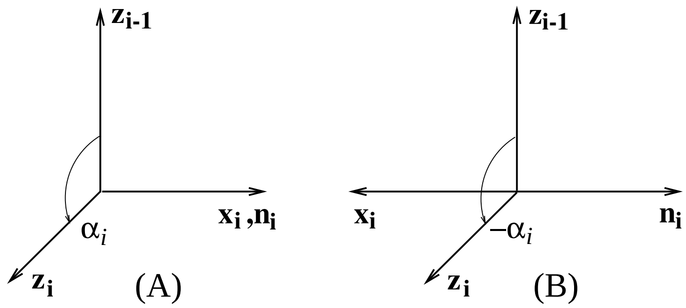


Figure 4.10: For intersecting joint axes, the coordinate axis $\mathbf{x}_i$ can be made either in the same direction as $\mathbf{n}_i$ (A) or oppositely directed (B). The DH parameters in (B) are the negative of those in (A).

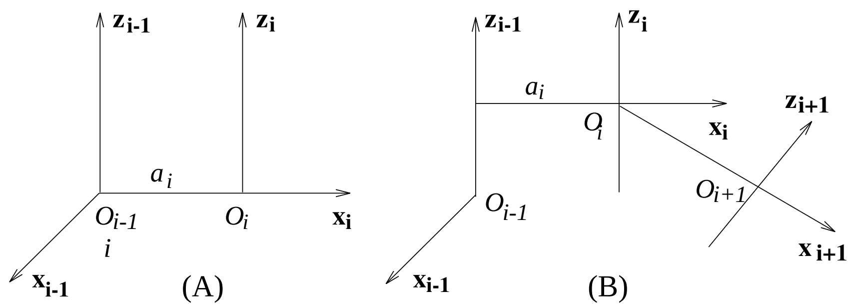


Figure 4.11: For parallel joint axes, common choices for $\mathbf{x}_i$ are either to intersect $\mathbf{x}_{i-1}$ (A) or $\mathbf{x}_{i+1}$ (B).

1. Choose the $\mathbf{x}_i$ that intersects $\mathbf{x}_{i-1}$, i.e., the proximal origin O_{i-1} determines $\mathbf{x}_i$ (Figure 4.11(A)). Then $d_i = 0$. This choice is recommended by [5].

2. Choose the $\mathbf{x}_i$ that intersects $\mathbf{x}_{i+1}$, i.e., the distal origin O_{i+1} determines $\mathbf{x}_i$ (Figure 4.11(B)). Then $d_{i+1} = 0$. This choice is recommended by [4].

We adopt the first convention, Figure 4.11(A), here.

For kinematic calibration, we have to abandon the DH parameters for parallel joint axes because slight misalignments cause the common normal to vary wildly. A parameterization called the Hayati coordinates is used in that case, but is outside the scope of this course.

4.3.3 Prismatic Joints

Since a prismatic joint axis is a free vector, its location is arbitrary. Thus two of the four DH parameters, the length parameters, can be chosen arbitrarily and for convenience. One simple possibility is to locate the origin O_{i-1} of a prismatic joint i at the previous origin O_{i-2} (Figure 4.12). This sets $a_{i-1} = d_{i-1} = 0$ as the arbitrary DH parameter choices. The DH parameters for frame i are then defined in the normal way. Note that d_i is the joint variable; as it varies, O_i slides up and down axis $\mathbf{z}_{i-1}$.

Paul [4] suggests that the frame associated with the prismatic joint should intersect the distal joint axis $\mathbf{z}_i$ rather than the proximal joint axis $\mathbf{z}_{i-2}$. In this construction, a plane is defined in which axes $\mathbf{z}_{i-2}$ and $\mathbf{x}_{i-1}$ lie, centered at origin O_{i-2}. This plane is intersected with axis $\mathbf{z}_i$ to locate O_{i-1}.

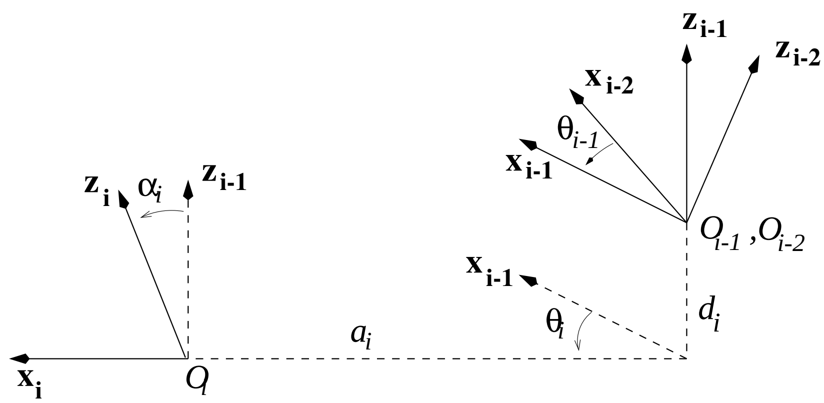


Figure 4.12: Prismatic joint i and associated frame $i-1$ is located at origin O_{i-2}.

4.3.4 Base Coordinates

At the robot base, although the first axis $\mathbf{z}_0$ is a fixed line in space, the location of the base coordinates O_0 along $\mathbf{z}_0$ and the $\mathbf{x}_0$ axis direction are arbitrary. Hence the values of d_1 and θ_1 are arbitrary. Choices can be made based upon convenience or simplicity, or external factors may dictate these choices if the robot base is located relative to environmental frames. Manufacturers often define these choices in controllers provided with their robots.

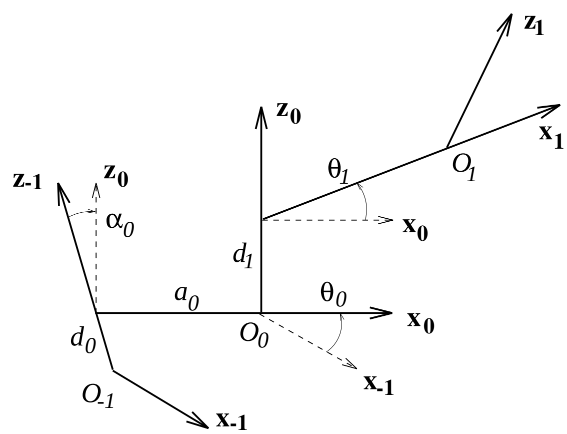


Figure 4.13: Location of robot base frame 0 relative to environmental frame -1.

Suppose there is an environmental reference frame that has been previously established, represented as -1 for mathematical convenience; hence the origin is O_{-1} and the axes are $\mathbf{x}_{-1}$, $\mathbf{y}_{-1}$, and $\mathbf{z}_{-1}$ (Figure 4.13). For example, this environmental frame may represent the coordinate system of a stereo vision system.

- $\mathbf{x}_0$ is now determined by the common normal between $\mathbf{z}_{-1}$ and $\mathbf{z}_0$.
- Hence θ_1 and d_1 are determined.
- New DH parameters α_0, d_0, a_0 and θ_0 are now required to relate frame 0 to frame -1.

Note that the establishment of the relationship between the environmental frame and the first manipulator frame required 6 parameters to be specified. This is not surprising, as we know we require six numbers to arbitrarily locate a frame.

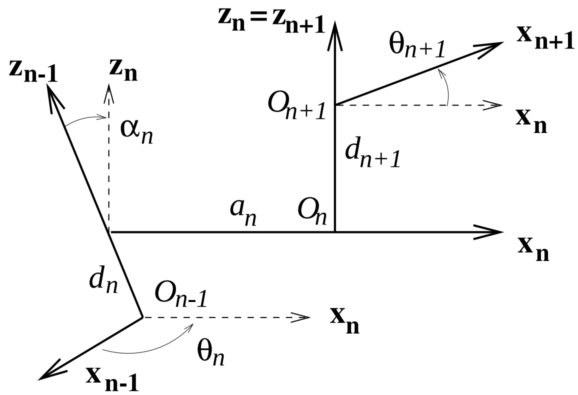


Figure 4.14: Locating an intermediate frame n between the measured end-link frame $n + 1$.

4.3.5 End Link Coordinates

Whereas for the internal links the placement of the coordinate system is forced, for the end link the placement of its frame is arbitrary. This arbitrariness may be resolved through simplifying choices or by placement at a distinguished location in the gripper. Consider first the case where the end effector frame is specified. Then 6 numbers are required to locate it relative to the previous link. This cannot be done with a single set of 4 DH parameters. Hence introduce an addtional frame $n+1$ which is the pre-specified end effector frame. An intermediate frame n in the end link provides four parameters, while frame $n + 1$ provides four parameters also. The resulting 8 parameters are 2 too many, and hence two parameters may be set arbitrarily. The assignment of coordinate system n in Figure 4.14 is one possibility.

- $\mathbf{z}_n$ is made coincident with $\mathbf{z}_{n+1}$. This arbitrary choice sets $\alpha_{n+1} = 0$ and $a_{n+1} = 0$.

- $\mathbf{x}_n$ and O_n are then defined in the normal manner, as are the DH parameters α_n, d_n, a_n, θ_n, d_{n+1}, and θ_{n+1}.

For rotary joints, now θ_n is the joint angle while θ_{n+1} is a constant. This method of defining the intermediate frame is convenient but arbitrary.

In the simplest case, there are no constraints on this placement and we can choose a placement based solely on convenience. Suppose our manipulator has n links. The next-to-last link has a frame $n - 1$ placed at its distal end, where the joint is between links $n - 1$ and n.

- The simplest placement is to make frame n coincident with frame $n - 1$ when $\theta_n = 0$. Then $\alpha_n = 0$, $d_n = 0$, and $a_n = 0$. Figure 4.20 shows an example of an RPR wrist where the last frame $n = 6$ follows this placement.

- Another simple placement is to make $\mathbf{z}_n$ coincident with $\mathbf{z}_{n-1}$, but to place the origin O_n a certain distance d_n from O_{n-1} towards the endpoint. Then $\alpha_n = 0$ and $a_n = 0$, while $d_n \neq 0$ is specified. A common situation is an RPR wrist with an axis-aligned gripper. Figure 4.15 shows a gripper frame offset from frame 5 by d_6 along axis $\mathbf{z}_5$, such that $\mathbf{z}_6 = \mathbf{z}_5$. The other DH parameters for the last frame are $a_6 = 0$ and $\alpha_6 = 0$.

In both examples, we presume that θ_n is the rotary joint variable.

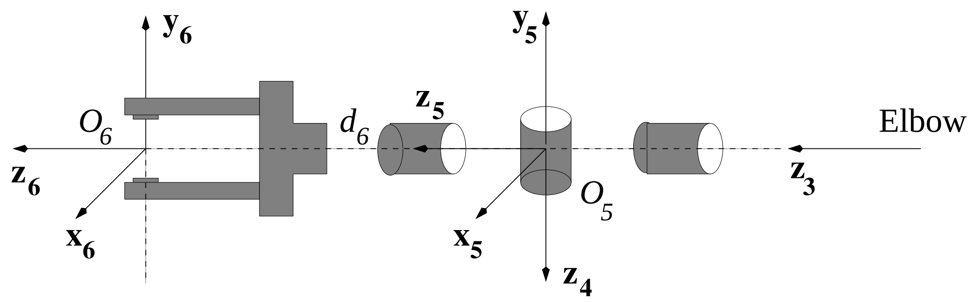


Figure 4.15: End frame offset from an RPR wrist along the $\mathbf{z}_5$ axis.

4.4 DH Parameters to Manipulators

Any serial manipulator can be represented by a table of DH parameters, such as in Table 4.1(A). The asterick (*) under the θ_i column will mean a joint variable, so in this case the DH parameters represent a 2-link rotary manipulator. For all-rotary manipulators, we will usually omit the θ_i column, it is only needed in case there are prismatic joints (Table 4.1(B)). One of the lengths, a_2, is zero, the other length parameters are some non-zero constants which we will assume are positive unless explicitly indicated otherwise.

i	a_i	d_i	α_i	θ_i
1	a_1	d_1	$-\frac{\pi}{2}$	$*$
2	0	d_2	$\frac{\pi}{2}$	$*$

(A)

i	a_i	d_i	α_i
1	a_1	d_1	$-\frac{\pi}{2}$
2	0	d_2	$\frac{\pi}{2}$

(B)

Table 4.1: (A) A spatial 2-DOF rotary manipulator. (B) The θ_i column is usually omitted for an all-rotary manipulator.

4.4.1 Zero-Angle Position

Given such a table, we wish to draw a diagram of the manipulator to understand its geometric structure. We adopt the convention of drawing a manipulator in its zero-angle position, i.e., $\theta_i = 0$ for each rotary joint i. When joint i is prismatic, the angle θ_i is a fixed constant that cannot be changed. The zero-angle position will be considered the manipulator's reference or home position. We do not set prismatic variables d_i to zero in the reference position because the structure will become incomprehensible, as seen in the next section.

The drawing of the structure begins by drawing the first coordinate system in terms of the origin O_0 and the $\mathbf{x}_0$ and $\mathbf{z}_0$ axes. In Figure 4.16(A), we have chosen $\mathbf{z}_0$ pointing up and $\mathbf{x}_0$ pointing to the right. A different placement, such as the $\mathbf{x}_0$ axis pointing out of the page, may or may not yield a diagram that is easier to understand, it is somewhat of an art and personal preference. Then coordinate systems are placed recursively by following the order of DH parameters indicated in Figure 4.7 and from a coordinate transformation viewpoint of

$$^{i-1}\mathbf{T}_i = \text{Trans}(d_i\ ^{i-1}\mathbf{z}_{i-1})\text{Rot}(\mathbf{R}_z(\theta_i))\text{Trans}(a_i\ ^{i}\mathbf{x}_i)\text{Rot}(\mathbf{R}_x(\alpha_i)) \tag{4.2}$$

Starting from coordinate system $i-1$,

Trans$(d_i\ ^{i-1}\mathbf{z}_{i-1})$ Index d_i units along $\mathbf{z}_{i-i}$ to find where $\mathbf{x}_i$ intersects $\mathbf{z}_{i-1}$.

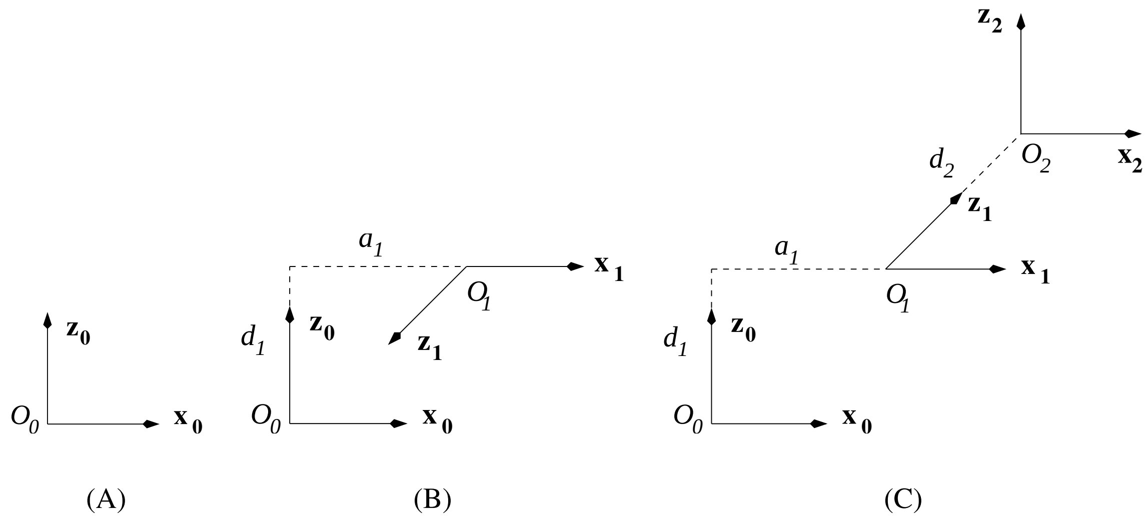


Figure 4.16: Construction of the spatial 2-DOF rotary manipulator by successive placements of coordinate systems.

Rot$(\mathbf{R}_z(\theta_i))$ Find the direction of $\mathbf{x}_i$. If a joint i is rotary, then $\mathbf{x}_i$ equals $\mathbf{x}_{i-1}$ in the zero-angle position. If all joints are rotary, then all $\mathbf{x}_i$ axes are parallel and point in the same direction. In case of a prismatic joint i, the $\mathbf{x}_i$ and $\mathbf{x}_{i-1}$ axes will have a fixed relation to each other defined by the constant value of θ_i; they only point in the same direction if it happens that $\theta_i = 0$.

Trans$(a_i\ {}^i\mathbf{x}_i)$ Index a_i units in the $\mathbf{x}_i$ direction to find origin O_i.

Rot$(\mathbf{R}_x(\alpha_i))$ Rotate $\mathbf{z}_{i-1}$ by α_i to find $\mathbf{z}_i$.

This procedure is illustrated for the manipulator of Table 4.1,

1. For this all-rotary manipulator, axes $\mathbf{x}_0$, $\mathbf{x}_1$, and $\mathbf{x}_2$ all point in the same direction. Origin O_1 is located by indexing up $\mathbf{z}_0$ by length d_1 and to the right by length a_1 (Figure 4.16(B)). Axis $\mathbf{z}_1$ is oriented to point into the page since $\alpha_1 = -\pi/2$.

2. Origin O_2 is placed d_2 units along the extension of the $\mathbf{z}_1$ axis, since $a_2 = 0$. Since $\alpha_2 = \pi/2$, the $\mathbf{z}_2$ axis points upwards.

4.4.2 A Cylindrical Manipulator

An example of a manipulator with prismatic joints is the 3-DOF manipulator with DH parameters in Table 4.2. As indicated by the astericks, the first and third joints are prismatic, the second is rotary. In the zero-angle position, we do not also set the prismatic variables d_1 and d_3 to zero but leave some linear extension to see (Figure 4.17(A)).

1. Origin O_1 is located d_1 units along the extension of $\mathbf{z}_0$, since $a_1 = 0$. Axis $\mathbf{z}_1$ is parallel to $\mathbf{z}_0$ since $\alpha_1 = 0$, and $\mathbf{x}_1$ points in the same direction as $\mathbf{x}_0$ since $\theta_1 = 0$.

i	a_i	d_i	α_i	θ_i
1	0	$*$	0	0
2	0	0	$-\frac{\pi}{2}$	$*$
3	0	$*$	$\frac{\pi}{2}$	0

Table 4.2: A 3-DOF cylindrical manipulator.

2. Origin O_2 is coincident with origin O_1 since both a_2 and d_2 are zero. Axis $\mathbf{x}_2$ points in the same direction as axis $\mathbf{x}_1$ since joint 2 is rotary. Axis $\mathbf{z}_2$ points into the page since $\alpha_2 = -\pi/2$.

3. Origin O_3 is located d_3 units along the extension of $\mathbf{z}_2$. Axis $\mathbf{x}_3$ points in the same direction as axis $\mathbf{x}_2$ since $\theta_3 = 0$. Finally, axis $\mathbf{z}_3$ points up since $\alpha_3 = \pi/2$.

The reason for not setting the prismatic variables d_1 and d_3 in the zero-angle or reference position is shown in Figure 4.17(B). When the prismatic variables are set to zero, the coordinate systems have collapsed together and one can't tell anything about the structure.

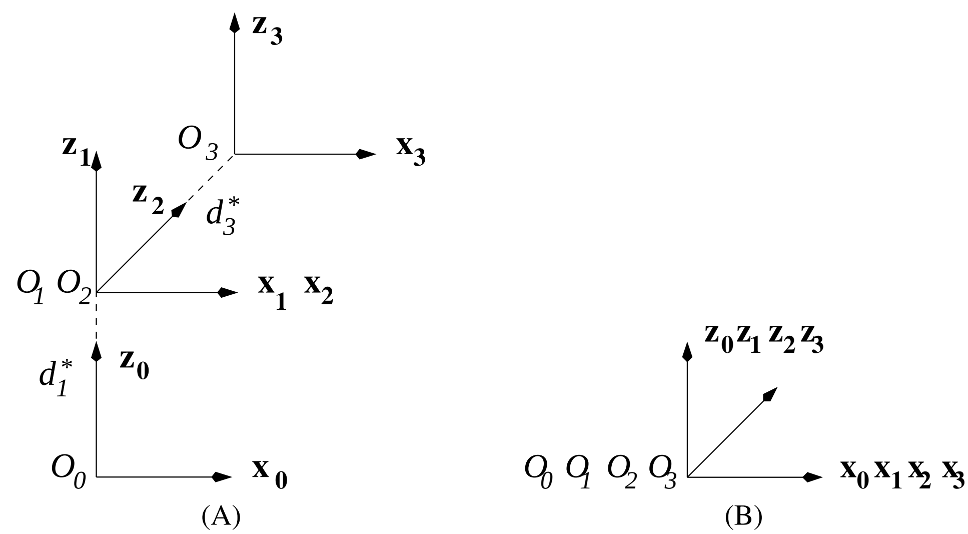


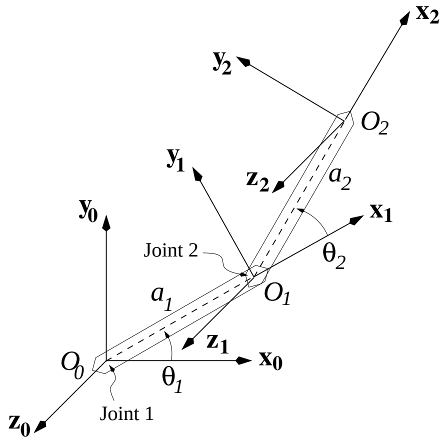
Figure 4.17: (A) The 3-DOF cylindrical manipulator. (B) The manipulator when the prismatic variables are zero.

4.5 Manipulators to DH Parameters

The converse problem is to take an actual manipulator and derive the DH parameters for it. Below, we'll discuss how to set up DH parameters for several common robot kinematic configurations.

4.5.1 Two-Link Planar Manipulator

We already discussed how to set up coordinate axes for the two-link planar manipulator in Chapter 2. Now we do so formally with DH parameters. Figure 4.18 indicates the link labeling and a table of the DH parameters. We reemphasize the following points about this parameterization.



i	a_i	d_i	α_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2

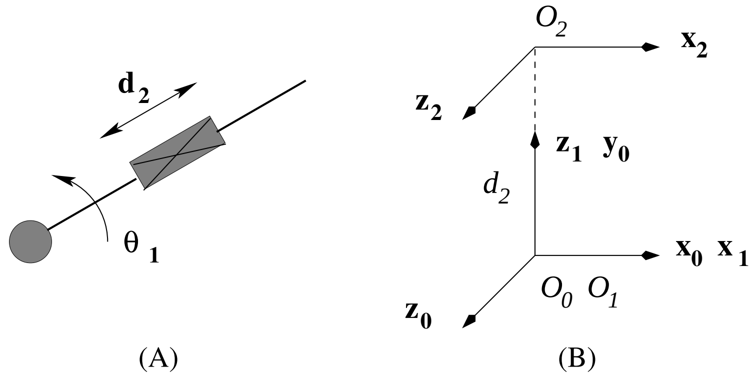
Figure 4.18: DH parameterization of two-link planar manipulator.

- Axes are located at the distal ends of each link.
- By locating link 2 coordinates at the end, we avoid another transformation to locate the endpoint.
- We made $\mathbf{x}_2$ the approach vector in this special case.
- Joint i is located at coordinate origin $i-1$.

The zero position of this manipulator is with the arm straight out to the right.

4.5.2 Polar Manipulator

The 2-DOF polar manipulator in Figure 4.19(A) is another planar manipulator, but with a prismatic joint as the second joint.



i	a_i	d_i	α_i	θ_i
1	0	0	$-\pi/2$	$*$
2	0	$*$	$\pi/2$	0

(C)

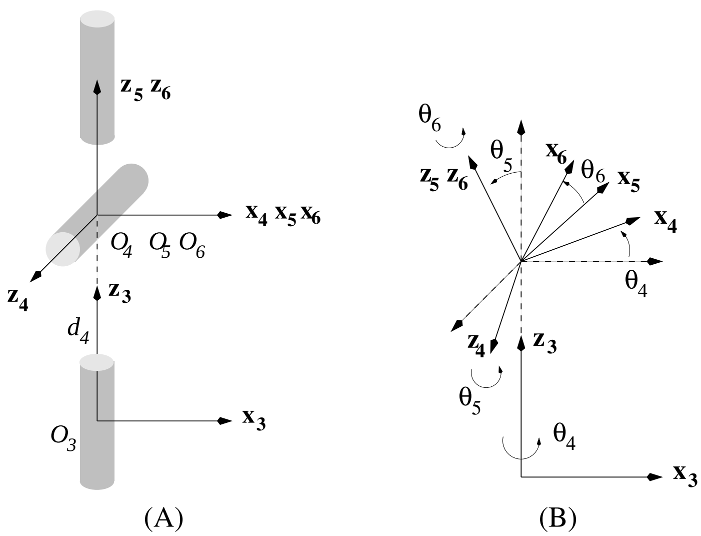
Figure 4.19: (A) A polar manipulator. (B) Coordinate system placement and zero-angle position. (C) DH parameters.

The first coordinate system is placed as for the two-link planar manipulator, with the $\mathbf{z}_0$ axis pointing out of the page and $\mathbf{x}_0$ pointing to the right (Figure 4.19(B)). The prismatic axis $\mathbf{z}_1$ intersects $\mathbf{z}_0$, so the origin O_1 is at the same location as O_0. The prismatic axis is at right angles to the rotary axis, and in the zero-angle position has to be such that the $\mathbf{x}_1$ axis is coincident with $\mathbf{x}_0$. Choosing $\alpha_1 = \pi/2$ would cause the $\mathbf{z}_1$ axis to point down, while choosing $\alpha_1 = -\pi/2$ causes $\mathbf{z}_1$ to point up as in the figure. The latter should be chosen if the preference is for the axis to point up in the zero-angle position. A reason might be that the polar arm is sitting on a table, and a zero-angle position with the axis pointing down would cause the arm to go through the table.

Lastly, the link 2 coordinate system has been chosen with $\mathbf{z}_2$ pointing out of the page in order to be consistent with the direction of $\mathbf{z}_0$. This choice yields $\mathbf{x}_2$ pointing to the right, a fixed joint angle $\theta_2 = 0$, and a skew angle $\alpha_2 = \pi/2$. The joint variable d_2 is shown with a positive non-zero length to separate origin O_2 from O_0. The final DH parameters are summarized in the table in Figure 4.19(C). Astericks have been placed for θ_1 and d_2 to show they are the joint variables.

4.5.3 Spherical Wrist

The DH parameterization of a spherical joint is illustrated in Figure 4.20. The first axes are labeled 3, because we will be adding this spherical joint as a wrist at the end of the elbow robot (Figure 4.4(A)). Origin O_3 will be interpreted as the elbow joint, and $\mathbf{z}_3$ will point along the "forearm" of the robot. Figure 4.20(A) shows the wrist in the zero position, when all joint angles are zero.



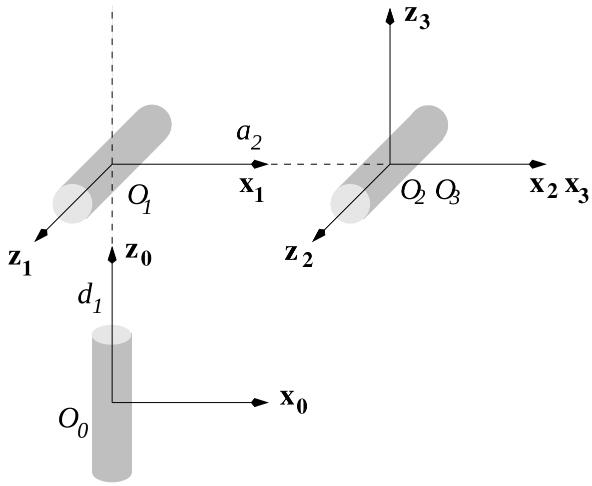
i	a_i	d_i	α_i	θ_i
4	0	d_4	$\pi/2$	θ_4
5	0	0	$-\pi/2$	θ_5
6	0	0	0	θ_6

Figure 4.20: (A) Spherical joint in zero position. (B) Spherical joint after displacements in all angles.

The last frame 6, which will be interpreted as the end link, has been chosen conveniently so that $\mathbf{z}_5$ and $\mathbf{z}_6$ are coincident. Axis $\mathbf{z}_6$ can be considered as pointing along the gripper, i.e., the approach vector $\mathbf{a}$ (4.23). We have chosen $\mathbf{x}_5 = \mathbf{z}_5 \times \mathbf{z}_4$ and $\alpha_5 = -\pi/2$ because when $\theta_5 = 0$ we wish the hand to point straight up from the forearm. The alternative would have the hand fold back into the forearm, which while kinematically possible is not physically possible.

4.5.4 Elbow Robot

The elbow robot (Figure 4.4(A)) has the spherical wrist just developed. The DH parameterization for the first three joints is shown in Figure 4.21. Note that $\alpha_3 = -\pi/2$ and that $\mathbf{x}_3 = \mathbf{z}_3 \times \mathbf{z}_2$. This choice was



i	a_i	d_i	α_i	θ_i
1	0	d_1	$\pi/2$	θ_1
2	a_2	0	0	θ_2
3	0	0	$-\pi/2$	θ_3

Figure 4.21: First three joints of the elbow robot.

made so that in the zero position, the elbow points up rather than down. There is no reason to do this, other than for convenience in thinking where the manipulator is when all joints are zero.

4.5.5 SCARA Robot

The DH parameters for the SCARA robot appear below. Note that for joint 3, the displacement d_3 is the joint variable whereas $\theta_3 = 0$ is constant.

- The location of the sliding axis $\mathbf{z}_2$ is arbitrary, since it is a free vector. For simplicity, we make it coincident with $\mathbf{z}_3$. Thus a_2 and d_2 are arbitrarily set.

- The placement of O_3 and $\mathbf{x}_3$ along $\mathbf{z}_3$ is arbitrary, since $\mathbf{z}_2$ and $\mathbf{z}_3$ are coincident. Once we choose O_3, however, then the joint displacement d_3 is defined.

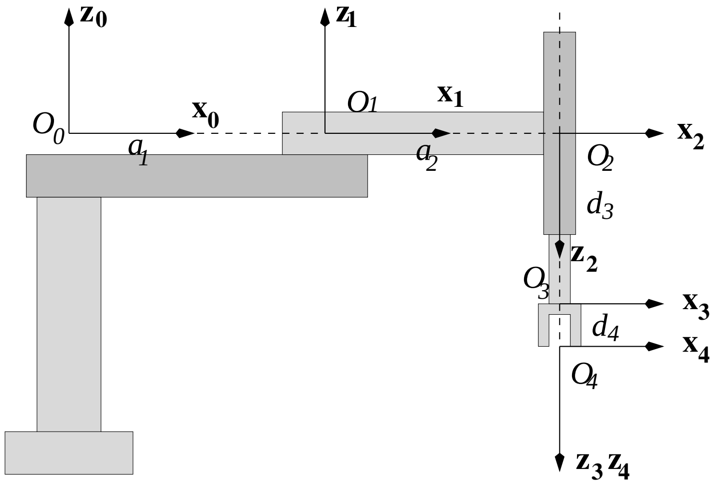
- We have also placed the end link frame in a convenient manner, with the $\mathbf{z}_4$ axis coincident with the $\mathbf{z}_3$ axis and the origin O_4 displaced down into the gripper by d_4.

When drawing a manipulator from the DH parameterization, we usually draw it in the zero-angle position (all joint angles or variable displacements 0). All $\mathbf{x}_i$ axes will be parallel and pointing in the same direction!

4.6 Forward Kinematics

4.6.1 Link Coordinate Transformation

The coordinate transformation from the link i frame to the link $i-1$ frame will now be derived with the aid of Figure 4.7. The rotational transformation ${}^{i-1}\mathbf{R}_i$ first rotates about the $\mathbf{z}_{i-1}$ axis by θ_i, then about the $\mathbf{x}_i$



i	a_i	d_i	α_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	π	θ_2
3	0	d_3	0	0
4	0	d_4	0	θ_4

Figure 4.22: The SCARA robot.

axis by α_i:

$$^{i-1}\mathbf{R}_i = \mathbf{R}_z(\theta_i)\mathbf{R}_x(\alpha_i) = \begin{bmatrix} \mathrm{c}\theta_i & -\mathrm{s}\theta_i & 0 \\ \mathrm{s}\theta_i & \mathrm{c}\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\alpha_i & -\mathrm{s}\alpha_i \\ 0 & \mathrm{s}\alpha_i & \mathrm{c}\alpha_i \end{bmatrix} = \begin{bmatrix} \mathrm{c}\theta_i & -\mathrm{s}\theta_i\mathrm{c}\alpha_i & \mathrm{s}\theta_i\mathrm{s}\alpha_i \\ \mathrm{s}\theta_i & \mathrm{c}\theta_i\mathrm{c}\alpha_i & -\mathrm{c}\theta_i\mathrm{s}\alpha_i \\ 0 & \mathrm{s}\alpha_i & \mathrm{c}\alpha_i \end{bmatrix} \tag{4.3}$$

The translational component referenced to axes $i-1$ is:

$$^{i-1}\mathbf{d}_{i-1,i} = d_i\,{}^{i-1}\mathbf{z}_{i-1} + a_i\,{}^{i-1}\mathbf{x}_i = d_i\,{}^{i-1}\mathbf{z}_{i-1} + a_i\,{}^{i-1}\mathbf{R}_i\,{}^{i}\mathbf{x}_i = d_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + a_i \begin{bmatrix} \mathrm{c}\theta_i \\ \mathrm{s}\theta_i \\ 0 \end{bmatrix} = \begin{bmatrix} a_i\mathrm{c}\theta_i \\ a_i\mathrm{s}\theta_i \\ d_i \end{bmatrix} \tag{4.4}$$

Hence the homogeneous transformation $^{i-1}\mathbf{T}_i$ from frame i to frame $i-1$ is:

$$^{i-1}\mathbf{T}_i = \begin{bmatrix} ^{i-1}\mathbf{R}_i & ^{i-1}\mathbf{d}_{i-1,i} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{c}\theta_i & -\mathrm{s}\theta_i\mathrm{c}\alpha_i & \mathrm{s}\theta_i\mathrm{s}\alpha_i & a_i\mathrm{c}\theta_i \\ \mathrm{s}\theta_i & \mathrm{c}\theta_i\mathrm{c}\alpha_i & -\mathrm{c}\theta_i\mathrm{s}\alpha_i & a_i\mathrm{s}\theta_i \\ 0 & \mathrm{s}\alpha_i & \mathrm{c}\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.5}$$

4.6.2 Relating Any Two Link Frames

By composition of intervening coordinate transformations, we can relate any two nonadjacent frames i and j, where say $i > j$. Using the elementary homogeneous transformations between adjacent frames,

$$^{i}\mathbf{T}_j = {}^{i}\mathbf{T}_{i+1}\,{}^{i+1}\mathbf{T}_{i+2}\cdots{}^{j-2}\mathbf{T}_{j-1}\,{}^{j-1}\mathbf{T}_j, \qquad i < j \tag{4.6}$$

For efficiency reasons during computer implementation, it is usually best to decompose the coordinate transformations into separate rotations and translations.

$$^{i}\mathbf{R}_j = {}^{i}\mathbf{R}_{i+1}\,{}^{i+1}\mathbf{R}_{i+2}\cdots{}^{j-2}\mathbf{R}_{j-1}\,{}^{j-1}\mathbf{R}_j \tag{4.7}$$

$$^{i}\mathbf{d}_{ij} = \sum_{k=i+1}^{j} {}^{i}\mathbf{R}_{k-1}\,{}^{k-1}\mathbf{d}_{k-1,k} \tag{4.8}$$

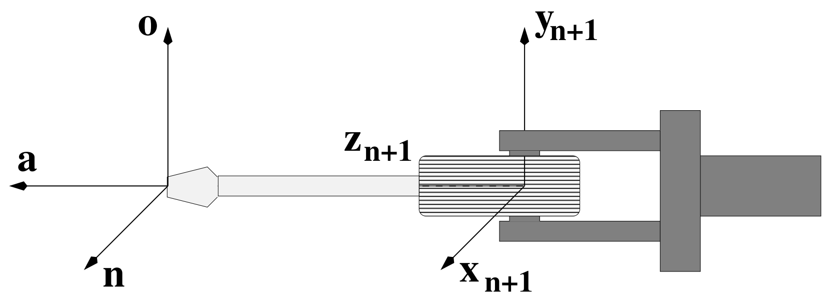


Figure 4.23: Approach $\mathbf{a}$, orientation $\mathbf{o}$, and normal $\mathbf{n}$ axes in the tip of a screwdriver.

where we define ${}^{i}\mathbf{R}_i = \mathbf{I}$, the identity matrix. An alternative is to write the translation vectors in terms of their components, which is often preferable in practice:

$$ {}^{i}\mathbf{d}_{ij} = \sum_{k=i+1}^{j} d_k \, {}^{i}\mathbf{R}_{k-1} \, {}^{k-1}\mathbf{z}_{k-1} + a_k \, {}^{i}\mathbf{R}_k \, {}^{k}\mathbf{x}_k \tag{4.9} $$

4.6.3 Forward Kinematics Computation

The forward kinematics computation is to find the position and orientation of the last frame n with respect to the base frame 0. Composing homogeneous transformations as in (4.6),

$$ {}^{0}\mathbf{T}_n = {}^{0}\mathbf{T}_1 \, {}^{1}\mathbf{T}_2 \cdots {}^{n-1}\mathbf{T}_n \tag{4.10} $$

Composing the rotations and translations separately as in (4.7)-(4.8),

$$ {}^{0}\mathbf{R}_n = {}^{0}\mathbf{R}_1 \, {}^{1}\mathbf{R}_2 \cdots {}^{n-1}\mathbf{R}_n \tag{4.11} $$

$$ {}^{0}\mathbf{d}_{0n} = \sum_{k=1}^{n} d_k \, {}^{0}\mathbf{R}_{k-1} \, {}^{k-1}\mathbf{z}_{k-1} + a_k \, {}^{0}\mathbf{R}_k \, {}^{k}\mathbf{x}_k \tag{4.12} $$

To simplify computationally, one could do the following:

- ${}^{0}\mathbf{R}_{k-1} \, {}^{k-1}\mathbf{z}_{k-1}$ just copies the last column.
- ${}^{0}\mathbf{R}_k \, {}^{k}\mathbf{x}_k$ copies the first column.
- ${}^{0}\mathbf{R}_k = {}^{0}\mathbf{R}_{k-1} \, {}^{k-1}\mathbf{R}_k$ is computed recursively.

If additional frames are added at the beginning or end, for example the frame -1 as in Figure 4.13 or $n+1$ in Figure 4.14, the forward kinematics computation is extended to include them.

4.6.4 The Tool Transform

A robot will be frequently picking up objects or tools, and motion is most easily planned in terms of natural coordinates in the gripper or objects or tools held in the gripper. As mentioned previously, two DH frames n and $n+1$ are generally required to locate an arbitrary frame in the last link. Rather than redefining frames n and $n+1$ every time a different object is picked up, standard practice is to keep these frames the same and to add an extra homogeneous transformation ${}^{n+1}\mathbf{T}_{tool}$ that relates the frame of the object or tool to a fixed frame in the end effector. For example, Figure 4.23 shows a gripper holding a screwdriver. It is natural to plan in terms of a coordinate system fixed in the tip of the blade. Coordinate system $n+1$ is fixed in the gripper as constructed previously, and another coordinate system with axes $\mathbf{x} = \mathbf{n}$, $\mathbf{y} = \mathbf{o}$, and $\mathbf{z} = \mathbf{a}$ is located at the blade tip. This special nomenclature corresponds to $\mathbf{a}$ as the approach vector, $\mathbf{o}$ as the orientation vector, and $\mathbf{n}$ as the normal vector.

The forward kinematics for the robot holding the object is

$$^{0}\mathbf{T}_{tool} = \ ^{0}\mathbf{T}_{n+1}\ ^{n+1}\mathbf{T}_{tool}$$

Once an object is rigidly grasped, then $^{n+1}\mathbf{T}_{tool}$ is constant. For practical purposes, the object can be considered to comprise part of the last link. The transform $^{n}\mathbf{T}_{n+1}$ is also constant. The two constant transforms can be grouped together:

$$^{n}\mathbf{T}_{tool} = \ ^{n}\mathbf{T}_{n+1}\ ^{n+1}\mathbf{T}_{tool} \tag{4.13}$$

The constant transformation $^{n}\mathbf{T}_{tool}$ is referred to as the *tool transform.* Even if an additional frame $n+1$ is not defined and an object frame is related directly to frame n, the composite transformation (4.13) is still referred to as the tool transform. It represents the frame operated on by joint n.

4.6.5 Kinematic Calibration

Kinematic calibration is the process of inferring the DH parameters by experimental measurement. Typically, an external metrology system such as a stereo camera system measures the 3D position of one or more points on the end effector [3]. The error between the measured position and the position predicted by the nominal kinematic parameters is employed in an iterative nonlinear estimation procedure to adjust the kinematic parameters to their "true" values. Kinematic calibration needs to be performed on all manipulators, due to variations in manufacturing and assembly that cause these parameters to vary from the initial specifications. The effect of inaccurate kinematic parameters is usually significant. For example, a commercial robot such as the PUMA robot may have a positioning accuracy of only a few millimeters. Calibration can improve the accuracy by a factor of 10.

Another aspect of kinematic calibration is to determine the tool transform. Sometimes we are lucky, as in stereotyped assembly line operations where the geometry of objects and their location in the grasp is predefined. Otherwise, the tool transform has to be discovered experimentally. The same holds for environmental frame transformations to the robot base.

Parallel neighboring joint axes $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$ offer special problems in calibrating the DH parameters. No joint axes are in reality exactly parallel, and hence they will be nearly parallel at a skew angle α_i to be determined. The location of the common normal, and hence of origin O_i, is extremely sensitive to slight changes of relative orientation: it can vary wildly with changes of just fractions of a degree. Thus the identification of the skew angle is poorly conditioned; the iterative nonlinear estimation procedure mentioned above is likely not to converge.

To circumvent this problem, an alternative link parameterization must be employed for the nearly parallel case. The Hayati parameters [2] have become accepted as this alternative parameterization. The origin O_i is fixed by the intersection of $\mathbf{z}_i$ with the $\mathbf{x}_{i-1}$/$\mathbf{y}_{i-1}$ plane (Figure 4.24). This plane passes through O_{i-1}, and $\mathbf{z}_{i-1}$ is its normal. The origin O_i is robustly fixed by this construction for the nearly parallel case.

The vector $\mathbf{x}_{i''}$ connects O_{i-1} to O_i, but it is not the $\mathbf{x}$-axis for coordinate system i. The double prime ($''$) represents one of two intermediate coordinate system between coordinate systems $i-1$ and i, with $\mathbf{z}_{i''} = \mathbf{z}_{i-1}$; $\mathbf{y}_{i-1''}$ completes the right-hand set. The two intermediate coordinate systems are required to

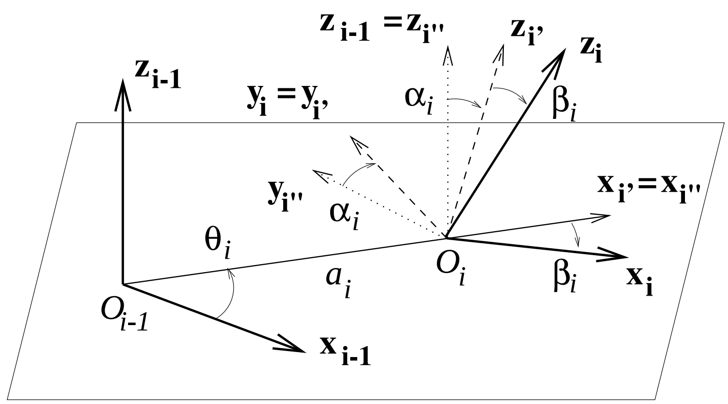


Figure 4.24: Hayati parameterization.

represent the two rotational transformations between $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$. The single prime ($'$) coordinate system is related to the double prime coordinates by rotation about $\mathbf{x}_{i''} = \mathbf{x}_{i'}$ by α_i:

$$\mathbf{z}_{i''} = \mathbf{R}_x(\alpha_i)\mathbf{z}_{i'}, \quad \mathbf{y}_{i''} = \mathbf{R}_x(\alpha_i)\mathbf{y}_{i'} \tag{4.14}$$

Finally, $\mathbf{z}_i$ and $\mathbf{x}_i$ are related to $\mathbf{z}_{i'}$ and $\mathbf{x}_{i'}$ by rotation about $\mathbf{y}_{i'} = \mathbf{y}_i$ by β_i:

$$\mathbf{z}_{i'} = \mathbf{R}_y(\beta_i)\mathbf{z}_i, \quad \mathbf{x}_{i'} = \mathbf{R}_y(\beta_i)\mathbf{x}_i \tag{4.15}$$

In the figure, $\mathbf{x}_i$ would lie below the plane, as is indicated by the dotted line.

The angle from $\mathbf{x}_{i-1}$ to $\mathbf{x}_{i''}$ about $\mathbf{z}_{i-1}$ is θ_i, while the distance from O_{i-1} to O_i along $\mathbf{x}_{i''}$ is a_i. The four parameters α_i, β_i, θ_i, and a_i are the Hayati parameters. Although the definitions of α_i, θ_i, and a_i are not the same as for the DH parameters, the same symbols are used because their definitions are in the same spirit. The Hayati parameter β_i replaces d_i as the fourth parameter. The rotational transformation from i to $i-1$ is:

$$^{i-1}\mathbf{R}_i \quad = \quad \mathbf{R}_z(\theta_i)\mathbf{R}_x(\alpha_i)\mathbf{R}_y(\beta_i) \tag{4.16}$$

while the vector from O_{i-1} to O_i is:

$$^{i-1}\mathbf{d}_{i-1,i} = a_i\, ^{i-1}\mathbf{x}_{i''} = a_i\mathbf{R}_z(\theta_i)\begin{bmatrix}1\\0\\0\end{bmatrix} = a_i\begin{bmatrix}\mathrm{c}\theta_i\\ \mathrm{s}\theta_i\\ 0\end{bmatrix} \tag{4.17}$$

The subsequent coordinate system $i+1$ is then defined with respect to this coordinate system i. This next coordinate system may involve DH parameters, or Hayati parameters again.

The Hayati parameters themselves have a problem for nearly perpendicular joint axes: the intersection of the $\mathbf{x}_{i-1}$/$\mathbf{y}_{i-1}$ plane with $\mathbf{z}_i$ is poorly defined. For calibration purposes, one therefore chooses the DH parameters for nearly perpendicular axes and the Hayati parameters for nearly parallel axes. For in between cases, one can choose a value such as $\alpha_i = 30$ degrees as the cutoff. Once calibration has been performed, one may then transform from Hayati parameters to DH parameters and thereafter use only DH parameters for kinematic transformations. The relation between the two may be derived straightforwardly. Alternatively, one can keep the Hayati parameters and employ the appropriate rotational and translational transformations (4.16) and (4.17).

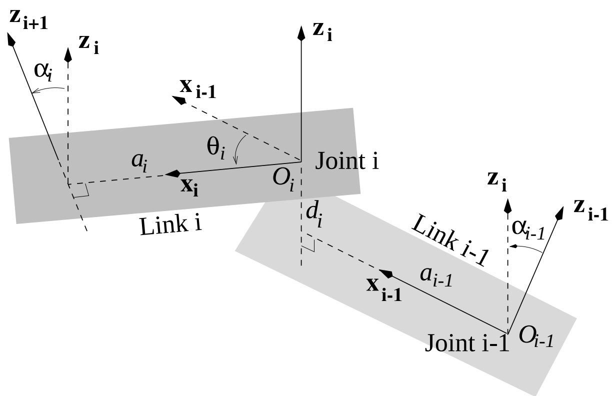


Figure 4.25: Craig's version of the DH parameters.

4.7 Craig's Convention

What has been presented so far is the set of standard DH parameters, espoused for example by Paul [4]. It is important to know that there is a different set of DH parameters that is commonly used, due to Craig [1]. Many people in robotics are actually unaware that there are two different conventions in use.

The big difference in Craig's convention is that coordinate origin i for link i is located at the proximal joint. The parameterization follows from this decision (Figure 4.25).

- $\mathbf{x}_{i-1}$ lies along the common normal from $\mathbf{z}_{i-1}$ to $\mathbf{z}_i$.
- a_i is the distance along $\mathbf{x}_i$ from $\mathbf{z}_i$ to $\mathbf{z}_{i+1}$.
- d_i is the distance along $\mathbf{z}_i$ from $\mathbf{x}_{i-1}$ to $\mathbf{x}_i$.
- α_{i-1} is the angle from $\mathbf{z}_{i-1}$ to $\mathbf{z}_i$ about $\mathbf{x}_{i-1}$.
- θ_i is the angle from $\mathbf{x}_{i-1}$ to $\mathbf{x}_i$ about $\mathbf{z}_i$.

The transformation from frame i to frame $i-1$ is defined as follows:

$$
\begin{aligned}
{}^{i-1}\mathbf{R}_i &= \mathbf{R}_x(\alpha_{i-1})\mathbf{R}_z(\theta_i) \\
{}^{i-1}\mathbf{d}_{i-1,i} &= d_i\,{}^{i-1}\mathbf{z}_i + a_{i-1}\,{}^{i-1}\mathbf{x}_{i-1} \\
&= d_i\mathbf{R}_x(\alpha_{i-1})\mathbf{R}_z(\theta_i)\,{}^{i}\mathbf{z}_i + a_{i-1}\,{}^{i-1}\mathbf{x}_{i-1} \\
&= d_i\mathbf{R}_x(\alpha_{i-1})\,{}^{i}\mathbf{z}_i + a_{i-1}\,{}^{i-1}\mathbf{x}_{i-1}
\end{aligned}
$$

An advantage of Craig's convention is the proximal placement of the origin for a link. Also the rotation θ_i is about $\mathbf{z}_i$ and the joint number is the same as the coordinate number, which seem more natural. Torque exerted about joint i is also at the same place as at link i's coordinate system, to which inertial parameters such as center of mass are likely to be referenced. A disadvantage is that the transform mixes $i-1$ and i parameters.

As an example, a set of Craig parameters is developed for the first three joints of the elbow robot (Figure 4.21) in Figure 4.26. The axes of rotation are now $\mathbf{z}_1$, $\mathbf{z}_2$, and $\mathbf{z}_3$. A coordinate system 0, fixed in the ground, has to be added as a reference for coordinate system 1. This has been done by by making $\mathbf{z}_0$ parallel

$$\left[\begin{array}{ccccc} i & a_i & d_i & \alpha_i & \theta_i \\ \hline 0 & 0 & - & 0 & - \\ 1 & 0 & d_1 & \pi/2 & * \\ 2 & a_2 & 0 & 0 & * \\ 3 & - & 0 & - & * \end{array}\right]$$

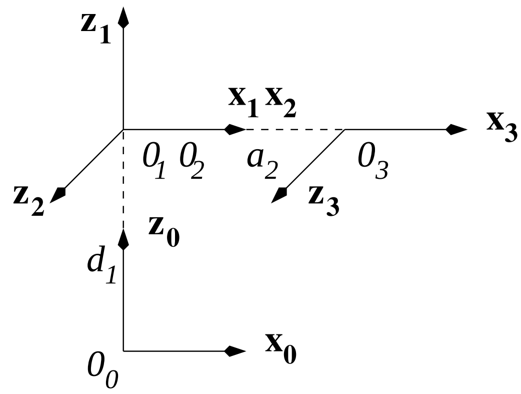


Figure 4.26: Craig parameters developed for the first three joints of the elbow robot.

to and coincident with $\mathbf{z}_1$, and placing O_0 a distance d_1 from O_1. The arbitrary choices by this placement of coordinate system 0 are thus d_1, $\alpha_0 = 0$ (parallel axes), and $a_0 = 0$ (coincident axes). Parameters d_0 and θ_0 are undefined. For the last link, parameters a_3 and α_3 are undefined.

Both Craig's convention and the standard DH convention are equally valid. The choice of one over the other is merely a matter of taste or habit.

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