ANALISI MATEMATICA 1 - LEZIONE 23

ESEMPI

- $\displaystyle\int \frac{1}{x(x^2+2x+2)}\,dx = ?$ $\quad \Delta<0$

Sia

$$f(x)=\frac{1}{x(x^2+2x+2)}=\frac{A}{x}+\frac{Bx+C}{x^2+2x+2}$$

allora

$$A(x^2+2x+2)+x(Bx+C)=1 \iff \begin{cases} A+B=0 & x^2 \\ 2A+C=0 & x^1 \\ 2A=1 & x^0 \end{cases}$$

da cui $A=\frac{1}{2}$, $B=-\frac{1}{2}$, $C=-1$. Così

$$\int \frac{1}{x(x^2+2x+2)}\,dx=\int\left(\frac{1/2}{x}-\frac{1/2\,x+1}{x^2+2x+2}\right)dx$$

$$\overset{x=t-1}{=}\frac{1}{2}\log|x|-\int\frac{1/2(t-1)+1}{(t-1)^2+2(t-1)+2}\,dt$$

$$t^2-\cancel{2t}+1+\cancel{2t}-\cancel{2}+\cancel{2}$$

$$=\frac{1}{2}\log|x|-\frac{1}{2}\int\frac{t+1}{t^2+1}\,dt$$

$$=\frac{1}{2}\log|x|-\frac{1}{2}\int\frac{t}{t^2+1}\,dt-\frac{1}{2}\int\frac{1}{t^2+1}\,dt$$

$$=\frac{1}{2}\log|x|-\frac{1}{4}\int\frac{1}{t^2+1}\,d(t^2+1)-\frac{1}{2}\int\frac{1}{t^2+1}\,dt$$

$$=\frac{1}{2}\log|x|-\frac{1}{4}\log(t^2+1)-\frac{1}{2}\operatorname{arctg}(t)+c$$

$$=\frac{1}{2}\log|x|-\frac{1}{4}\log(x^2+2x+2)-\frac{1}{2}\operatorname{arctg}(x+1)+c$$

$1\,(\log|t-1|-\log|t+1|)$ $\qquad t^2+1=0 \to t^2=-1$

$t = \pm\sqrt{1} \Rightarrow t = \pm 1$

$(t+1)(t-1)$

- $$\int \frac{1}{\sin(x)}\,dx = \int \frac{\sin(x)}{1-\cos^2(x)}\,dx$$

funz. razionale

$t=\cos(x)$
$dt=\sin(x)\,dx$

$$= \int \frac{-dt}{1-t^2} = \int\left(\frac{\overset{=1/2}{A}}{t-1} + \frac{\overset{=-1/2}{B}}{t+1}\right)dt$$

$$\boxed{= \frac{1}{2}\log|t-1| - \frac{1}{2}\log|t+1| + c}$$

$$= \frac{1}{2}\log\left(\frac{1-\cos(x)}{1+\cos(x)}\right) + c.$$

- $$\int \frac{1}{x\log^2(x)(\log^2(x)+1)}\,dx$$

$t=\log(x)$
$dt=\frac{dx}{x}$

$$= \int \frac{1}{t^2(t^2+1)}\,dt = \int\left(\frac{\overset{=0}{A}}{t} + \frac{\overset{=1}{B}}{t^2} + \frac{\overset{=0}{C}t+\overset{=-1}{D}}{t^2+1}\right)dt$$

$$= -\frac{1}{t} - \operatorname{arctg}(t) + c = -\frac{1}{\log(x)} - \operatorname{arctg}(\log(x)) + c$$

- $$\int \frac{e^{2x}+2e^{-x}}{1+e^{-x}}\,dx$$

$t=e^x$
$x=\log(t)$
$dx=\frac{dt}{t}$

$$= \int \frac{t^2+2t^{-1}}{1+t^{-1}}\cdot\frac{dt}{t} = \int \frac{t^3+2}{t(t+1)}\,dt$$

$$= \int\left(t-1+\frac{\overset{=2}{A}}{t} + \frac{\overset{=-1}{B}}{t+1}\right)dt$$

$$= \frac{t^2}{2} - t + 2\log|t| - \log|t+1| + c$$

$$= \frac{e^{2x}}{2} - e^x + 2x - \log(e^x+1) + c$$

$$\frac{1+t+t^2-1}{1+t+t^2} = \frac{t+t^2}{1+t+t^2} = \frac{t(1+t)}{1+t+t^2} = t\cdot\frac{1+t}{1+t+t^2}$$

- $\displaystyle\int \frac{1+\sqrt{x}}{1+\sqrt{x}+x}\,dx$

$t=\sqrt{x}$
$t^2=x$
$2t\,dt=dx$

$$= \int \frac{1+t}{1+t+t^2}\,2t\,dt$$

$$= 2\int\left(1-\frac{1}{1+t+t^2}\right)dt$$

$t=s-\frac{1}{2}$
$dt=ds$

$$= 2t - 2\int \frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,ds$$

$$= 2t - 2\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{s}{\sqrt{3}/2}\right)+c$$

$$= 2\sqrt{x} - \frac{4}{\sqrt{3}}\operatorname{arctg}\left(\frac{2\sqrt{x}+1}{\sqrt{3}}\right)+c$$

- $\displaystyle\int \frac{1}{\sqrt{1+e^x}}\,dx$

$t=e^x,\ dx=\frac{dt}{t}$

$x=\log(t)$

$$= \int \frac{1}{\sqrt{1+t}}\,\frac{dt}{t}$$

$s=\sqrt{1+t}$
$s^2=1+t$
$2s\,ds=dt$

$$\frac{2}{(s+1)(s-1)}\,ds$$

$$= \int \frac{1}{\cancel{s}}\cdot\frac{2\cancel{s}\,ds}{s^2-1} = \int\left(\frac{1}{s-1}-\frac{1}{s+1}\right)ds$$

$$= \log|s-1| - \log|s+1| + c$$

$$= \log\left(\frac{|s-1|}{|s+1|}\right)+c$$

$$= \log\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right)+c$$

OSSERVAZIONE

Se g' è continua in $[a,b]$ e f è continua in $g([a,b])$ allora

$$\int_a^b f(g(x))g'(x)\,dx \overset{t=g(x)}{=} \int_{g(a)}^{g(b)} f(t)\,dt.$$

$\boxed{\text{ESEMPI}}$

- $$\int_0^{\pi/8} \operatorname{sen}^2(x)\,dx = \frac{1}{2}\int_0^{\pi/8}(1-\cos(2x))\,dx \overset{t=2x,\ dt=2dx}{=} \frac{1}{2}\int_0^{\pi/4}(1-\cos(t))\frac{dt}{2}$$

$$= \frac{1}{4}\Big[t-\operatorname{sen}(t)\Big]_0^{\pi/4} = \frac{\pi}{16} - \frac{\sqrt{2}}{8}$$

Perché $\cos(2x) = \cos^2(x) - \operatorname{sen}^2(x) = 1 - 2\operatorname{sen}^2(x)$ implica $\operatorname{sen}^2(x) = \frac{1}{2}(1-\cos(2x))$.

- $$\int_0^8 \frac{1}{1+\sqrt[3]{x}}\,dx = \int_0^2 \frac{1}{1+t}3t^2\,dt$$

$t=\sqrt[3]{x}$, $t^3=x$, $3t^2\,dt$, $\sqrt[3]{0}=0$, $\sqrt[3]{8}=2$

$$= 3\int_0^2 \left(t-1+\frac{1}{1+t}\right)dt = 3\left[\frac{t^2}{2} - t + \log|1+t|\right]_0^2$$

$$= 3\log(3)$$

$$\bullet \int_{\frac{\pi}{2}}^{\pi} \underbrace{\operatorname{sen}(2x)}_{2\operatorname{sen}(x)\cos(x)} \log(1-\cos(x))\,dx = -2\int_{0}^{-1} t\log(1-t)\,dt$$

$t=\cos(x)$, $dt=-\operatorname{sen}(x)\,dx$, $\cos(\frac{\pi}{2})=0$, $\cos(\pi)=-1$

$$= \int_{-1}^{0} \log(1-t)\,d(t^2) = \Big[t^2\log(1-t)\Big]_{-1}^{0} - \int_{-1}^{0} -\frac{t^2}{1-t}\,dt$$

$$= -\log(2) - \int_{-1}^{0}\left(t+1+\frac{1}{t-1}\right)dt$$

$$= -\log(2) - \left[\frac{t^2}{2}+t+\log|t-1|\right]_{-1}^{0}$$

$$= -\cancel{\log(2)} + \frac{1}{2} - 1 + \cancel{\log(2)} = -\frac{1}{2}.$$

$$\bullet \int_{0}^{1} \sqrt{x}\,\operatorname{arcsen}(2x-1)\,dx = \int_{0}^{1} \operatorname{arcsen}(2x-1)\,d\left(\tfrac{2}{3}x^{3/2}\right)$$

$$= \frac{2}{3}\Big[x^{3/2}\operatorname{arcsen}(2x-1)\Big]_{0}^{1} - \frac{2}{3}\int_{0}^{1} x^{3/2}\frac{2}{\sqrt{1-(2x-1)^2}}\,dx$$

$\cancel{1}-4x^2+4x\cancel{+1}=4x(1-x)$

$$= \frac{\pi}{3} - \frac{2}{3}\int_{0}^{1}\frac{x}{\sqrt{1-x}}\,dx$$

$t=\sqrt{1-x}$

$t^2=1-x$

$2t\,dt=-dx$

$\sqrt{1-0}=1$, $\sqrt{1-1}=0$

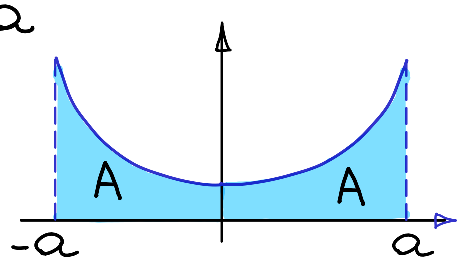
$$= \frac{\pi}{3} - \frac{2}{3}\int_{1}^{0}\frac{1-t^2}{\cancel{t}}(-2\cancel{t}\,dt)$$

$$= \frac{\pi}{3} - \frac{4}{3}\left[t-\frac{t^3}{3}\right]_{0}^{1} = \frac{\pi}{3} - \frac{8}{9}$$

OSSERVAZIONE

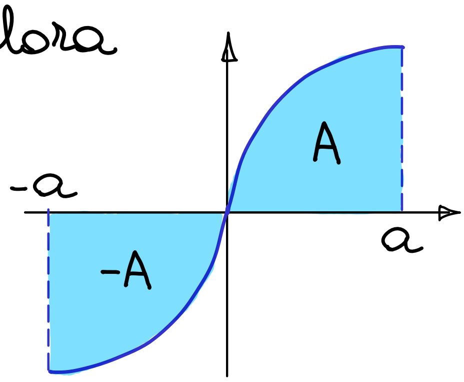
Se f è pari in $[-a,a]$ allora

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$



Se f è dispari in $[-a,a]$ allora

$$\int_{-a}^{a} f(x)\,dx = 0$$



- $$\int_{-1}^{1} e^{-x^2}(|x|+\operatorname{sen}(x))\,dx = \int_{-1}^{1} \underbrace{e^{-x^2}|x|}_{\text{pari}}\,dx + \int_{-1}^{1} \underbrace{e^{-x^2}\operatorname{sen}(x)}_{\text{dispari}}\,dx$$

$$= 2\int_{0}^{1} e^{-x^2}\underbrace{|x|}_{x}\,dx + 0 = \left[-e^{-x^2}\right]_0^1 = 1 - e^{-1}.$$

- $$\int_{-3}^{3} \lfloor x \rfloor\,dx = (1+2)-(1+2+3) = -3$$