5)

a) $(3+i)z = 2-4i$

$$z = \frac{2-4i}{3+i} = 2-4i \cdot \frac{3-i}{|3+i|^2} = \frac{(2-4i)(3-i)}{9+1} =$$

$4 \cdot (-1) = -4$

$$= \frac{6-2i-12i+4i^2}{10} = \frac{4i^2-14i+6}{10} = \frac{-14i+2}{10} =$$

$$= \frac{1}{5} - \frac{7}{5}i$$

b) $(2-i)\bar{z} - 5 = (1+2i)^3$

$$(2-i)\bar{z} = (1+2i)^3 + 5$$

$1^3 \quad 3 \cdot 1^2 \cdot 2i \quad 3 \cdot 1 \cdot (2i)^2 \quad (2i)^3$

$$\bar{z} = \frac{(1+2i)^3+5}{2-i} = \frac{1+6i+12i^2+8i^3+5}{2-i} =$$

$$= \frac{1+6i-12-8i+5}{2-i} = \frac{-2i-6}{2-i} = -\frac{2i+6}{2-i} =$$

$$= -2\,\frac{i+3}{2-i} = -2\left(i+3 \cdot \frac{2+i}{|2-i|^2}\right) = -2\left(\frac{(i+3)(2+i)}{4+1}\right) =$$

$$= -2\left(\frac{2i+i^2+6+3i}{5}\right) = -2\left(\frac{5i+5}{5}\right) = -2i-2$$

alors $z = \overline{(\overline{z})} = \overline{-2-2i} = -2+2i$

c) $2z(z+1) = -|3-4i|$

$$2z^2+2z+|3-4i| = 0$$

$$2z^2+2z+\sqrt{3^2+(-4)^2} = 0$$

$$2z^2+2z+\sqrt{9+16} = 0$$

$$2z^2+2z+5 = 0$$

$$z_{1-2} = \frac{-2 \pm \sqrt{4-4(2)(5)}}{4} = \frac{-2 \pm \sqrt{4-40}}{4} = \frac{-2 \pm \sqrt{-36}}{4} =$$

$$= \frac{-2+6i}{4} = \frac{-1+3i}{2} = -\frac{1}{2}+\frac{3i}{2} = z_1$$

$$\frac{-2-6i}{4} = \frac{-1-3i}{2} = -\frac{1}{2} - \frac{3}{2}i = z_2$$

d) $$z^2(z^2+13) = -36$$

$$z^4 + 13z^2 + 36 = 0$$

PONENDO $w = z^2$ ABBIAMO

$$w^2 + 13w + 36 = 0$$

$$w_{1-2} = \frac{-13 \pm \sqrt{169-4(36)}}{2} = \frac{-13 \pm \sqrt{169-144}}{2} =$$

$$= \frac{-13 \pm 5}{2} = \begin{cases} -4 \\ -9 \end{cases}$$

QUINDI

$$z^2 = -4 \quad e \quad z^2 = -9$$

$$z_{1-2} = \pm\sqrt{-4} = \begin{cases} +2i = z_1 \\ -2i = z_2 \end{cases}$$

$$z_{3,4} = \pm\sqrt{-9} = \begin{cases} 3i = z_3 \\ -3i = z_4 \end{cases}$$

e)

$$||z| - 3i|^2 = 4$$

$$|z|^2 + (-3)^2 = 4$$

$$|z|^2 = -5 \qquad \nexists z \in \mathbb{C}$$

$$\boxed{|z|^2 \geq 0}$$

f)

$$(1+i)^2\left((z+4i)^2 - i\right) = 6$$

$$(\cancel{1} + 2i \cancel{-1})(z^2 + 8zi - 16 - i) = 6$$

$$2iz^2 - 16z - 32i + 2 - 6 = 0$$

$$2iz^2 - 16z + (-32i - 4) = 0$$

$$z_{1,2} = \frac{16 \pm \sqrt{256 - 4(2i)(-32i-4)}}{4i} =$$

$$= \frac{16 \pm \sqrt{256 - 8i(-32i-4)}}{4i} = \frac{16 \pm \sqrt{\cancel{256} - \cancel{256} + 32i}}{4i} =$$

$$= \frac{16 \pm (4+4i)}{4i} =$$

$$z_1 = \frac{16 + 4 + 4i}{4i} = \frac{20+4i}{4i} = \frac{5}{i} + 1 = 1 - 5i$$

$$z_2 = \frac{16 - 4 - 4i}{4i} = \frac{12 - 4i}{4i} = \frac{3}{i} - 1 = -1 - 3i$$

g) $(z+3)^3 = 64$

SIA $w = z+3$, ALLORA

$$w^3 = 64 = \underset{|z|}{2^6} \cdot e^{i0}$$

$$w_k = 2^{6/3} \cdot e^{i\frac{(0+2k\pi)}{3}} \quad \text{PER } k = 0, 1, 2$$

$$w_0 = 4 \cdot e^{i \cdot \frac{0}{3}} = 4$$

$$w_1 = 4 \cdot e^{i\left(\frac{2\pi}{3}\right)} = 4 \cdot \left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right) =$$

$$= 4\left(-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right) = \cancel{4}\left(\frac{-1+i\sqrt{3}}{\cancel{2}}\right) =$$

$$= -2 + 2\sqrt{3}\,i$$

$$w_2 = 4 \cdot e^{i\left(\frac{4\pi}{3}\right)} = 4\left(\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)\right) =$$

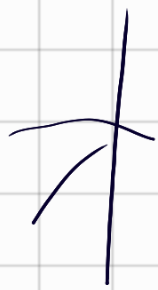
$$= 4\left(\left(-\frac{1}{2}\right) + i\left(-\frac{\sqrt{3}}{2}\right)\right) = -2 - 2\sqrt{3}\,i$$

b)

$$(z^4 + 16)(z^2 - 2z + 3 - 2i\sqrt{3})i = 0$$

PRIMO FATTORE

$$z^4 = -16 = 2^4 e^{i\pi}$$



QUINDI

$$z_k = 2^{e^{i\frac{(\pi + 2k\pi)}{4}}}$$

$$z_0 = 2e^{i\frac{\pi}{4}} = 2\left(\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right) = \cancel{2}\left(\frac{\sqrt{2}}{\cancel{2}} + j\frac{\sqrt{2}}{\cancel{2}}\right) =$$

$$= \sqrt{2} + j\sqrt{2} = \sqrt{2}(1+j)$$

$$z_1 = 2e^{i\frac{3\pi}{4}} = 2\left(\cos\left(\frac{3\pi}{4}\right) + j\sin\left(\frac{3\pi}{4}\right)\right) = \cancel{2}\left(-\frac{\sqrt{2}}{\cancel{2}} + j\left(\frac{\sqrt{2}}{\cancel{2}}\right)\right) =$$

$$= -\sqrt{2} + \sqrt{2}j = \sqrt{2}(-1+j)$$

$$z_2 = 2e^{i\frac{5\pi}{4}} = 2\left(\cos\left(\frac{5}{4}\pi\right) + j\sin\left(\frac{5}{4}\pi\right)\right) = \cancel{2}\left(-\frac{\sqrt{2}}{\cancel{2}} + j\left(-\frac{\sqrt{2}}{\cancel{2}}\right)\right) =$$

$$= -\sqrt{2} - j\sqrt{2} = \sqrt{2}(-1-j$$

$$z_3 = 2e^{i\frac{7\pi}{4}} = 2\left(\cos\left(\frac{7\pi}{4}\right) + j\sin\left(\frac{7\pi}{4}\right)\right) = \cancel{2}\left(\frac{\sqrt{2}}{\cancel{2}} + j\left(-\frac{\sqrt{2}}{\cancel{2}}\right)\right) =$$

$$= \sqrt{2} - \sqrt{2}j = \sqrt{2}(1-j)$$

SECONDO FATTORE

$$z^2 - 2z + 3 - 2i\sqrt{3} = 0$$

$$\Delta = 4 - 4(3 - 2i\sqrt{3}) = 4 - 12 + 8i\sqrt{3} = -8 + 8i\sqrt{3} =$$

$$= 16e^{i\left(\frac{2\pi}{3}\right)}$$

LE CUI RADICI SON

$$4e^{i\frac{\pi}{3}} = 4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = 2 + 2i\sqrt{3}$$

$\cdot -4e^{i\frac{\pi}{3}} = -2-2i\sqrt{3}$

Quindi

$$z_4 = \frac{2+(2+2i\sqrt{3})}{2} = 2+i\sqrt{3}$$

$$z_5 = \frac{2-(2+2i\sqrt{3})}{2} = -i\sqrt{3}$$

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2)

a) $$\int_0^{\frac{\pi}{4}} \sin(2x)\, \lg^2(\sin^2(x))\, dx$$

$\sin(2x) = 2\sin(x)\cos(x)$.

$t = \sin^2(x)$, $dt = 2\sin(x)\cos(x)\,dx$

Quindi

$$\int_0^{\frac{\pi}{4}} \sin(2x)\, \lg^2(\sin^2(x))\, dx =$$

$$= \int^{\frac{1}{2}} \lg^2(t)\, dt = \Big[t \cdot \lg^2(t)\Big]^{\frac{1}{2}} - \int^{\frac{1}{2}} t \cdot d(\lg^2(t))$$

$$= \frac{1}{2}\left(-\log(2)\right)^2 - 2\int_0^{\frac{1}{2}} \log(t)\,dt =$$

$$= \frac{\log^2(2)}{2} - 2\left[t\log(t) - t\right]_0^{\frac{1}{2}}$$

$$= \frac{\log^2(2)}{2} - \cancel{2}\left(\frac{1}{\cancel{2}}\cdot\left(-\log(2)\right) - \frac{1}{\cancel{2}}\right) =$$

$$= \frac{\log^2(2)}{2} + \log(2) + 1$$

b) $$\int_1^{+\infty} \frac{\log(x^2-x)}{x^3}\,dx =$$

PER PARTI

$$= \left[-\frac{x^{-2}}{2}\cdot\log(x^2-x)\right]_1^{+\infty} + \int_1^{+\infty} \frac{\left(\log(x^2-x)\right)'}{2x^2}\,dx =$$

$$= \left[-\frac{x^2}{\dots}\cdot\log(x^2-x)\right]^{+\infty} + \int^{+\infty} \frac{x-\frac{1}{2}}{\dots}\,dx$$

$$\int_1 \quad \int_1 \quad x^2(x^2-x)$$

$$= \frac{x-\frac{1}{2}}{x^4-x^3} = \frac{x-\frac{1}{2}}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1}$$

$$D = \lim_{x\to 1} \frac{x-\frac{1}{2}}{x^3} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$C = \lim_{x\to 0} \frac{x-\frac{1}{2}}{x-1} = +\frac{1}{2}$$

$$A(x^2)(x-1) + Bx(x-1) + C(x-1) + D(x^3) =$$

$$= Ax^3 - Ax^2 + Bx^2 - Bx + Cx - C + Dx^3$$

$$(A+D)x^3 + (-A+B)x^2 + \cancel{(-B+C)x - C}$$

$$\begin{cases} A+D=0 \end{cases} \qquad \begin{cases} A=-D \end{cases} \qquad \begin{cases} A=-\frac{1}{2} \end{cases}$$

$$\begin{cases} -A+B=0 \end{cases} \quad \begin{cases} B=A \end{cases} \quad \begin{cases} B=-\frac{1}{2} \end{cases}$$

Dunque

$$\int \frac{x-\frac{1}{2}}{x^2(x^2-x)}\,dx = \frac{1}{2x} - \frac{1}{4x^2} + \frac{1}{2}\log\left|\frac{x-1}{x}\right| + c$$

Quindi

$$\int_1^{+\infty} \frac{\log(x^2-x)}{x^3} = \left[-\frac{\log(x^2-x)}{2x^2}\right]_1^{+\infty} + \left[\frac{1}{2x} - \frac{1}{4x^2} + \frac{1}{2}\log\left|\frac{x-1}{x}\right|\right]_1^{+\infty}$$

$$= -\frac{1}{2}\left(\lim_{x\to 1^+}\left(-\frac{\log(x^2-x)}{x^2} + \log\left(\frac{x-1}{x}\right)\right) + 1 - \frac{1}{2}\right) =$$

$$= -\frac{1}{2}\left(\lim_{x\to 1^+}\left(\frac{-\log(x^2-x) + x^2\log(x-1)}{x^2}\right) + \frac{1}{2}\right) =$$

$$= -\frac{1}{2}\left(0 + \frac{1}{2}\right) = -\frac{1}{4}$$

---

4) $$\sum^{\infty} \frac{\log(k^2+3k) - 2\log(k)}{(\log(\sqrt{\ldots}+1))^{\ldots}}$$

$$k=1 \qquad \left(\log\left(\sqrt{k+1}\right)\right)$$

PER $k \to +\infty$

$$\frac{\log(k^2+3k)-2\log(k)}{\left(\log\left(\sqrt{k+1}\right)\right)^{3\alpha}} = \frac{\log\left(1+\frac{3}{k}\right)+2\log(k)-2\log(k)}{\left(\log\left(1+\frac{1}{\sqrt{k}}\right)\right)^{3\alpha}} =$$

$$= \frac{\log\left(1+\frac{3}{k}\right)}{\left(\frac{1}{2}\log(k)\right)\left(\log\left(1+\frac{1}{\sqrt{k}}\right)\right)^{3\alpha}} \sim \frac{\frac{3}{k}}{\left(\log(k)\right)^{3\alpha}} = \frac{C}{k\log^{3\alpha}(k)}$$

COSÌ PER CONFRONTO ASINTOTICO, LA SERIE CONVERGE SOLO E SOLTANTO SE $3\alpha>1$ $(\alpha=1,\ \beta=3\alpha)$, QUINDI SE $\alpha>\frac{1}{3}$

b) $$\sum_{k=1}^{\infty}\frac{(4^k+k)\,a^{2k}}{3^k-1}$$

PER $k\to+\infty$

$$\frac{(4^k+k)\,a^{2k}}{3^k-1} \sim \frac{4^k\cdot a^{2k}}{3^k} = \left(\frac{4\cdot a^2}{3}\right)^k$$

Allora per confronto asintotico, la serie converge solo se

$$\sum_{k=1}^{\infty} \left(\frac{4a^2}{3}\right)^k$$ converge, ossia se $\left|\frac{4a^2}{3}\right| < 1$

$$|4a^2| < 3 \to |a^2| < \frac{3}{4} \to |a| < \frac{\sqrt{3}}{2}$$

quindi se $|a| < \frac{\sqrt{3}}{2}$

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3)

a) $$\int_1^{+\infty} \frac{\log^3(x)}{(x-1)^{a} \log^5(1+x^4)}\,dx$$

D

① $(x-1)^{a} \neq 0 \to x \neq 1$

② $\log^5(1+x^4) \neq 0$

$1+x^4 \neq 1 \to x^4 \neq 0 \to x \neq 0$

Sia

$$f(x) = \frac{\log^3(x)}{(x-1)^{a} \log^5(1+x^4)}$$

nell'intervallo $(1, +\infty)$ i punti da indagare sono 1^+ e $+\infty$

PER $x \to 1^+$, $t = x - 1 \to 0^+$

$$f(x) \sim \frac{\log^3(1+t)}{t^\alpha \log^5(2)} \sim \frac{t^3}{t^\alpha} = \frac{1}{t^{\alpha-3}}$$

COSÌ PER LA CONVERGENZA $\alpha - 3 < 1$ OSSIA $\alpha < 4$

PER $x \to +\infty$ ABBIAMO

$$f(x) \sim \frac{\log^3(x)}{x^\alpha \log^5(x^5)} = \frac{\cancel{\log^3}(x)}{x^\alpha \cdot x^5 \cdot \log^5(x)} = \frac{1}{x^{\alpha+5} \cdot \log^2(x)}$$

QUINDI PER LA CONVERGENZA, $\alpha + 5 \geq 1$

OSSIA $\alpha \geq -4$

QUINDI L'INTEGRALE IMPROPRIO È CONVERGENTE

SE E SOLO SE $-4 \leq \alpha < 4$

b) $$\int_0^{+\infty} \frac{\arctan(x^3)}{x^\alpha \log^2(1+x^3)}\,dx$$

SIA

$$f(x) = \frac{\arctan(x^{8})}{x^{a} \log^{2}(1+x^{2})}$$

NELL'INTERVALLO $(0, +\infty)$ I PUNTI DA CONTROLLARE SONO 0^{+} E $+\infty$

PER $x \to 0^{+}$

$$f(x) \sim \frac{x^{8}}{x^{a}\,(x^{2})^{2}} = \frac{x^{\cancel{8}\,2}}{x^{a}} = \frac{1}{x^{a-2}}$$

QUINDI PER LA CONVERGENZA $a - 2 < 1$ OSSIA $a < 3$

PER $x \to +\infty$

$$f(x) \sim \frac{\frac{\pi}{2}}{x^{a} \log^{2}(x^{2})}$$

COSÌ PER LA CONVERGENZA $a \geq 1$

Quindi l'integrale improprio è convergente

solo e soltanto se $1 \leq a < 3$

c) $$\int_0^{\frac{\pi}{2}} \frac{\sin^2(x)}{(1-\cos^3(x))^a}\,dx$$

Sia

$$f(x) = \frac{\sin^2(x)}{(1-\cos^3(x))^a}$$

nell'intervallo $\left(0, \frac{\pi}{2}\right)$ c'è un solo

punto da controllare ossia 0^+

Per $x \to 0^+$

$$f(x) \sim \frac{x^2}{\left(1-\left(1-\frac{x^2}{2}\right)^3\right)^a} \sim \frac{x^2}{\left(1-\left(1-\frac{3x^2}{2}\right)\right)^a} = \frac{1}{\left(\frac{3}{2}\right)^a} \cdot \frac{x^2}{x^{2a}} =$$

$$1^3 - 3\cdot(1)^2\left(\frac{x^2}{2}\right) + \cancel{\left(\frac{x^2}{2}\right)^3 - 3(1)\left(\frac{x^2}{2}\right)^2}$$

$$= \frac{C}{x^{2a-2}}$$

Quindi per l'integrabilità, $2\alpha - 2 < 1$

e l'integrale improprio converge se

e solo se $\alpha < \frac{3}{2}$

d) $$\int_0^{\pi/2} \frac{\arctan\left(\sqrt{\sin(x)}\right)}{\sin^\alpha(2x)\sqrt{\cos(x)}}\,dx$$

Sia

$$f(x) = \frac{\arctan\left(\sqrt{\sin(x)}\right)}{\sin^\alpha(2x)\sqrt{\cos(x)}}$$

nell'intervallo $\left(0, \frac{\pi}{2}\right)$ i punti di

interesse sono 0^+ e $\left(\frac{\pi}{2}\right)^-$

per $x \to 0^+$

$$f(x) \sim \frac{\arctan(\sqrt{x})}{2x^\alpha} \sim \frac{x^{\frac{1}{2}}}{x^\alpha} = \frac{1}{x^{\alpha - \frac{1}{2}}}$$

Quindi, per l'integrabilità $\alpha - \frac{1}{2} < 1$ ossia

$\alpha < \frac{3}{2}$

PER $x \to \left(\frac{\pi}{2}\right)^-$, $t = \frac{\pi}{2} - x \to 0^+$ E

$$f(x) \sim \frac{\frac{\pi}{4}}{\sin^{\alpha}(\pi - 2t)\sqrt{\sin(t)}} \sim \frac{\frac{\pi}{4}}{\sin^{\alpha}(2t) \cdot t^{\frac{1}{2}}} \sim \frac{c}{t^{\alpha + \frac{1}{2}}}$$

PER L'INTEGRABILITÀ $\alpha + \frac{1}{2} < 1$ OSSIA

$\alpha < \frac{1}{2}$

L'INTEGRALE IMPROPRIO DATO È CONVERGENTE

SE E SOLO SE $\alpha < \frac{1}{2}$