Homework 4

Time Series Analysis (STAT 6391)

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STAT 6391

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Problem 3.4

Consider a process consisting of a linear trend with an additive noise term w_t

$$x_t = \beta_0 + \beta_1 t + w_t$$

where β_0, β_1 are fixed constants and $w_t \sim WN(0, \sigma_w^2)$.

(a) Proove x_t is nonstationary

Proof:

It suffices to show that the mean function, μ_{x_t}, depends on t.

We can compute the mean function by

$$\begin{aligned}
\mu_{x_t} &= E(x_t) \\
&= E(\beta_0 + \beta_1 t + w_t) \\
&= \beta_0 + \beta_1 t + E(w_t) \\
&= \beta_0 + \beta_1 t \qquad (\text{Because } E(w_t) = 0)
\end{aligned}$$

Clearly, μ_{x_t} depends on the time t, which proves that x_t is nonstationary.

$\blacksquare$

Part (b)

Prove that the first difference series $\nabla x_t = x_t - x_{t-1}$ is stationary by finding its mean and autocovariance functions.

Proof:

Let $z_t = \nabla x_t = x_t - x_{t-1}$

$\Rightarrow z_t = \beta_0 + \beta_1 t + w_t - [\beta_0 + \beta_1(t-1) + w_{t-1}]$

$z_t = \beta_1 + w_t - w_{t-1}$

Then, the mean function can be computed as

$E(z_t) = E(\beta_1 + w_t - w_{t-1})$

$= \beta_1 + E(w_t) - E(w_{t-1})$

$= \beta_1$ (since $E(w_t) = E(w_{t-1}) = 0$ for a white noise process)

Also, for the covariance function, we have

$\gamma_z(h) = Cov(z_{t+h}, z_t) = Cov(w_{t+h} - w_{t+h-1}, w_t - w_{t-1})$

$\Rightarrow \gamma_z(0) = Cov(w_t - w_{t-1}, w_t - w_{t-1})$

$= Cov(w_t, w_t) + Cov(w_{t-1}, w_{t-1})$ (Because w_t and w_{t-1} are uncorrelated)

$= 2\sigma_w^2$

$\Rightarrow \gamma_z(1) = Cov(w_{t+1} - w_t, w_t - w_{t-1})$

$= - Cov(w_t, w_t) = -\sigma_w^2$

$\gamma_z(-1) = Cov(w_{t-1} - w_{t-2}, w_t - w_{t-1})$

$= - Cov(w_{t-1}, w_{t-1})$

$= -\sigma_w^2$

$$\Rightarrow \gamma_z(2) = Cov(w_{t+2} - w_{t+1}, w_t - w_{t-1})$$

$$= 0 = Cov(w_{t-2} - w_{t-3}, w_t - w_{t-1}) = \gamma_z(-2)$$

It turns out $\gamma_z(h) = 0$ for all $|h| > 1$, so in general

$$\gamma_z(h) = \begin{cases} 2\sigma_w^2 & h = 0 \\ -\sigma_w^2 & h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence, we conclude that the first difference series based on x_t is stationary because the mean function is a constant independent of t and the autocovariance function only depends on the time difference h. ▨

Part (c)

Now, we repeat (b), replacing w_t by a general stationary process y_t with mean function μ_y and autocovariance function $\gamma_y(h)$, that is

$$x_t = \beta_0 + \beta_1 t + y_t,$$

and the first difference series is given by

$$z_t = \beta_1 + y_t - y_{t-1}.$$

The mean function now becomes

$$E(z_t) = E(\beta_1 + y_t - y_{t-1})$$

$$= \beta_1 + E(y_t) - E(y_{t-1})$$

$$= \beta_1 + \mu_y - \mu_y = \beta_1 \quad (\text{a constant independent of } t)$$

and the autocovariance is computed as

$$\gamma_z(h) = Cov(z_{t+h}, z_t) = Cov(y_{t+h} - y_{t+h-1}, y_t - y_{t-1})$$

$$= Cov(y_{t+h}, y_t) - Cov(y_{t+h}, y_{t-1}) - Cov(y_{t+h-1}, y_t) + Cov(y_{t+h-1}, y_{t-1})$$

$$= \gamma_y(h) - \gamma_y(h+1) - \gamma_y(h-1) + \gamma_y(h)$$

$$= 2\gamma_y(h) - \gamma_y(h+1) - \gamma_y(h-1)$$

which is only dependent on the lag h since y_t is stationary.

Our results prove that the first difference series is again stationary if we replace w_t by a general stationary process. $\blacksquare$

Problem 4.2

Let $\{w_t; t=0,1,\cdots\}$ be a white noise process with variance σ_w^2 and let $|\phi| < 1$ be a constant. Consider the process $x_0 = w_0$ and

$$x_t = \phi x_{t-1} + w_t, \quad t = 1,2,\ldots$$

Part (a)

We wish to show that

$$x_t = \phi x_{t-1} + w_t = \sum_{j=0}^{t} \phi^j w_{t-j} \quad \text{for any } t = 0,1,\ldots \qquad (*)$$

Proof:

We prove the above result by induction as follows.

We know that $x_0 = w_0$ for $t=0$.

When $t=1$, we have

$$x_1 = \phi x_0 + w_1 \qquad \text{(left hand side)}$$

$$x_1 = \sum_{j=0}^{1} \phi^j w_{1-j} = \phi^0 w_1 + \phi^1 w_0 \qquad \text{(right hand side)}$$
$$= w_1 + \phi w_0$$

But $w_0 = x_0$, so $x_1 = w_1 + \phi x_0 \Rightarrow x_1 = \phi x_0 + w_1$, showing that the result is true for $t=1$.

Assume it holds for some arbitrary time $k \geq 0$ $(t=k)$ such that

$$x_k = \phi x_{k-1} + w_k = \sum_{j=0}^{k} \phi^j w_{k-j} \qquad (1)$$

Now, it remains to show that it also holds true for $t = k+1$.
For $t=k+1$, the left hand side is

$$x_{k+1} = \phi x_k + w_{k+1} \qquad (2)$$

Putting ① into ② leads to the following result.

$$x_{k+1} = \phi\left(\sum_{j=0}^{k} \phi^j w_{k-j}\right) + w_{k+1}$$

$$\Rightarrow x_{k+1} = \phi\left(w_k + \phi w_{k-1} + \phi^2 w_{k-2} + \cdots + \phi^k w_0\right) + w_{k+1}$$

$$= \left(\phi w_k + \phi^2 w_{k-1} + \phi^3 w_{k-2} + \cdots + \phi^{k+1} w_0\right) + w_{k+1}$$

$$= w_{k+1} + \phi w_k + \phi^2 w_{k-1} + \phi^3 w_{k-2} + \cdots + \phi^{k+1} w_0$$

$$= \sum_{j=0}^{k+1} \phi^j w_{k+1-j}$$

At this point, we have established that

$$x_{k+1} = \phi x_k + w_{k+1} = \sum_{j=0}^{k+1} \phi^j w_{k+1-j},$$

which completes the induction step.

Since the base case $(t=1)$ and the induction step $(t=k \Rightarrow t=k+1)$ have been proved as true, by mathematical induction the statement in ⊛ holds for every $t = 0, 1, \cdots$ ▨

Part (b)

Using the new representation for x_t, we have that

$$E(x_t) = E\left(\sum_{j=0}^{t}\phi^j w_{t-j}\right)$$

$$= \sum_{j=0}^{t}\phi^j E(w_{t-j})$$

$$= 0 \qquad (\text{since } E(w_t) = 0 \text{ for any time index } t)$$

Part (c)

We want to show that for $t = 0, 1, \dots,$

$$Var(x_t) = \frac{\sigma_w^2}{1-\phi^2}(1-\phi^{2(t+1)})$$

__Proof__:

Variance of x_t is given by

$$Var(x_t) = Var\left[\sum_{j=0}^{t}\phi^j w_{t-j}\right]$$

$$= \sum_{j=0}^{t}\phi^{2j} Var(w_{t-j})$$

$$= \sigma_w^2 \sum_{j=0}^{t}\phi^{2j} \qquad (\text{since } Var(w_{t-j}) = \sigma_w^2 \text{ for all } j = 0, 1, \dots, t)$$

$$= \sigma_w^2 \sum_{j=0}^{t}(\phi^2)^j \qquad \left(\text{Note that } \sum_{j=0}^{t}(\phi^2)^j \text{ is a finite geometric series}\right)$$

$$= \sigma_w^2\left[\frac{1-(\phi^2)^{t+1}}{1-\phi^2}\right] = \sigma_w^2\left[\frac{1-\phi^{2(t+1)}}{1-\phi^2}\right]$$

$$= \frac{\sigma_w^2}{1-\phi^2}\left(1-\phi^{2(t+1)}\right)$$

as required. ▨

Part (d)

W.T.S.: For $h \geq 0$,

$$\text{cov}(x_{t+h}, x_t) = \phi^h \text{Var}(x_t).$$

Proof:

By definition

$t=3,\ h=2$

$$\gamma_x(h) = \text{cov}(x_{t+h}, x_t)$$

$$= \text{cov}\left(\sum_{j=0}^{t+h} \phi^j w_{t+h-j},\ \sum_{j=0}^{t} \phi^j w_{t-j} \right)$$

$$= \text{cov}\left[\phi^0 w_{t+h} + \phi w_{t+h-1} + \cdots + \phi^h w_t + \phi^{h+1} w_{-1} + \cdots + \phi^{t+h} w_0,\ \phi^0 w_{-} + \phi w_{t-1} + \cdots + \phi^t w_0 \right]$$

$$= \sigma^2 \phi^h \text{cov}(w_t, w_t) + \phi^{h+1}\phi \,\text{cov}(w_{t-1}, w_{t-1}) + \cdots + \phi^{t+h}\phi^t \text{cov}(w_0, w_0)$$

$$= \sigma_w^2 \left[\phi^0\phi^h + \phi^{h+1}\phi + \cdots + \phi^{t+h}\phi^t \right]$$

$$= \sigma_w^2 \left[\sum_{j=0}^{t} \phi^j \phi^{h+j} \right]$$

$$= \sigma_w^2 \phi^h \left(\sum_{j=0}^{t} \phi^{2j} \right)$$

$$= \phi^h \underbrace{\left(\sigma_w^2 \sum_{j=0}^{t} \phi^{2j} \right)}_{(i)}$$

From part (c), we recognize that the expression (i) is nothing but the variance of x_t.

$\Rightarrow \text{cov}(x_{t+h}, x_t) = \phi^h \text{Var}(x_t)$, for $h \geq 0$ since $\gamma_x(h) = \gamma_x(-h)$

which completes the proof. ▨

Part (e)

Our results so far suggest that x_t is <u>not stationary</u>.

Specifically, we found in part (a) that

$E(x_t) = 0$, which is a constant independent of t alright.

But the autocovariance function $\gamma_x(h)$ depends on t when we combine our results from parts (c) and (d). That is

$$\gamma_x(h) = \phi^h \operatorname{var}(x_t)$$
$$= \frac{\sigma_w^2 \phi^h}{1-\phi^2}\left(1-\phi^{2(t+1)}\right)$$

We have a violation of one of the conditions for stationarity.

Therefore, x_t is not stationary as already pointed out. ▨

Part (f)

We want to show that x_t becomes stationary as $t \to \infty$.

Proof:

It is sufficient to show that the autocovariance function is independent of t as $t \to \infty$.

From part (e), we showed that

$$\gamma_x(h) = \frac{\sigma_w^2 \phi^h (1 - \phi^{2(t+1)})}{1-\phi^2}$$

Taking limits as $t \to \infty$, we have

$$\lim_{t \to \infty} \gamma_x(h) = \lim_{t \to \infty} \left(\frac{\sigma_w^2 \phi^h (1 - \phi^{2(t+1)})}{1-\phi^2} \right)$$

$$= \frac{\sigma_w^2 \phi^h}{1-\phi^2} \left(1 - \lim_{t \to \infty} (\phi^{2(t+1)}) \right)$$

$$= \frac{\sigma_w^2 \phi^h}{1-\phi^2} (1) \quad \left(\text{Because } |\phi| < 1, \lim_{t \to \infty} \phi^{2(t+1)} \approx 0 \right)$$

showing that $\gamma_x(h) \longrightarrow \frac{\sigma_w^2 \phi^h}{1-\phi^2}$, $|\phi^2| < 1$, as $t \to \infty$, which depends on t only through the lag h.

It follows that x_t is stationary as $t \to \infty$. ▨

Part (g)

Given a value for ϕ and a simulated white noise process, $w_1, w_2, \cdots, w_n + \cdots + w_{n+m} \overset{iid}{\sim} N(0,1)$, we can generate $(n+m)$ data recursively from the relation

$$x_t = \sum_{j=0}^{t} \phi^j w_{t-j}$$

with initial values $x_0 = w_0 = 0$, where we have deliberately generated m extra data points to account for potential problems with startup values. We will then discard the first m observations, leaving us with the desired simulated n observations of a stationary Gaussian AR(1) model.

Part (h)

Suppose $x_0 = \frac{w_0}{\sqrt{1-\phi^2}}$. Want to show that x_t is stationary.

Proof:

It suffices to show that $var(x_t)$ is constant.

We have already established that

$$x_t = \sum_{j=0}^{t} \phi^j w_{t-j}$$

which can also be written as

$$x_t = \sum_{j=0}^{t-1} \phi^j w_{t-j} + \phi^t w_0$$

Now

$$\Rightarrow Var(x_t) = Var\left(\sum_{j=0}^{t-1} \phi^j w_{t-j} + \phi^t w_0 \right)$$

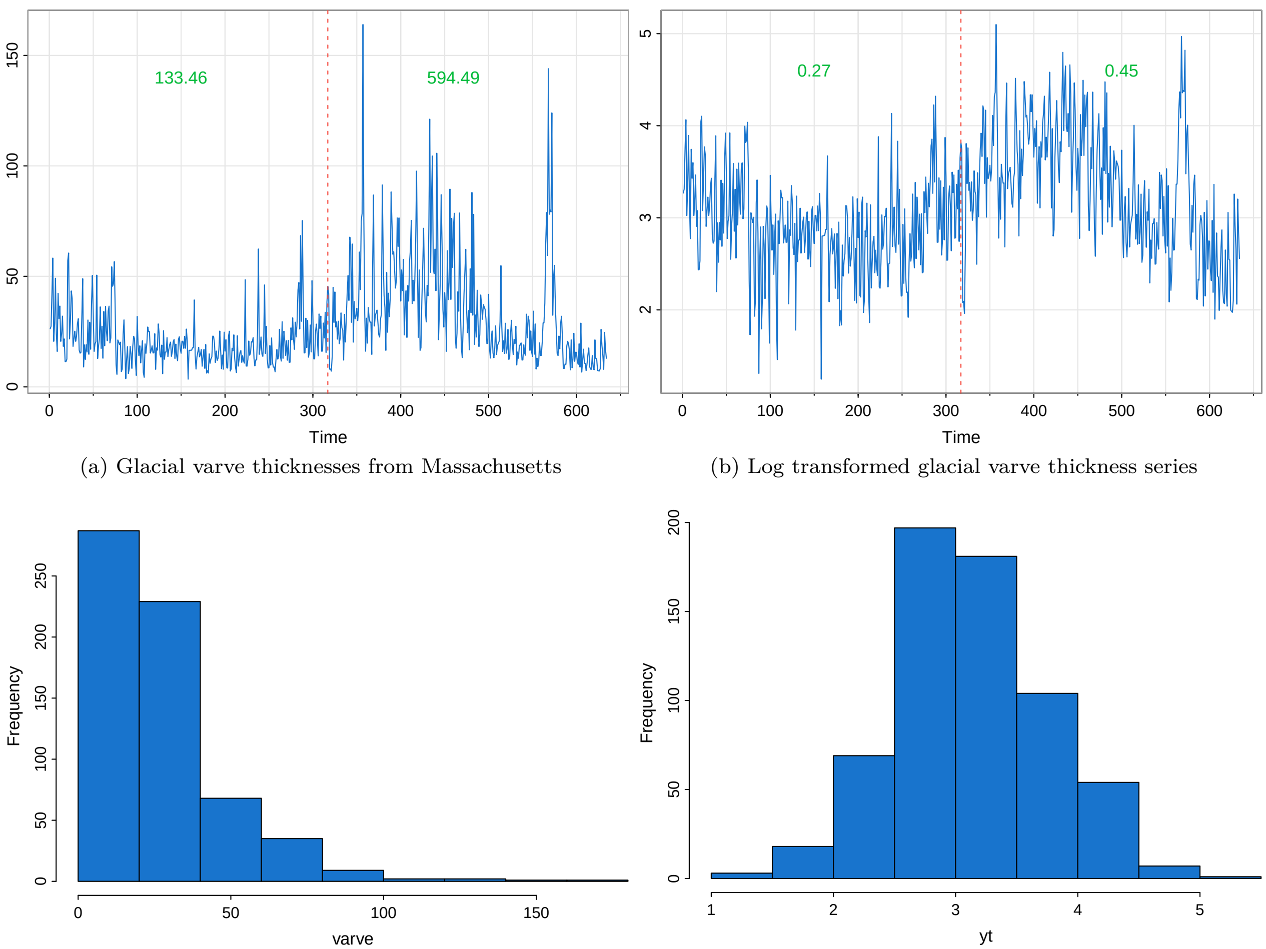
$$= \phi^{2t} var(w_0) + \sum_{j=0}^{t-1} \phi^{2j} var(w_{t-j}) \quad \left(\text{because } w_t\text{s are uncorrelated}\right)$$

Problem 3.6

Part (a)

Figure 1 is a graph of the glacial varve data (a) along with histograms of the original data (c) and a log transformed version (b). The computed sample variances are shown in green, from which we see that the variance of the second half of the Glacial data far exceeds (more than 4 times) the first half of the data. This is a clear sign of unequal variance over the course of the series, thus supporting the fact that the series exhibits heteroscedasticity.

We can indeed argue from plot (b) that the log transformation stabilizes the data as the variance of the second half is now about twice as large as that of the first half. And again, comparing the histogram in plot (c) to the one in plot (d), we can argue that the approximation to normality is improved by the the log transformation as the histogram of y_t is roughly symmetric.



(a) Glacial varve thicknesses from Massachusetts

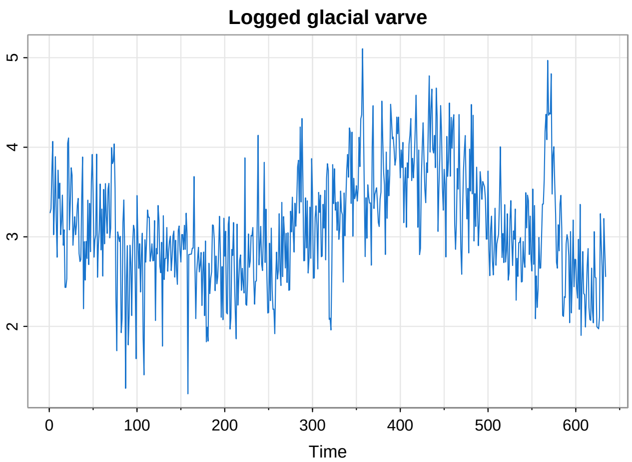
(b) Log transformed glacial varve thickness series

(c) Histogram of original varve data

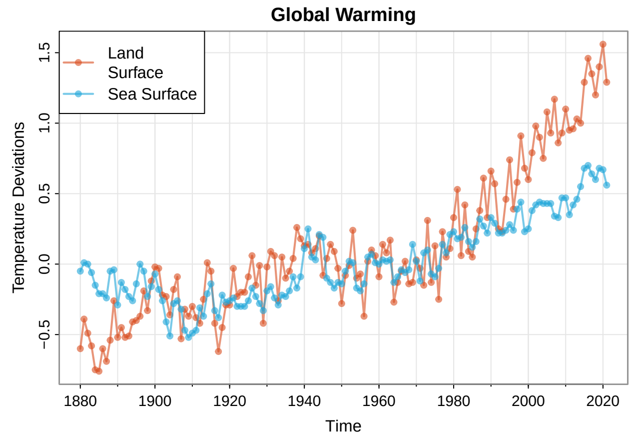
(d) Histogram of log transformed varve data

Figure 1: Glacial varve series and histograms. Numbers in green represent the sample variances over the first and second halves of the data on the left and right side of the dotted red line, respectively.

Part (b)



(a)



(b)

Figure 2: Log transformed varve series along with the global temperature records (only shown here for comparison).

A plot of the series y_t (log transformed glacial varve) is again shown in in Figure 2 along with the global temperature records. From this plot (a), we can say that some time intervals, of the order 100 years, exists especially between times 200 and 450 with an upward trend similar to the upward trend observed in the global temperature records during the latter part of the twentieth century as seen in plot (b).

Part (c)

We examined the sample ACF of the log transformed series using an ACF plot as shown in 3. The autocorrelation at different lags are significant but decays as the lag increases. It is clear from this plot that the data is not stationary.

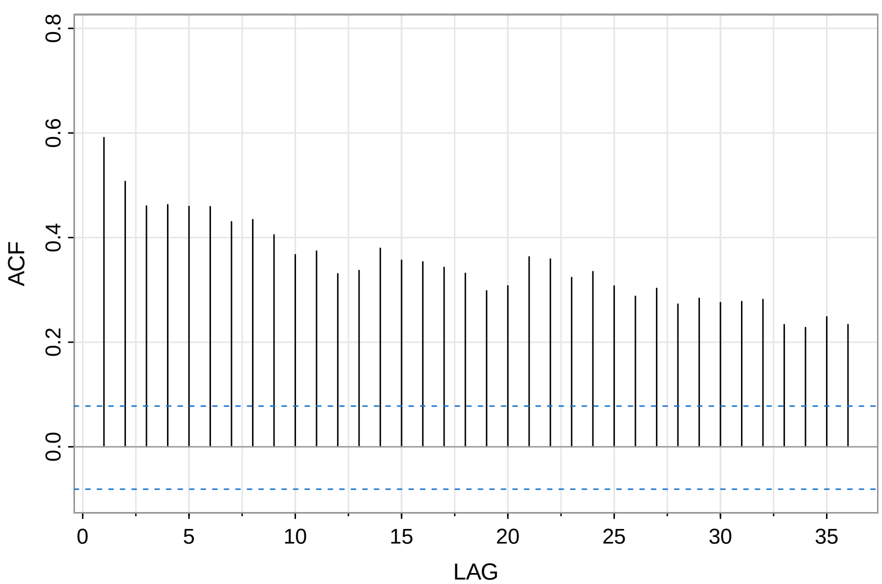
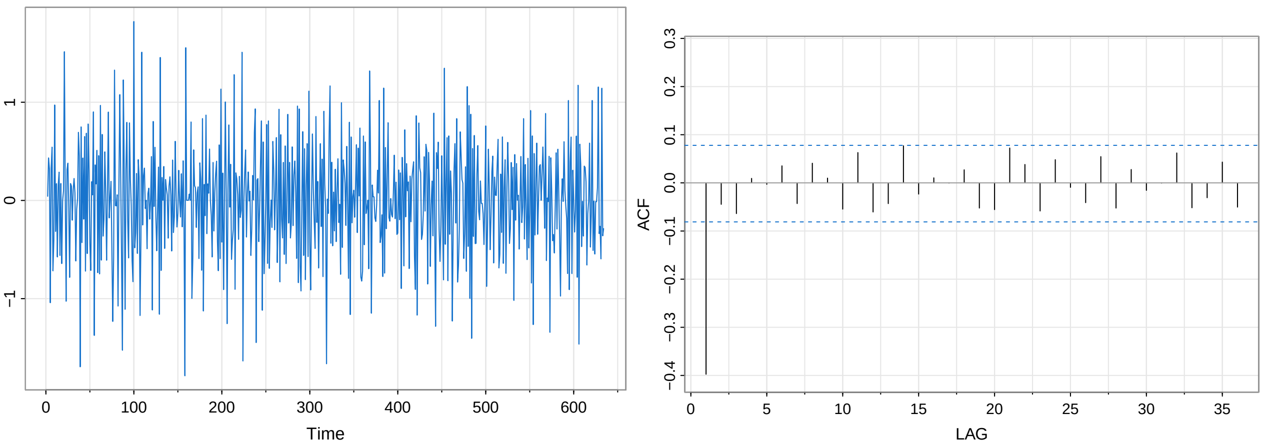


Figure 3: Sample ACF of the log transformed glacial varve

Part (d)

The computed differenced data is shown in Figure 4 along with the corresponding sample ACF. The differenced series appear centered around zero with a stable variance, and virtually all the autocorrelations appear non-significant which resemble that of a white noise process. It is therefore evident that differencing the logged varve data produced a reasonably stationary series.

A practical interpretation of $\mu_t = y_t - y_{t-1}$ is that it coerces a nonstationary series into a stationary one.



(a) Differenced series

(b) Sample ACF

Figure 4: Differenced loggged glacial varve series and its sample ACF.

Problem 3.8

In this problem, our goal is to investigate the presence or otherwise of a strong 4-year cycle.

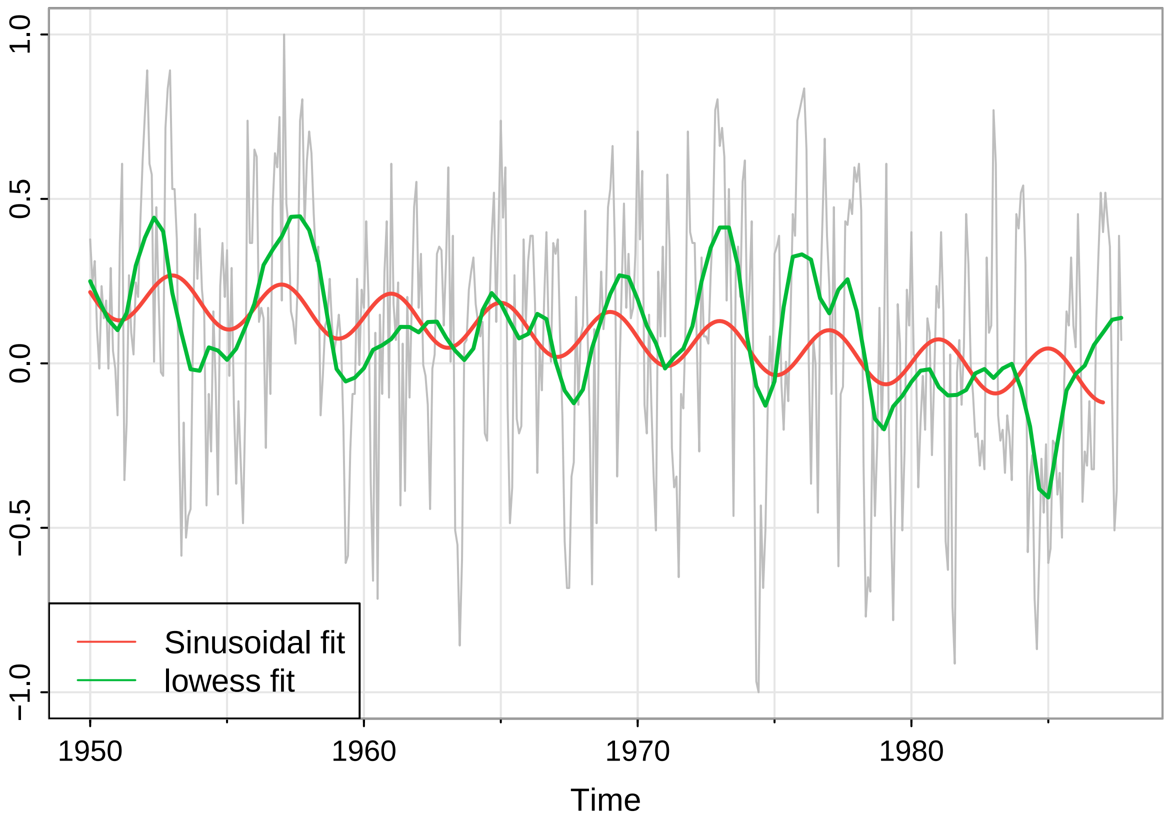


Figure 5: The Southern Oscilation Index (SOI) series with a sinusoidal fit to the SOI (red line) and a lowess fit of SOI (green line)

According to Figure 5, the sinusoidal fit compares reasonably well to the lowess fit (except at the tail end of the last decade where the trends are in opposite directions) in terms of the rises and falls, accentuating the presence of a 4-year cycle. There is approximately one cycle every four years. Thus, there is a strong 4-year cycle in the SOI series.

References

- Robert H. Shumway, & David S. Stoffer. (2019). Time Series: A Data Analysis Approach Using R.

Appendix: R codes

```
# Set global options for output rendering
knitr::opts_chunk$set(eval = T, echo = F, warning = F, message = F,
                      fig.pos = "H", out.extra = "", fig.align = "center",
                      cache = F)

#----------------- Load required packages
# library(dplyr)
# library(knitr)
# library(kableExtra)
# library(broom)
library(stats)
library(astsa)


#----------------- set the current working directory to the file path
setwd(dirname(rstudioapi::getSourceEditorContext()$path))




#---------- Problem 3.6 codes

x1 <- varve[1:317]      # first half
x2 <- varve[318:634]    # second half
# var(x1); var(x2)

# log transformation
yt <- log(varve)
y1 <- yt[1:317]      # first half
y2 <- yt[318:634]    # second half
# plot the varve series
tsplot(varve, col = 4, ylab = "")
abline(v=length(varve)/2, col=2, lty=2)
text(x=150, y=140, round(var(x1),2), col=3)
text(x=460, y=140, round(var(x2),2), col=3)

# plot the log transformed series
tsplot(yt, col = 4, ylab = "")
abline(v=length(yt)/2, col=2, lty=2)
text(x=150, y=4.6, round(var(y1),2), col=3)
text(x=500, y=4.6, round(var(y2),2), col=3)

# histograms
hist(varve, main = "", col=4)
hist(yt, main = "", col=4)


# plot the log transformed series
```

```
tsplot(yt, col = 4, ylab = "", main = "Logged glacial varve")

# bring in the global temperature figure for comparison
culer = c(rgb(.85,.30,.12,.6), rgb(.12,.65,.85,.6))
tsplot(gtemp_land, col=culer[1], lwd=2, type="o", pch=20,
ylab="Temperature Deviations", main="Global Warming")
lines(gtemp_ocean, col=culer[2], lwd=2, type="o", pch=20)
legend("topleft", col=culer, lty=1, lwd=2, pch=20, legend=c("Land
Surface", "Sea Surface"), bg="white")



# examine the sample ACF of yt and comment
sample_acf <- acf1(yt, plot = T, main = "")


# compute the difference
ut <- diff(yt, lag=1)

# generate plot of the series
tsplot(ut, col=4, ylab = "")

# create the ACF plot
sample_acf <- acf1(ut, plot = T, main = "")

#---------- END
#---------- Problem
# A sinusoidal fit to the SOI data
t <- time(soi)
z1 <- cos(2*pi*t/4) # time in months
z2 <- sin(2*pi*t/4)
sinu_fit <- lm(soi ~  t + z1 + z2)
# summary(sinu_fit)
tsplot(soi, col="gray", ylab = "")
fitted_vals <- ts(fitted(sinu_fit), frequency = 12, start = 1950, end = 1987)
# lines(fitted(sinu_fit), col=2, lwd=2)
lines(fitted_vals, col=2, lwd=2)
lines(lowess(soi, f=.05), lwd=2, col=3) # El Niño cycle
legend("bottomleft", legend = c("Sinusoidal fit", "lowess fit"),
       col = c(2, 3), lty = 1)
#---------- END
```