Chapter 20: THE MARKET MODEL AND OPTIONS' VOLATILITY DYNAMICS

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20.1 THE BLACK FORMULA FOR CAPS AND FLOORS PRICING

- In this chapter we review the quoting conventions used by market dealers to trade such securities; these rely on the use of a particular option pricing formula originally designed by Fischer Black to price options on commodity futures
- The Black formula was initially adopted by traders mainly as a simple way to quote the prices of caps, floors and swaptions, recently it has been established that this formula follows from a no arbitrage argument under some assumptions about the dynamics of stochastic variables
- Table 20.1 reports quotes for swaptions, caps and floors on November 1, 2004 obtained from Bloomberg; besides the swap rates in Column 2, all of the quantities appearing in the table are expressed in volatility units
- For instance, a 1-year cap was trading on November 1, 2004 at a 23.5% volatility, while a 2-year cap was trading at the much higher 29.89% volatility
- Similarly, a European swaption with 3 months to maturity written on a 1-year swap was trading at 27.115% volatility.

Table 20.1 Swaptions, Caps and Floors Quotes on November 1, 2004

Maturity	Swap Rates	Swaption Vols 3M	Swaption Vols 6M	Swaption Vols 1Y	Volatilities Caps	Volatilities Floors
1 Y	2.555	27.115	30.234	31.750	23.50	23.50
2 Y	2.932	32.210	32.327	31.258	29.89	29.89
3 Y	3.254	31.011	30.937	29.801	30.55	30.55
4 Y	3.520	29.901	29.622	28.491	29.86	29.86
5 Y	3.751	28.719	28.513	27.404	28.62	28.62
7 Y	4.118	25.337	25.332	24.711	26.48	26.48
10 Y	4.505	21.889	21.833	21.570	23.68	23.68

Source: Bloomberg.

20.1 THE BLACK FORMULA FOR CAPS AND FLOORS PRICING

- The Black formula for a caplet with maturity T_{i+1} and strike rate r_K is:

  $$Caplet(0;T_{i+1}) = N \times \Delta \times Z(0,T_{i+1}) \times [f_n(0,T_i,T_{i+1})\, N(d_1) - r_K N(d_2)]$$

  where $f_n(0,T_i,T_{i+1})$ is the n−times compounded forward rate at time 0 for an investment at T_i and maturity T_{i+1}, and

  $$d_1 = \frac{1}{\sigma_f \sqrt{T_i}} \log\left(\frac{f_n(0,T_i,T_{i+1})}{r_K}\right) + \frac{1}{2}\sigma_f \sqrt{T_i}$$

  $$d_2 = d_1 - \sigma_f \sqrt{T_i}$$

  σ_f is a volatility parameter that is related to the volatility of forward rates
- The value of the cap itself is given by the sum of its caplets
- The Black formula to value a floorlet with maturity T_{i+1} and strike rate r_K is

  $$Floorlet(0;T_{i+1}) = N \times \Delta \times Z(0,T_{i+1}) \times [r_K N(-d_2) - f_n(0,T_i,T_{i+1})\, N(-d_1)]$$
- The value of the floor itself is given by the sum of its floorlets

20.1 THE BLACK FORMULA FOR CAPS AND FLOORS PRICING

- An example:
  - Today is November 1, 2004: consider a 1-year quarterly cap with strike rate r_K = 2.555% and let the volatility be σ_f = 23.5% (see Table 20.1)
  - To compute the price of the cap, we need to compute the price of the three caplets that make up the cap, that is, those that expire at $T_2 = 0.5$, $T_3 = 0.75$, and $T_4 = 1$
  - Note that we do not need to price the first caplet (expiring at $T_1 = 0.25$), as the cash flow would depend on the current interest rate $r_n(0)$ and its payment can be subtracted from the cost of the cap
  - In other words, payments (if any) will start at $T = 0.5$
  - We need to know the forward rates $fn(0,T_{i-1},T_i)$, for $i = 1, .., 4$
  - We can extract those from the LIBOR discount factors:
    - Let the LIBOR discount factors be as in the second column of Table 20.2
    - We then obtain the quarterly compounded forward rates as follows…

20.1 THE BLACK FORMULA FOR CAPS AND FLOORS PRICING

- An example (cont'd):
  - Define the forward discount factor

$$F\left(0,T_{i-1},T_i\right)=\frac{Z\left(0,T_i\right)}{Z\left(0,T_{i-1}\right)}$$

  - and then compute:

$$f_4\left(0,T_{i-1},T_i\right)=4\times\left(\frac{1}{F\left(0,T_{i-1},T_i\right)}-1\right)$$

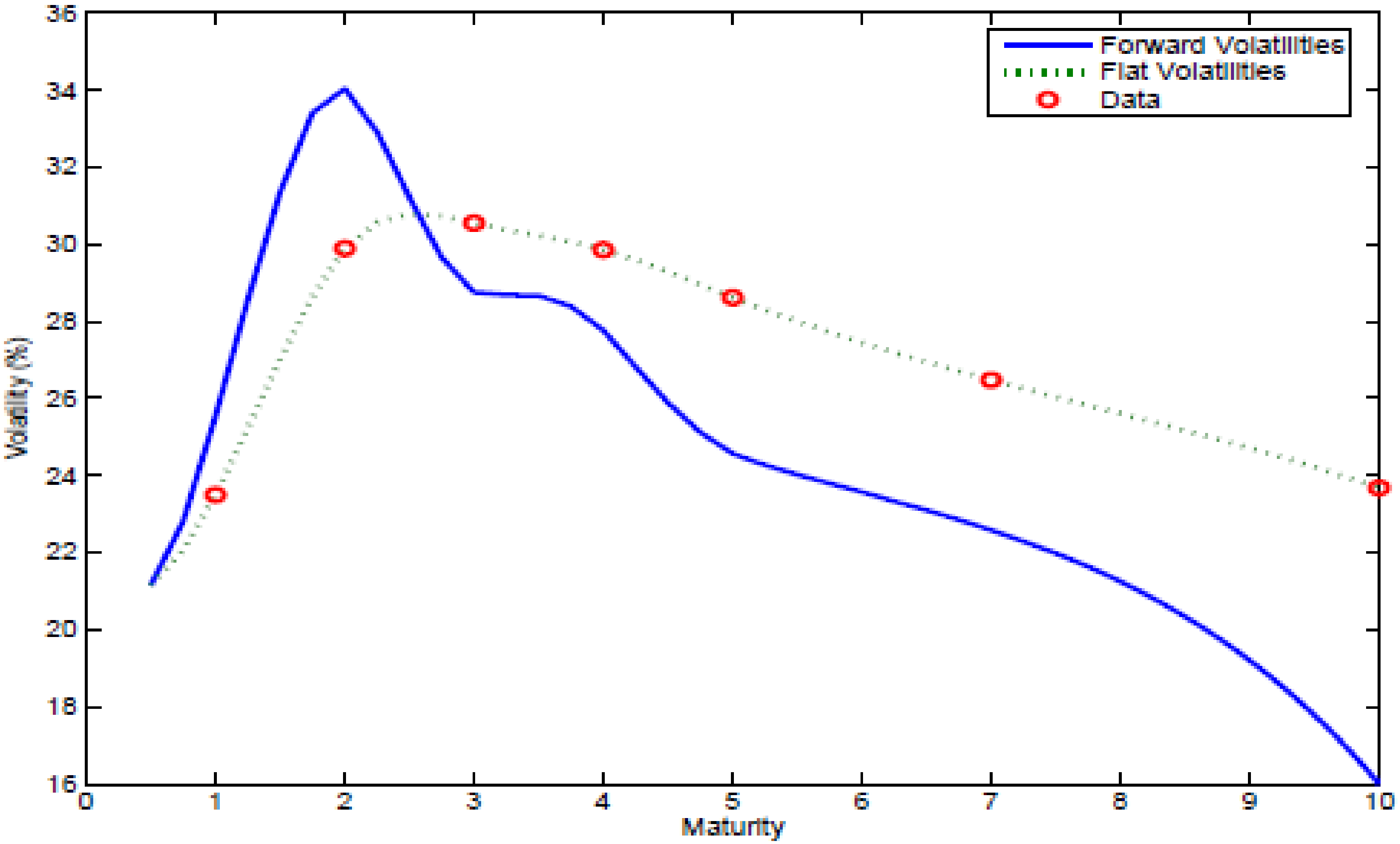
  - The third column of Table 20.2 contains the forward rates
    - We now have all the inputs to apply the formula for caplets
  - Column 4 reports the quantity $\sigma_f \times (T_{i-1})^{½}$, where recall that $\sigma_f = 23.5\%$
  - When we price the caplet with maturity T_i we need to compute the volatility at time $T_{i-1} = T_i - \Delta$; note that the price of a caplet expiring at T_{i+1} depends on volatilities $\sigma_f \times (T_i)^{½}$
  - The next two columns report d_1 and d_2
  - The last column reports the value of each caplet
  - The value of the cap is then

$$Cap(1Y) = \$0.0184 + \$0.0617 + \$0.1057 = \$0.1859$$

20.1.1 Flat and Forward Volatilities

- The **flat volatility** of a cap with maturity T is the quoted volatility $\sigma_f(T)$ that must be inserted in the Black formula for each and every caplet that make up the cap, in order to obtain a dollar price for the cap
  - Given each volatility quote, a trader would translate the volatility into a price by using the Black formula
  - It is important to note that the same volatility is used for each caplet in the cap, even if different caplets have different maturities
  - Table 20.3 translates the quotes in Table 20.1 into dollar prices for every maturity at quarterly intervals
  - The swap rates are interpolated, and then the discount factors are obtained through a bootstrap procedure; the volatility for maturities shorter than one year is extrapolated from the subsequent volatilities
  - Note the hump shape in the flat volatility
  - Note that the same caplet has different volatilities depending on which cap it is part of, this may suggest at first that there is a large inconsistency in the traders' quotes of caps, but this is in fact not correct

Figure 20.1 The Flat and Forward Volatility on November 1st, 2004



Data Source: Bloomberg.

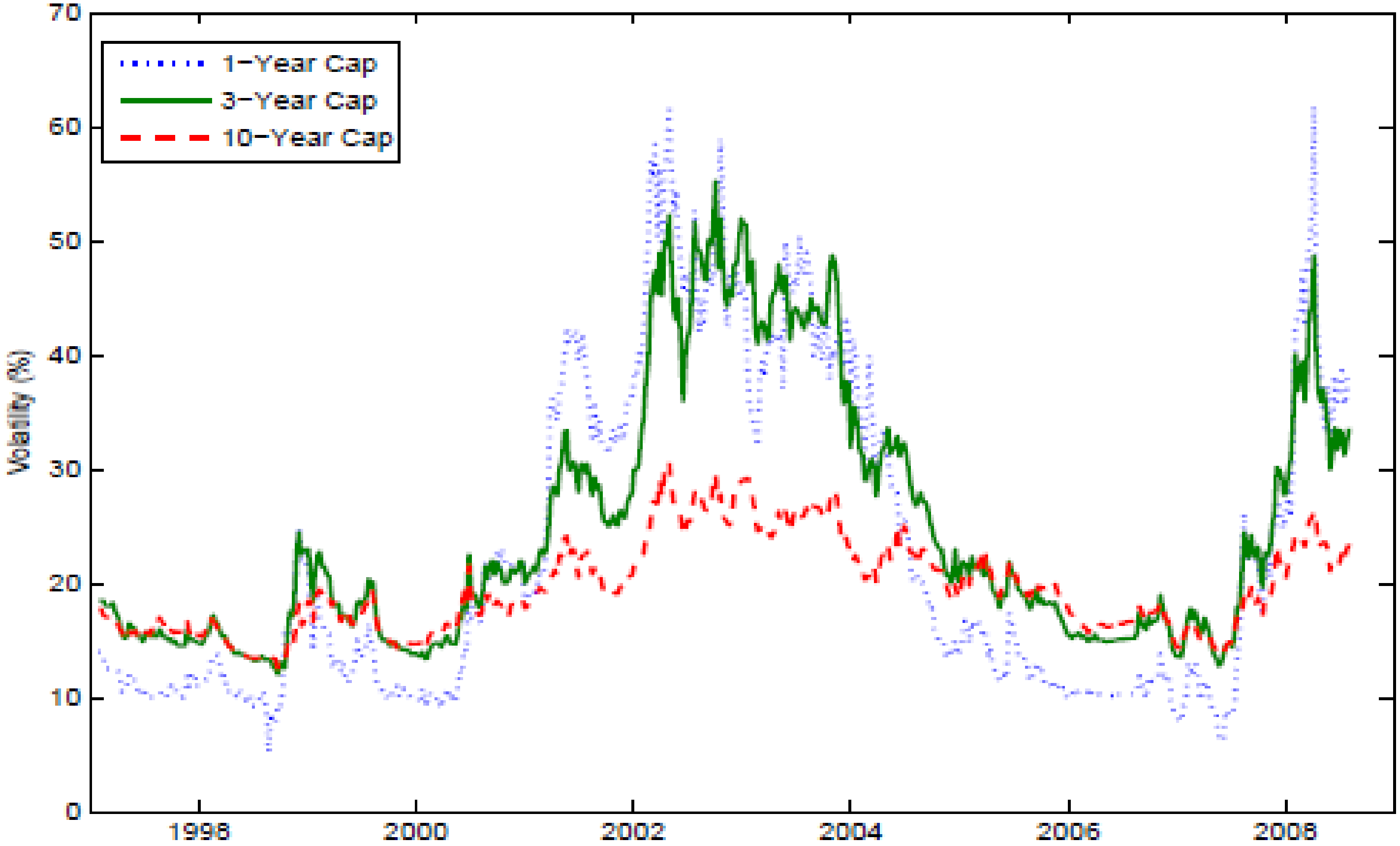
20.1.1 Flat and Forward Volatilities

- The **forward volatility** of a caplet with maturity T and strike rate r_K is the volatility $\sigma_f^{Fwd}(T)$ that characterizes that particularly caplet, independent of which cap the caplet belongs to
  - Forward volatility is applied to caplets, while the flat volatility is applied to caps
  - The logic then runs as follows: given $\sigma_f^{Fwd}(T)$ for every maturity, we can compute the value of each caplet in Table 20.4, and thus the value of each cap for each maturity
  - For quoting purposes traders like to re-express the value of the cap in terms of a single implied volatility, finding a constant volatility $\sigma_f(T)$ that can be applied to all of the caplets, and use that to quote and trade caps
  - The flat volatility stems then from simply a quoting convention, and it is not a reflection of no arbitrage or an inefficiency
  - In a sense, the flat volatility of a cap with maturity T is an average of forward volatilities for the caplets with maturities up to T
- In the cap market, traders prefer to trade in terms of volatilities instead of the dollar price of caps, because such volatilities have the same unit independent on the number of caplets

Table 20.4 The Flat and Forward Volatility of Caps

Flat Volatility			Forward Volatility							
			$\sigma_f^{Fwd}(0.50)$		$\sigma_f^{Fwd}(0.75)$		$\sigma_f^{Fwd}(1.00)$		$\cdots$	$\sigma_f^{Fwd}(10.0)$
			$\Downarrow$		$\Downarrow$		$\Downarrow$		$\cdots$	$\Downarrow$
$\sigma_f(0.50) \Longrightarrow$	Cap(0.50)	=	caplet(0.50)		-		-		$\cdots$	-
$\sigma_f(0.75) \Longrightarrow$	Cap(0.75)	=	caplet(0.50)	+	caplet(0.75)		-		$\cdots$	-
$\sigma_f(1.00) \Longrightarrow$	Cap(1.00)	=	caplet(0.50)	+	caplet(0.75)	+	caplet(1.00)		-	-
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\ddots$	-
$\sigma_f(10.0) \Longrightarrow$	Cap(10.0)	=	caplet(0.50)	+	caplet(0.75)	+	caplet(1.00)	+	$\cdots$	caplet(10.0)

Figure 20.2 Cap Volatility Over Time



Data Source: Bloomberg.

20.1.2 Extracting Forward Volatilities from Flat Volatilities

- For caps and floors we need to extract the sequence of forward volatilities from (quoted) flat volatilities by employing a bootstrap methodology:
  1. Use the quoted flat volatilities to obtain cap prices for all maturities from (see Table 20.3)
  $$Cap(T_i) = \sum_{j=1}^{i} Caplet(T_j, r_{K,i}, \sigma_f(T_i))$$
  2. The shortest ($T_1 = 0.5$) cap is made up of only one caplet, which implies
  $$\sigma_f^{Fwd}(0.5) = \sigma_f(0.5)$$
  3. For every $i = 2, \ldots, n$ use the following three-step procedure:
     (a) Use the previously extracted forward volatilities $\sigma_f^{Fwd}(T_j)$ for $j = 1, \ldots, i-1$ to compute the caplets up to T_j, $Caplet(T_j, r_{K,i}, \sigma_f^{Fwd}(T_j))$
     (b) Obtain the dollar value of the remaining caplet T_i as the difference between the cap price for T_i (obtained in Step 1) and the sum of caplets up to T_{i-1}
     $$\text{Dollar value of } T_i \text{ caplet} = Cap(T_i) - \sum_{j=1}^{i-1} Caplet(T_j, r_{K,i}, \sigma_f^{Fwd}(T_j))$$
     (c) Find the (forward) volatility $\sigma_f^{Fwd}(T_i)$ such that:
     $$Caplet(T_i, r_{K,i}, \sigma_f^{Fwd}(T_i)) = \text{Dollar value of } T_i \text{ caplet}$$

20.1.2 Extracting Forward Volatilities from Flat Volatilities

- In general, then, from the following equation

$$Cap(T_i) = \sum_{j=1}^{i} Caplet\left(T_j, r_{K,i}, \sigma_f^{Fwd}(T_j)\right)$$

- the value of the last caplet T_i of the cap with maturity T_i is given by the formula

$$Caplet\left(T_i, r_{K,i}, \sigma_f^{Fwd}(T_i)\right) = Cap(T_i) - \sum_{j=1}^{i-1} Caplet\left(T_j, r_{K,i}, \sigma_f^{Fwd}(T_j)\right)$$

- The forward volatility in step (c), above, is the volatility $\sigma_f^{Fwd}(T_i)$ that has to be inserted into the Black formula to make the previous equation

20.1.2 Extracting Forward Volatilities from Flat Volatilities

- An example:
  - Consider the data in Table 20.3; these are obtained from the quotes on November 1,2004, after an interpolation of the LIBOR yield curve at the quarterly horizon and of the quoted volatility [the original quotes are in Table 20.1, and the 3-month LIBOR was $r_4(0;0.25) = 2.1800\%$]
  - As mentioned, the volatility for maturity less than one year was extrapolated
  - Because these quotes are for at-the-money instruments, the strike rate used for each cap is given by the swap rate in the second column
    - 1. The first cap ($T_1 = 0.5$) has only one caplet, which implies that

$$\sigma_f^{Fwd}(T_1) = \sigma_f(T_1) = 21.1564\%$$

20.1.2 Extracting Forward Volatilities from Flat Volatilities

- An example (cont'd):
  - 2. The second cap ($T_2 = 0.75$) has a dollar price of Cap(T_2) = \$0.1059, as can be seen from the last column in Table 20.3
  - We now use the three-step procedure in Step 3 above:

    (a) Compute the T_1 caplet using forward volatility $\sigma_f^{Fwd}(T_1) = 21.1564\%$

    Note that the value of this caplet is not equal to the one computed in the previous step, because the strike rate has changed from $r_{K,1} = 2.3177\%$ to the current one $r_{K,2} = 2.4420\%$

    Using the Black formula, we obtain:

$$Caplet(T_1, r_{K,2}, \sigma_f^{Fwd}(T_1)) = \$0.0273$$

    (b) The dollar value of the T_2 caplet is then

$$\begin{aligned}\text{Dollar value of } T_2 \text{ caplet} &= Cap(T_2) - Caplet(T_1, r_{K,2}, \sigma_f^{Fwd}(T_1)) \\ &= \$0.1059 - \$0.0273 = \$0.0786\end{aligned}$$

    (c) Use the Black formula to $\sigma_f^{Fwd}(T_2)$ such that

$$Caplet(T_2, r_{K,2}, \sigma_f^{Fwd}(T_2)) = \$0.0786 \Rightarrow \sigma_f^{Fwd}(T_2) = 22.81\%$$

20.1.2 Extracting Forward Volatilities from Flat Volatilities

- An example (cont'd):
  - 3. And so on… for instance, for the T_3 forward volatility

    (a) Compute the T_1 and T_2 caplet using the forward volatilities $\sigma_f^{Fwd}(T_1)$ = 21.1564% and $\sigma_f^{Fwd}(T_2)$ = 22.81%

    Using the Black formula with strike rate $r_{K,3}$ = 2.4420%:

$$Caplet(T_1,r_{K,3},\sigma_f^{Fwd}(T_1)) = \$0.0157;\ Caplet(T_2,r_{K,3},\sigma_f^{Fwd}(T_2)) = \$0.0605$$

    (b) The dollar value of the T_3 caplet is:

$$\text{Dollar value of } T_3 \text{ caplet} = Cap(T_3) - \sum_{j=1}^{2} Caplet\left(T_j, r_{K,2}, \sigma_f^{Fwd}(T_j)\right)$$

$$= \$0.1859 - (\$0.0157 + \$0.0605) = \$0.1096$$

    (c) Use the Black formula to find the (forward) volatility $\sigma_f^{Fwd}(T_3)$ such that

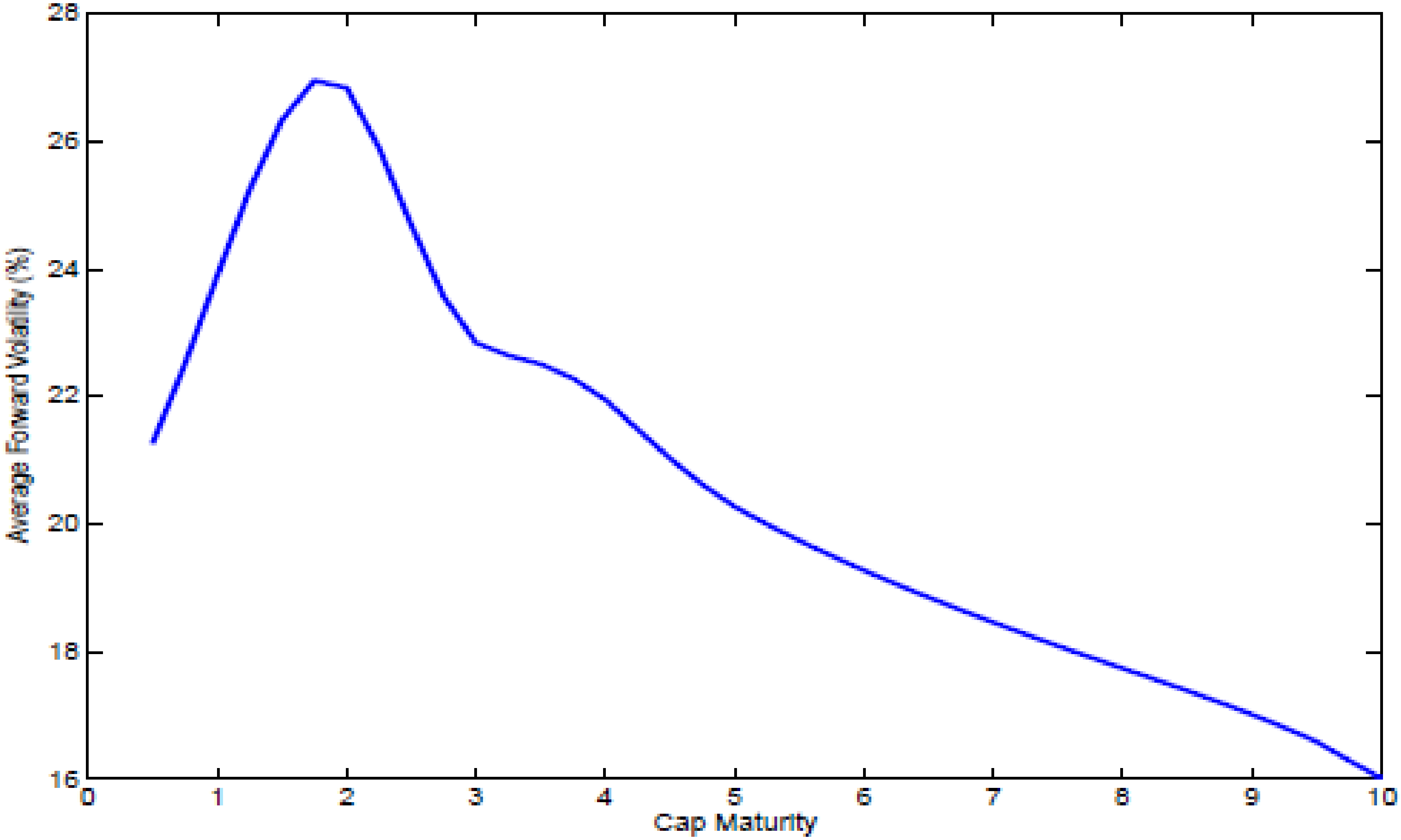
$$Caplet(T_3,r_{K,2},\sigma_f^{Fwd}(T_3)) = \$0.1096 \Rightarrow \sigma_f^{Fwd}(T_2) = 25.54\%$$

- The forward volatility increases more sharply than the flat volatility (see Table 20.3) as the latter is increasing, and then it also declines more sharply

20.1.3 The Behavior of the Implied Forward Volatility

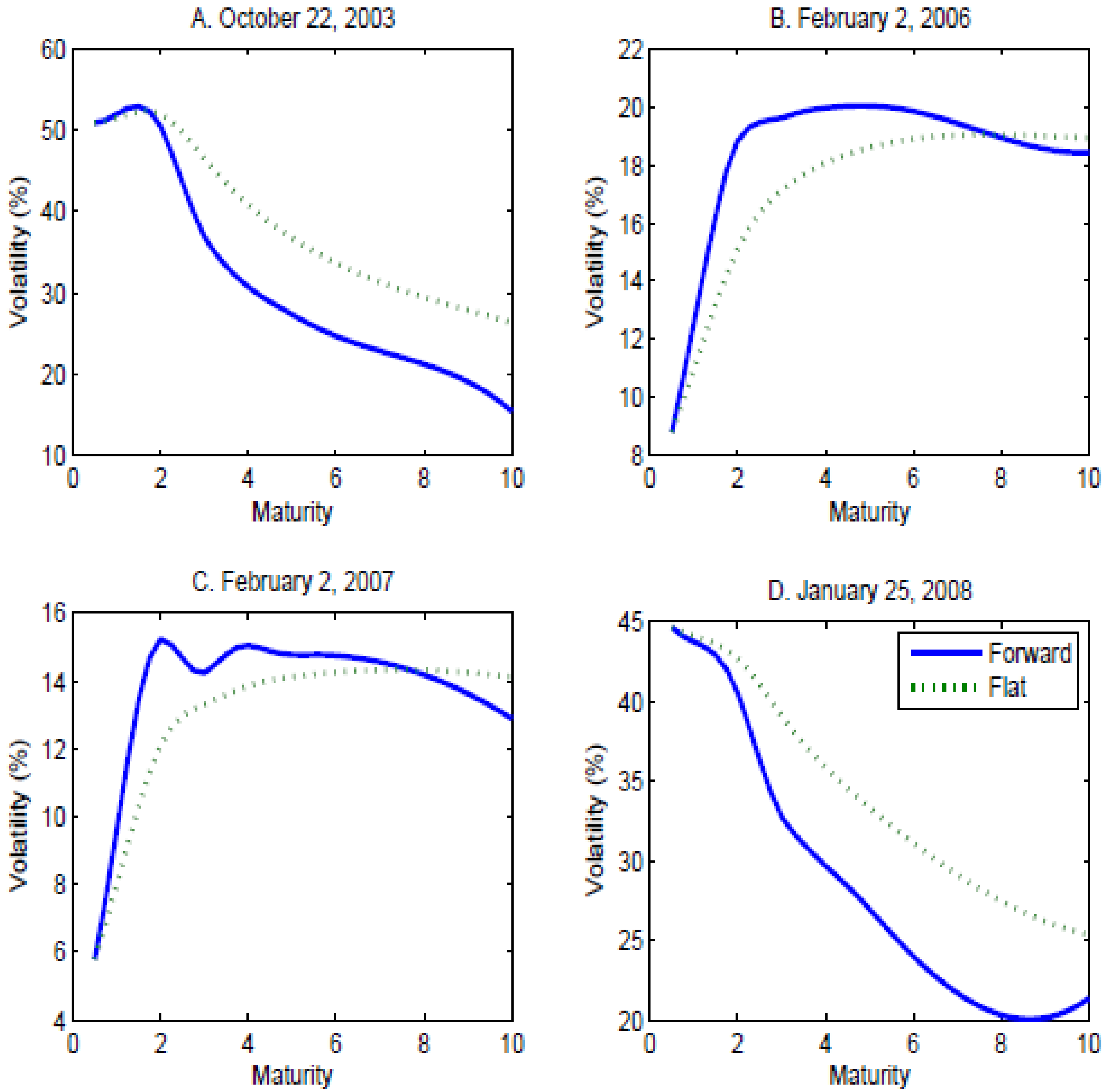
- Figure 20.3 plots average forward volatility from 1997 - 2008
  - Although it is common to see a “hump”, the shape of the forward volatility curve changes over time, exactly as the term structure of interest rates changes
- Figure 20.4 plots some of the other shapes that the forward volatility curve took in the past
  - On October 22, 2003 we observe that the forward volatility curve was declining, with a small hump around T = 2
  - On February 2, 2006, the forward volatility curve was mainly increasing, as it was one year later, on February 2, 2007
  - On January 25, 2008, the forward volatility curve was strongly decreasing
- The level of the volatility varies dramatically over time: From Figure 20.4, for instance, in Panel A (October 22, 2003) and Panel D (January 25, 2008) the forward volatility was quite high, over 40% at the short end, while in Panel B (February 2, 2006) and Panel C (February 2, 2007), the volatility was much lower, below 10% at the short end, and never above 20%

Figure 20.3 The Average Forward Volatility 1997 - 2008



Data Source: Bloomberg.

Figure 20.4 Some Shapes of Forward and Flat Volatility



Data Source: Bloomberg.

20.1.3 The Behavior of the Implied Forward Volatility

- On Figure 20.5:
  - Panel A plots the average forward volatility across maturities from 1997 to 2008

    The variation in average forward volatility is in fact quite large as it ranges from just above 10% to over 35%
  - Panel B reports the slope of the forward volatility curve, simply defined here as the difference between the 10-year and 1-year forward volatility

    As can be seen, the difference between the long and the short end of the forward curve is not always negative, as we could surmise from the average forward volatility curve in Figure 20.3

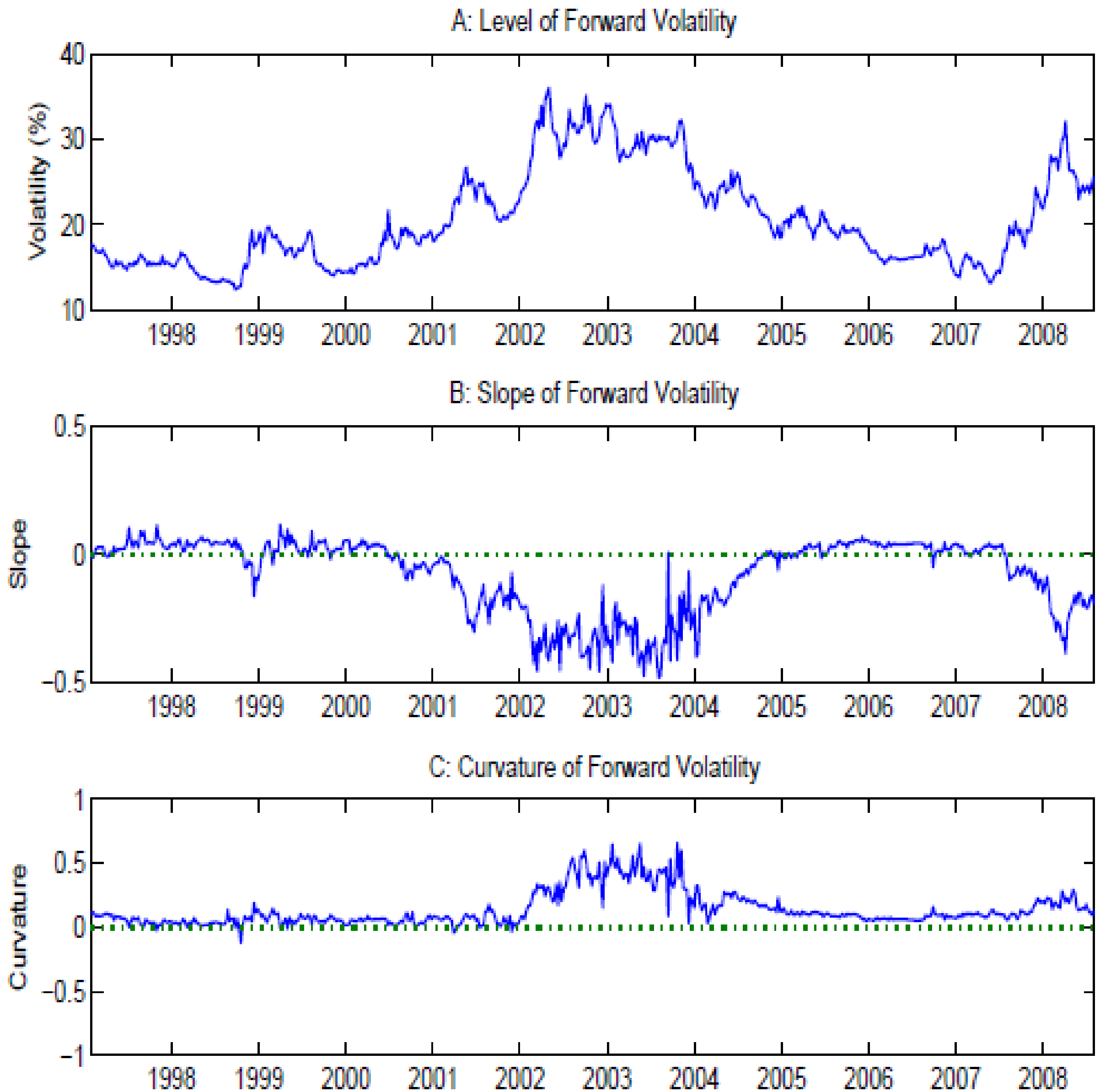
    Instead, it is often positive, albeit not very much so.
  - Panel C plots the curvature of the forward volatility curve, defined as

    $$\text{Curvature} = 2 \times \sigma_f^{Fwd}(2) - \sigma_f^{Fwd}(1) - \sigma_f^{Fwd}(10)$$

    When the volatility of the 2-year caplet is higher than the short and the long end, we say that the forward volatility curve has a strong curvature

    Panel C of Figure 20.5 shows that the curvature shows a good deal of variation as well, and that it is always very low

Figure 20.5 Level, Slope and Curvature of Forward Volatility



Data Source: Bloomberg.

20.1.3 The Behavior of the Implied Forward Volatility

- What determines the variation of the forward volatility and the shape of the forward volatility curve? Recall that the forward volatility embedded in caps reflects an insurance premium, namely, the amount of money that an investor is willing to pay to be covered against a run up in interest rates
- Such insurance is more valuable the higher the uncertainty about future interest rates: if for instance there is a large uncertainty about the action of the Federal Reserve at an upcoming Federal Open Market Committee meeting, we could expect that the forward volatility of short-term options would be higher than in other times
- The average hump in the term structure of volatility may be due to uncertainty about medium term interest rates generated by uncertainty about business cycle variation or inflation
- The average low forward volatility at the long end, instead, is due to the mean reversion of interest rates: Interest rates are unlikely to be very high or very low for very long periods of time, and thus there is relatively less uncertainty about the average interest rates over a ten-year span rather than over a two-year span

20.1.4 Forward Volatilities and the Black, Derman and Toy Model

- There is a relation between the forward volatilities obtained from the Black model and the forward volatility from the BDT model
- In particular, recall that the BDT model indeed implies that the future spot rates are (approximately) log-normally distributed
- It turns out that we can use the forward volatility obtained in the bootstrap procedure directly as input in the BDT tree; since both curves should be identical
- We see instead a little difference in levels, which is due to the fact that in discrete time we use a relatively coarse time grid for cap prices (the time-step there is one quarter, $dt = 0.25$)
- As we decrease the time step dt in the BDT tree, the implied volatility from the tree and from the Black model converge
- A standard practice to obtain a BDT tree is to use the Black formula to compute $\sigma_f^{Fwd}(T)$ for every maturity, and then, given the forward volatility for every maturity, fit the BDT tree to match the current term structure of interest rates
- This second methodology is much faster than the previous one

20.2 THE BLACK FORMULA FOR SWAPTION PRICING

- Swaptions are options to enter into a swap, as either fixed rate payers (payer swaption) or fixed rate receiver (receiver swaption)
- To introduce the Black model to quote swaptions, consider first the payoff of a receiver swaption with strike swap rate r_K, that can be expressed as a call option on a coupon bond, with coupon rate r_K

  $$\text{Payoff receiver swaption} = \max(P_c(T_O;T_S) - N, 0)$$

  where N is the notional, T_O is the maturity of the swaption, and T_S is the maturity of the underlying swap
- $P_c(T_O;T_S)$ is the value of a coupon bond with coupon rate r_K:

  $$P_c\left(T_O;T_S\right) = N \times r_K \times \Delta \times \sum_{i=1}^{n} Z\left(T_O;T_i\right) + N \times Z\left(T_O;T_n\right)$$

  where T_i are the swap fixing dates, and $T_n = T_S$
- It is convenient to rewrite this payoff in terms of the future swap rate at time T_O
- We denote this future swap rate by $c(T_O;T_S)$, where T_S is the maturity date of the swap underlying the option

20.2 THE BLACK FORMULA FOR SWAPTION PRICING

- By definition, the swap rate at time T_O for a swap with maturity T_S is that rate that makes the value of the swap equal to zero, that is, $c(T_O;T_S)$ satisfies:

$$N = N \times c(T_O;T_S) \times \Delta \times \sum_{i=1}^{n} Z(T_O;T_i) + N \times Z(T_O;T_n)$$

- Substitute $P_c(T_O;T_S)$ and N into the swaption payoff and rearrange to obtain the alternative, but equivalent, expression for the receiver swaption payoff:

$$\text{Payoff receiver swaption} = N \times \Delta \times \left[\sum_{i=1}^{n} Z(T_O;T_i)\right] \times \max\left(r_K - c(T_O;T_S), 0\right)$$

- This payoff structure has a simple interpretation: suppose that the swap rate at maturity of the receiver swaption $c(T_O;T_S)$ is below r_K
- Then, the holder of the swaption can exercise the option, and receive the higher (strike) rate r_K instead of the market swap rate $c(T_O;T_S)$
- At every payment date of the swap underlying the option, the option holder (who exercised the option at T_O) gains the constant spread: $r_K - c(T_O;T_S)$
- Thus, the total payoff from exercising the swaption is given by the present value of these gains, each implicitly paid at a different maturity T_i

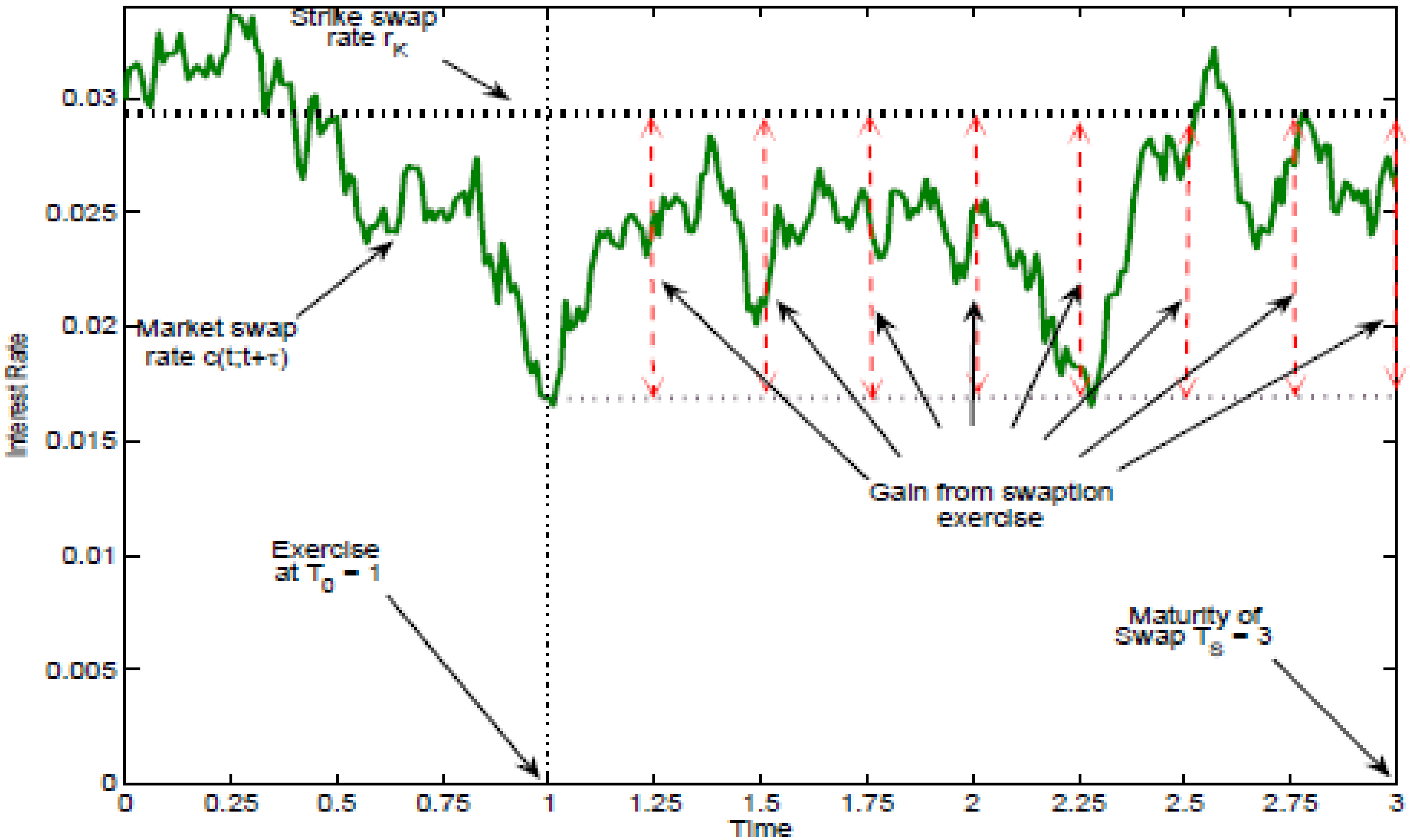
20.2 THE BLACK FORMULA FOR SWAPTION PRICING

- An example:
  - Today is November 1, 2004, and the Treasurer of a large corporation forecasts that in one year the corporation will need to enter into a quarterly fixed-for-floating 2-year swap in which it will receive fixed coupons and pay floating coupons
  - The Treasurer is worried that the 2-year swap rate will decline in the next year from its current value of 2.93% (see Table 20.1)
  - To hedge against the possible decline in the 2-year swap rate, she considers purchasing a receiver swaption defined on a 2-year swap, with strike rate $r_K = 2.931\%$ and maturity $T_O = 1$
  - If at maturity $T_O = 1$ the 2-year swap rate is above the strike rate $r_K = 2.931\%$, for instance $c(T_O,T_S) = 4\%$ where $T_S = T_O + 2 = 3$, then the corporation is better off entering into a 2-year swap at the current market rate $c(T_O,T_S) = 4\%$
  - In fact, by letting the option expire worthless, the corporation will receive the fixed rate 4% per year in exchange of the 3-month LIBOR

20.2 THE BLACK FORMULA FOR SWAPTION PRICING

- An example (cont'd):
  - If at maturity $T_O = 1$ the 2-year swap rate is below the strike rate $r_K = 2.931\%$, for instance $c(T_O,T_S) = 1.58\%$, then the Treasurer can exercise the option and have the corporation receive the strike rate $r_K = 2.931\%$ every quarter for the next two years in exchange of the 3-month LIBOR
  - The corporation than gains the difference between the strike rate r_K and the current market rate $c(T_O,T_S)$ every quarter from $T_O + 0.25$ until the maturity of the underlying swap $T_S = T_O + 2$
  - Figure 20.6 illustrates the payoff for this swaption under an hypothetical path of the 2-year swap rate $c(t,t+\tau)$, where $\tau = 2$ is the tenor of the underlying swap
  - In the figure, at maturity of the swaption, $T_O = 1$, the 2-year swap rate is $c(T_O,T_S) = 1.58\%$
  - By exercising the receiver swaption, then, the Treasurer would receive $r_K = 3\%$ up to the maturity of the swap, instead of the market swap rate $c(T_O,T_S) = 1.58\%$
  - Therefore, its gain per period is $N \times \Delta \times (r_K - c(T_O,T_S)) = 100 \times 0.25 \times 0.0142 = 0.35$ for $N = 100$

Figure 20.6 The Payoff of a Receiver Swaption



20.2 THE BLACK FORMULA FOR SWAPTION PRICING

- In a similar manner, the payoff of a payer swaption can be written as:

$$\text{Payoff payer swaption} = N \times \Delta \times \left[\sum_{i=1}^{n} Z(T_O, T_i)\right] \times \max\left(c(T_O, T_S) - r_K, 0\right)$$

- The Black formula to value a European receiver swaption with strike rate r_K and maturity T_O on a swap with maturity T_S is given by

$$V(0, T_O; T_S) = N \times \Delta \times \left[\sum_{i=1}^{n} Z(T_O, T_i)\right] \times \left[r_K N(-d_2) - f_n^s(0, T_O, T_S) N(-d_1)\right]$$

where $f_n^s(0,T_O,T_S)$ is the n–times compounded forward swap rate at time 0 to enter at T_O into a swap with maturity T_S

$$d_1 = \frac{1}{\sigma_f^s(T_O; T_S)\sqrt{T_O}} \ln\left(\frac{f_n^s(0, T_O; T_S)}{r_K}\right) + \frac{1}{2}\sigma_f^s \sqrt{T_O};\ \ d_2 = d_1 - \sigma_f^s(T_O; T_S)\sqrt{T_O}$$

σ^s_f is a volatility parameter that is related to the volatility of forward rates

- Similarly, the Black formula to value a European payer swaption with maturity T_O and strike swap rate r_K on a swap with maturity T_S, is given by

$$V(0, T_O; T_S) = N \times \Delta \times \left[\sum_{i=1}^{n} Z(T_O, T_i)\right] \times \left[f_n^s(0, T_O, T_S) N(d_1) - r_K N(d_2)\right]$$

20.2 THE BLACK FORMULA FOR SWAPTION PRICING

- An example:
  - Today: November 1, 2004; we have the LIBOR zero coupon curve, interpolated at quarterly frequency (Table 20.3)
  - Consider a 1-year receiver swaption defined on a 5-year swap ($T_O = 1$, $T_S = 6$) with $r_K = 3.751\%$, and $\sigma^s_f = 27.404\%$
  - If exercised, the swap underlying the swaption will generate cash flows at quarterly intervals starting on $T_1 = 1.25$
  - The discounts at these times are given in Column 2 in Table 20.5, so that:

$$\Delta \times \sum_{i=1}^{n} Z(0, T_i) = 4.4046$$

  - To apply Black formula, we must compute the forward swap rate $f_4^s(0,T_O,T_S)$
  - The forward discount factor for each maturity T_i is given by the formula $F(0,T_O,T_i) = Z(0,T_i) / Z(0,T_O)$, and is reported in Column 5 of Table 20.5
  - The forward swap rate is $f_4^s(0,T_O,T_S) = 4.26\%$
  - We then obtain $d_1 = 0.6023$ and $d_2 = 0.3282$
  - The Black formula for $N = 100$ yields:

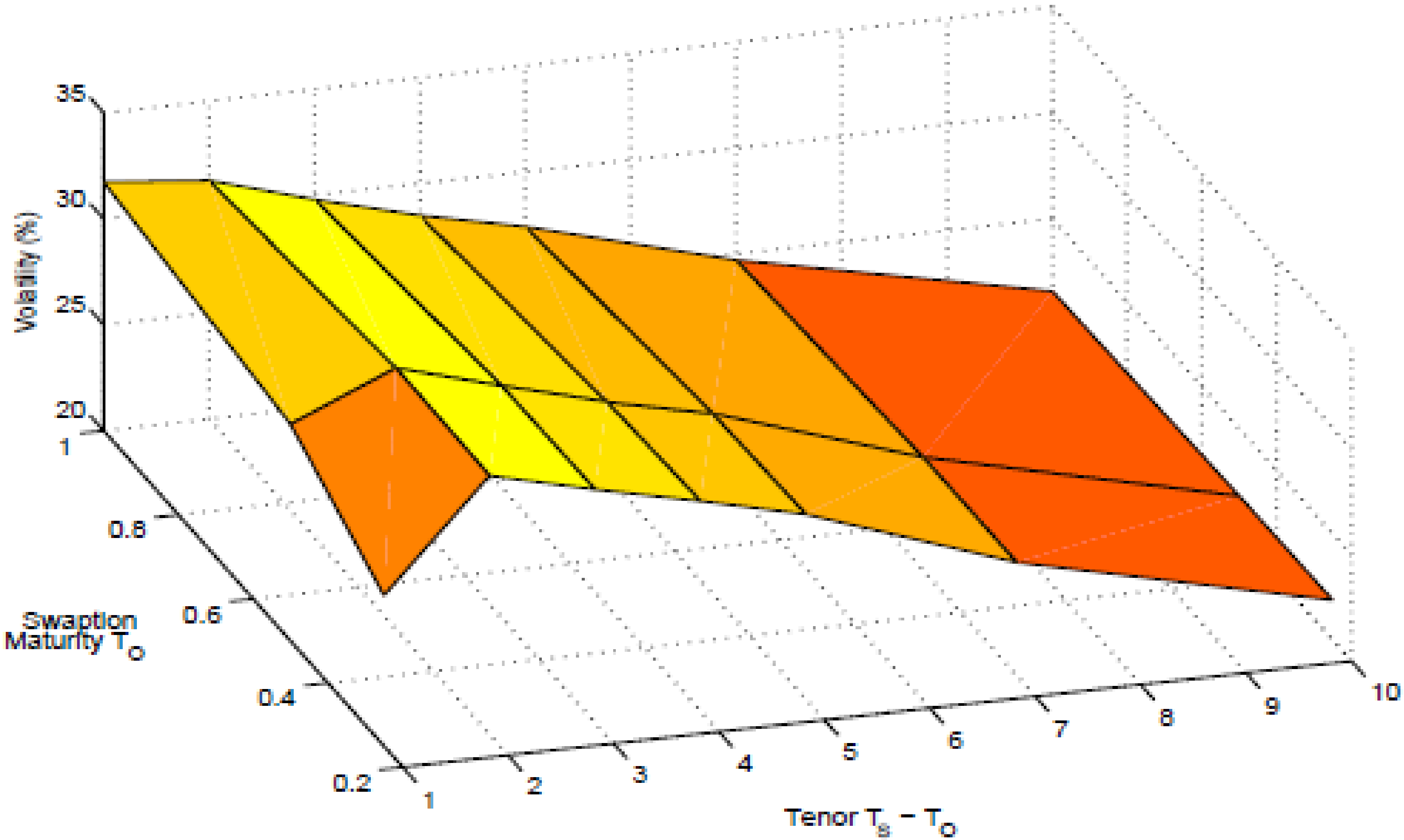
$$V(0,1;6) = \$1.0026$$

Table 20.5 LIBOR Discount and Forward Discount on November 1, 2004

Maturity	Discount ($\times 100$)	Forward Discount ($\times 100$)
1.2500	96.7402	99.2367
1.5000	95.9608	98.4372
1.7500	95.1491	97.6045
2.0000	94.3075	96.7412
2.2500	93.4385	95.8498
2.5000	92.5449	94.9331
2.7500	91.6294	93.9940
3.0000	90.6949	93.0354
3.2500	89.7441	92.0600
3.5000	88.7793	91.0703
3.7500	87.8027	90.0686
4.0000	86.8162	89.0566
4.2500	85.8211	88.0358
4.5000	84.8186	87.0075
4.7500	83.8100	85.9728
5.0000	82.7963	84.9329
5.2500	81.7786	83.8890
5.5000	80.7584	82.8424
5.7500	79.7370	81.7947
6.0000	78.7161	80.7474

Original Swap Data Source: Bloomberg.

Figure 20.7 Swaption Quoted Volatility. November 1, 2004



Data Source: Bloomberg.