Homework 6

Time Series Analysis (STAT 6391)

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Problem 4.5

Part (e)

The forecasts over a four-week horizon based on the AR(1) model fitted to the differenced mortality series, x_t, are shown in Figure 1 (See forecasted values in Table 1) with a 95% prediction band around them. The solid horizontal blue line represents the mean of the series. It can be seen that the forecasts appear to be leveling off after from the third week.

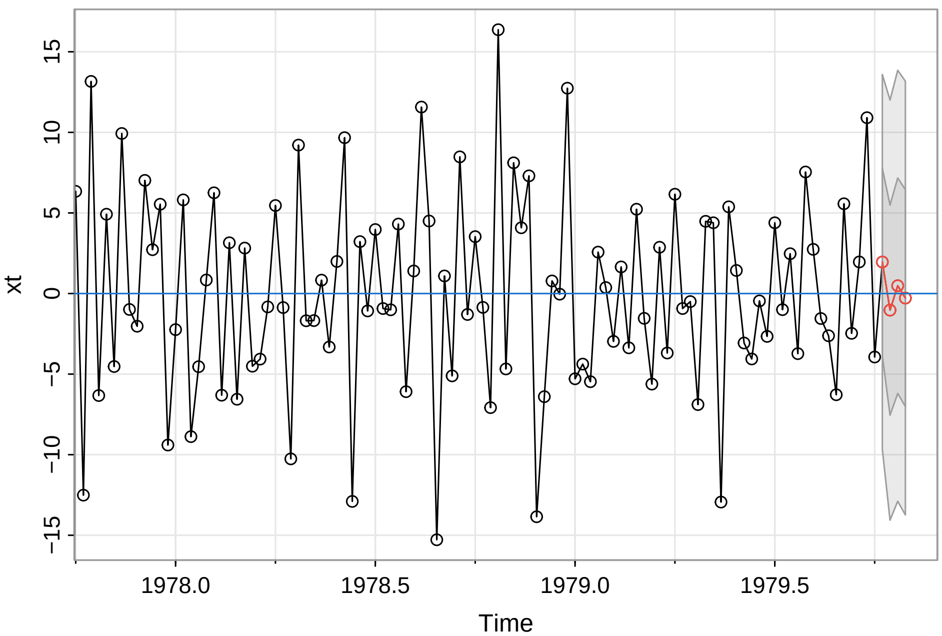

Figure 1: Four-week forcasts for the first-order differenced cardiovascular mortality series (x_t). The forecasts are displayed in red with 95% prediction intervals.

The forecasts are presented in Table 1 below.

m	1	2	3	4
x_{n+m}^n	1.955542	-1.029884	0.4818973	-0.2836494

Table 1: Forecasts over a four-week horizon

Part (f)

In the previous assignment (Homework 5), we obtained the following estimates from the fitted AR(1) model.

	Estimate	SE	t.value	p.value
Coefficients				
AR1	-0.5064	0.0383	-13.2224	0
Other metrics				
Sigma^2	33.8106			
AIC	6.3671			
BIC	6.3838			

Table 2: Maximum likelihood (unconditional least squares) estimates for the AR(1) model fitted to x_t

The forecasted values in part (e) were obtained as follows:

We know that for an AR(1) model, the m-step-ahead forecast is given by

$$x^n_{n+m} = \phi^m x_n$$

So, for m = 1, 2, 3, 4, $x_n = -3.94$, and using the estimate of ϕ obtained from the fitted model (see Table 2) we have

$$x^n_{n+1} = \phi x_n = \hat{\phi} x_n = -0.5064 \times -3.94 = 1.995216$$

$$x^n_{n+2} = \phi^2 x_n = \hat{\phi}^2 x_n = -0.5064^2 \times -3.94 = -1.010377$$
$$x^n_{n+3} = \phi^3 x_n = \hat{\phi}^3 x_n = -0.5064^3 \times -3.94 = 0.5116551$$
$$x^n_{n+4} = \phi^4 x_n = \hat{\phi}^4 x_n = -0.5064^4 \times -3.94 = -0.2591021$$

We note a slight variation between the results we have here and the ones obtained in part (e).

Part (g)

Denote the actual values of the cardiovascular mortality by $c_t, \quad t = 1, 2, \ldots$.

Recall that $x_t = c_t - c_{t-1}$. So $x_{n+1} = c_{n+1} - c_n \Rightarrow x^n_{n+1} = c^n_{n+1} - c^n_n$. Hence

$$\begin{aligned} c^n_{n+1} &= x^n_{n+1} + c^n_n \\ &= 1.95554 + 85.49 = 87.44554. \end{aligned}$$

Therefore, the one-step-ahead forecast of the actual value of cardiovascular mortality is `87.44554`.

Problem 4.8

The MLEs of the three parameters for each one of the 10 generated realizations are presented in Table 3, from which we observe that the estimates (MLEs) of the three parameters are close to their corresponding true values. The averages of the 10 estimates for each parameter are approximately equal (within one decimal place) to the true values.

	ϕ	θ	σ^2
MLEs	0.813	0.613	0.986
	0.830	0.588	1.184
	0.910	0.485	0.876
	0.889	0.413	0.976
	0.837	0.512	0.919
	0.832	0.598	0.874
	0.931	0.469	1.043
	0.864	0.487	1.041
	0.868	0.465	1.010
	0.913	0.457	0.859
Average	0.869	0.509	0.977
True values	0.900	0.500	1.000

Table 3: MLEs obtained for each of the 10 realizations of n=200 each of an ARMA(1,1) process with $\phi = .9$, $\theta = .5$ and $\sigma^2 = 1$ together with their averages over the 10 values.

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Problem 4.6

For an ~~AR~~AR(1) model, we wish to determine the general form of the m-step ahead forecast x_{n+m}^{n}.

The AR(1) model is given by

$$x_t = \phi x_{t-1} + w_t \quad , \quad t = 1, 2, \cdots, n \quad \text{—— ①}$$

When $m = 1$, we have

$$x_{n+1} = \phi x_n + w_{n+1} \qquad (\text{By ①})$$

$$\Rightarrow x_{n+1}^{n} = \phi x_n^n + w_{n+1}^{n}, \text{ where } x_n^n = x_n$$

Note that

$$w_{n+1}^{n} = E(w_{n+1} | w_1, \cdots, w_n) = E(w_{n+1}) = 0$$

$$\Rightarrow x_{n+1}^{n} = \phi x_n^n = \phi x_n \quad \text{—— ②}$$

When $m = 2$:

$$x_{n+2} = \phi x_{n+1} + w_{n+2}$$

$$\Rightarrow x_{n+2}^{n} = \phi x_{n+1}^{n} + w_{n+2}^{n}$$

$$\Rightarrow x_{n+2}^{n} = \phi x_{n+1}^{n} \qquad (\text{Since } w_{n+2}^{n} = 0)$$

$$\Rightarrow x_{n+2}^{n} = \phi(\phi x_n) \qquad (\text{By eqn ②})$$

$$= \phi^2 x_n \quad \text{—— ③}$$

when $m = 3$:

$$x_{n+3} = \phi x_{n+2} + w_{n+3}$$

$$\Rightarrow x^n_{n+3} = \phi x^n_{n+2} + w^n_{n+3}$$

But $w^n_{n+3} = 0$ since it's a future error, and so

$$x^n_{n+3} = \phi x^n_{n+2}$$

$$= \phi(\phi^2 x_n) = \phi^3 x_n$$

From the 3 examples, we can see a pattern emerging. That is, for any $m = 1, 2, \cdots$, we have

$$x^n_{n+m} = \phi x^n_{n+(m-1)}$$

$$= \phi\left(\phi^{(m-1)} x_n\right)$$

$$= \phi^{m-1+1} x_n$$

$$= \phi^m x_n$$

Therefore, the general form of the m-step ahead forecast for the AR(1) model is given by

$$\boxed{x^n_{n+m} = \phi^m x_n}$$

Next, we show that the corresponding MSPE for the m-step ahead forecast x^n_{n+m} is given by

$$E\left[(x_{n+m} - x^n_{n+m})^2\right] = \sigma_w^2 \frac{1-\phi^{2m}}{1-\phi^2}.$$

Denote the MSPE by P^n_{n+m}. Let's investigate the behavior for some values of m.

When $m = 1$

$$\begin{aligned}
P^n_{n+1} &= E(x_{n+1} - x^n_{n+1})^2 \\
&= E(x_{n+1} - \phi x_n)^2 \\
&= E(w_{n+1})^2 \\
&= E(w^2_{n+1}) = Var(w_{n+1}) = \sigma_w^2
\end{aligned}$$

For $m = 2$, we have

$$\begin{aligned}
P^n_{n+2} &= E(x_{n+2} - x^n_{n+2})^2 \\
&= E(x_{n+2} - \phi^2 x_n)^2 \\
&= E(\phi x_{n+1} + w_{n+2} - \phi^2 x_n)^2 \\
&= E\left[\phi(x_{n+1} - \phi x_n) + w_{n+2}\right]^2 \\
&= E(\phi w_{n+1} + w_{n+2})^2 \\
&= E(\phi^2 w^2_{n+1} + 2\phi w_{n+1} w_{n+2} + w^2_{n+2}) \\
&= \phi^2 E(w^2_{n+1}) + E(w^2_{n+2}) \quad (\text{since } E(2\phi w_{n+1} w_{n+2}) = 0)
\end{aligned}$$

$$\Rightarrow \quad P_{n+2}^n = \sigma^2\phi^2 + \sigma^2$$
$$= \sigma_w^2(\phi^2+1) = \sigma_w^2(1+\phi^2)$$

For $m = 3$

$$P_{n+3}^n = E\left(x_{n+3} - x_{n+3}^n\right)^2$$
$$= E\left(x_{n+3} - \phi^3 x_n\right)^2$$
$$= E\left[(\phi x_{n+2} + w_{n+3}) - \phi^3 x_n\right]^2$$
$$= E\left[\phi(x_{n+2} - \phi^2 x_n) + w_{n+3}\right]^2$$
$$= E\left[\phi(\phi x_{n+1} + w_{n+2} - \phi^2 x_n) + w_{n+3}\right]^2$$
$$= E\left[\phi^2(x_{n+1} - \phi x_n) + \phi w_{n+2} + w_{n+3}\right]^2$$
$$= E\left[\phi^2 w_{n+1} + \phi w_{n+2} + w_{n+3}\right]^2$$
$$= E\left[(\phi^2 w_{n+1} + \phi w_{n+2} + w_{n+3})(\phi^2 w_{n+1} + \phi w_{n+2} + w_{n+3})\right]$$
$$= \phi^4 E(w_{n+1}^2) + \phi^2 E(w_{n+2}^2) + E(w_{n+3}^2)$$
$$= \sigma_w^2(\phi^4 + \phi^2 + 1) = \sigma_w^2\left(1 + \phi^2 + \phi^{2(3-1)}\right)$$

It appears that for any $m \in \mathbb{N}$, the MSPE is

$$P_{n+m}^n = E\left[x_{n+m} - x_{n+m}^n\right]^2$$
$$= E\left[x_{n+m} - \phi^m x_n\right]^2$$
$$= \sigma_w^2\left(1 + \phi^2 + \phi^4 + \cdots + \phi^{2(m-1)}\right)$$

$$\Rightarrow \quad P^n_{n+m} = \sigma_w^2 \left(\phi^{2(1-1)} + \phi^{2(2-1)} + \phi^{2(3-1)} + \cdots + \phi^{2(m-1)} \right)$$

$$= \sigma_w^2 \left(\sum_{i=0}^{m-1} \phi^{2i} \right)$$

The summation is a finite geometric series with first term 1 and common ratio ϕ^2. Thus, requiring $|\phi| < 1$ implies that

$$P^n_{n+m} = \sigma_w^2 \sum_{i=0}^{m-1} \phi^{2i}$$

$$= \sigma_w^2 \, \frac{1 - \phi^{2(m-1+1)}}{1 - \phi^2}$$

$$= \sigma_w^2 \, \frac{1 - \phi^{2m}}{1 - \phi^2}$$

as required. $\blacksquare$

Problem 4.9

Given the AR(1) model

$$x_t = \phi x_{t-1} + w_t,$$

we wish to find the Gauss-Newton procedure for estimating the autoregressive parameter ϕ.

Solution

Let

$$x_t = \phi x_{t-1} + w_t \quad \text{——} \quad (1)$$

Derive the errors as

$$w_t(\phi) = x_t - \phi x_{t-1}, \quad t = 1, 2, \cdots, n \quad \text{——} \quad (2)$$

conditioned on $x_0 = 0$

Our goal is to find the value of ϕ that minimizes the sum of squared errors denoted by

$$S_c(\phi) = \sum_{t=1}^{n} w_t^2(\phi) \quad \text{——} \quad (3)$$

And by first-order Taylor expansion at ϕ_0 of $w_t(\phi)$ we have

$$S_c(\phi) = \sum_{t=1}^{n} w_t^2(\phi) \approx \sum_{t=1}^{n} \left[w_t(\phi_0) - (\phi - \phi_0) z_t(\phi_0) \right]^2$$

$w_t(\phi) = x_t - \phi x_{t-1}$ where

$$z_t(\phi_0) = - \left. \frac{\partial w_t(\phi)}{\partial \phi} \right|_{\phi = \phi_0}, \quad \text{——} \quad (4)$$

and

$$\frac{\partial w_t(\phi)}{\partial \phi} = - x_{t-1} \quad \text{——} \quad (5)$$

(The negative derivative is taken for computational convenience).

③ implies that

$$z_t(\phi) = x_{t-1} \qquad \text{——} \quad (6)$$

The problem now becomes one of minimizing the following criterion w.r.t ϕ.

$$Q(\phi) = \sum_{t=1}^{n}\left[w_t(\phi_0) - (\phi-\phi_0)z_t(\phi_0)\right]^2 \qquad \text{——} \quad (7)$$

If we let $\beta = (\phi-\phi_0)$, $y_t = w_t(\phi)$, and $z_t = z_t(\phi_0)$, then ⑦ is the same minimization problem for a simple linear regression of the form $y_t = \beta z_t + \varepsilon$ with least squares estimate of β given as

$$\hat{\beta} = \frac{\sum_{t=1}^{n} y_t z_t}{\sum_{t=1}^{n} z_t^2}$$

$$\Rightarrow \hat{\phi} = \phi_0 + \frac{\sum_{t=1}^{n} y_t z_t}{\sum_{t=1}^{n} z_t^2}$$

$$= \phi_0 + \frac{\sum_{t=1}^{n} w_t(\phi) z_t(\phi)}{\sum_{t=1}^{n} z_t^2(\phi)}$$

Combining all the results we have so far, the Gauss–Newton procedure proceeds as follows

$$\phi_{j+1} = \phi_j + \frac{\sum_{t=1}^{n} w_t(\phi_j) z_t(\phi_j)}{\sum_{t=1}^{n} z_t^2(\phi_j)}, \quad t = 1, 2, \cdots, n, \text{ and } j = 0, 1, 2, \cdots$$

where

$$\left.\begin{aligned} w_t(\phi_j) &= x_t - \phi_j x_{t-1} \\ z_t(\phi_j) &= x_{t-1} \end{aligned}\right\} \quad t = 1, 2, \cdots, n$$

$x_0 = 0$ (we condition on this),

and with termination criteria $|\phi_{j+1} - \phi_j| < \varepsilon$ or $|Q(\phi_{j+1}) - Q(\phi_j)| < \varepsilon$ for a reasonably small number ε.

The initial parameter value ϕ_0 can be obtained using the method of moment estimate.

We notice that this procedure is non-recursive since the values $w_t(\phi_j)$ and $z_t(\phi_j)$ are calculated based on the observed data $\{x_1, \cdots, x_n\}$ with an appropriately chosen initial value for ϕ_j.

▨

Appendix: R codes

```
# Set global options for output rendering
knitr::opts_chunk$set(eval = T, echo = F, warning = F, message = F,
                      fig.pos = "H", out.extra = "", fig.align = "center",
                      cache = F, comment="")

#----------------- Load required packages
# library(dplyr)
library(knitr)
library(kableExtra)
# library(broom)
library(stats)
library(astsa)


#----------------- set the current working directory to the file path
setwd(dirname(rstudioapi::getSourceEditorContext()$path))



#---------- Problem 4.5 (e): forecasting

# obtain the differenced mortality series
xt <- diff(cmort, lag = 1)
sarima_forcast <- sarima.for(xt, n.ahead = 4, p=1, d=0, q=0)
abline(h=0, col=4) # display the zero mean


# A table of the forecased values
forecasts <- rbind(sarima_forcast$pred) # get the forcasts
forecasts <- data.frame(desc="$x_{n+m}^n$", forecasts)
names(forecasts) <- c("m", 1:4)
kable(forecasts, booktabs=T, linesep="", align = "lcccc", escape = F,
          caption = "Forecasts over a four-week horizon") |>
    kable_styling(latex_options = c("HOLD_position")) |>
    kable_classic()

#---------- Problem 4.5 (f): model fitting
ar_fit <- sarima(xt, p=1, d=0, q=0, no.constant = TRUE)

# collect relevant model outputs into a table
result_tbl <- data.frame(rbind(
    ar_fit$ttable,
    c(ar_fit$fit$sigma2, rep(NA, 3)),
    c(ar_fit$AIC, rep(NA, 3)),
    c(ar_fit$BIC, rep(NA, 3))
))
rownames(result_tbl) <- c("AR1", "Sigma^2", "AIC", "BIC")
## display table of results
options(knitr.kable.NA='')  # suppress NAs from table output
result_tbl |>
    kable(booktabs=T, linesep="", align = "lcccc", digits=4,
          caption = "Maximum likelihood (unconditional least squares) estimates for the AR(1) model fit
```

```
    pack_rows("Coefficients", 1, 1) |>
    pack_rows("Other metrics", 2, 4) |>
    kable_styling(latex_options = c("HOLD_position")) |>
    kable_classic()
# generating the forecasts manually
phi_hat <- -0.5064
xn <- xt[length(xt)] # -3.94
for(i in 1:4){
    print(xn*phi_hat^i)
}


##---------- Problem 4.8 codes
set.seed(1234) # seed for reproducibility of results
replicates <- 10
# initialize container to store MLEs
param_estimate <- matrix(NA, nrow = replicates, ncol = 3)
for (i in 1:replicates) {
    # Generate 10 realizations of length n=200 each of an ARMA(1,1) process
    arma11 <- arima.sim(list(order=c(1,0,1), ar=.9, ma=.5), n=200, sd=1)
    # Fit arima model to obtain MLEs of the parameters
    arima_fit <- arima(arma11, order = c(1,0,1))
    param_estimate[i,1] <- arima_fit$coef[1]
    param_estimate[i,2] <- arima_fit$coef[2]
    param_estimate[i,3] <- arima_fit$sigma2
}

avg <- apply(param_estimate, 2, mean) # compute the means for comparsion
param_estimate <- data.frame(type=c("MLEs", rep("",9), "Average", "True values"),
                             rbind(param_estimate, avg, c(0.9, 0.5,1)))
names(param_estimate) <- c("","phi", "theta", "sigma2")

rownames(param_estimate) <- NULL

# display output in a nice formatted tables
kable(param_estimate, booktab=T, align = "c", digits = 3, linesep="", escape = F,
     col.names = c("","$\\phi$", "$\\theta$", "$\\sigma^2$"),
     caption = "MLEs obtained for each of the 10 realizations of n=200 each of an ARMA(1,1) process wi
    pack_rows("",1,10) |>
    pack_rows("",11,11) |>
    pack_rows("", 12,12) |>
    kable_styling(latex_options = c("HOLD_position")) |>
    kable_classic()
```

References

Shumway, Robert, and David Stoffer. 2019. *Time Series: A Data Analysis Approach Using r*. CRC Press.