William Ofosu Agyapong
STAT 6391

Homework 2

Problem 2.3

Consider the simple two-sided moving smoother of the form

$$v_t = \frac{1}{4}\left(w_{t-1} + 2w_t + w_{t+1}\right), \quad w_t \sim N(0, \sigma_w^2)$$

Determine the autocovariance and autocorrelation functions as a function of lag h.

Autocovariance function

$$\gamma_v(h) = Cov(v_{t+h}, v_t)$$

$$\Rightarrow \gamma_v(0) = Cov(v_t, v_t) = \frac{1}{4^2} Cov\left[w_{t-1} + 2w_t + w_{t+1},\ w_{t-1} + 2w_t + w_{t+1}\right]$$

$$= \frac{1}{16}\left[Cov(w_{t-1}, w_{t-1}) + 4Cov(w_t, w_t) + Cov(w_{t+1}, w_{t+1})\right]$$

$$= \frac{1}{16}\left(\sigma_w^2 + 4\sigma_w^2 + \sigma_w^2\right)$$

$$= \frac{1}{16} \cdot 6\sigma_w^2 = \frac{3}{8}\sigma_w^2$$

$$\gamma_v(1) = Cov(v_{t+1}, v_t) = \frac{1}{16} Cov\left[w_t + 2w_{t+1} + w_{t+2},\ w_{t-1} + 2w_t + w_{t+1}\right]$$

$$= \frac{1}{16}\left\{2Cov(w_t, w_t) + 2Cov(w_{t+1}, w_{t+1})\right\}$$

$$= \frac{4}{16}\sigma_w^2 = \frac{1}{4}\sigma_w^2 = \gamma_v(-1)$$

$$\gamma_v(2) = Cov(v_{t+2}, v_t) = \frac{1}{16} Cov\left[w_{t+1} + 2w_{t+2} + w_{t+3},\ w_{t-1} + 2w_t + w_{t+1}\right]$$

$$= \frac{1}{16}\left[Cov(w_{t+1}, w_{t+1})\right] = \frac{\sigma_w^2}{16} = \gamma_v(-2)$$

$\gamma_v(h) = 0$, for $|h| > 2$. Hence,

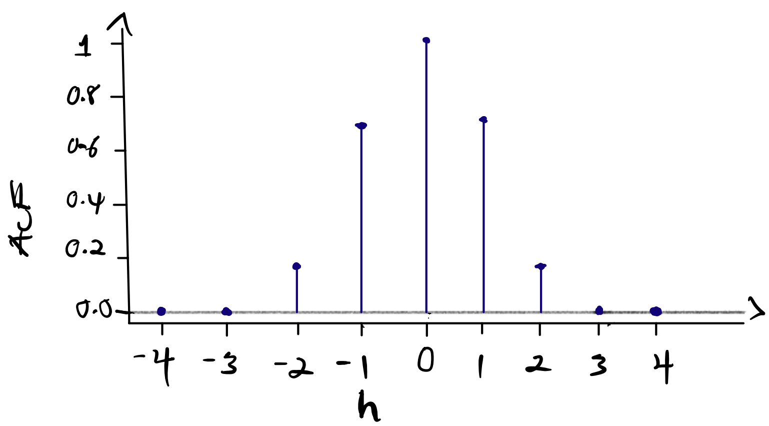
$$\gamma_v(h) = \begin{cases} \frac{6}{16}\sigma_w^2 & h = 0 \\ \frac{4}{16}\sigma_w^2 & |h| = 1 \\ \frac{1}{16}\sigma_w^2 & |h| = 2 \\ 0 & |h| > 2 \end{cases}$$

Autocorrelation function

$$\rho_v(h) = \frac{\gamma(h)}{\gamma(0)}$$

$$\Rightarrow \rho_v(h) = \begin{cases} 1 & h = 0 \\ 2/3 & h = \pm 1 \\ 1/6 & h = \pm 2 \\ 0 & |h| > 2 \end{cases}$$

A sketch of the ACF as a function of h



Problem 2.5

Consider the random walk with drift model

$$x_t = \delta + x_{t-1} + w_t$$

for $t = 1, 2, \ldots$, with $x_0 = 0$, where w_t is white noise with variance σ_w^2

Part (a)

W.T.S: The above model can be written as

$$x_t = \delta t + \sum_{k=1}^{t} w_k .$$

Proof

We have

$$x_t = \delta + x_{t-1} + w_t \quad \text{—— } ①$$, and let

$$x_t = \delta t + \sum_{k=1}^{t} w_k . \quad \text{—— } ②$$

Using eqn ②, x_{t-1} can be expressed as

$$x_{t-1} = \delta(t-1) + \sum_{k=1}^{t-1} w_k \quad \text{—— } ③$$

Now, put ③ into ①

$$\Rightarrow x_t = \delta + \left[\delta(t-1) + \sum_{k=1}^{t-1} w_k \right] + w_t$$

$$= \delta + \delta t - \delta + \sum_{k=1}^{t-1} w_k + w_t$$

$$= \delta t + \sum_{k=1}^{t} w_k$$

Thus,

$$x_t = \delta t + \sum_{k=1}^{t} w_k$$

is indeed an alternative way to write the random walk with drift model under consideration.

▨

Part (b): Mean and autocovariance functions of x_t.

$$x_t = \delta t + \sum_{k=1}^{t} w_k, \quad t = 1, 2, 3, \ldots$$

For the mean function, we have

$$\mu_{x_t} = E(x_t) = E\left[\delta t + \sum_{k=1}^{t} w_k\right]$$

$$= \delta t + \sum_{k=1}^{t} E(w_k)$$ by the linearity property of $E(\cdot)$ since w_ks are independent.

$$= \delta t \quad (\text{since } E(w_k) = 0 \ \forall k)$$

Therefore $\mu_{x_t} = \delta t$, which depends on the time, t. ▨

The autocovariance function of x_t is given by

$$\gamma_x(s,t) = \operatorname{cov}\left(\delta s + \sum_{k=1}^{s} w_k, \ \delta t + \sum_{j=1}^{t} w_j\right)$$

$$= \operatorname{Cov}\left(\sum_{k=1}^{s} w_k, \ \sum_{j=1}^{t} w_j\right)$$

$$= \min\{s, t\} \, \sigma_w^2$$

which also depends on the times, s and t. ▨

Part (c)

By definition, a time series x_t is stationary if the following two conditions hold:

(i) The mean function, μ_{x_t}, is constant and does not depend on t.

(ii) The autocorrelation function, $\gamma_x(s,t)$, depends on times s and t only through their lag or time difference.

It is therefore clear from the results we obtained in part (b) that x_t is not stationary, because the value of μ_{x_t} depends on t and moreover $\gamma_x(s,t)$ depends on the times s and t, which voilate the two conditions stated above.

$\blacksquare$

Part (d)

W.T.S: $\rho_x(t-1,t) = \sqrt{\frac{t-1}{t}} \to 1$ as $t \to \infty$.

Proof

First, we show that $\rho_x(t-1,t) = \sqrt{\frac{t-1}{t}}$.

By definition,

$$\rho_x(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma_x(s)\cdot\gamma_x(t)}},$$

and we found in part (b) that $\gamma_x(s,t) = \min\{s,t\}\sigma_w^2$.

$$\Rightarrow \rho_x(t-1,t) = \frac{(t-1)\sigma_w^2}{\sqrt{(t-1)\sigma_w^2 \cdot t\sigma_w^2}}$$

$$= \frac{(t-1)\sigma_w^2}{\sqrt{t(t-1)(\sigma_w^2)^2}} = \sqrt{\frac{(t-1)^2}{t(t-1)}} = \sqrt{\frac{t-1}{t}}$$

For the remaining part, we have to take the limit of $\rho_x(\cdot,\cdot)$ as $t \to \infty$.

$$\Rightarrow \lim_{t\to\infty} \rho_x(t-1,t) = \lim_{t\to\infty}\left(\sqrt{\frac{t-1}{t}}\right)$$

$$= \sqrt{1}$$

$$= \underline{1}$$

Hence,

$$\rho_x(t-1,t) = \sqrt{\frac{t-1}{t}} \to \underline{1} \text{ as } t \to \infty,$$

as required. ▨

This implies that for large t the series at t x_t can be (linearly) predicted perfectly using only the value at t-1 (x_{t-1}).

<u>Part (e)</u>

I suggest using a transformation based on differencing as follows.

$$x_t = \delta + x_{t-1} + w_t$$

$$\Rightarrow x_t - x_{t-1} = \delta + w_t \quad —— \quad (*)$$

We now prove that the transformed series in (*) is stationary.

Let $z_t = x_t - x_{t-1}$

$$\Rightarrow E(z_t) = E(\delta + w_t) = \delta + E(w_t) = \delta$$

Moreover,

$$\gamma_z(h) = Cov(z_{t+h}, z_t) = Cov(w_{t+h}, w_t)$$

$$\Rightarrow \gamma_z(0) = Cov(w_t, w_t) = \sigma_w^2$$

$$\Rightarrow \gamma_z(1) = Cov(w_{t+1}, w_t) = 0 \quad (w_t\text{'s are uncorrelated})$$

$$= \gamma_z(-1)$$

$$\gamma_z(h) = \begin{cases} \sigma_w^2 & h = 0 \\ 0 & h \neq 0 \end{cases},$$

Hence, we have proved that the transformed series is stationary because the mean function does not depend on t and the autocovariance function only depends on h.

Problem 2.7

Given

$$x_t = U_1 \sin(2\pi \omega_0 t) + U_2 \cos(2\pi \omega_0 t),$$

where U_1 and U_2 are independent random variables with zero means and $E(U_1^2) = E(U_2^2) = \sigma^2$.
The constant ω_0 determines the period or time it takes the process to make one complete cycle.

W.T.S: The above series is weakly stationary with autocovariance function

$$\gamma(h) = \sigma^2 \cos(2\pi \omega_0 h).$$

Proof:
It suffices to find the mean function and the autocovariance of the series.
The mean function is given by

$$\mu_{x_t} = E(x_t) = E\big(U_1 \sin(2\pi \omega_0 t) + U_2 \cos(2\pi \omega_0 t)\big)$$

$$= E(U_1) \sin(2\pi \omega_0 t) + E(U_2) \cos(2\pi \omega_0 t)$$

$$= 0, \quad \text{since } E(U_1) = E(U_2) = 0 \qquad \text{— (1)}$$

The autocovariance function is also given by

$$\gamma_x(h) = Cov(x_{t+h}, x_t)$$

$$= Cov\left[U_1 \sin(2\pi \omega_0 (t+h)) + U_2 \cos(2\pi \omega_0 (t+h)),\; U_1 \sin(2\pi \omega_0 t) + U_2 \cos(2\pi \omega_0 t)\right]$$

For simplicity;
Let $A = 2\pi \omega_0 t$ and $B = 2\pi \omega_0 h$. Then

$$\gamma_x(h) = \sin A \sin(A+B)\, Cov(U_1, U_1) + \cos A \cos(A+B)\, Cov(U_2, U_2)$$

Note that $E(U_1^2) = Var(U_1) = \sigma^2$ and $E(U_2^2) = Var(U_2) = \sigma^2$.

$$\Rightarrow \gamma_x(h) = \sigma^2\left[\sin A \sin(A+B) + \cos A \cos(A+B)\right]$$

$$= \sigma^2\left[\sin A(\sin A \cos B + \cos A \sin B) + \cos A(\cos A \cos B - \sin A \sin B)\right]$$

$$= \sigma^2\left[\sin^2 A \cos B + \cancel{\sin A \sin B \cos A} + \cos^2 A \cos B - \cancel{\sin A \sin B \cos A}\right]$$

$$= \sigma^2\left[\sin^2 A \cos B + \cos^2 A \cos B\right]$$

$$= \sigma^2\left[\cos B(\sin^2 A + \cos^2 A)\right],$$ but $\sin^2 A + \cos^2 A = 1$ by the Pythagorean identity

$$= \sigma^2 \cos B$$

$$= \sigma^2 \cos(2\pi \omega_0 h) \quad \text{———} \quad 2$$

From 1 and 2, it can be seen that the mean and autocovariance functions are independent of time, which shows that the series, x_t, is weakly stationary with autocovariance function given by 2 as required.

Problem 2.8

Given the two series

$$x_t = w_t$$
$$y_t = w_t - \theta w_{t-1} + u_t,$$

where w_t and u_t are independent white noise series with variances σ_w^2 and σ_u^2, respectively, and θ is an unspecified constant.

Part (a):

We wish to express the ACF, $\rho_y(h)$, for $h = 0, \pm 1, \pm 2, \ldots$ of the series y_t as a function of σ_w^2, σ_u^2, and θ.

To this end, we need to first find the autocovariance function of y_t.

$$\gamma_y(h) = Cov(y_{t+h}, y_t) = Cov(w_{t+h} - \theta w_{t+h-1} + u_{t+h},\ w_t - \theta w_{t-1} + u_t)$$

$$\Rightarrow \gamma_y(0) = Cov(w_t - \theta w_{t-1} + u_t,\ w_t - \theta w_{t-1} + u_t)$$

$$= Cov(w_t, w_t) + \theta^2 cov(w_{t-1}, w_{t-1}) + cov(u_t, u_t)$$
$$= \sigma_w^2 + \theta^2 \sigma_w^2 + \sigma_u^2$$
$$= \sigma_w^2 (1 + \theta) + \sigma_u^2$$

$$\Rightarrow \gamma_y(1) = cov(w_{t+1} - \theta w_t + u_{t+1},\ w_t - \theta w_{t-1} + u_t)$$

$$= -\theta\, cov(w_t, w_t)$$
$$= -\theta \sigma_w^2 = \gamma_y(-1)$$

When $|h| > 1$, $\gamma_y(h) = 0$.

Thus,

$$\gamma_y(h) = \begin{cases} \sigma_w^2 (1+\theta) + \sigma_u^2 & h = 0 \\ -\theta \sigma_w^2 & h = \pm 1 \\ 0 & |h| > 1, \end{cases}$$

which implies that the ACF, $\rho_y(h)$ is given by

$$\rho_y(h) = \begin{cases} 1 & h = 0 \\ \dfrac{-\theta\sigma_w^2}{(1+\theta)\sigma_w^2 + \sigma_y^2} & h = \pm 1 \\ 0 & |h| > 1 \end{cases},$$

for the fact that $\rho_y(h) = \dfrac{\gamma_y(h)}{\gamma_y(0)}$. ▨

Part (b):

We want to determine CCF, $\rho_{xy}(h)$ relating x_t and y_t.

The cross-covariance function, $\gamma_{xy}(h)$ can be computed as

$$\gamma_{xy}(h) = \text{cov}(x_{t+h}, y_t) = \text{cov}(w_{t+h}, w_t - \theta w_{t-1} + u_t)$$

$$\Rightarrow \gamma_{xy}(0) = \text{cov}(w_t, w_t - \theta w_{t-1} + u_t)$$
$$= \text{cov}(w_t, w_t) \quad \text{since } w_ts \text{ and } w_t \text{ and } u_t \text{ are uncorrelated.}$$
$$= \sigma_w^2$$

$$\gamma_{xy}(1) = \text{cov}(w_{t+1}, w_t - \theta w_{t-1} + u_t)$$
$$= 0 \quad \text{and}$$
$$\gamma_{xy}(-1) = \text{cov}(w_{t-1}, w_t - \theta w_{t-1} + u_t)$$
$$= -\theta\,\text{cov}(w_{t-1}, w_{t-1}) = -\theta\sigma_w^2$$

We notice that $\gamma_{xy}(h) = 0$ for $|h| \geq 2$. So, generally, we have

$$\gamma_{xy}(h) = \begin{cases} \sigma_w^2 & h = 0 \\ -\theta\sigma_w^2 & h = -1 \\ 0 & h = 1,\ |h| \geq 2. \end{cases}$$

From part (a), $\gamma_y(0) = (1+\theta)\sigma_w^2 + \sigma_u^2$ and for x_t we know that

$$\gamma_x(0) = \sigma_w^2.$$

Thus,

$$\rho_{xy}(h) = \frac{\gamma_{xy}(h)}{\sqrt{\gamma_x(0)\gamma_y(0)}}, \quad h = 0, \pm 1, \pm 2, \dots$$

$$= \begin{cases} \dfrac{\sigma_w^2}{\sqrt{\sigma_w^2((1+\theta)\sigma_w^2 + \sigma_u^2)}} & h = 0 \\ 0 & h = 1 \\ \dfrac{-\theta\sigma_w^2}{\sqrt{\sigma_w^2((1+\theta)\sigma_w^2 + \sigma_u^2)}} & h = -1 \\ 0 & |h| \geq 2 \end{cases}$$

$\blacksquare$

Part (c)

W.T.S: x_t and y_t are jointly stationary.

First, we notice that x_t is stationary because

$$E(x_t) = 0 \quad \text{and} \quad \gamma_x(h) = \begin{cases} \sigma_w^2 & h = 0 \\ 0 & h \neq 0 \end{cases},$$

which are independent of time t.

Moreover, from part (a), $\gamma_y(h)$ only depends on the lag h

and $E(y_t) = E(w_t) - \theta E(w_{t-1}) + E(u_t)$
$= 0$,
showing that y_t is also stationary.

Finally, the cross-covariance function, $\gamma_{xy}(h)$, in part (b) only depends on the lag h.

Therefore,
we conclude that x_t and y_t are jointly stationary since they are each stationary and their cross-covariance function $\gamma_{xy}(h)$ is a function only of lag h. $\blacksquare$

Problem 2.9

Let w_t, for $t = 0, \pm 1, \pm 2, \ldots$ be a normal white noise process, and consider the series

$$x_t = w_t w_{t-1}.$$

We wish to determine the mean and autocovariance functions of x_t, and state whether it is stationary.

$$w_t \sim N(0, \sigma_w^2), \quad t = 0, \pm 1, \pm 2, \ldots$$

The mean function of x_t is computed as

$$\mu_{x_t} = E(x_t) = E(w_t w_{t-1})$$

$$= E(w_t) \cdot E(w_{t-1}) \quad \text{(by the independence of } w_t \text{ and } w_{t-1}\text{)}$$

$$= 0.$$

In addition, the autocovariance function is

$$\gamma_x(h) = Cov(x_{t+h}, x_t)$$

$$= E[(x_{t+h} - \mu_{x_{t+h}})(x_t - \mu_{x_t})]$$

$$= E[x_{t+h} x_t] \quad (\text{because } \mu_{x_{t+h}} = \mu_{x_t} = 0)$$

$$= E[(w_{t+h} w_{t+h-1})(w_t w_{t-1})]$$

when $h = 0$

$$\gamma_x(0) = E[w_t w_{t-1} w_t w_{t-1}]$$

$$= E[w_t^2 w_{t-1}^2] = E(w_t^2) \cdot E(w_{t-1}^2)$$

$$= Var(w_t) \cdot Var(w_{t-1})$$

$$= (\sigma_w^2)^2$$

w_{t+1} and w_{t-1} are uncorrelated

$$\gamma_x(1) = E[w_{t+1} w_t w_t w_{t-1}] = E(w_t^2) \cdot E(w_{t+1} w_{t-1}) = 0 = \gamma_x(-1)$$

Now, since w_t is a white noise process $\gamma_x(h) = 0$ for $h \neq 0$.

Hence, the covariance function of x_t is

$$\gamma_x(h) = \begin{cases} (\sigma_w^2)^2 & h = 0 \\ 0 & h \neq 0 \end{cases}$$

From the results we have so far, we can state with certainty that x_t is stationary since the mean function is independent of time t and the autocovariance function depends only on the lag h.

$\blacksquare$