Computational Finance

Outline

- Binomial model: option pricing and optimal investment
- Monte Carlo techniques for pricing of options
  - pricing of non-standard options
  - improving accuracy of Monte Carlo method
  - solving Stochastic Differential Equations (SDE)
  - pricing of exotic options
  - pricing of options in stochastic volatility models
- Partial Differential Equations (PDE)
  - numerical methods
  - the Black-Scholes equation for non-standard options

Literature

- Rüdiger Seydel, *Tools for Computational Finance*, Third Edition, Springer, 2006
- John C. Hull, *Options, Futures, and Other Derivatives*, Sixth Edition, Prentice Hall, 2006
- John C. Hull, *Options, Futures, and Other Derivatives*, Fifth Edition, Prentice Hall, 2003
- Justin London, *Modeling Derivatives in C++*, Wiley Finance, 2005
- Paul Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer, 2004

Lecture 1

Discrete Time Finance

Lecture Notes

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with additions
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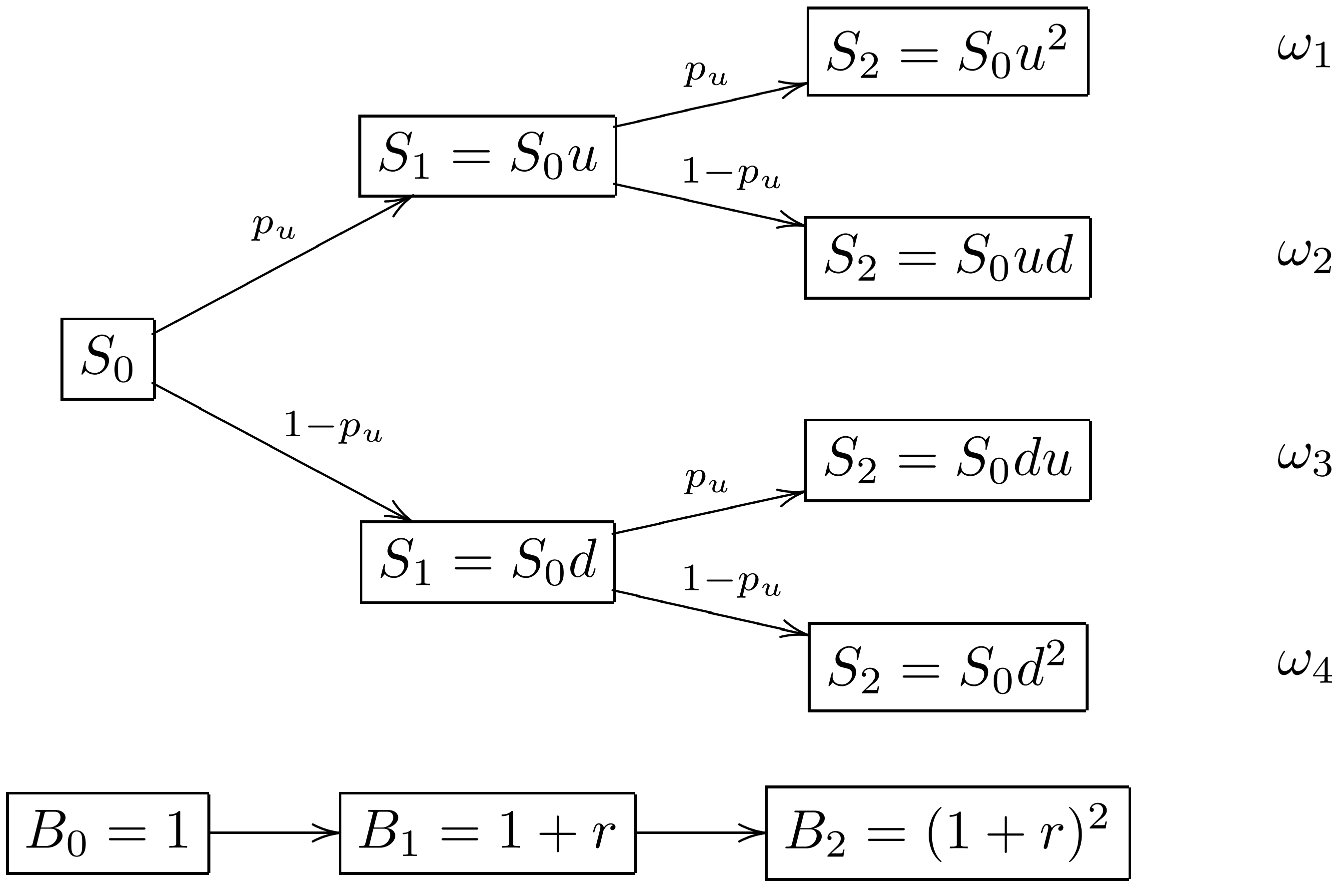
Discrete Time Finance

1. Time $t = 0, 1, 2, \ldots, T$.
2. A probability space $(\Omega, \mathbb{P})$ with the filtration $(\mathcal{F}_t)_{t=0,\ldots,T}$.
3. A money market account (B_t), which evolves according to $B_t = (1 + r)^t$ or $B_t = \mathrm{e}^{rt}$.
4. A risky financial asset whose price is given by a stochastic process

   $$\left(S_t\right)_{t=0,\ldots,T}$$

   which is assumed to be $(\mathcal{F}_t)$-adapted.
5. A family $\mathcal{T}$ of adapted self-financing trading strategies.
6. Risk neutral probability measure.

Binomial model



Parameters: u, d, r, S_0, p.

We change the Binomial Model slightly. We do not assume that the time elapsing between two consecutive stock price movements is 1, but we choose a time step, which we denote by Δt. The terminal time T is constant. Thus, the smaller Δt the more accurate the model (it has more time periods squeezed into the interval $[0, T]$). These are our time points:

$$i\ \Delta t, \qquad i = 0, 1, 2, \ldots, \frac{T}{\Delta t}.$$

We will use the notation $S_i = S_{i\,\Delta t}$, $B_i = B_{i\,\Delta t}$.

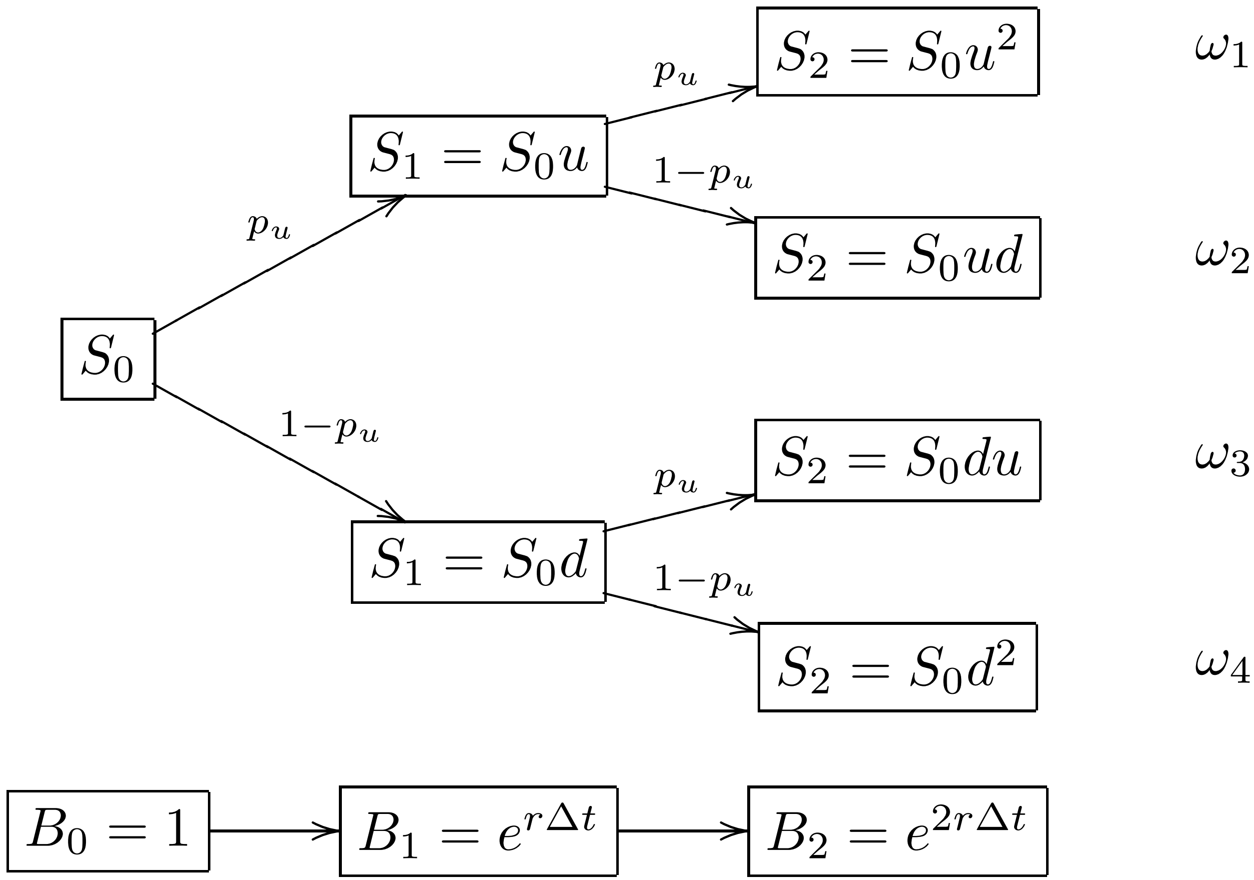
For the sake of consistency we shall assume that

$$B_i = e^{r\,i\Delta t},$$

so

$$\frac{B_{i+1}}{B_i} = e^{r\,\Delta t}.$$

A new binomial model



Parameters: u, d, r, S_0, p.

Properties

In the Binomial Market Model

$$\frac{S_i(\omega)}{S_{i-1}(\omega)} \in \{u, d\}, \qquad \omega \in \Omega.$$

We assume that $u > d$.

Theorem. The model is arbitrage free if and only if

$$\boxed{u > e^{r\Delta t} > d.}$$

Theorem. If the model is arbitrage free, then it is complete.

Explanation

If there is only one risky asset in the multi period market model then:

- There are no arbitrage if and only if all the constituent single period models are arbitrage-free.
- All risk neural measures can be computed by considering constituent single period market models.
- Single period market models give conditional probabilities for risk neutral measures.
- Single period market models can be used to compute prices of contingent claims.

Contingent claims

- European path-independent options
  - payoff given by $h(S_T)$
  - European call and put options
  - European binary call and put options
- European path-dependent options
  - Asian options (calls and puts on the average of the stock price)
  - Barrier options
  - Lookback options
- American options
  - they can be exercised anytime before or at maturity
  - the payoff at the exercise moment t is $h(S_t)$

Pricing of European options - first draft

Input: u, d, r, S_0, T, M (number of periods).

Compute one-period risk neutral probability p^*.

Construct a tree.

Compute values in the leaves of the tree (payoffs).

For i from $M-1$ to 0 do

  compute values in each node in the period i using one-period binomial formula (using p^*) and using values computed in the previous step

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Output: the value in the root of the tree is the price of the option

In above sketch of the algorithm the period length Δt is equal to $\frac{T}{M}$.

Computational complexity

Computational complexity plays a major role in the construction of numerical algorithms. It says how the amount of resources required by the algorithm grows with the size of the problem (eg. number of periods in the Binomial Model). There are two main types of resources:

- memory (RAM)
- time

For the algorithm from the previous slide

- memory requirement is proportional to 2^{M+1}
- time is proportional to 2^{M+1}

Computational complexity made practical

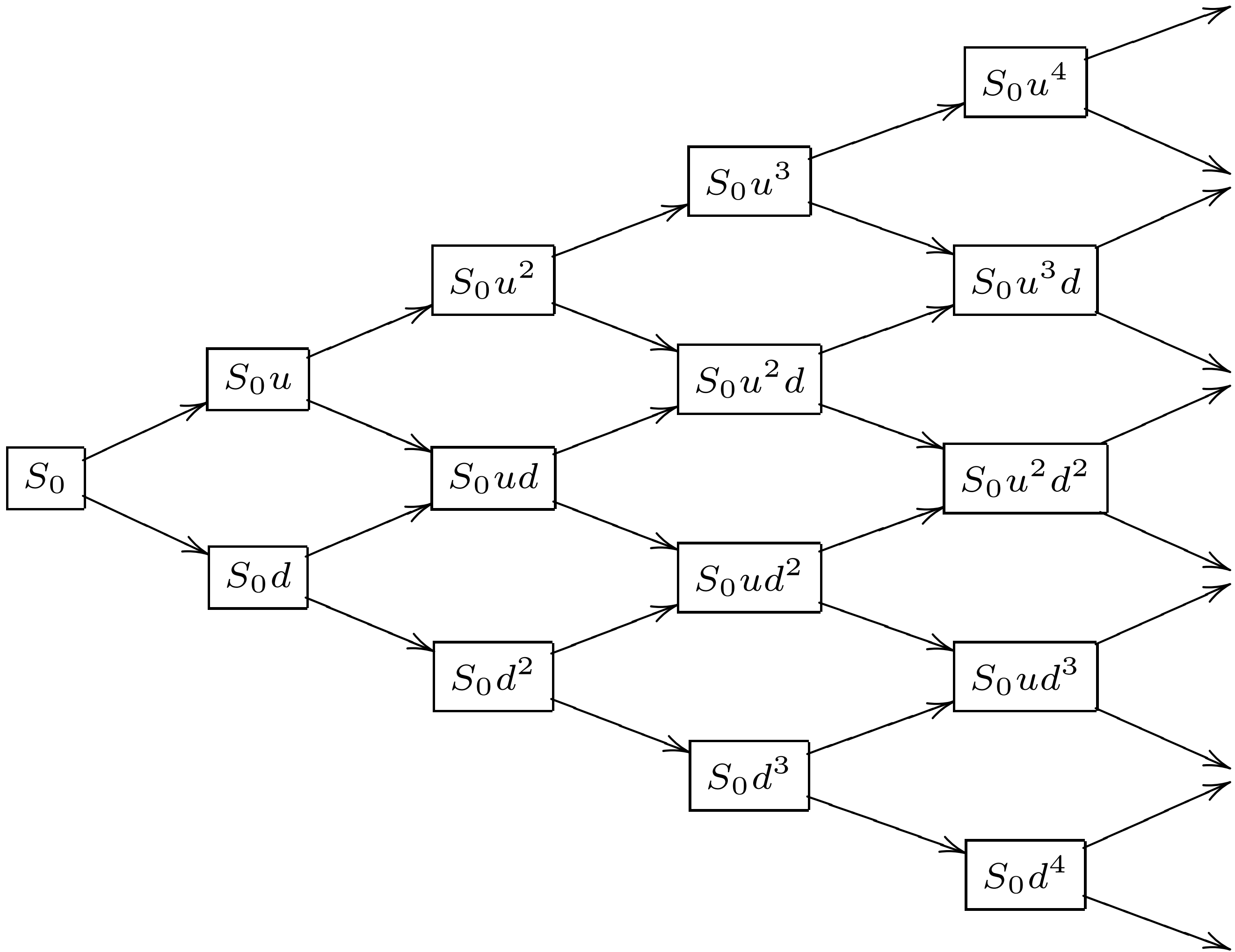
Why is the computational complexity so important?

Assume that one byte is enough for each node. To simulate $M = 100$ periods we need

$$2^{101} \text{ Bytes} = 2^{91} \text{ kBytes} = 2^{81} \text{ MBytes} = 2^{61} \text{ TBytes (!)}$$

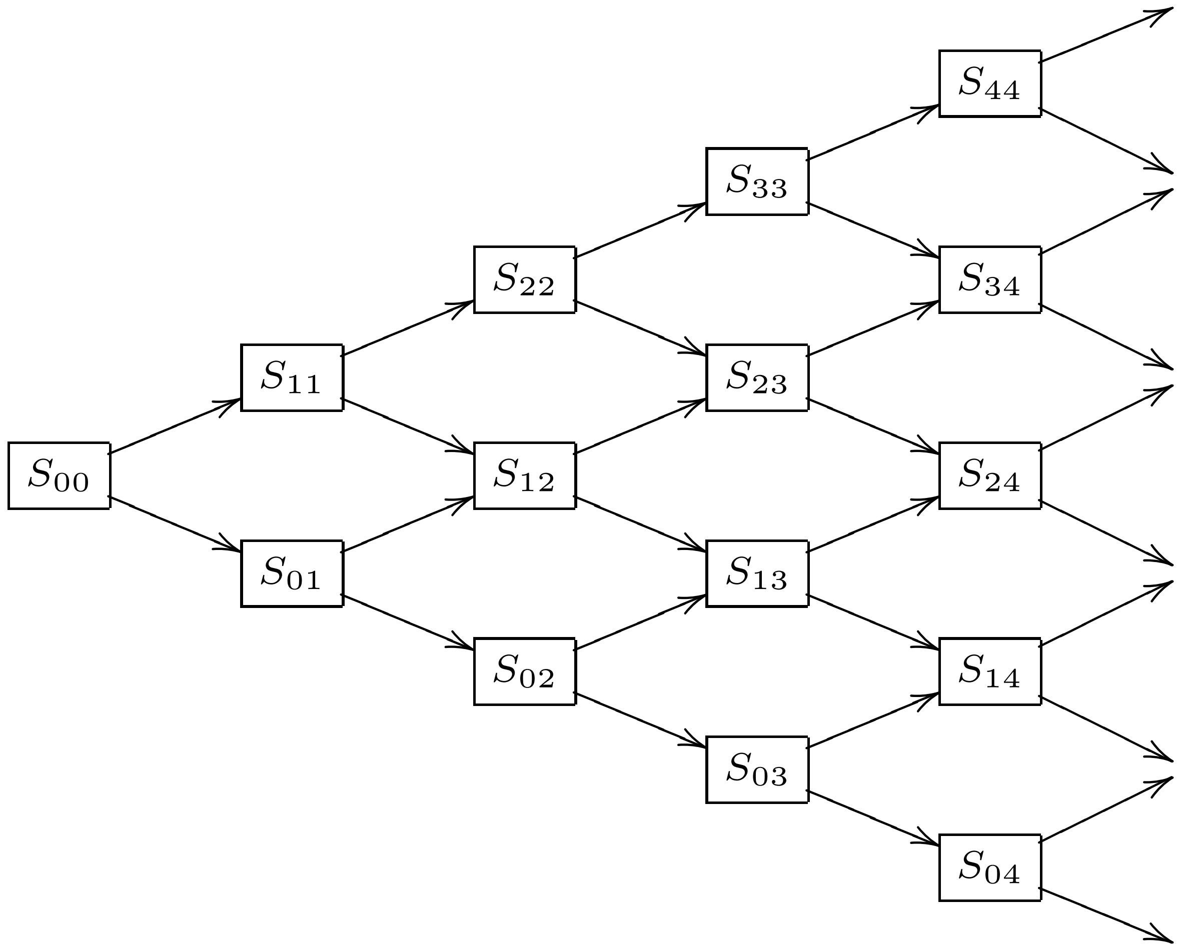
Path-independent options: the values in many of nodes of the tree are identical (for a simple example draw a tree for $n = 3$). We can glue identical nodes into one node -> recombining binomial tree.

Recombination of the binomial tree



Recombining tree can only be used to price path independent options and American options!!!!!

Recombination of the binomial tree



Recombining tree can only be used to price path independent options and American options!!!!!

Properties

Denote by S_{ji} the price after i periods with j being the number of up moves:

$$S_{ji} = S_0 u^j d^{i-j}.$$

Theorem. There exists a risk neutral measure (probability) given by the formula

$$p^* = \frac{e^{r\Delta t} - d}{u - d}$$

Under this measure disconted stock prices $\mathrm{e}^{-ri\Delta t} S(i\Delta t)$ form a martingale, i.e.

$$S_{ji} = \mathrm{e}^{-r\Delta t}\Big(p^* S_{j+1,i+1} + (1-p^*) S_{j,i+1}\Big)$$

Pricing contingent claims

Let h be the payoff function and V_{ji} denote the value process of the replicating strategy as known in the node S_{ji}. Then

$$V_{jM} = h(S_{jM}), \qquad j = 0, 1, \ldots, M.$$

European options: computation:

$$V_{ji} = \mathsf{e}^{-r\Delta t}\Big(p^* V_{j+1,i+1} + (1-p^*)V_{j,i+1}\Big)$$

American options: computation:

$$V_{ji} = \max\Big(h(S_{ji}), e^{-r\Delta t}\Big(p^* V_{j+1,i+1} + (1-p^*)V_{j,i+1}\Big)\Big).$$

European options

Input: u, d, r, S_0, T, M (number of periods).

$$\Delta t = T/M, \qquad p^* = \frac{e^{r\Delta t} - d}{u - d}$$

$$S_{00} = S_0$$

$$S_{jM} = S_{00} u^j d^{M-j}, \quad j = 0, 1, \ldots, M$$

For j from 0 to M do $\quad V_{jM} = h(S_{jM})$

For i from $M-1$ to 0 do

$$V_{ji} = e^{-r\Delta t}\Big(p^* V_{j+1,i+1} + (1-p^*)V_{j,i+1}\Big)$$

Output: V_{00}

American options

Input: u, d, r, S_0, T, M (number of periods).

$$\Delta t = T/M, \qquad p^* = \frac{e^{r\Delta t} - d}{u - d}$$

$S_{00} = S_0$

For i from 0 to M

 For j from 0 to i do $\quad S_{ji} = S_{00} u^j d^{i-j}$

For j from 0 to M do $\quad V_{jM} = h(S_{jM})$

For i from $M-1$ to 0 do

$$V_{ji} = \max\left(h(S_{ji}), e^{-r\Delta t}\left(p^* V_{j+1,i+1} + (1-p^*)V_{j,i+1}\right)\right)$$

Output: V_{00}

Computational complexity

For the algorithm from the previous slide

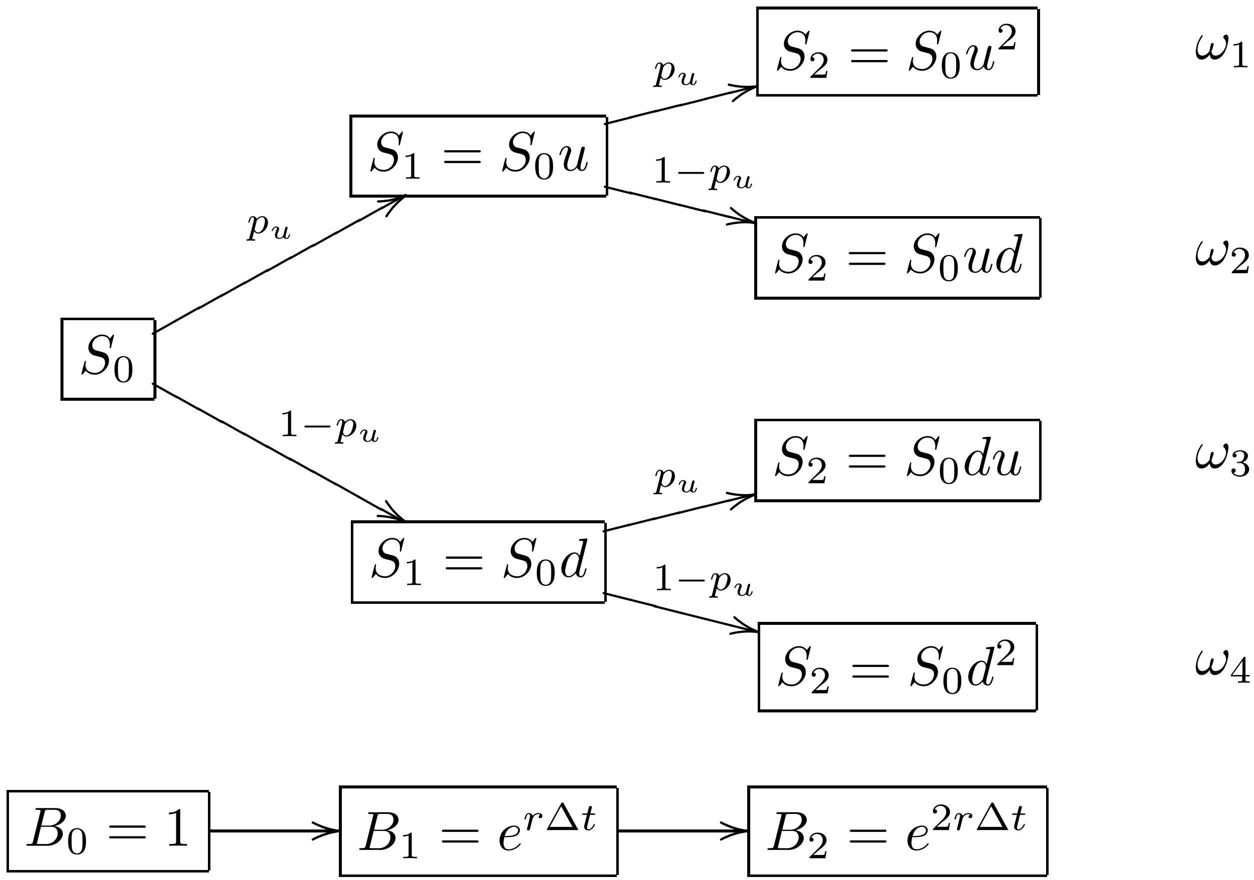
- memory size is proportional to $\frac{(M+1)(M+2)}{2}$, which means it grows like M^2
- time is proportional to $\frac{(M+1)(M+2)}{2}$, which means it grows like M^2

To simulate $M = 100$ periods we need

$$101 * 102/2 = 5151 \text{ nodes.}$$

Is there any hope for path-dependent options? Yes, Monte Carlo methods.

Binomial model in practice



Parameters:

unknown: u, d, p_u known: r, S_0

Calibration of the model

The parameters r and S_0 can be observed on financial markets, but the parameters u and d are only model idealizations and cannot be directly determined by observation of the real world data. We need to construct an algorithm to find u and d. It is called “calibration of the model”.

People trading in stock markets study a different parameter, which they call volatility, and which reflects a property of the continuous time model, known as the Black-Scholes model. In the Black-Scholes model asset prices follow the dynamics

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}, \qquad S_0 > 0,$$

where W_t is a Brownian Motion, σ is a volatility and μ is a growth rate.

Distribution of returns

For $\tau > 0$

$$\boxed{\log\left(\frac{S_{t+\tau}}{S_t}\right) \sim \mathcal{N}\big((\mu - \frac{1}{2}\sigma^2)\tau, \sigma^2\tau\big),}$$

where $\mathcal{N}\big((\mu - \frac{1}{2}\sigma^2)\tau, \sigma^2\tau\big)$ stands for a normal distribution with the mean

$$(\mu - \frac{1}{2}\sigma^2)\tau$$

and the variance

$$\sigma^2\tau.$$

We wish to compute the volatility σ (in the risk neutral world the knowledge of μ is not needed).

Estimation of the volatility of a stock price: The stock price is usually observed at fixed intervals of time (e.g. every day, week, or month). Define

- $n+1$ – the number of observations (numbered $0, 1, \ldots, n$);
- $\hat{S}_i$ – observed stock price at the end of i-th interval, $i = 0, 1, \ldots, n$;
- τ – length of time interval in years (can be different from Δt in the binomial model).

Let

$$u_i = \log \frac{\hat{S}_i}{\hat{S}_{i-1}}, \qquad i = 1, \ldots, n.$$

The usual estimate of volatility requires computation of the empirical variance:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2,$$

where

$$\bar{u} = \frac{1}{n} \sum_{i=1}^{n} u_i.$$

From the equation

$$\log\left(\frac{S_{t+\tau}}{S_t}\right) \sim \mathcal{N}\big((\mu - \frac{1}{2}\sigma^2)\tau, \sigma^2\tau\big),$$

the empirical variance s^2 is an estimate for $\sigma^2\tau$. Therefore, we have

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

with the standard error approximately equal to

$$\frac{\hat{\sigma}}{\sqrt{2n}}.$$

Standard error measures the precision of computation. It is a main building block for confidence intervals.

The rule of thumb: to get the precision of e.g. 0.1% we need the standard error smaller than $0.1\%/1.96$.

Trading Days vs. Calendar Days

An important issue is whether time should be measured in calendar days or trading days when volatility parameters are estimated and used. Practitioners tend to ignore days when the exchange is closed while estimating volatility from historical data. The number of trading days in a year is approximately equal to 252. Therefore,

- when using prices from consecutive trading days, take $\tau = \frac{1}{252}$
- when using prices from consecutive weeks (Mondays, Tuesdays, etc.), take $\tau = \frac{5}{252}$

Binomial Model Calibration

- so far we have computed the volatility of the stock price in the Black-Scholes model
- we do not know the growth rate
- we do not need the growth rate, because pricing is done with respect to a risk-neutral measure under which:

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t},$$

where r is a risk-free interest rate.

Find u and d such that under a risk-neutral measure p^* the one period expectation and variance in the Binomial Model agree with the Black-Scholes model under its risk-neutral measure.

$\mathbb{E}$ and Var in Black-Scholes

Let Δt be the length of one period in the Binomial Model.

The Black-Scholes model under a risk-neutral measure:

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

The conditional variance of $S_{t+\Delta t}$ given S_t is equal to

$$Var(S_{t+\Delta t}|S_t) = S_t^2 e^{2r\Delta t}\Big(e^{\sigma^2 \Delta t} - 1\Big).$$

The conditional expectation of $S_{t+\Delta t}$ given S_t is equal to

$$\mathbb{E}\left(S_{t+\Delta t}|S_t\right) = S_t e^{r\Delta t}.$$

$\mathbb{E}$ and Var in Binomial Model

Remember that Δt is the length of one period in the Binomial Model.

The conditional expectation of $S_{t+\Delta t}$ given S_t is

$$\mathbb{E}\left(S_{t+\Delta t}|S_t\right) = S_t\big(p^*u + (1-p^*)d\big).$$

The conditional variance of $S_{t+\Delta t}$ given S_t in a binomial model with the period length Δt, parameters r, u, d and the risk neutral probability of the up move equal to p^* is

$$\begin{aligned} Var(S_{t+\Delta t}|S_t) &= \mathbb{E}\left(S_{t+\Delta t}^2|S_t\right) - \Big(\mathbb{E}\left(S_{t+\Delta t}|S_t\right)\Big)^2 \\ &= p^*(S_t u)^2 + (1-p^*)(S_t d)^2 - S_t^2\big(p^*u + (1-p^*)d\big)^2. \end{aligned}$$

How they meet

Expectation

$$S_t\big(p^*u + (1-p^*)d\big) = S_t e^{r\Delta t}$$

Variance

$$p^*(S_t u)^2 + (1-p^*)(S_t d)^2 - S_t^2\big(p^*u + (1-p^*)d\big)^2 = S_t^2 e^{2r\Delta t}\Big(e^{\sigma^2\Delta t} - 1\Big)$$

After simplification:

$$\boxed{\begin{aligned} p^*u + (1-p^*)d &= e^{r\Delta t} \\ p^*u^2 + (1-p^*)d^2 &= e^{2r\Delta t + \sigma^2\Delta t} \end{aligned}}$$

The story is not finished yet

$$\boxed{\begin{aligned} p^* u + (1 - p^*) d &= e^{r \Delta t} \\ p^* u^2 + (1 - p^*) d^2 &= e^{2r\Delta t + \sigma^2 \Delta t} \end{aligned}}$$

There are two equations with three unknowns, so we may expect more than one solution.

The following relation follows from the first equation

$$p^* = \frac{e^{r\Delta t} - d}{u - d}.$$

Possible solutions:

- $u = d^{-1}$ leads to the Cox-Ross-Rubinstein model – industry standard
- $p^* = 1 - p^* = 1/2$ leads to the Jarrow-Rudd model – also quite popular

Cox-Ross-Rubinstein (CRR)

Assume that $u = d^{-1}$. We obtain the following set of parameters

$$u = \beta + \sqrt{\beta^2 - 1},$$
$$d = 1/u = \beta - \sqrt{\beta^2 - 1},$$

where

$$\beta = \frac{1}{2}\Big(e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}\Big).$$

In practice, there is a tendency to use a simplified version of the CRR model. It can be shown that up to the terms of higher order, we can set

$$u = e^{\sigma\sqrt{\Delta t}}, \qquad d = 1/u = e^{-\sigma\sqrt{\Delta t}},$$

and p^* computed as the risk neutral measure for these u and d.

Jarrow-Rudd (JR)

Assume now that $p^* = \frac{1}{2}$. This leads to the following parameters

$$u = e^{r\Delta t}\left(1 + \sqrt{e^{\sigma^2 \Delta t} - 1}\right),$$

$$d = e^{r\Delta t}\left(1 - \sqrt{e^{\sigma^2 \Delta t} - 1}\right).$$

The most popular approximate version for the Jarrow-Rudd model is

$$u = e^{(r-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}},$$

$$d = e^{(r-\sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}}.$$

Observe that the CRR tree is symmetric (due to $ud = 1$) and the JR tree is skewed since $ud = e^{(2r-\sigma^2)\Delta t}$.

General binomial tree

Assume

$$ud = e^{2\nu\Delta t},$$

where ν is some scalar.

This leads to the following approximate solution (exact solution is rather complicated):

$$u = e^{\nu\Delta t + \sigma\sqrt{\Delta t}},$$

$$d = e^{\nu\Delta t - \sigma\sqrt{\Delta t}}$$

$$p^* = \frac{1}{2} + \frac{1}{2}\left(\frac{\mu - \nu}{\sigma}\right)\sqrt{\Delta t}.$$

where $\mu = r - \sigma^2/2$.

Observe that for $\nu = 0$ we obtain the approximate version of the CRR model and for $\nu = \mu$, the JR model.

Input: market data, number of periods M, payoff function h, exercise time T

$\Delta t = T/M$

Calibrate the model to the market data: S_0, r, u, d, p^*

$S_{00} = S_0$

$S_{jM} = S_{00} u^j d^{M-j}, \quad j = 0, 1, \ldots, M$

(for American options) $S_{ji} = S_{00} u^j d^{i-j}$ for $i = 0, 1, \ldots, M-1$, and $j = 0, 1, \ldots i$

For j from 0 to M do $\quad V_{jM} = h(S_{jM})$

For i from $M-1$ to 0 do

(European) $V_{ji} = e^{-r\Delta t}\Big(p^* V_{j+1,i+1} + (1-p^*)V_{j,i+1}\Big)$

(American) $V_{ji} = \max\Big(h(S_{ji}), e^{-r\Delta t}\Big(p^* V_{j+1,i+1} + (1-p^*)V_{j,i+1}\Big)\Big)$

Output: V_{00} – the price