

# Homework 5

Time Series Analysis (STAT 6391)

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## Problem 4.3

We consider the following two models:

$$(i) \quad x_t = 0.80x_{t-1} - 0.15x_{t-2} + w_t - 0.30w_{t-1}$$

$$(ii) \quad x_t = x_{t-1} - 0.50x_{t-2} + w_t - w_{t-1}.$$

### Part (a)

Checking the models for parameter redundancy.

**For model (i):**

$$\begin{aligned} x_t &= 0.80x_{t-1} - 0.15x_{t-2} + w_t - 0.30w_{t-1} \\ \Rightarrow x_t - 0.80x_{t-1} + 0.15x_{t-2} &= w_t - 0.30w_{t-1} \\ \Rightarrow (1 - 0.80B + 0.15B^2)x_t &= (1 - 0.30B)w_t \\ \Rightarrow (1 - 0.5B)(1 - 0.30B)X_t &= (1 - 0.30B)w_t \end{aligned}$$

The common term  $(1 - 0.30B)$  on each side of the last line indicates that the model is over-parameterized or suffers from parameter redundancy, and can be reduced by canceling out the factor  $(1 - 0.30B)$  as follows.

$$\begin{aligned} \Rightarrow (1 - 0.5B)X_t &= w_t \\ \Rightarrow x_t - 0.50Bx_t &= w_t \\ \Rightarrow x_t &= 0.50x_{t-1} + w_t, \end{aligned}$$

which suggests that model (i) is actually an AR(1) model.

**For model (ii):**

$$\begin{aligned} x_t &= x_{t-1} - 0.50x_{t-2} + w_t - w_{t-1} \\ \Rightarrow x_t - x_{t-1} + 0.50x_{t-2} &= w_t - w_{t-1} \\ \Rightarrow (1 - B + 0.50B^2)x_t &= (1 - B)w_t \end{aligned}$$

We notice that it is not possible to factorize the left hand side to have a term in  $(1 - B)$ . Thus, we conclude that model (ii) is not over-parameterized (there is no parameter redundancy). Consequently, model (ii) cannot be reduced any further and so it remains an ARMA (2,1) model.

For confirmation, we also obtained the roots of the two models in R. Considering the roots for model (i), there is one common factor with root 3.33 and hence the model is over-parameterized and can be reduced. There is no common root for model (ii) and hence the model is not over-parameterized and cannot be reduced any further as noted above.

```
### checking for parameter redundancy

# Model (i)
AR <- c(1, -.8, .15)
MA <- c(1, -.3)
polyroot(AR)
```

```
[1] 2.000000-0i 3.333333+0i
```

```
polyroot(MA)
```

```
[1] 3.333333+0i
```

```
# Model (ii)
```

```
AR <- c(1, -1, .5)
```

```
MA <- c(1, -1)
```

```
polyroot(AR)
```

```
[1] 1+1i 1-1i
```

```
polyroot(MA)
```

```
[1] 1+0i
```

```
#
```

## Part (b)

Next, we determine if the two models are causal and/or invertible.

```
# ----- Problem 4.3 (b)
```

```
# model (i)
```

```
AR <- c(1, -.5)      # new AR coefs on the left
```

```
polyroot(AR)
```

```
[1] 2+0i
```

The above result shows that the reduced form of model (i) is causal since the AR term has no roots that are less than or equal to one in magnitude. It seems that we cannot draw a firm conclusion about invertibility using the root principle since the MA part was shown to be redundant.

However, If we assume that the MA part is zero (in the case of the reduced model), then the root of the MA would be zero which suggests that the model is not invertible according to the root criteria.

```
# Model (ii)
```

```
AR <- c(1, -1, .5)
```

```
MA <- c(1, -1)
```

```
polyroot(AR)
```

```
[1] 1+1i 1-1i
```

```
polyroot(MA)
```

```
[1] 1+0i
```

The roots for the AR term and the MA term indicate that model (ii) is causal but not invertible because the roots of the AR part are greater than 1 in magnitude ( $\approx 1.41$ ) while the single root of the MA part is equal to 1.

### Part (c)

We found the first 50 coefficients of the causal  $[MA(\infty)]$  and invertible  $[AR(\infty)]$  representations of models (i) and model (ii), respectively, and plotted the results as shown in Figure 1.

The results suggest that the reduced form of model (i),  $AR(1)$ , is both causal and invertible since the coefficients for the causal and invertible representations converge to zero. However, model (ii) appears to be only causal since the causal representation converges to zero but the invertible part does not.

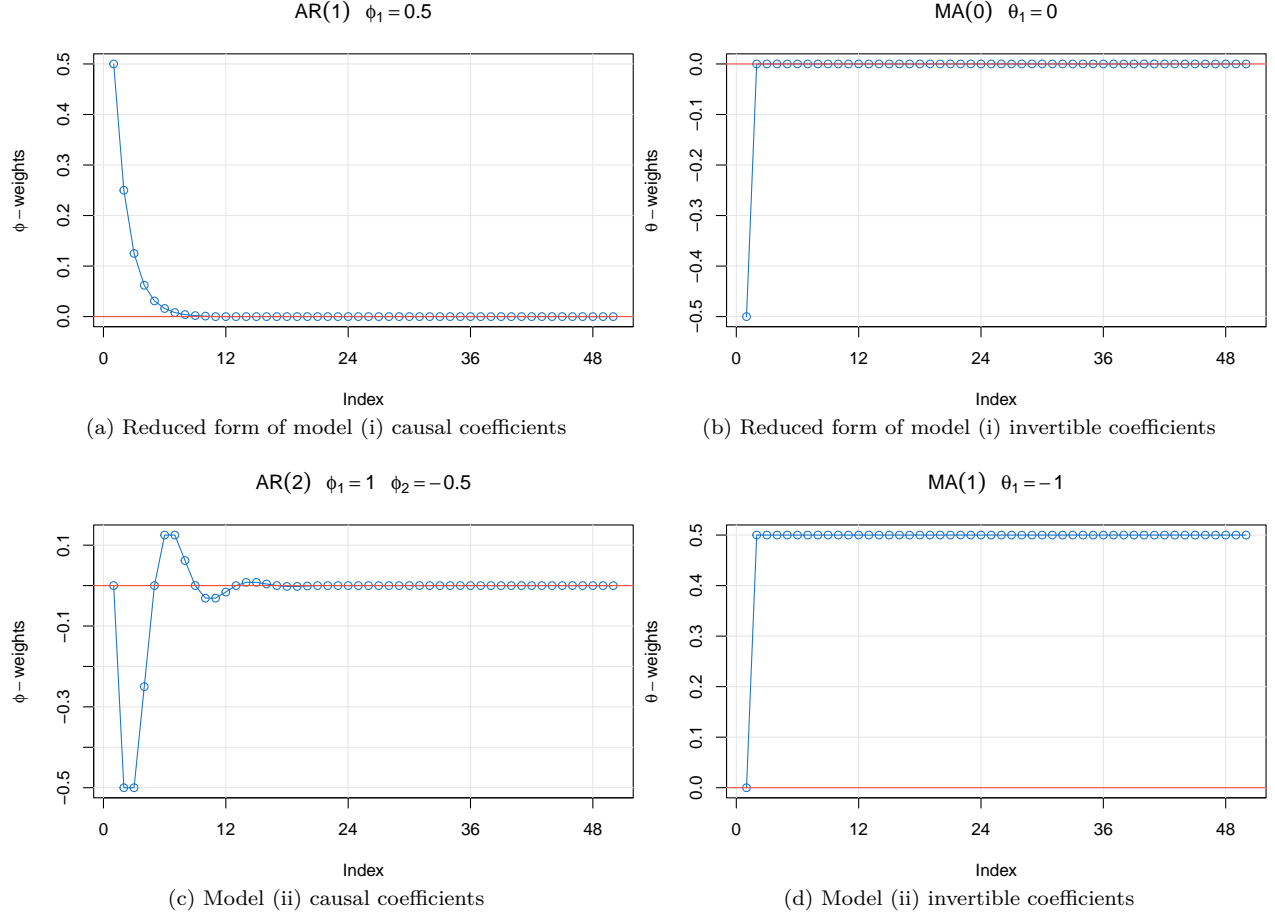


Figure 1: First 50 coefficients of the causal and invertible representations of the two models. For model (i), we showed the representations for the reduced model,  $AR(1)$ .

## Problem 4.4

### Part (a)

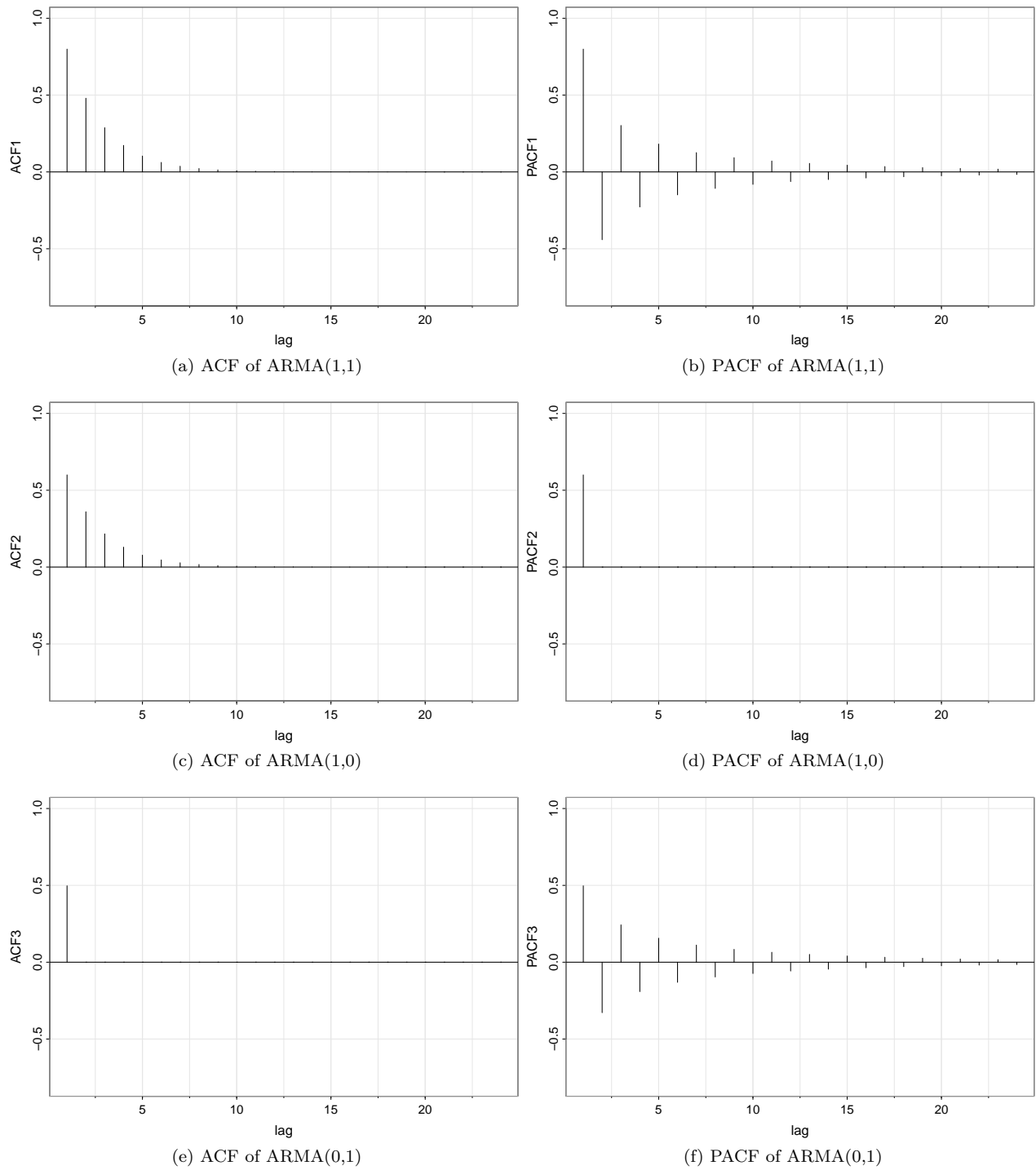
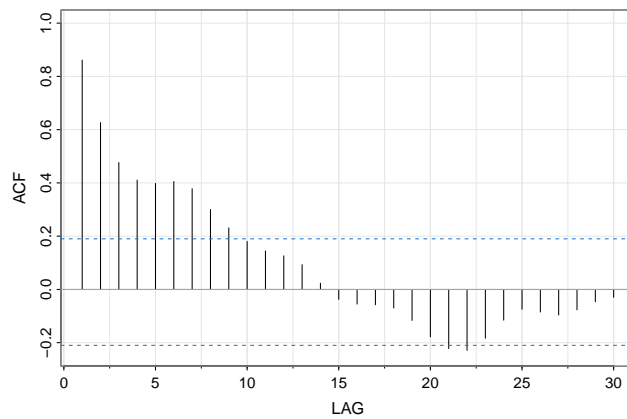
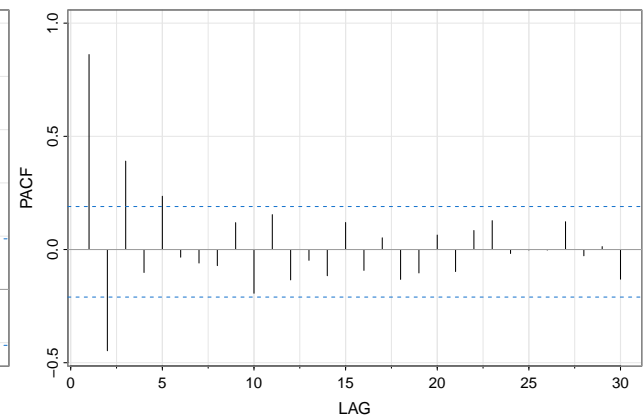


Figure 2: Comparing theoretical ACF and PACF of ARMA(1,1), an ARMA(1,0), and an ARMA(0,1) for  $\phi = 0.6$ ,  $\theta = 0.9$ .

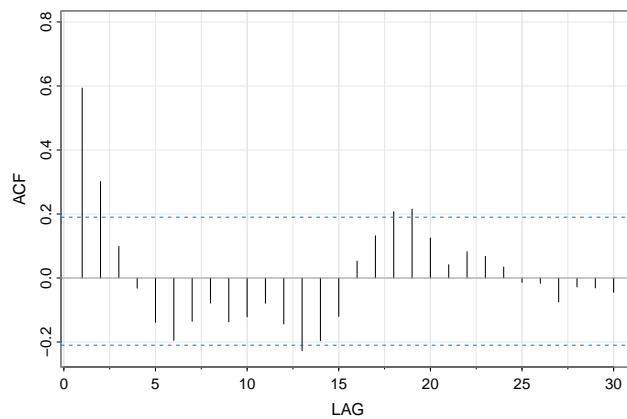
Based on Table 4.1 in (Shumway and Stoffer 2019), it is clear from Figure 2 that the theoretical ACFs and PACFs adequately determine the order of their respective models. Both ACF and PACF for the ARMA(1,1) model do not cut off as expected. On the other hand, the ACF for the ARMA(1,0) model tails off while the corresponding PACF cuts off at lag 1, signifying an autoregressive model of order 1, and the opposite is true for the ARMA(0,1) model where its ACF cuts off at lag 1 whereas its PACF tails off, signifying a moving average model of order 1.

**Part (b)**

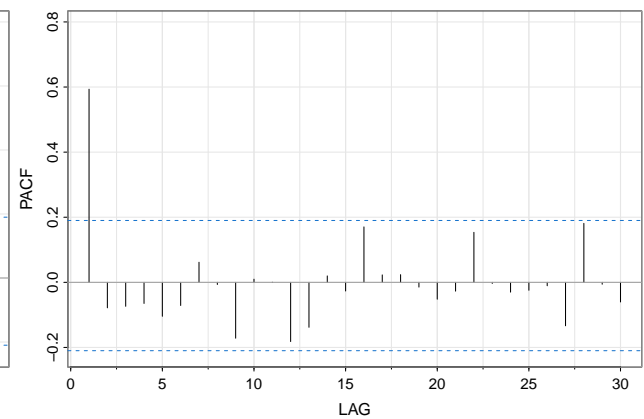
(a) Sample ACF of series simulated from ARMA(1,1)



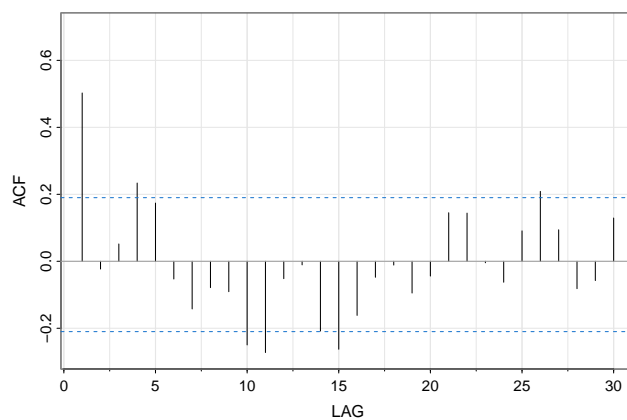
(b) Sample PACF of series simulated from ARMA(1,1)



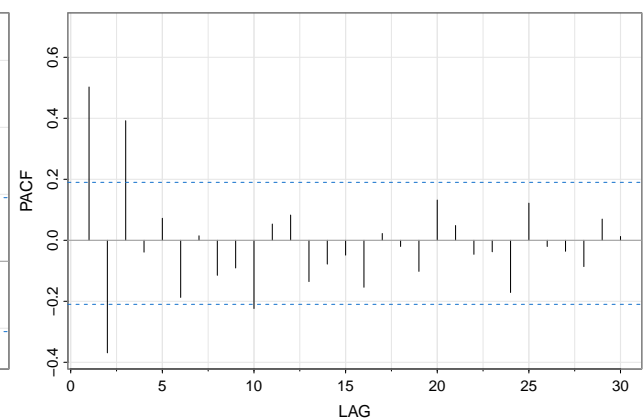
(c) Sample ACF of series simulated from ARMA(1,0)



(d) Sample PACF of series simulated from ARMA(1,0)



(e) ACF of ARMA(0,1)



(f) Sample PACF of series simulated from ARMA(0,1)

Figure 3: Comparing sample ACF and PACF of simulated series from ARMA(1,1), an ARMA(1,0), and an ARMA(0,1) for  $\phi = 0.6$ ,  $\theta = 0.9$ .  $n = 100$  observations were generated.

The results appear somewhat identical to the results we obtained for the theoretical ACFs and PACFs, but the confidence bounds extend much wider from zero. Specifically;

- **ARMA(1,1):**

Apart from the cyclical behavior observed, the sample ACF and PACF for the ARMA(1,1) look roughly the same as their corresponding theoretical counterparts in part (a). We do see more spikes but these additional spikes are essentially not significant since they fall within the confidence bounds.

- **ARMA(1,0):**

The sample PACF is approximately identical to the corresponding theoretical PACF. Both sample and theoretical PACFs cut off at lag 1, while the theoretical ACF tails off but the sample ACF appear to cut off after lag 2.

- **ARMA(0,1):**

While the theoretical ACF clearly cuts off at lag 1, the sample ACF appears to cut off at lag 2. However, the sample PACF is roughly identical to the theoretical PACF.

### **Part (c)**

Generally, the variations around the sample ACFs and PACFs for  $n = 100$  are larger than the variations around the sample ACFs and PACFs when  $n = 500$ . When we increased the sample size to 500, we obtained tight confidence bounds around zero. In this case, the sample ACFs and PACFs look very much identical to the corresponding theoretical ACFs and PACFs for all three models.

Overall, the results from parts (a) - (c) confirm the large sample distributional properties of the ACF and the PACF we have studied.



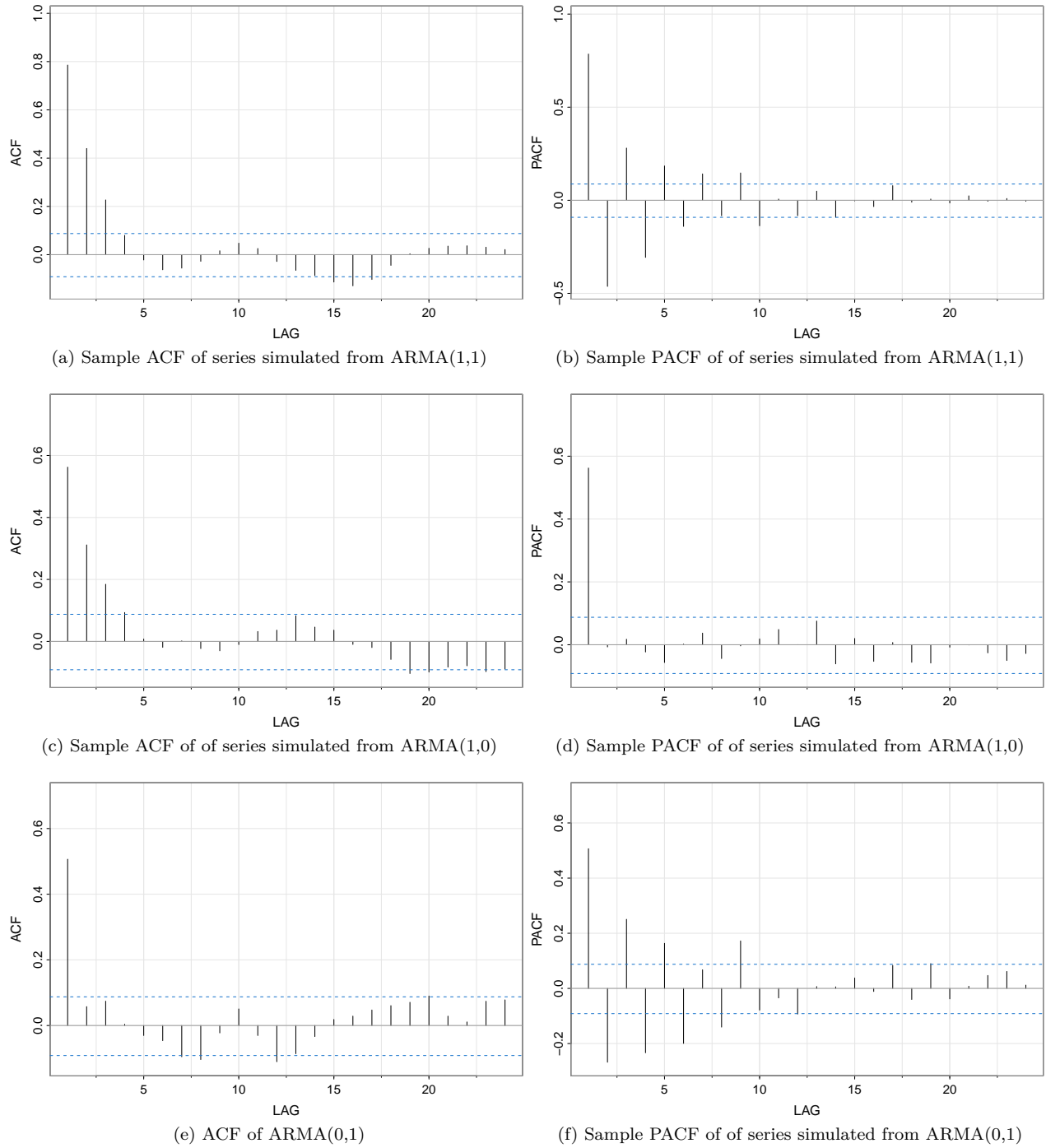


Figure 4: Comparing sample ACF and PACF of simulated series from ARMA(1,1), an ARMA(1,0), and an ARMA(0,1) for  $\phi = 0.6$ ,  $\theta = 0.9$ .  $n = 500$  observations were generated.

## Problem 4.5

### Part (a)

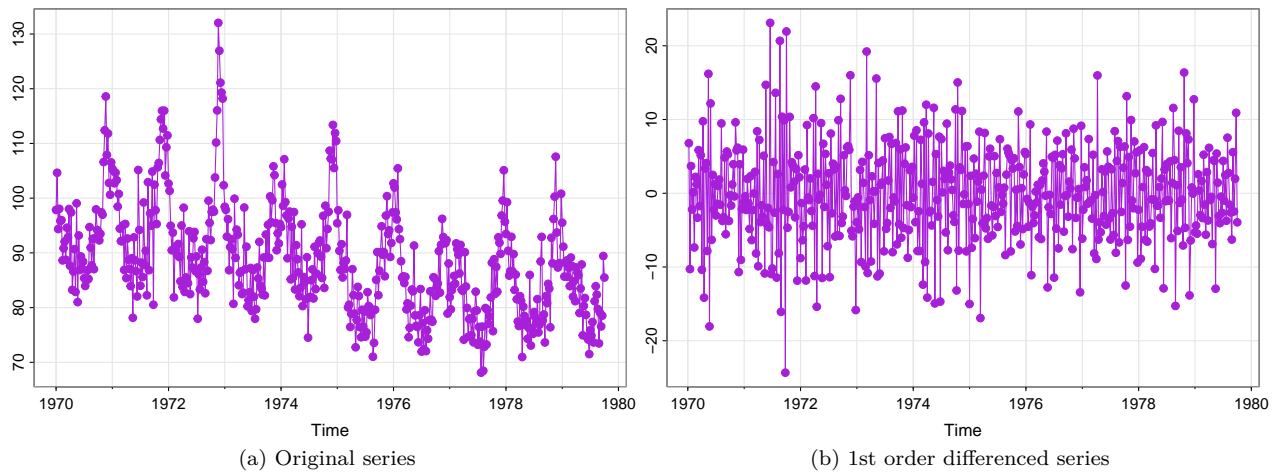


Figure 5: Original cardiovascular mortality series versus its differenced series

The original series appears to exhibit cyclical behavior (has strong seasonal component) with a downward trend over time on top of it. Here, differencing appears reasonable because it succeeded in eliminating the trend and at the same time coercing the series to stationarity as the plot of the differenced series resembles a white noise process.

### Part (b)

Plots of the calculated ACF and PACF values are shown in Figure 6.

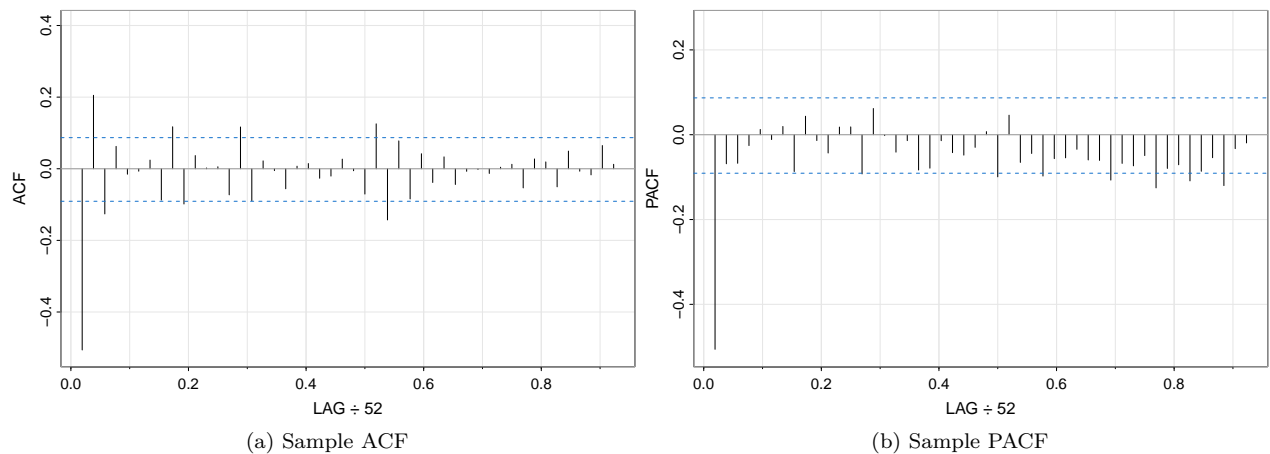


Figure 6: Sample ACF and PACF plots of the differenced cardiovascular mortality series.

According to Table 4.1 in (Shumway and Stoffer 2019), the results in Figure 6 suggest that an AR(1) is appropriate for  $x_t$ , since the ACF tails off but the PACF cuts off at lag 1 (that is, it only has a large value for  $h=1$  and then it is essentially zero for higher-order lags).

### Part (c)

We fitted an AR(1) model to  $x_t$  (first order differenced cardiovascular mortality) using the maximum likelihood via unconditional least squares with the help of the `sarima` script from `astsa`. We are working with a differenced data so we set  $d = 0$ . Additionally, the differenced data appear to have a zero mean function (see plot (b) in Figure 5) so we did not fit a mean to the data by specifying `no.constant=TRUE`.

The first output that follows shows the number of iterations performed until convergence. Diagnostic plots are shown in Figure 7, while the relevant estimates for the model parameters are presented in Table 1.

```
initial value 1.908720
iter 2 value 1.760373
iter 3 value 1.760373
iter 4 value 1.760373
iter 4 value 1.760373
final value 1.760373
converged
initial value 1.760679
iter 1 value 1.760679
final value 1.760679
converged
```

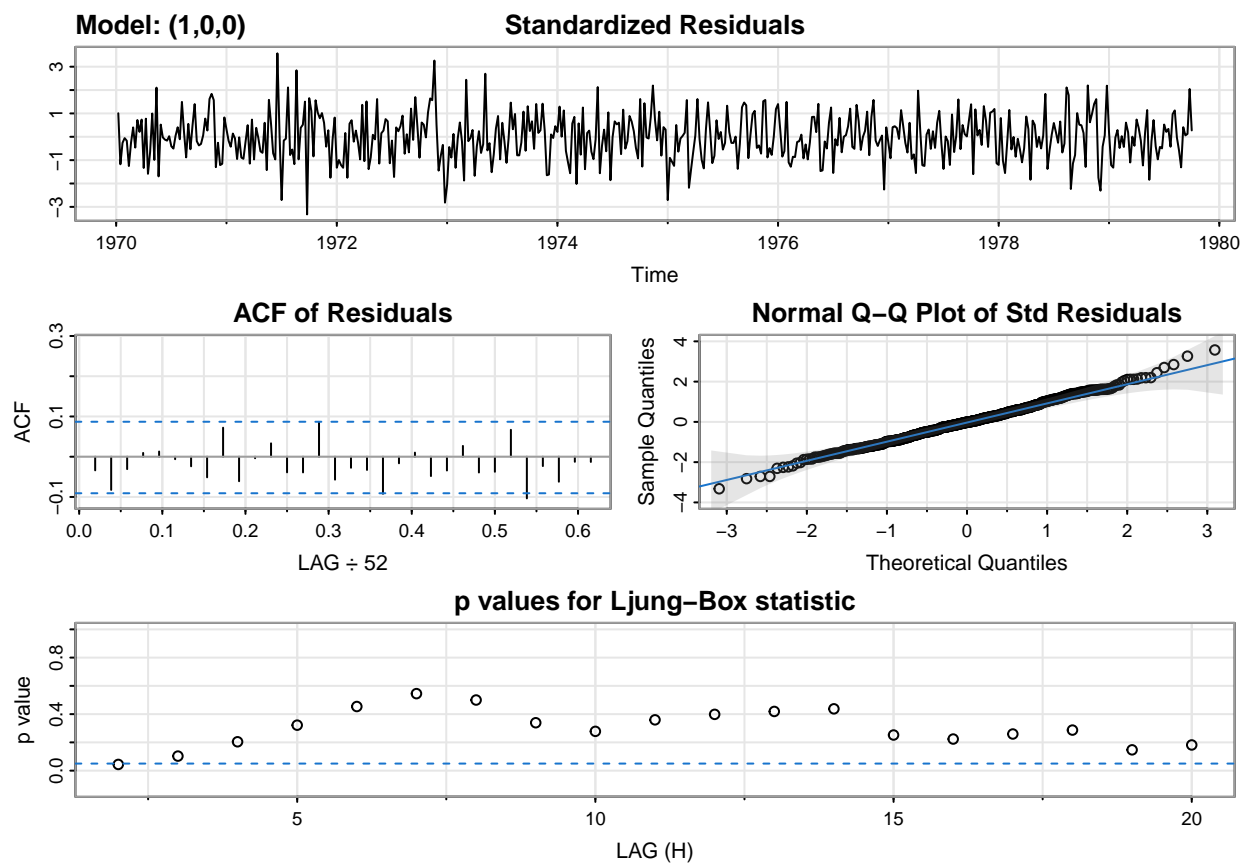


Figure 7: Residual diagnostics plots for the AR(1) model.

	Estimate	SE	t.value	p.value
<b>Coefficients</b>				
AR1	-0.5064	0.0383	-13.2224	0
<b>Other metrics</b>				
Sigma <sup>2</sup>	33.8106			
AIC	6.3671			
BIC	6.3838			

Table 1: Maximum likelihood (unconditional least squares) estimates for the AR(1) model fitted to  $x_t$

According to Table 1, the final estimates are  $\hat{\phi} = -0.5064$  with standard error 0.0383 and  $\hat{\sigma}_w^2 = 33.81$ . The ratio of the estimated coefficient to its standard error is substantially larger than 0 (p-value  $\approx 0$ ), so the regression parameter estimate is significant. The white noise variance is estimated as 33.81.

### Part (d)

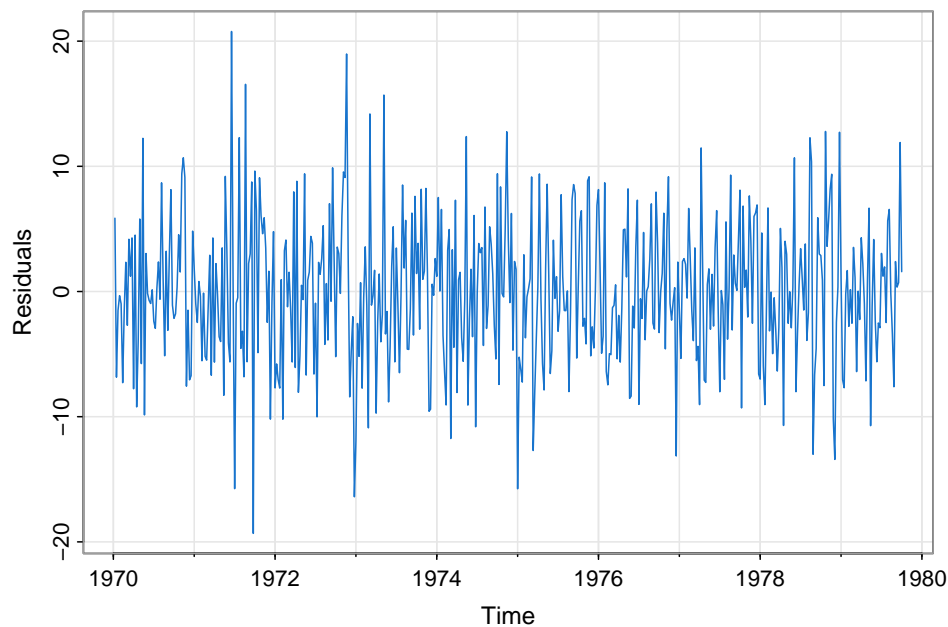


Figure 8: Plot of the residuals of the fit of AR(1) to  $x_t$ .

We examined the residuals using a plot of the residuals over time and the ACF of the residuals plot as presented in 7. The plot of the standardized residuals behave like a white noise process, which is also evident from the corresponding ACF plot of the residuals where the autocorrelation values after lag 0 are significantly not different from zero. In this case, we conclude that the residuals are white.

## Appendix: R codes

```

# Set global options for output rendering
knitr::opts_chunk$set(eval = T, echo = F, warning = F, message = F,
                        fig.pos = "H", out.extra = "", fig.align = "center",
                        cache = F, comment="")

#----- Load required packages
# library(dplyr)
library(knitr)
library(kableExtra)
# library(broom)
library(stats)
library(astsa)

#----- set the current working directory to the file path
setwd(dirname(rstudioapi::getSourceEditorContext()$path))

#--- checking for parameter redundancy

# Model (i)
AR <- c(1, -.8, .15)
MA <- c(1, -.3)
polyroot(AR)
polyroot(MA)

# Model (ii)
AR <- c(1, -1, .5)
MA <- c(1, -1)
polyroot(AR)
polyroot(MA)
#

# ----- Problem 4.3 (b)
# model (i)
AR <- c(1, -.5)      # new AR coefs on the left

polyroot(AR)

# Model (ii)
AR <- c(1, -1, .5)
MA <- c(1, -1)
polyroot(AR)
polyroot(MA)

# ----- Problem 4.3 (c)

# model (i) - reduced form
digits <- 3
psi <- round(ARMAtoMA(ar=.5, ma=0, 50), digits)
theta <- round(ARMAtoAR(ar=.5, ma=0, 50), digits)

```

```

plot(psi, xaxp=c(0,144,12), type="n", col=4,
      ylab=expression(phi-weights),
      main=expression(AR(1)~~~phi[1]==0.5))
abline(v=seq(0,48,by=12), h=seq(min(psi), max(psi),.1), col=gray(.9))
lines(psi, type="o", col=4)
abline(h=0, col=2)

plot(theta, xaxp=c(0,144,12), type="n", col=4,
      ylab=expression(theta-weights),
      main=expression(MA(0)~~~theta[1]==0.0))
abline(v=seq(0,48,by=12), h=seq(min(theta), max(theta),.1), col=gray(.9))
lines(theta, type="o", col=4)
abline(h=0, col=2)

# Model (ii) - original model
psi <- round(ARMAtoMA(ar=c(1,-.5), ma=-1, 50), digits)
theta <- round(ARMAtoAR(ar=c(1,-.5), ma=-1, 50), digits)

plot(psi, xaxp=c(0,144,12), type="n", col=4,
      ylab=expression(phi-weights),
      main=expression(AR(2)~~~phi[1]==1~~~phi[2]==-0.5))
abline(v=seq(0,48,by=12), h=seq(min(psi), max(psi),.1), col=gray(.9))
lines(psi, type="o", col=4)
abline(h=0, col=2)

plot(theta, xaxp=c(0,144,12), type="n", col=4,
      ylab=expression(theta-weights),
      main=expression(MA(1)~~~theta[1]==-1))
abline(v=seq(0,48,by=12), h=seq(min(theta), max(theta),.1), col=gray(.9))
lines(theta, type="o", col=4)
abline(h=0, col=2)

# ----- Problem 4.4 (a)

# compute the theoretical ACFs of the three models
ACF1 <- ARMAacf(ar=.6, ma=.9, 24)[-1] # discard the first
ACF2 <- ARMAacf(ar=.6, ma=0, 24)[-1]
ACF3 <- ARMAacf(ar=0, ma=.9, 24)[-1]

# compute the theoretical PACFs of the three models
PACF1 <- ARMAacf(ar=.6, ma=.9, 24, pacf=TRUE)
PACF2 <- ARMAacf(ar=.6, ma=0, 24, pacf=TRUE)
PACF3 <- ARMAacf(ar=0, ma=.9, 24, pacf=TRUE)
# plot the results
tsplot(ACF1, type="h", xlab="lag", ylim=c(-.8,1))
abline(h=0)
tsplot(PACF1, type="h", xlab="lag", ylim=c(-.8,1))
abline(h=0)

tsplot(ACF2, type="h", xlab="lag", ylim=c(-.8,1))
abline(h=0)

```

```

tsplot(PACF2, type="h", xlab="lag", ylim=c(-.8,1))
abline(h=0)

tsplot(ACF3, type="h", xlab="lag", ylim=c(-.8,1))
abline(h=0)
tsplot(PACF3, type="h", xlab="lag", ylim=c(-.8,1))
abline(h=0)

# ----- Problem 4.4 (b)
set.seed(1234) # seed for reproducibility of results

# Generate n=100 observations from each of the 3 models in part (a)
arma11 <- arima.sim(list(order=c(1,0,1), ar=.6, ma=.9), n=100)
arma10 <- arima.sim(list(order=c(1,0,0), ar=.6), n=100)
arma01 <- arima.sim(list(order=c(0,0,1), ma=.9), n=100)

# compute and plot the sample ACFs and PACFs of the three series
max_lag <- 30
ACF1 <- acf1(arma11, max_lag, main = "")
PACF1 <- acf1(arma11, max_lag, pacf = T, main = "")

ACF2 <- acf1(arma10, max_lag, main = "")
PACF1 <- acf1(arma10, max_lag, pacf = T, main = "")

ACF3 <- acf1(arma01, max_lag, main = "")
PACF1 <- acf1(arma01, max_lag, pacf = T, main = "")

# ----- Problem 4.4 (c)

# Generate n=100 observations from each of the 3 models in part (a)
arma11 <- arima.sim(list(order=c(1,0,1), ar=.6, ma=.9), n=500)
arma10 <- arima.sim(list(order=c(1,0,0), ar=.6), n=500)
arma01 <- arima.sim(list(order=c(0,0,1), ma=.9), n=500)

# compute and plot the sample ACFs and PACFs of the three series
ACF1 <- acf1(arma11, 24, main = "")
PACF1 <- acf1(arma11, 24, pacf = T, main = "")

ACF2 <- acf1(arma10, 24, main = "")
PACF1 <- acf1(arma10, 24, pacf = T, main = "")

ACF3 <- acf1(arma01, 24, main = "")
PACF1 <- acf1(arma01, 24, pacf = T, main = "")

# difference the mortality series
xt <- diff(cmort, lag = 1)

culer = c(rgb(.66,.12,.85), rgb(.12,.66,.85), rgb(.85,.30,.12))
tsplot(cmort, col = culer[1], type="o", pch=19, ylab="") # origianl
tsplot(xt, col = culer[1], type="o", pch=19, ylab="") # differenced

```

```

# calculate and plot the sample ACF and PACF
ACF <- acf1(xt, 48, main = "")
PACF <- acf1(xt, 48, pacf = T, main = "")

#----- Problem 4.5 (c): model fitting
ar_fit <- sarima(xt, p=1, d=0, q=0, no.constant = TRUE)

# collect relevant model outputs into a table
result_tbl <- data.frame(rbind(
  ar_fit$table,
  c(ar_fit$fit$sigma2, rep(NA, 3)),
  c(ar_fit$AIC, rep(NA, 3)),
  c(ar_fit$BIC, rep(NA, 3))
))
rownames(result_tbl) <- c("AR1", "Sigma^2", "AIC", "BIC")

## display table of results
options(knitr.kable.NA='') # suppress NAs from table output
result_tbl |>
  kable(booktabs=T, linesep="", align = "lcccc", digits=4,
        caption = "Maximum likelihood (unconditional least squares) estimates for the AR(1) model fit")
  pack_rows("Coefficients", 1, 1) |>
  pack_rows("Other metrics", 2, 4) |>
  kable_styling(latex_options = c("HOLD_position")) |>
  kable_classic()
tsplot(resid(ar_fit$fit), col = 4, ylab = "Residuals")

```

## References

Shumway, Robert, and David Stoffer. 2019. *Time Series: A Data Analysis Approach Using r*. CRC Press.