An introductory examination of the Finite Element Method

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Agenda

PDE og grænsebetingelser
- Basis om PDE
- Dirichlet
- Neumann

Finite Element Method
- Opdeling
- Finite Elements
- Konstruktion

Teoretiske fejl

Numerisk Analyse
- Plot af fejl

Basis om PDE

$$f(x) = c(x)u + \sum_{i=1}^{n} b_i(x)u_{x_i} - \sum_{i,k=1}^{n} a_{ik}(x)u_{x_i x_k}$$

Elliptisk i x, hvis $A(x)$ er positiv definit

Differential operator

Lad L være den partielle differential operator defineret som

$$Lu = -\sum_{i,k=1}^{n} \partial_i (a_{ik} \partial_k u) + a_0 u, \tag{1}$$

hvor a_{ik} er indgange i en matrix $A(x)$.

Sobolev rum

Lad $m \geq 0$ være et helttal, så er $H^m(\Omega)$

$$H^m(\Omega) = \{f \in L_2(\Omega) \mid \partial^\alpha f \in L_2(\Omega) \quad \forall |\alpha| \leq m\}.$$

**** Skal ikke med **** In other words, $H^m(\Omega)$ is the set of functions in $L_2(\Omega)$ which possess weak derivatives up to degree m.

Friedrichs ulighed

Lad Ω være indeholdt i en hyperkube med sidelængde s. Så gælder

$$\|v\|_k \leq s|v|_{k+1}, \forall v \in H_0^1(\Omega), C \in \mathbb{R}.$$

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Homogene Dirichlet Grænsebetingelser

$$
\begin{aligned}
Lu &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Minimal egenskab

Let an elliptic PDE be given, and let a_{ik} be the entries in the positive definite matrix A for the PDE. Every classical solution of the boundary-value problem given by

$$-\sum_{i,k} \partial_i(a_{ik}\partial_k u) + a_0 u = f \quad \text{in } \Omega \tag{2}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{3}$$

is a solution to the variational problem given by

$$J(v) = \int_\Omega \left[\frac{1}{2}\sum_{i,k} a_{ik}\partial_i v \partial_k v + \frac{1}{2}a_0 v^2 - fv\right] dx \longrightarrow \min,$$

among all functions in $C^2(\Omega) \cap C^0(\bar{\Omega})$ with zero boundary values.

Eskistens sætning

Let L be the second order uniformly elliptic partial differential operator from (**??**). Then the homogeneous Dirichlet problem, always has a weak solution in $H_0^1(\Omega)$. It is a minimum of the variational problem

$$J(v) = \frac{1}{2}a(v, v) - (f, v)_0 \to \min \tag{4}$$

over $H_0^1(\Omega)$.

Neumann Grænsebetingelser

Let $\frac{\partial u}{\partial \mathbf{n}} = \mathbf{n} \cdot \nabla u$ be the normal derivative. Then the Neumann boundary condition is given by

$$\frac{\partial u}{\partial \mathbf{n}} = g \quad \text{on } \partial\Omega. \tag{5}$$

Neumann Grænsebetingelser

Let Ω be bounded, have a piecewise smooth boundary, and satisfy the cone condition. Let $f \in L_2(\Omega)$ and $g \in L_2(\partial\Omega)$. There exists a unique $u \in H^1(\Omega)$ that solves the variational problem

$$J(v) = \frac{1}{2}a(v,v) - (f,v)_{0,\Omega} - (g,v)_{0,\partial\Omega} \to \min .$$

Also, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ if and only if a classical solution of

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ \sum_{i,k} \mathbf{n}_i a_{ik} \partial_k u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{6}$$

exists, in which case these 2 solutions are the same. Here **n** is outward pointing normal on $\partial\Omega$, defined almost everywhere.

Finite Element Method
General Idea

- Partition the domain Ω.
- Define subspace with finite dimension S_h.
- Elements/Cells.

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S_h

Ud fra $H^m(\Omega)$ or $H_0^m(\Omega)$ definer et endeligt underrum

- Splicing functions over each cell.
- Edge restrictions.
- Solve the varational problem over S_h.

$$J(v) = \frac{1}{2}a(v,v) - \ell(v) \to \min_{S_h} . \tag{7}$$

- Solution $u_h \in S_h$,

$$a(u_h, v) = \ell(v) \quad \forall v \in S_h. \tag{8}$$

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Basisvektorer

- Let $\{\psi_1, \psi_2, \ldots, \psi_N\}$ be a basis for S_h
- $a(u_h, \psi_i) = \ell(\psi_i) \quad i = 1, 2, \ldots, N$
- $\sum_{k=1}^{N} z_k a(\psi_k, \psi_i) = \ell(\psi_i) \quad i = 1, 2, \ldots, N$

$$Az = b$$

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Sammenhæng mellem H^k og C^{k-1}

Theorem

Let $k \geq 1$ and suppose Ω is bounded. Then a piecewise infinitely differentiable function $v : \bar{\Omega} \to \mathbb{R}$ belongs to $H^k(\Omega)$ if and only if $v \in C^{k-1}(\bar{\Omega})$.

Bevis

Vi begrænser os til $\mathbb{R}^2$ og starter med $k = 1$. Antag $v \in C^0(\bar{\Omega})$. Lad $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned}\int_\Omega \phi w_i dxdy &= \sum_j \int_{T_j} \phi \partial_i v dxdy \\ &= \sum_j \left(-\int_{T_j} \partial_i \phi v dxdy + \int_{\partial T_j} \phi v \mathbf{n}_i ds \right).\end{aligned} \tag{9}$$

Hvilket resulterer i:

$$\int_\Omega \phi w_i dxdy = -\int_\Omega \partial_i \phi v dxdy. \tag{10}$$

Bevis

- Antag $v \in H^1(\Omega)$
- Roter kanten så vi ligger på y-aksen
- Definer $[y_1 - \delta, y_2 + \delta]$, $y_1 < y_2$ på kanten

Definer

$$\psi(x) = \int_{y_1}^{y_2} v(x, y)dy. \tag{11}$$

$$|\psi(x_2) - \psi(x_1)|^2 = \left| \int_{x_1}^{x_2} \int_{y_1}^{y_2} \partial_1 v dx dy \right|^2 \tag{12}$$

$$\leq \left| \int_{x_1}^{x_2} \int_{y_1}^{y_2} 1 dx dy \right|^2 \cdot |v|_{1,\Omega}^2 \tag{13}$$

$$\leq |x_2 - x_1|^2 \cdot |y_2 - y_1|^2 \cdot |v|_{1,\Omega}^2, \tag{14}$$

A finite element is a triple (T, Π, Σ) which has the following properties:

1. $T \subset \mathbb{R}^d$ is a polyhedron
2. $\Pi \subset C(T)$ with finite dimension s
3. Σ is a set of s linearly independent functionals on Π. Every $p \in \Pi$ is uniquely defined by the values of the s functionals in Σ.

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Finite Element Method

Reference Finite Element

- Why do we need a Reference Finite Element?
- Let $(T_{\text{ref}}, \Pi_{\text{ref}}, \Sigma_{\text{ref}})$ be a finite element with $T_{\text{ref}} \in \mathcal{T}$ for som admissible partition of Ω, and F an affine transformation. Assume that for $T_i \in \mathcal{T}$ the following is true for the corresponding finite element (T_i, Π_i, Σ_i):
  - $F(T_{\text{ref}}) = T_i$
  - $\{f \circ F \mid f \in \Pi_i\} = \Pi_{\text{ref}}$
  - $\{s(f \circ F) \mid f \in \Pi_i, s \in \Sigma_{\text{ref}}\} = \Sigma_i$

  If the previous equalities are true for all $T_i \in \mathcal{T}$ we call $(T_{\text{ref}}, \Pi_{\text{ref}}, \Sigma_{\text{ref}})$ the finite reference element.

Eksempel på konstruktion

1. Lad $\Omega \subset \mathbb{R}^2$, og $\mathcal{T}$ være en admissible partition for trekanter
2. Lad $t > 0$ og $\Pi = \mathcal{P}_t$ for alle finite elements
3. Placer $s = (t+1)(t+2)/2$ punkter, med $t+1$ på hver kant

Kontinuitet vs. differentiabilitet

Sætning 4.6

Theorem

Let $t \geq 2$, and suppose $\mathcal{T}_h$ is a shape-regular triangulation of Ω. Then there exists a constant c such that

$$\|u - I_h u\|_{m,h} \leq c h^{t-m} |u|_{t,H^t(\Omega)}, \quad \forall u \in H^t(\Omega), \quad 0 \leq m \leq t, \tag{15}$$

where $I_h u$ denotes the interpolation by a piecewise polynomial of degreee $t - 1$.

Bevis

$$||u - I_h u||_{m,T_j} \leq ch^{t-m}|u|_{t,T_j}, \quad \forall u \in H^t(T_j) \tag{16}$$

$$\begin{aligned} ||u - I_h u||_{m,T_j} &\leq c||B^{-1}||^m |\det B|^{1/2} |\hat{u} - I_h \hat{u}|_{m,T_{\text{ref}}} \\ &\leq c||B^{-1}||^m |\det B|^{1/2} c |\hat{u}|_{t,T_{\text{ref}}} \\ &\leq c||B^{-1}||^m |\det B|^{1/2} \cdot c||B||^t \cdot |\det B|^{-1/2} |u|_{t,T} \\ &\leq c\big(||B|| \cdot ||B^{-1}||\big)^m ||B||^{t-m} |u|_{t,T}. \end{aligned} \tag{17}$$

Bevis

$$||B|| \leq \frac{r_2}{\rho_1}. \tag{18}$$

Shape regularity: $r_i/\rho_i \leq \kappa$ for all T_i in $\mathcal{T}$

$$||B|| \cdot ||B^{-1}|| \leq \frac{r_i \hat{r}}{\rho_i \hat{\rho}} \leq \left(2+\sqrt{2}\right)\kappa$$

Og

$$||B|| \leq h/\hat{\rho} \leq 4h$$

$$||u - I_h u||_{m,T_j} \leq ch^{t-m}|u|_{t,T} \tag{19}$$

Aubin–Nitsche

Theorem

Let H be a Hilbert space with norm $|\cdot|$ and a scalar product $(\cdot,\cdot)$. Let $V \subset H$ be a Hilbert space for another norm $\|\cdot\|$, let the imbedding $V \hookrightarrow H$ be continuous, and $\forall g \in H$, let $\varphi_g \in V$ denote the unique weak solution to

$$a(w, \varphi_g) = (g, w) \quad \forall w \in V. \tag{20}$$

Here $a(\cdot,\cdot)$ is a bilinear continuous form. Then the finite element solution $u_h \in S_h \subset V$ obeys

$$|u - u_h| \leq C\|u - u_h\| \sup_{g \in H} \left\{ \frac{1}{|g|} \inf_{v \in S_h} \|\varphi_g - v\| \right\},$$

where $\sup$ *is over all $g \in H$ such that $|g| \neq 0$.*

Bevis

$$a(u, v) = f(v) \quad \forall v \in V,$$
$$a(u_h, v) = f(v) \quad \forall v \in S_h$$
$$\Downarrow$$
$$a(u - u_h, v) = 0 \quad \forall v \in S_h$$

Bevis

$$
\begin{aligned}
(g, u-u_h) &= a(u-u_h, \varphi_g) \\
&= a(u-u_h, \varphi_g - g) \\
&\leq C\|u-u_h\| \cdot \|\varphi_g - v\|.
\end{aligned}
$$

$$
\begin{aligned}
|u-u_h| &= \sup_{g\in H} \frac{(g, u-u_h)}{|g|} \\
&\leq C\|u-u_h\| \sup_{g\in H}\left\{ \frac{1}{|g|} \inf_{v\in S_h} \|\varphi_g - v\| \right\}.
\end{aligned}
$$

Anvendt Aubin-Nitsche

Theorem

Assume $\mathcal{T}_h$ is a family of shape regular triangulations of Ω. If $u \in H^1(\Omega)$ is the solution of the variational problem, then

$$\|u - u_h\|_0 \leq cCh\|u - u_h\|_1.$$

If $f \in L_2(\Omega)$ and $u \in H^2(\Omega)$, then

$$\|u - u_h\|_0 \leq cC^2h^2\|f\|_0.$$

Bevis

$$H = H^0(\Omega) = L_2(\Omega) \quad \text{and} \quad V = H_0^1(\Omega).$$

$$|\cdot| = \|\cdot\|_0 \quad \text{and} \quad \|\cdot\| = \|\cdot\|_1,$$

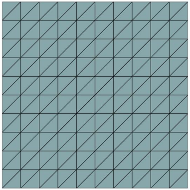
$$\|u - u_h\|_0 \leq C\|u - u_h\|_1 \sup_{g \in H} \left\{ \frac{1}{|g|} \inf_{v \in S_h} \|\varphi_g - v\|_1 \right\},$$

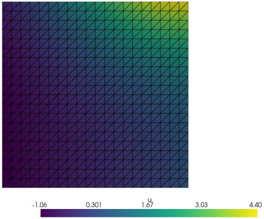
Opstilling af problem

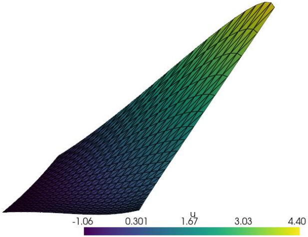
$$k(x,y) = \mathrm{e}^{x+y}\cos(x)\sin(y) + x \quad \text{on } \Omega$$

$$\begin{aligned} Lu &= -\operatorname{div}\nabla k && \text{on } \Omega \\ u &= k && \text{on } \partial\Omega, \end{aligned}$$

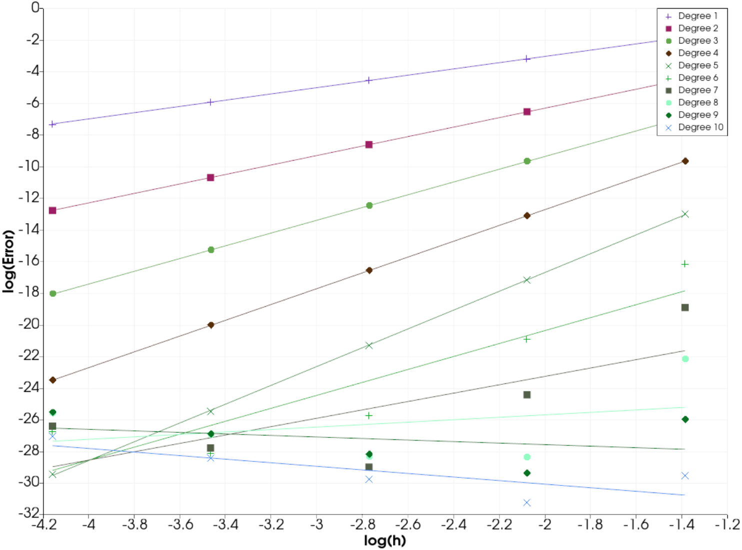
Domæne





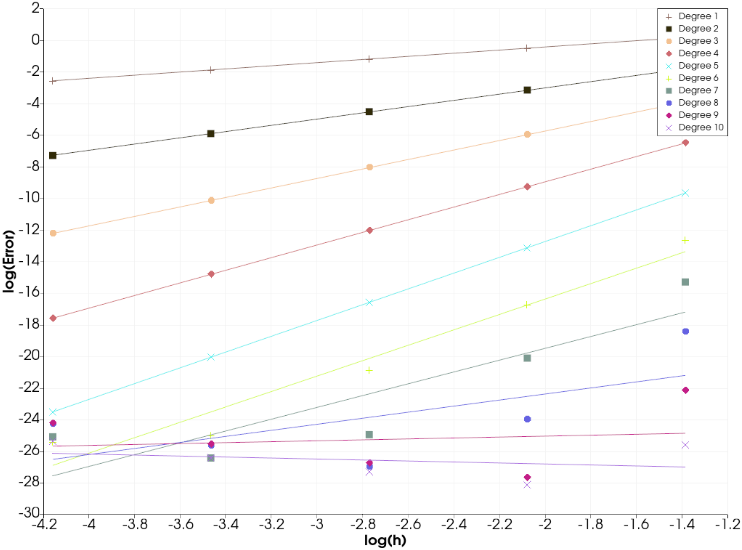


Brug af L_2



Degree 1
Degree 2
Degree 3
Degree 4
Degree 5
Degree 6
Degree 7
Degree 8
Degree 9
Degree 10
log(Error)
log(h)
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
-22
-24
-26
-28
-30
-32
-4.2
-4
-3.8
-3.6
-3.4
-3.2
-3
-2.8
-2.6
-2.4
-2.2
-2
-1.8
-1.6
-1.4
-1.2

Brug af H^1



Degree 1
Degree 2
Degree 3
Degree 4
Degree 5
Degree 6
Degree 7
Degree 8
Degree 9
Degree 10
log(Error)
log(h)
2
0
-2
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-3
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-2.2
-2
-1.8
-1.6
-1.4
-1.2