Chapter 13: MONTE CARLO SIMULATIONS ON TREES

13.1 MONTE CARLO SIMULATIONS ON A ONE-STEP BINOMIAL TREE

- Consider a one-step risk neutral binomial tree in which the interest rate has an equal chance to move up or down the tree
- We want to price an interest rate option that pays:

$$c_1 = 100 \times \max(r_1 - r_K, 0) \text{ at time } T = 0.5 \text{ (i.e. } i = 1)$$

- Using risk neutral pricing, the value of the option is:

$$c_0 = E^*[e^{-r0 \times T} \times c_1] = e^{-r0 \times T}\,[½ \times c_{1,u} + ½ \times c_{1,d}]$$

- An alternative way of computing the expected future payoff is to simulate upward and downward movements in the tree using a computer
- In Excel this is done through the RAND() function:
  - when RAND() < 0.5 then we say interest rates moved up
  - when RAND() ≥ 0.5 then we say interest rates moved down
- For each realization of RAND(), s, we can compute the value of interest rates at time $i = 1$ ($r^s_1 = r_{1,u}$ or $r_{1,d}$) and compute the payoff: $c^s_1 = 100 \times \max(r^s_1 - r_K, 0)$
- The expected discounted payoff can be approximated as the average of simulations:

$$\hat{c}_0 = average\left\{e^{-r_0 \times T} \times c_1^1, e^{-r_0 \times T} \times c_1^2, \ldots, e^{-r_0 \times T} \times c_1^N\right\} = \frac{1}{N}\sum_{s=1}^{N} e^{-r_0 \times T} \times c_1^s$$

Table 13.2 Ten Simulations on the Binomial Tree

Simulation Number	Realization of $RAND()$	Move on the Tree	Interest Rate Realization	Payoff at Maturity T	Discounted Payoff
1	0.67901	down	1.30%	0.000	0.000
2	0.222179	up	3.75%	1.747	1.732
3	0.684549	down	1.30%	0.000	0.000
4	0.761836	down	1.30%	0.000	0.000
5	0.140407	up	3.75%	1.747	1.732
6	0.092252	up	3.75%	1.747	1.732
7	0.999465	down	1.30%	0.000	0.000
8	0.472856	up	3.75%	1.747	1.732
9	0.521622	down	1.30%	0.000	0.000
10	0.575471	down	1.30%	0.000	0.000
				Average	0.693

13.1 MONTE CARLO SIMULATIONS ON A ONE-STEP BINOMIAL TREE

- The value of the security is, approximately, the average of the discounted payoffs, that is, the average of the numbers in the final column
- In this case, $\hat{c}_0 = 0.693$, with only 10 simulations
- This is still far from the value obtained from the tree ($c_0 = 0.866$), however as the number of simulations N increases, the value from simulations becomes more and more accurate
- For example, with 500 simulations we can get to $\hat{c}_0 = 0.897$, and with 1,000 simulations $\hat{c}_0 = 0.888$

13.2 MONTE CARLO SIMULATIONS ON A TWO-STEP BINOMIAL TREE

- **13.2.1 Example: Non-Recombining Trees in Asian Interest Rate Options**

- **13.2.2 Monte Carlo Simulations for Asian Interest Rate Options**

13.2 MONTE CARLO SIMULATIONS ON A TWO-STEP BINOMIAL TREE

- To understand Monte Carlo simulations on the two-step binomial tree it is important to recall that the binomial tree backward computation methodology is equivalent to the outright calculation of the value of the security as the risk neutral discounted present value of the payoff at maturity, discounted at the risk free rate:

$$c_0 = E^*[(e^{-r_0 \times 0.5}) \times (e^{-r_1 \times 0.5}) \times c_2] = E^*[(e^{-(r_0+r_1) \times 0.5}) \times c_2]$$

- Or equivalently:

$$c_0 = \tfrac{1}{4} \times e^{-(r_0+r_{1,u}) \times 0.5} \times c_{2,uu} + \tfrac{1}{4} \times e^{-(r_0+r_{1,u}) \times 0.5} \times c_{2,ud} + \tfrac{1}{4} \times e^{-(r_0+r_{1,d}) \times 0.5} \times c_{2,du} + \tfrac{1}{4} \times e^{-(r_0+r_{1,d}) \times 0.5} \times c_{2,dd}$$

- The following tree summarizes these results

Table 13.3 The 1-year Option

$i=0$	$i=1$	$i=2$
		$r_{2,uu} = 6.06\%$ $c_{2,uu} = 4.0616$
	$r_{1,u} = 3.75\%$ $c_{1,u} = e^{-3.75\%/2}$ $\times[\frac{1}{2}(4.0616 + 1.6150)]$ $= 2.7856$	
$r_0 = 1.74\%$ $c_0 = e^{-1.74\%/2}\times$ $\times[\frac{1}{2}(2.7856 + 0.8022)]$ $= 1.7784$		$r_{2,ud} = r_{2,du} = 3.61\%$ $c_{2,ud} = c_{2,du} = 1.6150$
	$r_{1,d} = 1.30\%$ $c_{1,d} = e^{-1.30\%/2}$ $\times\frac{1}{2} \times 1.6150$ $= 0.8022$	
		$r_{2,dd} = 1.17\%$ $c_{2,dd} = 0$

13.2 MONTE CARLO SIMULATIONS ON A TWO-STEP BINOMIAL TREE

- Performing the same simulation procedure as for the one-step binomial tree, we now take two random realizations for the interest rate movements
  - If $\text{RAND}()_1 < 0.5$ and $\text{RAND}()_2 < 0.5 \rightarrow r_{2,uu}$
  - If $\text{RAND}()_1 < 0.5$ and $\text{RAND}()_2 \geq 0.5 \rightarrow r_{2,ud}$
  - If $\text{RAND}()_1 \geq 0.5$ and $\text{RAND}()_2 < 0.5 \rightarrow r_{2,du}$
  - If $\text{RAND}()_1 \geq 0.5$ and $\text{RAND}()_2 \geq 0.5 \rightarrow r_{2,dd}$
- Note that this also has an effect on discount rates used to obtain present value, since we must use realized discounts
  - $\text{RAND}()_1$ determines whether we use $r_{1,u}$ or $r_{1,d}$
- We perform this for various realizations of RAND() and compute the average
- Once again, as we increase the number of simulations we will be closer to the value obtained in the model

Table 13.4 Ten Simulations on the Two-Step Binomial Tree

Simulation Number	First Realization of $RAND()$	First Move on the Tree	Interest Rate Realization at Time $i = 1$	Second Realization of $RAND()$	Second Move on the Tree	Interest Rate Realization at Time $i = 2$	Payoff at Maturity	Discounted Payoff
1	0.468692	up	3.75%	0.513012	down	3.61%	1.615	1.571
2	0.103819	up	3.75%	0.164345	up	6.06%	4.062	3.952
3	0.848307	down	1.30%	0.340712	up	3.61%	1.615	1.591
4	0.702089	down	1.30%	0.254097	up	3.61%	1.615	1.591
5	0.001839	up	3.75%	0.118779	up	6.06%	4.062	3.952
6	0.444781	up	3.75%	0.241821	up	6.06%	4.062	3.952
7	0.066274	up	3.75%	0.55081	down	3.61%	1.615	1.571
8	0.386942	up	3.75%	0.765509	down	3.61%	1.615	1.571
9	0.769553	down	1.30%	0.627444	down	1.17%	0.000	0.000
10	0.888142	down	1.30%	0.97949	down	1.17%	0.000	0.000
							Average	1.975

13.2.1 Example: Non-Recombining Trees in Asian Interest Rate Options

- An Asian interest rate call option is an option whose payoff at maturity is given by:

$$\max(\text{average rate from 0 to } T - r_K,\ 0)$$

- An Asian interest rate put option is an option whose payoff at maturity is given by:

$$\max(r_K - \text{average rate from 0 to } T,\ 0)$$

- Where the average rate from 0 to T is:

$$\bar{r} = \frac{1}{n}\sum_{i=0}^{n} r_i$$

- The problem for pricing these securities through simple trees is that the interest rate path "up and down" leads to a different average interest rate than the path "down and up"

$$\left(r_0 = 1.74\% \rightarrow r_{1,u} = 3.75\% \rightarrow r_{2,d} = 3.61\%\right) \Rightarrow \bar{r}_{2,ud} = 3.03\%$$

$$\left(r_0 = 1.74\% \rightarrow r_{1,d} = 1.30\% \rightarrow r_{2,u} = 3.61\%\right) \Rightarrow \bar{r}_{2,du} = 2.22\%$$

- The tree for the average interest rate does not recombine

13.2.2 Monte Carlo Simulations for Asian Interest Rate Options

- Monte Carlo Simulations are especially useful to compute the value of path dependent securities, that is, securities whose value at maturity does not depend only on the interest rate at maturity, but on the whole history of interest rates
- Consider an Asian interest rate call option with $r_K = 2\%$
- As explained before standard tree methodologies don’t work well for these securities, since they are path dependent
- In order to find the price, we now compute the average of each path of interest rates and then follow the same procedure as with standard options, using Monte Carlo Simulations

Table 13.5 An Asian Interest Rate Option on a Binomial Tree

$i = 0$	$i = 1$	$i = 2$	
		$r_{2,uu} = 6.06\%$ $\overline{r}_{2,uu} = 3.85\%$	$\Longrightarrow$ $c_{2,uu} = 1.8495$
	$r_{1,u} = 3.75\%$ $\overline{r}_{1,u} = 2.7\%$		
		$r_{2,ud} = 3.61\%$ $\overline{r}_{2,ud} = 3.03\%$	$\Longrightarrow$ $c_{2,ud} = 1.0340$
$r_0 = 1.74\%$			
		$r_{2,du} = 3.61\%$ $\overline{r}_{2,ud} = 2.22\%$	$\Longrightarrow$ $c_{2,du} = 0.2185$
	$r_{1,d} = 1.30\%$ $\overline{r}_{1,d} = 1.52\%$		
		$r_{2,dd} = 1.17\%$ $\overline{r}_{2,ud} = 1.40\%$	$\Longrightarrow$ $c_{2,dd} = 0$

Table 13.6 The Valuation of the Asian Interest Rate Option on a Binomial Tree

$i = 0$	$i = 1$	$i = 2$
		$c_{2,uu} = 1.8495$
	$c_{1,u} = e^{-3.75\%/2} \times \left[\frac{1}{2} \times 1.8495 + \frac{1}{2} \times 1.0340\right] = 1.4150$	
		$c_{2,ud} = 1.0340$
$c_0 = e^{-1.74\%/2} \times \left[\frac{1}{2} \times 1.4150 + \frac{1}{2} \times 0.1085\right] = 0.7552$		
		$c_{2,du} = 0.2185$
	$c_{1,d} = e^{-1.30\%/2} \times \left[\frac{1}{2} \times 0.2185 + \frac{1}{2} \times 0\right] = 0.1085$	
		$c_{2,dd} = 0$

Table 13.7 Ten Simulations for the Asian Options

Simulation Number	First Realization of $RAND()$	First Move on the Tree	Interest Rate Realization at Time $i = 1$	Second Realization of $RAND()$	Second Move on the Tree	Interest Rate Realization at Time $i = 2$	Payoff at Maturity	Discounted Payoff
1	0.468692	up	3.75%	0.513012	down	3.61%	1.034	1.006
2	0.103819	up	3.75%	0.164345	up	6.06%	1.84953	1.799
3	0.848307	down	1.30%	0.340712	up	3.61%	0.21847	0.215
4	0.702089	down	1.30%	0.254097	up	3.61%	0.21847	0.215
5	0.001839	up	3.75%	0.118779	up	6.06%	1.84953	1.799
6	0.444781	up	3.75%	0.241821	up	6.06%	1.84953	1.799
7	0.066274	up	3.75%	0.55081	down	3.61%	1.034	1.006
8	0.386942	up	3.75%	0.765509	down	3.61%	1.034	1.006
9	0.769553	down	1.30%	0.627444	down	1.17%	0	0.000
10	0.888142	down	1.30%	0.97949	down	1.17%	0	0.000
							Average :	0.885

13.3 MONTE CARLO SIMULATIONS ON MULTI-STEP BINOMIAL TREES

- **13.3.1 Does This Procedure Work?**
- **13.3.2 Illustrative Example: Long-Term Interest Rate Options**
- **13.3.3 How Many Simulations are Enough?**

13.3 MONTE CARLO SIMULATIONS ON MULTI-STEP BINOMIAL TREES

- The approach can readily be extended to multi-period binomial trees, which is an important extension as it allows us to evaluate relatively complex securities
- Consider the Ho-Lee model that postulates that interest rates dynamics follow:

$$r_{i+1,j} = r_{i,j} + \theta_i \times \Delta + \sigma \times (\Delta)^{½} \text{ with } p^* = ½$$

$$r_{i+1,j+1} = r_{i,j} + \theta_i \times \Delta - \sigma \times (\Delta)^{½} \text{ with } p^* = ½$$

- It is relatively straightforward to simulate the paths on the tree:
  - Obtain random realizations for the interest rate process up to maturity (for each time step use the RAND() function to determine whether you go up or down the tree)
  - From these rates we can obtain discount rates for each path:

$$Z^s(0,T_i) = e^{-(r_0+r^s_1+r^s_2+\ldots+r^s_{i-1}) \times \Delta}$$

- For each maturity we can compute the average discount across simulations:

$$\hat{Z}(0,T_i) = \frac{1}{N}\sum_{s=1}^{N} Z^s(0,T_i)$$

Table 13.8 Ten Monte Carlo Simulations of Ho-Lee Interest Rate Tree

Time T	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
Period i	0	1	2	3	4	5	6	7	8	9	10
$\theta_i (\times 100)$	1.5675	2.1824	1.4374	1.7324	0.7873	0.0423	-0.0628	0.4322	0.9271	0.1202	
Panel A: The Ho-Lee Interest Rate Tree											
j											
0	1.74	3.75	6.06	8.00	10.09	11.71	12.95	14.15	15.59	17.27	18.56
1		1.30	3.61	5.56	7.65	9.26	10.51	11.70	13.14	14.83	16.11
2			1.17	3.11	5.20	6.82	8.06	9.25	10.69	12.38	13.66
3				0.66	2.75	4.37	5.61	6.81	8.25	9.93	11.22
4					0.31	1.92	3.17	4.36	5.80	7.49	8.77
5						-0.52	0.72	1.91	3.35	5.04	6.32
6							-1.73	-0.53	0.91	2.59	3.88
7								-2.98	-1.54	0.15	1.43
8									-3.99	-2.30	-1.02
9										-4.75	-3.46
10											-5.91

Panel B: Ten Simulated Paths of Ho-Lee Tree

	Period i										
Simulation	0	1	2	3	4	5	6	7	8	9	10
1	1.74	1.30	3.61	5.56	5.20	6.82	8.06	6.81	8.25	7.49	6.32
2	1.74	3.75	3.61	3.11	2.75	4.37	5.61	6.81	5.80	5.04	3.88
3	1.74	3.75	3.61	3.11	5.20	6.82	8.06	9.25	8.25	9.93	8.77
4	1.74	3.75	6.06	5.56	7.65	9.26	8.06	6.81	8.25	7.49	8.77
5	1.74	1.30	3.61	3.11	5.20	6.82	5.61	6.81	5.80	7.49	6.32
6	1.74	1.30	3.61	3.11	5.20	4.37	5.61	6.81	5.80	7.49	6.32
7	1.74	1.30	3.61	5.56	5.20	4.37	3.17	4.36	3.35	2.59	1.43
8	1.74	1.30	1.17	0.66	2.75	1.92	0.72	1.91	3.35	5.04	3.88
9	1.74	3.75	3.61	3.11	5.20	6.82	8.06	6.81	5.80	7.49	8.77
10	1.74	3.75	6.06	8.00	10.09	9.26	8.06	9.25	10.69	12.38	11.22

Panel C: Ten Simulated Discounts $Z^s(0, T_i)$

	Maturity T_i										
Simulation	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5
1	0.9913	0.9849	0.9673	0.9408	0.9166	0.8859	0.8509	0.8224	0.7892	0.7602	0.7366
2	0.9913	0.9729	0.9555	0.9408	0.9279	0.9078	0.8827	0.8532	0.8288	0.8082	0.7927
3	0.9913	0.9729	0.9555	0.9408	0.9166	0.8859	0.8509	0.8124	0.7796	0.7418	0.7100
4	0.9913	0.9729	0.9439	0.9180	0.8836	0.8436	0.8103	0.7832	0.7515	0.7239	0.6929
5	0.9913	0.9849	0.9673	0.9523	0.9279	0.8968	0.8720	0.8428	0.8187	0.7886	0.7641
6	0.9913	0.9849	0.9673	0.9523	0.9279	0.9078	0.8827	0.8532	0.8288	0.7983	0.7735
7	0.9913	0.9849	0.9673	0.9408	0.9166	0.8968	0.8827	0.8637	0.8493	0.8384	0.8324
8	0.9913	0.9849	0.9792	0.9759	0.9626	0.9534	0.9499	0.9409	0.9253	0.9022	0.8849
9	0.9913	0.9729	0.9555	0.9408	0.9166	0.8859	0.8509	0.8224	0.7989	0.7696	0.7366
10	0.9913	0.9729	0.9439	0.9069	0.8622	0.8232	0.7907	0.7549	0.7156	0.6727	0.6360

13.3.1 Does This Procedure Work?

- One way to check whether the Monte Carlo simulations methodology provides the correct value of derivative securities is to use the same simulations to also compute the values of zero coupon bonds
- Recall that the Ho-Lee interest rate tree was computed so that the tree would price exactly the zero coupon bond prices
- Since Monte Carlo simulations are designed to provide the same answers as any security computed on the tree, with an approximation that depends on the number of simulations, it follows that the zero coupon bond prices obtained should match the zero coupon bonds we used to estimate the model
- The prices are indeed quite close to each other, and the discrepancy is mainly due to the low number of simulations used

Table 13.9 Simulated Zero Coupon Bonds versus Data (January 8, 2002)

Maturity	Data	Simulated Zeros
0.5	99.1338	99.1338
1.0	97.8925	97.9140
1.5	96.1462	96.2376
2.0	94.1011	94.3031
2.5	91.7136	92.0217
3.0	89.2258	89.6090
3.5	86.8142	87.2894
4.0	84.5016	85.0911
4.5	82.1848	82.8855
5.0	79.7718	80.5609
5.5	77.4339	78.3302

13.3.2 Illustrative Example: Long-Term Interest Rate Options

- Consider an interest rate option that pays at maturity T_i: $100 \times \max(r_i - r_K, 0)$
- Risk neutral pricing implies that :
  $c_0(T_i) \quad = E^*[\text{Present Value of payoff at } T_i]$
  $\quad = E^*[e^{-(r_0+r_1+\ldots+r_{i-1})\times\Delta} \times 100 \times \max(r^s_i - r_K, 0)$
- Note that now the payoff also depends on the interest rate
- However for each simulation of path s we must compute the discounted payoff
  $c^s_0(T_i) \quad = e^{-(r_0+r^s_1+r^s_2+\ldots+r^s_{i-1})\times\Delta} \times 100 \times \max(r^s_i - r_K, 0)$
  $\quad = Z_s(0,T_i) \times 100 \times \max(r^s_i - r_K, 0)$
- The value of $c_0(T_i)$ is approximately the average of $c^s_0(T_i)$

$$\hat{c}_0(T_i) = \frac{1}{N}\sum_{s=1}^{N} c_0^s(T_i)$$

Table 13.10 Monte Carlo Simulations for Long-Term Interest Rate Options

$r_K = 1.74\%$

Panel A: Simulated Interest Rate Call Options

Maturity T_i									
0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
0.9650	1.8857	2.4959	3.1892	3.4691	3.4134	3.2866	3.4007	3.6378	3.5060

Panel B: Ten Simulated Discounted Payoffs

	Maturity T_i									
Simulation	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
1	0.00	1.85	3.69	3.25	4.65	5.60	4.31	5.35	4.53	3.48
2	1.99	1.82	1.31	0.95	2.44	3.52	4.47	3.46	2.73	1.73
3	1.99	1.82	1.31	3.25	4.65	5.60	6.39	5.29	6.39	5.21
4	1.99	4.20	3.60	5.42	6.65	5.33	4.11	5.10	4.32	5.09
5	0.00	1.85	1.33	3.29	4.71	3.47	4.42	3.42	4.70	3.61
6	0.00	1.85	1.33	3.29	2.44	3.52	4.47	3.46	4.76	3.66
7	0.00	1.85	3.69	3.25	2.41	1.28	2.31	1.39	0.72	0.00
8	0.00	0.00	0.00	0.99	0.18	0.00	0.16	1.52	3.05	1.93
9	1.99	1.82	1.31	3.25	4.65	5.60	4.31	3.34	4.59	5.41
10	1.99	4.20	5.91	7.58	6.49	5.20	5.94	6.76	7.61	6.37

13.3.3 How Many Simulations are Enough?

- We can think of a price of an interest rate security computed by Monte Carlo Simulations as an estimate of the true price
  - Using simulations we are essentially generating a sample of observations of the price of the security, under various scenarios for (risk neutral) future interest rates
- As with any statistical estimate, we can compute a measure that calculates how confident we are in the number we estimated:

$$\text{Standard error} = \text{std.dev.}\{c^1{}_0, c^2{}_0, c^3{}_0, \ldots, c^N{}_0\} \,/\, (N)^{1/2}$$

- Using the standard error we can compute confidence intervals which state that given a number of simulations N, there is 95% probability that the true value of the security is between the upper and lower boundaries:

$$\text{Confidence Interval} = [\hat{c}_0 - 2 \times \text{St.Err.},\ \hat{c}_0 + 2 \times \text{St.Err.}]$$

Table 13.11 Standard Errors and Confidence Intervals

	Maturity T_i									
	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
Price	0.9650	1.8857	2.4959	3.1892	3.4691	3.4134	3.2866	3.4007	3.6378	3.5060
Standard Errors	0.0315	0.0464	0.0562	0.0639	0.0708	0.0715	0.0751	0.0737	0.0758	0.0745
Upper C.I.	1.0279	1.9784	2.6083	3.3170	3.6107	3.5563	3.4368	3.5482	3.7893	3.6550
Lower C.I.	0.9021	1.7930	2.3835	3.0613	3.3274	3.2705	3.1363	3.2532	3.4862	3.3569
True Price (Tree)	1.000	1.976	2.612	3.309	3.542	3.507	3.361	3.463	3.687	3.596

13.4 PRICING PATH DEPENDANT OPTIONS

- **13.4.1 Illustrative Example: Long-Term Asian Options**

- **13.4.1 Case Study: Banc One AIRS**

13.4.1 Illustrative Example: Long-Term Asian Options

- We can use exactly the same methodology of simulated paths interest rate paths to compute the value of Asian options
- The convenient feature of Monte Carlo simulations is that we need to change only the payoff at maturity to obtain the price of the new security, but the simulated paths can be the same
- We use the same paths as the interest rate options discussed previously
- Asian options, as it can be seen, are cheaper than standard options, as the underlying (the average of interest rates) is less volatile than interest rates themselves

Table 13.12 Monte Carlo Simulations for Asian Interest Rate Options

Panel A: Simulated Interest Rate Asian Call Options

	Maturity T_i									
	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
price	0.4825	0.9079	1.2579	1.6022	1.8616	2.0189	2.1100	2.1750	2.2386	2.2647
standard errors	0.0157	0.0243	0.0308	0.0351	0.0382	0.0401	0.0416	0.0424	0.0428	0.0426
Upper C.I.	0.5140	0.9566	1.3194	1.6723	1.9380	2.0991	2.1932	2.2598	2.3242	2.3500
Lower C.I.	0.4510	0.8592	1.1964	1.5320	1.7851	1.9386	2.0269	2.0902	2.1529	2.1794

Panel B: Ten Simulated Discounted Payoffs

	Maturity T_i									
simulation #	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
1	0.00	0.47	1.27	1.64	2.11	2.55	2.68	2.90	2.95	2.90
2	0.99	1.26	1.25	1.18	1.38	1.66	1.97	2.08	2.09	2.01
3	0.99	1.26	1.25	1.64	2.11	2.55	2.94	3.08	3.30	3.33
4	0.99	2.05	2.39	2.95	3.47	3.60	3.54	3.61	3.55	3.57
5	0.00	0.47	0.68	1.19	1.75	1.95	2.21	2.28	2.46	2.49
6	0.00	0.47	0.68	1.19	1.38	1.66	1.97	2.08	2.29	2.34
7	0.00	0.47	1.27	1.64	1.73	1.64	1.70	1.63	1.52	1.34
8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.29	0.44
9	0.99	1.26	1.25	1.64	2.11	2.55	2.68	2.67	2.79	2.94
10	0.99	2.05	2.97	3.80	4.09	4.09	4.18	4.30	4.43	4.36

13.4.1 Case Study: Banc One AIRS

- In 1993 Banc One invented a new security to increase the yield on its investments, the amortizing interest rate swap (AIRS); the idea was to synthetically replicate an investment in mortgage backed securities
- AIRS is a swap in which the notional decreases as the interest rate decreases
- How could such an investment help Banc One increase the yield on its investment? To understand the logic, consider a plain vanilla swap
- Recall that the market swap rate, call it $c(0,T)$, is determined at time 0 such that the value of the swap with maturity T is equal to zero at initiation
- If we use the plain vanilla swap rate $c(0,T)$ in an AIRS, then the fixed rate receiver is worse off
- This is because when interest rates go up, the notional is constant, and thus the fixed rate receiver must make high payments
- When interest rates go down, in contrast, the fixed rate receiver would receive a positive net payment $(c(0,T) - r_4(t))$, but this amount is now multiplied by a lower notional, because the latter decreases when interest rates decline
- To make the value of the swap equal to zero at time 0, we must increase the swap rate above the market rate of plain vanilla swaps

Table 13.13 Index Amortizing Interest Rate Swap Term Sheet

Counterparties	Banc One and Investment Bank XYZ
Notional Amount	$500 million, subject to amortization schedule
Maximum Maturity Date	3 years
Early Maturity Date (Cleanup Provision)	On any fixing date leading to a notional amount less than or equal to 10% of original notional
Banc One Pays	in USD, 3m LIBOR, paid quarterly (current LIBOR =3.25%)
Banc One Receives	Fixed rate $r_{\text{Fxd}} = 4.5\%$
Amortization Schedule	see Amortization Schedule below
Lockout period	1 year (included)

Amortization Schedule

USD 3m LIBOR	Notional Reduction
$\leq$ 3.35%	Total Amortization
4.35%	Reduced by 31%
5.35%	Reduced by 10.5%
$\geq$ 6.35 %	No Amortization

(If the spot rate falls between two entries, the amortization amount is interpolated.)

13.4.1 Case Study: Banc One AIRS

- The difficulty in pricing AIRS is that the payoffs are highly path dependent
- The amortization schedule implies that every three months (time t) the Banc One receives the net flow: $CF(t) = ¼ \times (r_{Fxd} - r^L_4(t - 0.25)) \times \text{Notional}_{t\text{-}0.25}$
- For example, suppose the LIBOR follows the sequence in Table 13.14; then, the following cash flows are generated:
  - At time t = 1.25, Banc One receives:

$$CF(1.25) = ¼ \times (r_{Fxd} - 4\%) \times 500 \text{ million}$$

  - At time t = 1.5, Banc One receives:

$$CF(1.50) = ¼ \times (r_{Fxd} - 4.35\%) \times 0.69 \times 500 \text{ million}$$

  - That is the notional has been reduced by 31% to 69% of its previous value
- The notional level is path dependent
  - At time t = 1.75, the Notional would be:

$$\text{Notional}_{1.75} = (0.69 \times 500 \text{ million}) \times 0.895 = 308.775 \text{ million}$$

  - Now imagine that instead at t = 1.25, LIBOR was 5.35% instead of 4.35%:

$$\text{Notional}_{1.75} = (0.895 \times 500 \text{ million}) \times 0.895 = 400.5125 \text{ million}$$

- We can use Monte Carlo simulations on a binomial tree to price this security

Table 13.14 A Hypothetical Path of the 3-Months LIBOR

Time	LIBOR
0.75	3.5%
1.00	4%
1.25	4.35%
1.50	5.35%

13.4.1 Case Study: Banc One AIRS (cont.)

- We can use Monte Carlo simulations on a binomial tree to price this security
- We simulate interest rates at a quarterly frequency, and for every interest rate path s $(r_0, r^s_1, r^s_2, \ldots, r^s_n)$ we define the cash flow in period i as:

$$CF(t) = \tfrac{1}{4} \times (r_{Fxd} - r^L_4(t - 0.25)) \times \text{Notional}_{t-0.25}$$

where $r^s_4(i-1) = 4 \times (e^{r^s_{i-1} \times 0.25} - 1)$ is the quarterly compounded rate corresponding to r^s_{i-1}

- We can compute the notional to be applied in the next period $i + 1$; this cash flow will depend on the Notional^s_i, which is computed as:

$$\text{Notional}^s_i = \text{Notional}^s_{i-1} \times Adj(r^s_i)$$

we compute the adjustment $Adj(r^s_i)$ according to the amortization schedule in the term sheet

- For every simulation s, we obtain the time-0 value of future cash-flows as

$$P_s = e^{-\frac{1}{4} \times r_0} \times CF^s_1 + e^{-\frac{1}{4} \times (r_0 + r^s_1)} \times CF^s_2 + \ldots + e^{-\frac{1}{4} \times (r_0 + r^s_1 + \ldots + r^s_{n-1})} \times CF^s_n$$

- As before, the simulated value of the future cash flows is

$$P = \frac{1}{J} \sum_{j=1}^{J} P_j$$

Table 13.15 A Schematic Representation of Simulated Interest Rates and Cash Flows

		period			
Simulation number	0	1	2	...	n
Simulation 1	r_0	r_1^1	r_2^1	...	r_n^1
		CF_1^1	CF_2^1	...	CF_n^1
Simulation 2	r_0	r_1^2	r_2^2	...	r_n^2
		CF_1^2	CF_2^2	...	CF_n^2
$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$
Simulation N	r_0	r_1^N	r_2^N	...	r_n^N
		CF_1^N	CF_2^N	...	CF_n^N

13.4.1 Case Study: Banc One AIRS (cont.)

- AIRS Pricing by Monte Carlo Simulations (Table 13.16)
  - **What Data? (Panel A)** Because we are pricing a LIBOR-linked asset, we should use LIBOR-based plain vanilla swaps; from swap data, we can compute the discount up to the maturity of the AIRS
  - **What Model? (Panel B)**We use the Ho-Lee model for this exercise, as interest rates were already very low, and we want to give them chance to get even lower
  - **Simulate Interest Rates: (Panel C)** the important caveat is that cash flows computations and amortization trigger rates are expressed with quarterly compounding, so we must then convert the interest rates into quarterly compounding

13.4.1 Case Study: Banc One AIRS (cont.)

- AIRS Pricing by Monte Carlo Simulations (Table 13.16)
  - **Simulate Notional: (Panel D)** We have the dynamics of the notional over 10 paths:
    - Since there is a 1-year lockout period the notional doesn't change in the first five periods
    - Note also that we must interpolate in order to find the level of amortization: let $r^s_4(i)$ be the simulated interest rate, r_{lw} and r^{up} be the two thresholds surrounding it, and let A_{dn} and A^{up} be their corresponding amortization level; then:

$$\text{Amortization} = (1 - w_i) \times A_{dn} + w_i \times A^{up}$$

    - where

$$w_i = \frac{r_4^s(i) - r_{dn}}{r^{up} - r_{dn}}$$

  - **Simulate Discounted Cash Flows: (Panel E)** Given the paths of notional and interest rates, we can compute the discounted cash flows; we calculate first the discounts $Z^s(0,T_i)$ path-by-path and use them to discount each of the cash flows

Table 13.16 Pricing the Index Amortizing Interest Rate Swap by Monte Carlo Simulations

Panel A: Data on September 30, 1993

Maturity	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
Swap Rate	3.25	3.3578	3.4291	3.5625	3.674	3.7829	3.8885	3.99	4.0875	4.1817	4.2725	4.36
Discount	99.1940	98.3418	97.4710	96.5130	95.5265	94.5002	93.4383	92.3459	91.2266	90.0827	88.9176	87.7343

Panel B: The Fitted Interest Rate Tree

i / j	0	1	2	3	4	5	6	7	8	9	10	11
0	3.24	4.03	4.71	5.68	6.41	7.20	7.98	8.74	9.49	10.24	10.99	11.72
1		2.88	3.56	4.53	5.26	6.05	6.83	7.59	8.35	9.10	9.84	10.57
2			2.41	3.38	4.12	4.90	5.68	6.45	7.20	7.95	8.69	9.42
3				2.23	2.97	3.76	4.53	5.30	6.05	6.80	7.54	8.28
4					1.82	2.61	3.39	4.15	4.90	5.65	6.40	7.13
5						1.46	2.24	3.00	3.76	4.51	5.25	5.98
6							1.09	1.86	2.61	3.36	4.10	4.83
7								0.71	1.46	2.21	2.95	3.69
8									0.31	1.06	1.81	2.54
9										-0.08	0.66	1.39
10											-0.49	0.24
11												-0.90

Panel C: Ten Simulated Paths of the Quarterly Interest Rate

Simulation	Period 0	1	2	3	4	5	6	7	8	9	10	11
1	3.25	2.89	2.42	3.39	4.14	4.93	4.56	5.33	4.93	4.53	4.12	4.86
2	3.25	2.89	3.58	4.55	5.30	4.93	5.72	5.33	6.10	5.69	5.28	6.03
3	3.25	2.89	3.58	4.55	5.30	4.93	5.72	6.50	7.26	8.03	7.61	7.19
4	3.25	2.89	2.42	3.39	4.14	3.77	3.40	3.01	3.77	3.37	4.12	3.70
5	3.25	4.05	4.73	4.55	5.30	6.10	6.89	7.67	8.43	8.03	7.61	8.36
6	3.25	4.05	4.73	4.55	5.30	6.10	6.89	6.50	6.10	5.69	5.28	4.86
7	3.25	2.89	3.58	3.39	2.98	3.77	4.56	4.17	3.77	3.37	4.12	3.70
8	3.25	2.89	2.42	3.39	4.14	3.77	4.56	5.33	6.10	5.69	6.45	6.03
9	3.25	4.05	4.73	5.72	5.30	4.93	4.56	5.33	4.93	5.69	6.45	6.03
10	3.25	2.89	3.58	4.55	4.14	3.77	3.40	4.17	4.93	5.69	5.28	4.86

Panel D: The Simulated Notional

Simulation												
1	500.00	500.00	500.00	500.00	500.00	404.96	296.92	264.72	214.37	155.94	83.05	66.06
2	500.00	500.00	500.00	500.00	500.00	404.96	378.30	337.28	328.35	305.76	269.45	260.31
3	500.00	500.00	500.00	500.00	500.00	404.96	378.30	378.30	378.30	378.30	378.30	378.30
4	500.00	500.00	500.00	500.00	500.00	146.61	0.00	0.00	0.00	0.00	0.00	0.00
5	500.00	500.00	500.00	500.00	500.00	486.79	486.79	486.79	486.79	486.79	486.79	486.79
6	500.00	500.00	500.00	500.00	500.00	486.79	486.79	486.79	473.89	441.28	388.88	309.31
7	500.00	500.00	500.00	500.00	500.00	146.61	107.50	60.98	0.00	0.00	0.00	0.00
8	500.00	500.00	500.00	500.00	500.00	146.61	107.50	95.84	93.30	86.88	86.88	83.94
9	500.00	500.00	500.00	500.00	500.00	404.96	296.92	264.72	214.37	199.62	199.62	192.85
10	500.00	500.00	500.00	500.00	500.00	146.61	0.00	0.00	0.00	0.00	0.00	0.00

Panel E: The Simulated Discounted Cash Flow

Simulation	Time of Cash Flow 0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
1	1.55	1.98	2.55	1.34	0.44	-0.42	-0.04	-0.51	-0.21	-0.01	0.07	-0.05
2	1.55	1.98	1.13	-0.06	-0.95	-0.41	-1.07	-0.64	-1.18	-0.81	-0.46	-0.86
3	1.55	1.98	1.13	-0.06	-0.95	-0.41	-1.07	-1.72	-2.34	-2.93	-2.54	-2.16
4	1.55	1.98	2.55	1.34	0.44	0.25	0.00	0.00	0.00	0.00	0.00	0.00
5	1.55	0.56	-0.28	-0.06	-0.95	-1.81	-2.67	-3.47	-4.22	-3.71	-3.21	-3.90
6	1.55	0.56	-0.28	-0.06	-0.95	-1.81	-2.67	-2.19	-1.68	-1.16	-0.66	-0.24
7	1.55	1.98	1.13	1.34	1.83	0.25	-0.02	0.05	0.00	0.00	0.00	0.00
8	1.55	1.98	2.55	1.34	0.44	0.25	-0.02	-0.19	-0.34	-0.23	-0.38	-0.28
9	1.55	0.56	-0.28	-1.45	-0.94	-0.41	-0.04	-0.50	-0.21	-0.53	-0.85	-0.63
10	1.55	1.98	1.13	-0.06	0.43	0.25	0.00	0.00	0.00	0.00	0.00	0.00

Data source: Bloomberg.

13.4.1 Case Study: Banc One AIRS (cont.)

- In order to obtain the value of the AIRS, we must sum all of the rows in Panel E of Table 13.16, and then take the average of the resulting sums to obtain the average (i.e., expected) discounted value of future cash flows:

$$\hat{P}_0 = \frac{1}{N}\sum_{s=1}^{N}\left[\sum_{i=0}^{n-1} e^{-\left(r_0+\ldots+r_i^s\right)\times\Delta} \times CF^s(i+1)\right]$$

- Using 1,000 simulations, as in the earlier examples, we find:

AIRS price = 0.1389

Standard error = 0.3393

Confidence interval = [−0.5396, 0.8175]

- Notice, however, that if we were to use the 3-year swap rate of plain vanilla swaps, 4.36% from Panel A of Table 13.16, the value of AIRS according to our simulations would be strongly negative:

AIRS price (with r_{Fxd} = 4.36%) = −1.1137

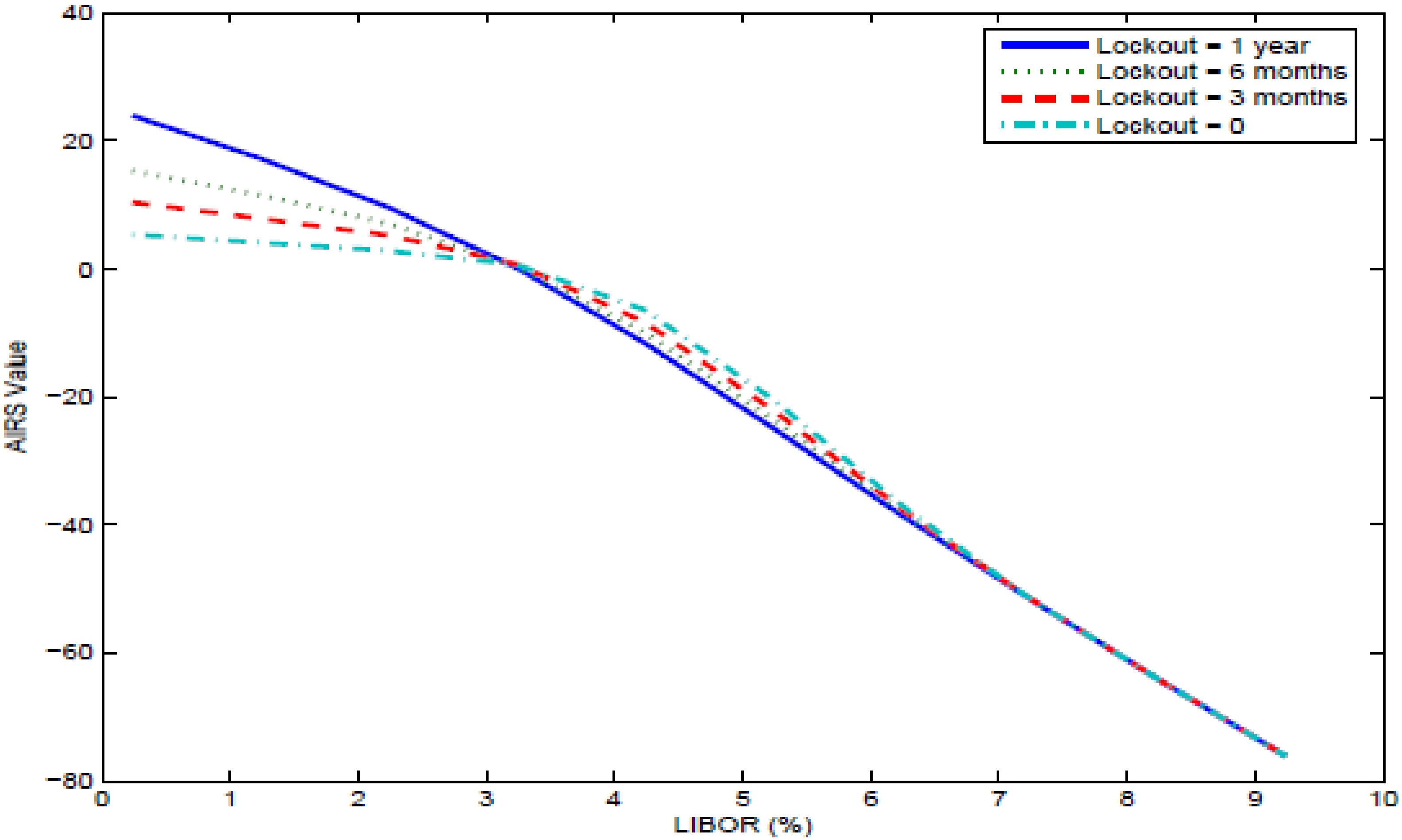
Standard error = 0.3503

Confidence interval = [−1.8144,−0.4131]

13.4.1 Case Study: Banc One AIRS (cont.)

- After the calculation of the AIRS, we can perform some additional analysis to understand in what sense this security is supposed to mimic the behavior of MBS
  - First, we observe that an investment in AIRS is in fact an investment in a security with negative convexity, in the sense that as the interest rate drops, the increase in value of the security is much smaller than the drop in value in case of an interest rate hike
  - Second, the negative convexity becomes stronger the shorter the lockout period is; in other words, as time goes by, the negative convexity becomes more and more important
- Both properties are common to callable bonds, although the second one isn't as noticeable in MBS

Figure 13.1 AIRS Value versus LIBOR and Lockout Period



13.5 SPOT RATE DURATION BY MONTE CARLO SIMULATIONS

- How can we compute some measures of risk of a security, such its duration, if we use Monte Carlo simulations to compute its value?
- Recall that this risk measure is given by:

$$Spot.rate.duration = -\frac{1}{P}\frac{dP}{dr}$$

- We proceed as follows:
  - We perform two Monte Carlo simulations:
    - The first starts at r_0, our root of the tree
    - The second Monte Carlo simulation starts instead slightly higher, at r_{0+dr}, where *dr* is a small number, such as one basis point
  - The spot rate duration can then be approximated by

$$Spot.rate.duration \approx -\frac{1}{\hat{P}(r_0)}\frac{\hat{P}(r_0 + dr) - \hat{P}(r_0)}{dr}$$

  - To avoid introducing simulation errors in the approximation, use the same realizations of the RAND() function for the computation of the two prices

Table 13.17 Spot Rate Duration by Monte Carlo Simulations

Maturity	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
				Panel A: Simulated Spot Rate Duration of Zero Coupon Bonds						
$\widehat{Z}(r_0, T_i)$	99.1338	97.9105	96.2084	94.2228	91.8927	89.4423	87.0761	84.8009	82.5165	80.1298
$\widehat{Z}(r_0 + dr, T_i)$	99.1288	97.9007	96.1940	94.2039	91.8698	89.4155	87.0456	84.7670	82.4794	80.0897
Spot Rate Duration	0.5000	1.0000	1.4999	1.9998	2.4997	2.9996	3.4994	3.9992	4.4990	4.9988
				Panel B: Simulated Spot Rate Duration of Long-Term Call Options						
$\widehat{c}(r_0)$	0.9650	1.8857	2.4959	3.1892	3.4691	3.4134	3.2866	3.4007	3.6378	3.5060
$\widehat{c}(r_0 + dr)$	0.9697	1.8927	2.5038	3.1973	3.4770	3.4202	3.2935	3.4064	3.6434	3.5107
Spot Rate Duration	-49.3222	-37.1823	-31.5883	-25.4278	-22.9155	-19.8689	-21.0933	-16.8563	-15.5984	-13.4163
				Panel C: Simulated Spot Rate Duration of Asian Call Options						
$\widehat{c^A}(r_0)$	0.4825	0.9079	1.2579	1.6022	1.8616	2.0189	2.1100	2.1750	2.2386	2.2647
$\widehat{c}(r_0 + dr)$	0.4873	0.9150	1.2659	1.6099	1.8693	2.0264	2.1171	2.1816	2.2450	2.2708
Spot Rate Duration	-99.1445	-78.3049	-64.1551	-48.3325	-41.7412	-37.4040	-33.7344	-30.3088	-28.5699	-26.9281

13.6 PRICING RESIDENTIAL MORTGAGE BACKED SECURITIES

- **13.6.1 Simulating the Prepayment Decision**
- **13.6.2 Additional Factors Affecting the Prepayment Decision**
- **13.6.3 Residential Mortgage Backed Securities**
- **13.6.4 Prepayment Models**

13.6.1 Simulating the Prepayment Decision

- In a previous chapter, we found that for each period *i* there is a trigger interest rate r_i such that if the interest rate drops below the trigger rate at *i*, the homeowner prepays the mortgage
- This fact suggests a way to use Monte Carlo simulations on the tree and incorporate the optimal decision of the homeowner:
  - We can simulate the interest rates on the tree, and whenever the simulated interest rate drops below the trigger rate, the whole remaining principal is paid back
  - Before the trigger rate is hit the mortgage owner only pays interest plus scheduled principal
- By summing up all the entries in each row we find the present value of all the cash flows for each simulated path *s*
- The average of all of these cash flows gives us the value of the mortgage:

$$V_0(10) = \frac{1}{N}\sum_{s=1}^{N}\left[\sum_{i=0}^{n-1} e^{-\left(r_0 + \ldots + r_i^s\right)\times\Delta} \times CF^s(i+1)\right]$$

- The value of the mortgage that we obtain using 1,000 simulations is V0(10) = \$100,096, with confidence interval C.I. = [99, 998; 100, 195],
- Recall that previously we had found that the mortgage was worth \$100,000
  - This seems to be a good approximation

Table 12.8 The Mortgage Value without Prepayment Option and the Outstanding Principal

Panel A: The Mortgage Value without Prepayment Option Tree

i / j	0	1	2	3	4	5	6	7	8	9	10
0	102220	90816	80008	69644	59617	49838	40216	30550	20912	10791	0
1		95314	84165	73363	62826	52484	42263	32007	21764	11142	0
2			87420	76278	65340	54552	43858	33135	22419	11408	0
3				78528	67280	56147	45084	33999	22916	11609	0
4					68760	57362	46017	34654	23292	11760	0
5						58280	46721	35147	23574	11872	0
6							47249	35517	23784	11956	0
7								35793	23941	12018	0
8									24058	12064	0
9										12098	0
10											0

Panel B: Computation of Outstanding Balance

Interest Paid	0	3782	3464	3134	2791	2435	2066	1683	1285	873	0
Principal Paid	0	8414	8732	9062	9405	9761	10130	10513	10910	11323	0
Outstanding Principal	100000	91586	82855	73792	64388	54627	44497	33985	23074	11751	0

Panel C: Prepayment Option Tree

i / j	0	1	2	3	4	5	6	7	8	9	10
0	2220	845	294	81	12	0	0	0	0	0	0
1		3728	1460	534	159	25	1	0	0	0	0
2			4566	2485	952	307	52	2	0	0	0
3				4735	2892	1519	587	107	4	0	0
4					4372	2735	1519	669	218	8	0
5						3653	2223	1163	500	121	0
6							2752	1532	710	204	0
7								1808	867	267	0
8									984	313	0
9										347	0
10											0

Panel D: The Interest Rate that Triggers Prepayment

i	0	1	2	3	4	5	6	7	8	9	10
Trigger rate		5.53%	4.92%	5.76%	6.64%	5.62%	6.28%	5.55%	6.08%	7.28%	–

Table 13.18 The Prepayment Decision in Simulations

	Period i										
	0	1	2	3	4	5	6	7	8	9	10
Panel A: Mortgage Payments and Outstanding Principal											
Interest Payment		3782	3464	3134	2791	2435	2066	1683	1285	873	444
Scheduled Principal		8414	8732	9062	9405	9761	10130	10513	10910	11323	11751
Outstanding Principal	100000	91586	82855	73792	64388	54627	44497	33985	23074	11751	0
Panel B: Trigger Rate and Ten Simulated Paths											
Trigger rate $\underline{r}_i \Longrightarrow$		5.53	4.92	5.76	6.64	5.62	6.28	5.55	6.08	7.28	–
Simulation											
1	5.86	$\boxed{5.53}$	6.66	7.79	6.64	7.61	8.50	7.51	8.24	7.28	5.95
2	5.86	7.49	6.66	$\boxed{5.76}$	4.90	5.62	6.28	7.51	6.08	5.38	4.39
3	5.86	7.49	6.66	$\boxed{5.76}$	6.64	7.61	8.50	10.17	8.24	9.86	8.05
4	5.86	7.49	9.02	7.79	8.99	10.30	8.50	7.51	8.24	$\boxed{7.28}$	8.05
5	5.86	$\boxed{5.53}$	6.66	5.76	6.64	7.61	6.28	7.51	6.08	7.28	5.95
6	5.86	$\boxed{5.53}$	6.66	5.76	6.64	5.62	6.28	7.51	6.08	7.28	5.95
7	5.86	$\boxed{5.53}$	6.66	7.79	6.64	5.62	4.64	5.55	4.49	3.97	3.24
8	5.86	$\boxed{5.53}$	4.92	4.25	4.90	4.15	3.43	4.10	4.49	5.38	4.39
9	5.86	7.49	6.66	$\boxed{5.76}$	6.64	7.61	8.50	7.51	6.08	7.28	8.05
10	5.86	7.49	9.02	10.55	12.17	10.30	8.50	10.17	11.15	13.35	10.90

Panel C: Discounted Cash Flows

Simulation	Maturity T_i 0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
1	100784	0	0	0	0	0	0	0	0	0
2	11843	11408	77799	0	0	0	0	0	0	0
3	11843	11408	77799	0	0	0	0	0	0	0
4	11843	11408	10905	10488	10027	9524	9128	8791	16566	0
5	100784	0	0	0	0	0	0	0	0	0
6	100784	0	0	0	0	0	0	0	0	0
7	100784	0	0	0	0	0	0	0	0	0
8	100784	0	0	0	0	0	0	0	0	0
9	11843	11408	77799	0	0	0	0	0	0	0
10	11843	11408	10905	10344	9734	9245	8861	8421	7965	7450

13.6.2 Additional Factors Affecting the Prepayment Decision

- We now add additional factors and insert them into our Monte Carlo simulation methodology:
  - **Random Event**: Homeowners may prepay their mortgages for reasons completely independent of interest rates, the most common of these begin the sale of the house
    - To capture this, we insert in our Monte Carlo simulations a probability p_t that prepayment occurs independently of the level of the interest rate
    - Using another uniform distribution (RAND()), we can simulate the prepayment event by requiring that the prepayment occurs when RAND() < p_t
  - **Seasonality**: Homeowners tend to move much more frequently in summer than in any other season in the year:
    - We can capture this by requiring the probability p_t to be higher during the summer
  - **Non Optimal Exercise**: Homeowners may also not take advantage of a refinancing opportunity, either because they do not pay attention to current rates, or because there are some additional costs to refinancing that make them hesitate
    - We can capture this behavior by assuming that even if it is optimal to refinance homeowners do so only with a probability $q(r^s_i)$ that depends on the level of interest rates:

$$q^s_i = a \times e^{-b \times r^s_i} \qquad \text{if } r^s_i < \underline{r}_i$$

    - where a and b are two constants and r_i is the trigger rate
  - Note that if $a = 1$ and $b = 0$, homeowners are refinancing optimally

Table 13.19 Pricing MBS by Monte Carlo Simulations with Additional Factors

Panel A: Assumptions about Additional Factors

100%PSA	1.20%	2.40%	3.60%	4.80%	6.00%	6.00%	6.00%	6.00%	6.00%	6.00%
PSA% Used in Simulations	50									
Season	1	2	1	2	1	2	1	2	1	2
Monthly Probability	0.30%	1.20%	0.90%	2.41%	1.51%	3.02%	1.51%	3.02%	1.51%	3.02%

Parameters for Prepayment Probability when Optimal to Exercise $q = a \times e^{-b \times r_i^s}$:

a	0.8	b	20

Panel B: Discounted Cash Flows

Simulation	Maturity T_i 0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
1	11843	11520	11143	67297	0	0	0	0	0	0
2	11843	11408	11034	10721	57320	0	0	0	0	0
3	11843	11408	11034	10721	10371	9984	9569	9094	8727	8307
4	11843	11408	10905	10488	10027	44274	0	0	0	0
5	11843	11520	11143	10827	10473	10082	9771	27215	0	0
6	11843	11520	78564	0	0	0	0	0	0	0
7	100784	0	0	0	0	0	0	0	0	0
8	11843	89786	0	0	0	0	0	0	0	0
9	11843	11408	11034	67323	0	0	0	0	0	0
10	11843	11408	10905	10344	9734	9245	8861	8421	7965	7450

13.6.3 Residential Mortgage Backed Securities

- Given the simulated cash flows, and especially prepayment times, we can finally divide these cash flows across the various types of securities
- Using the same example (the $100,000 mortgage) now assume that is a $100 million mortgage pool (with 7% PT rate)
  - Pass-through price = $99,996; C.I. = [$99,848, $100,144]
  - IO strip price = $12,540; C.I. = [$12,211, $12,868]
  - PO strip price = $87,456; C.I. = [$87,057, $87,855]
- We can obtain the price of collateralized mortgage obligations with sequential structure
  - In such a structure, we have to assign individual principal payments, whether scheduled or unscheduled, sequentially to various tranches (A, B and C)
  - Monte Carlo simulations prove their usefulness here: As the interest rate moves in each simulation and principal prepayment occurs, whether scheduled or unscheduled, we simply assign the principal payments to the various tranches, in the order above

Table 13.20 Pass-Through MBS and IO and PO Strips

Panel A: Pass-Through Interest, Scheduled Principal, and Outstanding Principal

	Maturity T_i									
	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
Interest (PT)	3500	3206	2900	2583	2254	1912	1557	1189	808	411
Principal	8414	8732	9062	9405	9761	10130	10513	10910	11323	11751
Outstanding Principal	91586	82855	73792	64388	54627	44497	33985	23074	11751	0

Panel B: Pass-Through MSB – Ten Simulated Discounted Cash Flows

	Maturity T_i									
Simulation	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
1	11569	11276	10929	67114	0	0	0	0	0	0
2	11569	11166	10823	10538	57164	0	0	0	0	0
3	11569	11166	10823	10538	10217	9858	9470	9023	8681	8285
4	11569	11166	10696	10309	9878	44153	0	0	0	0
5	11569	11276	10929	10642	10317	9955	9670	27141	0	0
6	11569	11276	78350	0	0	0	0	0	0	0
7	100510	0	0	0	0	0	0	0	0	0
8	11569	89542	0	0	0	0	0	0	0	0
9	11569	11166	10823	67140	0	0	0	0	0	0
10	11569	11166	10696	10168	9589	9129	8769	8355	7922	7430

Panel C: IO Strip – Ten Simulated Discounted Cash Flows

Simulation	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
	Maturity T_i									
1	3399	3028	2650	2270	0	0	0	0	0	0
2	3399	2998	2624	2270	1933	0	0	0	0	0
3	3399	2998	2624	2270	1916	1565	1222	887	578	280
4	3399	2998	2593	2221	1853	1493	0	0	0	0
5	3399	3028	2650	2293	1935	1581	1248	918	0	0
6	3399	3028	2650	0	0	0	0	0	0	0
7	3399	0	0	0	0	0	0	0	0	0
8	3399	3028	0	0	0	0	0	0	0	0
9	3399	2998	2624	2270	0	0	0	0	0	0
10	3399	2998	2593	2191	1799	1449	1132	821	527	251

Panel D: PO Strip – Ten Simulated Discounted Cash Flows

Simulation	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
	Maturity T_i									
1	8171	8248	8280	64844	0	0	0	0	0	0
2	8171	8168	8199	8268	55231	0	0	0	0	0
3	8171	8168	8199	8268	8300	8293	8248	8136	8103	8005
4	8171	8168	8103	8088	8025	42660	0	0	0	0
5	8171	8248	8280	8349	8382	8374	8422	26223	0	0
6	8171	8248	75701	0	0	0	0	0	0	0
7	97111	0	0	0	0	0	0	0	0	0
8	8171	86514	0	0	0	0	0	0	0	0
9	8171	8168	8199	64870	0	0	0	0	0	0
10	8171	8168	8103	7977	7790	7679	7638	7534	7395	7179

Table 13.21 Collateralized Mortgage Obligation

Panel A: Tranche A – Ten Simulated Discounted Cash Flows

	Maturity T_i										
	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
Interest		1750	1456	1150	833	504	162	0	0	0	0
Scheduled Principal		8414	8732	9062	9405	9761	4627	0	0	0	0
Outstanding Principal	50,000	41586	32855	23792	14388	4627	0	0	0	0	0
Simulation											
1		9870	9623	9330	21639	0	0	0	0	0	0
2		9870	9529	9239	9000	12774	0	0	0	0	0
3		9870	9529	9239	9000	8728	3921	0	0	0	0
4		9870	9529	9131	8804	8439	3740	0	0	0	0
5		9870	9623	9330	9088	8814	3959	0	0	0	0
6		9870	9623	31068	0	0	0	0	0	0	0
7		50255	0	0	0	0	0	0	0	0	0
8		9870	40658	0	0	0	0	0	0	0	0
9		9870	9529	9239	21648	0	0	0	0	0	0
10		9870	9529	9131	8684	8192	3631	0	0	0	0

Panel B: Tranche B – Ten Simulated Discounted Cash Flows

Interest		1050	1050	1050	1050	1050	1050	857	489	108	0
Scheduled Principal		0	0	0	0	0	5503	10513	10910	3074	0
Outstanding Principal	30000	30000	30000	30000	30000	30000	24497	13985	3074	0	0

Simulation										
1	1020	992	959	27285	0	0	0	0	0	0
2	1020	982	950	923	26634	0	0	0	0	0
3	1020	982	950	923	893	5364	8921	8501	2277	0
4	1020	982	939	903	863	24248	0	0	0	0
5	1020	992	959	932	902	5417	9109	11169	0	0
6	1020	992	28369	0	0	0	0	0	0	0
7	30153	0	0	0	0	0	0	0	0	0
8	1020	29330	0	0	0	0	0	0	0	0
9	1020	982	950	27296	0	0	0	0	0	0
10	1020	982	939	891	838	4967	8261	7872	2078	0

Panel C: Tranche C – Ten Simulated Discounted Cash Flows

Interest		700	700	700	700	700	700	700	700	700	411
Scheduled Principal		0	0	0	0	0	0	0	0	8249	11751
Outstanding Principal	20000	20000	20000	20000	20000	20000	20000	20000	20000	11751	0

Simulation										
1	680	661	640	18190	0	0	0	0	0	0
2	680	655	633	615	17756	0	0	0	0	0
3	680	655	633	615	595	573	549	522	6404	8285
4	680	655	626	602	576	16165	0	0	0	0
5	680	661	640	621	601	579	561	15973	0	0
6	680	661	18913	0	0	0	0	0	0	0
7	20102	0	0	0	0	0	0	0	0	0
8	680	19554	0	0	0	0	0	0	0	0
9	680	655	633	18197	0	0	0	0	0	0
10	680	655	626	594	559	531	509	483	5844	7430

13.6.4 Prepayment Models

- For each mortgage pool we still assume that if prepayment occurs, the whole principal is paid back
- This approach, albeit sufficiently simple to understand how to use Monte Carlo simulations to price MBS, is unsatisfactory; we would like to know not only whether prepayments occurs, but also how much of the prepayment will occur
- More complicated models of prepayment can be set up to take this additional quantity (amount of prepayment) into account:
  - A widespread methodology used by investment banks and hedge funds is to estimate a prepayment model, that is, a model that helps predict the amount of prepayment that will be realized, depending on market conditions
  - A prepayment model combines historical data on external factors together with historical prepayments, often within a regression framework, to obtain a forecast of the amount of prepayment for each possible future scenario
  - This methodology is then made operational by using Monte Carlo simulations on these factors, and computing the price of mortgage backed securities quite in the same way as discussed in the previous sections