George Pólya & problem solving ... An appreciation

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GEORGE PÓLYA & PROBLEM SOLVING ...AN APPRECIATION

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ABSTRACT. George Pólya belongs to an extremely rare breed of persons: he was a front rank mathematician who maintained a deep interest in mathematics education all through his life and contributed significantly to that field. Over a period of several decades he returned over and over again to the question of how the culture of problem solving could be nurtured among students, and how mathematics could be experienced 'live'. He wrote many books now regarded as masterpieces: *Problems and Theorems in Analysis* (with Gábor Szegö), *How to Solve It*, *Mathematical Discovery*, ... This article is a tribute to Pólya, and a celebration of this aspect of his work.

Mathematicians are sometimes loosely categorized as *theory builders* or *problem solvers*. In his article "The Two Cultures of Mathematics" ([2]), Tim Gowers suggests that if one were to ask mathematicians to comment on the truth of the following statements: (i) the point of solving problems is to understand mathematics better, (ii) the point of understanding mathematics is to become better able to solve problems, the answers would point to two kinds of mathematicians who may loosely be called 'theory builders' and 'problem solvers' depending on their priorities. Sir Michael Atiyah would almost certainly belong to the first category, while Paul Erdős would belong to the second one. Of course, such categorizations are not meant to be taken too seriously, but they does tell us something about the different personalities that mathematicians have.

In this context, George Pólya occupies a special place. Not only does he straddle both categories, he is one of that exceedingly rare breed: a front-rank mathematician who maintained a deep interest in education and pedagogy all through his life and contributed significantly to that field. Over a period of several decades he returned repeatedly to the question of how the "know-how" of mathematics could be conveyed to students. The following extract from the Preface of *How To Solve It* conveys this quest eloquently:

> The author remembers the time when he was a student himself, a somewhat ambitious student, eager to understand a little mathematics and physics. He listened to lectures, read books, tried to take in the solutions . . . , but there was a question that disturbed him again and again: “Yes, the solution seems to work, . . . ; but how is it possible to invent such a solution?” . . . Today the author is teaching mathematics in a university; he thinks or hopes that some of his more eager students ask similar questions and he tries to satisfy their curiosity. Trying to understand not only the solution of this or that problem but also the motives and procedures of the solution, and trying to explain these motives and procedures to others, he was finally led to write the present book. He hopes that it will be useful to teachers who wish to develop their students’ ability to solve problems, and to students who are keen on developing their own abilities.

He adds later in the same Preface:

> Studying the methods of solving problems, we perceive another face of mathematics. Yes, mathematics has two faces; it is the rigorous science of Euclid but it is also something else. Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears as an experimental, inductive science. Both aspects are as old as the science of mathematics itself. But the second aspect is new in one respect; mathematics . . . in the process of being invented has never before been presented in quite this manner . . . .

These lines are prophetic, for they point to the making of what is now a respected field with its own journal: ‘Experimental Mathematics’.

Pólya’s work in problem-solving dates to the time when he and fellow Hungarian Gábor Szegö collaborated to produce the finest problem books of all, *Problems and Theorems in Analysis* (1925). We quote again, this time from the Preface to this book:

> The chief aim of this book . . . is to accustom advanced students of mathematics, through systematically arranged problems . . . to the ways and means

> of independent thought and research. It is intended to serve the need for individual active study on the part of both the student and the teacher. . . . This book is no mere collection of problems. Its most important feature is the systematic arrangement of . . . material . . . to suggest useful lines of thought. . . . The imparting of factual knowledge is for us a secondary consideration. Above all we aim to promote in the reader a correct attitude . . . which would appear to be of even more essential importance in mathematics than in other scientific disciplines.

The Preface ends with a typically witty touch: "[There] are two kinds of generalisation: . . . by dilution [and] by concentration. . . . The unification of concepts which . . . appear far removed from each other is concentration. [For] example, group theory has concentrated ideas which formerly were found scattered in algebra, number theory, geometry and analysis and which appeared to be very different. Examples of generalisation by dilution would be . . . easier to quote, but this would be at the risk of offending sensibilities."

Over the course of several decades, Pólya authored many books in the field of problem-solving. Other than the books written with Szegö he also wrote *How to Solve It* (1945); *Mathematics and Plausible Reasoning* (1954), in two volumes, subtitled "Induction and Analogy in Mathematics" and "Patterns of Plausible Inference"; and *Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving* (1962), also in two volumes.

Pólya contributed generously to mathematics, with beautiful theorems in fields such as Combinatorics, Complex Analysis, Numerical Analysis, Probability Theory and Inequalities. But some would say that one of his great contributions was in bringing energy and attention to bear on the field of Pedagogy, and to leave behind so many beautiful insights in this area.

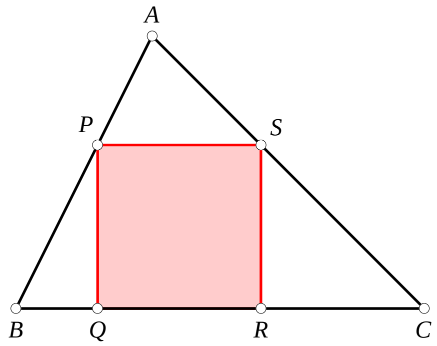
HOW TO SOLVE IT

As noted, Pólya's aim is to communicate and share the know-how of mathematics and the heuristics which most mathematicians instinctively use without ever bothering to talk about them. This is the book in which he sets about this task. Along the way he also has many wise things to say to the teacher of mathematics.

Extracts from the Preface. "A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime. Thus, a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking."

Suggested steps in approaching a problem. (1) First, understand the problem; ask yourself, what is the unknown, what has to be proved? (2) Make a plan. (3) Carry out the plan. (4) Look back on your work. Ask yourself: How could it be better?

For part 2 (making a plan), Pólya suggests numerous heuristics: guess and check; make an orderly list (the order matters, so plan it out); eliminate possibilities; use analogies; look for symmetry; consider special cases; look for a pattern; solve a simpler problem. Various questions are suggested: (i) Can you find a problem analogous to yours and solve that? (ii) Can you find a more general problem? (iii) Can you solve your problem by deriving a generalization from some examples? Can you vary or change your problem to create a new problem whose solution will help solve the original one? And so on. Pólya even introduces a variational 'upper bound' element: "If you can't solve a problem, then there is an easier problem you can't solve: find it!"



PQRS is a square inscribed in $\triangle ABC$, with *P* on side *AB*, *Q* and *R* on side *BC*, and *S* on side *AC*.

FIGURE 1. Square in a triangle. How can it be constructed using compass and ruler?

THREE CASE STUDIES

Let us illustrate the suggested use of heuristics using two examples discussed by Pólya. More such problems can easily be listed, either from his books or from one's own teaching experience.

Square in a triangle. How would one inscribe a square in an arbitrary triangle, with two vertices on one side of the triangle, and one vertex each on the remaining two sides, using compass and ruler? (See Figure 1. The problem is from *How To Solve It*, Section 18, pages 23–25; it is also studied in *Mathematical Discovery*, Vol 1, Chapter 1, problem 1.36.) For simplicity we shall take the triangle to be acute-angled.

From experience the author can say that this problem does indeed challenge the young student, as it does not yield to a linear approach in which one starts at the beginning and muscles ones way to the end. What would be Pólya's suggestion? "Relax the conditions", he would say. "Why not drop the condition that *S* lies on side *AC*?" This change turns a challenging problem into a simple one! The problem is now: Construct a square *PQRS* such that *P* lies on *AB*, and *Q* and *R* lie on *BC*. How many such squares can be drawn? What pattern do they give rise to, and what can be deduced from this observation? How does this solve the given problem?

Planes in space. This problem is the subject of a famous Pólya lecture (see [8]). Imagine 5 planes in space, in 'general position'. How many disjoint regions do they create? (The phrase 'general position' means: no two planes are parallel, no three meet in a line, and no four meet in a point. So the planes create the maximum possible number of regions.)

Here, the sheer visual complexity is off-putting. What would be Pólya's approach now? It seems natural to simplify the problem, to start with something significantly simpler, to look for a pattern perhaps, and build towards the given problem. In this case we could *reduce the dimensionality*: consider lines on a plane rather planes in space. Or, even simpler: points on a line! And this is just what Pólya does in the lecture. Rather than spoil the reader's fun by giving away the solution, we warmly recommend watching the video.

The Basel problem. This is the problem of finding the following sum in closed form:

$$\sum_{1}^{\infty} \frac{1}{n^2} \ = \ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots .$$

The greatest minds of the time tried their hand at it, but with no success, leading Jacques (also known as Jakob) Bernoulli to plaintively exclaim, "If anyone finds and communicates to us that which thus far has eluded our efforts, great will be our gratitude". It is instructive to see Pólya's account (see [5]) of how Euler tackled the problem. "He [started by finding] various expressions for the desired sum (definite integrals, other series), none of which satisfied him. He used one of these expressions to compute the sum numerically to seven places (1.644934). Yet this is only an approximate value and his goal was to find the exact value. He discovered it, eventually. Analogy led him to an extremely daring conjecture."

Here is how analogy enters the scene: Euler knew very well how to express various functions as power series. Also, the theory of equations and of polynomials in one variable were well understood subjects. Can we 'import' the knowledge about polynomials into the domain of power series? (So we are treating power series as *analogous* to polynomials.) Here's how Euler took the analogy forward. It is easy to show that if $f(x)$ is a polynomial such that $f(0) \neq 0$, and its roots are α_1, α_2, $\ldots \alpha_n$, then $\sum_i 1/\alpha_i = -f'(0)/f(0)$. Now consider the function $g(x) = \sin x/x$ for $x \neq 0$, $g(0) = 1$. It has infinitely many roots, $x = \pm n\pi$ for $n = 1,2,3,\ldots$. Hence the function $f(x) = \sin\sqrt{x}/\sqrt{x}$ for $x > 0$, $f(0) = 1$ has as roots the following: $n^2\pi^2$ for $n = 1,2,3,\ldots$. Since the power series for $g(x)$ is

$1-x^2/6+x^4/120-\ldots$, the power series for $f(x)$ is $1-x/6+x^2/120-\ldots$, and so for this function we have $f(0)=1$ and $f'(0)=-1/6$. Invoking the result about polynomials we may now jump to the conclusion that

$$\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}=-\frac{f'(0)}{f(0)}=\frac{1}{6},$$

and hence that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6};$$

an incredible result. Pólya writes: “This is the series that withstood the efforts of Jacques Bernoulli—but it was a daring conclusion. …Euler knew very well that his conclusion was daring. …He saw some objections himself and many objections were raised by his mathematical friends when they recovered from their first admiring surprise. Yet Euler had his reasons to trust his discovery. First of all, the numerical value for the sum of the series which he has computed before, agreed to the last place with $\pi^2/6$. Comparing further coefficients …he found the sum of other remarkable series, as that of the reciprocals of the fourth powers,

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{\pi^4}{90}.$$

Again, he examined the numerical value and again he found agreement. …Euler also tested his method on other examples. Doing so he succeeded in re-deriving the sum $\pi^2/6$ for Bernoulli’s series …. He succeeded also in rediscovering by his method the sum of an important series due to Leibnitz. …[Thus his] daring procedure led to a known result. …Yet Euler kept on doubting. He continued the numerical verifications, …examined more series and more decimal places, and found agreement in all cases examined. He tried other approaches, too, and, finally, he succeeded in verifying not only numerically, but exactly, the value $\pi^2/6$ for Jacques Bernoulli’s series. He found a new proof. This proof …was based on more usual considerations and was accepted as completely rigorous. Thus, the most conspicuous consequence of Euler’s discovery was satisfactorily verified. These arguments, it seems, convinced Euler that his result was correct.”

PÓLYA'S COMMANDMENTS TO MATH TEACHERS

We now turn to Pólya's suggestions to teachers. Some of these have become sufficiently famous as to be referred to as Pólya's Ten Commandments. The content below is adapted from Section 14.8 of *Mathematical Discovery* (reference [6]).

The aims of teaching. I have an old fashioned idea about [the aim of teaching]: first and foremost, it should teach young people to THINK. ... "Teaching to think" means that the teacher should not merely impart information, but should try also to develop the ability of the students to use the information imparted: he should stress knowledge, useful attitudes, desirable habits of mind.

The art of teaching. Teaching obviously has much in common with the theatrical art. For instance, you have to present to your class a proof which you know thoroughly having presented it already so many times in former years in the same course. You really cannot be excited about the proof—but, please, do not show that to your class; if you appear bored, the whole class will be bored. Pretend to be excited about the proof when you start it, pretend to have bright ideas when you proceed, pretend to be surprised and elated when the proof ends. You should do a little acting for the sake of your students who may learn, occasionally, more from your attitudes than from the subject matter presented. ... Now and then, teaching may approach poetry, and now and then it may approach profanity. ... Nothing is too good or too bad, too poetical or too trivial to clarify your abstractions. As Montaigne put it: The truth is such a great thing that we should not disdain any means that could lead to it. Therefore, if the spirit moves you to be a little poetical or a little profane in your class, do not have the wrong kind of inhibition.

The teacher's attitude. On what authority are these commandments founded? Dear fellow teacher, do not accept any authority except your own well-digested experience and your own well-considered judgment. Try to see clearly what the advice means in your particular situation, try the advice in your classes, and judge after a fair trial.

(1) *Be interested in your subject.*

There is just one infallible teaching method: if the teacher is bored by his subject, his whole class will be infallibly bored by it.

(2) *Know your subject.*

If a subject has no interest for you, do not teach it, because you will not be able to teach it acceptably. Interest is an indispensable necessary condition; but, in itself, it is not a sufficient condition. No amount of interest, or teaching methods, or whatever else will enable you to explain clearly a point to your students that you do not understand clearly yourself.

Between points #1 and #2, I put interest first because with genuine interest you have a good chance to acquire the necessary knowledge, whereas some knowledge coupled with lack of interest can easily make you an exceptionally bad teacher.

(3) *Know about the ways of learning: the best way to learn anything is to discover it by yourself.*

(4) *Try to read the faces of your students, try to see their expectations and difficulties, put yourself in their place.*

(5) *Give them not only information, but "know-how", attitudes of mind, the habit of methodical work.*

(6) *Let them learn guessing.*

(7) *Let them learn proving.*

(8) *Look out for such features of the problem at hand as may be useful in solving the problems to come—try to disclose the general pattern that lies behind the present concrete situation.*

(9) *Do not give away your whole secret at once—let the students guess before you tell it—let them find out by themselves as much as is feasible.*

Voltaire expressed it more wittily: *The art of being a bore consists in telling everything.*

(10) *Suggest it, do not force it down their throats.*

In other words: Let your students ask the questions; or ask such questions as they may ask for themselves. Let your students give the answers; or give such answers as they may give by themselves. At any rate avoid asking questions that nobody has asked, not even yourself.

CRITICISMS

Pólya's endeavour was to put a toolkit in the hands of the would be problem solver. How effective is it? Does it really empower? How successful has been this endeavour? This question has been debated at length by math educators. (See [4] for an excellent account of the debate.) While Pólya's tips and suggestions to mathematics teachers are in the nature of timeless classics, and countless teachers have benefited from his words, the effect of his suggestions concerning heuristics for problem solving is much less clear. The fact that they hugely enhance one's *appreciation* of the art and craft of problem solving can hardly be disputed. One only has to study his masterful analysis of Euler's handling of the Basel problem to see this. But to what extent does deep internalization of Pólya's heuristics make one a better problem solver? It surely does help to some extent; but one would be hard put to quantify the change. But whether great ideas can emerge that way seems doubtful. It is sobering to read the critical words of Douglas Hofstadter: "Nobody can ever learn to be a better mathematician from reading Pólya's volumes. You'll know more math after reading them, but they are not going to turn you from a class B to a class A mathematician. You won't suddenly have great, deep insights that you didn't already have. Nothing is going to teach any body to have great, deep insights. There's just no way you can do it. You either have them or you don't." However, it may never have been Pólya's intention to create great mathematicians with deep insights, so this extravagant criticism seems misplaced. It may be more accurate to call Pólya's books an attempt to enhance students' and teachers' appreciation of the problem solving process; an attempt to demystify something that tends to stay hidden; and an attempt to make the learning of mathematics that much more fun. One only has to re-read the Preface of *How To Solve It* to see that Pólya is not attempting something impossible. Indeed, at no stage does he leave the ground and go off into flights of fantasy. He is nothing if not immensely practical and down-to-earth.

So let us conclude by stating that while the jury may still be out on the extent to which the problem solver actually benefits from Pólya's work, there is not the slightest doubt that for mathematics teachers and students, his books have enormous value. The mathematics teacher community will be eternally indebted to him.

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