1) a) $f(x) = \dfrac{\sqrt{2x - x^2}}{1 - x}$

DOMINIO

$$\begin{cases} 2x - x^2 \geq 0 \\ 1 - x \neq 0 \end{cases} \qquad \begin{cases} x^2 - 2x \leq 0 \\ x \neq 1 \end{cases}$$

$$\begin{cases} x(2 - x) \leq 0 \\ x \neq 1 \end{cases} \qquad \begin{cases} x \in [0, 2] \\ x \neq 1 \end{cases}$$

$$D = [0, 1) \cup (1, 2]$$

ASINTOTOTI

$$\lim_{x \to 1^{\pm}} \frac{\sqrt{2x - x^2}}{1 - x} = \frac{2 - 1}{1 - 1^{\pm}} = \frac{2 - 1}{0^{\pm}} =$$

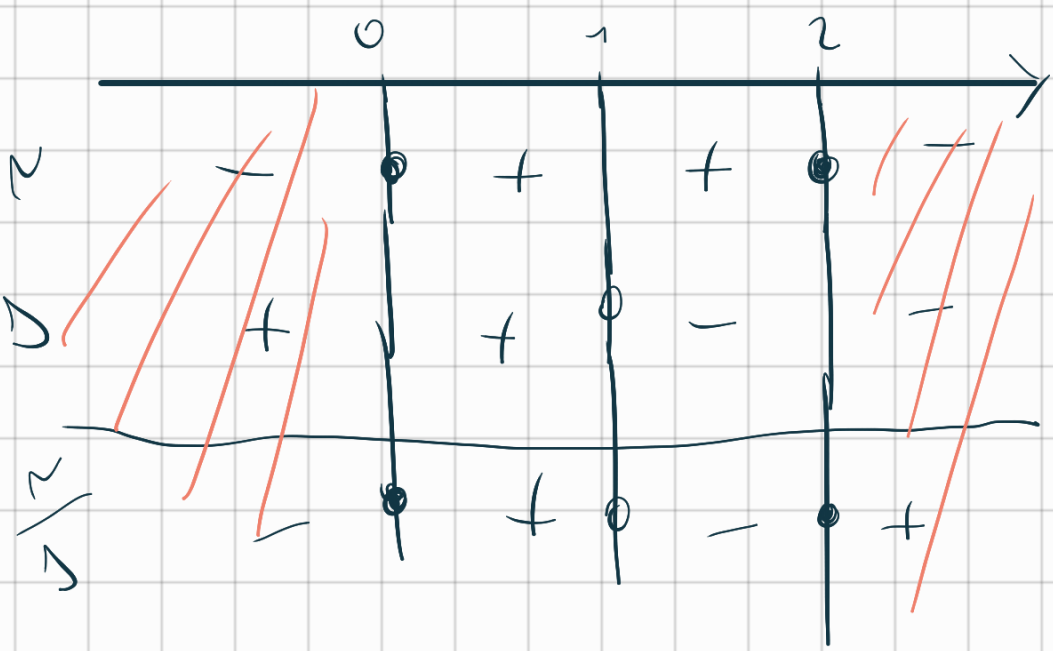
$$= \pm \infty$$

SCHEMA

$$\frac{\sqrt{2x-x^2}}{1-x} \geq 0$$

N: $2x - x^2 \geq 0 \Rightarrow x \in [0,2]$

D: $1 - x > 0 \Rightarrow x < 1$



LA FUNZIONE È POSITIVA IN $(0,1)$

È NEGATIVA IN $(1,2)$

VALE ZERO IN $x=0$ E $x=2$

MONOTONIA

f È DERIVABILE IN $(0,1) \cup (1,2)$

$$f'(x) = \frac{\left(\sqrt{2x-x^2}\right)' \cdot (1-x) - \left[\left(\sqrt{2x-x^2}\right) \cdot (1-x)'\right]}{(1-x)^2}$$

$$\left[(2x-x^2)^{\frac{1}{2}}\right]' = \frac{1}{2}(2x-x^2)^{-\frac{1}{2}} \cdot (2-2x) =$$

$$= \frac{(1-x)}{(2x-x^2)^{\frac{1}{2}}}$$

$$= \frac{\frac{(1-x)(1-x)}{(2x-x^2)^{\frac{1}{2}}} + \left(\sqrt{2x-x^2}\right)}{(1-x)^2} =$$

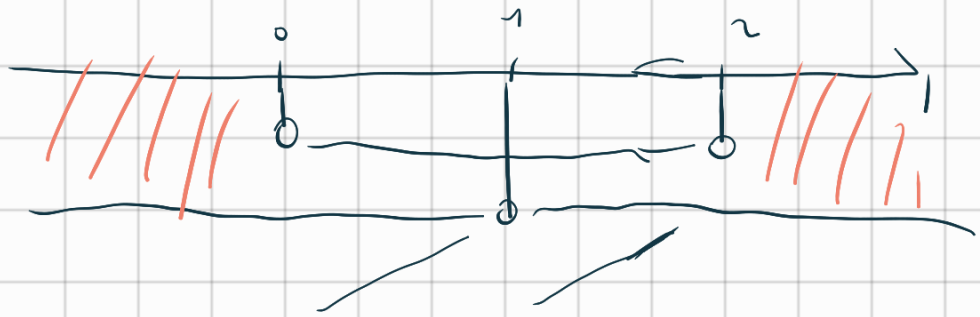
$$= \frac{1-2x+x^2 + \left(\sqrt{2x-x^2}\right)\cdot\left(\sqrt{2x-x^2}\right)}{\sqrt{2x-x^2}\cdot(1-x)^2} =$$

$$= \frac{1-\cancel{2x}+\cancel{x^2}+\cancel{2x}-\cancel{x^2}}{\sqrt{2x-x^2}\cdot(1-x)^2} = \frac{1}{\sqrt{2x-x^2}\cdot(1-x)^2}$$

$$f'(x) \geq 0 \rightarrow \frac{1}{\sqrt{2x-x^2}\cdot(1-x)^2} \geq 0$$

D_1: $\sqrt{2x-x^2} > 0 \rightarrow 2x-x^2 > 0 \Rightarrow x \in (0,2)$

Dx: $(1-x)^2 > 0 \Rightarrow \forall x \in \mathbb{R} \setminus \{1\}$



f è sempre crescente nel dominio

CONVESSITÀ - CONCAVITÀ

$$f''(x) = \left((2x - x^2)^{-\frac{1}{2}} \cdot (x-1)^{-2} \right)' =$$

$$= \left(-\frac{1}{2}\right) \cdot (2x - x^2)^{-\frac{3}{2}} \cdot (2 - 2x) \cdot (x-1)^{-2} +$$

$$+ (2x - x^2)^{-\frac{1}{2}} \cdot -2 (x-1)^{-3} \cdot 1 =$$

$$= \frac{(-1 + x)}{(2x - x^2)^{\frac{3}{2}} \cdot (x-1)^2} - \frac{2}{(2x - x^2)^{\frac{1}{2}} (x-1)^3} =$$

$$= \frac{(-1+x) \cdot (x-1) - 2 \cdot (2x - x^2)}{(2x - x^2)^{\frac{3}{2}} \cdot (x-1)^3} =$$

$$= \frac{-2x+1+x^2 - 4x + 2x^2}{(2x-x^2)^{\frac{2}{3}} \cdot (x-1)^3} = \frac{3x^2-6x+1}{(2x-x^2)^{\frac{2}{3}} \cdot (x-1)^3}$$

$$f''(x) \geq 0 \rightarrow \frac{3x^2-6x+1}{(2x-x^2)^{\frac{2}{3}} \cdot (x-1)^3} \geq 0$$

N: $3x^2-6x+1 \geq 0 \rightarrow x \in \left(-\infty, \frac{3-\sqrt{6}}{3}\right] \cup \left[\frac{3+\sqrt{6}}{3}, +\infty\right)$

D_1: $(2x-x^2)^{\frac{3}{2}} > 0 \rightarrow 2x-x^2 > 0 \rightarrow x \in (0, 2)$

D_2: $(x-1)^3 > 0 \rightarrow x-1 > 0 \rightarrow x > 1$

		0		$\frac{3-\sqrt{6}}{3}$		1		$\frac{3+\sqrt{6}}{3}$		2	
N	$+$		$+$	0	$-$		$-$	0	$+$		$+$
D_1	$-$	0	$+$		$+$		$+$		$+$		$-$
D_2	$-$		$-$		$-$	0	$+$		$+$		$+$
$\frac{N}{D}$	$+$	0	$-$	0	$+$	0	$-$	0	$+$	0	$-$
		$\cap$	$\uparrow$		$\cup$		$\cap$	$\uparrow$		$\cup$	

PUNTO DI FLESSO

PUNTO DI FLESSO

La funzione è convessa in $\left[\frac{3-\sqrt{6}}{3}, 1\right) \cup \left[\frac{3+\sqrt{6}}{3}, 2\right]$ è concava in $\left(0, \frac{3-\sqrt{6}}{3}\right] \cup \left(1, \frac{3+\sqrt{6}}{3}\right]$.

$x = \frac{3-\sqrt{6}}{3}$ e $x = \frac{3+\sqrt{6}}{3}$ sono punti di flesso

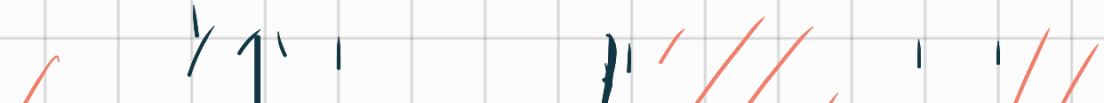
Minimi e massimi

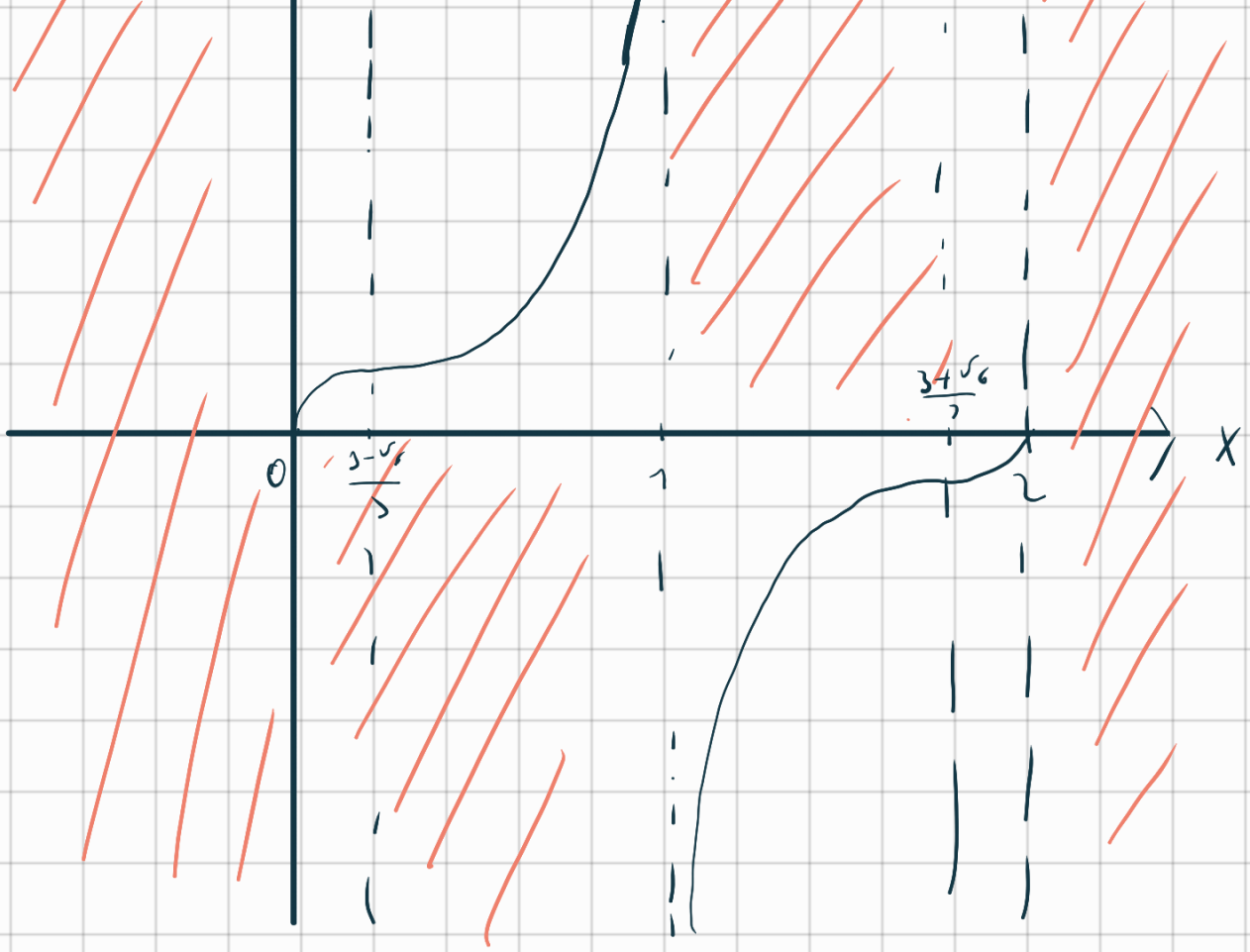
$x = 2$ è un punto di massimo relativo

$x = 0$ è un punto di minimo relativo

Non ci sono punti di minimo o massimo assoluto

Grafico





b) $f(x) = \dfrac{|x^2-4|-4}{(x-2)^2}$

Dominio

$D: \mathbb{R} \setminus \{2\}$

Sinal

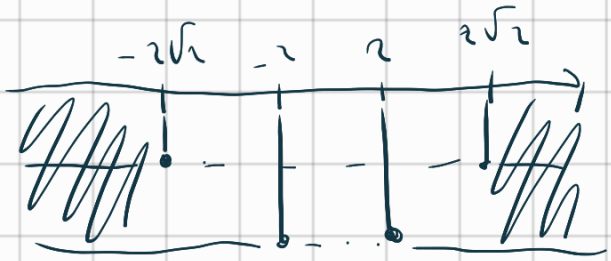
$f(x) \geq 0 \rightarrow \dfrac{|x^2-4|-4}{(x-2)^2} \geq 0$

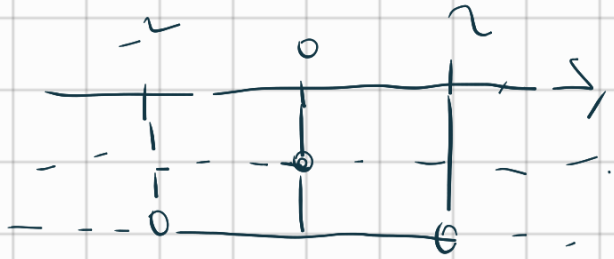
$r: |x^2-4| - 4 \geq 0 \rightarrow$

$$\rightarrow \begin{cases} x^2-4-4 \geq 0 \\ x^2-4 \geq 0 \end{cases} \cup \begin{cases} -x^2+4-4 \geq 0 \\ x^2-4 < 0 \end{cases}$$

$$\begin{cases} x^2 \geq 8 \\ x^2 \geq 4 \end{cases} \cup \begin{cases} x^2 \leq 0 \\ x^2 < 4 \end{cases}$$

$$\begin{cases} x \in (-\infty, -2\sqrt{2}] \cup [2\sqrt{2}, +\infty) \\ x \in (-\infty, -2] \cup [2, +\infty) \end{cases} \cup \begin{cases} x = 0 \\ x \in (-2, 2) \end{cases}$$

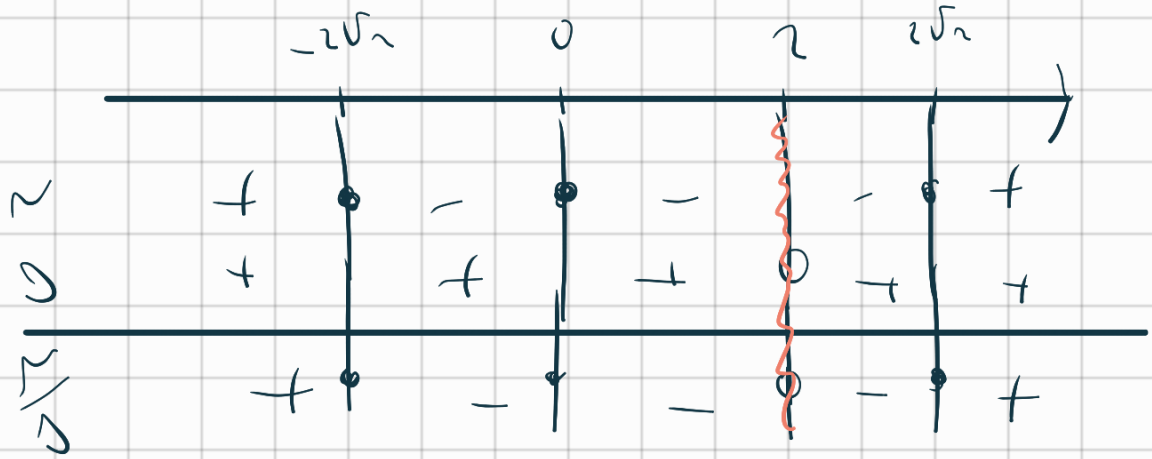




$$\begin{cases} x \in (-\infty, -2\sqrt{2}] \cup [2\sqrt{2}+8) \\ x = 0 \end{cases}$$

$r: x \in (-\infty, -2\sqrt{2}] \cup [2\sqrt{2}+8) \cup \{0\}$

$D: (x-2)^2 > 0 \rightarrow \forall x \in \mathbb{R} \setminus \{2\}$



f è positiva in $(-\infty, -2\sqrt{2}) \cup (2\sqrt{2}, +\infty)$

e si annulla in $x = \pm 2\sqrt{2}$ e $x = 0$

ASINTOTI

$$\lim_{x \to 2^{\pm}} \frac{|x^2-4|-4}{(x-2)^2} = \frac{-4}{0^{-}} = -\infty$$

$$\lim_{x \to +\infty} \frac{|x^2-4|-4}{(x-2)^2} = \lim_{x \to +\infty} \frac{x^2-8}{x^2+4-4x} =$$

$$= \lim_{|x| \to +\infty} \frac{\cancel{x^2}\overbrace{\left(1-\frac{8}{x^2}\right)}^{\to 1}}{\cancel{x^2}\underbrace{\left(1+\frac{4}{x^2}-\frac{4}{x}\right)}_{\to 1}} = 1$$

QUINDI L'ASINTOTO A $\pm\infty$ È $y=1$

MONOTONIA

$$f'(x) = \frac{(|x^2-8|)' \cdot (x-2)^2 - (|x^2-8|-8) \cdot 2 \cdot (x-2)}{(x-2)^2}$$

$x \geq 0$

$x < 0$

$$\begin{cases} \dfrac{(2x)\cdot(x-2)^2 - (x^2-8)\cdot 2\cdot(x-2)^1}{(x-2)^4} & \text{SE } x \in (-\infty;2) \cup (2,+\infty) \\[2ex] \dfrac{-2x\cdot(x-2)^2 - (-x^2)\cdot 2\cdot(x-2)^1}{(x-2)^4} & \text{SE } x \in (-2,2) \end{cases}$$

$$\begin{cases} \dfrac{\cancel{(x-2)}\cdot\left[2x\cdot(x-2) - 2(x^2-8)\right]}{(x-2)^{\cancel{4}\,3}} & \text{SE } x \in (-\infty,-2) \cup (2,+\infty) \\[2ex] \dfrac{\cancel{(x-2)}\cdot\left[-2x\cdot(x-2) + 2x^2\right]}{(x-2)^{\cancel{4}\,3}} & \text{SE } x \in (-2,2) \end{cases}$$

$$\begin{cases} \dfrac{\cancel{2x^2} - 4x - \cancel{2x^2} + 16}{(x-2)^3} & \text{si } x \in (-\infty, -2) \cup (2, +\infty) \\[2ex] \dfrac{\cancel{-2x^2} + 4x + \cancel{2x^2}}{(x-2)^3} & \text{si } x \in (-2, 2) \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{-4(x-4)}{(x-2)^3} & \text{si } x \in (-\infty, -2) \cup (2, +\infty) \\[2ex] \dfrac{4x}{(x-2)^3} & \text{si } x \in (-2, 2) \end{cases}$$

$$\frac{-4(x-4)}{(x-2)^3} \geq 0$$

N: $-4(x-4) \geq 0 \rightarrow 4(x-4) \leq 0 \rightarrow x \leq 4$

D: $(x-2)^3 > 0 \rightarrow x > 2$

2 4

N	+		+		−
D	−	0	+		+
N/D	−	0	+	0	−

$[2, 4]$

$$\frac{4x}{(x-2)^2} \geq 0 \rightarrow N: x \geq 0$$

$$D: x > 2$$

		0		2	
N:	−	0	+		+
D:	−		−	0	+
N/D:	+	0	−	0	+

$(-\infty, 0] \cup (2, +\infty)$

		-2		0		2		4
$f'(x)$	−	0	+	0	−	0	+	0

$f(x)$

f è crescente in $[-2, 0)$ e $(2, 4]$

è decrescente in $(-\infty, -2)$, in $(0, 2)$ e in $[4, +\infty)$

Convessità - concavità

$$f''(x) = \begin{cases} -\dfrac{4(x-4)}{(x-2)^3} & \text{se } x \in (-\infty, -2) \cup (2, +\infty) \\[2ex] \dfrac{4x}{(x-2)^3} & \text{se } x \in (-2, 2) \end{cases}$$

$$\frac{-4 \cdot (x-2)^3 - (-4(x-4)) \cdot 3(x-2)^2}{(x-2)^6} =$$

$$= \frac{-4(x-2)^3 + (12x - 48)(x-2)^2}{(x-2)^6} =$$

$$= \frac{\cancel{(x-2)} \cdot \left(-4(x-2) + (12x-48)\right)}{(x-2)^{\cancel{6}\,4}} =$$

$$= \frac{-4x + 8 + 12x - 48}{(x-2)^4} = \frac{8x - 40}{(x-2)^4} = \frac{8(x-5)}{(x-2)^4}$$

$$\frac{4 \cdot (x-2)^3 - 4x \cdot 3(x-2)^2}{(x-2)^6} =$$

$$= \frac{\cancel{(x-2)^2}\left(4x - 8 - 12x\right)}{(x-2)^{\cancel{6}\,4}} =$$

$$= \frac{-8x - 8}{(x-2)^4} = -\frac{8(x+1)}{(x-2)^4}$$

סוף מן

$$f''(x) = \begin{cases} \dfrac{8(x-5)}{(x-2)^4} & \text{si } x \in (-\infty, -2) \cup (2, +\infty) \\[2ex] -\dfrac{8(x+1)}{(x-2)^4} & \text{si } x \in (-2, 2) \end{cases}$$

$$f''(x) \geq 0$$

$$\frac{8(x-5)}{(x-2)^4} \geq 0 \rightarrow N: x \geq 5$$

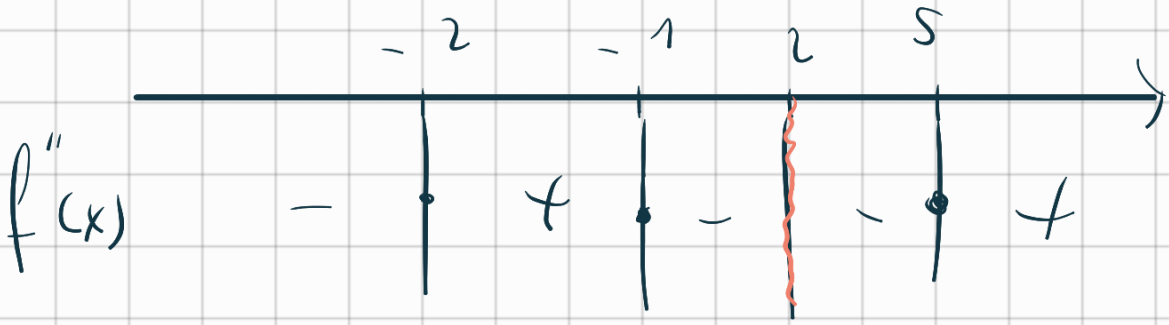
$$D: \forall x \in \mathbb{R} \setminus \{2\}$$

$$x \in [5, +\infty)$$

$$-\frac{8(x+1)}{(x-2)^4} \geq 0 \qquad N: -(x+1) \geq 0 \Rightarrow x \leq -1$$

$$D: \forall x \in \mathbb{R} \setminus \{2\}$$

$$x \in (-\infty, -1]$$



f è convessa in $[-2, -1]$ e in $[5, +\infty)$

ed è concava in $(-\infty, -2)$, in $[-1, 2)$ e in

$(2, 5]$

PUNTI MAX, MIN, FLESSO

$x=-2$, $x=-1$, $x=5$ sono punti di flesso

$$f(4) = \frac{16-4-4}{2^2} = 2$$

$$f(-2) = \frac{|-2-4|-4}{(-4)^2} = \frac{2}{16} = \frac{1}{8}$$

$$f(0) = \frac{0}{(-2)^2} = 0$$

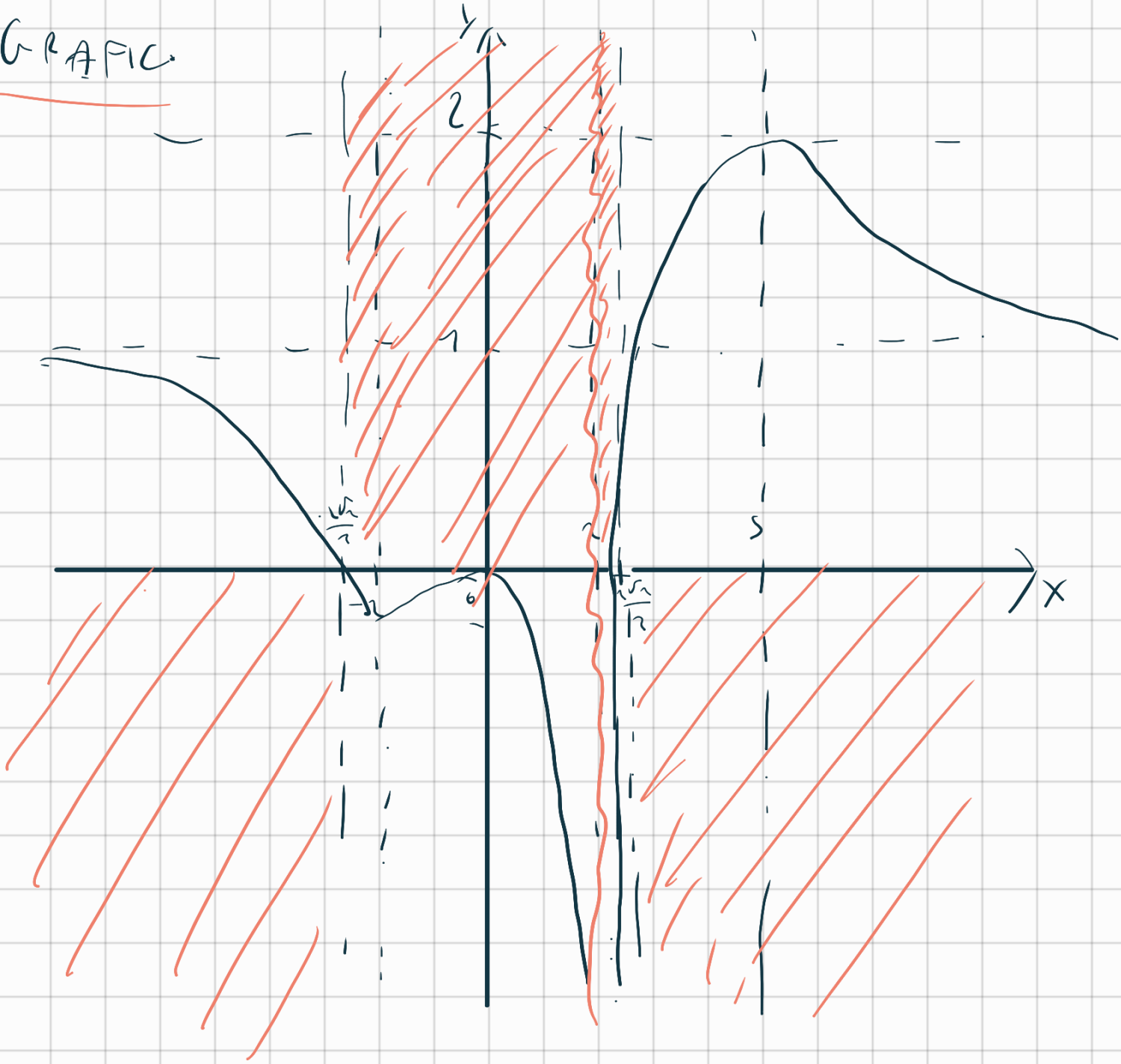
$x=4$ è il massimo assoluto

$x=-2$ è il massimo relativo

$x=0$ è minimo relativo

Poiché $f(2^-) = -\infty$ non ci sono punti di minimo assoluto

GRAFICO



2)

a) $f(x) = \log\left(\frac{x}{2-x^2}\right)$

$y_0 = 1$

$m = 3$

$\log\left(\frac{x}{2-x^2}\right) = \log(x) - \log(2-x^2)$

$$\log(2-x^2) = \log(x) - \log(2-x^2) =$$

pour $x = t+1$

$$= \log(1+t) - \log\left(2-(t+1)^2\right) =$$

$$= \log(1+t) - \log(2-t^2-1-2t) =$$

$$= \log(1+t) - \log(-t^2-2t+1) =$$

$$= \log(1+t) - \log(1-(2t+t^2)) =$$

$$= t - \frac{t^2}{2} + \frac{t^3}{3} + (2t+t^2) + \frac{(2t+t^2)^2}{2} + \frac{(2t+t^2)^3}{3} + o(t^3)$$

$$= (1+2)t + \left(-\frac{1}{2}+1+2\right)t^2 + t^3\left(\frac{1}{3}+2+\frac{8}{3}\right)t^3 + o(t^3)$$

$$= 3t + \frac{5t^2}{2} + 5t^3 + o(t^3)$$

donc

$$T_3 = 3(x-1) + \frac{5(x-1)^2}{2} + 5(x-1)^3$$

c) $f(x) = \dfrac{\log(\cos(x))}{1+\sin^2(x)}$ $\qquad x_0 = 0$

$n = 5$

DATO CHE LA FUNZIONE È PARI,

I COEFFICIENTI DELLE POTENZE DISPARI DI X

SONO ZERO, QUINDI BASTA SVILUPPARE FINO A $n=4$

$$\frac{\log(\cos(x))}{1+\sin^2(x)} = \frac{\log\left(1+\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)+o(x^4)\right)}{1+(x+o(x))^2} =$$

$$= \frac{\left(-\frac{x^2}{2}+\frac{x^4}{24}+o(x^4)\right) - \frac{\left(-\frac{x^2}{2}+o(x^2)\right)^2}{2} + o(x^4)}{1+x^2+o(x^2)} =$$

$$= \frac{\left(-\frac{x^2}{2}+\frac{x^4}{24}+o(x^4)\right) - \left(\frac{\frac{x^4}{4}+o(x^4)}{2}\right) + o(x^4)}{1+x^2+o(x^2)} =$$

$$= -\frac{x^2}{2}+\frac{x^4}{24}+o(x^4) - \frac{x^4}{8}+o(x^4) + o(x^4)$$

$$1 + x^2 + o(x^2)$$

$$= \frac{-\frac{x^2}{2} + \left(\frac{1}{24} - \frac{1}{8}\right)x^4 + o(x^4)}{1 + x^2 + o(x^2)}$$

$$= \frac{-\frac{x^2}{2} - \frac{1}{12}x^4 + o(x^4)}{1 + x^2 + o(x^2)}$$

$$\frac{1-3}{24} = -\frac{2}{\cdot}$$

e) $f(x) = x^x$

$x_0 = 1$

$\sim = 3$

$$x^x = e^{x\log(x)}$$

Pin : $x = t + 1$

$$e^{x\log(x)} = e^{(t+1)\cdot\log(1+t)} = \exp\left((1+t)\cdot\log(1+t)\right) =$$

$$= \exp\left((1+t)\left(t - \frac{t^2}{2} + \frac{t^3}{3} + o(t^3)\right)\right)$$

$$= \exp\left(t - \frac{t^2}{2} + \frac{t^3}{3} + t^2 - \frac{t^3}{2} + \frac{t^4}{3} + o(t^4)\right)$$

$$= \exp\left(t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{3} + o(t^4)\right)$$

$$= 1 + \left(t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{3} + o(t^4)\right) +$$

$$+ \frac{\left(t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{3} + o(t^4)\right)^2}{2} +$$

$$+ \frac{\left(t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{3} + o(t^4)\right)^3}{6} =$$

$$= 1 + t + \left(\frac{1}{2} + \frac{1}{2}\right)t^2 + \left(-\frac{1}{6} + \frac{1}{2} + \frac{1}{6}\right)t^3 + o(t^3)$$

$$= 1 + t + t^2 + \frac{t^3}{2} + o(t^3)$$

Quindi

$$T_3(x) = 1 + (x-1) + (x-1)^2 + \frac{(x-1)^3}{2}$$

---

3) a) $\lim$ $\quad x \cos(\sin(x)) - x^2$

$x \geq 0$ $\qquad \sqrt{1+3x^4} - 1$

Per $x \to 0$

$\sin(x) = x + \frac{x^3}{6} + o(x^3)$ e.

$\sqrt{1+3x^4} = 1 + \frac{1}{2} \cdot 3x^4 + o(x^4) \to (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + o(x^2)$

Quindi

$$\lim_{x \to 0} \frac{x \cdot \left(\cancel{x} + \frac{x^3}{6} + o(x^3)\right) \cancel{-x^2}}{\cancel{1} + \frac{3x^4}{2} + o(x^4) \cancel{-1}} =$$

$$= \lim_{x \to 0} \frac{\frac{x^4}{6} + o(x^4)}{3\frac{x^4}{2} + o(x^4)} = \frac{2x^4 + o(x^4)}{3 \cdot 6 \cdot 3x^4 + o(x^4)} \to \frac{1}{9}$$

b) $$\lim_{x \to 1} \left(\frac{x^x - x}{\log^2(x)} - \frac{1}{\log(x)} \right)$$

Per $x \to 1$, $t = x - 1 \to 0$

Quindi

$\log(x) = \log(1+t) = t - \frac{t^2}{2} + o(t^2)$

DALL' ES 2.i

$$x^x = 1 + t + t^2 + \frac{t^3}{2} + o(t^3)$$

QUINDI

$$\frac{x^x - x}{\log^3(x)} - \frac{1}{\log(x)} = \frac{x^x - x - \log^2(x)}{\log^3(x)} =$$

$t-1$

$$= \frac{1 + t + t^2 + \frac{t^3}{2} + o(t^3) - (t+1) - \left(\overset{a}{t} - \overset{b}{\frac{t^2}{2}} + o(t^2)\right)^2}{\left(t - \frac{t}{2} + o(t^2)\right)^3} =$$

$$-2 \cdot t \cdot \frac{t^2}{2}$$

$$= \frac{\cancel{1} + \cancel{t} + \cancel{t^2} + \frac{t^3}{2} + o(t^3) - \cancel{t} - \cancel{1} - \cancel{t^2} + t^3 + o(t^3)}{t^3 + o(t^3)} =$$

$$= \frac{+\left(\frac{1}{2} + 1\right)t^3 + o(t^3)}{t^3 + o(t^3)} =$$

$$= \frac{3}{\ } t^3 + o(t^3) = \frac{3t^3 + o(t^3)}{\ } \frac{3}{\ }$$

$$\frac{\quad}{t^3+o(t^3)} \qquad \frac{\quad}{2t^3+o(t^3)} \to \frac{1}{2}$$