Recursion

COMS W1007
Introduction to Computer Science

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The Fibonacci Sequence

The Fibonacci numbers are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

We can calculate the nth Fibonacci number ($n \geq 2$) using the formula:

$$F_n = F_{n-1} + F_{n-2}$$

Recursion

- Defining a function in terms of itself is called *recursion*. We call a method that calls itself a *recursive method*.
- We don't have to do anything special to write a recursive method in Java; any method can call itself.
- Each recursive call has its own distinct set of parameters and local variables. A recursive call is a separate entry on the execution stack.

The Basis Case

- In order for recursion to work correctly, every recursive method must have a *basis case*.
- The basis case is an input for which the method does not make a recursive call. The basis case prevents *infinite recursion*.
- It's not enough for there to simply *be* a basis case; the values of the input must reliably *approach* the basis value.

The Fibonacci Sequence Redux

In the Fibonacci sequence, the basis cases are $n = 0$ and $n = 1$. Since the sequence is only defined for nonnegative integers n, the recursive definition will always approach 0.

Basis cases:

$$F_0 = 1$$
$$F_1 = 1$$

Recursive case ($n \geq 2$):

$$F_n = F_{n-1} + F_{n-2}$$

The Factorial Function

The factorial function is:

$$n! = n(n-1)\cdots 1$$

We can define the factorial function recursively as:

Basis case:

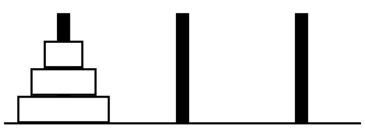
$$1! = 1$$

Recursive case ($n > 1$):

$$n! = n \cdot (n-1)!$$

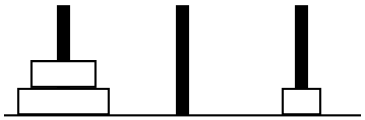
The Towers of Hanoi

The Towers of Hanoi problem is to move a stack of plates from the first post to the third. You may only move one plate at a time and a larger plate cannot be stacked on top of a smaller one.

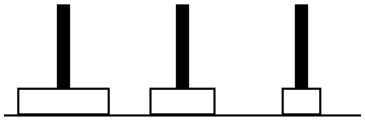


The Towers of Hanoi

Step 1:

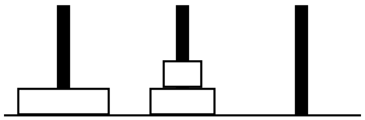


Step 2:

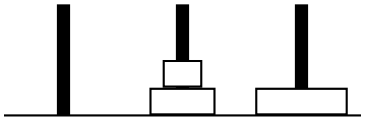


The Towers of Hanoi

Step 3:

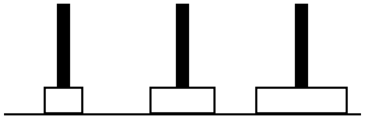


Step 4:

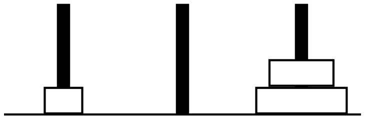


The Towers of Hanoi

Step 5:

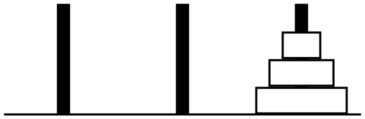


Step 6:



The Towers of Hanoi

Step 7:



The Towers of Hanoi

The Towers of Hanoi is a classic example of a recursive problem. To solve it for n plates:

1. Move $n-1$ plates from the first post to the extra post.
2. Move the largest plate to the destination post.
3. Move $n-1$ plates from the extra post to the destination.

The basis case (1 plate) is trivial.

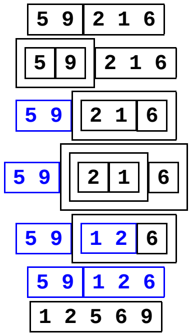
Merge Sort

Input: A list of numbers $X_1, X_2, \ldots, X_n$ and a range $i..j$ to sort. (i and j are initially 1 and n, respectively).

Output: A list in ascending order.

1. If $i = j$, goto Step 6.
2. $k := (i + j) \div 2$.
3. $Y :=$ sort $X_{i..k}$.
4. $Z :=$ sort $X_{k+1..j}$.
5. Merge Y with Z into X'.
6. Return sorted list X'.

Merge Sort: Example



Merge Sort: Efficiency

From the top down, a merge sort of a list of length n will:

- Merge two lists of length $n/2$: approximately n operations.
- Merge four lists of length $n/4$: n operations.
- Merge eight lists of length $n/8$: n operations.
- And so on, until we have n lists of length one: n operations.

If each step takes n operations, the question is: how many steps until we reach the basis case?

Merge Sort: Efficiency, 2

- If we continually divide n by 2, it takes $\lfloor \log_2 n \rfloor$ steps to reach 1.
- Since we use $\log_2 n$ a lot in computer science, we like to abbreviate it $\lg n$.
- Merge sort performs n operations in every one of $\lfloor \lg n \rfloor$ steps. The running time is:

$$T(n) \approx n \lg n$$

- Recall that the other sorts we studied took approximately n^2 operations. $n \lg n$ is much better than n^2. (Compare them for $n = 100$ or $n = 1000$.)

Binary Search

Input: A sorted list $X_1, X_2, \ldots, X_n$ and a number to find a.

Output: A boolean value $Found$ indicating whether a is contained in X.

1. $Found := 0$, $i := 1$, $j := n$.
2. $k := (i + j) \div 2$.
3. If $j \le i$, go to Step 6.
4. If $X_k > a$, $j := k - 1$, go to Step 2.
5. If $X_k < a$, $i := k + 1$, go to Step 2.
6. If $X_k = a$, $Found := 1$.

Binary Search: Efficiency

- Each step of the binary search divides the list in 2. The number of steps it will take to find a number in a list of length n is:

$$T(n) \approx \lg n$$

- The log function grows quite slowly:

$$\lg 100 \approx 5$$
$$\lg 1,000 \approx 10$$
$$\lg 1,000,000 \approx 20$$

Calculating Fibonacci Numbers

Consider a recursive method for calculating the nth Fibonacci number:

```
int fibo(int n) {
  if( n==0 || n==1 )
    return 1 ;
  else
    return fibo(n-1) + fibo(n-2) ;
}
```

What is the running time of **`fibo`** on an input n?

Calculating Fibonacci Numbers, 2

From the top down, a call to **`fibo`** will:

- Add the result of two recursive method calls: 1 operation.
- Each recursive call will add the value of two further recursive calls (4 in all): 2 operations.
- Each of those calls will add the value of two further recursive calls (8 in all): 4 operations.
- And so on, until we reach **`fibo(1)`** and **`fibo(0)`**.

It will take $n-1$ steps to reach the basis case.

Calculating Fibonacci Numbers, 3

The running time of **`fibo`** grows exponentially with n:

$$\begin{aligned} T(n) &= 1 + 2 + 4 + \cdots + 2^{n-2} \\ &= \sum_{i=0}^{n-2} 2^i \\ &= 2^{n-1} - 1 \\ &\approx 2^n \end{aligned}$$

This is bad. Exponential growth is worse than n^2. In fact, its worse than n^c for any c.

Iteration vs. Recursion

Consider an iterative method for calculating Fibonacci numbers:

```
int fibo2(int n) {
  int n = 1, n2 = 1 ;
  for( int i=2 ; i < n ; i++ ) {
    int tmp = n2 ;
    n2 = n ;
    n = tmp + n2 ;
  }
  return n ;
}
```

Iteration vs. Recursion: 2

- **`fibo2`** performs one addition for each number in the sequence, from 2 to n. Thus, it is linear in n.

$$T(n) \approx n$$

- Recursion is not always the best solution to a problem. Even when the problem itself is defined recursively.
- We can usually solve a problem iteratively (i.e., using loops) or recursively. Which one we choose depends on the particular problem and personal taste.

Orders of Magnitude

- When we say the running time of an algorithm is approximately $f(n)$, what we really mean is it is on the same *order of magnitude* as $f(n)$.
- We express orders of magnitude using the notation $\Theta(f(n))$. $T(n) = \Theta(f(n))$ means that an algorithm grows neither faster nor slower than $f(n)$.
- The orders of magnitude are related as follows:

$$c \prec \lg n \prec n \prec n \lg n \prec n^c \prec c^n$$

  where c is a constant.

Orders of Magnitude, 2

- Constant-time algorithms ($\Theta(1)$) are as good as it gets. That means we can calculate the result in a fixed number of steps, irregardless of the input.
- Exponential algorithms ($\Theta(c^n)$) are just about as bad as it gets. We call exponential algorithms *intractable*—it is not practical to solve them for anything but very small inputs.
- Quadratic and higher polynomial algorithms ($\Theta(n^c)$) are tractable but slow. Linear ($\Theta(n)$), logarithmic ($\Theta(\lg n)$) and linear-logarithmic ($\Theta(n \lg n)$) algorithms are what we shoot for.