

CS4533 Lecture 3

Slides/Notes

HW1 Discussion; 3D Transformations (Notes, Ch 2,3,4)

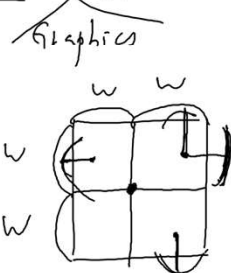
By Prof. Yi-Jen Chiang
CSE Dept., Tandon School of Engineering
New York University

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HW1 Part (d)

Input file circles (c_{x_i}, c_{y_i}, k_i) for each circle i .
in world coord. system.

Sol. Window width & height are each $2w$.



From input, find

$$\max_x = \max_i \{ c_{x_i} + k_i, |c_{x_i} - k_i| \}$$

$$\max_y = \max_i \{ c_{y_i} + k_i, |c_{y_i} - k_i| \}$$

$$\max_abs_coord = \max \{ \max_x, \max_y \}$$

If $\max_abs_coord > w$, then we need to scale down. (c_x, c_y, k)

for all circles.
 $c_x \leftarrow (c_x \cdot w) / \max_abs_coord$ $k \leftarrow (k \cdot w) / \max_abs_coord$
 $c_y \leftarrow (c_y \cdot w) / \max_abs_coord$

*

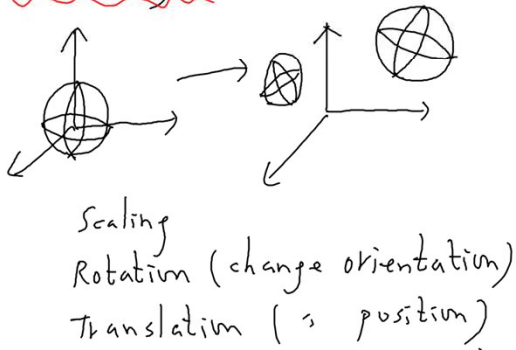
check for values
< 0 to see if
the scaling is
as desired.

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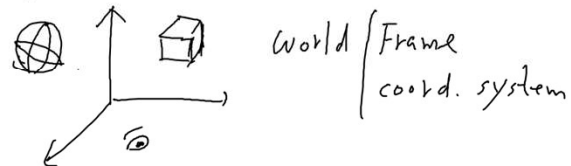
3D Transformations

Motivations: Why do we study transformations?

① Modeling Transformation



② Coordinate System Change



"What should be the final graphics image when seen from the viewer?"

World Frame $\xrightarrow{\text{transf.}}$ Eye Frame

3

I. Vector Space, Basis & Linear Transformations.

(1) Basis Vectors

Let V be a vector space. $\vec{v} \in V$

A set of vectors $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n$ is a basis for the vector space V if

① $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n$ are linearly indept. i.e. $a_1 \vec{b}_1 + a_2 \vec{b}_2 + \dots + a_n \vec{b}_n = \vec{0}$ for scalars a_1, \dots, a_n
 $\Leftrightarrow a_1 = a_2 = \dots = a_n = 0$.

($\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n$ span V)

② Any vector $\vec{v} \in V$ can be expressed as a linear combination of $\vec{b}_1, \dots, \vec{b}_n$

i.e. $\vec{v} = \sum_{i=1}^n c_i \vec{b}_i$ for some scalars c_1, c_2, \dots, c_n

[meaning: any \vec{b}_i can **NOT** be expressed as a linear combination of the other $n-1$ vectors $\vec{b}_j, j \neq i$]

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* $\vec{b}_1, \dots, \vec{b}_n$ are called the basis vectors.

* The number of basis vectors, n , is called the dimension of the vector space V . In graphics, $n=3$.

* Consider dimension $n=3$.

For any vector $\vec{v} \in V$, \exists scalars c_1, c_2, c_3 s.t. $\vec{v} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3$

$\Rightarrow [\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3]$: row of basis vectors

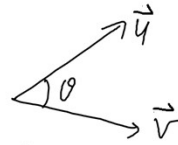
$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$: (column) coordinate vector

eg. $\vec{v} = (1, 2, 3)$
 $\begin{matrix} \nearrow & \downarrow \\ \text{row matrix} & \text{column matrix} \end{matrix}$
 $\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$
 basis: $\vec{b}_1 = \vec{e}_1 = (1, 0, 0)$ x
 $\vec{b}_2 = \vec{e}_2 = (0, 1, 0)$ y
 $\vec{b}_3 = \vec{e}_3 = (0, 0, 1)$ z

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(2) Inner Product & Cross Product.

A. Inner Product. \vec{u}, \vec{v} are vectors



inner product: $\vec{u} \cdot \vec{v}$

$\vec{u} \cdot \vec{v}$ returns a scalar value (real number)

* It allows to define the square length (or length) of a vector \vec{v} .

$|\vec{v}|^2 \stackrel{\text{def}}{=} \vec{v} \cdot \vec{v}$ (i.e. $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$)

* $\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos \theta$

$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$

Two vectors \vec{u}, \vec{v} are orthogonal ($\vec{u} \perp \vec{v}$) if $\theta = 90^\circ$

$\Leftrightarrow \cos \theta = 0 = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$

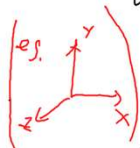
* A basis is orthonormal if

① all basis vectors are of length 1, and
 ② all \vec{v} \vec{w} are pairwise orthogonal



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* In orthonormal basis: $\vec{u} = \sum_i c_i \vec{b}_i$, $\vec{v} = \sum_j d_j \vec{b}_j$ for vectors \vec{u}, \vec{v}

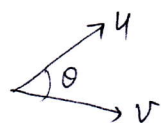
Usually we use such basis  where $|\vec{b}_i| = 1 \quad \forall i$; $\vec{b}_i \cdot \vec{b}_j = 0 \quad \forall i \neq j$.

$$\vec{u} \cdot \vec{v} = \left(\sum_i c_i \vec{b}_i \right) \cdot \left(\sum_j d_j \vec{b}_j \right) = \sum_i \sum_j (c_i d_j) (\vec{b}_i \cdot \vec{b}_j) = \sum_i c_i d_i \cdot 1$$

In 3D $\vec{u} = (c_1, c_2, c_3)$ $\vec{u} \cdot \vec{v} = c_1 d_1 + c_2 d_2 + c_3 d_3$
 $\vec{v} = (d_1, d_2, d_3)$

(2) Inner Product & Cross Product

A. Inner Product: u, v are vectors



$\vec{u} \cdot \vec{v}$ returns a scalar value (real number)

* It allows to define the squared length (or simply length) of a vector:

$$|\vec{v}|^2 := \vec{v} \cdot \vec{v} \quad (\text{i.e. } |\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}})$$

* $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$ $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$

Two vectors are orthogonal if $\theta = 90^\circ \Leftrightarrow \cos \theta = 0 = \vec{u} \cdot \vec{v}$

* A basis is orthonormal if all the basis vectors are unit length and pairwise orthogonal. Note: Usually we use orthonormal basis

* In orthonormal basis: $\vec{u} = \sum_i c_i \vec{b}_i$ $\vec{v} = \sum_i d_i \vec{b}_i$ where $\vec{b}_i \cdot \vec{b}_i = |\vec{b}_i|^2 = 1$
 $\vec{b}_i \cdot \vec{b}_j = 0$ when $i \neq j$

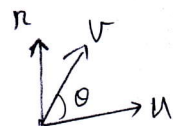
$$\begin{aligned} \vec{u} \cdot \vec{v} &= \left(\sum_i c_i \vec{b}_i \right) \cdot \left(\sum_j d_j \vec{b}_j \right) = \sum_i \sum_j (c_i d_j) (\vec{b}_i \cdot \vec{b}_j) \\ &= \sum_i c_i d_i \end{aligned}$$

In 3D: $\vec{u} = (c_1, c_2, c_3)$
 $\vec{v} = (d_1, d_2, d_3)$

$\vec{u} \cdot \vec{v} = c_1 d_1 + c_2 d_2 + c_3 d_3$

Geometric Meaning:

* $\vec{u} \times \vec{v}$ gives a third vector \perp to both \vec{u} and \vec{v}



B. Cross Product: In 3D, two vectors $\vec{u} = \sum_i c_i \vec{b}_i$ $\vec{v} = \sum_i d_i \vec{b}_i$

$\vec{u} \times \vec{v} = (|\vec{u}| |\vec{v}| \sin \theta) \vec{n}$. \vec{n} is the unit vector

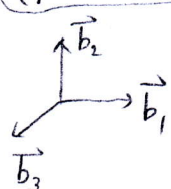
In a right-handed orthonormal basis

$$\begin{aligned} \vec{u} \times \vec{v} &= \left(\sum_i c_i \vec{b}_i \right) \times \left(\sum_j d_j \vec{b}_j \right) \\ &= \sum_i \sum_j c_i d_j (\vec{b}_i \times \vec{b}_j) \end{aligned}$$

$$\begin{cases} \vec{b}_1 \times \vec{b}_2 = \vec{b}_3 \\ \vec{b}_2 \times \vec{b}_3 = \vec{b}_1 \\ \vec{b}_3 \times \vec{b}_1 = \vec{b}_2 \\ \vec{b}_i \times \vec{b}_i = 0 \quad \forall i \end{cases}$$

perpendicular to the plane spanned by \vec{u} & \vec{v}
in the direction by right-hand rule

(4 fingers curling from \vec{u} to \vec{v} , the thumb points to the direction of \vec{n})



$$(c_1, c_2, c_3)$$

$$\times) (d_1, d_2, d_3)$$

$$(c_2 d_3 - c_3 d_2, c_3 d_1 - c_1 d_3, c_1 d_2 - c_2 d_1)$$

(3) Linear Transformations

A linear transformation \mathcal{L} is a transformation from V to V satisfying:

- ① $\mathcal{L}(\vec{u} + \vec{v}) = \mathcal{L}(\vec{u}) + \mathcal{L}(\vec{v}) \quad \forall \text{ vectors } \vec{u}, \vec{v} \in V, \text{ and}$
- ② $\mathcal{L}(a\vec{u}) = a\mathcal{L}(\vec{u}) \quad \forall \text{ vector } \vec{u} \in V \text{ and real number (scalar) } a.$

Now let $[\vec{b}_1, \vec{b}_2, \vec{b}_3]$ be a basis of the 3D vector space V .

For any $\vec{v} \in V$, we have $\vec{v} = \sum_i c_i \vec{b}_i = [\vec{b}_1, \vec{b}_2, \vec{b}_3] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$$\mathcal{L}(\vec{v}) = \mathcal{L}\left(\sum_i c_i \vec{b}_i\right) = \sum_i c_i \mathcal{L}(\vec{b}_i) = [\mathcal{L}(\vec{b}_1) \quad \mathcal{L}(\vec{b}_2) \quad \mathcal{L}(\vec{b}_3)] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

for some c_1, c_2, c_3
scalar values
(real numbers)

But $\mathcal{L}(\vec{b}_i)$ is still a vector in V , thus $\mathcal{L}(\vec{b}_1) = \sum_j M_{j1} \vec{b}_j = [\vec{b}_1, \vec{b}_2, \vec{b}_3] \begin{bmatrix} M_{11} \\ M_{21} \\ M_{31} \end{bmatrix}$

for some real numbers M_{11}, M_{21}, M_{31}

Similarly for $\mathcal{L}(\vec{b}_2)$ and $\mathcal{L}(\vec{b}_3)$

$$\mathcal{L}(\vec{v}) = [\mathcal{L}(\vec{b}_1) \quad \mathcal{L}(\vec{b}_2) \quad \mathcal{L}(\vec{b}_3)] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = [\vec{b}_1, \vec{b}_2, \vec{b}_3] \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

ie Linear transformation

$$\vec{v} \Rightarrow \mathcal{L}(\vec{v})$$

can be expressed by matrix multiplication:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}}_{3 \times 3 \text{ matrix } M} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

(4) Standard Transformations:

Scaling, Rotation, Translation

We like to express each of them by matrix multiplication so that they can be easily supported.

Plan:

- ① Define the transformation
- ② Derive $P \rightarrow Q$ using def. and simple geometry
- ③ Derive matrix M s.t. $Q = MP$

① Scaling: $P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, Q = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ where $\begin{cases} x' = p_x x \\ y' = p_y y \\ z' = p_z z \end{cases}$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} p_x & 0 & 0 \\ 0 & p_y & 0 \\ 0 & 0 & p_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

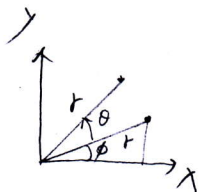
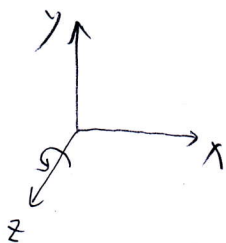
$$S = S(p_x, p_y, p_z) =$$

$$S^{-1}(p_x, p_y, p_z) = S\left(\frac{1}{p_x}, \frac{1}{p_y}, \frac{1}{p_z}\right)$$

② Rotation :

(i) Rotation about the z -axis by an angle of θ

$R_z(\theta)$:



$$\begin{cases} x = r \cos \phi \\ y = r \sin \phi \end{cases}$$

$$\begin{cases} x' = r \cos(\phi + \theta) \\ y' = r \sin(\phi + \theta) \\ z' = z \end{cases}$$

$$x' = r \cos(\phi + \theta) = r [\cos \phi \cos \theta - \sin \phi \sin \theta] = (r \cos \phi) \cos \theta - (r \sin \phi) \sin \theta \\ = (\cos \theta) x - (\sin \theta) y$$

$$y' = r \sin(\phi + \theta) = r [\sin \phi \cos \theta + \cos \phi \sin \theta] = (r \sin \phi) \cos \theta + (r \cos \phi) \sin \theta \\ = (\sin \theta) x + (\cos \theta) y$$

$$\therefore \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} (\cos \theta) x - (\sin \theta) y \\ (\sin \theta) x + (\cos \theta) y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_z(\theta)} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

* Similarly, we can derive $R_x(\theta)$ (rotation about x -axis by an angle θ)
 $R_y(\theta)$ (" " y -axis " ")

(See next page for details can skip)

* We can undo the rotation by rotating $-\theta$:

$$R^{-1}(\theta) = R(-\theta) \quad [R = \text{any of } R_x, R_y, R_z \text{ and in fact any rotation}]$$

* All $\cos \theta$'s are on the diagonal. $\sin \theta$'s are off diagonal

$$\cos(-\theta) = \cos \theta \quad \sin(-\theta) = -\sin \theta \quad \Rightarrow \quad R^{-1}(\theta) = R(-\theta) = R^T(\theta) \quad \text{transpose}$$

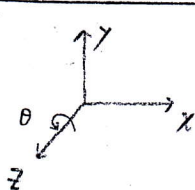
* Any rotation about the origin can be expressed as $R = R_z R_y R_x$

$$\underline{R^{-1}} = (R_z R_y R_x)^{-1} = R_x^{-1} R_y^{-1} R_z^{-1} = R_x^T R_y^T R_z^T = (R_z R_y R_x)^T = \underline{R^T}$$

$$\therefore \text{For any rotation matrix } \underline{R^{-1} = R^T}$$

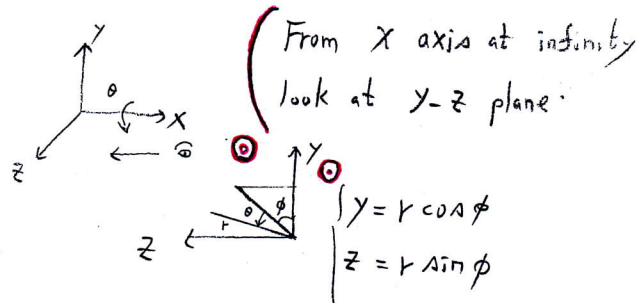
* Not commutative: $R_x R_y \neq R_y R_x$ (use textbook as an example)

Derivation of $R_x(\theta)$, $R_y(\theta)$, $R_z(\theta)$:



(Supplementary)
skip!

(2) $R_x(\theta)$:



(1) $R_z(\theta)$:

$$\begin{cases} x = r \cos \phi \\ y = r \sin \phi \end{cases}$$

$$\begin{cases} x' = r \cos(\phi + \theta) \\ y' = r \sin(\phi + \theta) \\ z' = z \end{cases}$$

$$x' = r [\cos \phi \cos \theta - \sin \phi \sin \theta]$$

$$= (r \cos \phi) \cos \theta - (\sin \theta) (r \sin \phi)$$

$$= (\cos \theta) x - (\sin \theta) y$$

$$y' = r [\sin \phi \cos \theta + \cos \phi \sin \theta]$$

$$= (\cos \theta) y + (\sin \theta) x$$

$$= (\sin \theta) x + (\cos \theta) y$$

$$\therefore R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$y' = r \cos(\phi + \theta) = r [\cos \phi \cos \theta - \sin \phi \sin \theta]$$

$$= (r \cos \phi) \cos \theta - (\sin \theta) (r \sin \phi)$$

$$= (\cos \theta) y - (\sin \theta) z$$

$$z' = r \sin(\phi + \theta) = r [\sin \phi \cos \theta + \cos \phi \sin \theta]$$

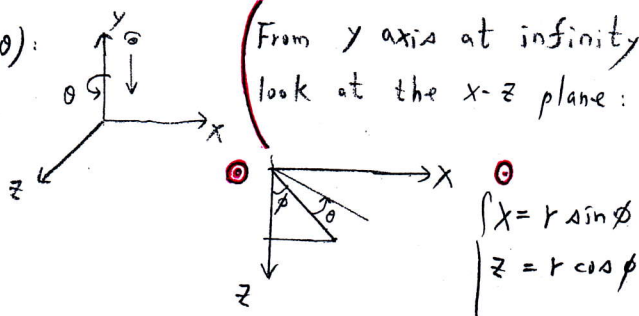
$$= (r \sin \phi) \cos \theta + (r \cos \phi) \sin \theta$$

$$= (\sin \theta) y + (\cos \theta) z$$

$$x' = x$$

$$\therefore R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(3) $R_y(\theta)$:



$$x' = r \sin(\phi + \theta) = r [\sin \phi \cos \theta + \cos \phi \sin \theta]$$

$$= (r \sin \phi) \cos \theta + (r \cos \phi) \sin \theta$$

$$= (\cos \theta) x + (\sin \theta) z$$

$$z' = r \cos(\phi + \theta) = r [\cos \phi \cos \theta - \sin \phi \sin \theta]$$

$$= (r \cos \phi) \cos \theta - (\sin \theta) (r \sin \phi)$$

$$= (-\sin \theta) x + (\cos \theta) z$$

$$y' = y$$

$$\therefore R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

③ Translation: $P \xrightarrow{d=(d_x, d_y, d_z)} Q$ $P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $Q = P + d = \begin{bmatrix} x + d_x \\ y + d_y \\ z + d_z \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

Matrix representation: $Q = M_P : \begin{bmatrix} x + d_x \\ y + d_y \\ z + d_z \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow x + d_x = ax + by + cz$
 $a=1 \quad b=c=0$

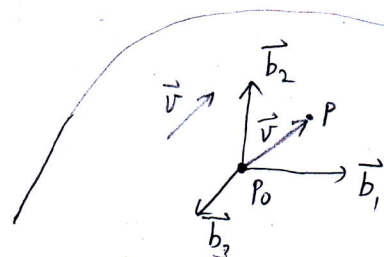
How about d_x ?

\Rightarrow There is NOT enough expressive power to make it work

We need new concepts: Affine space & Homogeneous coordinate system

II. Affine Space and Homogeneous Coordinate System

* Affine Space: vector space + points



Frame (or coordinate system): Basis vectors + origin point P_0 . $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$: basis vectors

* Vectors are NOT fixed w.r.t origin. But each point p is fixed relative to the origin P_0

$$P = P_0 + \vec{v} \quad (\vec{v} = P - P_0)$$

(Vector \vec{v} is uniquely expressed by: $\vec{v} = x\vec{b}_1 + y\vec{b}_2 + z\vec{b}_3$ (+0. P_0))

Point P

$$P = P_0 + \vec{v} = x\vec{b}_1 + y\vec{b}_2 + z\vec{b}_3 + 1 \cdot P_0$$

$\vec{v} = (x, y, z)$ the same? (origin point P_0 is NOT shown)
 $P = (x, y, z)$

sol: Use Homogeneous coord. system: $P = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & P_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & P_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$

ie Extending to 4D 4th component: $\begin{cases} 0 & \text{if vector} \\ 1 & \text{if point} \end{cases}$

Consistency check: Vector + Vector = Vector Vector + pt = pt pt + pt = ? (Never happens!) pt - pt = Vector
 4th component: $0 + 0 = 0 \checkmark$ $0 + 1 = 1 \checkmark$ $1 + 1 = 2$ $1 - 1 = 0 \checkmark$

* Translation: $Q = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} x + d_x \\ y + d_y \\ z + d_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 3 \times 3 \\ \vec{z} \end{bmatrix} & \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ (\vec{I} : identity matrix)

* $T^{-1}(d_x, d_y, d_z) = T(-d_x, -d_y, -d_z)$

Note: * Using Homogeneous Coord. System and 4x4 matrix multiplication to perform a 3D transformation is called an affine transformation

* Translation is NOT a linear transformation, but is an affine transformation

* We can use affine transformation to perform linear transformation (eg. scaling, rotation) on points:

Let l be a 3×3 matrix for linear transformation (such as scaling, rotation)

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix}_{3 \times 3, l} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ 0 \ 0 \ 0 \ 1 \end{bmatrix}_{3 \times 3, l} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_P$$

* Scaling

$S(\beta_x, \beta_y, \beta_z)$:
Fixed pt is at the origin.

* Rotation

$R(\theta, \vec{v})$:
rotation about axis (whose vector is \vec{v}) going thru origin:
Fixed pt is the origin.

∴ The corresponding 4×4 matrix L is

$$L = \begin{bmatrix} l & 0 \\ 0 & 1 \end{bmatrix}$$

* If l is rotation (resp. scaling), the properties of rotation (resp. scaling) stay the same for L .

$$\begin{bmatrix} x'+0 \\ y'+0 \\ z'+0 \\ 0+1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

cf: The 4×4 matrix T for translation is:

$$T = \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ 0 \ 0 \ 0 \ 1 \end{bmatrix}_{3 \times 3, i} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}_t = \begin{bmatrix} i & t \\ 0 & 1 \end{bmatrix}$$

where i is a 3×3 identity matrix

t is a column vector $\begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$

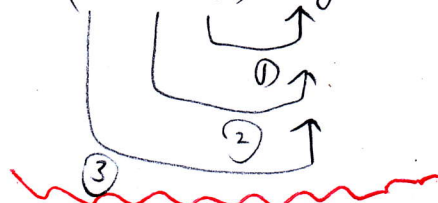
* In general, a 4×4 matrix for affine transformation is

$$M = \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ 0 \ 0 \ 0 \ 1 \end{bmatrix}_{3 \times 3, l} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}_t = \begin{bmatrix} l & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} i & \text{---} \\ 0 & 0 \ 0 \ 1 \end{bmatrix} \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ 0 \ 0 \ 0 \ 1 \end{bmatrix}_l = T \cdot L \quad (\neq L \cdot T)$$

ie. decomposed into eg. ① L (scaling rotation with fixed pt at the origin) and/or ② T (translation).

* Concatenation of Transformations

es. $M \leftarrow M_1 M_2 M_3$ Applying M to obj: $M \cdot \text{obj} = (M_1 M_2 M_3) \cdot \text{obj}$



Order is very important, since $M_1 M_2 \neq M_2 M_1$

* Standard Transformation Functions:

$\text{glRotatef}(\text{angle}, V_x, V_y, V_z) \longleftrightarrow \text{Rotate}(\text{angle}, V_x, V_y, V_z)$

in degrees

right hand

(The rotation axis goes thru the origin
→ origin is the fixed pt)

(specifies the vector of the rotation axis)

$\text{glScalef}(P_x, P_y, P_z) \longleftrightarrow \text{Scale}(P_x, P_y, P_z)$

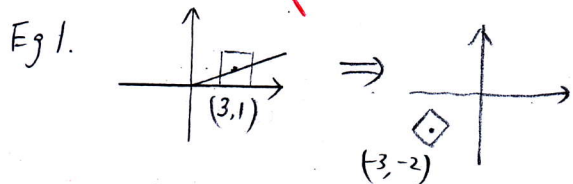
Fixed pt at the origin

$\text{glTranslatef}(dx, dy, dz) \longleftrightarrow \text{Translate}(dx, dy, dz)$

* Helper functions
 $\text{Rotate}()$, $\text{Scale}()$,
 $\text{Translate}()$
each returns a
4x4 matrix

* They can be multiplied together

* In general, first translate so that the object center is at the origin,
perform rotations / scalings (with center at the origin)
then translate the obj center to the final location.



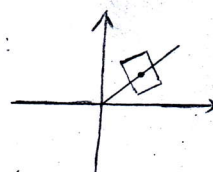
Correct:

- ① $T(-3, -1)$
- ② $R(45^\circ)$
- ③ $T(-3, -2)$

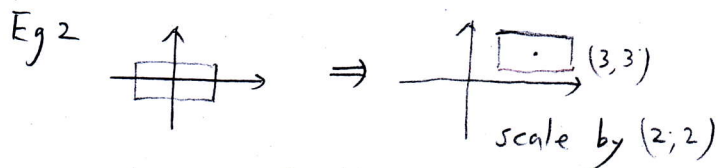
R: rotation about the origin (fixed at origin)

Common mistake:

- ① $R(45^\circ)$



center also moves



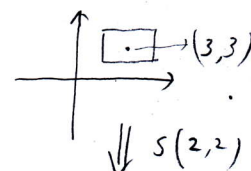
Correct:

- ① $S(2, 2)$
- ② $T(3, 3)$

S: scaling with fixed pt at the origin

Wrong:

- ① $T(3, 3)$
- ② $S(2, 2)$



center also moves

Note: Textbook Sec. 3.6 (Transformation for Normal Vectors)

is skipped here

Cover it later when discussing (shading)