algorithm - What is a plain English explanation of "Big O" notation? - Stack Overflow

Big-O notation (also called "asymptotic growth" notation) is *what functions "look like" when you ignore constant factors and stuff near the origin*. We use it to talk about **how thing scale**.

Basics

for "sufficiently" large inputs...

- `f(x) ∈ O(upperbound)` means `f` "grows no faster than" `upperbound`
- `f(x) ∈ Θ(justlikethis)` mean `f` "grows exactly like" `justlikethis`
- `f(x) ∈ Ω(lowerbound)` means `f` "grows no slower than" `lowerbound`

big-O notation doesn't care about constant factors: the function `9x²` is said to "grow exactly like" `10x²`. Neither does big-O *asymptotic* notation care about *non-asymptotic* stuff ("stuff near the origin" or "what happens when the problem size is small"): the function `10x²` is said to "grow exactly like" `10x² - x + 2`.

Why would you want to ignore the smaller parts of the equation? Because they become completely dwarfed by the big parts of the equation as you consider larger and larger scales; their contribution becomes dwarfed and irrelevant. (See example section.)

Put another way, it's all about the **ratio** as you go to infinity. *If you divide the actual time it takes by the `O(...)`, you will get a constant factor in the limit of large inputs*. Intuitively this makes sense: functions "scale like" one another if you can multiply one to get the other. That is, when we say...

```
actualAlgorithmTime(N) ∈ O(bound(N))
                                        e.g. "time to mergesort N elements
```

is O(N log(N))"

... this means that ***for "large enough" problem sizes N*** (if we ignore stuff near the origin), there exists some constant (e.g. 2.5, completely made up) such that:

$$\frac{\texttt{actualAlgorithmTime(N)}}{\texttt{bound(N)}} < \texttt{constant} \qquad \text{e.g. } \frac{\texttt{"mergesort\_duration(N)"}}{\texttt{N log(N)}} < 2.5$$

There are many choices of constant; often the "best" choice is known as the "constant factor" of the algorithm... but we often ignore it like we ignore non-largest terms (see Constant Factors section for why they don't usually matter). You can also think of the above equation as a bound, saying "*In the worst-case scenario, the time it takes will never be worse than roughly `N*log(N)`, within a factor of 2.5 (a constant factor we don't care much about)*".

In general, `O(...)` is the most useful one because we often care about worst-case behavior. If `f(x)` represents something "bad" like processor or memory usage, then "`f(x) ∈ O(upperbound)`" means "`upperbound` is the worst-case scenario of processor/memory usage".

Applications

As a purely mathematical construct, big-O notation is not limited to talking about processing time and memory. You can use it to discuss the asymptotics of anything where scaling is meaningful, such as:

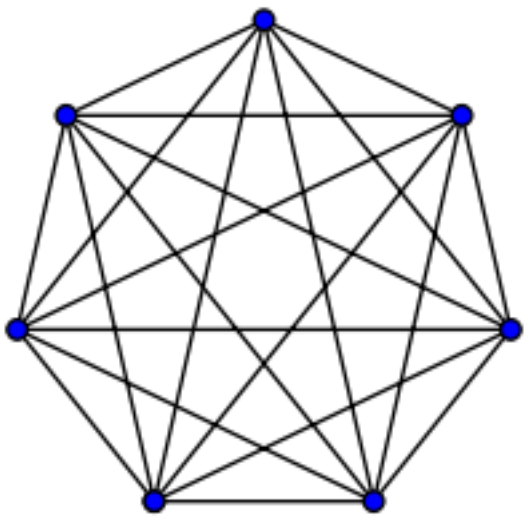
- the number of possibly handshakes among `N` people at a party (`Θ(N²)`, specifically `N(N-1)/2`, but what matters is that it "scales like" `N²`)
- probabilistic expected number of people who have seen some viral marketing as a function of time
- how website latency scales with the number of processing units in a CPU or GPU or computer cluster

- how heat output scales on CPU dies as a function of transistor count, voltage, etc.
- how much time an algorithm needs to run, as a function of input size
- how much space an algorithm needs to run, as a function of input size

Example

For the handshake example above, everyone in a room shakes everyone else's hand. In that example, `#handshakes ∈ Θ(N²)`. Why?

Back up a bit: the number of handshakes is exactly n-choose-2 or `N*(N-1)/2` (each of N people shakes the hands of N-1 other people, but this double-counts handshakes so divide by 2):



	1	2	3	4	5	6	7	8	9	10	11	...
1		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
2	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
3	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓
4	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓
5	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓
6	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓
7	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓
8	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓
9	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓
10	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓
11	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓
...	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	

However, for very large numbers of people, the linear term `N` is dwarfed and effectively contributes 0 to the ratio (in the chart: the fraction of empty boxes on the diagonal over total boxes gets smaller as the number of participants becomes larger). Therefore the scaling behavior is `order N²`, or the number of handshakes "grows like N²".

$$\frac{\#\text{handshakes}(N)}{N^2} \approx 1/2$$

It's as if the empty boxes on the diagonal of the chart (N*(N-1)/2 checkmarks) weren't even there (N^2 checkmarks asymptotically).

(temporary digression from "plain English":) If you wanted to prove this to yourself, you could perform some simple algebra on the ratio to split it up into multiple terms (`lim` means "considered in the limit of", just ignore it if you haven't seen it, it's just notation for "and N is really really big"):

$$\lim_{N\to\infty} \frac{N^2/2 - N/2}{N^2} = \lim_{N\to\infty} \left(\frac{(N^2)/2}{N^2} - \underbrace{\frac{N/2}{N^2}}_{\text{this is 0 in the limit of } N\to\infty\text{:}} \right) = \lim_{N\to\infty} \frac{1/2}{1} = 1/2$$

graph it, or plug in a really large number for N

tl;dr: The number of handshakes 'looks like' x^2 so much for large values, that if we were to write down the ratio #handshakes/x^2, the fact that we don't need *exactly* x^2 handshakes wouldn't even show up in the decimal for an arbitrarily large while.

> e.g. for x=1million, ratio #handshakes/x^2: 0.499999...

Building Intuition

This lets us make statements like...

> "For large enough inputsize=N, no matter what the constant factor is, if I ***double*** *the input size...*

- ... I double the time an O(N) ("linear time") algorithm takes."

  > $\mathbf{N} \rightarrow (2N) = 2(\mathbf{N})$

- ... I double-squared (quadruple) the time an $O(N^2)$ ("quadratic time") algorithm takes." *(e.g. a problem 100x as big takes 100^2=10000x as long... possibly unsustainable)*

> $\mathbf{N^2} \rightarrow (2N)^2 = 4(\mathbf{N^2})$

- ... I double-cubed (octuple) the time an $O(N^3)$ ("cubic time") algorithm takes." *(e.g. a problem 100x as big takes 100^3=1000000x as long... very unsustainable)*

> $\mathbf{cN^3} \rightarrow c(2N)^3 = 8(\mathbf{cN^3})$

- ... I add a fixed amount to the time an O(log(N)) ("logarithmic time") algorithm takes." *(cheap!)*

> $\mathbf{c\ log(N)} \rightarrow c\ log(2N) = (c\ log(2))+(\mathbf{c\ log(N)}) = (\text{fixed amount})+(\mathbf{c\ log(N)})$

- ... I don't change the time an O(1) ("constant time") algorithm takes." *(the cheapest!)*

> $\mathbf{c*1 \rightarrow c*1}$

- ... I "(basically) double" the time an O(N log(N)) algorithm takes." *(fairly common)*

> it's less than $O(N^{1.000001})$, which you might be willing to call basically linear

- ... I ridiculously increase the time a $O(2^N)$ ("exponential time") algorithm takes." *(you'd double (or triple, etc.) the time just by increasing the problem by a single unit)*

> $\mathbf{2^N} \rightarrow 2^{2N} = (4^N)$.....put another way..... $\mathbf{2^N} \rightarrow 2^{N+1} = 2^N 2^1 = 2\ \mathbf{2^N}$

[for the mathematically inclined, you can mouse over the spoilers for minor sidenotes]

(with credit to https://stackoverflow.com/a/487292/711085)

(technically the constant factor could maybe matter in some more esoteric examples, but I've phrased things above (e.g. in log(N)) such that it doesn't)

These are the bread-and-butter orders of growth that programmers and applied computer scientists use as reference points. They see these all the time. (So while you could technically think "Doubling the input makes an O(√N) algorithm 1.414 times slower," it's better to think of it as "this is worse than logarithmic but better than linear".)

Constant factors

Usually we don't care what the specific constant factors are, because they don't affect the way the function grows. For example, two algorithms may both take `O(N)` time to complete, but one may be twice as slow as the other. We usually don't care too much unless the factor is very large, since optimizing is tricky business (When is optimisation premature?); also the mere act of picking an algorithm with a better big-O will often improve performance by orders of magnitude.

Some asymptotically superior algorithms (e.g. a non-comparison `O(N log(log(N)))` sort) can have so large a constant factor (e.g. `100000*N log(log(N))`), or overhead that is relatively large like `O(N log(log(N)))` with a hidden `+ 100*N`, that they are rarely worth using even on "big data".

Why O(N) is sometimes the best you can do, i.e. why we need datastructures

`O(N)` algorithms are in some sense the "best" algorithms if you need to read all your data. The **very act of reading** a bunch of data is an `O(N)` operation. Loading it into memory is usually `O(N)` (or faster if you have hardware support, or no time at all if you've already read the data). However if you touch or even *look* at every piece of data (or even every

other piece of data), your algorithm will take `O(N)` time to perform this looking. Nomatter how long your actual algorithm takes, it will be at least `O(N)` because it spent that time looking at all the data.

The same can be said for the **very act of writing**. All algorithms which print out N things will take N time, because the output is at least that long (e.g. printing out all permutations (ways to rearrange) a set of N playing cards is factorial: `O(N!)`).

This motivates the use of **data structures**: a data structure requires reading the data only once (usually `O(N)` time), plus some arbitrary amount of preprocessing (e.g. `O(N)` or `O(N log(N))` or $O(N^2)$) which we try to keep small. Thereafter, modifying the data structure (insertions / deletions / etc.) and making queries on the data take very little time, such as `O(1)` or `O(log(N))`. You then proceed to make a large number of queries! In general, the more work you're willing to do ahead of time, the less work you'll have to do later on.

For example, say you had the latitude and longitude coordinates of millions of roads segments, and wanted to find all street intersections.

- Naive method: If you had the coordinates of a street intersection, and wanted to examine nearby streets, you would have to go through the millions of segments each time, and check each one for adjacency.
- If you only needed to do this once, it would not be a problem to have to do the naive method of `O(N)` work only once, but if you want to do it many times (in this case, `N` times, once for each segment), we'd have to do $O(N^2)$ work, or $1000000^2=1000000000000$ operations. Not good (a modern computer can perform about a billion operations per second).
- If we use a simple structure called a hash table (an instant-speed lookup table, also known as a hashmap or dictionary), we pay a small cost by preprocessing everything in `O(N)` time. Thereafter, it only takes constant time on average to look up something by its key (in this case, our key is the latitude and longitude coordinates, rounded into a grid;

we search the adjacent gridspaces of which there are only 9, which is a constant).
- Our task went from an infeasible $O(N^2)$ to a manageable $O(N)$, and all we had to do was pay a minor cost to make a hash table.
- **analogy**: The analogy in this particular case is a jigsaw puzzle: We created a data structure which exploits some property of the data. If our road segments are like puzzle pieces, we group them by matching color and pattern. We then exploit this to avoid doing extra work later (comparing puzzle pieces of like color to each other, not to every other single puzzle piece).

The moral of the story: a data structure lets us speed up operations. Even more advanced data structures can let you combine, delay, or even ignore operations in incredibly clever ways. Different problems would have different analogies, but they'd all involve organizing the data in a way that exploits some structure we care about, or which we've artificially imposed on it for bookkeeping. We do work ahead of time (basically planning and organizing), and now repeated tasks are much much easier!

Practical example: visualizing orders of growth while coding

Asymptotic notation is, at its core, quite separate from programming. Asymptotic notation is a mathematical framework for thinking about how things scale, and can be used in many different fields. That said... this is how you *apply* asymptotic notation to coding.

The basics: Whenever we interact with every element in a collection of size A (such as an array, a set, all keys of a map, etc.), or perform A iterations of a loop, that is a multiplcative factor of size A. Why do I say "a multiplicative factor"?--because loops and functions (almost by definition) have multiplicative running time: the number of iterations, times work done in the loop (or for functions: the number of times you call the function, times work done in the function). (This holds if we don't do anything fancy, like skip loops or exit the loop early, or change control flow in the function based on arguments, which is very common.) Here are

some examples of visualization techniques, with accompanying pseudocode.

(here, the `x`s represent constant-time units of work, processor instructions, interpreter opcodes, whatever)

```
for(i=0; i<A; i++)         // A x ...
    some O(1) operation      // 1

--> A*1 --> O(A) time

visualization:

|<------ A ------->|
1 2 3 4 5 x x ... x

other languages, multiplying orders of growth:
  javascript, O(A) time and space
    someListOfSizeA.map((x,i) => [x,i])
  python, O(rows*cols) time and space
    [[r*c for c in range(cols)] for r in range(rows)]
```

Example 2:

```
for every x in listOfSizeA:   // A x ...
    some O(1) operation         // 1
    some O(B) operation         // B
    for every y in listOfSizeC: // C x ...
        some O(1) operation       // 1

--> O(A*(1 + B + C))
    O(A*(B+C))         (1 is dwarfed)

visualization:

|<------ A ------->|
1 x x x x x x ... x

2 x x x x x x ... x ^
3 x x x x x x ... x |
4 x x x x x x ... x |
5 x x x x x x ... x B  <-- A*B
x x x x x x x ... x |
.................... |
```

```
x x x x x x x ... x v

x x x x x x x ... x ^
x x x x x x x ... x |
x x x x x x x ... x |
x x x x x x x ... x C  <-- A*C
x x x x x x x ... x |
................... |
x x x x x x x ... x v
```

Example 3:

```
function nSquaredFunction(n) {
    total = 0
    for i in 1..n:          // N x
        for j in 1..n:      // N x
            total += i*k      // 1
    return total
}
// O(n^2)

function nCubedFunction(a) {
    for i in 1..n:                // A x
        print(nSquaredFunction(a))  // A^2
}
// O(a^3)
```

If we do something slightly complicated, you might still be able to imagine visually what's going on:

```
for x in range(A):
    for y in range(1..x):
        simpleOperation(x*y)

x x x x x x x x x x |
x x x x x x x x x   |
x x x x x x x x     |
x x x x x x x       |
x x x x x x         |
x x x x x           |
x x x x             |
x x x               |
x x                 |
x___________________|
```

Here, the smallest recognizable outline you can draw is what matters; a triangle is a two dimensional shape (0.5 A^2), just like a square is a two-dimensional shape (A^2); the constant factor of two here remains in the asymptotic ratio between the two, however we ignore it like all factors... (There are some unfortunate nuances to this technique I don't go into here; it can mislead you.)

Of course this does not mean that loops and functions are bad; on the contrary, they are the building blocks of modern programming languages, and we love them. However, we can see that the way we weave loops and functions and conditionals together with our data (control flow, etc.) mimics the time and space usage of our program! If time and space usage becomes an issue, that is when we resort to cleverness, and find an easy algorithm or data structure we hadn't considered, to reduce the order of growth somehow. Nevertheless, these visualization techniques (though they don't always work) can give you a naive guess at a worst-case running time.

Here is another thing we can recognize visually:

```
<----------------------------- N ----------------------------->
x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x
x x x x x x x x x x x x x x x x
x x x x x x x x
x x x x
x x
x
```

We can just rearrange this and see it's O(N):

```
<----------------------------- N ----------------------------->
x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x
x x x x x x x x x x x x x x x x|x x x x x x x x|x x x x|x x|x
```

Or maybe you do log(N) passes of the data, for O(N*log(N)) total time:

```
   <----------------------------- N ----------------------------->
 ^  x x x x x x x x x x x x x x x x|x x x x x x x x x x x x x x x x
```

```
 |  x x x x x x x x|x x x x x x x x|x x x x x x x x|x x x x x x x x
lgN x x x x|x x x x|x x x x|x x x x|x x x x|x x x x|x x x x|x x x x
 |  x x|x x|x x|x x|x x|x x|x x|x x|x x|x x|x x|x x|x x|x x|x x|x x
 v  x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x|x
```

Unrelatedly but worth mentioning again: If we perform a hash (e.g. a dictionary / hashtable lookup), that is a factor of O(1). That's pretty fast.

```
[myDictionary.has(x) for x in listOfSizeA]
 \----- O(1) ------/

--> A*1 --> O(A)
```

If we do something very complicated, such as with a recursive function or divide-and-conquer algorithm, ~~you can use the Master Theorem (usually works), or in ridiculous cases the Akra-Bazzi Theorem (almost always works)~~ you look up the running time of your algorithm on Wikipedia.

But, programmers don't think like this because eventually, algorithm intuition just becomes second nature. You will start to code something inefficient, and immediately think "am I doing something **grossly inefficient?**". If the answer is "yes" AND you foresee it actually mattering, then you can take a step back and think of various tricks to make things run faster (the answer is almost always "use a hashtable", rarely "use a tree", and very rarely something a bit more complicated).

Amortized and average-case complexity

There is also the concept of "amortized" and/or "average case" (note that these are different).

Average Case: This is no more than using big-O notation for the expected value of a function, rather than the function itself. In the usual case where you consider all inputs to be equally likely, the average case is just the average of the running time. For example with quicksort, even though the worst-case is `O(N^2)` for some really bad inputs, the average case is the usual `O(N log(N))` (the really bad inputs are very small in number, so few

that we don't notice them in the average case).

Amortized Worst-Case: Some data structures may have a worst-case complexity that is large, but guarantee that if you do many of these operations, the average amount of work you do will be better than worst-case. For example you may have a data structure that normally takes constant `O(1)` time. However, occasionally it will 'hiccup' and take `O(N)` time for one random operation, because maybe it needs to do some bookkeeping or garbage collection or something... but it promises you that if it does hiccup, it won't hiccup again for N more operations. The worst-case cost is still `O(N)` per operation, but the amortized cost *over many runs* is `O(N)/N = O(1)` per operation. Because the big operations are sufficiently rare, the massive amount of occasional work can be considered to blend in with the rest of the work as a constant factor. We say the work is "amortized" over a sufficiently large number of calls that it disappears asymptotically.

> The analogy for amortized analysis:
>
> You drive a car. Occasionally, you need to spend 10 minutes going to the gas station and then spend 1 minute refilling the tank with gas. If you did this every time you went anywhere with your car (spend 10 minutes driving to the gas station, spend a few seconds filling up a fraction of a gallon), it would be very inefficient. But if you fill up the tank once every few days, the 11 minutes spent driving to the gas station is "amortized" over a sufficiently large number of trips, that you can ignore it and pretend all your trips were maybe 5% longer.

Comparison between average-case and amortized worst-case:

- If you use a data structure many times, the running time will tend to the average case... eventually... assuming your assumptions about what is 'average' were correct (if they aren't, your analysis will be wrong).
- If you use an amortized worst-case data structure, the running is guaranteed to be within the amortized worst-case... eventually (even if

the inputs are chosen by an evil demon who knows everything and is trying to screw you over). (However unless your data structure has upper limits for much outstanding work it is willing to procrastinate on, an evil attacker could perhaps force you to catch up on the maximum amount of procrastinated work all-at-once. Though, if you're reasonably worried about an attacker, there are many other algorithmic attack vectors to worry about besides amortization and average-case.)

Both average-case and amortization are incredibly useful tools for thinking about and designing with scaling in mind.

(See Difference between average case and amortized analysis if interested on this subtopic.)

Multidimensional big-O

Most of the time, people don't realize that there's more than one variable at work. For example, in a string-search algorithm, your algorithm may take time `O([length of text] + [length of query])`, i.e. it is linear in two variables like `O(N+M)`. Other more naive algorithms may be `O([length of text]*[length of query])` or `O(N*M)`. Ignoring multiple variables is one of the most common oversights I see in algorithm analysis, and can handicap you when designing an algorithm.

The whole story

Keep in mind that big-O is not the whole story. You can drastically speed up some algorithms by using caching, making them cache-oblivious, avoiding bottlenecks by working with RAM instead of disk, using parallelization, or doing work ahead of time -- these techniques are often *independent* of the order-of-growth "big-O" notation, though you will often see the number of cores in the big-O notation of parallel algorithms.

Also keep in mind that due to hidden constraints of your program, you

might not really care about asymptotic behavior. You may be working with a bounded number of values, for example:

- If you're sorting something like 5 elements, you don't want to use the speedy `O(N log(N))` quicksort; you want to use insertion sort, which happens to perform well on small inputs. These situations often comes up in divide-and-conquer algorithms, where you split up the problem into smaller and smaller subproblems, such as recursive sorting, fast Fourier transforms, or matrix multiplication.
- If some values are effectively bounded due to some hidden fact (e.g. the average human name is softly bounded at perhaps 40 letters, and human age is softly bounded at around 150). You can also impose bounds on your input to effectively make terms constant.

In practice, even among algorithms which have the same or similar asymptotic performance, their relative merit may actually be driven by other things, such as: other performance factors (quicksort and mergesort are both `O(N log(N))`, but quicksort takes advantage of CPU caches); non-performance considerations, like ease of implementation; whether a library is available, and how reputable and maintained the library is.

Programs will also run slower on a 500MHz computer vs 2GHz computer. We don't really consider this as part of the resource bounds, because we think of the scaling in terms of machine resources (e.g. per clock cycle), not per real second. However, there are similar things which can 'secretly' affect performance, such as whether you are running under emulation, or whether the compiler optimized code or not. These might make some basic operations take longer (even relative to each other), or even speed up or slow down some operations asymptotically (even relative to each other). The effect may be small or large between different implementation and/or environment. Do you switch languages or machines to eke out that little extra work? That depends on a hundred other reasons (necessity, skills, coworkers, programmer productivity, the monetary value of your time, familiarity, workarounds, why not assembly or GPU, etc...), which may be more important than performance.

The above issues, like programming language, are almost never considered as part of the constant factor (nor should they be); yet one should be aware of them, because *sometimes* (though rarely) they may affect things. For example in cpython, the native priority queue implementation is asymptotically non-optimal (`O(log(N))` rather than `O(1)` for your choice of insertion or find-min); do you use another implementation? Probably not, since the C implementation is probably faster, and there are probably other similar issues elsewhere. There are tradeoffs; sometimes they matter and sometimes they don't.

(*edit*: The "plain English" explanation ends here.)

Math addenda

For completeness, the precise definition of big-O notation is as follows: `f(x) ∈ O(g(x))` *means that "f is asymptotically upper-bounded by const*g": ignoring everything below some finite value of x, there exists a constant such that* `|f(x)| ≤ const * |g(x)|`*. (The other symbols are as follows: just like* `o` *means ≤,* `Ω` *means ≥. There are lowercase variants:* `o` *means <, and* `ω` *means >.)* `f(x) ∈ Θ(g(x))` *means both* `f(x) ∈ O(g(x))` *and* `f(x) ∈ Ω(g(x))` *(upper- and lower-bounded by g): there exists some constants such that f will always lie in the "band" between* `const1*g(x)` *and* `const2*g(x)`*. It is the strongest asymptotic statement you can make and roughly equivalent to ==. (Sorry, I elected to delay the mention of the absolute-value symbols until now, for clarity's sake; especially because I have never seen negative values come up in a computer science context.)*

People will often use `= O(...)`*. It is technically more correct to use* `∈ O(...)`*.* `∈` *means "is an element of".* `O(N²)` *is actually an equivalence class, that is, it is a set of things which we consider to be the same. In this particular case,* `O(N²)` *contains elements like* `{2 N², 3 N², 1/2 N², 2 N² + log(N), - N² + N^1.9, ...}` *and is infinitely large, but it's still a set. People will know what you mean if you use* `=` *however. Additionally, it is often the case that in a casual setting, people will say* `O(...)` *when they mean*

`Θ(...)`; this is technically true since the set of things `Θ(exactlyThis)` is a subset of `O(noGreaterThanThis)`... and it's easier to type. ;-)