Econometrics
Part I: Basic regression.
Session 3.

References on Bruce Hansen's textbook will be abbreviated with H. Say, H 1.1 means chapter 1.1 from Bruce Hansen's textbook.

Distribution of the OLS estimator. (H 5.8-5.11)
Tests of linear hypotheses. (H 5.12-5.16, 8.1-8.4, 9.1-9.7, 9.14)
Tests of inequalities. (These notes).

Preliminaries

Let Z be an n-dimensional random vector with the joint density of its elements given by $f_Z(z)$. Consider a transformed vector $W = g(Z)$. Suppose that g is a one-to-one transformation of $\mathbb{R}^n$ to $\mathbb{R}^n$. Then the joint density of the elements of W, $f_W(w)$, is given by

$$f_W(w) = f_Z\left(g^{-1}(w)\right)|J|,$$

where J is the Jacobian of the inverse transformation. That is, the determinant of the matrix of partial derivatives of the elements of $g^{-1}(w)$ with respect to the elements of w. (For a derivation see Casella and Berger's (1990) textbook).

For example, if $W = AZ$ for some invertible matrix A, then

$$f_W(w) = f_Z\left(A^{-1}w\right)\left|\det\left(A^{-1}\right)\right|.$$

In particular, let Z be an n-dimensional normal vector with distribution $N(0, \sigma^2 I)$. Consider $W = UZ$, where U (that is, such that $U'U = I$) is an orthogonal matrix. Then, because $|\det U| = 1$, we have

$$\begin{aligned} f_W(w) &= f_Z(U'w) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left\{-\frac{1}{2\sigma^2}(U'w)'(U'w)\right\} \\ &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left\{-\frac{1}{2\sigma^2} w'w\right\} \end{aligned}$$

Hence, W also has distribution $N(0, \sigma^2 I)$. We say that the normal distribution with covariance matrix proportional to identity is invariant with respect to rotations. More generally, if $Z \sim N(\mu, \Sigma)$ and $W = AZ$ with not necessarily orthogonal A, then by following the same steps, we can show that

$$f_W(w) = \frac{1}{(2\pi)^{n/2}|A\Sigma A'|^{1/2}} \exp\left\{-\frac{1}{2}(w - A\mu)'(A\Sigma A')^{-1}(w - A\mu)\right\}.$$

In other words, $W \sim N(A\mu, A\Sigma A')$.

Conditional distribution of the OLS estimator.

Theorem 1 *Under GM1-4, we have*

$$\hat{\beta}_{OLS}|X \sim N\left(\beta, \sigma^2 (X'X)^{-1}\right)$$

Indeed,

$$\hat{\beta}_{OLS} = (X'X)^{-1} X'Y \text{ and } Y|X \sim N\left(X\beta, \sigma^2 I\right).$$

Therefore,

$$\hat{\beta}_{OLS}|X \sim N\left(AX\beta, \sigma^2 AA'\right),$$

where $A = (X'X)^{-1} X'$. But $AA' = (X'X)^{-1} X'X (X'X)^{-1} = (X'X)^{-1}$, and $AX = (X'X)^{-1} X'X = I_k$.$\square$

Further, consider

$$\hat{\sigma}^2 = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n-k} = \frac{1}{n-k} Y'M_X Y = \frac{1}{n-k} Y'\left(I - X(X'X)^{-1}X'\right)Y$$

Denote $X(X'X)^{-1/2}$ as U_1. Note that $U_1'U_1 = I_k$. Let U_2 be such that $U = [U_1, U_2]$ is an orthogonal matrix. In other words, let the columns of U_2 represent an orthonormal basis in the space orthogonal to that spanned by the columns of X. Then,

$$I - X(X'X)^{-1}X' = UU' - U_1U_1' = U_1U_1' + U_2U_2' - U_1U_1' = U_2U_2'$$

so that

$$\hat{\sigma}^2 = \frac{1}{n-k} Y'U_2U_2'Y.$$

On the other hand, $U_2'Y$ is the vector of last $n-k$ coordinates of $U'Y \sim N(U'X\beta, \sigma^2 I_n)$. We have

$$U'X\beta = \begin{pmatrix} U_1'X\beta \\ U_2'X\beta \end{pmatrix} = \begin{pmatrix} U_1'X\beta \\ 0 \end{pmatrix},$$

so

$$U_2'Y \sim N\left(0, \sigma^2 I_{n-k}\right)$$

and

$$\frac{Y'U_2U_2'Y}{\sigma^2} \sim \chi^2(n-k).$$

Hence, we have.

Theorem 2 *Under GM1-4, we have*

$$\hat{\sigma}^2|X \sim \frac{\sigma^2}{n-k}\chi^2(n-k).$$

In particular, the variance of $\hat{\sigma}^2|X$ is

$$\left(\frac{\sigma^2}{n-k}\right)^2 Var\left(\chi^2(n-k)\right) = \frac{\sigma^4 2(n-k)}{(n-k)^2} = \frac{2\sigma^4}{n-k}$$

Hence the unbiased estimator $\hat{\sigma}^2$ does not attain the CRLB:

$$\frac{2\sigma^4}{n-k} > \frac{2\sigma^4}{n}.$$

Finally, note that $\hat{\beta}_{OLS}$ has form $(X'X)^{-1/2} U_1'Y$. Since, as follows from the above derivation, $U_1'Y$ and $U_2'Y$ are independent (uncorrelated variables that are jointly normal must be independent normals), we must have

Theorem 3 *Under GM1-4, $\hat{\beta}_{OLS}$ and $\hat{\sigma}^2$ are independent conditionally on X.*

Tests of linear hypotheses

To test linear hypotheses of form $R\beta = q$, assume GM1-4 hold.

<u>Test:</u> $H_0 : R\beta = q$ vs. $H_1 : R\beta \neq q$, where R is $p \times k$ and $\text{rank}(R) = p$.

$$\hat{\beta}|X \sim N\left(\beta, \sigma^2 (X'X)^{-1}\right)$$

so

$$R\hat{\beta} - q|X \sim N\left(R\beta - q, \sigma^2 R(X'X)^{-1} R'\right) \overset{H_0}{\sim} N\left(0, \sigma^2 R(X'X)^{-1} R'\right)$$

Hence,

$$\left(\sigma^2 R\left(X'X\right)^{-1}R'\right)^{-1/2}\left(R\hat{\beta}-q\right)|X \overset{H_0}{\sim} N\left(0, I_p\right) \text{ and}$$
$$\left(R\hat{\beta}-q\right)'\left(\sigma^2 R\left(X'X\right)^{-1}R'\right)^{-1}\left(R\hat{\beta}-q\right)|X \overset{H_0}{\sim} \chi^2(p)$$

Unfortunately, this test statistic contains σ^2, which we generally do not know. We can consider replacing σ^2 with its estimator, $\hat{\sigma}^2$, to obtain the so-called Wald statistic

$$W=\left(R\hat{\beta}-q\right)'\left(\hat{\sigma}^2 R\left(X'X\right)^{-1}R'\right)^{-1}\left(R\hat{\beta}-q\right)$$

where we recall that

$$\frac{n-k}{\sigma^2}\hat{\sigma}^2|X \sim \chi^2\left(n-k\right) \text{ and is independent from } \hat{\beta}|X.$$

Since the ratio of two independent chi-squares divided by their degrees of freedom is distributed as an F, we have

$$\begin{aligned}\frac{W}{p} &= \frac{1}{p}\frac{\sigma^2}{\hat{\sigma}^2}\left(R\hat{\beta}-q\right)'\left(\sigma^2 R\left(X'X\right)^{-1}R'\right)^{-1}\left(R\hat{\beta}-q\right)\\ &= \frac{\left(R\hat{\beta}-q\right)'\left(\sigma^2 R\left(X'X\right)^{-1}R'\right)^{-1}\left(R\hat{\beta}-q\right)/p}{\frac{n-k}{\sigma^2}\hat{\sigma}^2/\left(n-k\right)}\end{aligned}$$

so

$$\frac{W}{p}|X \overset{H_0}{\sim} F(p, n-k)$$

Note the importance of the independence of the estimators $\hat{\beta}$ and $\hat{\sigma}^2$ in forming this test statistic.

Special case: Single Hypothesis restriction

In a special case, a single hypothesis is tested

$$\begin{aligned}R &= (0, ..., 1, ..., 0)\\ q &= \text{scalar}\end{aligned}$$

The null and the alternative hypotheses become

$$H_0 : \beta_j = q \text{ vs. } H_1 : \beta_j \neq q$$

$$
\begin{aligned}
\hat{\beta}|X &\sim N\left(\beta, \sigma^2 \left(X'X\right)^{-1}\right) \\
\implies \hat{\beta}_j|X &\sim N\left(\beta_j, \sigma^2 \left(X'X\right)^{-1}_{jj}\right) \\
\implies \hat{\beta}_j|X &\overset{H_0}{\sim} N\left(q, \sigma^2 \left(X'X\right)^{-1}_{jj}\right)
\end{aligned}
$$

and

$$
\frac{\hat{\beta}_j - q}{\sqrt{\sigma^2 \left(X'X\right)^{-1}_{jj}}}|X \overset{H_0}{\sim} N(0,1)
$$

Again, we do not know σ^2, and so substitute $\hat{\sigma}^2$ as above, denoting this new statistic t :

$$
\begin{aligned}
t &= \frac{\hat{\beta}_j - q}{\sqrt{\hat{\sigma}^2 \left(X'X\right)^{-1}_{jj}}} \\
&= \frac{\left(\hat{\beta}_j - q\right) / \sqrt{\sigma^2 \left(X'X\right)^{-1}_{jj}}}{\sqrt{\frac{(n-k)\hat{\sigma}^2}{\sigma^2} / (n-k)}}
\end{aligned}
$$

Conditional on X, the numerator is standard normal and the denominator is the square root of a chi-square. Further, the numerator and denominator are independent conditionally on X. Hence

$$
t|X \overset{H_0}{\sim} t(n-k)
$$

Note that for large n, the t distribution approaches a normal distribution.

Another form of the linear test statistic

Assume GM1-4 hold. We are testing the following hypothesis, where R is $p \times k$ and $\operatorname{rank}(R) = p$:

$$
H_0 : R\beta = q \text{ vs. } H_1 : R\beta \neq q
$$

Previously, we showed that

$$
W/p = \left(R\hat{\beta} - q\right)' \left(\hat{\sigma}^2 R \left(X'X\right)^{-1} R'\right)^{-1} \left(R\hat{\beta} - q\right) / p \overset{H_0}{\sim} F(p, n-k)
$$

Suppose that we impose the null hypothesis restrictions when we minimize the sum of squared residuals. call the solution to this problem the “restricted least squares estimator," $\tilde{\beta}$.

$$\min_{\beta} (Y - X\beta)' (Y - X\beta) \text{ s.t. } R\beta = q$$

The Lagrangian for this minimization is

$$L(\beta) = (Y - X\beta)' (Y - X\beta) + \lambda' (R\beta - q)$$

The first order conditions are:

$$\begin{aligned} \frac{\partial L}{\partial \beta} &= -2X' \left(Y - X\tilde{\beta}\right) + R'\lambda = 0 && (1) \\ \frac{\partial L}{\partial \lambda} &= R\tilde{\beta} - q = 0 && (2) \end{aligned}$$

From FOC (1), we have

$$\tilde{\beta} = \hat{\beta} - (X'X)^{-1} R' \left(\frac{\lambda}{2}\right)$$

where $\hat{\beta}$ is the usual (unrestricted) OLS estimator. Now, by FOC (2) $R\tilde{\beta} = q$, so

$$\begin{aligned} 0 &= R\hat{\beta} - q - R(X'X)^{-1} R' \left(\frac{\lambda}{2}\right) \\ &\Longrightarrow \frac{\lambda}{2} = \left(R(X'X)^{-1} R'\right)^{-1} \left(R\hat{\beta} - q\right) \end{aligned}$$

Thus,

$$\tilde{\beta} = \hat{\beta} - (X'X)^{-1} R' \left(R(X'X)^{-1} R'\right)^{-1} \left(R\hat{\beta} - q\right)$$

Now from the corresponding restricted and unrestricted residuals,

$$\begin{aligned} \hat{\varepsilon} &= Y - X\hat{\beta} \\ \tilde{\varepsilon} &= Y - X\tilde{\beta} = X\hat{\beta} + \hat{\varepsilon} - X\tilde{\beta} = \hat{\varepsilon} + X\left(\hat{\beta} - \tilde{\beta}\right). \end{aligned}$$

Since $\hat{\varepsilon}'X = 0$,

$$\tilde{\varepsilon}'\tilde{\varepsilon} = \hat{\varepsilon}'\hat{\varepsilon} + \left(\hat{\beta} - \tilde{\beta}\right)' X'X \left(\hat{\beta} - \tilde{\beta}\right)$$

and substituting $\hat{\beta} - \tilde{\beta} = (X'X)^{-1} R' \left(R(X'X)^{-1} R'\right)^{-1} \left(R\hat{\beta} - q\right)$,

$$\tilde{\varepsilon}'\tilde{\varepsilon} - \hat{\varepsilon}'\hat{\varepsilon} = \left(R\hat{\beta} - q\right)' \left(R(X'X)^{-1} R'\right)^{-1} \left(R\hat{\beta} - q\right)$$

Now,

$$
\begin{aligned}
W/p &= \frac{\left(R\hat{\beta}-q\right)'\left(R\left(X'X\right)^{-1}R'\right)^{-1}\left(R\hat{\beta}-q\right)}{\hat{\sigma}^2 p} = \frac{\tilde{\varepsilon}'\tilde{\varepsilon}-\hat{\varepsilon}'\hat{\varepsilon}}{\frac{\hat{\varepsilon}'\hat{\varepsilon}}{n-k}p} \\
\implies \frac{W}{p} &= \frac{\left(\tilde{\varepsilon}'\tilde{\varepsilon}-\hat{\varepsilon}'\hat{\varepsilon}\right)/p}{\hat{\varepsilon}'\hat{\varepsilon}/\left(n-k\right)} = \frac{\left(RSS_r - RSS_u\right)/p}{RSS_u/\left(n-k\right)} \overset{H_0}{\sim} F\left(p, n-k\right)
\end{aligned}
$$

We see that in the context of testing linear restrictions under GM1-4, the W/p statistic is nothing else as the familiar F statistic.

Likelihood ratio test

Suppose that the likelihood function is in general given by $\mathcal{L}\left(Z,\theta\right) \equiv f\left(Z,\theta\right),$ where Z is a vector of data and θ is a vector of parameters. Consider testing

$$H_0 : \theta \in S_0 \text{ vs. } H_1 : \theta \in S_1.$$

The likelihood ratio test is defined by the following procedure:

Reject H_0 if

$$LR = -2\log\frac{\max_{\theta\in S_0} f\left(Z,\theta\right)}{\max_{\theta\in S_0\cup S_1} f\left(Z,\theta\right)} > c,$$

where c is chosen so as to satisfy $\max_{\theta\in S_0}\Pr\left(LR > c\right) = \alpha$ for a given significance level $\alpha < 1$ (probability of the type I error).

Neyman-Pearson Lemma. When $S_0 = \theta_0$ and $S_1 = \theta_1$ (so both null and alternative specify just one value of the parameter vector), the likelihood ratio test is the most powerful test at significance level α.

Proof (not examinable): Let $\phi_{LR}\left(Z\right)$ be the indicator function of the event that the likelihood ratio test rejects. Consider any other test, say T, with significance level α. Let $\phi_T\left(Z\right)$ be the indicator function that this test rejects. Let S^+ be the region of the sample space where $\phi_{LR}\left(Z\right) > \phi_T\left(Z\right)$ (LR rejects and T does not). Similarly, let S^- be the region where $\phi_{LR}\left(Z\right) < \phi_T\left(Z\right)$. We have

$$
\begin{aligned}
&\int\left(\phi_{LR}(z) - \phi_T\left(z\right)\right)\left(f\left(z,\theta_1\right) - \exp\left\{\frac{c}{2}\right\}f\left(z,\theta_0\right)\right)dz \\
=& \int_{S^+\cup S^-}\left(\phi_{LR}(z) - \phi_T\left(z\right)\right)\left(f\left(z,\theta_1\right) - \exp\left\{\frac{c}{2}\right\}f\left(z,\theta_0\right)\right)dz \geq 0
\end{aligned}
$$

Therefore, the difference in power between LR and T satisfies

$$\int \left(\phi_{LR}(z) - \phi_T(z)\right) f(z,\theta_1)\, dz \geq \exp\left\{\frac{c}{2}\right\} \int \left(\phi_{LR}(z) - \phi_T(z)\right) f(z,\theta_0)\, dz \geq 0. \square$$

In addition to being most powerful under the Neyman-Pearson assumptions, the LR test has asymptotically optimal properties (e.g. Wald, 1943).

Likelihood ratio test of linear restrictions under GM1-4.

Consider a hypothesis $R\beta = r$ about coefficients of linear regression with normal errors

$$Y = X\beta + \varepsilon, \text{ where } \varepsilon|X \sim N\left(0, \sigma^2 I\right).$$

The unconstrained ML estimates of β and σ^2 in such a model are $\hat{\beta}_{OLS}$ and $\hat{\sigma}^2_{ML} = RSS_{\mathrm{u}}/n$. We have

$$\begin{aligned}
& \log(\max \mathcal{L}(Y,\theta|X) \text{ without the restrictions}) \\
= & -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\left(\hat{\sigma}^2_{ML}\right) \\
& -\frac{1}{2\hat{\sigma}^2_{ML}}\left(Y - X\hat{\beta}_{OLS}\right)'\left(Y - X\hat{\beta}_{OLS}\right) \\
= & -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\left(\frac{RSS_{\mathrm{u}}}{n}\right) - \frac{n}{2}.
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
& \log(\max \mathcal{L}(Y,\theta|X) \text{ under the restrictions}) \\
= & -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\left(\frac{RSS_{\mathrm{r}}}{n}\right) - \frac{n}{2}
\end{aligned}$$

Therefore, the log likelihood ratio statistic for the test of $R\beta = r$ against $R\beta \neq r$ is

$$\begin{aligned}
LR &= -2\left[-\frac{n}{2}\log\left(\frac{RSS_{\mathrm{r}}}{n}\right) + \frac{n}{2}\log\left(\frac{RSS_{\mathrm{u}}}{n}\right)\right] \\
&= n\left[\log(RSS_{\mathrm{r}}) - \log(RSS_{\mathrm{u}})\right] = n\left[\log\left(\frac{RSS_{\mathrm{r}}}{RSS_{\mathrm{u}}}\right)\right] \\
&= n\left[\log\left(\frac{p}{n-k}\frac{(RSS_{\mathrm{r}} - RSS_{\mathrm{u}})/p}{RSS_{\mathrm{u}}/(n-k)} + 1\right)\right] \\
&= n\left[\log\left(\frac{p}{n-k}\frac{W}{p} + 1\right)\right].
\end{aligned}$$

We see that the LR statistic is a monotone transformation of the F statistic, so the LR test and the F test must be equivalent in the context of testing the linear restrictions under GM1-4.

Testing inequality constraints

Sometimes, the interest lies in testing inequality rather than equality restrictions. There is a large literature on this. A couple of early important works are Chernoff (1954) and Wolak (1989). Here we would like to give a flavour of the techniques involved using a very stylized framework. Consider a model with two explanatory variables

$$Y = \beta_1 X_1 + \beta_2 X_2 + \varepsilon$$

where $E(\varepsilon | X_1, X_2) = 0$. Assume that GM1-4 hold. In addition, assume that $\sigma^2 = 1$ is known and that X_1 and X_2 are orthonormal, that is, $X'X = I_2$ where $X = [X_1, X_2]$. This will simplify formulae below.

Suppose that we would like to test

$$H_0 : \beta_1 \geq 0 \text{ and } \beta_2 \geq 0 \text{ vs. } H_1 : \beta_j < 0 \text{ for some } j.$$

Let us derive the LR statistic. Recall that we assumed that it is known that $\sigma^2 = 1$. We have

$$\begin{aligned} & \log(\max \mathcal{L}(Y, \theta | X) \text{ without the restrictions}) \\ = \; & -\frac{n}{2} \log(2\pi) - \frac{1}{2} \left(Y - X\hat{\beta}_{OLS}\right)' \left(Y - X\hat{\beta}_{OLS}\right). \end{aligned}$$

$$\begin{aligned} & \log(\max \mathcal{L}(Y, \theta | X) \text{ under the restrictions}) \\ = \; & -\frac{n}{2} \log(2\pi) - \frac{1}{2} \min_{b_1, b_2 \geq 0} (Y - Xb)' (Y - Xb). \end{aligned}$$

Hence,

$$LR = \min_{b_1, b_2 \geq 0} \left\{ (Y - Xb)' (Y - Xb) - \left(Y - X\hat{\beta}_{OLS}\right)' \left(Y - X\hat{\beta}_{OLS}\right) \right\}.$$

Note that

$$\begin{aligned}
(Y-Xb)'(Y-Xb) &= Y'Y-2b'X'Y+b'X'Xb \\
&= Y'Y-2b'X'X\hat{\beta}_{OLS}+b'X'Xb \\
&= Y'Y-2b'\hat{\beta}_{OLS}+b'b
\end{aligned}$$

and, similarly,

$$\left(Y-X\hat{\beta}_{OLS}\right)'\left(Y-X\hat{\beta}_{OLS}\right)=Y'Y-\hat{\beta}_{OLS}'\hat{\beta}_{OLS}.$$

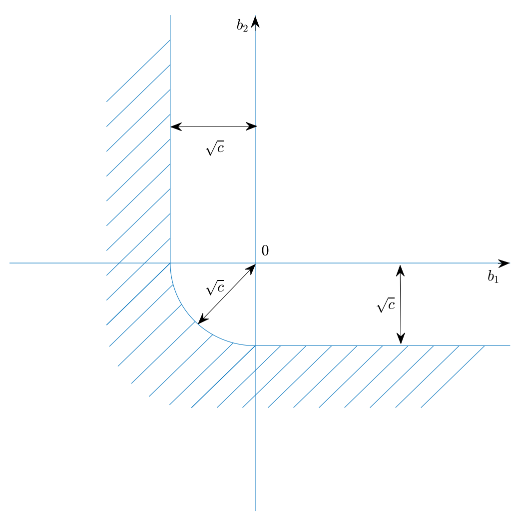
Thus, the LR simplifies

$$LR=\min_{b_1,b_2\geq 0}\left\{\left(\hat{\beta}_{OLS}-b\right)'\left(\hat{\beta}_{OLS}-b\right)\right\}$$

so that

$$LR=\begin{cases} 0 & \text{if } \hat{\beta}_{1,OLS}\geq 0 \text{ and } \hat{\beta}_{2,OLS}\geq 0 \\ \hat{\beta}_{1,OLS}^2 & \text{if } \hat{\beta}_{1,OLS}< 0 \text{ and } \hat{\beta}_{2,OLS}\geq 0 \\ \hat{\beta}_{2,OLS}^2 & \text{if } \hat{\beta}_{1,OLS}\geq 0 \text{ and } \hat{\beta}_{2,OLS}< 0 \\ \hat{\beta}_{1,OLS}^2+\hat{\beta}_{2,OLS}^2 & \text{if } \hat{\beta}_{1,OLS}< 0 \text{ and } \hat{\beta}_{2,OLS}< 0 \end{cases} \tag{3}$$

In words, the LR statistic equals the squared distance from $\hat{\beta}_{OLS}$ to the positive quadrant in $\mathbb{R}^2$. Here is the picture for a critical region of the LR test.



If $\hat{\beta}_{OLS}$ ends up in the striped region (call it Ω), LR test rejects. For the test with 5% significance level, we need to choose the critical value c so that

$$\max_{\beta_1 \geq 0, \beta_2 \geq 0} \Pr(LR > c) = 0.05.$$

Recall that in the special case that we consider

$$\hat{\beta}_{OLS}|X \sim N\left(\beta, \sigma^2 (X'X)^{-1}\right) \sim N(\beta, I_2).$$

Therefore,

$$\begin{aligned} \Pr(LR > c) &= \int_{\Omega} \frac{1}{2\pi} \exp\left\{-\frac{(z-\beta)'(z-\beta)}{2}\right\} dz \\ &= \int_{\Omega-\beta} \frac{1}{2\pi} \exp\left\{-\frac{x'x}{2}\right\} dx, \end{aligned}$$

where $\Omega - \beta$ is the region obtained from Ω by a linear shift. Note that, for any β from the positive quadrant (consistent with H_0), $\Omega - \beta \subseteq \Omega$ with equality achieved only at $\beta = 0$. Therefore,

$$\max_{\beta_1 \geq 0, \beta_2 \geq 0} \Pr(LR > c) = \Pr(LR > c)|_{\beta_1 = 0, \beta_2 = 0.}$$

On the other hand, when $\beta_1 = 0, \beta_2 = 0$, we have $\hat{\beta}^2_{1,OLS} \sim \chi^2(1)$, $\hat{\beta}^2_{2,OLS} \sim \chi^2(1)$, and $\hat{\beta}^2_{1,OLS} + \hat{\beta}^2_{2,OLS} \sim \chi^2(2)$. So from (3),

$$LR = \begin{cases} 0 & \text{with probability } 1/4 \\ \chi^2(1) & \text{with probability } 1/2 \\ \chi^2(2) & \text{with probability } 1/4 \end{cases}$$

So c must be the 0.95 quantile of the mixture of chi-squared distributions. This can be found numerically.

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