Q1.

$$\hat{y} = \begin{cases} c_1 e^{-2x} + c_2 & \text{if } x < 1 \\ c_1 e^{-x} + c_3 & \text{if } x \geq 1 \end{cases}$$

a)

$$\phi_1(x) = \begin{cases} e^{-2x} & \text{if } x < 1 \\ e^{-x} & \text{if } x \geq 1 \end{cases}$$

$$\phi_2(x) = \begin{cases} 1 & \text{if } x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$

$$\phi_3(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

$$\hat{y} = \beta_1 \phi_1(x) + \beta_2 \phi_2(x) + \beta_3 \phi_3(x)$$

where $\beta_1 = c_1$ $\beta_2 = c_2$ $\beta_3 = c_3$

b)

$f(x)$ contihous at $x = 1$ means

$$c_1 e^{-2} + c_2 = c_1 e^{-1} + c_3 \qquad x = 1$$

$$c_2 = c_1 e^{-1} - c_1 e^{-2} + c_3 = c_1\left(\frac{1}{e} - \frac{1}{e^2}\right) + c_3$$
$$= c_1\left(\frac{e}{e^2} - \frac{1}{e^2}\right) + c_3$$

$$c_2 = c_1\left(\frac{e-1}{e^2}\right) + c_3$$

$$\phi_1(x) = \begin{cases} e^{-2x} + \frac{e-1}{e^2} & \text{if } x < 1 \\ e^{-x} & \text{if } x \geq 1 \end{cases}$$

$$\hat{y} = \begin{cases} c_1 e^{-2x} + c_1\left(\frac{e-1}{e^2}\right) + c_3 \\ c_1 e^{-x} + c_3 \end{cases}$$

$$\phi_2(x) = 1$$

$\hat{y} = \beta_1 \phi_1(x) + \beta_2 \phi_2(x)$ where $\beta_1 = c_1$ $\beta_2 = c_3$

Q2

a) Model 1: The minimum parameter is 2 since $\hat{y} = \beta_0 + \beta_1 x_1$

$\beta_0 = b \quad \beta_1 = \omega$

Model 2: The min. parameter is 3 since

$$\hat{y} = \beta_0 + \beta_1 \phi_1(x) + \beta_2 \phi_2(x)$$

$$\beta_0 = b \quad \beta_1 = \omega_1 \quad \beta_2 = \omega_2$$

Model 3: The min. parameter is 5 since

$$\hat{y} = \beta_0 + \beta_1 \phi_1(x) + \beta_2 \phi_2(x) + \beta_3 \phi_3(x) + \beta_4 \phi_4(x)$$

$$\beta_0 = b \quad \beta_1 = W_1 \quad \beta_2 = W_2 \quad \beta_3 = W_3 \quad \beta_4 = W_4$$

b) Model 3 since it has more features and parameters to fit training data better.

c) Model 3 since the higher the model order the lower the bias

d) Model 1 since the more features the higher the variance.

e)

Normal Rule: $\underset{p}{\operatorname{argmin}}$ MSE of testing = Model 2

SE Rule: $S_{tgt} = (MSE + SE)_{\text{testing of Model 2}} = 4.1 + 0.5 = 4.6$

Find $\min\{p \mid MSE_p \leq S_{tgt}\}$ = Model 2.

Q3.

$$P(y=1|x) = \frac{1}{1+e^{-z}} \qquad z = b + wx$$

a)

$P(y=1|x) = 0.8$ at $x=3$

and

$P(y=0|x) = 0.8$ at $x=1$

$$P(y=0|x) = 1 - P(y=1|x) = 1 - \frac{1}{1+e^{-z}} = 0.8 \quad \text{at } x=1.$$

$$\downarrow$$

$$\frac{1}{1+e^{-z}} = 0.8 \quad \text{s.t } x=3 \qquad\qquad \frac{e^{-z}}{1+e^{-z}} = 0.8 \quad \text{st } x=1$$

$$1 = 0.8 + 0.8e^{-z} \qquad\qquad e^{-z} = 0.8 + 0.8e^{-z}$$

$$\frac{0.2}{0.8} = e^{-z} \qquad\qquad 0.2e^{-z} = 0.8$$

$$e^{-z} = 4$$

$$\ln\left(\tfrac{1}{4}\right) = -z \qquad\qquad -z = \ln(4)$$

$$\cancel{-}\ln(4) = \cancel{-}z$$

$$b + wx = -\ln(4)$$

$$\ln(4) = b + wx = b + 3w \qquad\qquad b + w = -\ln(4)$$

$$
\begin{array}{rl}
 & \begin{cases} b + 3w = \ln(4) \\ b + w = -\ln(4) \end{cases} \\
- & \\
\hline
 & 2w = 2\ln(4) \\
 & w = \ln(4)
\end{array}
\qquad
\begin{aligned}
b + \ln(4) &= -\ln(4) \\
b &= -2\ln(4)
\end{aligned}
$$

$$W = \ln(4) \qquad b = -2\ln(4)$$

b)

$$P(y=1|x) = \frac{1}{1+e^{-z}} \qquad z = b + wx \qquad x=4$$

$$= -2\ln(4) + 4\ln(4)$$

$$= \frac{1}{1+e^{2\ln(4)-4\ln(4)}} = \frac{1}{1+\frac{e^{\ln(16)}}{e^{\ln(256)}}}$$

$$= \frac{1}{1+\frac{16}{256}} = \frac{1}{1+0.0625}$$

$$= \frac{1}{1.0625}$$

$$= 0.94$$

Midterm 1: Python Problems

There are three python problems. Answer all the sections marked `#TODO` . Print to PDF. Submit the PDF only.

Loading Packages and Data

For the problems, you can use the following packages

In [411…

```
import numpy as np
import matplotlib.pyplot as plt
from sklearn.model_selection import train_test_split
from sklearn.linear_model import LinearRegression
import pickle
```

Run the following code to download the data for the midterm. This will retrieve three files -- one for each problem.

In [412…

```
import requests

def download_file_from_google_drive(id, destination):
    URL = "https://docs.google.com/uc?export=download"

    session = requests.Session()

    response = session.get(URL, params = { 'id' : id }, stream = True)
    token = get_confirm_token(response)

    if token:
        params = { 'id' : id, 'confirm' : token }
        response = session.get(URL, params = params, stream = True)

    save_response_content(response, destination)

def get_confirm_token(response):
    for key, value in response.cookies.items():
        if key.startswith('download_warning'):
            return value

    return None

def save_response_content(response, destination):
    CHUNK_SIZE = 32768

    with open(destination, "wb") as f:
        for chunk in response.iter_content(CHUNK_SIZE):
            if chunk: # filter out keep-alive new chunks
                f.write(chunk)

file_path = 'https://drive.google.com/file/d/10_1PxDIoSiuuOFC_iyVaoU9bDiQYHcTT/view?
file_id = '10_1PxDIoSiuuOFC_iyVaoU9bDiQYHcTT'

dst = 'midterm_data.zip'
download_file_from_google_drive(file_id, dst)


# Unzip the files
import zipfile
```

```
with zipfile.ZipFile(dst, 'r') as zip_ref:
    zip_ref.extractall('data')


# Move them to the top directory
import shutil
for fn in ['prob_linear.p', 'prob_model.p', 'prob_logistic.p']:
    src = 'data/midterm1_data/%s' % fn
    shutil.move(src, fn)
    print('%s loaded' % fn)
```

```
prob_linear.p loaded
prob_model.p loaded
prob_logistic.p loaded
```

Problem 1. Linear Regression

Run the following code to load the data

In [413...

```
with open('prob_linear.p', 'rb') as fp:
    X,y = pickle.load(fp)
```

Split the data into training and test. You may use the `train_test_split` function.

In [414...

```
# TODO
Xtr, Xts, ytr, yts = train_test_split(X, y, test_size=0.3)
```

In [415...

```
print(X.shape)
print(y.shape)
```

```
(500, 2)
(500,)
```

Suppose we want to fit a model of the form:

```
yhat[i] = b + w[0]*X[i,0] + w[1]*X[i,1] + w[2]*X[i,0]*X[i,1] +
w[3]*X[i,0]**2 + w[4]*X[i,1]**2
```

Complete the function `transform` below that creates a matrix `Z` whose columns are the basis functions for this model. You may use the `np.column_stack()` function. For example,

```
Z = np.column_stack((col1, col2, col3))
```

creates a matrix `Z` with columns `col1` , `col2` , and `col3` .

In [416...

```
def transform(X):
    # TODO
    Z = np.column_stack((X[:,0], X[:,1], X[:,0] * X[:,1], X[:,0]**2, X[:,1]**2))
    return Z
```

In [417...

```
# Testing on transform function
Z = transform(X)
print(Z.shape)
```

```
print(X[0, :])
print(Z[0, :])
```

```
(500, 5)
[-0.54002778 -0.81326207]
[-0.54002778 -0.81326207  0.43918411  0.29163     0.66139519]
```

Now fit and evaluate the model:

- Fit the model on the training data. You may use the `LinearRegression` object and the `transform` function above.
- Predict the values `y` on the test data
- Print the test MSE

In [418…

```
# TODO
# Transform Xtr and Xts
Ztr, Zts = transform(Xtr), transform(Xts)
# Fit Xtr with ytr
reg = LinearRegression()
reg.fit(Ztr, ytr)
# predict yhat from Zts
yhat = reg.predict(Zts)
# Compute MSE with yhat and yts
MSE = np.mean((yhat - yts)**2)
print("The Mean Squared Error is: ", MSE)
```

```
The Mean Squared Error is:  0.022120120586006018
```

Problem 2. Model Selection

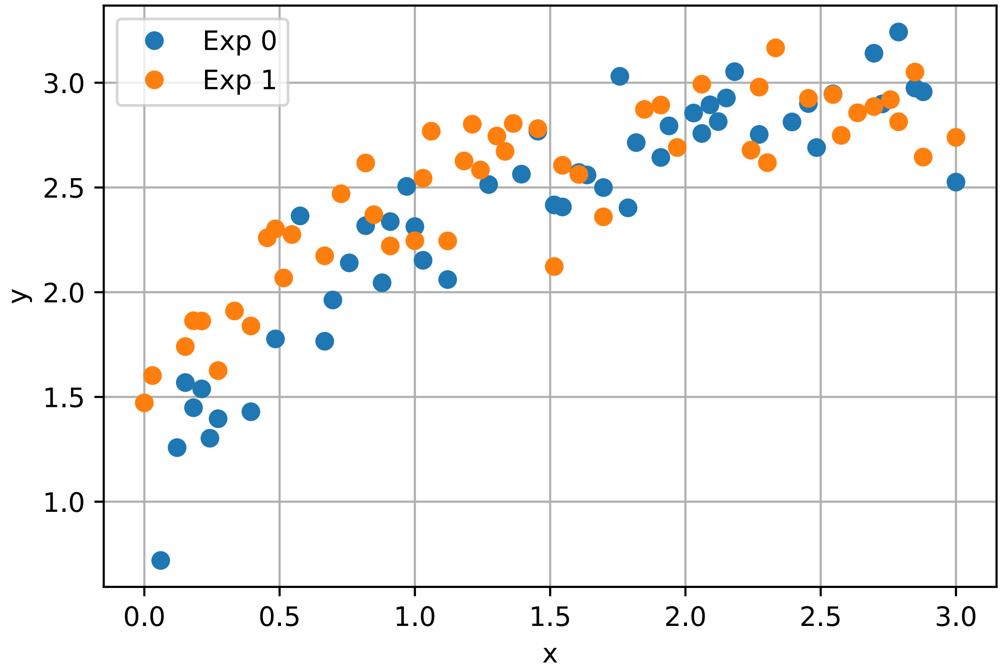
Run the code below to load and plot the data. The data is from two experiments:

- `Xtr[:,0], Ytr[:,0]` is the training data from experiment 0
- `Xtr[:,1], Ytr[:,1]` is the training data from experiment 1
- `Xts[:,0], Yts[:,0]` is the test data from experiment 0
- `Xts[:,1], Yts[:,1]` is the test data from experiment 1

In [419…

```
with open('prob_model.p', 'rb') as fp:
    Xtr, Xts, Ytr, Yts = pickle.load(fp)

plt.plot(Xtr[:,0], Ytr[:,0], 'o')
plt.plot(Xtr[:,1], Ytr[:,1], 'o')
plt.legend(['Exp 0', 'Exp 1'])
plt.xlabel('x')
plt.ylabel('y')
plt.grid()
```



In [420…

```
# Checking the shapes of objects
print(Xtr.shape, Xts.shape)
print(Ytr.shape, Yts.shape)
```

```
(50, 2) (50, 2)
(50, 2) (50, 2)
```

You want to learn the relation between `y` vs. `x` .

First, fit two *separate* models for each experiment of the form:

```
Y[:,0] ~= a0 + b0*exp(-X[:,0])
Y[:,1] ~= a1 + b1*exp(-X[:,1])
```

For the data in each experiment, fit the model and pint the test MSE.

You may use the `LinearRegression` function for the fitting. But, if `z` is a vector (not a matrix), you cannot use:

```
reg = LinearRegression()
reg.fit(z, y)    # WILL NOT WORK if z is a vector.
```

You must reshape `z` to a `n x 1` matrix first:

```
reg = LinearRegression()
reg.fit(z[:,None], y)    # This will work
```

In [421…

```
# TODO
nexp = Xtr.shape[1]  # number of experiments = 2
reg = LinearRegression()
for i in range(nexp):
    Ztr = np.exp(-Xtr[:,i])
    Zts = np.exp(-Xts[:,i])
    Ztr = Ztr[:,None]
```

```
    Zts = Zts[:,None]
    reg.fit(Ztr, Ytr[:,i])
    yhat = reg.predict(Zts)
    mse = np.mean((yhat-Yts[:,i])**2)
    print("The test Mean Square Error for experiment {} is {}".format(i, mse))
```

```
The test Mean Square Error for experiment 0 is 0.0339311556185114
The test Mean Square Error for experiment 1 is 0.041306971069426573
```

Now, fit a model of the form:

```
Y[:,0] = a + b0*exp(-X[:,0])
Y[:,1] = a + b1*exp(-X[:,1])
```

So, the two experiments have the same intercept term. Fit the model on the training data and measure the test MSE.

For training, you will want to combine the data into a single feature matrix `Z` using `Xtr[:,0]` and `Xtr[:,1]` and single target vector `b` from `Ytr[:,0]` and `Ytr[:,1]` .

In [422…

```
# This is wrong
# Ztr = np.exp(-Xtr)
# Zts = np.exp(-Xts)

# reg.fit(Ztr, Ytr)
# Yhat = reg.predict(Zts)
# print(reg.coef_)
# print(reg.intercept_)
# MSE = np.mean((Yhat-Yts)**2)
# print("The Mean Square Error for two experiements having same intercept term is ",
```

In [ ]:

```
```

In [ ]:

```
```

Problem 3. Logistic Regression

Run the following code to load the data as follows:

In [423…

```
with open('prob_logistic.p', 'rb') as fp:
    X,y = pickle.load(fp)
```

In [424…

```
# Checking shape and content of X y
print(X.shape)
print(y.shape)
print(X[0:5, :])
print(y[:5])
```

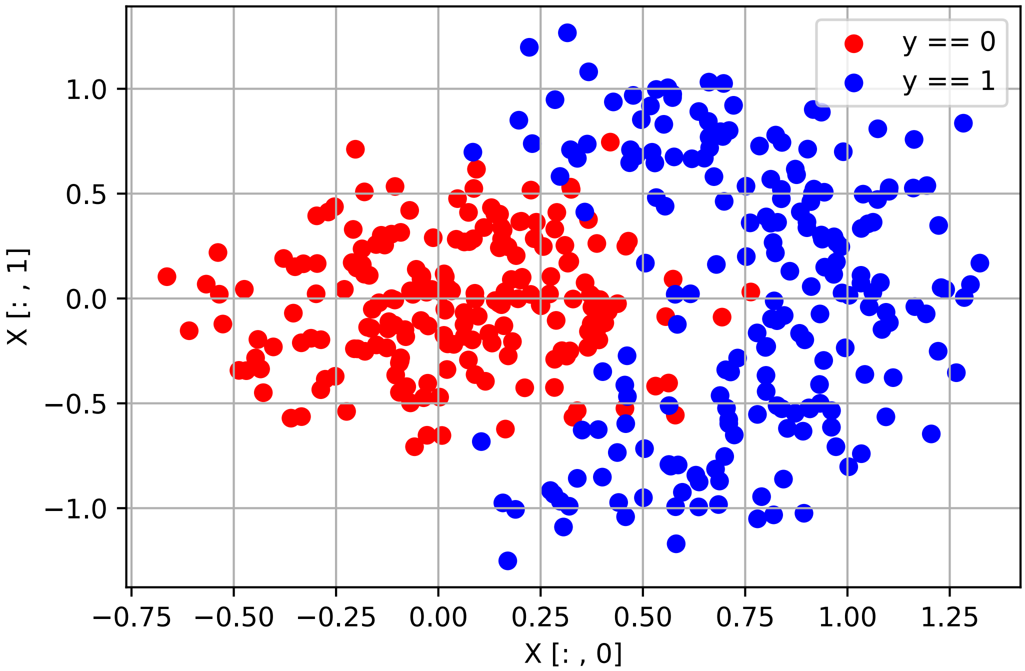
```
(400, 2)
(400,)
[[ 0.08294982 -0.19753769]
 [-0.06222682  0.01981546]
 [ 0.36398206  0.73624195]
 [ 0.13820735  0.40317062]
```

```
 [ 0.98679706  0.02633604]]
[0. 0. 1. 0. 1.]
```

Plot a scatter plot of the data with different colors for the two classes. You may use the `plt.scatter` function.

In [425…

```
# TODO
plt.scatter(X[(y==0),0], X[(y==0), 1], c='r')
plt.scatter(X[(y==1),0], X[(y==1), 1], c='b')
plt.legend(['y == 0','y == 1'],loc='upper right')
plt.grid(True)
plt.xlabel("X [: , 0]")
plt.ylabel("X [: , 1]")
plt.show()
```



Split the data into training and test. You may use the `train_test_split` method. Use `test_size=0.5`.

In [426…

```
# TODO
Xtr, Xts, ytr, yts = train_test_split(X, y, test_size=0.5)
```

Consider a classifier of the form:

```
yhat[i] = 1 when z[i] > t
yhat[i] = 0 when z[i] <= t
```

where `z[i] = X[i,0] + np.abs(X[i,1])`.

For each value `t` in `ttest`, compute the accuracy of the classifier on the *training* data. Plot the training accuracy as a function of `t`.

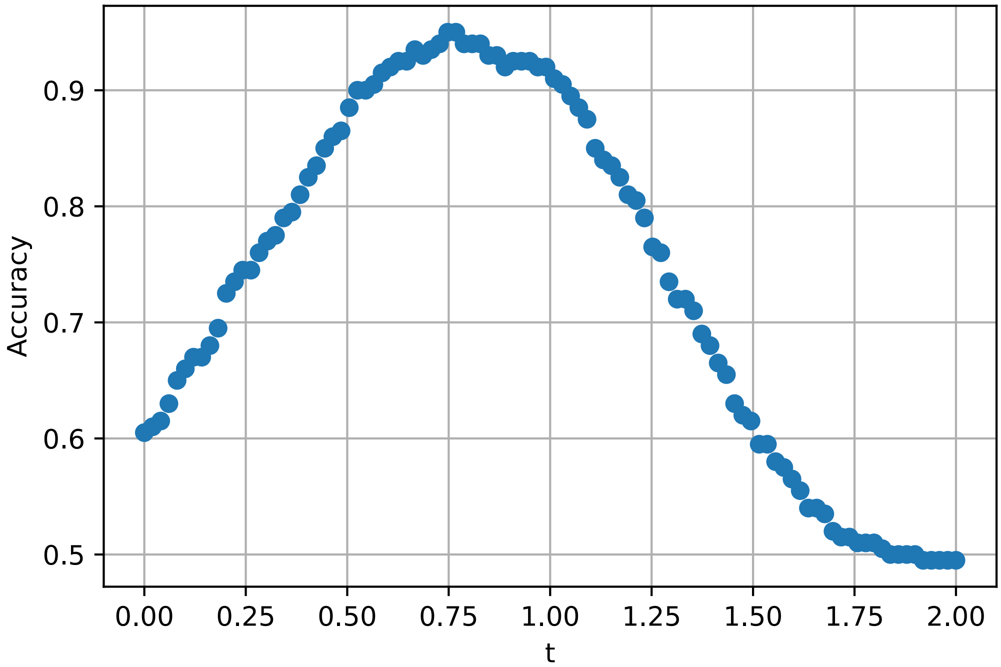
In [427…

```
ttest = np.linspace(0,2,100)
# vector of accuracy with size = len(ttest)
accuracy = np.zeros((100))
# make z vector
z = Xtr[:,0] + np.abs(Xtr[:,1])
```

```python
for i, t in enumerate(ttest):
    yhat = z > t
    accuracy[i] = np.mean(yhat == ytr)

plt.plot(ttest, accuracy, 'o')
plt.grid()
plt.xlabel("t")
plt.ylabel("Accuracy")
```

Out[427… Text(0, 0.5, 'Accuracy')



Find the value of `t` with the highest training accuracy. Print the test accuracy for the classifier with that value of `t`.

In [428…

```python
# TODO:
iopt = np.argmax(accuracy)
topt = ttest[iopt]

# TODO.
z = Xts[:,0] + np.abs(Xts[:,1])
yhat = z > topt
acc_ts = np.mean(yhat == yts)
print("The test accuracy with t {} is {}".format(topt, acc_ts))
```

```
The test accuracy with t 0.7474747474747475 is 0.965
```

In [ ]:

In [ ]: