1) $$f(x) = \frac{(\sqrt{2} - \sqrt{x}) \log\left|\sin\left(\frac{\pi x}{2}\right)\right|}{\left((x-2)^2 + \log\left(\frac{x}{2}\right)\right) \log|x-2|}$$

Per $x \to 0^+$

$$f(x) \sim \frac{\sqrt{2} \log\left(\frac{\pi x}{2}\right)}{\log\left(\frac{x}{2}\right) \cdot \log(2)} = \frac{\sqrt{2}\left(\cancel{\log(x)} + \log\left(\frac{\pi}{2}\right)\right)}{\cancel{\log(x)} \cdot \log(2)}$$

$$\to \frac{\sqrt{2}}{\log(2)}$$

Per $x \to 2$, $t = x - 2 \to 0$

$$f(x) = \frac{(\sqrt{2} - \sqrt{t+2}) \log\left|\sin\left(\frac{\pi(t+2)}{2}\right)\right|}{\left(t^2 + \log\left(\frac{t+2}{2}\right)\right) \log|t|} =$$

$$= \sqrt{2}\, \frac{\left(1 - \sqrt{\frac{t}{2} + 1}\right) \log\left|\sin\left(\frac{\pi t}{2}\right)\right|}{\left(t^2 + \log\left(\frac{t}{2} + 1\right)\right) \log|t|}$$

Per $t \to 0$

$$\cdot\ 1-\sqrt{\frac{t}{2}+1} = 1-\left(1+\frac{t}{4}+o(t)\right) \sim \frac{t}{4}$$

$$\cdot\ t^2+\log\left(\frac{t}{2}+1\right) = t^2+\frac{t}{2}+o(t) \sim \frac{t}{2}$$

$$\cdot\ \frac{\log\left|\sin\left(\frac{\pi-t}{2}\right)\right|}{\log|t|} = \frac{\log\left(\frac{\pi|t|}{2}+o(t)\right)}{\log|t|} \sim$$

$$\sim \frac{\log|t|+\log\left(\frac{\pi}{2}\right)}{\log|t|} \to 1$$

quindi

$$f(x) \sim \sqrt{2}\cdot\frac{-\frac{t}{4}}{\frac{t}{2}}\cdot 1 \to -\frac{1}{\sqrt{2}}$$

e dunque

$$\lim_{x\to 2} f(x) = -\frac{1}{\sqrt{2}}$$

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3) $\displaystyle\int_0^2 \arctan\left(\frac{x}{2-x}\right)dx =$ $\qquad \left(\frac{x}{2-x}\right)'$

$$= \left[x \cdot \arctan\left(\frac{x}{2-x}\right) \right]_0^2 - \int_0^1 x \cdot \frac{1}{1+\left(\frac{x}{2-x}\right)^2} \cdot \boxed{\frac{2-\cancel{x}+\cancel{x}}{(2-x)^2}} \, dx$$

$$\lim_{x \to 2} \arctan\left(\frac{x}{2-x}\right) = \arctan(\infty) = \frac{\pi}{2}$$

$$= 2 \cdot \frac{\pi}{\cancel{2}} - \int_0^2 \frac{2x}{1+\frac{x^2}{(2-x)^2} \cdot (2-x)^2} \, dx =$$

$$= \pi - \int_0^2 \frac{2x}{\frac{(2-x)^2+x^2}{\cancel{(2-x)^2}} \cdot \cancel{(2-x)^2}} \, dx =$$

$$= \pi - \int_0^2 \frac{2x}{4+x^2-4x+x^2} \, dx =$$

$$= \pi - \int_0^2 \frac{2x}{2x^2-4x+4} \, dx = \pi - \int \frac{x}{x^2-2x+2} \, dx$$

$$t = x-1 \Rightarrow x = t+1 \, , \, dt = dx$$

$$= \pi - \int_{-1}^1 \frac{t+1}{(t+1)^2-2(t+1)+2} \, dt = \pi - \int_{-1}^1 \frac{t+1}{t^2+1+2\cancel{t}-2t-2+2} \, dt =$$

$$= \pi - \int^1 \frac{t+1}{\quad} \, dt =$$

$$\int_{-1} \frac{}{t^2+1}$$

$$= \pi \cdot \left(\int_{-1}^{1} \frac{t}{t^2+1}\, dt + \int_{-1}^{1} \frac{1}{t^2+1}\, dt \right) =$$

$$= \pi \cdot \left(\left[\frac{1}{2} \ln(t^2+1) \right]_{-1}^{1} + \left[\arctan(t) \right]_{-1}^{1} \right) =$$

$$= \pi \cdot \left(\frac{1}{2} \cancel{\ln(2)} - \frac{1}{2} \cancel{\ln(2)} + 2 \left[\arctan(t) \right]_{0}^{1} \right) =$$

$$= \pi \cdot \cancel{2} \cdot \frac{\pi}{\cancel{4}\,2} = \frac{\pi}{2}$$

4) $$\sum_{k=1}^{\infty} \left(\frac{\cos^2\left(\frac{1}{k}\right) - \sin^2\left(\frac{2}{k}\right)}{\cos\left(\frac{3}{k}\right)} \right)^{k^2}$$

Pt. $k \to \infty$

$$\frac{\cos^2\left(\frac{1}{k}\right) - \sin^2\left(\frac{2}{k}\right)}{\cos\left(\frac{3}{k}\right)} =$$

$$= \frac{\left(1-\frac{1}{2k^2}+o\left(\frac{1}{k^2}\right)\right)^2-\left(\frac{2}{k}+o\left(\frac{1}{k^2}\right)\right)^2}{1-\frac{9}{2k^2}+o\left(\frac{1}{k^2}\right)} =$$

$$= \left(\left(1-\frac{2}{2k^2}+o\left(\frac{1}{k^2}\right)\right)-\left(\frac{4}{k^2}+o\left(\frac{1}{k^2}\right)\right)\right)\left(1+\frac{9}{2k^2}+o\left(\frac{1}{k^2}\right)\right)$$

$$= \left(1-\frac{\cancel{10}\,5}{\cancel{2}k^2}+o\left(\frac{1}{k^2}\right)\right)\left(1+\frac{9}{2k^2}+o\left(\frac{1}{k^2}\right)\right) =$$

$$= 1-\frac{5}{k^2}+\frac{9}{2k^2}-\cancel{\frac{45}{2k^4}}+o\left(\frac{1}{k^2}\right) =$$

$$= 1+\frac{-10+9}{2k^2}+o\left(\frac{1}{k^2}\right) = 1-\frac{1}{2k^2}+o\left(\frac{1}{k^2}\right)$$

c: Quindi

$$\sqrt[k]{a_k} = \left(\frac{\cos^2\left(\frac{1}{k}\right)-\sin^2\left(\frac{2}{k}\right)}{\cos\left(\frac{3}{k}\right)}\right)^{\frac{k^2}{k}} =$$

$$= \left(1 - \frac{1}{2k^2} + o\left(\frac{1}{k^2}\right)\right)^{k^2} \to e^{-\frac{1}{2}}$$

ALLORA PER IL CRITERIO DELLA RADICE, DATO CHE $e^{-\frac{1}{2}} < 1$ LA SERIE È CONVERGENTE

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6) $$f(x,y) = \frac{y^2\sqrt{x}}{1+xy}$$

$(1,2)$

$(1,2,f(1,2))$

$$f_x(x,y) = \frac{(y^2 \cdot x^{\frac{1}{2}})'(1+xy) - (y^2\sqrt{x})(1+xy)'}{(1+xy)^2} =$$

$$= \frac{\frac{y^2}{2\sqrt{x}} \cdot 1+xy - (y^2\sqrt{x}) \cdot y}{(1+xy)^2} = \frac{\frac{y^2+xy^3}{2\sqrt{x}} - y^3\sqrt{x}}{(1+xy)^2} =$$

$$= \frac{y^2+xy^3-(y^3\sqrt{x})(2\sqrt{x})}{2\sqrt{x}(1+xy)^2} = \frac{y^2+xy^3-2y^3x}{2\sqrt{x}(1+xy)^2} =$$

$$= \frac{y^2(1+xy-2xy)}{2\sqrt{x}(1+xy)^2} = \frac{y^2(1-xy)}{2\sqrt{x}(1+xy)}$$

$$f_y(x,y) = \frac{(y^2\sqrt{x})'(1+xy) - (y^2\sqrt{x})(1+xy)'}{(1+xy)^2} =$$

$$= \frac{2y\sqrt{x}(1+xy) - (y^2\sqrt{x})(1+x)}{(1+xy)^2} = \frac{2y\sqrt{x} + 2y^2\sqrt{x}\cdot x - y^2\sqrt{x} - y^2\sqrt{x}\cdot x}{(1+xy)^2}$$

$$= \sqrt{x} \cdot \frac{2y + 2y^2x - y^2 - y^2x}{(1+xy)^2} = \sqrt{x}\,\frac{y + y^2x}{(1+xy)^2} =$$

$$= \frac{y\sqrt{x}(1+yx)}{(1+xy)^2}$$

Or.

$$\nabla f(1,1) = \left(f_x(1,2), f_y(1,2)\right) =$$

$$= \left(\frac{2^2(1-2)}{2\sqrt{1}(1+2)^2}, \frac{2\sqrt{1}(2+2)}{(1+2)^2}\right) = \left(-\frac{4}{18}, \frac{8}{9}\right) =$$

$$= \left(-\frac{2}{9} , \frac{8}{9}\right)$$

così il gradiente nel punto $(1,2,f(1,2))$ è

$$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) =$$

$$= \frac{4}{3} - \frac{2}{9}(x-1) + \frac{8}{9}(y-2) =$$

$$= \frac{4}{3} - \frac{2}{9}x + \frac{2}{9} + \frac{8}{9}x - \frac{16}{9} = \frac{12 - 2x + 2 + 8x - 16}{9} =$$

$$= \frac{-2 - 2x + 8x}{9} = -\frac{2}{9} - \frac{2x}{9} + \frac{8x}{9}$$

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5)

$$|z| + i z \, Re(z) = z^2$$

$$\sqrt{x^2+y^2} + i(x+iy) \cdot x = (x+iy)^2$$

$$\sqrt{x^2+y^2} + (ix - y)x = x^2 - y^2 + 2xiy$$

$$\sqrt{x^2+y^2} + i(x^2 - xy) = x^2 + y^2 + 2xiy$$

$$\begin{cases} \sqrt{x^2+y^2} - xy = x^2 - y^2 \\ x^2 = 2xy \end{cases}$$

$\Rightarrow x(x-2y) = 0 \Rightarrow x = 0$ oppure $x = 2y$

$$\begin{cases} |y| = -y^2 \\ x = 0 \end{cases} \cup \begin{cases} \sqrt{4y^2+y^2} - 2y^2 = 4y^2 - y^2 \\ x = 2y \end{cases}$$

$$\begin{cases} |y|(1+|y|) = 0 \\ x = 0 \end{cases} \cup \begin{cases} \sqrt{5y^2} - 2y^2 - 3y^2 = 0 \\ x = 2y \end{cases}$$

$$\begin{cases} y = 0 \\ x = 0 \end{cases} \cup \begin{cases} \sqrt{5y^2} - 5y^2 = 0 \\ x = 2y \end{cases}$$

$$\begin{cases} x = 0 \end{cases} \cup \begin{cases} |y|(1-\sqrt{5}\,|y|) = 0 \end{cases}$$

$$y = 0 \qquad\qquad x = 2y$$

$$\cancel{\begin{cases} x = 0 \\ y = 0 \end{cases}} \cup \begin{cases} y = 0 \\ x = 0 \end{cases} \cup \begin{cases} y = \frac{1}{\sqrt{5}} \\ x = \frac{2}{\sqrt{5}} \end{cases} \cup \begin{cases} y = -\frac{1}{\sqrt{5}} \\ x = -\frac{2}{\sqrt{5}} \end{cases}$$

Così le soluzioni sono

$$0, \quad \frac{2+i}{\sqrt{5}}, \quad -\frac{2+i}{\sqrt{5}}$$