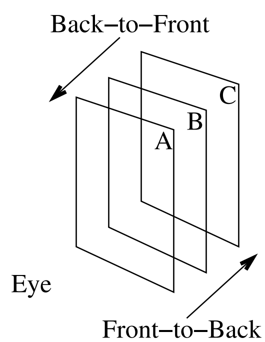
Handout: The Over Operation in Compositing and Its Associativity

CS4533 Interactive Computer Graphics
By Prof. Yi-Jen Chiang
CSE Dept., Tandon School of Engineering, New York University

Note: There are 3 pages.

0. Overview: The Over Operation, Back-to-Front/Front-to-Back Rendering, and Associativity

Suppose there are 3 overlapping fragments/polygons A, B, C **from front to back** (i.e., A is the closest to the eye, then B, then C; see the figure below) with tuples (color, opacity) as follows:



$A : (C_A, \alpha_A)$
$B : (C_B, \alpha_B)$
$C : (C_C, \alpha_C)$.

The **over** operation is used in compositing/α-blending to simulate the physical phenomena of transparency. Originally, the over operation should be performed in the **back-to-front (farthest-to-closest)** rendering, i.e., rendering C first, then B (in front of C) with operation (B over C), and finally A (in front of B), with operation (A over (the result of (B over C))). The final effect is (A over (B over C)). Note that if the over operation is **associative**, i.e., if

$$(A \text{ over } (B \text{ over } C)) = ((A \text{ over } B) \text{ over } C),$$

then we can perform **front-to-back (closest-to-farthest)** rendering, i.e., first rendering A, then B (behind A) with operation (A over B), and finally C (behind B) with operation ((the result of (A over B)) over C); the final effect is ((A over B) over C).

The **front-to-back** rendering is more desirable, since in the process the resulting opacity increases; if in the middle of the rendering the opacity already reaches 1 (completely opaque), then anything behind the current fragment cannot be seen and thus we can stop the rendering. Such **early termination** can save the computing cost. (In the back-to-front rendering, however, we always need to complete the whole process since the closest fragment is always visible.)

In the following, we first in Sec. 1 give the original formula for the over operation (Eq. (1)), which only works for **back-to-front** rendering. We then in Sec. 2 show the back-to-front rendering result, (A over (B over C)), using the original formula. In Sec. 3 we present a modified formula for the over operation (Eq. (8)), and show that it gives the **same** result of the back-to-front rendering as using the original formula (Eq. (1)). Finally, in Sec. 4 we present the **front-to-back** rendering result, $((A \text{ over } B) \text{ over } C)$, using the modified formula, and verify that it is the **same** as its **back-to-front** result. This proves that the **modified formula** for the over operation is **associative**, as desired.

1. The Original Formula for the Over Operation

The original formula for the **over** operation is as follows:

$$\begin{aligned} C_{new} &= \alpha_s C_s + (1-\alpha_s) C_d \\ \alpha_{new} &= \alpha_s + (1-\alpha_s)\alpha_d \end{aligned} \tag{1}$$

where (C_s, α_s), (C_d, α_d) and (C_{new}, α_{new}) are the (color, opacity) tuples of the source, destination, and the result. Note that this formula must be applied in the **back-to-front** order. Let us define C'_x to be $C'_x = \alpha_x C_x$ for any fragment/polygon x (i.e., x can be A, B or C).

2. Back-To-Front Rendering using the Original Formula

Assume that the background, denoted by $\square$, has $(C_\square, \alpha_\square) = (0, 0)$. First look at $(C \text{ over } \square)$:

$$\begin{aligned} C_{C\square} &= \alpha_C C_C + (1-\alpha_C)\cdot 0 = C'_C \\ \alpha_{C\square} &= \alpha_C + (1-\alpha_C)\cdot 0 = \alpha_C. \end{aligned}$$

Next, look at $(B \text{ over } C) = (B \text{ over } C\square)$:

$$C_{BC} = C_{B\_C\square} = \alpha_B C_B + (1-\alpha_B) C_{C\square} = C'_B + (1-\alpha_B) C'_C \tag{2}$$

$$\alpha_{BC} = \alpha_{B\_C\square} = \alpha_B + (1-\alpha_B)\alpha_{C\square} = \alpha_B + (1-\alpha_B)\alpha_C. \tag{3}$$

Now, look at $(A \text{ over } (B \text{ over } C))$:

$$C_{A\_BC} = \alpha_A C_A + (1-\alpha_A) C_{BC} = C'_A + (1-\alpha_A) C_{BC} \tag{4}$$

$$= C'_A + (1-\alpha_A)[C'_B + (1-\alpha_B) C'_C]$$

$$= C'_A + (1-\alpha_A) C'_B + (1-\alpha_A)(1-\alpha_B) C'_C \tag{5}$$

$$\alpha_{A\_BC} = \alpha_A + (1-\alpha_A)\alpha_{BC} \tag{6}$$

$$= \alpha_A + (1-\alpha_A)[\alpha_B + (1-\alpha_B)\alpha_C]$$

$$= \alpha_A + (1-\alpha_A)\alpha_B + (1-\alpha_A)(1-\alpha_B)\alpha_C. \tag{7}$$

3. Modified Formula for the Over Operation and its Back-To-Front Rendering

Note that the right-most parts of Eqs. (2) and (3) are of the same pattern. Similarly, the right-most parts of Eqs. (4) and (6) are of the same pattern, so are the right sides of Eqs. (5) and (7). Let us re-write Eq. (1) as follows:

$$\begin{aligned} C_{new} &= C_{current} + (1-\alpha_{current}) C_{back} \\ \alpha_{new} &= \alpha_{current} + (1-\alpha_{current})\alpha_{back}, \end{aligned} \tag{8}$$

where for C_x (x is "current" or "back"), if x is a **"single"** fragment then we **use** $C'_x = \alpha_x C_x$ **in its place**, and if x is a **"compound"** fragment then we **use** C_x **in its place**. See the examples below for more details.

Example (I) (B over C):
In Eq. (2), for C_{BC}, "current" is B and "back" is C; B and C are each a single fragment. So in the first part of Eq. (8), $C_{current}$ is C'_B and C_{back} is C'_C, and we get $C_{BC} = C'_B + (1-\alpha_B)C'_C$, which is exactly Eq. (2).
(Also, in Eq. (3), for α_{BC}, in the second part of Eq. (8), we have $\alpha_{current} = \alpha_B$ and $\alpha_{back} = \alpha_C$, and thus we get $\alpha_{BC} = \alpha_B + (1-\alpha_B)\alpha_C$, which is exactly Eq. (3).)
Therefore (B over C) gets the same result no matter whether we use the modified equation Eq. (8) or the original equation Eq. (1).

Example (II) (A over (B over C)):
In Eq. (4), for C_{A_BC}, "current" is A (*single* fragment) and "back" is BC (*compound* fragment). So in the first part of Eq. (8), $C_{current}$ is C'_A and C_{back} is C_{BC} (note: **not** $C'_{BC}(= \alpha_{BC}C_{BC})$), and we get $C_{A\_BC} = C'_A + (1-\alpha_A)C_{BC}$, which is exactly Eq. (4).
(Also, in Eq. (6), for α_{A_BC}, in the second part of Eq. (8), we have $\alpha_{current} = \alpha_A$ and $\alpha_{back} = \alpha_{BC}$, and thus we get $\alpha_{A\_BC} = \alpha_A + (1-\alpha_A)\alpha_{BC}$, which is exactly Eq. (6).)
Therefore (A over (B over C)) gets the same result no matter whether we use the modified equation Eq. (8) or the original equation Eq. (1). This also shows that using the modified equation Eq. (8), the final results of (A over (B over C)) are given in Eqs. (5) and (7).

4. Front-to-Back Rendering using the Modified Formula and the Associativity

First look at (A over B). Note that "current" is A and "back" is B, each a *single* fragment, so $C_{current}$ is C'_A and C_{back} is C'_B. Also, $\alpha_{current}$ is α_A and α_{back} is α_B. Thus we have

$$\begin{aligned} C_{AB} &= C'_A + (1-\alpha_A)C'_B \\ \alpha_{AB} &= \alpha_A + (1-\alpha_A)\alpha_B. \end{aligned}$$

Next, look at ((A over B) over C). Note that "current" is AB (*compound* fragment) and "back" is C (*single* fragment), so $C_{current}$ is C_{AB} and C_{back} is C'_C. Thus we have

$$\begin{aligned} C_{AB\_C} &= C_{AB} + (1-\alpha_{AB})C'_C \\ &= [C'_A + (1-\alpha_A)C'_B] + (1-[\alpha_A + (1-\alpha_A)\alpha_B])C'_C \\ &= C'_A + (1-\alpha_A)C'_B + (1-\alpha_A)(1-\alpha_B)C'_C \qquad (9) \\ \alpha_{AB\_C} &= \alpha_{AB} + (1-\alpha_{AB})\alpha_C \\ &= [\alpha_A + (1-\alpha_A)\alpha_B] + (1-[\alpha_A + (1-\alpha_A)\alpha_B])\alpha_C \\ &= \alpha_A + (1-\alpha_A)\alpha_B + (1-\alpha_A)(1-\alpha_B)\alpha_C. \qquad (10) \end{aligned}$$

We can see that the right sides of Eq. (9) and of Eq. (5) are the same, and also the right sides of Eq. (10) and of Eq. (7) are the same, namely, $C_{AB\_C} = C_{A\_BC}$ and $\alpha_{AB\_C} = \alpha_{A\_BC}$. This means that (($A$ over B) over C) = (A over (B over C)) under the **modified formula (Eq. (8))**, i.e., such over operation is **associative**, as desired.