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Numerical Simulation using Runge-Kutta Method for the System with Uncertain Discontinuous Change

Masanobu Koga, Nachi Tanimura, Tsuyoshi Sato

Kyushu Institute of Technology, 680-4 Kawazu, Iizuka, Fukuoka, Japan

koga@ces.kyutech.ac.jp, tanimura@mk.ces.kyutech.ac.jp, and sato@mk.ces.kyutech.ac.jp

Abstract: In the Runge-Kutta method, the action of the system at the change time is not strictly taken into consideration. In this paper, we propose the new simulation method in which discontinuous change is considered strictly by improving the Runge-Kutta method.

Keywords: Numerical simulation, Discontinuous change, Runge-Kutta method

1. Introduction

There are some systems whose components (dynamics on states) change suddenly. For example, the system (change of agreement, reversal of speed, etc.) on whose state change discontinuously such as the system which speed reverses by rebounding of a ball(Fig.1, Fig.2) or the system from which a robot's order changes before and after a jump in an instant like a jump robot[1].

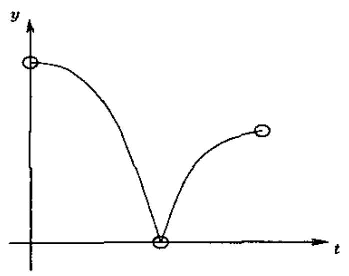

Figure 1: Rebounding of a ball

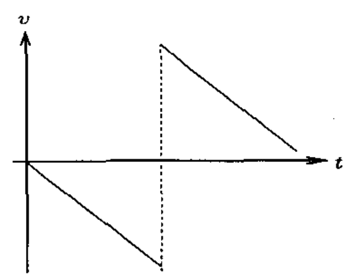

Figure 2: The speed change with rebounding of a ball

When we do the simulation of those systems, the simulator must detect a change precisely which takes place suddenly and discontinuously.

Conventionally, the variable time step method, which calculates by making the time-step smaller when the change is sharp, and calculate by making the time-step larger when the change is in the loose, has been applied to the numerical simulation[2]. By the popular Runge-Kutta method the solution is acquired in the same accuracy as the limited approximation of Taylor series expansion. In the method the action of the system at the change time is not strictly taken into consideration. Thus it is difficult to obtain the strict solution at the change time though a certain amount of reliance is obtained.

In this paper, we propose a new simulation method in which the discontinuous change is considered strictly by improving the Runge-Kutta method.

2. Runge-Kutta method

In the Runge-Kutta method, the solution is obtained in the same accuracy of the limited approximation of Taylor series expansion. In stead of calculating the coefficients of high order derivatives, we calculate some coefficients of the first derivative at some points between the current sample time t and the next sample time $t+h$, where h is the time step of the simulation.

Consider the following system

$$\dot{x} = \frac{dx}{dt} = f(t,x)$$

where, x is the state of the system and $f(t,x)$ is a differentiable function of t and x. The formula of $(p+1)$-th order Runge-Kutta method[3] is written in the form:

$$\begin{aligned}
x_{i+h} &= x_i + \Delta x \\
\Delta x &= \lambda_0 k_0 + \lambda_1 k_1 + \lambda_2 k_2 + \cdots\cdots + \lambda_p k_p \\
k_0 &= hf(t_i, x_i) \\
k_1 &= hf(t_i + \alpha_1 h, x_i + \beta_1 k_0) \\
k_2 &= hf(t_i + \alpha_2 h, x_i + \beta_2 k_0 + \gamma_2 k_1) \\
&\cdots\cdots \\
k_p &= hf(t_i + \alpha_p h, x_i + \beta_p k_0 + \gamma_p k_1 + \cdots) \quad (1)
\end{aligned}$$

where $\lambda_1, \cdots, \lambda_p$, $\alpha_1, \cdots, \alpha_p$, $\beta_1, \cdots, \beta_p$ and $\gamma_1, \cdots, \gamma_p$ are determined according to the coefficients of Taylor series expansion, and the order of the error is $O(h^{p+1})$. The method acquires the same accuracy as the $(p+1)$-th order approximation of Taylor series expansion.

The amount of the calculation step increases as the Runge-Kutta method becomes high order. RKF45[3] method was proposed in order to reduce the problem.

By RKF45 method, the solution whose accuracy is $O(h^6)$, is obtained by

$$\Delta x_i = \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 \quad \frac{9}{50}k_5 + \frac{2}{55}k_6$$

where

$$\begin{aligned}
k_1 &= hf(t_i, x_i) \\
k_2 &= hf\left(t_i + \frac{h}{4}, x_i + \frac{k_1}{4}\right) \\
k_3 &= hf\left(t_i + \frac{3h}{8}, x_i + \left(\frac{3}{32}k_1 + \frac{9}{32}k_2\right)\right) \\
k_4 &= hf\left(t_i + \frac{12h}{13}, x_i + \left(\frac{1932}{2197}k_1 \quad \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right)\right) \\
k_5 &= hf\left(t_i + h, x_i + \left(\frac{439}{216}k_1 \quad 8k_2 + \frac{3680}{513}k_3 + \frac{845}{4104}k_4\right)\right) \\
k_6 &= hf\left(t_i + \frac{h}{2}, x_i + \left(\quad \frac{8}{27}k_1 + 2k_2\right.\right. \\
&\qquad \left.\left. \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 \quad \frac{11}{40}k_5\right)\right) \qquad (2)
\end{aligned}$$

The number of derivative calculations is 6 while by Runge-Kutta method the number is 10. By choosing four k_i in (2), the solution $O(h^5)$ is obtained by

$$\Delta x_i^* = \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 \quad \frac{1}{5}k_5$$

3. Determination of the change time

3.1 Conventional method

The RKF45[3] method saves the number of the calculation of the derivative, and makes the error becomes small enough by adjusting the time step automatically. However, the method adjusts the time step by comparing the value of current time with the value of the last sample time, and guarantees the accuracy of the solution.

Figure 3 shows a simulation result by RKF45 method when the state of the system changes discontinuously. With a small change of the state, the time step is large (a). While the change of the state is large, time step is fine around the change time. It means that the accuracy of the change time depends on the size of change. That is, it is impossible to determine the change time strictly. Moreover, when the change is smaller than the size of the accuracy of RKF45, the discontinuous change is missed.

We proposed the algorithm to determine the change time for the systems which change discontinuously [4].

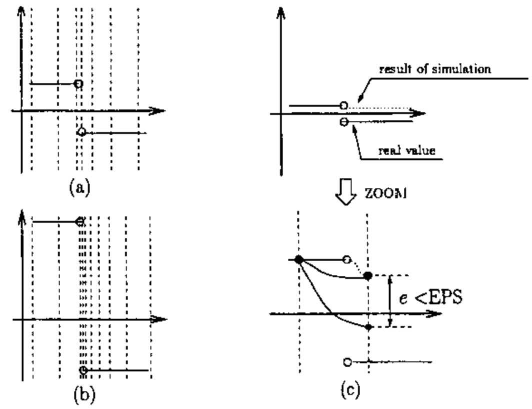

Figure 3: Auto adjusting time step using RKF45

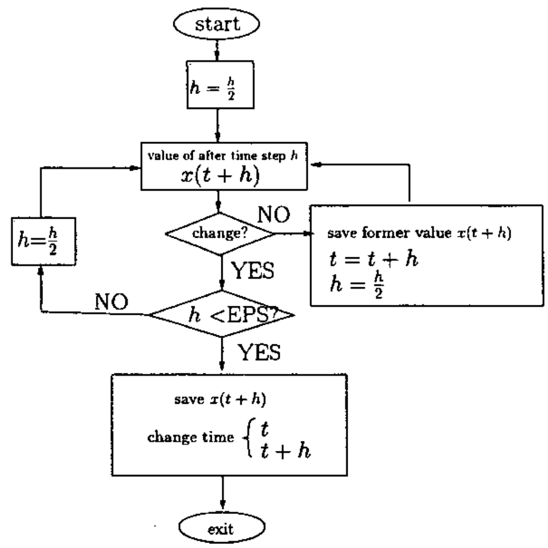

Figure 4: Flow chart to determine change time

First, the current state $x(t)$ and the time step h are given, and $x(t+h)$ is calculated by Runge-Kutta method. If $x(t+h)$ doesn't meet the condition that the change has occurred, we save the value and set forward with time step h. When $x(t+h)$ meets the condition, since it means that the system has changed, we make the time step h half ($h = \frac{h}{2}$), and we go back to the first calculation. Here, if the time step h becomes smaller than the specified accuracy, we regard the time as the change time. The flow chart of the algorithm is shown in Fig 4.

3.2 Example

In order to check the validity of the algorithm, we do a simulation of a rebounding of ball.

The results are shown in Fig.5 and Fig.6.

In Fig.5, x_1 is the height, x_2 is the speed. x_2 changes

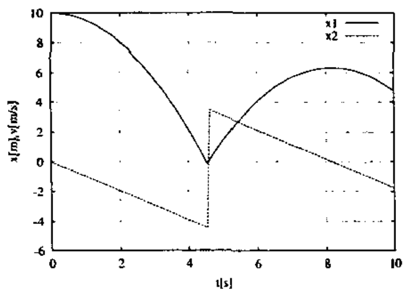

Figure 5: Position and speed of a rebounding of ball

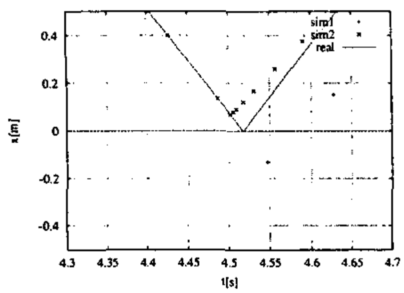

Figure 6: Comparison with the analytical solution

discontinuously at the moment the ball contacts the floor. In Fig.6, the sllid line is the analytical solution. "siml" is the result by the conventional method (only adjusting the time step automatically), and "sim2" is the result by the proposed method. It turns out that the proposed method gives closer solution to the analytical solution.

3.3 Problem of the algorithm

When the change time is determined by the method in the previous section, the actual discontinuous change may happen between the time step of the Runge-Kutta method, as shown in Fig.7.

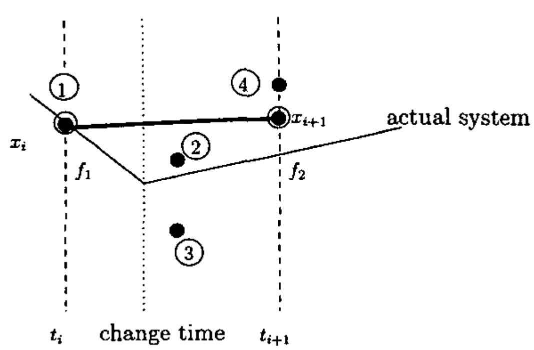

Figure 7: Change time and Runge-Kutta method

When we calculate the solution x_{i+1} from x_i using the 4th order Runge-Kutta method, it is given by

$$x_{i+1} = x_i + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)h \quad (3)$$

where

$$k_0 = f_1(t_i, x_i) \quad (4)$$

$$k_1 = f_1\left(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_0\right) \quad (5)$$

$$k_2 = f_1\left(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_1\right) \quad (6)$$

$$k_3 = f_1(t_i + h, x_i + hk_2) \quad (7)$$

Here,(4)~(7) correspond to ① ~ ④ in Fig.7. The change happens between ① and ②. Although (4)~(7) are altogether calculated using the old function f_1, (5)~(7) should be calculated using the new function f_2.

4. Runge-Kutta method for the uncertain discontinuous change

4.1 New RKF45 method

In this section a new RKF45 method for the uncertain discontinuous change is proposed. The method determines the change time by the method in the previous section, and when it passes the change time, it uses the fixed time step h_c only once.

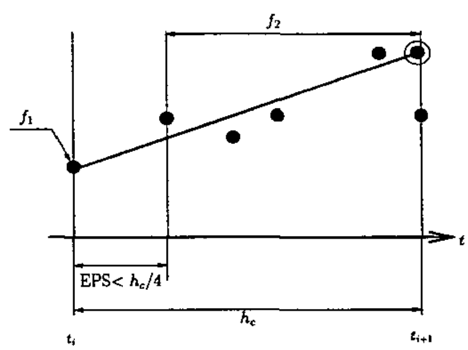

Figure 8: New RKF45 method

The solution is calculated with the accuracy of EPS as shown in Fig.9(a).
where

$$\text{EPS} < \frac{h_c}{4} \quad (8)$$

It is guaranteed that the change happens between t_i and $t_i+h/4$. In RKF45 method, k_2, k_3, k_4, k_5 and k_6 are calculated after $h/4$. When (8) is assumed, k_1 should

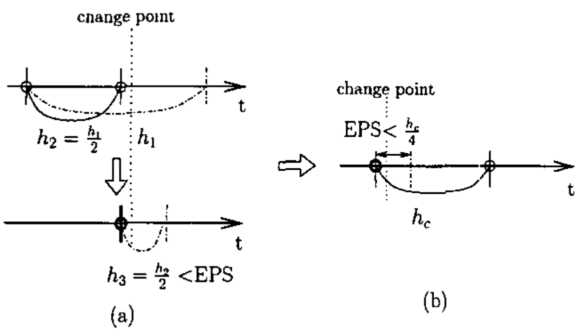

Figure 9: The accuracy of change time and Runge-Kutta method

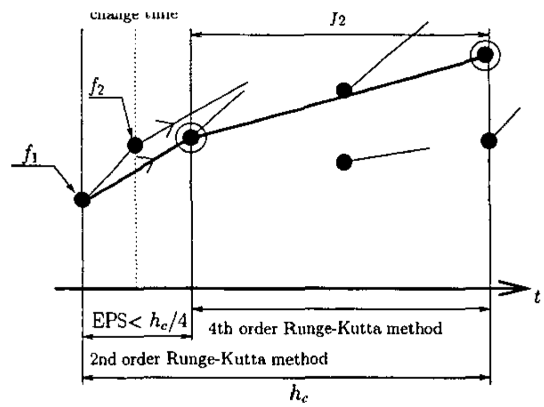

Figure 10: Modified Euler's method + 4th order Runge-Kutta method

use the old function, and k_2, k_3, k_4, k_5 and k_6 should use the new function.

The method seems good, but it loses the mathematical guarantee since the method uses multiple functions in Runge-Kutta method based on the Taylor series expansion.

4.2 New modified Euler's method

We proposed a new modified Euler's method which calculates the solution by using the values before and after the change time as .

$$x_{i+1} = x_i + hk_1 \tag{9}$$

where

$$\begin{aligned} k_0 &= f(t_i, x_i) \\ k_1 &= f\left(t_i + \frac{h}{2}, x_n + \left(\frac{h}{2}k_0\right)\right) \end{aligned}$$

The change time is determined by the proposed method and it passes the change time with the fixed time step h_c as shown in Fig.10. $x(t_i + h_c/4)$ is calculated by the modified Euler's method, and the 4th order Runge-Kutta method is used to calculate $x(t + h_c)$. In the step of the 4th order Runge-Kutta method, the new function is used.

This method assumes that the change time happens in the middle of $h_c/4$, i.e. $t = h_c/8$.

4.3 Example

We consider the simulation of a rebounding ball. The result by the 5th order Runge-Kutta method and the modified Euler's method are shown in Fig.11. They have the same number of calculation of the derivative of the state. The dotted line is analytical solution, the points with '+' is the solution by the 5th order Runge-Kutta method, the line with '×' is the result of the 4th order Runge-Kutta method with the modified Euler's method. The change is detected around $t = 1.4391$, and the value near $t = 1.4395$ is calculated. It turns out that the modified Euler's method gives a the closer solution to the analytical solution.

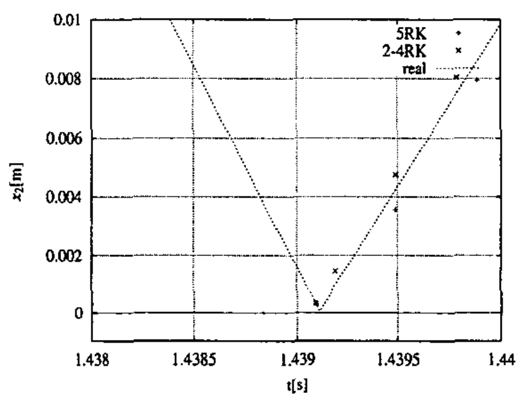

Figure 11: Comparing two method about position

5. Summary

In this paper, we proposed a new numerical method for the simulation of the system which changes discontinuously. The method determines the interval where the change happens and uses the 5th order Runge-Kutta method which changes the model (expression) of the system. And assuming that the change happens at the center of the interval, a modified Euler's method was proposed. The latter method is useful when the value at the change time is to be calculated like the collision.

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