1 Background

A *representation* of an algebraic structure A is, generally speaking, a vector space V with an action of A on V that respects the structure of A. For instance, there is a natural group representation of the symmetric group on 3 letters S_3 on $\mathbb{C}^3$: we view $\sigma \in S_3$ as a linear function on $\mathbb{C}^3$ that permutes basis vectors. This assignment of permutations to functions on $\mathbb{C}^3$ is defined in a way such that the group operation in S_3 is compatible with the action on $\mathbb{C}^3$. Formally, one defines this representation as a group homomorphism $\phi : S_3 \to GL_3(\mathbb{C})$. Thus, the function associated to σ is $\phi(\sigma)$.

Next, we translate this idea to a different context. Our algebraic structure will be a Lie algebra of vector fields $\mathfrak{s} := \operatorname{span}_{\mathbb{C}}\{\partial_x,\ x\partial_x,\ x^2\partial_x\}$, and the vector space receiving the action will be the polynomial ring $\mathbb{C}[x]$. We remark in passing that $\mathfrak{s}$ is isomorphic as a Lie algebra to $\mathfrak{sl}_2(\mathbb{C})$, whose representations have been well-studied. Now, $\mathfrak{s}$ comes equipped with a bilinear product called a bracket. For polynomials p and q of degree ≤ 2, we have $[p\partial_x, q\partial_x] = (pq' - p'q)\partial_x$, but the bracket's explicit formula is largely unimportant for the purposes of this statement. More relevant is the natural action of $\mathfrak{s}$ on $\mathbb{C}[x]$: if f has degree ≤ 2 and $g \in \mathbb{C}[x]$, we write

$$f\partial_x(g) := fg'.$$

Similar to the previous paragraph, the bracket in $\mathfrak{s}$ is compatible with the action on $\mathbb{C}[x]$. And again, we have analogous formalism in the sense that this representation of $\mathfrak{s}$ is a homomorphism of Lie algebras $\pi_0 : \mathfrak{s} \to \operatorname{End}(\mathbb{C}[x])$.

Note the subscript of the homomorphism π_0. We use this notation since this representation is in fact a member of a 1-parameter family of representations. For each $\lambda \in \mathbb{C}$, we may define $\pi_\lambda : \mathfrak{s} \to \operatorname{End}(\mathbb{C}[x])$ by the rule

$$\pi_\lambda(f\partial_x)(g) := fg' + \lambda f'g. \tag{1}$$

Naturally, one would like to consider compositions of such maps (e.g. $\pi_\lambda(f_1\partial_x) \circ \cdots \circ \pi_\lambda(f_r\partial_x)$ for some positive integer r) and their effects on $\mathbb{C}[x]$. Thus, we extend π_λ to an associative algebra homomorphism defined on the *universal enveloping algebra* $\mathfrak{U}(\mathfrak{s})$ of $\mathfrak{s}$, simply writing π_λ for this extension. The universal enveloping algebra is the quotient of the tensor algebra $\otimes\mathfrak{s}$ by the two-sided ideal generated by all $X \otimes Y - Y \otimes X - [X, Y]$ where $X, Y \in \mathfrak{s}$. Taking a quotient of $\otimes\mathfrak{s}$ by this ideal encodes into the tensor algebra the Lie structure of the underlying Lie algebra. The unfamiliar reader may make the conceptual correspondence "$\otimes = \circ$" to represent the assignment $\pi_\lambda(f_1\partial_x \otimes \cdots \otimes f_r\partial_x) = \pi_\lambda(f_1\partial_x) \circ \cdots \circ \pi_\lambda(f_r\partial_x)$.

A question that arises after making this extension is the following: what is the kernel of π_λ? This leads us to the following definition:

Definition 1. Let (π, V) be a representation of a Lie algebra $\mathfrak{g}$. The *annihilator* of V is

$$\operatorname{Ann}_{\mathfrak{U}(\mathfrak{g})}(V) := \{\Omega \in \mathfrak{U}(\mathfrak{g}) : \pi(\Omega)v = 0 \text{ for all } v \in V\}.$$

It is not hard to show that $\mathrm{Ann}_{\mathfrak{U}(\mathfrak{g})}(V)$ is a two-sided ideal in $\mathfrak{U}(\mathfrak{g})$. Consequently, we arrive at one of our major themes.

Problem 1. Given an arbitrary representation (π, V) of $\mathfrak{g}$, describe $\mathrm{Ann}_{\mathfrak{U}(\mathfrak{g})}(V)$.

The following example will show us that the annihilator of $\mathbb{C}[x]$ under the action of π_λ is non-trivial.

Example 1. Put $Q := \big(x\partial_x\big)^2 + x\partial_x - \partial_x\, x^2\partial_x$. The familiar reader may recognize Q as a scalar multiple of the *Casimir operator* of $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{C})$. One checks by direct computation that $\pi_\lambda(Q)(g) = (\lambda^2 - \lambda)g$ for every $g \in \mathbb{C}[x]$. Thus, the differential operator $\pi_\lambda(Q - \lambda^2 + \lambda)$ kills all $g \in \mathbb{C}[x]$.

It happens that the annihilator is generated by $Q - \lambda^2 + \lambda$, which is a consequence of a result of Duflo: if $\mathfrak{g}$ is a finite-dimensional complex semi-simple Lie algebra, then the annihilators of these modules are generated by their intersection with the center of the universal enveloping algebra (see [1]). It is a fact that the center of $\mathfrak{U}(\mathfrak{s})$ is $\mathbb{C}[Q]$.

Problem 2. Suppose that $\mathfrak{g}$ is an infinite-dimensional complex semi-simple Lie algebra whose universal enveloping algebra has trivial center. Is there an appropriate analogue of Duflo's theorem?

As stated above, $\mathfrak{U}(\mathfrak{s})$ has non-trivial center. However, $\mathfrak{s}$ is a maximal subalgebra of the infinite-dimensional Lie algebra of vector fields on the line, which we now define:

Definition 2. Set $\mathrm{Vec}(\mathbb{R}) := \mathrm{span}_{\mathbb{C}}\{\partial_x,\ x\partial_x,\ x^2\partial_x,\ x^3\partial_x,\ \ldots\}$. The vector space $\mathbb{C}[x]$ together with the action π_λ determined by

$$\pi_\lambda(f\partial_x)g = fg' + \lambda f'g$$

is called the *tensor density module* of degree λ and denoted $\mathcal{F}_\lambda$.

Note that $\mathrm{Vec}(\mathbb{R})$ has the same bracket formula as $\mathfrak{s}$, which is still unimportant for our purposes. It is also a simple Lie algebra, and its universal enveloping algebra has trivial center. The π_λ are similarly extended to $\mathfrak{U}\big(\mathrm{Vec}(\mathbb{R})\big)$. In 2007, Conley and Martin [5] made a preliminary step toward Problem 2 by computing the annihilating ideal of $\mathcal{F}_\lambda$ in $\mathfrak{U}\big(\mathrm{Vec}(\mathbb{R})\big)$ for each λ.

Theorem 1 (Conley and Martin)**.** *Generically, the annihilating ideal of $\mathcal{F}_\lambda$ is generated by the Duflo generator $Q - \lambda^2 + \lambda$ and one additional vector: a cubic element not contained in $\mathfrak{U}(\mathfrak{s})$. In the cases of $\lambda = 0$ and 1, (the roots of the Casimir eigenvalue) they are principal ideals generated by a quadratic element which is not contained in $\mathfrak{U}(\mathfrak{s})$.*

2 Current Work

My thesis is a variation on the problem tackled by Conley and Martin. Instead of the Lie algebra $\mathrm{Vec}(\mathbb{R})$, I consider tensor density modules of the infinite-dimensional Lie *superalgebra* of contact vector fields on the supersymmetric line. These objects are discussed at length in [3], but we will give a brief introduction here.

Here, ξ is a symbol that satisfies $\xi^2 = 0$, and $\mathbb{R}^{1|1}$ is the supersymmetric line with even coordinate x and odd coordinate ξ. The space of polynomial vector fields on the supersymmetric line is $\mathrm{Vec}(\mathbb{R}^{1|1}) := \mathrm{span}_{\mathbb{C}[x,\xi]}\{\partial_x, \partial_\xi\}$. Let us write $\mathcal{W}_s$ for $\mathrm{Vec}(\mathbb{R}^{1|1})$. Again, it is a Lie superalgebra: an algebra with a $\mathbb{Z}_2$-grading written as $\mathcal{W}_s = \mathcal{W}_0 \oplus \mathcal{W}_1$. The *parity* of a vector field X is 0 if $X \in \mathcal{W}_0$ and 1 if $X \in \mathcal{W}_1$. This quantity is denoted by $|X|$ and the map sending X to $(-1)^{|X|}X$ is called the *parity endomorphism.* The super-bracket of $\mathcal{W}_s$ is defined as

$$[X, Y] = XY - (-1)^{|X||Y|}YX.$$

There is a *contact form* $\omega := dx + \xi d\xi$ which induces a contact structure on $\mathbb{R}^{1|1}$. Now, $\mathcal{W}_s$ acts on the kernel of ω via the adjoint action, and we denote the stabilizer of this kernel as $\mathscr{K}$. It may also be defined as the image of the even linear injection

$$\mathbb{X} : \mathbb{C}[x,\xi] \to \mathrm{Vec}(\mathbb{R}^{1|1}) \qquad \mathbb{X}(f) := f\partial_x + \tfrac{1}{2}f'\xi\partial_\xi, \qquad \mathbb{X}(f\xi) := \tfrac{1}{2}f(\partial_\xi + \xi\partial_x)$$

where $f \in \mathbb{C}[x]$. For a vector field $X \in \mathscr{K}$, the polynomial $\mathbb{X}^{-1}(X)$ is called the *contact Hamiltonian* of X. Finally, we may introduce the tensor density modules of degree λ for $\mathscr{K}$.

Definition 3. Let $F, G \in \mathbb{C}[x,\xi]$ and $\lambda \in \mathbb{C}$. Write F' for $\partial_x(F)$. The *tensor density module* of $\mathscr{K}$ of degree λ is the polynomial ring $\mathbb{C}[x,\xi]$ together with the action

$$\pi_\lambda\big(\mathbb{X}(F)\big)(G) := \mathbb{X}(F)G + \lambda F'G.$$

It is denoted as $\mathbb{F}_\lambda$.

The π_λ are lifted to the universal enveloping algebra $\mathfrak{U}(\mathscr{K})$ just as before. Of note is the finite-dimensional maximal conformal subalgebra of $\mathscr{K}$ isomorphic to $\mathfrak{osp}(1|2)$, which has its own Casimir operator $Q_{\mathfrak{osp}}$. Remarkably, $Q_{\mathfrak{osp}}$ has a square root $T_{\mathfrak{osp}}$ in the universal enveloping algebra, and this square root is called its "ghost". One may find further reading about this in [4]. The ghost satisfies $T_{\mathfrak{osp}}^2 + \frac{1}{16} = Q_{\mathfrak{osp}}$. The image of $Q_{\mathfrak{osp}}$ under π_λ is $\lambda^2 - \frac{1}{2}\lambda$. My thesis describes the annihilating ideal of $\mathbb{F}_\lambda$ in $\mathfrak{U}(\mathscr{K})$ for each λ. The main result is as follows:

Theorem 2. *Generically, the annihilating ideals of $\mathbb{F}_\lambda$ are generated by the super-analog of the Duflo generator $Q_{\mathfrak{osp}} - \lambda^2 + \frac{1}{2}\lambda$ and two additional vectors: both cubic elements not contained in $\mathfrak{U}(\mathfrak{osp})$. One acts by a scalar and the other acts by a scalar multiple of the parity endomorphism. In the cases of $\lambda = 0$ and $\frac{1}{2}$, (the roots of the Casimir eigenvalue) they are principal ideals generated by a quadratic element which is not contained in $\mathfrak{U}(\mathfrak{osp})$. In the case of $\lambda = \frac{1}{4}$, the annihilating ideal is generated by the ghost.*

Theorem 2 tracks with the answer we expected to see based on the results of [5] for $\lambda \neq \frac{1}{4}$. The only unexpected phenomenon was in the so-called *self-dual* case of $\lambda = \frac{1}{4}$, where the ghost played a significant role.

The tactics used to achieve the results of my thesis are blunt instruments: they require lots of brute force computation and counting up dimensions of weightspaces. However, there is a more systematic approach, and the overall strategy is generalizable. At the moment, the goal is to finish developing this systematic strategy for Lie algebras, submit that as a separate result, and then write up the results of my thesis using that final product, simultaneously extending the result to Lie superalgebras.

3 Future Work

There are several dials one can adjust on Problem 1. For instance, it would be very interesting to investigate these ideas for Lie algebras of higher rank. First, a discussion on $\mathrm{Vec}(\mathbb{R}^2)$ — it would be difficult to go directly to $\mathrm{Vec}(\mathbb{R}^n)$ and answer Problem 1 for their *tensor field modules*. For more information on these objects, see [2]. Similarly, one may increase the number of odd variables. In fact, for a non-negative integer m, one may induce a contact structure on $\mathbb{R}^{2n+1|m}$. Again, it would be difficult to go directly to $\mathrm{Vec}(\mathbb{R}^{2n+1|m})$. The jumping-off point would be an examination of the contact vector fields of $\mathrm{Vec}(\mathbb{R}^{1|2})$.

There is also another angle which interests me. The tensor density modules are irreducible for $\lambda \neq 0$. Thus, their annihilating ideals are *primitive ideals*. For an algebra A, the collection of primitive ideals of A is denoted $\mathrm{Prim}(A)$. One may endow this space with the *Jacobson topology.*

Definition 4. Let A be an algebra and S be a non-empty subset of $\mathrm{Prim}(A)$. We define the *closure* of S to be the set

$$\overline{S} := \left\{J \in \mathrm{Prim}(A) : J \supseteq \cap_{I \in S} I\right\}.$$

We define $\overline{\varnothing} := \varnothing$. The Jacobson topology is the topology with closed sets as given by this closure operator.

Problem 3. Describe $\mathrm{Prim}\big(\mathfrak{U}(\mathfrak{g})\big)$ as a set and as a topological space.

The statement of this problem is taken directly from [6], the article that initally sparked my interest in the topological properties. Now since the annihilators of the tensor density modules are primitive ideals, they inherit the subspace topology from $\mathrm{Prim}\big(\mathfrak{U}(\mathfrak{g})\big)$. An additional result included in my thesis but not discussed at length here is the description of the set of annihilators topologically: it is homeomorphic to $\mathbb{C}^\times$ with the co-finite topology via the map $\mathrm{Ann}(\mathbb{F}_\lambda) \mapsto \lambda$ for $\lambda \neq 0$. There is some computational evidence to suggest that in the case of $\mathrm{Vec}(\mathbb{R}^n)$, one finds that the space of annihilators is homeomorphic to some subset of affine n-space with the Zariski topology.

Returning to those primitive ideals described in 2007 by Conley and Martin, it's not currently known if the annihilators of the $\mathcal{F}_\lambda$ constitute all of $\mathrm{Prim}\big(\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))\big)$. If a representation is well-behaved in certain capacities, it is referred to as *admissible*. The $\mathcal{F}_\lambda$ essentially comprise the collection of irreducible admissibe representations of $\mathrm{Vec}(\mathbb{R})$ [7], but there are irreducible representations of $\mathrm{Vec}(\mathbb{R})$ that are less well-behaved: in particular, there are irreducible representations with infinite-dimensional weightspaces [8] and irreducible representations without weights at all [9].

The most immediate new goal is to analyze representations from the less well-behaved class and compute their annihilators. Currently, the examples I have worked suggest that the annihilators of certain poorly-behaved representations are either zero or the same as those of the well-behaved representations. That said, more data on this is required: I intend to determine annihilators of representations with varying characteristics to get an idea of what might constitute a workable proof of that claim or how one might construct a counter example.

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