# Tensor Density Modules of Contact Vector Fields


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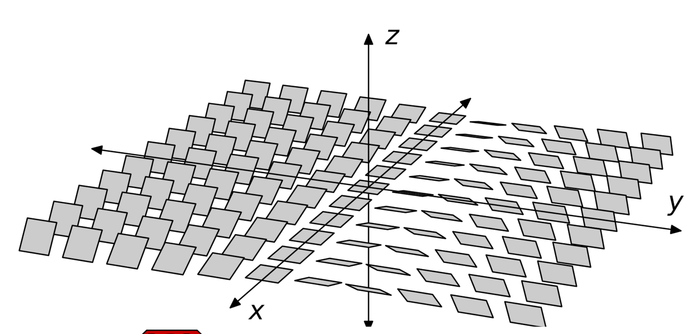
## Geometric underpinnings

We start by reviewing some fundamental ideas in contact geometry. Here, $M$ is an orientable real manifold of odd dimension.

- A 1-form $\alpha$ is a **contact form** if for each $p \in M$, the restriction of $d\alpha_p$ to the subspace Ker $\alpha_p \subset T_pM$ is non-degenerate.
- The **contact structure** associated to $\alpha$ is a completely non-integrable distribution of codimension 1 given by Ker $\alpha$.
- A **contact vector field** $X$ preserves the contact distribution. That is, if $\mathcal{L}_X(\alpha)$ is the Lie derivative of $\alpha$ with respect to $X$, then each contact vector field has the property that $\mathcal{L}_X(\alpha) = f_X\alpha$ for some smooth function $f_X$.
- There is a one-to-one correspondence between contact vector fields on $M$ and smooth functions on $M$. It is the map $X \mapsto \alpha(X)$.
- Given a vector field $X$, we call $\alpha(X)$ the **contact Hamiltonian of** $X$.

## A quick example

The **standard contact structure** on $\mathbb{R}^3$ is defined via the form $\alpha = dz - ydx$. It is easy to check that $d\alpha = dx \wedge dy$. The corresponding hyperplane distribution is shown to the right. It is given by Ker $\alpha$ = span$\{\partial_y, \partial_x + y\partial_z\}$ and is completely non-integrable, meaning that $\alpha \wedge d\alpha$ is a *volume form*. Indeed, $\alpha \wedge d\alpha = dx \wedge dy \wedge dz \neq 0$ on $\mathbb{R}^3$.




## $\mathcal{K}$ is really something super

Now we move to the main event. Recall that a **superspace** $V = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space. For $v \in V$, we define the **parity** to be $|v| = 0$ if $v \in V_0$ and $|v| = 1$ if $v \in V_1$.

- Consider the **supermanifold** $\mathbb{R}^{1|1}$ with even coordinate $x$ and odd coordinate $\xi$ where $\xi^2 = 0$.
- The **Lie superalgebra of polynomial vector fields on** $\mathbb{R}^{1|1}$ is Vec$(\mathbb{R}^{1|1}) :=$ Span$_{\mathbb{C}[x,\xi]}\{\partial_x, \partial_\xi\}$. It has bracket $[X, Y] = XY - (-1)^{|X||Y|}YX$.
- The **contact form** we are interested in is $\omega = dx + \xi d\xi$. We define vector fields $D := \partial_\xi + \xi\partial_x$ and $\overline{D} := \partial_\xi - \xi\partial_x$. Observe that $\omega(\overline{D}) = 0$. Here, our contact distribution is Ker $\omega = \mathbb{C}[x,\xi]\overline{D}$.
- We consider $\mathcal{K}$, the collection of **contact vector fields on** $\mathbb{R}^{1|1}$. They inherit the Lie superalgebra structure from Vec$(\mathbb{R}^{1|1})$. For $f \in \mathbb{C}[x]$, we make the assignment

$$\mathbb{X} : f \mapsto f\partial_x + \frac{1}{2}f'\xi\partial_\xi \quad \text{and} \quad \mathbb{X} : \xi f \mapsto fD$$

  so that $\mathbb{X}$ takes the contact hamiltonian to its vector field.

This is already a lot of moving parts! Luckily, $\mathcal{K}$ has a convenient basis with concise brackets. For $n \in \mathbb{N}$ we define the following elements of $\mathcal{K}$:

$$e_{n-1} := \mathbb{X}(x^n), \qquad e_{n-1/2} := 2\mathbb{X}(\xi x^n)$$

with brackets

$$\begin{aligned} [e_n, e_m] &= (m-n)e_{n+m} && \text{if } n, m \in \mathbb{N} - 1 \\ [e_n, e_m] &= (m-n/2)e_{n+m} && \text{if } n \in \mathbb{N} - 1, m \in \mathbb{N} - 1/2 \\ [e_n, e_m] &= 2e_{n+m} && \text{if } n, m \in \mathbb{N} - 1/2 \end{aligned}$$

## Weights weights weights!

Here, we state definitions of weights and weight spaces within the context of $\mathcal{K}$. Let $(\pi, V)$ be any representation of $\mathcal{K}$.

- A **weight vector** is an eigenvector of $\pi(e_0)$. For a fixed weight $\lambda$, the $\lambda$-**weightspace** is the $\lambda$-eigenspace of $\pi(e_0)$.

- Under the **adjoint action**, the weight of each basis vector $e_n$ is its index, $n$.
- A **lowest weight vector** or **LWV** is a vector annihilated by $\pi(e_{-1/2})$.
- For certain well-behaved representations $\pi$ of $\mathcal{K}$, the transformation $\pi(e_{-1/2})$ **is a surjective map** from the $\lambda$ weightspace to the $\lambda - \frac{1}{2}$ weightspace.

## Neither tense nor dense

Finally we arrive at the so-called **tensor density modules** or **TDMs**. $\mathcal{K}$ has a natural action on $\mathbb{C}[x,\xi]$: applying the vector field $X$ to the polynomial $f + g\xi$. The vector space $\mathbb{C}[x,\xi]$ with this action under $\mathcal{K}$ is referred to as the tensor density module of degree zero, and is denoted by $(\pi_0, \mathbb{F}_0)$. There is a 1-parameter family of deformations of $\mathbb{C}[x,\xi]$:

Let $\lambda$ be a complex number and set $\mathbb{F}_\lambda := \mathbb{C}[x,\xi]\omega^\lambda$. Define a corresponding family of actions $\pi_\lambda$ via the assignment

$$\pi_\lambda(e_{n-1}) = x^n\partial_x + nx^{n-1}(\lambda + \frac{1}{2}\xi\partial_\xi), \qquad \pi_\lambda(e_{n-1/2}) = x^nD + 2n\lambda\xi x^{n-1}.$$

The **tensor density module** of degree $\lambda$ is the representation $(\pi_\lambda, \mathbb{F}_\lambda)$. For $\lambda \neq 0$, it is irreducible as a $\mathcal{K}$-module and $(\mathcal{K}, \text{ad})$ is $\mathcal{K}$-equivalent to $\mathbb{F}_{-1}$.

Recall the definitions of the **universal enveloping algebra** and **symmetric algebra**. We have analagous definitions that encode the supersymmetry of our algebra:

$$\mathfrak{U} := {}^{\otimes\mathcal{K}}\big/_{\langle v \otimes w - (-1)^{|v||w|}w \otimes v - [v,w]\rangle} \qquad \mathcal{S}\mathcal{K} := {}^{\otimes\mathcal{K}}\big/_{\langle v \otimes w - (-1)^{|v||w|}w \otimes v\rangle}$$

Henceforth, we write $vw$ for $v \otimes w$, dropping the tensor symbol. Given a representation $(\pi, V)$ of $\mathcal{K}$, we lift $\pi$ to $\mathfrak{U}$ in the natural way.

**Thesis topic**: Can we describe Ann$_{\mathfrak{U}}(\pi_\lambda)$? What are its generating vectors?

## Casimir and his ghost

$\mathcal{K}$ contains a maximal subalgebra

$$\mathfrak{s} := \text{span}_{\mathbb{C}}\{e_{-1}, e_{-1/2}, e_0, e_{1/2}, e_1\}$$

which is isomorphic to the orthosymplectic Lie superalgebra $\mathfrak{osp}_{1|2}$. Its even part is $\mathfrak{sl}_2$.

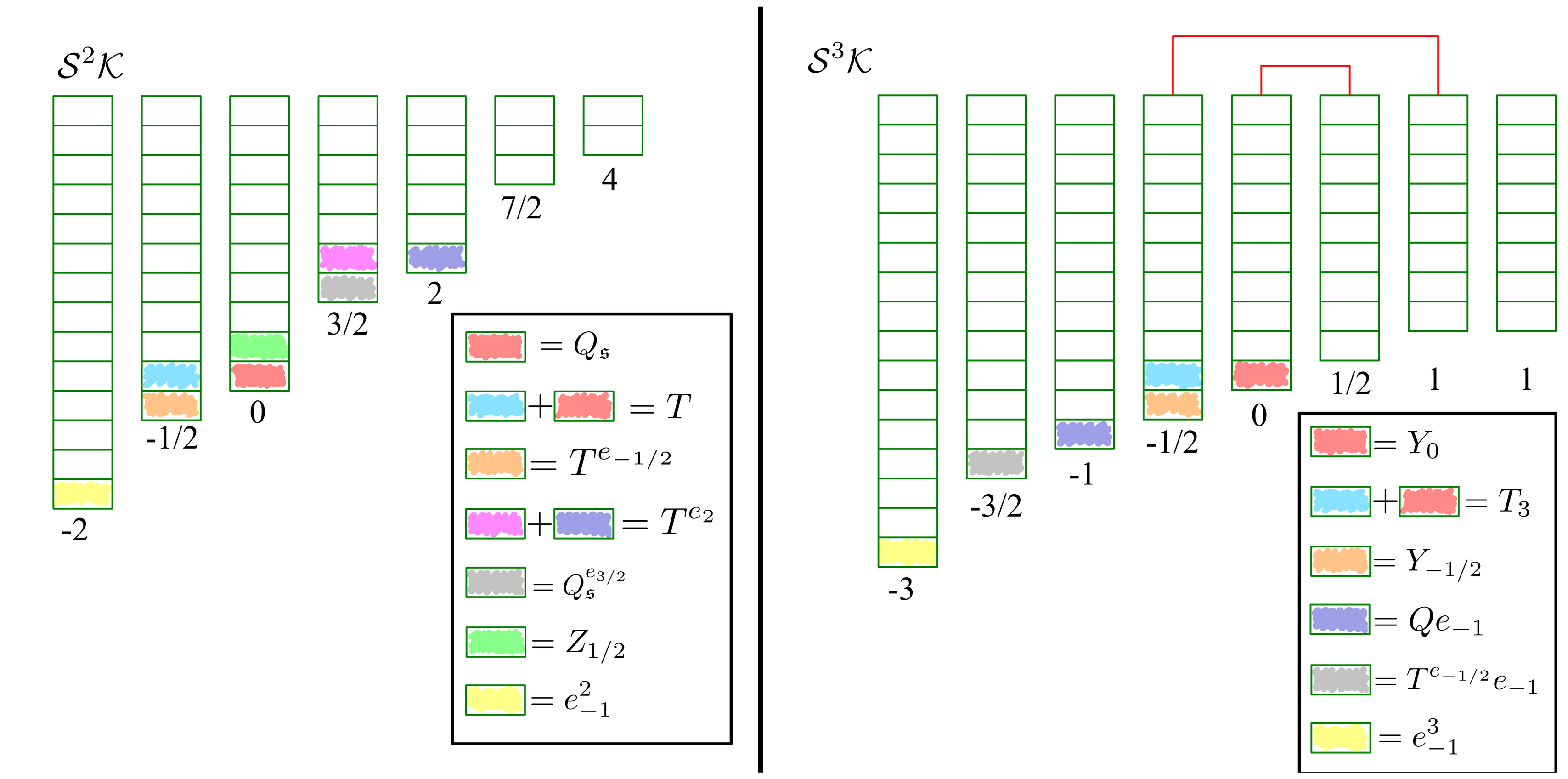
- The **Casimir operator** $Q_\mathfrak{s}$ is a quadratic element of $\mathfrak{U}(\mathfrak{s})$.
- **Casimir's ghost** is the element $T = e_0 - e_{1/2}e_{-1/2} - 1/4$. It commutes with $\mathfrak{s}_0$ and anti-commutes with $\mathfrak{s}_1$.
- $T$ satisfies $T^2 = Q_\mathfrak{s} + \frac{1}{16}$. Ergo it is a quadratic element that **squares to a quadratic element**.
- The center of $\mathfrak{U}(\mathfrak{s})$ is $\mathbb{C}[Q_\mathfrak{s}]$ and the ghost center of $\mathfrak{U}(\mathfrak{s})$ is $\mathbb{C}[T]$.
- We have $\pi_\lambda(Q_\mathfrak{s}) = \lambda^2 - \frac{1}{2}\lambda$ and $\pi_\lambda(T) = (\lambda - \frac{1}{4})(1 - 2\xi\partial_\xi)$.
- Put $q_\lambda = \lambda^2 - \frac{1}{2}\lambda$. Then $Q_\mathfrak{s} - q_\lambda \in$ Ann$_{\mathfrak{U}}(\pi_\lambda)$. Note that $T$ only annihilates $\mathbb{F}_\lambda$ for $\lambda = 1/4$.

## Super powers? No, supersymmetric powers...

In this section, we consider $\mathcal{S}^2\mathcal{K}$ and $\mathcal{S}^3\mathcal{K}$, the supersymmetric square and supersymmetric cube of $\mathcal{K}$, respectively. By counting dimensions of weight spaces and decomposing along Casimir eigenspaces, we can find that under the adjoint action

$$\mathcal{S}^2\mathcal{K} \overset{\mathfrak{s}}{\cong} \mathbb{F}_{-2} \oplus \mathbb{F}_{-1/2} \oplus \mathbb{F}_0 \oplus \mathbb{F}_{3/2} \oplus \mathbb{F}_2 \oplus \cdots \quad \text{and} \quad \mathcal{S}^3\mathcal{K} \overset{\mathfrak{t}}{\cong} \bigoplus_{i,j \in \mathbb{N},\, b \in \{0, \frac{3}{2}, \frac{5}{2}, 4\}} m_j\mathbb{F}_{b+2j+3(i-1)}$$

where $m_j$ is the multiplicity and $\mathfrak{t}$ = span$_{\mathbb{C}}\{e_{-1}, e_{-1/2}, e_0\}$. We have a visual representation of these ideas:




Each column is a copy of a TDM, and each ▭ represents a weight vector. A ▭ at the bottom of a column is a LWV. The numbers along the bottom correspond to the weight of this LWV. The red lines along the top indicate that the linked TDMs are **indecomposable** as $\mathfrak{s}$-modules.

- $Y_{-1/2}$ and $Y_0$ are the unique cubic LWVs of weight -1/2 and 0, resp. They act by $q_\lambda\overline{D}$ and $y_\lambda := (\lambda - 1/4)q_\lambda$, resp.
- $T_3$ is a cubic vector bridging the $\mathbb{F}_{-1/2}$ and $\mathbb{F}_0$ in $\mathcal{S}^3\mathcal{K}$. It commutes with $\mathfrak{t}_0$ and anti-commutes with $\mathfrak{t}_1$. It has the same action as $T$, up to scalar. Its square is quartic, which is funny.

  **Conjecture:** Ann$_{\mathfrak{U}}(\pi_\lambda) = \langle Q_\mathfrak{s} - q_\lambda, Y_0 - y_\lambda, T_3 - q_\lambda T\rangle_{\mathfrak{U}}$ for $\lambda \neq 0, \frac{1}{4}, \frac{1}{2}$.

- $T^{e_{-1/2}}$ is the unique quadratic LWV of weight $-1/2$. It acts by $(\lambda - 1/4)\overline{D}$.

  **Conjecture**: Ann$_{\mathfrak{U}}(\pi_{1/4}) = \langle T\rangle_{\mathfrak{U}}$

- $Z_{1/2}$ has the property that $Z_{1/2}^{e_{-1/2}} = Q_\mathfrak{s}$.

  **Conjecture**: Ann$_{\mathfrak{U}}(\pi_0)$ = Ann$_{\mathfrak{U}}(\pi_{1/2}) = \langle Z_{1/2}\rangle_{\mathfrak{U}}$.

- $T^{e_2} = Q^{e_2} +$ (the LWV of weight 2). It generates the LWVs of positive weights in $\mathcal{S}^2\mathcal{K}$.

  **Conjecture:** $\bigcap_{\lambda \in \mathbb{C}}$ Ann$_{\mathfrak{U}}(\pi_\lambda) = \langle T^{e_2}\rangle_{\mathfrak{U}}$.

## References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[3] M. Gorelik. On the ghost center of Lie superalgebras, *Ann. Inst. Fourier (Grenoble)* 50 (2000), no. 6, 1745-1764.