AN INTRODUCTION TO COMBINATORICS OF DETERMINACY

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1. Introduction

This article is an introduction to combinatorics under the axiom of determinacy. The main topics are partition properties and ∞-Borel codes. To illustrate the important ideas, the article will focus on the simplest settings. This will mean that one will work at simple pointclass such as $\mathbf{\Sigma}_1^1$, small cardinals such as ω_1 or ω_2, or very natural models of determinacy such as $L(\mathbb{R})$.

Despite the simplicity, there are still many interesting combinatorial questions in these settings. One purpose of this article is to serve as a reference for some of the notations and backgrounds that will be helpful for some results of Jackson, Trang, and the author on combinatorics around ω_1, ω_2, and $\mathbb{R}$ under AD and AD^+. (Section 2 to 6 were written for [4] and [7]. Section 7 and 8 will be helpful for reading [3] and [5].)

The article should be accessible with very basic knowledge of descriptive set theory and general set theory. Some familiarity with determinacy and games, the pointclass $\mathbf{\Sigma}_1^1$, and the bounding principle for $\mathbf{\Sigma}_1^1$ should suffice for most of the article. Knowledge of club sets, ultrafilters, measures, and basic constructibility will also be helpful. There will be occasional mentions of the theory of prewellorderings and scales such as the Moschovakis coding lemma or the Kunen-Martin theorem. In a few places, one will refer to the result of Kechris that AD implies $L(\mathbb{R}) \models \mathsf{DC}_\mathbb{R}$. The last two sections will require more familiarity with topics in general set theory such as HOD and iterated forcing.

The article has essentially two parts. The first part deals with various partition properties which are very powerful tools of AD. The second part introduces the ∞-Borel code and the Vopénka forcing which are useful AD^+ tool. Except possibly for a few minor observations, no results in this article are due to the author.

Section 2 introduces the partition property in its various forms. It also defines the associated measures induced by the partition properties.

Section 3 develops the notation and theory around good codings of functions. This section proves Martin's criterion for establishing partition properties.

Section 4 gives some examples of good coding of functions on ω_1. This is used to give a simple proof of the weak partition property on ω_1 and two proofs of the strong partition property on ω_1. One will give Martin's original proof of the strong partition property using sharps. Then one gives a proof due to Kechris of the strong partition property on ω_1 which uses category arguments and the simple generic coding function for ω_1.

Section 5 develops the theory around the Kunen function which is named after the eponymous Kunen tree. Here it will be shown that every function from ω_1 to ω_1 has a Kunen function by using the original Kunen tree. The Kunen functions are especially useful for choosing representatives of certain ultrapowers. This will be used to establish the identity of ω_2 as the ultrapower $^{\omega_1}\omega_1/\mu$, where μ is the club ultrafilter on ω_1. Then one will prove a useful sliding lemma and use this to establish the weak partition property on ω_2.

Section 6 is dedicated to proving a result of Martin and Paris which shows that ω_2 is not a strong partition cardinal. However, this section presents a proof of Jackson which produces an explicit partition without a homogeneous subset.

Section 7 proves some results of Woodin about the structure of $L(\mathbb{R})$ and the nature of ∞-Borel codes in $L(\mathbb{R})$. In particular, it will be shown that in $L(\mathbb{R})$, there is an "ultimate" ∞-Borel code that can be used to generate all other ∞-Borel codes for sets of reals. This is quite useful for cardinality results in $L(\mathbb{R})$.

Section 8 will present some further descriptive set theoretic applications due to Woodin of the Vopénka forcing. The countable section uniformization and the Woodin's perfect set dichotomy theorem will be proved here.

2. Partition Properties

The following is the usual partition property:

Definition 2.1. Let κ be an ordinal, $\lambda \leq \kappa$, and $\gamma < \kappa$, then $\kappa \rightarrow (\kappa)^\lambda_\gamma$ indicates that for every $P : [\kappa]^\lambda \rightarrow \gamma$, there is some $\beta < \gamma$ and some $A \subseteq \kappa$ with $|A| = \kappa$ so that for all $f \in [A]^\lambda$, $P(f) = \beta$.

The most frequent situation is when $\gamma = 2$.

κ is a strong partition cardinal if and only if $\kappa \rightarrow (\kappa)^\kappa_2$.

κ is a weak partition cardinal if and only if $\kappa \rightarrow (\kappa)^\alpha_2$ for all $\alpha < \kappa$.

Definition 2.2. Let κ be an ordinal. Let $S : \kappa \rightarrow \kappa$. Let $Y^S = \{(\alpha, \beta) : \alpha < \kappa \wedge \beta < S(\alpha)\}$.

Let $f : \kappa \rightarrow \kappa$. f is said to have uniform cofinality S if and only if there is a function $g : Y^S \rightarrow \kappa$ with the following properties

(1) For all $\alpha < \kappa$ and all $\beta < \gamma < S(\alpha)$, $g(\alpha, \beta) < g(\alpha, \gamma)$

(1) For all $\alpha < \kappa$, $f(\alpha) = \sup\{g(\alpha, \beta) : \beta < S(\alpha)\}$.

Let μ be a measure on κ. $f : \kappa \rightarrow \kappa$ is said to have uniform cofinality S μ-almost everywhere if and only if there is a $g : Y^S \rightarrow \kappa$ as above so that for μ-almost all $\alpha < \kappa$, $f(\alpha) = \sup\{g(\alpha, \beta) : \beta < S(\alpha)\}$.

A function $f : \kappa \rightarrow \kappa$ has uniform cofinality ω if and only if f has uniform cofinality S where $S(\alpha) = \omega$ for all $\alpha < \kappa$.

Definition 2.3. Let $f : \kappa \rightarrow \kappa$ be a function. f is discontinuous everywhere if and only if for all $\alpha < \kappa$, $f(\alpha) > \sup\{f(\beta) : \beta < \alpha\}$.

(Jackson) A function $f : \kappa \rightarrow \kappa$ has the correct type if and only if f has uniform cofinality ω and is discontinuous everywhere.

Let $[\kappa]^\kappa_*$ denote the subset of $[\kappa]^\kappa$ consisting of functions of the correct type.

There are partition properties formulated for functions of the correct type in which the homogeneous set can be chosen to be club. Club homogeneous sets are conceptually useful in various construction. Functions of the correct type seem to naturally appear in many proofs establishing the partition properties under AD.

It should be noted that throughout the survey an asterisk, $*$, will be used to denote the corresponding concept that involves functions of the correct type when there is an ordinary version of this concept.

Definition 2.4. Let κ be an ordinal. Let $\lambda \leq \kappa$ and $\gamma < \kappa$. Let $\kappa \rightarrow_* (\kappa)^\lambda_\gamma$ assert that for all $P : [\kappa]^\lambda_* \rightarrow \gamma$, there exists a club $C \subseteq \kappa$ and a $\beta < \lambda$ so that for all $f \in [C]^\lambda_*$, $P(f) = \beta$. Such a club C is said to be homogeneous for P taking value β (for functions of the correct type).

Definition 2.5. Let κ be an ordinal. If $A \subseteq \kappa$ is such that $|A| = \kappa$, then let $\mathsf{enum}_A : \kappa \rightarrow A$ denote the increasing enumeration of A. (In context, when one writes enum_A, it should be clear what κ is.)

Fact 2.6. *Let κ be an ordinal and $\lambda \leq \kappa$. Then $\kappa \rightarrow_* (\kappa)^\lambda_2$ implies $\kappa \rightarrow (\kappa)^\lambda_2$. $\kappa \rightarrow (\kappa)^{\omega \cdot \lambda}_2$ implies $\kappa \rightarrow_* (\kappa)^\lambda_2$.*

Proof. Assume $\kappa \rightarrow_* (\kappa)^\lambda_2$.

Let $P : [\kappa]^\lambda \rightarrow 2$. Then there is some $C \subseteq \omega_1$ club so that C is homogeneous for P for function of the correct type. Let $D = \{\alpha < \kappa : (\exists \beta \in \mathrm{Lim})(\alpha = \mathsf{enum}_C(\beta + \omega))\}$, where Lim refers to the class of limit ordinals. Note $|D| = \kappa$.

The next claim is that every $f \in [D]^\lambda$ is a function of the correct type. Let $g : \kappa \times \omega \rightarrow \kappa$ be defined as follows: Suppose $\gamma < \kappa$. Let β_γ be the unique limit ordinal β so that $f(\gamma) = \mathsf{enum}_C(\beta_\gamma + \omega)$. Then for each $n \in \omega$, define $g(\gamma, n) = \mathsf{enum}_C(\beta_\gamma + n)$. Then it is clear that $f(\gamma) = \sup\{g(\gamma, n) : n \in \omega\}$. This shows that f has uniform cofinality ω. Suppose $\gamma < \kappa$. There is a unique limit ordinal β_γ so that $f(\gamma) = C(\beta_\gamma + \omega)$. Then for all $\epsilon < \gamma$, $f(\epsilon) \leq \mathsf{enum}_C(\beta_\gamma) < f(\gamma)$. That is, $\sup\{f(\epsilon) : \epsilon < \gamma\} \leq \mathsf{enum}_C(\beta_\gamma) < f(\gamma)$. f is discontinuous everywhere.

Thus every $f \in [D]^\lambda$ belongs to $[C]^\lambda_*$. Since C is homogeneous for P for functions of the correct type, D is homogeneous for P in the ordinary sense. This establishes $\kappa \to (\kappa)^\lambda_2$.

Now suppose $\kappa \to (\kappa)^{\omega \cdot \lambda}_2$.

Suppose $P : [\kappa]^\lambda \to 2$. Let $\mathsf{block} : [\kappa]^{\omega \cdot \lambda} \to [\kappa]^\lambda$ be defined by $\mathsf{block}(f)(\gamma) = \sup\{\omega \cdot \gamma + n : n \in \omega\}$. Define $P' : [\kappa]^{\omega \cdot \lambda} \to 2$ by $P'(f) = P(\mathsf{block}(f))$. Let $A \subseteq \omega_1$ be such that $|A| = \kappa$ and A is homogeneous for P' in the ordinary sense. Without loss of generality, suppose $P'(h) = 0$ for all $h \in [A]^{\omega \cdot \lambda}$. Let C be the collection of limit points of A. That is, $C = \{\alpha \in \kappa : \alpha = \sup(C \cap \alpha)\}$. C is a club subset of ω_1.

Suppose $f \in [C]^\lambda_*$. Since f is of the correct type, let $g : \lambda \times \omega \to \omega_1$ witness that it has uniform cofinality ω. Let $\gamma < \kappa$. Since f is discontinuous everywhere, $\sup\{f(\alpha) : \alpha < \gamma\} < f(\gamma)$. Therefore there is some $N_\gamma \in \omega$ so that for all $n \geq N_\gamma$, $g(\gamma, n) > \sup\{f(\alpha) : \alpha < \gamma\}$. Since $f(\gamma)$ is a limit point of A, define by recursion, $h(\omega \cdot \gamma + n)$ to be the least element of A greater than

$$\max\{g(\gamma, N_\gamma + n), h(\omega \cdot \gamma + k) : k < n\}.$$

Then $h \in [A]^{\omega \cdot \gamma}$ and $\mathsf{block}(h) = f$. Thus $P(f) = P'(h) = 0$.

It has been shown that for all $f \in [C]^\lambda_*$, $P(f) = 0$. $\kappa \to_* (\kappa)^\lambda_2$ has been established. $\qquad \square$

Partition properties on a cardinal κ yield other very interesting properties on κ. Much of the material of the remainder of this section can be found in [20].

Fact 2.7. $\kappa \to (\kappa)^2_2$ *implies that κ is regular.*

Proof. Suppose $\eta < \kappa$ and $h : \eta \to \kappa$ is a cofinal function. Define $P : [\kappa]^2 \to 2$ by

$$P(\alpha, \beta) = \begin{cases} 0 & (\exists \gamma < \eta)(\alpha \leq h(\gamma) \leq \beta) \\ 1 & \text{otherwise} \end{cases}.$$

Let $A \subseteq \omega_1$ with $|A| = \kappa$ be homogeneous for P. If A is homogeneous for P taking value 0, then one can show that $\mathrm{ot}(\mathrm{rang}(h)) = \kappa$, which is a contradiction. If A is homogeneous for 1, then one can show that $\mathrm{rang}(h) \subseteq \min A$. This violates $\kappa \to (\kappa)^2_2$. $\qquad \square$

Definition 2.8. Let κ be a regular cardinal and $\eta < \kappa$ be a limit ordinal. A set $C \subseteq \kappa$ is η-closed if and only if for all $f : \eta \to C$ increasing, $\sup(f) \in C$. $C \subseteq \kappa$ is a η-club if and only if C is η-closed and unbounded.

Let W^κ_η denote the filter of sets containing an η-club as a subset.

Fact 2.9. *Let κ be a regular cardinal and $\eta < \kappa$ be a limit ordinal. Let $\delta < \kappa$ and $\langle C_\alpha : \alpha < \delta \rangle$ be a sequence of η-clubs. Then $\bigcap_{\alpha \in \delta} C_\alpha$ is an η-club.*

Proof. Clearly $\bigcap_{\gamma < \delta} C_\gamma$ is η-closed. One needs to show that $\bigcap_{\gamma < \delta} C_\gamma$ is unbounded.

Fix $\epsilon < \kappa$. Let β^0_0 be the least element of C_0 greater than ϵ.

Suppose for $\alpha < \eta$ and $\gamma < \delta$, β^ξ_α has been defined for all $\xi < \gamma$. Note $\sup\{\beta^\xi_\alpha : \xi < \gamma\} < \kappa$ by the regularity of κ. Let β^γ_α be the least element of C_γ which is larger than $\sup\{\beta^\xi_\alpha : \xi < \gamma\}$.

Suppose for $\alpha < \eta$, β^γ_ν has been defined for all $\nu < \alpha$, and $\gamma < \delta$. By the regularity of κ, $\sup\{\beta^\gamma_\nu : \nu < \alpha \wedge \gamma < \delta\} < \kappa$. Let β^0_α be the least element of C_0 greater than $\sup\{\beta^\gamma_\nu : \nu < \alpha \wedge \gamma < \delta\}$.

Note that for all $\gamma_0, \gamma_1 \in \delta$, $\sup\{\beta^{\gamma_0}_\alpha : \alpha < \eta\} = \sup\{\beta^{\gamma_1}_\alpha : \alpha < \eta\}$. Let λ denote this common value. $\lambda \in C_\gamma$ for all $\gamma < \delta$ since C_γ is an η-club. Thus $\lambda \in \bigcap_{\gamma < \delta} C_\gamma$ and $\lambda > \epsilon$. This shows that $\bigcap_{\gamma < \delta} C_\gamma$ is unbounded. $\qquad \square$

In ZF, Fact 2.9 does not imply that W^κ_η is κ-complete. Suppose $\langle A_\alpha : \alpha < \delta \rangle$ where $\delta < \kappa$ is a sequence in W^κ_η. For each A_α, there is an η-club $C \subseteq A_\alpha$. To apply Fact 2.9, one would need to produce a sequence of η-clubs, $\langle C_\alpha : \alpha < \delta \rangle$, so that $C_\alpha \subseteq A_\alpha$ for each $\alpha < \delta$. This appears to require some choice principle.

It will be shown next that appropriate partition properties imply that W^κ_γ is an ultrafilter, is κ-complete, and is normal.

Fact 2.10. *Let κ be a regular cardinal and $\eta < \kappa$ be a limit ordinal. Let $A \subseteq \kappa$ be an unbounded set. Let $\mathrm{Lim}^\eta(A) = \{\sup(f) : f \in [A]^\eta\}$. That is, $\mathrm{Lim}^\eta(A)$ is the collection of all $\alpha \in \kappa$ which are the supremum of an η-increasing sequence through A.*

Then $\mathrm{Lim}^\eta(A)$ is an η-club.

Proof. $\text{Lim}^{\eta}(A)$ is clearly unbounded since A is unbounded.

Suppose $f : \eta \to \text{Lim}^{\eta}(A)$ is an increasing function. For each $\alpha < \eta$, let $g(\alpha)$ be the least element $\gamma \in A$ so that $f(\alpha) < \gamma < f(\alpha + 1)$ which exists since $f(\alpha + 1) = \sup(h)$ for some $h \in [A]^{\eta}$. Thus $\sup(f) = \sup(g)$ and $g \in [A]^{\eta}$. Thus $\sup(f) \in \text{Lim}^{\eta}(A)$. $\text{Lim}^{\eta}(A)$ is an η-club. $\square$

Fact 2.11. *Let κ be a regular cardinal and $\eta < \kappa$ be a limit ordinal. Suppose $\kappa \to (\kappa)_2^{\eta}$. Then W_{η}^{κ} is an ultrafilter.*

Proof. Let $A \subseteq \kappa$. Define the partition $P : [\kappa]^{\eta} \to 2$ by $P(f) = 1 \Leftrightarrow \sup(f) \in A$.

By $\kappa \to (\kappa)_2^{\eta}$, let $B \subseteq \kappa$ with $|B| = \kappa$ be homogeneous for this partition. Without loss of generality, suppose B is homogeneous taking value 1. $\text{Lim}^{\eta}(B)$ is an η-club by Fact 2.10. Let $\alpha \in \text{Lim}^{\eta}(B)$. Let $f \in [B]^{\eta}$ be such that $\sup(f) = \alpha$. Then $P(f) = 1$ implies that $\alpha = \sup(f) \in A$. This shows that $\text{Lim}^{\eta}(B) \subseteq A$ which implies $A \in W_{\eta}^{\kappa}$.

If B was homogeneous for P taking value 0, then the same argument would have shown $\kappa \setminus A \in W_{\eta}^{\kappa}$. $\square$

Fact 2.12. *Assume κ is a regular cardinal and $\eta < \kappa$ is a limit ordinal. Let $\lambda < \kappa$ be infinite. If $\kappa \to (\kappa)_{\lambda}^{\eta}$ holds, then W_{η}^{κ} is λ^{+}-complete.*

Proof. Let $\langle A_{\alpha} : \alpha < \lambda \rangle$ be a sequence in W_{η}^{κ}. Suppose $\bigcap_{\alpha < \lambda} A_{\alpha} \notin W_{\eta}^{\kappa}$. Since W_{η}^{κ} is an ultrafilter by Fact 2.11, one may assume that $\bigcap_{\alpha < \lambda} A_{\alpha} = \emptyset$ by adding one further set to the sequence.

Let $P : [\kappa]^{\eta} \to \lambda$ be defined by $P(f)$ is the least $\alpha < \lambda$ so that $\sup(f) \notin A_{\alpha}$. By $\kappa \to (\kappa)_{\lambda}^{\eta}$, there is some α^* and a $B \subseteq \kappa$ so that $|B| = \kappa$ and $P(f) = \alpha^*$ for all $f \in [B]^{\eta}$.

By Fact 2.10, $\text{Lim}^{\eta}(B) \in W_{\eta}^{\kappa}$. However, $\text{Lim}^{\eta}(B) \cap A_{\alpha^*} = \emptyset$. This contradicts $A_{\alpha^*} \in W_{\eta}^{\kappa}$. $\square$

Fact 2.13. *Assume κ is a regular cardinal and $\eta < \kappa$ is a limit ordinal. $\kappa \to (\kappa)_2^{\eta+\eta}$ implies $\kappa \to (\kappa)_{\lambda}^{\eta}$ for all $\lambda < \kappa$. Therefore, $\kappa \to (\kappa)_2^{\eta+\eta}$ implies W_{η}^{κ} is a κ-complete measure on κ.*

If κ has the weak partition property, then W_{η}^{κ} is a κ-complete measure on κ for all limit ordinals $\eta < \kappa$.

Proof. Let $P : [\kappa]^{\eta} \to \lambda$. Define $Q : [\kappa]^{\eta+\eta} \to 2$ by $Q(f_0 \hat{\ } f_1) = 0$ if and only if $P(f_0) = P(f_1)$. By $\kappa \to (\kappa)_2^{\eta+\eta}$, let $A \subseteq \kappa$ with $|A| = \kappa$ be homogeneous for Q.

Suppose A is homogeneous for Q taking value 1. Let $Q' : [A]^{\eta+\eta} \to 2$ be defined by

$$Q'(f_0 \hat{\ } f_1) = \begin{cases} 0 & P(f_0) < P(f_1) \\ 1 & P(f_0) > P(f_1) \end{cases}$$

Again by $\kappa \to (\kappa)_2^{\eta+\eta}$, there is a $B \subseteq A$ with $|B| = \kappa$ which is homogeneous for Q'. One can check that if B is homogeneous for Q' taking value 1, then one would have an infinite descending sequences of ordinals. If B is homogeneous for Q' taking value 0, then one can produce an injection of κ into $\lambda < \kappa$. B can not be homogeneous. Contradiction. Therefore, A must have been homogeneous for Q taking value 0.

Now the claim is that A is homogeneous for P: Let $f_0, f_1 \in [A]^{\eta}$. Find $f_2 \in [A]^{\eta}$ such that $\min(f_2)$ is larger than both $\sup(f_0)$ and $\sup(f_1)$. Then $f_0 \hat{\ } f_2 \in [A]^{\eta+\eta}$ and $f_1 \hat{\ } f_2 \in [A]^{\eta+\eta}$. Since A is homogeneous for Q taking value 0, $Q(f_0 \hat{\ } f_2) = 0$ and $Q(f_1 \hat{\ } f_2) = 0$. This implies that $P(f_0) = P(f_2) = P(f_1)$. Thus A is homogeneous for P. $\square$

Fact 2.14. *Let κ be a regular cardinal and $\eta < \kappa$ be a limit ordinal. Suppose $\kappa \to (\kappa)_2^{\eta+\eta}$. Then κ is a normal κ-complete ultrafilter.*

If κ has the weak partition property, then W_{η}^{κ} is a normal κ-complete measure on κ for each limit ordinal $\eta < \kappa$.

Proof. By Fact 2.13, W_{η}^{κ} is a κ-complete ultrafilter.

Let $F : \kappa \to \kappa$ be a regressive function. That is, $A' = \{\alpha \in \kappa : F(\alpha) < \alpha\} \in W_{\eta}^{\kappa}$. Let $A \subseteq A'$ be a η-club set. Define $P : [\kappa]^{\eta} \to 2$ by

$$P(f) = \begin{cases} 0 & F(\sup(f)) < \min f \\ 1 & \text{otherwise} \end{cases}$$

Since $\kappa \to (\kappa)_2^{\eta+\eta}$ implies $\kappa \to (\kappa)_2^{\eta}$, let $B \subseteq A$ with $|B| = \kappa$ be homogeneous for P.

Let $\tilde{B} = \{\mathsf{enum}_B(\eta \cdot \alpha + \eta) : \alpha < \kappa\}$, where $\mathsf{enum}_B : \kappa \to \kappa$ is the increasing enumeration of B. Suppose $f \in [\tilde{B}]^\eta$. Since $\tilde{B} \subseteq B \subseteq A$ and A is η-closed, $\sup(f) \in A \subseteq A'$. Therefore, $F(\sup(f)) < \sup(f)$. Let $\gamma < \eta$ be least so that $F(\sup(f)) < f(\gamma)$. Since $f(\gamma + 1) \in \tilde{B}$, let $f_0 : \gamma + 1 \to B$ be such that for all $\nu < \gamma + 1$, $f(\gamma) < f_0(\nu) < f(\gamma + 1)$. Let $f_1 : (\eta - (\gamma + 1)) \to \tilde{B}$ be the tail of f above $f(\gamma)$. Note that $f_2 = f_0{}^\frown f_1$ is an η-sequence in B with the property that $\min(f_2) > f(\gamma) > F(\sup(f))$ and $\sup(f_2) = \sup(f)$. Thus $F(\sup(f_2)) = F(\sup(f)) < \min(f_2)$. Since $f_2 \in [B]^\eta$, one has that B must be homogeneous for P taking value 0.

Let $f \in [B]^\eta$. Let $f' : \eta \to B$ be the increasing enumeration of $\{\min B\} \cup \mathrm{rang}(f)$. Thus $P(f') = 0$ implies that $F(\sup(f)) = F(\sup(f')) < \min(f') = \min(B)$. It has been shown that for all $f \in [B]^\eta$, $F(\sup(f)) < \min(B)$. Thus for all $\alpha \in \mathrm{Lim}^\eta(B)$, $F(\alpha) < \min(B)$. Since $\mathrm{Lim}^\eta(B) \in W_\eta^\kappa$ and W_η^κ is κ-complete, there is some $\xi < \min(B)$ and $C \in W_\eta^\kappa$ so that $F[C] = \xi$. It has been shown that F is constant W_η^κ-almost everywhere. Normality has been established. $\qquad \square$

Under suitable circumstances on κ, one can determine all the normal κ-complete measures on κ.

Fact 2.15. *Let κ be a regular cardinal and $\eta < \kappa$ be a limit ordinal. Assume $\kappa \to (\kappa)_2^\eta$. Then $W_\eta^\kappa = W_{\mathrm{cof}(\eta)}^\kappa$.*

Suppose that $\eta_0 < \eta_1$ are two infinite regular cardinals less than κ. Then $W_{\eta_0}^\kappa \neq W_{\eta_1}^\kappa$.

Suppose the collection of infinite regular cardinals below κ has cardinality less than κ. Let μ be any κ-complete normal measure on κ. There is some infinite regular cardinal $\eta < \kappa$ so that μ is equivalent to W_η^κ.

Proof. The partition property $\kappa \to (\kappa)_2^\eta$ implies that W_η^κ and $W_{\mathrm{cof}(\eta)}^\kappa$ are both ultrafilters.

Suppose $A \subseteq \kappa$ is a $\mathrm{cof}(\eta)$-club. Let $f \in [A]^\eta$. Let $\rho : \mathrm{cof}(\eta) \to \eta$ be a cofinal increasing sequence. Note that $f \circ \rho \in [A]^{\mathrm{cof}(\eta)}$. Since A is a $\mathrm{cof}(\eta)$-club, $\sup(f \circ \rho) = \sup(f) \in A$. Thus A is also an η-club. This shows that $W_{\mathrm{cof}(\eta)}^\kappa \subseteq W_\eta^\kappa$.

Suppose that $\neg(W_\eta^\kappa \subseteq W_{\mathrm{cof}(\eta)}^\kappa)$. Let $A \in W_\eta^\kappa$ be such that $A \notin W_{\mathrm{cof}(\eta)}^\kappa$. Since $W_{\mathrm{cof}(\eta)}^\kappa$ is an ultrafilter, $\kappa \setminus A \in W_{\mathrm{cof}(\eta)}^\kappa$. It has already been shown that $W_{\mathrm{cof}(\eta)}^\kappa \subseteq W_\eta^\kappa$. Therefore, $\kappa \setminus A \in W_\eta^\kappa$. However, $A, \kappa \setminus A \in W_\eta^\kappa$ contradicts the fact that W_η^κ is an ultrafilter.

It has been established that $W_\eta^\kappa = W_{\mathrm{cof}(\eta)}^\kappa$.

Now suppose $\eta_0 < \eta_1$ are two regular cardinals less than κ. Let $A_i = \{\alpha < \kappa : \mathrm{cof}(\alpha) = \eta_i\}$. For $i \in 2$, A_i is an η_i-club. Therefore, $A_i \in W_{\eta_i}^\kappa$. If $W_{\eta_0}^\kappa = W_{\eta_1}^\kappa$, then $A_0 \cap A_1 = \emptyset \in W_{\eta_1}^\kappa$. This contradicts $W_{\eta_1}^\kappa$ is a filter.

Now suppose the collection K of infinite regular cardinals below κ has cardinality less than κ. Let μ be a κ-complete normal measure on κ.

Let T be the set of limit ordinals below κ. Since κ and T are in bijection, μ is equivalent to a measure concentrating on T. Therefore, assume for simplicity that $T \in \mu$. For each $\eta \in K$, let $A_\eta = \{\alpha \in \kappa : \mathrm{cof}(\alpha) = \eta\}$. $T = \bigcup_{\eta \in K} A_\eta$. Since μ is κ-complete and $|K| < \kappa$, there is some η_0 so that $A_{\eta_0} \in \mu$.

The claim is that $\mu = W_{\eta_0}^\kappa$: Let $B \in W_{\eta_0}^\kappa$. Let $C \subseteq B$ be an η_0-club set. Suppose $B \notin \mu$. Then $\kappa \setminus B \in \mu$. Let $Q = A_{\eta_0} \cap (\kappa \setminus B)$. Note $Q \in \mu$.

Let $\alpha \in Q$. Note that $\mathrm{cof}(\alpha) = \eta_0$ and $\alpha \notin B$ so $\alpha \notin C$. $\sup(C \cap \alpha) < \alpha$ since otherwise $\alpha \in C$ since $\mathrm{cof}(\alpha) = \eta_0$ and C is an η_0-club. Let $G : Q \to \kappa$ be defined $G(\alpha) = \sup(C \cap \alpha)$. G is a regressive function on Q. Since μ is normal, there is some $\gamma < \kappa$ so that $J = \{\alpha : G(\alpha) = \gamma\} \in \mu$. For any $\alpha \in J$, $C \cap \alpha \subseteq \gamma$. Since $J \in \mu$ implies that J is unbounded, this implies that $C \subseteq \gamma$. This is impossible since C is unbounded. $\qquad \square$

3. Good Coding of Functions

Definition 3.1. Let κ be a regular cardinal and $\lambda \leq \kappa$ be an ordinal. A good coding system for ${}^\lambda\kappa$ consists of Γ, decode, and $\mathsf{GC}_{\beta,\gamma}$ for each $\beta < \lambda$ and $\gamma < \kappa$ with the following properties:

(1) Γ is a pointclass closed under continuous substitution and $\exists^{\mathbb{R}}$. Let $\check{\Gamma}$ denote the dual pointclass. Let $\Delta = \Gamma \cap \check{\Gamma}$.

(2) $\mathsf{decode} : \mathbb{R} \to \mathscr{P}(\lambda \times \kappa)$. For all $f \in {}^\lambda\kappa$, there is some $x \in \mathbb{R}$ so that $\mathsf{decode}(x) = f$.

(3) For all $\beta < \lambda$ and $\gamma < \kappa$, $\mathsf{GC}_{\beta,\gamma} \subseteq \mathbb{R}$, $\mathsf{GC}_{\beta,\gamma} \in \Delta$, and $\mathsf{GC}_{\beta,\gamma}$ is defined by $x \in \mathsf{GC}_{\beta,\gamma}$ if and only if

$$\mathsf{decode}(x)(\beta, \gamma) \wedge (\forall \gamma' < \kappa)(\mathsf{decode}(x)(\beta, \gamma') \Rightarrow \gamma = \gamma').$$

For each $\beta < \lambda$, let $\mathsf{GC}_\beta = \bigcup_{\gamma < \kappa} \mathsf{GC}_{\beta,\gamma}$.

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(4) (Boundedness property) Suppose $A \in \exists^{\mathbb{R}}\Delta$ and $A \subseteq \mathsf{GC}_\beta$, then there exists some $\delta < \kappa$ so that $A \subseteq \bigcup_{\gamma < \delta} \mathsf{GC}_{\beta,\gamma}$.

(5) Δ is closed under less than κ length wellordered unions.

Suppose $x \in \mathbb{R}$, let $\mathsf{fail}(x)$ be the least $\beta < \lambda$ so that $x \notin \mathsf{GC}_\beta$ if it exists. Otherwise, let $\mathsf{fail}(x) = \infty$.

Let $\mathsf{GC} = \bigcap_{\beta < \lambda} \mathsf{GC}_\beta$. Note that if $x \in \mathsf{GC}$, then $\mathsf{decode}(x)$ is the graph of a function in $^\lambda\kappa$. If $x \in \mathsf{GC}$, then one will use function notations such as $\mathsf{decode}(x)(\beta) = \gamma$ to indicate $(\beta, \gamma) \in \mathsf{decode}(x)$.

Assuming AD, (5) follows from the other four conditions. This comes from a pointclass argument. See the end of the proof of Theorem 2.34 of [12]. Later, one will apply this to ω_1 with the associated pointclass $\boldsymbol{\Sigma}_1^1$. It is clear that (5) holds in this setting since $\boldsymbol{\Delta}_1^1$ is closed under countable unions.

Remark 3.2. The meaning of $x \in \mathsf{GC}_{\beta,\gamma}$ is that x is good at β in the sense that $\mathsf{decode}(x)$ has successfully mapped β to γ. One interprets $x \in \mathsf{GC}_\beta$ to be mean that x is good at β in the sense that $\mathsf{decode}(x)$ has successfully mapped β to some value. So a reals x belongs to GC means that x is a good code in the sense that $\mathsf{decode}(x)$ is truly a function from λ to κ.

Definition 3.3. Let κ be a regular cardinal and let $\lambda \leq \kappa$ be an ordinal. Let $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \lambda, \gamma < \kappa)$ be a coding system for $^\lambda\kappa$.

Let S^1 consists of reals r coding a Lipschitz continuous function $\Xi_r : \mathbb{R} \to \mathbb{R}$ which has the following property:

$$(\forall y)[\mathsf{fail}(y) \leq \mathsf{fail}(\Xi_r(y)) \wedge (\mathsf{fail}(y) < \infty \Rightarrow \mathsf{fail}(y) < \mathsf{fail}(\Xi_r(y)))].$$

Let S^2 consists of Player 2 strategies s so that its associated Lipschitz continuous function $\Xi_s : \mathbb{R} \to \mathbb{R}$ has the property that $(\forall x)(\mathsf{fail}(x) \leq \mathsf{fail}(\Xi_s(x)))$.

Fact 3.4. *Let κ be a regular cardinal and $\lambda \leq \kappa$ be an ordinal. Let $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \lambda, \gamma < \kappa)$ be a coding system for $^\lambda\kappa$.*

If $r \in S^1$, then there is a club $C \subseteq \kappa$ (obtained uniformly from r) so that for all $\delta \in C$, for all $\beta < \min\{\lambda, \delta\}$ and $\gamma < \delta$,

$$\Xi_r\left[\bigcap_{\beta' < \beta} \bigcup_{\gamma' < \gamma} \mathsf{GC}_{\beta',\gamma'}\right] \subseteq \bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < \delta} \mathsf{GC}_{\beta',\gamma'}$$

Suppose $s \in S^2$, then there is a club $C \subseteq \kappa$ so that for all $\delta \in C$, for all $\beta < \min\{\lambda, \delta\}$, for all $\gamma < \delta$,

$$\Xi_s\left[\bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < \gamma} \mathsf{GC}_{\beta',\gamma'}\right] \subseteq \bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < \delta} \mathsf{GC}_{\beta',\gamma'}.$$

Proof. Suppose $r \in S^1$. For each $\beta < \lambda$ and $\gamma < \kappa$, let

$$R_{\beta,\gamma} = \bigcap_{\beta' < \beta} \bigcup_{\gamma' < \gamma} \mathsf{GC}_{\beta',\gamma'}.$$

So if $x \in R_{\beta,\gamma}$, then $\mathsf{decode}(x)$ is good up to β by successfully mapping each ordinal less than β to some value less than γ. By property (5) of the good coding system, Δ is closed under less than κ wellordered union and also intersections. Property (3) states that $\mathsf{GC}_{\beta',\gamma'} \in \Delta$ for all $\beta' < \lambda$ and $\gamma' < \kappa$. Thus $R_{\beta,\gamma} \in \Delta$. Then $\Xi_r[R_{\beta,\gamma}]$ is $\exists^{\mathbb{R}}\Delta$.

Claim 1: For all $\beta' \leq \beta$, $\Xi_r[R_{\beta,\gamma}] \subseteq \mathsf{GC}_{\beta'}$.

To see Claim 1: Note that $x \in R_{\beta,\gamma}$ implies that $\mathsf{fail}(x) \geq \beta$. Since $r \in S^1$, $\mathsf{fail}(\Xi_r(x)) > \beta$. Thus for all $\beta' \leq \beta$, $\Xi_r(x) \in \mathsf{GC}_{\beta'}$. This shows Claim 1.

Therefore $\Xi_r[R_{\beta,\gamma}]$ is a $\exists^{\mathbb{R}}\Delta$ subset of $\mathsf{GC}_{\beta'}$ for each $\beta' \leq \beta$. By the boundedness property (4) of a good coding system, there is some $\epsilon_{\beta'} < \kappa$ so that $\Xi_r[R_{\beta,\gamma}] \subseteq \bigcup_{\gamma' < \epsilon_{\beta'}} \mathsf{GC}_{\beta',\gamma'}$.

Since κ is regular, let $\Upsilon(\beta, \gamma)$ be the least $\epsilon < \kappa$ so that $\beta < \epsilon$, $\gamma < \epsilon$, and for all $\beta' \leq \beta$, $\Xi_r[R_{\beta,\gamma}] \subseteq \bigcup_{\gamma' < \epsilon} \mathsf{GC}_{\beta',\gamma'}$. Thus $\Upsilon : \lambda \times \kappa \to \kappa$ is a well defined function.

Let $C = \{\delta : (\forall \beta < \delta)(\forall \gamma < \delta)(\Upsilon(\min\{\lambda, \beta\}, \gamma) < \delta)\}$.

Claim 2: C is a club.

To see Claim 2: Let $\alpha < \kappa$. Let $\alpha_0 = \alpha$. If α_n has been defined, let $\alpha_{n+1} = \Upsilon(\min\{\lambda, \alpha_n\}, \alpha_n)$. Let $\alpha_\infty = \sup\{\alpha_n : n \in \omega\}$. By definition of Υ, $\alpha_n < \alpha_{n+1}$ for all n. Let $\beta < \min\{\lambda, \alpha_\infty\}$ and $\gamma < \alpha_\infty$. Then there is some n so that $\beta < \alpha_n$ and $\gamma < \alpha_n$. Then $\Upsilon(\min\{\lambda, \beta\}, \gamma) \leq \Upsilon(\min\{\lambda, \alpha_n\}, \alpha_n) = \alpha_{n+1} < \alpha_\infty$. This shows that $\alpha_\infty \in C$. As $\alpha < \alpha_\infty$, C is unbounded. It is straightforward to show that C is closed. Claim 2 has been shown.

Fix a $\delta \in C$. Pick a $\beta < \min\{\lambda, \delta\}$ and a $\gamma < \delta$. Suppose $x \in R_{\beta,\gamma}$. Let $\beta' \leq \beta$. Then

$$\Xi_r(x) \in \bigcup_{\gamma' < \Upsilon(\beta,\gamma)} \mathsf{GC}_{\beta',\gamma'} \subseteq \bigcup_{\gamma' < \delta} \mathsf{GC}_{\beta',\gamma'}.$$

Since $\beta' \leq \beta$ was arbitrary,

$$\Xi_r(x) \in \bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < \delta} \mathsf{GC}_{\beta',\gamma'}.$$

This completes the proof of the result for S^1.

The argument for S^2 is similar. $\qquad\square$

Remark 3.5. Let $r \in S^1$. Let C be the club produced by Fact 3.4. This club C is called the club on which the opposite player (Player 2) has taken control of the output.

This means the following: In most applications, r codes a Lipschitz function coming from a Player 1 winning strategy. (Hence the notation S^1.) Let $\delta \in C$. Fact 3.4 states that for any code y so that $\mathsf{decode}(y)$ successfully defines a function up to some $\beta < \min\{\lambda, \delta\}$ and takes value some γ strictly below δ, then the Player 1 strategy coded by r when played against y produces some code x which successfully defines a function at and below β and still takes value below δ.

This observation is used in a game to show that although Player 1 may have a winning strategy which determines that a final sequence will land in a certain payoff set, there is a club C so that if the opposite player (Player 2) plays suitable codes for functions through this club, Player 2 actually determines the value of the final sequence. A similar statement holds if Player 2 has the winning strategy in this game.

This is the main idea of the following game of Martin to establish the partition properties and of an earlier game of Solovay to show the club filter is an ultrafilter.

Definition 3.6. Suppose κ is a regular cardinal and λ is such that $\omega \cdot \lambda < \kappa$. Suppose $f \in {}^{\omega \cdot \lambda}\kappa$. Let $\mathsf{block} : {}^{\omega \cdot \lambda}\kappa \to {}^{\lambda}\kappa$ be defined by $\mathsf{block}(f)(\alpha) = \sup\{f(\omega \cdot \alpha + k) : k \in \omega\}$.

Suppose $f, g \in {}^{\omega \cdot \lambda}\kappa$. Let $\mathsf{joint} : {}^{\omega \cdot \lambda}\kappa \times {}^{\omega \cdot \lambda}\kappa \to {}^{\lambda}\kappa$ be defined by

$$\mathsf{joint}(f, g)(\alpha) = \sup\{f(\omega \cdot \alpha + k), g(\omega \cdot \alpha + k) : k \in \omega\}.$$

Theorem 3.7. *(Martin) Assume* $\mathsf{ZF} + \mathsf{AD}$. *Suppose* λ, κ *are ordinals such that* $\omega \cdot \lambda \leq \kappa$. *Suppose there is a good coding system* $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta \in \omega \cdot \lambda, \gamma < \kappa)$ *for* ${}^{\omega \cdot \lambda}\kappa$. *Then* $\kappa \to_* (\kappa)_2^\lambda$ *holds.*

Proof. Let $P : [\kappa]_*^\lambda \to 2$. Consider the following game

$$
G \qquad
\begin{array}{c|ccccccc}
\text{I} & x_0 & & x_1 & & x_2 & & x_3 & & x \\
\hline
\text{II} & & y_0 & & y_1 & & y_2 & & y_3 & & y
\end{array}
$$

where Player 1 and 2 take turns playing integers. Player 1 produces the real x, and Player 2 produces the real y. Player 1 wins the game if and only the conjunction of the following two conditions hold:
(1) $\mathsf{fail}(x) > \mathsf{fail}(y) \vee \mathsf{fail}(x) = \mathsf{fail}(y) = \infty$.
(2) $(\mathsf{fail}(x) = \mathsf{fail}(y) = \infty) \Rightarrow P(\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(y))) = 0$.

(Case I) Suppose Player 1 has a winning strategy σ in this game.

The strategy σ induces a Lipschitz function Ξ_σ. Explicitly, $\Xi_\sigma(y)$ is just the response of Player 1 using σ when Player 2 plays using y. Note that $\sigma \in S^1$.

Let C be the club from Fact 3.4. On C, Player 2 takes control of the output in the sense below. Let D be the limit points of C.

Let $f \in [D]_*^\lambda$. Since f is of the correct type, let $F : \lambda \times \omega \to \kappa$ be a witness to f having uniform cofinality ω. For each $\alpha < \lambda$, let $\nu_\alpha = \sup\{f(\xi) : \xi < \alpha\}$. Let $g(\omega \cdot \alpha)$ be the least element of C greater than $\max\{\nu_\alpha, F(\alpha, 0)\}$. Suppose $g(\omega \cdot \alpha + k)$ has been defined for some $k \in \omega$. Let $g(\omega \cdot \alpha + k + 1)$ be the least

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element of C greater than $\max\{g(\omega \cdot \alpha + k), F(\alpha, k+1)\}$. Now $g \in [C]^{\omega \cdot \lambda}$, g is discontinuous everywhere, and $\mathsf{block}(g) = f$.

By property (2) of the good coding system, there is some $y \in \mathsf{GC}$ so that $\mathsf{decode}(y) = g$. Let Player 2 play this y against Player 1 using σ. Since $y \in \mathsf{GC}$, $\mathsf{fail}(y) = \infty$. Thus $\mathsf{fail}(\Xi_\sigma(r)) = \infty$.

For $\beta < \omega \cdot \lambda$, let $\epsilon_\beta = \sup\{g(\alpha) : \alpha < \beta\}$. Since g is discontinuous everywhere, $\epsilon_\beta < g(\beta)$. Then

$$\Xi_\sigma(y) \in \Xi_\sigma \left[\bigcap_{\beta' < \beta} \bigcup_{\gamma' < \epsilon_\beta} \mathsf{GC}_{\beta', \gamma'} \right] \subseteq \bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < g(\beta)} \mathsf{GC}_{\beta', \gamma'}$$

by the definition of C. Hence $\mathsf{decode}(\Xi_\sigma(y))(\beta) < g(\beta)$. Thus for all $\alpha < \lambda$,

$$\sup\{\mathsf{decode}(\Xi_\sigma(y))(\omega \cdot \alpha + k), \mathsf{decode}(y)(\omega \cdot \alpha + k) : k \in \omega\} = \sup\{g(\omega \cdot \alpha + k) : k \in \omega\} = f(\alpha).$$

This shows that $\mathsf{joint}(\mathsf{decode}(\Xi_\sigma(y)), \mathsf{decode}(y)) = f$. Since σ is a Player 1 winning strategy, one has that $P(f) = 0$. Since $f \in [D]^\lambda_*$ was arbitrary, D is homogeneous for P taking value 0.

(Case II) Suppose Player 2 has a winning strategy τ.

Note that Player 2 wins if and only if the disjunction of the following holds:
(1) $\mathsf{fail}(x) < \mathsf{fail}(y) \vee (\mathsf{fail}(x) = \mathsf{fail}(y) < \infty)$.
(2) $\mathsf{fail}(x) = \mathsf{fail}(y) = \infty \wedge P(\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(y))) = 1$.

Therefore, $\tau \in S^2$. Let C be the club coming from Fact 3.4. C is the club for which Player 1 takes control of the output. One may assume that C consists of only limit ordinals. Let D be the limit points of C.

Let $f \in [D]^\lambda_*$. As before, there is a $g \in [C]^{\omega \cdot \lambda}$ such that $\mathsf{block}(g) = f$. Let $x \in \mathsf{GC}$ be such that $\mathsf{decode}(x) = g$. Since $\mathsf{fail}(x) = \infty$, $\mathsf{fail}(\Xi_\tau(x)) = \infty$ and $\Xi_\tau(x) \in \mathsf{GC}$ as well.

Let $\beta < \omega \cdot \lambda$. Note that $g(\beta) + 1 < g(\beta + 1)$

$$\Xi_\tau(x) \in \Xi_\tau \left[\bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < g(\beta)+1} \mathsf{GC}_{\beta', \gamma'} \right] \subseteq \bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < g(\beta+1)} \mathsf{GC}_{\beta', \gamma'}$$

by definition of C. Hence $\mathsf{decode}(\Xi_\tau(x))(\beta) < g(\beta + 1)$. Thus for all $\alpha < \lambda$,

$$\sup\{\mathsf{decode}(x)(\omega \cdot \alpha + k), \mathsf{decode}(\Xi_\tau(x))(\omega \cdot \alpha + k) : k \in \omega\} = \sup\{g(\omega \cdot \alpha + k) : k \in \omega\} = f(\alpha).$$

This shows that $\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(\Xi_\tau(x))) = f$. Since τ is a Player 2 winning strategy, one has that $P(f) = 1$. D is homogeneous for P taking value 1.

The proof is complete. $\qquad\square$

Theorem 3.8. *(Almost everywhere uniformization on codes) Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let κ be a regular cardinal and $\lambda < \kappa$. Let $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \omega \cdot \lambda, \gamma < \kappa)$ be a good coding system for $^{\omega \cdot \lambda}\kappa$. Let $R \subseteq [\kappa]^\lambda_* \times \mathbb{R}$ be a relation.*

There is a club $C \subseteq \kappa$ and a Lipschitz continuous function $F : \mathbb{R} \to \mathbb{R}$ so that for all $x \in \mathsf{GC}$ with $\mathsf{decode}(x) \in [C]^{\omega \cdot \lambda}$ and $\mathsf{block}(\mathsf{decode}(x)) \in [C]^\lambda_* \cap \mathrm{dom}(R)$, $R(\mathsf{block}(\mathsf{decode}(x)), F(x))$.

Proof. Define a game G as follows:

	I	x_0		x_1		x_2		x_3		x
G										
	II		y_0, z_0		y_1, z_1		y_2, z_2		y_3, z_3	y, z

Player 2 wins if and only if the disjunction of the following hold:
(1) $\mathsf{fail}(x) < \mathsf{fail}(y) \vee \mathsf{fail}(x) = \mathsf{fail}(y) < \infty$.
(2) $(\mathsf{fail}(x) = \mathsf{fail}(y) = \infty) \wedge (\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(y)) \in \mathrm{dom}(R) \Rightarrow R(\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(y)), z))$.

Claim 1: Player 2 has a winning strategy.

To prove Claim 1: Suppose not. By AD, Player 1 has a winning strategy σ. Note that Player 1 winning condition is the conjunction of the follow:
(1) $\mathsf{fail}(y) < \mathsf{fail}(x) \vee \mathsf{fail}(x) = \mathsf{fail}(y) = \infty$.
(2) $\mathsf{fail}(x) = \mathsf{fail}(y) = \infty \Rightarrow [\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(y)) \in \mathrm{dom}(R) \wedge \neg R(\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(y)), z)]$.

Let Ξ_σ be the associated continuous function. Note that

$$(\forall y)(\forall z)[\mathsf{fail}(y) \leq \mathsf{fail}(\Xi_\sigma(y,z)) \wedge (\mathsf{fail}(y) < \infty \Rightarrow \mathsf{fail}(y) < \mathsf{fail}(\Xi_\sigma(y,z)))].$$

That is, if one fixes any z, the associated function $F_z(y) = \Xi_\sigma(y,z)$ belongs to S^1. With a small modification to the argument of Fact 3.4, there is a club C with the property that for all $\delta \in C$, for all $z \in \mathbb{R}$, for all $\beta < \min\{\lambda, \delta\}$, and $\gamma < \delta$

$$F_z \left[\bigcap_{\beta' < \beta} \bigcup_{\gamma' < \gamma} \mathsf{GC}_{\beta',\gamma'} \right] \subseteq \bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < \gamma} \mathsf{GC}_{\beta',\gamma'}.$$

That is, C is a club for which Player 2 controls the output against F_z for every $z \in \mathbb{R}$.

Let D be the limit points of C. Let $f \in [D]_*^\lambda$. As in the proof of Theorem 3.7, let $g \in [D]^{\omega \cdot \lambda}$ be such that $\mathsf{block}(g) = f$. Let $y \in \mathsf{GC}$ be such that $\mathsf{decode}(y) = g$. By the same argument as in Theorem 3.7, one has that for all z, $\mathsf{joint}(\mathsf{decode}(\Xi_\sigma(y,z)), \mathsf{decode}(y)) = f$. Since σ is a Player 1 winning strategy, one must have that $f \in \mathrm{dom}(R)$. Since $f \in \mathrm{dom}(R)$, there is some z^* so that $R(f, z^*)$. If Player 2 plays (y, z^*), then Player 1 loses using σ. This contradicts σ being a Player 1 winning strategy. Claim 1 has been shown.

Let τ be a Player 2 winning strategy in this game. Let $G(x) = \pi_1[\Xi_\tau(x)]$, where π_1 refers to the projection onto the first coordinate. Then G is a Lipschitz continuous function satisfying the condition from the definition of S^2. By Fact 3.4, let C be the club on which Player 2 takes controls of the output. Again one may assume C consists of only limit ordinals. Let D be the limit points of C.

Let $f \in [D]_*^\lambda \cap \mathrm{dom}(R)$. Let $g \in [C]^{\omega \cdot \lambda}$ be such that $\mathsf{block}(g) = f$. Let $x \in \mathsf{GC}$ be such that $\mathsf{decode}(x) = g$. By the argument in Theorem 3.7, $\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(G(x))) = f$. Since τ is a Player 2 winning strategy, one must have that $R(f, \pi_2(\Xi_r(x)))$.

Define $F(x) = \pi_2(\Xi_r(x))$. Then F is a Lipschitz function with the desired uniformization property. $\square$

Theorem 3.9. *Let κ be a regular cardinal and $\lambda \leq \kappa$. Suppose $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \lambda, \gamma < \kappa)$ is a good coding system for $^\lambda \kappa$. Let $M \models \mathsf{AD}$ be an inner model containing all the reals and within M, $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \lambda, \gamma < \kappa)$ is a good coding system.*

Then for any $\Phi : [\kappa]^\lambda \to \kappa$, there is a club D, necessarily in M by the Moschovakis coding lemma, so that $\Phi \upharpoonright [D]_*^\lambda \in M$.

Proof. The hypothesis implies that $\kappa < \Theta^M$. Let $\pi : \mathbb{R} \to \kappa$ be a surjection in M. Define a relation $R \subseteq [\kappa]_*^\lambda \times \mathbb{R}$ by

$$R(f,x) \Leftrightarrow \Phi(f) = \pi(x)$$

Let the Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ and club $C \subseteq \kappa$ be the objects given by Theorem 3.8. Let D be the set of limit points of C.

Let $f \in [D]_*^\lambda$. Let $x \in \mathsf{GC}$ be any function such that $\mathsf{decode}(x) \in [C]_*^{\omega \cdot \lambda}$ and $\mathsf{block}(\mathsf{decode}(x)) = f$. Then $R(\mathsf{block}(\mathsf{decode}(x)), F(x))$. This means that $\Phi(f) = \pi(F(x))$.

Note that for any x and x' so that $\mathsf{decode}(x), \mathsf{decode}(x') \in [C]_*^{\omega \cdot \lambda}$ and $\mathsf{block}(\mathsf{decode}(x)) = \mathsf{block}(\mathsf{decode}(x'))$, one has that $\pi(F(x)) = \pi(F(x')) = \Phi(\mathsf{block}(\mathsf{decode}(x)))$.

By the Moschovakis coding lemma and the fact that $\pi \in M$, C and hence D belongs to M. So within M, one can define $\Phi \upharpoonright [D]_*^\lambda$ as follows, $\Phi(f) = \gamma$ if and only if there exists some $x \in \mathsf{GC}$ so that $\mathsf{decode}(x) \in [C]_*^{\omega \cdot \lambda}$, $\mathsf{block}(\mathsf{decode}(x)) = f$, and $\pi(F(x)) = \gamma$. By the above, this is well defined and works. $\square$

One will say that a cardinal κ is "reasonable" if one is in the situation where there exists a good coding system:

Definition 3.10. (Jackson) Let κ be a regular cardinal and $\lambda \leq \kappa$. κ is λ-reasonable if and only if there is a good coding system for $^\lambda \kappa$.

4. Reasonableness at ω_1

Definition 4.1. Fix some recursive pairing function $\pi : \omega \times \omega \to \omega$. A real $x \in {}^\omega 2$ codes a relation R_x defined as follows: $R_x(m,n) \Leftrightarrow x(\pi(m,n)) = 1$.

Let the domain of x be $\mathrm{dom}(x) = \{n : (\exists m)(R_x(m,n) \vee R_x(m,n))\}$. Let LO be the set of reals x so that R_x is a linear ordering on its domain. Let WO be the set of reals x so that R_x is a well ordering. LO is an arithmetic set of reals. WO is Π_1^1.

If $x \in \mathrm{WO}$, then $\mathrm{ot}(x)$ is the order type of x. If $\beta < \mathrm{ot}(x)$, let n_β^x denote the unique element of $\mathrm{dom}(x)$ whose rank according to R_x is β.

If $\alpha < \omega_1$, then let $\mathrm{WO}_\alpha = \{x \in \mathrm{WO} : \mathrm{ot}(x) = \alpha\}$. $\mathrm{WO}_{<\alpha} = \{x \in \mathrm{WO} : \mathrm{ot}(x) < \alpha\}$. $\mathrm{WO}_{\leq\alpha}$, $\mathrm{WO}_{>\alpha}$, and $\mathrm{WO}_{\geq\alpha}$ are defined similarly. WO_α, $\mathrm{WO}_{<\alpha}$, and $\mathrm{WO}_{\leq\alpha}$ are $\mathbf{\Delta}_1^1$ (in any element of WO_α).

Definition 4.2. The ordertype function $\mathrm{ot} : \mathrm{WO} \to \omega_1$ is a Π_1^1 norm. Let $\preceq$ denote the induced prewellordering: $x \preceq y$ if and only if $\mathrm{ot}(x) \leq \mathrm{ot}(y)$.

Being a Π_1^1-norm implies that there is a Σ_1^1 relation $\leq_{\Sigma_1^1}$ and a Π_1^1 relation $\leq_{\Pi_1^1}$ so that

$$y \in \mathrm{WO} \Rightarrow (\forall x)[(x \in \mathrm{WO} \wedge x \preceq y) \Leftrightarrow (x \leq_{\Sigma_1^1} y) \Leftrightarrow (x \leq_{\Pi_1^1} y)].$$

It is useful to have a concrete coding of a nice collection of club subsets of ω_1.

Fact 4.3. *Let τ be a Player 2 strategy with the property that for all $w \in \mathrm{WO}$, $\tau(w) \in \mathrm{WO} \wedge \mathrm{ot}(w) < \mathrm{ot}(\tau(w))$. Let $C_\tau = \{\eta : (\forall w \in \mathrm{WO}_{<\eta})(\mathrm{ot}(\tau(w)) < \eta)\}$. C_τ is a club.*

Proof. Let $R_\alpha = \{\tau(w) : w \in \mathrm{WO}_{<\alpha}\}$. $R_\alpha \subseteq \mathrm{WO}$ is $\mathbf{\Sigma}_1^1$. By boundedness, $\sup\{\mathrm{ot}(v) : v \in R_\alpha\} < \omega_1$.

Let $\Phi(\alpha) = \sup\{\mathrm{ot}(\tau(w)) : w \in \mathrm{WO}_{<\alpha}\} + 1$. By the above, $\Phi(\alpha)$ is defined.

It is clear that C_τ is closed.

Let $\alpha < \omega_1$. Let $\alpha_0 = \alpha$. Let $\alpha_{n+1} = \Phi(\alpha_n)$. Let $\eta = \sup\{\alpha_n : n \in \omega\}$. Let $\beta < \eta$. There is some n so that $\beta < \alpha_n$. Let $w \in \mathrm{WO}_\beta \subseteq \mathrm{WO}_{<\alpha_n}$. Thus $\mathrm{ot}(\tau(w)) < \Phi(\alpha_n) = \alpha_{n+1} < \eta$. So $\eta \in C_\tau$. This shows that C_τ is unbounded. $\qquad\square$

Definition 4.4. Let $\mathsf{clubcode}_{\omega_1}$ be the set of Player 2 strategies so that for all $w \in \mathrm{WO}$, $\tau(w) \in \mathrm{WO} \wedge \mathrm{ot}(w) < \mathrm{ot}(\tau(w))$.

Note that $\mathsf{clubcode}_{\omega_1}$ is a Π_2^1 set of reals.

Fact 4.5. *Assume that $\tau \in \mathsf{clubcode}_{\omega_1}$. The relation $S(w)$ defined by $w \in \mathrm{WO} \wedge \mathrm{ot}(w) \in C_\tau$ is $\mathbf{\Pi}_1^1$.*

Assume $\mathsf{AC}_\omega^{\mathbb{R}}$. If $\alpha < \omega_1$, then the relation $T_\alpha(w)$ defined by $w \in \mathrm{WO}_{<\alpha} \wedge \mathrm{ot}(w) \in C_\tau$ is $\mathbf{\Delta}_1^1$.

Proof. Note that $S(w)$ holds if and only if $w \in \mathrm{WO} \wedge (\forall v)(v <_{\Sigma_1^1} w \Rightarrow \tau(v) <_{\Pi_1^1} w)$, where $<_{\Sigma_1^1}$ and $<_{\Pi_1^1}$ are Σ_1^1 and Π_1^1 relations, respectively, coming from the ordertype function, ot, being a Π_1^1-norm.

Note that $\alpha \cap C_\tau$ is a countable set. Using $\mathsf{AC}_\omega^{\mathbb{R}}$, let $r \in \omega$ be such that $\{\mathrm{ot}(r_n) : n \in \omega\} = \alpha \cap C_\tau$. (Here $r_n(m) = r(\langle n, m \rangle)$, where $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$ is a recursive bijective pairing function.) One can see that T_α is $\mathbf{\Delta}_1^1$ using as parameters r and some element of WO_α. $\qquad\square$

Fact 4.6. *Assume $\mathsf{ZF} + \mathsf{AD}$. Let $C \subseteq \omega_1$ be a club. There is a $\tau \in \mathsf{clubcode}_{\omega_1}$ so that $C_\tau \subseteq C$.*

Proof. Consider the game G_C:

$$
\begin{array}{cccccccc}
\mathrm{I} & w_0 & & w_1 & & w_2 & & w_3 & & w \\
G_C & & & & & & & & \\
\mathrm{II} & & v_0 & & v_1 & & v_2 & & v_3 & & v
\end{array}
$$

Player 2 wins this game if and only if

$$w \in \mathrm{WO} \Rightarrow (v \in \mathrm{WO} \wedge \mathrm{ot}(v) > \mathrm{ot}(w) \wedge \mathrm{ot}(v) \in C).$$

Claim 1: Player 2 has the winning strategy.

Suppose σ is a Player 1 winning strategy. Note that $\sigma[\mathbb{R}] \subseteq \mathrm{WO}$ and is $\mathbf{\Sigma}_1^1$. By boundedness, $\gamma = \sup\{\mathrm{ot}(w) : w \in \sigma[\mathbb{R}]\} < \omega_1$. Since C is unbounded, let $\delta \in C$ be such that $\delta > \gamma$. Let $v \in \mathrm{WO}_\delta$. If Player 2 plays v against σ, $v \in \mathrm{WO}$, $\mathrm{ot}(v) = \delta \in C$, and $\mathrm{ot}(\sigma(v)) \leq \gamma < \delta = \mathrm{ot}(v)$. Player 2 has won. This contradicts σ being a Player 1 winning strategy.

Let τ be a Player 2 winning strategy. It is clear that $\tau \in \mathsf{clubcode}_{\omega_1}$.

Claim 2: $C_\tau \subseteq C$.

Suppose $\eta \in C_\tau$. Let $\beta < \eta$. Let $w \in \mathrm{WO}_\beta$. Then $\beta < \mathrm{ot}(\tau(w)) < \eta$ and $\mathrm{ot}(\tau(w)) \in C$. Since $\beta < \eta$ was arbitrary and C is closed, $\eta \in C$. $\qquad\square$

Fact 4.7. *Suppose $A \subseteq \mathsf{clubcode}_{\omega_1}$ is $\mathbf{\Sigma}_1^1$. Then uniformly from A, there is a club C so that for all $\tau \in A$, $C \subseteq C_\tau$.*

Proof. For each $\beta < \omega_1$, let $R_\beta = \{\tau(w) : \tau \in A \wedge w \in \mathrm{WO}_{<\beta}\}$. $R_\beta \subseteq \mathrm{WO}$ and is $\mathbf{\Sigma}_1^1$. By the boundedness principle, $\sup\{\mathrm{ot}(v) : v \in R_\beta\} < \omega_1$.

Let $\Phi(\beta) = \sup\{\mathrm{ot}(v) : v \in R_\beta\} + 1$.

Let $C = \{\eta : (\forall \beta < \eta)(\Phi(\beta) < \eta)\}$. As before, one can check that C is a club subset of ω_1.

Fix $\tau \in A$. Let $\eta \in C$. Suppose $\beta < \eta$. Let $w \in \mathrm{WO}_\beta$. Then $\beta < \mathrm{ot}(\tau(w)) < \Phi(\beta) < \eta$. Thus $\eta \in C_\tau$. $\square$

Fact 4.8. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ *be such that each* $\mathcal{A}_\alpha$ *is a nonempty* $\subseteq$-*downward closed collection of club subsets of* ω_1. *Then there is a sequence* $\langle C_\alpha : \alpha < \omega_1 \rangle$ *with each* $C_\alpha \subseteq \omega_1$ *a club subset of* ω_1 *and* $C_\alpha \in \mathcal{A}_\alpha$.

In particular: Let μ *be the club measure on* ω_1. *Let* $\langle A_\alpha : \alpha < \omega_1 \rangle$ *be a sequence of sets in* μ, *that is each* A_α *contains a club subset of* ω_1. *Then there is a sequence* $\langle C_\alpha : \alpha < \omega_1 \rangle$ *with each* $C_\alpha \subseteq \omega_1$ *a club subset of* ω_1 *so that* $C_\alpha \subseteq A_\alpha$.

Proof. Consider the game

$$
\begin{array}{cccccccc}
& \mathrm{I} & w(0) & & w(1) & & w(2) & & w(3) & & & w \\
G & & & & & & & & & & \\
& \mathrm{II} & & z(0) & & z(1) & & z(2) & & z(3) & & z
\end{array}
$$

where Player 2 wins if and only if $w \in \mathrm{WO} \Rightarrow (z \in \mathrm{clubcode}_{\omega_1} \wedge C_z \in \mathcal{A}_{\mathrm{ot}(w)})$.

Claim 1: Player 2 has a winning strategy in this game.

To prove this, suppose otherwise that Player 1 has a winning strategy σ. Note that $\sigma[\mathbb{R}] \subseteq \mathrm{WO}$. Since $\sigma[\mathbb{R}]$ is a $\mathbf{\Sigma}_1^1$ subset of WO, by the boundedness principle, there is a $\zeta \in \omega_1$ so that for all $w \in \sigma[\mathbb{R}]$, $\mathrm{ot}(w) < \zeta$. By $\mathsf{AC}_\omega^{\mathbb{R}}$, pick $\langle C_\alpha : \alpha < \zeta \rangle$ with the property that for all $\alpha < \zeta$, $C_\alpha \in \mathcal{A}_\alpha$. Let $C = \bigcap_{\alpha < \zeta} C_\alpha$ which is also a club. By Fact 4.6, there is some $z \in \mathrm{clubcode}_{\omega_1}$ so that $C_z \subseteq C$. Note that since for each $\alpha < \zeta$, $C_z \subseteq C_\alpha$, $C_\alpha \in \mathcal{A}_\alpha$, and $\mathcal{A}_\alpha$ is $\subseteq$-downward closed, one has that $C_z \in \mathcal{A}_\alpha$ for each $\alpha < \zeta$. Thus Player 2 wins against σ by playing z. Contradiction.

Thus let τ be a Player 2 winning strategy in this game. For each $\alpha < \omega_1$, let $P_\alpha = \tau[\mathrm{WO}_\alpha]$. Note that $P_\alpha \subseteq \mathrm{clubcode}_{\omega_1}$ with the property that for all $z \in P_\alpha$, $C_z \in \mathcal{A}_\alpha$. Note that P_α is $\mathbf{\Sigma}_1^1$. By Fact 4.7, there is a uniform procedure to obtain from the set P_α, a club C_α with the property that $C_\alpha \subseteq C_z$ for all $z \in P_\alpha$. In particular since $\mathcal{A}_\alpha$ is $\subseteq$-downward closed, $C_\alpha \in \mathcal{A}_\alpha$. This completes the proof. $\square$

Fact 4.9. *(Martin) Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* $\lambda < \omega_1$. *Then* ω_1 *is* λ-*reasonable.*

Proof. Note that ω_1 is regular by $\mathsf{AC}_\omega^{\mathbb{R}}$. Let $\Gamma = \mathbf{\Sigma}_1^1$. Note that $\mathbf{\Delta}_1^1$ is closed under countable unions. Thus (1) and (5) hold.

For each $x \in \mathbb{R}$, let $x_n \in \mathbb{R}$ be defined by $x_n(k) = x(\langle n, k \rangle)$, where $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$ denotes a fixed recursive bijective pairing function.

Fix some $w \in \mathrm{WO}_\lambda$ with $\mathrm{dom}(w) = \omega$.

Define $\mathsf{decode}(x)(\alpha, \beta)$ if and only if $x_{n_\alpha^w} \in \mathrm{WO}_\beta$. Suppose $f \in {}^\lambda\omega_1$. By $\mathsf{AC}_\omega^{\mathbb{R}}$, for each $\beta < \lambda$, let $u_\beta \in \mathrm{WO}_{f(\beta)}$. Let $x \in \mathbb{R}$ be such that $x_{n_\beta^w} = u_\beta$. Thus $\mathsf{decode}(x) = f$. This shows (2).

Let $\mathsf{GC}_{\beta,\gamma} = \{x : x_{n_\beta^w} \in \mathrm{WO}_\gamma\}$. Note that $\mathsf{GC}_{\beta,\gamma}$ is $\mathbf{\Delta}_1^1$. It satisfies the property in (3).

Suppose $A \in \mathbf{\Sigma}_1^1$ and $A \subseteq \mathsf{GC}_\beta$. Let $R = \{x_{n_\beta^w} : x \in A\}$. Then R is a $\mathbf{\Sigma}_1^1$ subset of WO. By the usual boundedness principle, there is some $\delta < \omega_1$ so that for all $v \in R$, $\mathrm{ot}(v) < \delta$. Then $A \subseteq \bigcup_{\gamma < \delta} \mathsf{GC}_{\beta,\gamma}$.

It has been shown that $(\mathbf{\Sigma}_1^1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \lambda, \gamma < \omega_1)$ is a good coding system for ${}^\lambda\omega_1$. $\square$

Corollary 4.10. *(Martin) Assume* $\mathsf{ZF} + \mathsf{AD}$. ω_1 *is a weak partition cardinal, i.e. for all* $\lambda < \omega_1$, $\omega_1 \to_* (\omega_1)_2^\lambda$.

Corollary 4.11. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *The club filter* $\mu = W_\omega^{\omega_1}$ *is the unique countably complete normal measure on* ω_1.

Proof. By Fact 2.15. $\square$

Fact 4.12. *Let* $\alpha < \omega_1$. *Fix a good coding system for* ${}^{\omega \cdot \alpha}\omega_1$. *Suppose* $z \in \mathrm{clubcode}_{\omega_1}$. *Let* D *be the set of limit points of* C_z. *Let* $f \in [D]_*^\alpha$. *The set of* $\{x \in \mathbb{R} : \mathsf{decode}(x) \in [C_z]^{\omega \cdot \alpha} \wedge \mathsf{block}(\mathsf{decode}(x)) = f\}$ *is a* $\mathbf{\Delta}_1^1$ *set*

Proof. Fix any x^* so that $\mathsf{block}(\mathsf{decode}(x^*)) = f$. Since ω_1 is regular, $\sup(f) = \gamma < \omega_1$. Using x^* as a parameter, one can now check that the desired set is $\mathbf{\Delta}_1^1$ by unraveling the definition of the coding system in Theorem 4.9 and using Fact 4.5. $\qquad\square$

The original method of establishing that ω_1 is ω_1-reasonable involves sharps for reals which will be concisely reviewed below. See [23] for more details. The reader who would prefer to avoid sharps can skip ahead to Theorem 4.26 to see the argument of Kechris that uses the Banach-Mazur game and category.

Let $\mathscr{L} = \{\dot\in, \dot E\}$ be a language where $\dot\in$ is a binary relation symbol and $\dot E$ is a unary relation symbol.

For each $\mathscr{L}$-formula $\varphi(w, v_1, ..., v_k)$, whose free variable are exactly those listed, then let $h_\varphi(v_1, ..., v_k)$ be a formal function which will be called the Skolem function associated to φ.

Define the language $\mathscr{L}^I = \{\dot\in, \dot E, \dot c_n : n \in \omega\}$ where for each $n \in \omega$, $\dot c_n$ is a distinct constant symbol.

Let skolem denote the smallest class of function $h(v_1, ..., v_k)$ containing h_φ for all φ and closed under composition.

$\mathscr{L}^S$ consist of $\dot\in$, $\dot E$, constants c_n for all n, and new distinct constant symbols $h(c_{i_1}, ..., c_{i_k})$ for all $i_1 < ... < i_k$ in ω and $h \in \mathsf{skolem}$

One will assume one has fixed a recursive coding of formulas and terms of $\mathscr{L}^S$. One will identify terms or formulas with their associated integer code.

Let $\dot\prec$ denote the $\mathscr{L}$-formula that defines the canonical $L[\dot E]$ wellordering. Every formula ϕ of $\mathscr{L}^S$ can be converted into a $\mathscr{L}^I$ formula $\tilde\phi$ that roughly amounts to recursively replacing each Skolem term $h_\varphi(v_1, ..., v_k)$ with its intended meaning that it should represent the $\dot\prec$-least solution to φ if it exists and $\emptyset$ otherwise.

Definition 4.13. Let $A(T)$ assert the following:
(1) T is the set of integer code for a complete and consistent theory extending $\mathsf{ZF} + V = L[\dot E]$. The statement $(\forall x)(x \in \dot E \Rightarrow x \in \omega)$ belongs to T.
(2) (Indiscernibility) For each $\mathscr{L}$-formula $\varphi(v_1, ..., v_k)$ and increasing sequences of integers $(a_1, ..., a_k)$ and $(b_1, ..., b_k)$, the following sentence belongs to T:

$$\text{``}\varphi(c_{a_1}, ..., c_{a_k}) \Leftrightarrow \varphi(c_{b_1}, ..., c_{b_k})\text{''}.$$

(3) (Unbounded) Let $h(v_1, ..., v_k) \in \mathsf{skolem}$. The following sentence belongs to T:

$$\text{``}h(c_1, ..., c_k) \in \mathrm{ON} \Rightarrow h(c_1, ..., c_k) < c_{k+1}\text{''}.$$

(This is actually a $\mathscr{L}^S$-sentence ϕ. Precisely, one means that $\tilde\phi \in T$.)
(4) (Remarkability) Suppose $h(v_0, ..., v_k, w_1, ..., v_j) \in \mathsf{skolem}$. Then the following sentence belongs to T:

$$\text{``}[h(c_0, ..., c_k, c_{a_1}, ..., c_{a_j}) \in \mathrm{ON} \wedge h(c_0, ..., c_k, c_{a_1}, ..., c_{a_j}) < c_{a_1}]$$
$$\Rightarrow h(c_1, ..., c_k, c_{a_1}, ..., c_{a_j}) = h(c_1, ..., c_k, c_{b_1}, ..., c_{b_k})\text{''}$$

for all increasing sequences of integers $(a_1, ..., a_j)$ and $(b_1, ..., b_j)$ such that $k < a_1$ and $k < b_1$.

Note that $A(T)$ is an arithmetical statement asserting syntactical conditions on the $\mathscr{L}^I$-theory T.

Now assume T is a set of integers so that $A(T)$.

Let K be a linear ordering. For each $a \in K$, let c_a be distinct new constant symbols. Let $\mathscr{L}^{S,K}$ consists of $\dot\in$, $\dot E$, and new constant symbols $h(c_{a_1}, ..., c_{a_k})$ for each $h \in \mathsf{skolem}$ and increasing tuple $(a_1, ..., a_k)$ in K.

An equivalence relation is defined on the new constants of $\mathscr{L}^{S,K}$ by have $h_1(c_{a_1}, ..., c_{a_n}) \sim h_2(c_{a_1}, ..., c_{a_n})$ if and only if $\tilde\phi \in T$ where ϕ is the $\mathscr{L}^S$ formula $h_1(c_1, ..., c_n) = h_2(c_1, ..., c_n)$.

The membership relation can be defined on Skolem constant by referring to T in a similar manner.

By taking quotients, one obtains a structure denoted $\Gamma(T, K)$.

K embeds into $\Gamma(T, K)$ by the map $j(a) = [c_a]_\sim$.

Fact 4.14. *Let $T \subseteq \omega$ be such that $A(T)$. Then $\Gamma(T, \alpha)$ is wellfounded for all ordinals α if and only if $\Gamma(T, \alpha)$ is wellfounded for all $\alpha < \omega_1$.*

Suppose $w \in \mathrm{WO}$ and $(\omega, <_w)$ is its associated wellordering relation on ω. One can check that in this case one can find a structure on ω which is isomorphic to $\Gamma(T, \mathrm{ot}(w))$. This structure is recursive in T and w and produced uniformly from T and w. In such case that $w \in \mathrm{WO}$, one will denote $\Gamma(T, w)$ to be this structure on ω recursive in T and w.

Suppose M is an $\dot{\in}$ structure on ω. For each $k \in \omega$, there is a recursive function ST so that $\mathsf{ST}(M, k)$ is a structure on ω isomorphic to $(k, \dot{\in}^M)$.

Since $\Gamma(T, w)$ is considered as a structure on ω, for each $k \in \Gamma(T, w)$ such that $\Gamma(T, w) \models k$ is an ordinal, $\mathsf{ST}(\Gamma(T, w), k)$ gives (uniformly) a structure on ω isomorphic to ordinal k in $\Gamma(T, w)$.

Note that for each $\beta < \omega_1$, the set of (T, w) such that β is an initial segment of the ordinal of $\Gamma(T, w)$ is a $\mathbf{\Delta}_1^1$ set (using any element of WO_β as a parameter).

Consider formulas $\varphi(v, w)$ stating $v \in \mathrm{ON} \wedge \psi(v, w)$ where ψ is some other $\mathscr{L}$-formula. After fixing an enumeration of such formulas, one has an enumeration $\langle t_n : n \in \omega \rangle$ of Skolem constants whose intention is to name ordinals.

For each $\beta < \omega_1$ and $\gamma < \omega_1$, the set of (T, w, n) such that $\beta + \omega$ is an initial segment of $\Gamma(T, w)$ and $t_n^{\Gamma(T,w)}(\beta) < \gamma$ is a $\mathbf{\Delta}_1^1$ set (using any elements of WO_β and WO_γ). Similarly for $t_n^{\Gamma(T,w)}(\beta) = \gamma$.

Let $\mathrm{B}(T)$ assert that for all $\alpha < \omega_1$, $\Gamma(T, \alpha)$ is wellfounded. This is equivalent to

$$(\forall w)(w \in \mathrm{WO} \Rightarrow \Gamma(T, w) \text{ is wellfounded}).$$

This is Π_2^1.

Definition 4.15. A set T is a sharp of a real if and only if $A(T) \wedge B(T)$. Thus the statement that T is a sharp of a real is Π_2^1.

T is the sharp of a real x if and only if T is a sharp of a real and for all $n \in \omega$, "$n \in \dot{E}$" $\in T$ if and only if $n \in x$.

It can be shown that if there is a T such that T is a sharp of x, then this T is unique. Therefore $x^\sharp$ will denote this unique T for the real x.

Solovay showed that every subset of ω_1 is constructible from a real under AD:

Fact 4.16. *(Solovay) Assume* $\mathsf{ZF} + \mathsf{AD}$. *There is an single formula* $\theta(x, \alpha)$ *so that for all* $A \subseteq \omega_1$, *there is an* $x \in \mathbb{R}$ *so that* $\alpha \in A \Leftrightarrow L[x] \models \theta(x, \alpha)$.

Proof. Let $A \subseteq \omega_1$. Consider the game

	I	$x(0)$		$x(1)$		$x(2)$		$x(3)$			x
S_A											
	II		$y(1)$		$y(2)$		$y(3)$		$y(4)$		y

Player 2 wins if the disjunction of the following holds
(1) $x \notin \mathrm{WO}$.
(2) $x \in \mathrm{WO} \Rightarrow (\forall n)((y_n \in \mathrm{WO}) \wedge (\exists \gamma \geq \mathrm{ot}(x))(\{\mathrm{ot}(y_n) : n \in \omega\} = A \cap \gamma))$. (Here y_n is defined by $y_n(k) = y(\langle n, k \rangle)$.)

Suppose Player 1 has a winning strategy σ. Then for all $y \in \mathbb{R}$, $\sigma(y) \in \mathrm{WO}$. Then $E = \{v : (\exists y)(v = \sigma(y))\}$ is an $\mathbf{\Sigma}_1^1$ subset of WO. Thus there is some $\gamma < \omega_1$ so that $\mathrm{ot}(v) < \gamma$ for all $v \in E$. Let $y \in \mathbb{R}$ be such that for all n, $y_n \in \mathrm{WO}$ and $\{\mathrm{ot}(y_n) : n \in \omega\} = A \cap \gamma$. Then Player 1 using σ loses if Player 2 plays this y.

By AD, Player 2 must have a winning strategy τ. Note that $\alpha \in A$ if and only if $L[\tau]$ satisfies that $1_{\mathrm{Coll}(\omega, \alpha+1)}$ forces that for all $w \in \mathrm{WO}_{\alpha+1}$, there is some n so that $\mathrm{ot}(\tau(w)_n) = \alpha$. The formula θ is defined to assert this statement. The real associated to A is of course τ. $\square$

Fact 4.17. *Let* $f : \omega_1 \rightarrow \omega_1$. *There is some* $x \in \mathbb{R}$ *and an* $n \in \omega$ *so that for all* $\alpha < \omega_1$, $f(\alpha) = t_n^{\Gamma(x^\sharp, \beta)}(\alpha)$ *whenever* $\beta > \alpha$.

Proof. Consider $A = \{\langle \alpha, \beta \rangle : f(\alpha) = \beta\}$ where $\langle \cdot, \cdot \rangle$ denotes some constructible pairing function. By Fact 4.16, there is some real x so that $\langle \alpha, \beta \rangle \in A$ if and only if $L[x] \models \theta(x, \langle \alpha, \beta \rangle)$.

Consider the formula $\psi(\beta, \alpha)$ if and only if $\theta(\dot{E}, \langle \alpha, \beta \rangle)$. There is some n so that t_n is the Skolem function associated to this formula. Fix two ordinals $\alpha < \beta$. Since $\Gamma(x^\sharp, \beta)$ is of the form $L_{\beta'}[x]$ for some $\beta' \geq \beta$ and $\Gamma(x^\sharp, \beta) \prec_\omega L[x]$, $t_n^{\Gamma(x^\sharp, \beta)}(\alpha) = f(\alpha)$. $\square$

It should be noted that coding functions by simply coding their graph as a subset of ω_1 is generally not good enough. For each α, if one had to search for the corresponding β which is the value of $f(\alpha)$, the coding

system will be too complex. In this case, one uses the Skolem term to output the value of f when given α. This needs to be handled in the proof of Kechris as well. See Remark 4.25. There it is handled by a modified version of the Solovay game used above.

Theorem 4.18. *(Martin) Assume* $\mathsf{ZF} + \mathsf{AD}$. ω_1 *is* ω_1-*reasonable.*

Proof. If $x \in \mathbb{R} = {}^\omega\omega$, let $\mathsf{cut}(x) \in {}^\omega\omega$ be defined by $\mathsf{cut}(x)(k) = x(k+1)$.

One will now define a good coding system for ${}^{\omega_1}\omega_1$.

The pointclass of the coding system is $\boldsymbol{\Sigma}^1_1$.

For each $x \in \mathbb{R}$, let $\mathsf{decode}(x)(\alpha, \beta)$ holds if and only if $A(\mathsf{cut}(x)) \wedge t_{x(0)}^{\Gamma(\mathsf{cut}(x), \alpha + \omega)}(\alpha) = \beta$. Define $\mathsf{GC}_{\beta, \gamma}$ by $x \in \mathsf{GC}_{\beta, \gamma}$ if and only if $\mathsf{decode}(x)(\beta, \gamma)$ holds.

Suppose $f : \omega_1 \to \omega_1$. Let z be the real and n be the natural number obtained by applying Fact 4.17 to f. Let $x \in \mathbb{R}$ be such that $x(0) = n$ and $\mathsf{cut}(x) = z^\sharp$. Then $\mathsf{decode}(x) = f$. Such an x will be called a sharp code for f.

Let $\beta < \omega_1$ and $\gamma < \omega_1$, note that $\mathsf{GC}_{\beta, \gamma}$ consists of those $x \in \mathbb{R}$ such that the following holds
(i) $A(\mathsf{cut}(x))$
(ii) β is in the wellfounded part of $\Gamma(\mathsf{cut}(x), \beta + \omega)$
(iii) $t_{x(0)}^{\Gamma(\mathsf{cut}(x), \beta + \omega)}(\beta) = \gamma$.

By the discussion earlier, $\mathsf{GC}_{\beta, \gamma}$ is $\boldsymbol{\Delta}^1_1$. Note that since $t_{x(0)}$ is a Skolem function, $\tau_{x(0)}^{\Gamma(\mathsf{cut}(x), \beta + \omega)}$ is a function on its domain.

Now suppose $A \subseteq \mathsf{GC}_\beta$ is $\boldsymbol{\Sigma}^1_1$. Let $E = \{v \in \mathbb{R} : (\exists x)(x \in A \wedge v = \mathsf{ST}(\Gamma(\mathsf{cut}(x), \beta + \omega), t_{x(0)}^{\Gamma(\mathsf{cut}(x), \beta + \omega)})\}$. Then E is a $\boldsymbol{\Sigma}^1_1$ subset of WO. By the boundedness lemma, there is some $\delta < \omega_1$ so that for all $v \in E$, $\mathrm{ot}(v) < \delta$. Thus $A \subseteq \bigcup_{\gamma < \delta} \mathsf{GC}_{\beta, \gamma}$.

It has been shown that $(\boldsymbol{\Sigma}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta, \gamma} : \beta < \omega_1, \gamma < \omega_1)$ is a good coding system for ${}^{\omega_1}\omega_1$. $\square$

Next, one will give the argument of Kechris that ω_1 is a strong partition cardinal. This proof uses category and the Kechris-Woodin generic coding idea. However at ω_1, the generic coding function is essentially trivial and exists without even AD. For many other purposes, the generic coding function is very useful.

Note that in Martin's proof, the indiscernibility models are used to make complexity computations. In the argument of Kechris, category quantifiers will be used to ensure sets have the correct complexity.

First some results on the Banach-Mazur games. See [16] Section 21.C and 21.D for the proofs of these results:

Fact 4.19. *(Banach-Mazur Game) Assume* ZF. *Let* $A \subseteq {}^\omega\omega$. *Define* $G^*(A)$ *by*

$$
\begin{array}{c|ccccccc}
 & I & s_0 & & s_2 & & s_4 & \\
G^*_A & & & & & & & \\
 & II & & s_1 & & s_3 & & s_5
\end{array}
$$

Player 1 and 2 alternatingly play nonempty strings in ${}^{<\omega}\omega$. *Let* $x = s_0{}^\frown s_1{}^\frown s_2...$ *be the concatenation of the moves. Player 2 wins* G^*_A *if and only if* $x \in A$.

(1) A is comeager if and only if Player 2 has a winning strategy in G^*_A.

(2) There is some $s \in {}^{<\omega}\omega$ *so that* ${}^\omega\omega \setminus A$ *is comeager in* N_s *if and only if Player 1 has a winning strategy in* G^*_A.

Fact 4.20. *(Unfolded Banach-Mazur Game) Assume* ZF. *Let* $A \subseteq {}^\omega\omega$ *and* $B \subseteq {}^\omega\omega \times {}^\omega\omega$ *be such that for all* $x \in A$, *there exists a* $y \in {}^\omega\omega$ *so that* $B(x, y)$. *Let* $G^*_{A,B}$ *be the following game*

$$
\begin{array}{c|ccccccc}
 & I & s_0 & & s_2 & & s_4 & \\
G^*_{A,B} & & & & & & & \\
 & II & & s_1, y(0) & & s_3, y(1) & & s_5, y(2)
\end{array}
$$

Player 1 plays nonempty strings in ${}^{<\omega}\omega$. *Player 2 plays nonempty strings in* ${}^{<\omega}\omega$ *and an element of* ω. *Let* $x = s_0{}^\frown s_1{}^\frown s_2...$ *be the concatenation of the strings played by both player. Let* y *be the real produced by the extra elements of* ω *played by Player 2. Player 2 wins* $G^*_{A,B}$ *if and only if* $x \in A$ *and* $(x, y) \in B$.

If Player 2 wins $G^*_{A,B}$, then Player 2 wins G^*_A. If Player 1 wins $G^*_{A,B}$, then Player 1 wins G^*_A.

Corollary 4.21. *Assume* ZF. *Let* $A \subseteq {}^\omega\omega$ *be a* $\mathbf{\Sigma}^1_1$ *set. Let* $B \subseteq {}^\omega\omega \times {}^\omega\omega$ *be a* $\mathbf{\Pi}^0_1$ *set so that* $A = \pi_1[B]$. *A is comeager if and only if Player 2 has a winning strategy in* $G^*_{A,B}$.

Proof. Note that if $A = \pi_1[B]$, then the winning condition for Player 2 in $G^*_{A,B}$ is equivalent to merely $(x,y) \in B$. Thus in this case $G^*_{A,B}$ is a closed game. Thus the determinacy of such games hold under ZF. Now the result follows from Fact 4.19 and 4.20. $\qquad\square$

If $\alpha < \omega_1$, one can define a topology on ${}^\omega\alpha$ be declaring the basic open sets to be sets of the form $N^\alpha_s = \{f \in {}^\omega\alpha : f \supseteq s\}$, where $s \in {}^{<\omega}\alpha$. Using this topology, one can define the notions of comeagerness and meagerness for ${}^\omega\alpha$. As α is countable, ${}^\omega\alpha$ with this topology is homeomorphic to ${}^\omega\omega$. Observe that the set $\mathrm{surj}_\alpha = \{f \in {}^\omega\alpha : f \text{ is a surjection}\}$ is a comeager subset of ${}^\omega\alpha$. If $A \subseteq \mathbb{R} \times {}^\omega\alpha$, then one writes $(\forall^*_\alpha f)A(x,f)$ as an abbreviation for the statement that $A(x,f)$ holds for comeagerly many f in ${}^\omega\alpha$.

Corollary 4.22. *Assume* ZF. *Let* $A \subseteq {}^\omega\omega \times {}^\omega\omega$ *be a* $\mathbf{\Sigma}^1_1$ *set. Then* $A_0(x) \Leftrightarrow (\forall^*_\omega y)A(x,y)$ *is* $\mathbf{\Sigma}^1_1$ *and* $A_1(x) \Leftrightarrow (\forall^*_\omega y)\neg A(x,y)$ *is* $\mathbf{\Pi}^1_1$.

Proof. Let $B \subseteq {}^\omega\omega \times {}^\omega\omega \times {}^\omega\omega$ be such that $A = \{(x,y) : (\exists z)((x,y,z) \in B)\}$ where $B \in \mathbf{\Pi}^0_1$. For each $x \in {}^\omega\omega$, let $B_x = \{(y,z) : (x,y,z) \in B\}$ and $A_x = \{y : (x,y) \in A\}$. Note that $B_x \in \mathbf{\Pi}^0_1$ and $\pi_1[B_x] = A_x$.

If σ is a Player 1 strategy and τ is a Player 2 strategy for a game of the form $G^*_{C,D}$ for some appropriate C and D, then let $x_{\sigma*\tau}$ be the real produced by the concatenation of the finite strings played by each player. Let $y_{\sigma*\tau}$ be the auxiliary sequence produced by the moves of τ.

Note that $A_0(x)$ if and only if Player 2 has a winning strategy in $G^*_{A_x,B_x}$ if and only if $(\exists \tau)(\forall \sigma)((x_{\sigma*\tau}, y_{\sigma*\tau}) \in B_x)$ by Fact 4.20. Since $B_x \in \mathbf{\Pi}^0_1$ and $\mathbf{\Pi}^0_1$ is closed under $\forall^\mathbb{R}$, the latter expression is $\mathbf{\Sigma}^1_1$.

For $s \in {}^{<\omega}\omega$, let $\Phi_s : N_s \to {}^\omega\omega$ be the canonical homeomorphism between N_s and ${}^\omega\omega$. Let $A^s \subseteq {}^\omega\omega \times {}^\omega\omega$ be defined by $(x,y) \in A^s \Leftrightarrow (x, \Phi_s^{-1}(y)) \in A$. Let $B^s \subseteq {}^\omega\omega \times {}^\omega\omega \times {}^\omega\omega$ be defined by $(x,y,z) \in B^s$ if and only if $(x, \Phi_s^{-1}(y), z) \in B$.

For the second statement, note that $\neg A_1(x)$ if and only if $\neg((\forall^*_\omega y)\neg A(x,y))$. Since $\mathbf{\Pi}^1_1$ sets have Baire property, the latter holds if and only if there exists a $s \in {}^{<\omega}2$ so that A_x is comeager in N_s. Now there exists an $s \in {}^{<\omega}2$ so that A_x is comeager in N_s if and only if there exists an $s \in {}^{<\omega}2$ so that $\Phi_s[A_x \cap N_s]$ is comeager if and only if there exists an $s \in {}^{<\omega}2$ so that Player 2 has a winning strategy in $G^*_{(A^s)_x,(B^s)_x}$. As before, the last expression can be checked to be $\mathbf{\Sigma}^1_1$. Hence A_1 is $\mathbf{\Pi}^1_1$. $\qquad\square$

The following is the (essentially trivial) version of the Kechris-Woodin generic coding function for ω_1. (See [18].)

Fact 4.23. *There is a continuous function* $G : {}^\omega\omega_1 \to \mathrm{WO}$ *so that for all* $\ell \in {}^\omega\omega_1$ *such that* $\ell(0) = \{\ell(n+1) : n \in \omega\}$, $G(\ell) \in \mathrm{WO}$ *and* $\mathrm{ot}(G(\ell)) = \ell(0)$.

Proof. For each ℓ as above, let $A_\ell = \{n \in \omega \setminus \{0\} : (\forall m)(\ell(n) = \ell(m) \Rightarrow n \leq m)\}$. Let $G(\ell)$ be the wellordering with domain A_ℓ so that $m <_{G(\ell)} n$ if and only if $\ell(m) < \ell(n)$. $G(\ell) \in \mathrm{WO}$ and $\mathrm{ot}(G(\ell)) = \ell(0)$.

This function is continuous in the sense that for any n, $G(\ell) \restriction n$ is determined by $\ell \restriction m$ for some m. $\qquad\square$

Fact 4.24. *Assume* ZF + AD. *Let* $f : \omega_1 \to \omega_1$. *There is a Player 2 strategy so that for all* $v \in \mathrm{WO}$, $\{(\alpha,\beta) : (\exists n)((\tau(v)_n)_0 \in \mathrm{WO}_\alpha \wedge (\tau(v)_n)_1 \in \mathrm{WO}_\beta)\} = f \restriction \gamma$ *for some* $\gamma > \mathrm{ot}(v)$.

Here $\tau(v)$ *is* $(v * \tau)_{\mathrm{odd}}$, *i.e. the real produced by Player 2 when played against Player 1 playing the bits of* v. *Also* $f \restriction \gamma = \{(\alpha,\beta) \in f : \alpha < \gamma\}$. *Recall that if* $x \in \mathbb{R}$, x_n *is defined by* $x_n(k) = x(\langle n,k \rangle)$. *This is the* n^{th} *section of* x. *This result states that when given* $v \in \mathrm{WO}$, *the integer sections of* $\tau(v)$ *codes* f *up to some ordinal* γ *greater than* $\mathrm{ot}(v)$.

Proof. Consider the game S_f defined by

	I	$v(0)$		$v(1)$		$v(2)$		$v(3)$			v
S_f											
	II		$r(0)$		$r(1)$		$r(2)$		$r(3)$		r

Player 2 wins if and only if $v \in \mathrm{WO}$ implies that $\{(\alpha, \beta) : (\exists n)((r_n)_0 \in \mathrm{WO}_\alpha \wedge (r_n)_1 \in \mathrm{WO}_\beta)\} = f \upharpoonright \gamma$ for some $\gamma > \mathrm{ot}(v)$.

By essentially the same bounding argument of Fact 4.16, one has the Player 2 must have the winning strategy in this game. $\qquad \square$

Remark 4.25. In the above statement, it is very important that some section of $\tau(v)$ contains a code for the image of $f(\alpha)$. To search for $f(\alpha)$, which could be quite large in comparison to α, would push the complexity of $\mathsf{GC}_{\alpha,\beta}$ beyond $\boldsymbol{\Delta}_1^1$. Instead, a particular winning strategy τ will take a code v for α and ouput $\tau(v)$ which magically contains codes for $f(\alpha)$ among its integer sections, $\{(\tau(v)_n)_1 : n \in \omega\}$.

Theorem 4.26. *(Kechris' proof) Assume* $\mathsf{ZF} + \mathsf{AD}$. ω_1 *is* ω_1-*reasonable.*

Proof. One will define a good coding system for $^{\omega_1}\omega_1$ witnessing ω_1-reasonableness of ω_1.

The associated pointclass is $\boldsymbol{\Sigma}_1^1$.

Let $\mathsf{add} : \omega_1 \times {}^\omega\omega_1 \to {}^\omega\omega_1$ be defined by

$$\mathsf{add}(\alpha, g)(n) = \begin{cases} \alpha & n = 0 \\ g(n-1) & n > 0 \end{cases}$$

$\mathsf{add}(\alpha, g)$ simply inserts α at the beginning of g.

In some fix recursive manner, every real x codes a Player 2 strategy τ_x.

Now fix $\alpha, \beta \in \omega_1$. Define the formula $\varphi_\beta^\alpha(x, z)$ to state that

$$(\exists n)([(\tau_x(z)_n)_0 \in \mathrm{WO}_\alpha \wedge (\tau_x(z)_n)_1 \in \mathrm{WO}_\beta] \wedge (\forall m < n)((\tau_x(z)_m)_0 \notin \mathrm{WO}_\alpha)).$$

This formula defines a $\boldsymbol{\Delta}_1^1$ relation using elements of WO_α and WO_β as parameters.

Let $\phi_\beta^\alpha(x)$ be the statement $(\forall_\alpha^* g)\varphi_\beta^\alpha(x, G(\mathsf{add}(\alpha, g)))$, where G is the generic coding function of Fact 4.23. Note that since surj_α is comeager in $^\omega\alpha$, $(\forall_\alpha^* g)(G(\mathsf{add}(\alpha, g)) \in \mathrm{WO}_\alpha)$.

Now fix a bijection $B : \omega \to \alpha$. Let $T : {}^\omega\omega \to {}^\omega\alpha$ be defined by $T(r) = B \circ r$. Define $\tilde{G} : {}^\omega\omega \to \mathrm{WO}$ by $\tilde{G}(r) = G(\mathsf{add}(\alpha, T(r)))$. Now $\tilde{G}$ is a continuous function with the property that for comeagerly many $r \in {}^\omega\omega$ (in the usual sense of comeager), $\tilde{G}(r) \in \mathrm{WO}_\alpha$.

$\phi_\beta^\alpha(x)$ is equivalent to $(\forall_\omega^* r)\varphi_\beta^\alpha(x, \tilde{G}(r))$. Since φ_β^α defines a $\boldsymbol{\Delta}_1^1$ relation (in parameters from WO_α and WO_β), the collection of (x, r) satisfying $\varphi_\beta^\alpha(x, \tilde{G}(r))$ is $\boldsymbol{\Delta}_1^1$ using parameters from WO_α, WO_β, and a parameter coding the continuous function $\tilde{G}$. Now the set defined by $\phi_\beta^\alpha(x)$ is $\boldsymbol{\Delta}_1^1$ by Corollary 4.22. (Note this is not done uniformly in the α and β.)

If $x \in \mathbb{R}$, define $\mathsf{decode}(x)(\alpha, \beta)$ if and only if $\phi_\beta^\alpha(x)$.

Define $\mathsf{GC}_{\alpha,\beta} = \{x \in \mathbb{R} : \phi_\beta^\alpha(x)\}$. $\mathsf{GC}_{\alpha,\beta}$ is $\boldsymbol{\Delta}_1^1$.

For any $f : \omega_1 \to \omega_1$, Fact 4.24 states that there is some x so that $\mathsf{decode}(x) = f$.

Fix $\alpha < \omega_1$. For each $x \in {}^\omega\omega$, let $\psi_0(x, z)$ assert that $(\exists n)((\tau_x(z)_n)_0 \in \mathrm{WO}_\alpha)$. $\psi_0(x, z)$ is $\boldsymbol{\Delta}_1^1$ using any code of α as a parameter. Let $\psi_1(x, y, z)$ be the conjunction of the following statements
(1) $\psi_0(x, z) \wedge \psi_0(y, z)$.
(2) There exists $w_0, w_1 \in \mathbb{R}$, $n_0, n_1 \in \omega$ so that
    (2a) $(\tau_x(z)_{n_0})_0 \in \mathrm{WO}_\alpha$, $(\tau_x(z)_{n_0})_1 = w_0$, and for all $m < n_0$, $(\tau_x(z)_m)_0 \notin \mathrm{WO}_\alpha$.
    (2b) $(\tau_y(z)_{n_0})_0 \in \mathrm{WO}_\alpha$, $(\tau_y(z)_{n_1})_1 = w_1$, and for all $m < n_1$, $(\tau_y(z)_m)_0 \notin \mathrm{WO}_\alpha$.
    (2c) $w_0 <_{\Sigma_1^1} w_1$, where $<_{\Sigma_1^1}$ is the Σ_1^1 relation witnessing that the ordertype function is a Π_1^1-norm.

Observe that $\psi_1(x, y, z)$ is $\boldsymbol{\Sigma}_1^1$.

Let $A \subseteq \mathsf{GC}_\alpha$ and A is $\boldsymbol{\Sigma}_1^1$. Define a relation $\prec$ on $^\omega\omega$ by $x \prec y$ if and only if

$$x \in A \wedge y \in A \wedge (\forall_\alpha^* g)\psi_1(x, y, G(\mathsf{add}(\alpha, g))).$$

By Corollary 4.22, $\prec$ is a $\boldsymbol{\Sigma}_1^1$ relation on $^\omega\omega$.

By checking the various definitions, one can see that for all $x, y \in A$, $\mathsf{decode}(x)(\alpha) < \mathsf{decode}(y)(\alpha)$ if and only if $x \prec y$.

Since $\prec$ is a $\boldsymbol{\Sigma}_1^1$ relation, it is an ω-Suslin relation. By the Kunen-Martin theorem ([17] Section 7), the rank must be less than ω_1. Since ω_1 is regular, there is some $\gamma < \omega_1$ so that $A \subseteq \bigcup_{\beta < \gamma} \mathsf{GC}_\beta$.

(Some presentations of the Kunen-Martin theorem uses $\mathsf{DC}_\mathbb{R}$, but with some care, one can remove the use of $\mathsf{DC}_\mathbb{R}$. Another approach in the case of ω-Suslin relations is to observe that if K denotes a tree witnessing a relation is ω-Suslin, one can prove the Kunen-Martin theorem in $L[K] \models \mathsf{AC}$ and argue in $L[K]$ that the Kunen-Martin tree is wellfounded there and hence in the real world. Then one shows that this tree still works in the real world V. Alternatively, if one assumes Kechris's result that $L(\mathbb{R}) \models \mathsf{AD}$ implies $L(\mathbb{R}) \models \mathsf{DC}_\mathbb{R}$, then one can absorb this problem into $L(\mathbb{R})$ and apply the Kunen-Martin Theorem in $L(\mathbb{R})$.)

It has been shown that $(\boldsymbol{\Sigma}_1^1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \omega_1, \gamma < \omega_1)$ is a good coding system for $^{\omega_1}\omega_1$. $\qquad\square$

Corollary 4.27. *(Martin) Assume* $\mathsf{ZF} + \mathsf{AD}$. $\omega_1 \to_* (\omega_1)_2^{\omega_1}$.

Definition 4.28. Let $\lambda \leq \omega_1$. Let $\mu_{\omega_1}^\lambda$ consists of those subsets X of $[\omega_1]^\lambda$ which contain a set of the form $[C]_*^\lambda$ where C is a club subset of ω_1. Each $\mu_{\omega_1}^\lambda$ is a countably complete ultrafilter.

Corollary 4.29. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* $\lambda \leq \omega_1$. $\prod_{[\omega_1]^\lambda} \omega_1/\mu_{\omega_1}^\lambda = (\prod_{[\omega_1]^\lambda} \omega_1/\mu_{\omega_1}^\lambda)^{L(\mathbb{R})}$.
Since $L(\mathbb{R}) \models \mathsf{DC}$ *by a result of Kechris* [15], $\prod_{[\omega_1]^\lambda} \omega_1/\mu_{\omega_1}^\lambda$ *is an ordinal.*

Proof. Note that the good coding system for $^\lambda\omega_1$ constructed above belong to $L(\mathbb{R})$. Now apply Theorem 3.9. $\qquad\square$

Remark 4.30. This section presented two proofs of the strong partition property for ω_1. The original proof of Martin uses indiscernibility. Many early results were proved using this indiscernibility idea. For example, Kunen showed in $\mathsf{AD} + \mathsf{DC}$ that δ_3^1 is a weak partition cardinal and $\delta_3^1 = \aleph_{\omega+1}$.

The second proof of Kechris is perhaps the simplest proof using classical descriptive set theoretic ideas. The ideas of the Kechris-Woodin generic coding function is very useful in various different settings.

However, no proof in the flavor of these two arguments is known to establish the strong partition property at δ_3^1. There is another proof of the strong partition property for ω_1 due to Jackson. It uses ideas such as the Kunen tree and an analysis of all the measures on ω_1 due to Kunen. An exposition can be found in [10], [12], and [11]. More generally, Jackson developed the theory of description. This theory produces the only known proof that δ_3^1 and in fact all δ_{2n+1}^1 are strong partition cardinal under $\mathsf{AD} + \mathsf{DC}$. Jackson has also computed the identity of these cardinals. For example, $\delta_5^1 = \aleph_{\omega^{\omega^\omega}+1}$ and in fact there is a general formula for computing δ_{2n+1}^1. See [11] for more information. It should noted that Jackson's proof that ω_1 is a strong partition cardinal, Kunen's result that δ_3^1 is a weak partition cardinal, and all of Jackson's results mentioned above using description theory are proved using DC. The original Martin proof and the Kechris proof of the strong partition property of ω_1 are within $\mathsf{ZF} + \mathsf{AD}$.

5. Kunen Functions and Partition Properties at ω_2

Fact 5.1. *Let* μ *be a normal measure on a cardinal* κ. *Let* $A \in \mu$, *let* $\mathsf{enum}_A : \omega_1 \to A$ *be the increasing enumeration of* A. *Then* $\{\alpha \in A : \mathsf{enum}_A(\alpha) = \alpha\} \in \mu$.
Let $f : \kappa \to \kappa$ *and* $X \subseteq f[\kappa]$ *so that* $\{\alpha < \kappa : f(\alpha) \in X\} \in \mu$, *then* $\mathsf{enum}_X =_\mu f$.

Proof. Suppose not. Then $B = \{\alpha \in A : \alpha < \mathsf{enum}_A(\alpha)\} \in \mu$. Since enum_A is an order-preserving function, $B = \{\alpha \in A : \mathsf{enum}_A^{-1}(\alpha) < \alpha\}$. By normality, there is some $C \subseteq B$ and $\gamma < \kappa$ so that for all $\alpha \in C$, $\mathsf{enum}_A^{-1}(\alpha) = \gamma$. This contradicts the injectiveness of enum_A.

For the second statement: Let $A = \{\alpha : f(\alpha) \in X\}$. Note that $f \circ \mathsf{enum}_A = \mathsf{enum}_X$. Let $B \subseteq A$ with $B \in \mu$ be such that $\mathsf{enum}_A(\alpha) = \alpha$ for all $\alpha \in B$. Then for all $\gamma \in B$, $\mathsf{enum}_X(\alpha) = f(\mathsf{enum}_A(\alpha)) = f(\alpha)$. Hence $\mathsf{enum}_X =_\mu f$. $\qquad\square$

The next result states that an ultrapower of a strong partition cardinal by a measure is a cardinal as long as that ultrapower is well founded.

Fact 5.2. *(Martin) Let* κ *be a strong partition cardinal and* μ *be a measure on* κ. *If* $^\kappa\kappa/\mu$ *is wellfounded, then* $^\kappa\kappa/\mu$ *is a cardinal.*
Assuming $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$. *If* $\kappa < \Theta$ *is a strong partition cardinal and* μ *is a measure on* κ, *then* $^\kappa\kappa/\mu$ *is a cardinal.*

Proof. Suppose there is an $h \in {}^\kappa \kappa/\mu$ and an injection $\Phi : {}^\kappa \kappa/\mu \to \prod_\kappa h/\mu$, where $\prod_\kappa h/\mu$ is the collection of $[f]_\mu$ such that $f <_\mu h$. Note that $\prod_\kappa h/\mu$ is the initial segment of ${}^\kappa \kappa/\mu$ determined by $[h]_\mu$.

Let $\mathcal{T} = (\kappa \times 2, \sqsubset)$ denote the lexicographic ordering on $\kappa \times 2$. If $F : \kappa \times 2 \to \kappa$ is a $\mathcal{T}$-increasing function, then let $f_0, f_1 : \kappa \to \kappa$ be defined by $f_0(\alpha) = F(\alpha, 0)$ and $f_1(\alpha) = F(\alpha, 1)$. Note that f_0 and f_1 are increasing functions.

Define $P : [\kappa]^{\mathcal{T}} \to 2$ by

$$P(F) = \begin{cases} 0 & \Phi([f_0]_\mu) < \Phi([f_1]_\mu) \\ 1 & \text{otherwise} \end{cases}$$

Since $\mathrm{ot}(\mathcal{T}) = \kappa$, $\kappa \to (\kappa)_2^\kappa$ implies that there is an $E \subseteq \kappa$ with $|E| = \kappa$ which is homogeneous for this partition.

Suppose P is homogeneous taking value 1. For each $n \in \omega$, define $g_n : \kappa \to \kappa$ by $g_n(\alpha) = \mathsf{enum}_E(\omega \cdot \alpha + n)$. Note that if $m < n$, then $g_m <_\mu g_n$. For any $m < n$, define $G^{m,n} : \mathcal{T} \to \kappa$ by $G^{m,n}(\alpha, 0) = g_m(\alpha)$ and $G^{m,n}(\alpha, 1) = g_n(\alpha)$. Note that $G^{m,n}$ is $\mathcal{T}$-increasing and $G^{m,n} \in [\kappa]^{\mathcal{T}}$. Since $P(G^{m,n}) = 1$, one has that $\Phi([g_n]_\mu) = \Phi([G_1^{m,n}]_\mu) \leq \Phi([G_0^{m,n}]_\mu) = \Phi([g_m]_\mu)$. Since Φ is an injection, one must actually have $\Phi([g_n]_\mu) < \Phi([g_m]_\mu)$. Thus $\langle \Phi([g_k]_\mu) : k \in \omega \rangle$ is an infinite decreasing sequence in ${}^\kappa \kappa/\mu$. This contradicts the assumption that ${}^\kappa \kappa/\mu$ is wellfounded.

Thus E must be homogeneous for P taking value 0.

Let $\ell \in {}^\kappa \kappa$ be such that $[\ell]_\mu > [h]_\mu$ and $\ell(\alpha) > 0$ for all $\alpha \in \kappa$. Let $U = \{(\alpha, \beta) \in \kappa \times \kappa : \beta < \ell(\alpha)\}$. Let $\mathcal{U}$ denote $(U, \prec)$ where $\prec$ denotes the lexicographic ordering of U. Again, $\mathrm{ot}(\mathcal{U}) = \kappa$.

Let $F = \{\mathsf{enum}_E(\omega \cdot \alpha) : \alpha < \kappa\}$. Note that $|F| = \kappa$. Pick some $K : \mathcal{U} \to F$ which is order-preserving. For any $f <_\mu \ell$, define $k_f : \kappa \to \kappa$ by

$$k_f(\alpha) = \begin{cases} K(\alpha, f(\alpha)) & f(\alpha) < \ell(\alpha) \\ K(\alpha, 0) & \text{otherwise} \end{cases}.$$

Note that if $f <_\mu g <_\mu \ell$, then $k_f <_\mu k_g$ and that for all $f <_\mu \ell$, $k_f \in [F]^\kappa$.

Define $\Psi : \prod_\kappa \ell/\mu \to \prod_\kappa h/\mu$ as follows: Let $A < [\ell]_\mu$. Pick $f \in A$, i.e. a representative of A. Let $\Psi(A) = \Phi([k_f]_\mu)$. One can check that Ψ is well defined.

The next claim is that Ψ is order preserving: Suppose $A < B < [\ell]_\mu$. Let $f \in A$ and $g \in B$ be representatives for A and B, respectively. Since $k_f <_\mu k_g$, the set $C = \{\alpha \in \kappa : k_g(\alpha) \leq k_f(\alpha)\} \notin \mu$. Define $k_g' : \kappa \to E$ by $k_g'(\alpha) = k_g(\alpha)$ if $\alpha \notin C$ and $k_g'(\alpha)$ is the next element of E greater than $k_f(\alpha)$ if $\alpha \in C$. The purpose of defining F in the manner above was to ensure that between two successive elements of F, there are at least ω many point of E in between. From this, one can verify that $k_g' \in [E]^\kappa$ and $k_f(\alpha) < k_g'(\alpha) < k_f(\alpha + 1)$ for all $\alpha < \kappa$. Note that $[k_g']_\mu = [k_g]_\mu$ since k_g and k_g' agree off of $C \notin \mu$.

Let $F^{k_f, k_g'} : \mathcal{T} \to \kappa$ be defined by

$$F^{k_f, k_g'}(\alpha, i) = \begin{cases} k_f(\alpha) & i = 0 \\ k_g'(\alpha) & i = 1 \end{cases}$$

Again using the main property of F, one can verify that $F^{k_f, k_g'} \in [\kappa]^{\mathcal{T}}$. Thus $P(F^{k_f, k_g'}) = 0$. Thus $\Psi(A) = \Phi([k_f]_\mu) < \Phi([k_g']_\mu) = \Phi([k_g]_\mu) = \Psi(B)$.

It has been shown that Ψ is order preserving. However, this is not possible since $[h]_\mu < [\ell]_\mu$. This completes the proof. $\qquad\square$

Fact 5.3. *Let κ be a measurable cardinal possessing a normal κ-complete measure μ. Let $\epsilon < \kappa$. Let $\mathcal{T}^\epsilon = (\kappa \times \epsilon, \sqsubset)$ where $\sqsubset$ is the lexicographic ordering. For $F : \mathcal{T}^\epsilon \to \kappa$ which is order preserving and $\alpha < \epsilon$, let $F^\alpha \in [\kappa]^\kappa$ be defined by $F^\alpha(\gamma) = F(\gamma, \alpha)$.*

Let $\langle f_\alpha : \alpha < \epsilon \rangle$ be a sequence in $[\kappa]^\kappa$ such that for all $\alpha < \beta < \epsilon$, $f_\alpha <_\mu f_\beta$. Then there is an $F \in [\kappa]^{\mathcal{T}^\epsilon}$ so that for all $\alpha < \epsilon$, $F^\alpha =_\mu f_\alpha$.

Moreover, if $D \subseteq \kappa$ with $|D| = \kappa$ and each $f_\alpha \in [D]^\kappa$, then one can find $F \in [D]^{\mathcal{T}^\epsilon}$ with the above property.

Proof. Suppose $F(\alpha', \beta')$ has been defined for all $(\alpha', \beta') \sqsubset (\alpha, \beta)$. Then let $F(\alpha, \beta)$ be the least element in the range of f_β greater than $F(\alpha', \beta')$ for all $(\alpha', \beta') \sqsubset (\alpha, \beta)$.

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The claim is that F has the desired properties: This is proved by induction on $\beta < \epsilon$.

Consider $\beta = 0$. Suppose $A = \{\alpha < \kappa : F(\alpha, 0) \neq f_0(\alpha)\} \in \mu$. This means for each $\alpha \in A$, there is some $\alpha' < \alpha$, and some $\beta < \epsilon$ so that $F(\alpha', \beta) \geq f_0(\alpha)$. Let $g : \kappa \to \kappa$ be defined by $g(\alpha)$ is the least such α' with the property above if $\alpha \in A$ and 0 if $\alpha \notin A$. Note that $g(\alpha) < \alpha$ for all $\alpha \in A$. By the normality of μ, there is some $A' \subseteq A$ with $A' \in \mu$ and some δ so that $g(\alpha) = \delta$ for all $\alpha \in A'$. This means for all $\alpha \in A' \in \mu$, $f_0(\alpha) \leq \sup\{F(\delta, \beta) : \beta < \epsilon\}$. By the κ-completeness of μ, one may even find a β^* and a $A'' \in \mu$ so that for all $\alpha \in A''$, $f_0(\alpha) \leq F(\delta, \beta^*)$. This is clearly impossible. This shows that $F^0 =_\mu f_0$.

Suppose $\beta < \epsilon$ is such that for all $\beta' < \beta$, $F^{\beta'} = f_{\beta'}$. Suppose $\neg(F^\beta =_\mu f_\beta)$. Then $A = \{\alpha : F^\beta(\alpha) \neq f_\beta(\alpha)\} \in \mu$. Since $f_0(\alpha) = F^0(\alpha)$ for almost all α, it can not be the case that for almost all $\alpha \in A$, there is some $\alpha' < \alpha$ and some $\beta' < \epsilon$ so that $F^{\beta'}(\alpha') \geq f_\beta(\alpha)$. Thus for almost all $\alpha \in A$, there is some $\beta' < \beta$ so that $F^{\beta'}(\alpha) \geq f_\beta(\alpha)$. By the κ-additivity of μ, there is some $\beta^* < \beta$ so that for almost all α, $F^{\beta^*}(\alpha) \geq f_\beta(\alpha)$. By the induction hypothesis, one has that $F^{\beta^*} =_\mu f_{\beta^*}$. One has shown that $f_\beta \leq_\mu f_{\beta^*}$ despite the fact that $\beta^* < \beta$. Contradiction. The result has been established. $\square$

Fact 5.4. *(Martin) Let κ be a strong partition cardinal and μ is a normal κ-complete measure on a cardinal κ. If ${}^\kappa\kappa/\mu$ is wellfounded, then ${}^\kappa\kappa/\mu$ is a regular cardinal.*

Assuming $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$, if $\kappa < \Theta$ is a strong partition cardinal and μ is a normal κ-complete measure on κ, then ${}^\kappa\kappa/\mu$ is a regular cardinal.

Proof. By Fact 5.2, $\delta = \mathrm{ot}({}^\kappa\kappa/\mu)$ is a cardinal. Assume that ${}^\kappa\kappa/\mu$ is not a regular cardinal. There is some $\gamma < \delta$ and an increasing function $\Phi : \gamma \to \delta$ which is cofinal.

Let $\mathcal{T}^2$, F^0, F^1 come from the notation of Fact 5.3. Define a partition $P : [\kappa]^{\mathcal{T}^2} \to 2$ by

$$P(F) = \begin{cases} 0 & (\exists \beta < \gamma)([F^0]_\mu < \Phi(\beta) < [F^1]_\mu) \\ 1 & \text{otherwise} \end{cases}$$

Let $D \subseteq \kappa$ with $|D| = \kappa$ be a homogeneous set for P.

(Case I) Assume D is homogeneous for P taking value 0.

By Fact 5.1, every nonconstant function $f \in {}^\kappa D$ has some $g \in [D]^\kappa$ so that $f =_\mu g$. Thus one can show that $[D]^\kappa/\mu$ has ordertype δ.

For each α, let f_α and $f_{\alpha+1}$ be two elements of $[D]^\kappa$ which represents the elements of $[D]^\kappa/\mu$ of rank α and $\alpha + 1$ respectively. By Fact 5.3, there is some $F : [D]^{\mathcal{T}^2} \to \kappa$ so that $F^0 = f_\alpha$ and $F^1 = F_{\alpha+1}$. Then $P(F) = 0$. Hence let ν_α be the least ordinal less than γ so that $[f_\alpha] = [F^\alpha]_\mu < \Phi(\nu_\alpha) < [F^{\alpha+1}]_\mu = [f_{\alpha+1}]_\mu$. One can check that ν_α depends only on α and not on the choice of f_α and $f_{\alpha+1}$. Also if $\alpha \neq \alpha'$, then $\nu_\alpha < \nu_{\alpha'}$. This gives an order preserving map from δ into $\gamma < \delta$ which is impossible.

(Case II) Assume D is homogeneous for P taking value 1.

As argued above, $[D]^\kappa/\mu$ has ordertype δ so $[D]^\kappa/\mu$ is a cofinal subset of κ^κ/μ. Fix $f^* \in [D]^\kappa/\mu$. Since Φ is a cofinal map, there is some $\gamma^* < \gamma$ so that $[f^*]_\mu < \Phi(\gamma^*)$. Then there is some $f' \in [D]^\kappa$ so that $[f^*]_\mu < \Phi(\gamma^*) < [f']_\mu$. By Fact 5.3, there is some $F \in [D]^{\mathcal{T}^2}$ so that $F^0 =_\mu f^*$ and $F^1 =_\mu f'$. Thus $P(F) = 0$. This is impossible since D is homogeneous for P taking value 1.

The failure of both cases would imply P has no homogeneous subset of size κ violating the assumption that κ is a strong partition cardinal. This completes the proof. $\square$

Definition 5.5. Let κ be a cardinal. Let μ be a κ-complete normal ultrafilter on κ.

A function $f : \kappa \to \kappa$ is a block function if and only if $\{\alpha \in \kappa : f(\alpha) < |\alpha|^+\} \in \mu$.

Let $\Xi : \kappa \times \kappa \to \kappa$. For each $\alpha < \kappa$, let $\delta_\alpha^\Xi = \sup\{\Xi(\alpha, \beta) : \beta < \alpha\}$. Let $\Xi_\alpha : \alpha \to \delta_\alpha^\Xi$ be defined by $\Xi_\alpha(\beta) = \Xi(\alpha, \beta)$.

Ξ is a Kunen function for f with respect to μ if and only if $K_f^\Xi = \{\alpha < \kappa : f(\alpha) \leq \delta_\alpha^\Xi \wedge \Xi_\alpha \text{ is a surjection}\} \in \mu$. (Here, when one says that Ξ_α is a surjection, one is considering Ξ_α as a function $\Xi_\alpha : \alpha \to \delta_\alpha^\Xi$.) K_f^Ξ is the set of α on which Ξ provides a bounding for f.

For $\beta < \kappa$, let $\Xi^\beta : \kappa \to \kappa$ be defined by $\Xi^\beta(\alpha) = \Xi(\alpha, \beta)$ where $\alpha > \beta$ and 0 otherwise. Ξ^β is a block function provided that $\{\alpha < \kappa : \Xi_\alpha \text{ is a surjection}\} \in \mu$.

Fact 5.6. *Assume* ZF. *Let μ be a normal measure on a cardinal κ. Suppose $f : \kappa \to \kappa$ is a block function which possesses a Kunen function Ξ with respect to μ. Suppose $G \in \prod_{\alpha \in \kappa} f(\alpha)/\mu$. Then there is a $\beta < \kappa$ so that $[\Xi^\beta]_\mu = G$*

Proof. Take any $g \in G$. Let $A = \{\alpha \in K_f^\Xi : g(\alpha) < f(\alpha)\}$. Note that for $\alpha \in A$, $g(\alpha) < f(\alpha) < \delta_\alpha^\Xi$. Define $\Phi : A \to \kappa$ by $\Phi(\alpha)$ is the least $\beta < \alpha$ so that $g(\alpha) = \Xi(\alpha, \beta)$ which exists since Ξ_α is a surjection onto δ_α^Ξ. Thus on A, Φ is a regressive function. By normality, there is some $\beta < \kappa$ so that $\Phi(\alpha) = \beta$ for μ-almost all α. Then $\Xi^\beta =_\mu g$. Hence $[\Xi^\beta]_\mu = G$. One can check this β does not depend on the initial choice of $g \in G$. $\square$

Definition 5.7. Let κ be a cardinal. Let μ be a normal measure on κ. Let h be a block function. Suppose h possesses a Kunen function Ξ with respect to μ. An ordinal $\beta < \kappa$ is a minimal code (relative to Ξ) if and only if for all $\gamma < \beta$, $\neg(\Xi^\gamma =_\mu \Xi^\beta)$. Let J_h^Ξ be the collection of β which are minimal codes and $\Xi^\beta <_\mu h$. Define an ordering $\prec_h^\Xi$ on J_h^Ξ by $\alpha \prec_h^\Xi \beta$ if and only if $\Xi^\alpha <_\mu \Xi^\beta$. By Fact 5.6, for every $G < [h]_\mu$, there is a unique $\beta \in J_h^\Xi$ so that $\Xi^\beta \in G$. In this way, one says that β is a minimal code for G or for any $g \in G$ with respect to Ξ.

Fact 5.8. *Let μ be a normal measure on a cardinal κ. Let $f : \kappa \to \kappa$ be a block function possessing a Kunen function Ξ with respect to μ. Then $\prod_{\alpha \in \kappa} f(\alpha)/\mu$, i.e. the initial segment of $^\kappa\kappa/\mu$ determined by $[f]_\mu$, is a wellordering.*

If every block function has a Kunen function, then $\prod_{\alpha < \kappa} |\alpha|^+/\mu$ is wellfounded.

For each $F \in \prod_{\alpha < \kappa} |\alpha^+|/\mu$, $F < \kappa^+$. Thus $\prod_{\alpha < \kappa} |\alpha|^+/\mu \leq \kappa^+$.

Proof. Let f be a block function possessing a Kunen function Ξ. Every $G \in \prod_{\alpha \in \kappa} f(\alpha)/\mu$ has a unique $\beta \in J_f^\Xi$ so that $[\Xi^\beta]_\mu = G$. There is a bijection of J_f^Ξ with $\prod_{\alpha \in \kappa} f(\alpha)/\mu$ given by $\beta \mapsto [\Xi^\beta]_\mu$. This shows that $|\prod_{\alpha \in \kappa} f(\alpha)/\mu| \leq \kappa$.

Now suppose that $\prod_{\alpha \in \kappa} f(\alpha)/\mu$ is not wellfounded. Then $\prec_f^\Xi$ is an illfounded linear ordering on $J_f^\Xi \subseteq \kappa$. Let β_0 be the least (in the usual ordinal ordering) element of J_f^Ξ so that the initial segment determined by β_0 in $(J_f^\Xi, \prec_h^\Xi)$ has no $\prec$-minimal element. Suppose β_n has been defined, let β_{n+1} be the least ordinal in J_f^Ξ which is $\prec_f^\Xi$-below β_n. This process defines a sequence of ordinals $\langle \beta_n : n \in \omega \rangle$ in J_f^Ξ.

Then $\langle \Xi^{\beta_n} : n \in \omega \rangle$ is a sequence with the property that $[\Xi^{\beta_n}]_\mu$ is an infinite decreasing sequence in $\prod_{\alpha < \kappa} f(\alpha)/\mu$. Let $D_n = \{\alpha \in \kappa : \Xi^{\beta_{n+1}}(\alpha) < \Xi^{\beta_n}(\alpha)\} \in \mu$. Then $\bigcap_{n \in \omega} D_n \in \mu$. Let $\xi \in \bigcap_{n \in \omega} D_n$. Then $\langle \Xi^{\beta_n}(\xi) : n \in \omega \rangle$ is an infinite decreasing sequence of ordinals. This is impossible.

Since f was arbitrary, this shows that $\prod_{\alpha < \kappa} |\alpha|^+/\mu$ is wellfounded. By the first paragraph, each $F \in \prod_{\alpha < \kappa} |\alpha|^+/\mu$ has cardinality less than or equal to κ. Thus $F < \kappa^+$. Thus $\prod_{\alpha < \kappa} |\alpha|^+/\mu \leq \kappa^+$. $\square$

Definition 5.9. Let μ be a normal measure on κ. Let $h : \kappa \to \kappa$ be a block function. Let Ξ be a Kunen function for h with respect to μ. By Fact 5.6, for each $G < [h]_\mu$, there is a minimal code $\beta \in J_h^\Xi$ so that $\Xi^\beta \in G$. Thus $(J_h^\Xi, \prec_h^\Xi)$ has the same ordertype as $[h]_\mu$. By Fact 5.8, $[h]_\mu$ is a wellordering. Let $\epsilon_h^\Xi \in \mathrm{ON}$ denote the ordertype of $([h]_\mu, <)$ which is equal to the ordertype of $(J_h^\Xi, \prec_h^\Xi)$. Let $\pi_h^\Xi : \epsilon_h^\Xi \to (J_h^\Xi, \prec_h^\Xi)$ be the unique order-preserving isomorphism.

Note that every function $f : \omega_1 \to \omega_1$ is (everywhere) a block function.

Theorem 5.10. *(Kunen) Assume* ZF + AD. *Every function $f : \omega_1 \to \omega_1$ has a Kunen function with respect to the club measure μ.*

Proof. The Kunen function is derived from the Kunen tree which will be defined below.

Recall that each $x \in {}^\omega\omega$ codes a relation R_x defined in Definition 4.1. Also WF denotes the collection of x so that R_x is a wellfounded relation. Let $\pi_{\mathrm{pair}} : \omega \times \omega \to \omega$ denote a fixed recursive bijection.

Let $S \subseteq \omega \times \omega_1$ be the tree of partial rankings of relations on ω which is defined as follows: $(s, \bar\alpha) \in S$ if and only if $|s| = |\bar\alpha|$ and for all $i, j < |s|$, if $s(\pi_{\mathrm{pair}}(i, j)) = 1$, then $\bar\alpha(i) < \bar\alpha(j)$.

If $(x, f) \in [S] = \{(y, g) : (\forall n)(y \restriction n, g \restriction n) \in S)\}$, then $x \in \mathrm{WF}$ since f is a ranking of R_x into ω_1. Conversely, if $x \in \mathrm{WF}$, then R_x has a ranking f with image in ω_1. Thus $(x, f) \in [S]$. It has been shown that $\mathrm{WF} = \pi_1[[S]]$ is the projection of $[S]$ onto the first coordinate.

Let T be a recursive tree on $\omega \times \omega$ so that $\pi_1[[T]] = \{a \in {}^\omega\omega : (\exists b)((a, b) \in [T])\} = {}^\omega\omega \setminus \mathrm{WF}$. The important observation is that $^\omega\omega \setminus \mathrm{WF}$ is a Σ_1^1 set which is $\mathbf{\Sigma}_1^1$-complete.

Let $\pi_{\text{seq}} : {}^{<\omega}\omega \to \omega$ be a recursive bijection. For each $x \in {}^{\omega}\omega$, let $\sigma_x : {}^{<\omega}\omega \to \omega$ be a strategy in the usual integer game defined by $\sigma_x(s) = n$ if and $x(\pi_{\text{seq}}(s)) = n$. Suppose $p \in {}^{<\omega}\omega$ and $s \in {}^{<\omega}\omega$, then let $\sigma_p * s$ denote the partial play where Player 1 uses the partial strategy σ_p and Player 2 plays the bits of s turn by turn. The game goes on for as long as p codes a response to the partial play that is produced each turn.

Define a tree K on $\omega \times \omega \times \omega_1 \times \omega \times \omega$ as follows: $(p, s, \bar{\alpha}, t, u) \in K$ if and only if the conjunction of the following holds:

(1) $(s, \bar{\alpha}) \in S$.
(2) $(t, u) \in T$.
(3) $\sigma_p * s$ is a substring of t.

The meaning of K becomes clear if one looks at what a path through K represents: If $(x, y, f, v, w) \in [K]$, then $\sigma_x(y) = v$, $(y, f) \in [S]$, and $(v, w) \in [T]$. Therefore, $y \in \text{WF}$ and $v \in \mathbb{R} \setminus \text{WF}$ since S and T are trees that project onto WF and $\mathbb{R} \setminus \text{WF}$.

The above defines the tree K. One now introduces an arbitrary function $f : \omega_1 \to \omega_1$. Consider the game G_f defined as follows:

	I	$y(0)$		$y(1)$		$y(2)$		$y(3)$			y
G_f											
	II		$v(0)$		$v(1)$		$v(2)$		$v(3)$		v

Player 2 wins if and only if $y \in \text{WO} \Rightarrow (v \in \text{WF} \wedge \text{rk}(T_v) > \sup\{f(\alpha) : \alpha \leq \text{ot}(y)\})$. Here $T_v = \{u \in {}^{<\omega}\omega : (u, v \restriction |u|) \in T\}$. Note that since T projects onto ${}^{\omega}\omega \setminus \text{WF}$, if $v \in \text{WF}$, then T_v is a wellfounded tree.

Player 2 must have the winning strategy: Suppose ρ is a Player 1 winning strategy. $\rho[{}^{\omega}\omega] \subseteq \text{WO}$ since otherwise Player 2 can win against ρ. Thus $\rho[{}^{\omega}\omega]$ is a $\mathbf{\Sigma}_1^1$ subset of WO. By the bounding principle, let $\mu < \omega_1$ be such that $\text{ot}(r) < \mu$ for all $r \in \rho[{}^{\omega}\omega]$. Let $\zeta = \sup\{f(\alpha) : \alpha < \mu\}$. $\zeta < \omega_1$ because ω_1 is regular. Since the projection of T is $\mathbb{R} \setminus \text{WF}$ which is a $\mathbf{\Sigma}_1^1$-complete set, one must have that $\sup\{\text{rk}(T_r) : r \in \text{WF}\} = \omega_1$. Choose some $v \in \text{WF}$ so that $\text{rk}(T_v) > \zeta$. If Player 2 plays v against ρ, Player 2 will win. This contradicts the fact that ρ is a Player 1 winning strategy.

Thus there is a winning strategy σ for the game G_f. Let $x \in {}^{\omega}\omega$ be such that $\sigma_x = \sigma$.

Let K_x be the tree consisting of $(s, \bar{\alpha}, t, u)$ so that there is an $n \in \omega$ such that $|s| = |\bar{\alpha}| = |t| = |u| = n$ and $(x \restriction n, s, \bar{\alpha}, t, u) \in K$. For each $\alpha < \omega_1$, let $K_x \restriction \alpha$ denote the restrict of K_x to ${}^{<\omega}(\omega \times \alpha \times \omega \times \omega)$. Note that $K_x \restriction \alpha$ is wellfounded. To see this, suppose otherwise. This means there is some $y, v, w \in {}^{\omega}\omega$ and $f \in {}^{<\omega}\alpha$ so that $(x, y, f, v, w) \in [K]$. Thus $\sigma_x(y) = \sigma(y) = v$, $y \in \text{WO}$, and $v \in \mathbb{R} \setminus \text{WF}$. However, σ is a Player 2 winning strategy and $y \in \text{WO}$, so one must have that $v \in \text{WF}$. This is a contradiction.

Let $\alpha \geq \omega$. Suppose $y \in \text{WO}$ with $\text{ot}(y) = \alpha$. Then there is a ranking f of R_y using ordinals below α. Let $v = \sigma_x(y) = \sigma(y)$. Since σ is a Player 2 winning strategy, $\text{rk}(T_v) > f(\text{ot}(y))$. Note that $K_x \restriction \alpha$ has a subtree which is isomorphic to T_v. In particular, this subtree is

$$\hat{T}_v = \{(y \restriction n, f \restriction n, t, u) \in K_x : n \in \omega \wedge |t| = |u| = n \wedge (t, u) \in T_v\}.$$

This implies that $K_x \restriction \alpha$ is a wellfounded tree of rank greater than $\text{rk}(T_v) > f(\alpha)$.

Note that there is a club set of α so that α is closed under the Gödel pairing function. Using this pairing function on α, one can define uniformly a bijection $\pi_\alpha : \alpha \to {}^{<\omega}\alpha$. For all α closed under the Gödel pairing function, define $\Xi(\alpha, \beta)$ to be the rank of $\pi_\alpha(\beta)$ in $K_x \restriction \alpha$ whenever $\beta < \alpha$ and $\pi_\alpha(\beta) \in K_x \restriction \alpha$. (Technically in the definition of a Kunen function, Ξ should be a function on $\omega_1 \times \omega_1$; however, the value of Ξ on (α, β) where $\beta \geq \alpha$ is not relevant in any applications.) If $\pi_\alpha(\beta)$ does not belong to $K_x \restriction \alpha$ then let $\Xi(\alpha, \beta) = 0$.

Ξ is a Kunen function for f. For any $\alpha \geq \omega$ which is closed under the Gödel pairing function, Ξ_α is a surjection onto the $\text{rk}(K_x \restriction \alpha)$. Using the notation of Definition 5.5, $\text{rk}(K_x \restriction \alpha) = \delta_\alpha^\Xi$. Also, it was shown above that $\text{rk}(K_x \restriction \alpha) > f(\alpha)$ and hence $\delta_\alpha^\Xi > f(\alpha)$. This verifies that Ξ is a Kunen function for f with respect to μ. $\qquad \square$

Remark 5.11. The tree K produced in Theorem 5.10 is called the Kunen tree.

There are some simplifications of K that can be made. K is a tree on $\omega \times \omega \times \omega_1 \times \omega \times \omega$. One can merge the last four coordinates on $\omega \times \omega_1 \times \omega \times \omega$ into ω_1 to produce a tree on $\omega \times \omega_1$ with the same features. One can also Kleene-Brouwer order the various trees to produce linear orderings which can be used to produce the desired Kunen functions. See [12] for more details on the Kunen tree.

In this survey, one will only use the existence of Kunen functions for functions $f : \omega_1 \to \omega_1$; thus, it is not important here how uniformly these Kunen functions are obtained. However, the proof shows that all Kunen functions are index uniformly by reals. For instance, there is a single tree K, the Kunen tree, so that for any $f : \omega_1 \to \omega_1$, there is a section of this tree by some strategy which can be used to produce the Kunen function for f. The Kunen tree K is also Δ_1^1 in the codes. See [12] for the details and the precise meaning of these remarks.

Corollary 5.12. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* μ *denote the club measure on* ω_1. $|\prod_{\omega_1} \omega_1/\mu| \le \omega_2$.

Fact 5.13. *(Martin) Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* μ *denote the club measure on* ω_1. *Then* $\prod_{\omega_1} \omega_1/\mu = \omega_2$ *and is a regular cardinal.*

Proof. Note that the club filter on ω_1, μ, is a normal measure on ω_1. Therefore by Fact 5.4, the ultrapower is a regular cardinal. Thus it must be greater than or equal ω_2. Then Corollary 5.12 implies that the ultrapower is ω_2. $\qquad\square$

Definition 5.14. Let μ be a normal measure on a cardinal κ. Let $h : \kappa \to \kappa$ be a function so that $h(\alpha) > 0$ μ-almost everywhere. Let $T^h = \{(\alpha, \beta) \in \kappa \times \kappa : \beta < h(\alpha)\}$. Let $\mathcal{T}^h = (T^h, \sqsubset)$ where $\sqsubset$ is the lexicographic ordering. Note that $\mathrm{ot}(\mathcal{T}^h) = \kappa$.

Suppose $F : \mathcal{T}^h \to \kappa$ is an order-preserving function. Let $g \in {}^\kappa \kappa$ be such that $g <_\mu h$. Let $A^g = \{\alpha : g(\alpha) < h(\alpha)\}$. Let $F^g \in {}^\kappa \kappa$ be defined by

$$F^g(\alpha) = \begin{cases} F(\alpha, g(\alpha)) & \alpha \in A^g \\ F(\alpha, 0) & \text{otherwise} \end{cases}$$

Note that if $g_1 <_\mu g_2 <_\mu h$, then $F^{g_1} <_\mu F^{g_2}$.

Fix a Kunen function Ξ for h. Recall that ϵ_h^Ξ is the ordertype of the wellordering $(J_h^\Xi, \prec_h^\Xi)$ and $\pi_h^\Xi : \epsilon_h^\Xi \to (J_h^\Xi, \prec_h^\Xi)$ is the unique order isomorphism. If $\beta \in \epsilon_h^\Xi$, then let $F^{(\beta)} = F^{\Xi^{\pi_h^\Xi(\beta)}}$. Let $\mathsf{funct}(F) : \epsilon_h^\Xi \to \mathrm{ON}$, be defined by $\mathsf{funct}(F)(\alpha) = [F^{(\alpha)}]_\mu$.

$\mathcal{T}^h = (T^h, \sqsubset)$ is order isomorphic to κ. It is merely a reorganization of κ into successive blocks of length $h(\alpha)$. If $F : \mathcal{T}^h \to \kappa$ is order preserving and $\beta < \epsilon_h^\Xi$, then $F^{(\beta)} : \kappa \to \kappa$ is defined by letting $F^{(\beta)}(\alpha)$ be the value of F at $(\alpha, \Xi^{\pi_h^\Xi(\beta)}(\alpha))$, which can be construed to be the $\Xi^{\pi_h^\Xi(\beta)}(\alpha)^{\mathrm{th}}$ element in the αth block of length $h(\alpha)$.

Lemma 5.15. *(Sliding lemma) Let* μ *be a normal measure on a cardinal* κ. *Let* $h : \kappa \to \kappa$ *be a block function possessing a Kunen function* Ξ *with respect to* μ.

Let $\langle g_\alpha : \alpha \in \epsilon_h^\Xi \rangle$ *be an order preserving sequence elements of* ${}^\kappa\kappa$ *which are non-constant* μ-almost *everywhere: order preserving means that if* $\alpha < \beta < \epsilon_h^\Xi$, *then* $g_\alpha <_\mu g_\beta$.

Then there exists an order-preserving $F : \mathcal{T}^h \to \kappa$ *so that for all* $\beta < \epsilon_h^\Xi$, $F^{(\beta)} =_\mu g_\beta$.

Moreover, if $X \subseteq \kappa$ *and for all* $\beta < \epsilon_h^\Xi$, $\mathrm{rang}(g_\beta) \subseteq X$, *then* $\mathrm{rang}(F) \subseteq X$.

Proof. Fix some $X \subseteq \kappa$ so that $\mathrm{rang}(g_\beta) \subseteq X$ for all $\beta < \epsilon_h^\Xi$, $\mathrm{rang}(g_\beta) \subseteq X$.

Define $F : \mathcal{T}^h \to \kappa$ by recursion as follows:

Let $(\alpha, \beta) \in T^h$. Suppose $F(\alpha', \beta')$ has been defined for all $(\alpha', \beta') \sqsubset (\alpha, \beta)$. (Recall $\mathcal{T}^h = (T^h, \sqsubset)$ where $\sqsubset$ is the lexicographic ordering.)

(Case I) There is a $\gamma < \epsilon_h^\Xi$ so that $\beta = \Xi^{\pi_h^\Xi(\gamma)}(\alpha)$:

Let γ be least with this property. Let $F(\alpha, \beta)$ be the least element in the range g_γ which is greater than or equal to $\sup\{F(\alpha', \beta') : (\alpha', \beta') \sqsubset (\alpha, \beta)\}$.

(Case II) There is no $\gamma < \epsilon_h^\Xi$ so that $\beta = \Xi^{\pi_h^\Xi(\gamma)}(\alpha)$:

Let $F(\alpha, \beta)$ be the ωth-element of X above $\sup\{F(\alpha', \beta') : (\alpha', \beta') \sqsubset (\alpha, \beta)\}$.

Claim: For all $\gamma < \epsilon_h^\Xi$, $F^{(\gamma)} =_\mu g_\gamma$.

This will be proved by induction on ϵ_h^Ξ.

It is clear $B = \{\alpha \in \kappa : \Xi^{\pi_h^\Xi(0)}(\alpha) = 0\} \in \mu$. For $\alpha \in B$, $F^{(0)} = F(\alpha, \Xi^{\pi_h^\Xi(0)}(\alpha)) \in \mathrm{rang}(g_0)$. Thus on B, one must have that $F^{(0)}(\alpha) \ge g_0(\alpha)$. Suppose that $F^{(0)} >_\mu g_0$. Let $C = \{\alpha \in B : F^{(0)}(\alpha) > g_0(\alpha)\} \in \mu$.

Since for $\alpha \in C$, $\Xi^{\pi_{\bar{h}}^{\Xi}(0)}(\alpha) = 0$, there must be some $(\alpha', \beta') \sqsubset (\alpha, 0)$ with $\alpha' < \alpha$ so that $F(\alpha', \beta') \geq g_0(\alpha)$. Define $\Phi : C \to \kappa$ by letting $\Phi(\alpha)$ be the least $\alpha' < \alpha$ so that there exists some β' with $(\alpha', \beta') \sqsubset (\alpha, \beta)$ and $F(\alpha', \beta') \geq g_0(\alpha)$. Thus Φ is regressive on $C \in \mu$. There is some α' and a $D \in \mu$ so that $\Phi(\alpha) = \alpha'$ for all $\alpha \in D$. Thus for all $\alpha \in D$, $g_0(\alpha) \leq F(\alpha', \beta')$ for some $\beta' < h(\alpha')$. By κ-completeness, there is some $\bar{\beta} < h(\alpha')$ and an $E \subseteq D$ with $E \in \mu$ so that for all $\alpha \in E$, $g_0(\alpha) \leq F(\alpha', \bar{\beta})$. This is impossible since g_0 is not constant μ-almost everywhere. It has been shown that $F^{(0)} = g_0$.

Suppose $\gamma < \epsilon_h^{\Xi}$ and that it has been shown that $F^{(\gamma')} =_\mu g_{\gamma'}$ for all $\gamma' < \gamma$. One will seek to show that $F^{(\gamma)} = g_\gamma$.

Let $A = \{\alpha : (\forall \gamma' \in \gamma)(\Xi^{\pi_{\bar{h}}^{\Xi}(\gamma)}(\alpha) \neq \Xi^{\pi_{\bar{h}}^{\Xi}(\gamma')}(\alpha))\}$. If $A \notin \mu$, then by κ-completeness, there is some $\gamma' \in \gamma$ so that $\Xi^{\pi_{\bar{h}}^{\Xi}(\gamma')} =_\mu \Xi^{\pi_{\bar{h}}^{\Xi}(\gamma)}$. This is impossible since $\gamma' < \gamma$ implies $\Xi^{\pi_{\bar{h}}^{\Xi}(\gamma')} <_\mu \Xi^{\pi_{\bar{h}}^{\Xi}(\gamma)}$.

By induction, it has been shown that $F^{(0)} = g_0$. Let $B \subseteq A$ with $B \in \mu$ have the property that for all $\alpha \in B$, $\Xi^{\pi_{\bar{h}}^{\Xi}(0)}(\alpha) = 0$, $F^{(0)}(\alpha) = g_0(\alpha)$, and $g_\gamma(\alpha) > g_0(\alpha)$.

Suppose $F^{(\gamma)}$ is not equal to g_γ for μ-almost all α. For $\alpha \in B$, $F^{(\gamma)}(\alpha) \in \text{rang}(g_\gamma)$. Thus one must have that there is a set $C \subseteq B$ with $C \in \mu$ so that for all $\alpha \in C$, $F^{(\gamma)}(\alpha) > g_\gamma(\alpha)$. There must be some $(\alpha', \beta') \sqsubset (\alpha, \Xi^{\pi_{\bar{h}}^{\Xi}(\gamma)}(\alpha))$ so that $F(\alpha', \beta') \geq g_\gamma(\alpha)$. However, B was chosen so that for all $\alpha \in B$, $\Xi^{\pi_{\bar{h}}^{\Xi}(0)}(\alpha) = 0$ and $F^{(0)}(\alpha) = g_0(\alpha) < g_\gamma(\alpha)$. Therefore, for $\alpha \in C$, $F^{(0)}(\alpha) < g_\gamma(\alpha) < F^{(\gamma)}(\alpha)$. Thus the least $(\alpha', \beta') \sqsubset (\alpha, \Xi^{\pi_{\bar{h}}^{\Xi}(\gamma)}(\alpha))$ with $F(\alpha', \beta') \geq g_\gamma(\alpha)$ must have that $\alpha' = \alpha$.

Therefore, for all $\alpha \in C$, there is some $\beta' < \Xi^{\pi_{\bar{h}}^{\Xi}(\gamma)}(\alpha)$ so that $F(\alpha, \beta') \geq g_\gamma(\alpha)$. Let $\Phi(\alpha)$ be this β'. Then $\Phi <_\mu \Xi^{\pi_{\bar{h}}^{\Xi}(\gamma)} < h$. Thus there is some $\bar{\gamma} < \gamma$ so that $\Phi =_\mu \Xi^{\pi_{\bar{h}}^{\Xi}(\bar{\gamma})}$. By induction, $F^{(\bar{\gamma})} =_\mu g_{\bar{\gamma}}$. However by definition of Φ, $F^\Phi \geq_\mu g_\gamma$. Also $F^\Phi =_\mu F^{(\bar{\gamma})} =_\mu g_{\bar{\gamma}}$. Thus $g_{\bar{\gamma}} \geq_\mu g_\gamma$. Since $\langle g_\delta : \delta \in \epsilon_h^{\Xi} \rangle$ is an increasing sequence and $\bar{\gamma} < \gamma$, one has that $g_{\bar{\gamma}} <_\mu g_\gamma$. This is a contradiction. This shows that $F^\gamma = g_\gamma$.

The lemma has been proved. $\qquad\square$

It will be helpful to have the correct type version of the sliding lemma. A sketch of the necessary modification will be given:

Lemma 5.16. *(Correct-type sliding lemma) Let μ be a normal measure on a cardinal κ. Let $h : \kappa \to \kappa$ be a block function possessing a Kunen function Ξ with respect to μ.*

Let $\langle g_\alpha : \alpha < \epsilon_h^{\Xi} \rangle$ be an increasing sequence of functions from κ to κ of the correct type which are μ-almost everywhere non-constant. Further, suppose there is a sequence $\langle G_\alpha : \alpha < \epsilon_h^{\Xi} \rangle$ where each $G_\alpha : \kappa \times \omega \to \kappa$ is a function witnessing that g_α has uniform cofinality ω. Suppose $\langle [g_\alpha]_\mu : \alpha < \epsilon_h^{\Xi} \rangle$ is discontinuous everywhere.

Then there exists an order preserving function $F : \mathcal{T}^h \to \kappa$ which is of the correct type so that for all $\alpha < \epsilon_h^{\Xi}$, $F^{(\alpha)} =_\mu g_\alpha$.

Moreover, if $C \subseteq \kappa$ is an ω-club set such that for all $\beta \in \epsilon_h^{\Xi}$, $\text{rang}(g_\beta) \subseteq C$, then $\text{rang}(F) \subseteq C$.

Proof. Fix some $C \subseteq \kappa$ be an ω-club so that $\text{rang}(g_\beta) \subseteq C$ for all $\beta < \epsilon_h^{\Xi}$.

Define $F : \mathcal{T}^h \to \kappa$ by recursion as follows:

Let $(\alpha, \beta) \in T^h$. Suppose $F(\alpha', \beta')$ has been defined for all $(\alpha', \beta') \sqsubseteq (\alpha, \beta)$.

(Case I) There is some $\gamma < \epsilon_h^{\Xi}$ so that $\beta = \Xi^{\pi_{\bar{h}}^{\Xi}(\gamma)}(\alpha)$:

Let γ be least with this property. Let $F(\alpha, \beta)$ be the least element in the range of g_γ which is greater than $\sup\{F(\alpha', \beta') : (\alpha', \beta') \sqsubset (\alpha, \beta)\}$.

(Case II) There is no $\gamma < \epsilon_h^{\Xi}$ so that $\beta = \Xi^{\pi_{\bar{h}}^{\Xi}(\gamma)}(\alpha)$:

Find δ least so that $\mathsf{enum}_C(\delta) > \sup\{F(\alpha', \beta') : (\alpha', \beta') \sqsubset (\alpha, \beta)\}$. Let $F(\alpha, \beta) = \mathsf{enum}_C(\delta + \omega)$.

Claim 1: $F : \mathcal{T}^h \to C$ is a function of the correct type.

To prove this: Note that it is clear from the construction that F is discontinuous everywhere.

One will create a function $H : T^h \times \omega \to \kappa$ to witness the uniform cofinality of F.

Let $(\alpha, \beta) \in T^h$.

Suppose Case I had occurred at (α, β) with $\gamma < \epsilon_h^{\Xi}$ least so that $\beta = \Xi^{\pi_{\bar{h}}^{\Xi}(\gamma)}(\alpha)$. Let $\delta < \kappa$ be least so that $g_\gamma(\delta) > \sup\{F(\alpha', \beta') : (\alpha', \beta') \sqsubset (\alpha, \beta)\}$. Now let n be least so that $G_\gamma(\delta, n) > \sup\{F(\alpha', \beta') : (\alpha', \beta') \sqsubset (\alpha, \beta)\}$. Define $H((\alpha, \beta), m) = G_\gamma(\delta, n + m)$.

Suppose Case II had occurred at (α, β). Let $H((\alpha, \beta), m)$ be the mth element of C above $\sup\{F(\alpha', \beta') : (\alpha', \beta') \sqsubset (\alpha, \beta)\}$.

This function H witnesses that F has uniform cofinality ω. F has the correct type.

Claim 2: For all $\gamma < \epsilon_h^\Xi$, $F^{(\gamma)} =_\mu g_\gamma$.

To show Claim 2: This will be proved by induction on ϵ_h^Ξ. One will indicate some of the necessary modification from the proof of Lemma 5.15.

Suppose that $\gamma < \epsilon_h^\Xi$ and that it has been shown that $F^{(\gamma')} =_\mu g_{\gamma'}$ for all $\gamma' < \gamma$. Let $A = \{\alpha : (\forall \gamma' \in \gamma)(\Xi^{\pi_h^\Xi(\gamma)}(\alpha) \neq \Xi^{\pi_h^\Xi(\gamma')}(\alpha))\}$. As in Lemma 5.15, $A \in \mu$. Also just as before, one can show that $B \in \mu$ where $B \subseteq A$ and has the property that for all $\alpha \in B$, $\Xi^{\pi_h^\Xi(0)}(\alpha) = 0$, $F^{(0)}(\alpha) = g_0(\alpha)$, and $g_\gamma(\alpha) > g_0(\alpha)$.

Now suppose $F^{(\gamma)}$ is not equal to g_γ for μ-almost all α. In the present situation, there are two ways this can happen:

(i) There is a $D \subseteq B$ with $D \in \mu$ so that for all $\alpha \in D$, there exists some $(\alpha', \beta') \sqsubset (\alpha, \Xi^{\pi_h^\Xi(\gamma)}(\alpha))$ so that $F(\alpha', \beta') \geq g_\gamma(\alpha)$. As argued in Lemma 5.15, this can not occur.

(ii) There is a $D \subseteq B$ with $D \in \mu$ so that for all $\alpha \in D$, $g_\gamma(\alpha) = \sup\{F(\alpha', \beta') : (\alpha', \beta') \sqsubset (\alpha, \Xi^{\pi_h^\Xi(\gamma)}(\alpha))\}$.

For each $n \in \omega$, let $\Phi_n : D \to \omega_1$ be defined as follows: Since $\alpha \in D \subseteq B$, one has that $F^{(0)}(\alpha) = g_0(\alpha) < g_\gamma(\alpha)$. For each $n \in \omega$ and $\alpha \in D$, let $\Phi_n(\alpha)$ be the least $\beta' < \Xi^{\pi_h^\Xi(\gamma)}(\alpha)$ so that $F(\alpha, \beta') > G_\gamma(\alpha, n)$.

By Fact 5.6, there is some $\gamma_n < \gamma$ so that $\Phi_n =_\mu \Xi^{\gamma_n}$. By the induction hypothesis, $F^{\Xi^{\pi_h^\Xi(\gamma_n)}} = F^{(\gamma_n)} =_\mu g_{\gamma_n}$. By construction, $\sup\{[g_{\gamma_n}]_\mu : n \in \omega\} = [g_\gamma]_\mu$. However, by assumption, $\langle [g_\alpha]_\mu : \alpha < \epsilon_h^\Xi \rangle$ was discontinuous everywhere.

This complete the proof of the lemma. $\qquad\square$

Fact 5.17. *Let μ be a normal measure on a cardinal κ. Let $h : \kappa \to \kappa$ be a block function possessing a Kunen function Ξ with respect to μ. Suppose $F_0, F_1 \in [\omega_1]^{\mathcal{T}^h}$ have the property that $F_0^{(\beta)} =_\mu F_1^{(\beta)}$ for all $\beta < \epsilon_h^\Xi$. Then for μ-almost all α, $F_0(\alpha, \beta) = F_1(\alpha, \beta)$ for all $\beta < h(\alpha)$.*

Proof. Suppose $A = \{\alpha : (\exists \beta < h(\alpha))(F_0(\alpha, \beta) \neq F_1(\alpha, \beta))\} \in \mu$. Define $g : \omega_1 \to \omega_1$ by $g(\alpha)$ is the least $\beta < h(\alpha)$ so that $F_0(\alpha, \beta) \neq F_1(\alpha, \beta)$. Since $g <_\mu h$, Fact 5.6 implies there is some $\gamma < \epsilon_h^\Xi$ so that $g =_\mu \Xi^{\pi_h^\Xi(\gamma)}$. Then $\neg(F_0^{(\gamma)} =_\mu F_1^{(\gamma)})$. This contradicts the assumptions. $\qquad\square$

Fact 5.18. *Assume $\mathsf{ZF} + \mathsf{AD}$. Let μ denote the club measure on ω_1. Let $\alpha < \omega_2$. Let $A \subseteq \omega_1$ with $|A| = \omega_1$. Let $B = [A]^{\omega_1}/\mu$, which is a set of cardinality ω_2. Let $h : \omega_1 \to \omega_1$ have the property that $h(\beta) > 0$ for all $\beta < \omega_1$ and $[h]_\mu = \alpha$. Let Ξ be a Kunen function for h with respect to μ.*

For all $I \in [B]^\alpha$, there exists some $F : \mathcal{T}^h \to A$ which is order preserving and for all $\gamma < \alpha = \epsilon_h^\Xi$, $[F^{(\gamma)}]_\mu = I(\gamma)$.

Proof. Since $\omega_2 = {}^{\omega_1}\omega_1/\mu$ is regular by Fact 5.4, $\text{rang}(I) < \omega_2$. Let $k \in [\omega_1]^{\omega_1}$ be such that $\text{rang}(I) < [k]_\mu$. Let Ξ' be a Kunen function for k with respect to μ.

Let $J = \{\beta \in \epsilon_k^{\Xi'} : (\exists \gamma < \alpha)(I(\gamma) = [\Xi'^{\pi_k^{\Xi'}(\beta)}]_\mu)\}$. $J \subseteq \epsilon_k^{\Xi'}$ and $\text{ot}(J) = \alpha = \epsilon_h^\Xi$. Let $\rho : \epsilon_h^\Xi \to J$ be the unique order isomorphism. Define $g'_\gamma = \Xi'^{\pi_k^{\Xi'}(\rho(\gamma))}$. Since $[g'_\gamma]_\mu = I(\gamma)$ and $I(\gamma) \in B = [A]^{\omega_1}/\mu$, the set $B_\gamma = \{\alpha \in \omega_1 : g'_\gamma(\alpha) \in A\} \in \mu$. By Fact 5.1, $g'_\gamma =_\mu g'_\gamma \circ \mathsf{enum}_{B_\gamma}$. Let $g_\gamma = g'_\gamma \circ \mathsf{enum}_{B_\gamma}$. $[g_\gamma]_\mu = I(\gamma)$ and $\text{rang}(g_\gamma) \subseteq A$. The result now follows from Lemma 5.15. $\qquad\square$

Theorem 5.19. *(Martin-Paris) Assume $\mathsf{ZF} + \mathsf{AD}$. Let μ be the club measure on ω_1. Then for all $\alpha < \omega_2$, the partition relation $\omega_2 \to (\omega_2)_2^\alpha$ holds. That is, ω_2 is a weak partition cardinal.*

Proof. By Fact 5.13, ω_2 is isomorphic to ${}^{\omega_1}\omega_1/\mu$. One will identify ω_2 with ${}^{\omega_1}\omega_1/\mu$.

Let $\alpha < \omega_2$ and $P : [\omega_2]^\alpha \to 2$ be a partition.

Let $h : \omega_1 \to \omega_1$ be such that $[h]_\mu = \alpha$ and fix a Kunen function Ξ for h. Define a partition $Q : [\omega_1]^{\mathcal{T}^h} \to 2$ by $Q(F) = P(\mathsf{funct}(F)))$, where $\mathsf{funct}(F)$ is an α-sequence of ordinals in ω_2 defined in Definition 5.14 (relative to Ξ).

Since $\mathcal{T}^h$ is order isomorphic to ω_1, $\omega_1 \to (\omega_1)_2^{\omega_1}$ implies that there is a $A \subseteq \omega_1$ with $|A| = \omega_1$ which is homogeneous for Q. Without loss of generality, suppose A is homogeneous for Q taking value 0.

Note that $[A]^{\omega_1}/\mu$ has cardinality ${}^{\omega_1}\omega_1/\mu = \omega_2$. Let B denote $[A]^{\omega_1}/\mu$.

Let $I \in [B]^\alpha$. By Fact 5.18, let $F : \mathcal{T}^h \to A$ be such that $[F^{(\gamma)}]_\mu = I(\gamma)$ for all $\gamma < \alpha$. Since $F \in [A]^{\mathcal{T}^h}$, one has $0 = Q(F) = P(\mathsf{funct}(F)) = P(I)$ since $I = \mathsf{funct}(F)$. B is homogeneous for P taking value 0. The proof is complete. $\qquad\square$

By Fact 2.6, a suitable form of the ordinary partition property will imply the appropriate correct type version of the partition property which uses a club set as its version of the homogeneous set. In particular, the ordinary weak partition property for ω_2 proved in Theorem 5.19 implies the correct type club version of the weak partition property, that is $\omega_2 \to_* (\omega_2)_2^\alpha$ for all $\alpha < \omega_2$.

Corollary 5.20. *Assume* $\mathsf{ZF} + \mathsf{AD}$. $W_\omega^{\omega_2}$ *and* $W_{\omega_1}^{\omega_2}$ *are the only two* ω_2-*complete normal ultrafilters on* ω_2.

Proof. By Fact 2.15. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Fact 5.21. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* μ *be the club measure on* ω_1. *If* $C \subseteq \omega_1$ *is a club subset of* ω_1, *then* $[C]^{\omega_1}/\mu$ *is a club subset of* ω_2.
   If $D \subseteq \omega_2$ *is club, then there is a club* $C \subseteq \omega_1$ *so that* $[C]^{\omega_1}/\mu \subseteq D$.

Proof. Let $\epsilon < \omega_2$. Suppose $\langle \nu_\gamma : \gamma < \epsilon \rangle$ is an increasing sequence in $[C]^{\omega_1}/\mu$. Let $h : \omega_1 \to \omega_1$ with $h(\alpha) > 0$ for all $\alpha \in \omega_1$ be such that $\epsilon = [h]_\mu$. Let Ξ be a Kunen function for h with respect to μ.
   By Fact 5.18, there is a function $F : \mathcal{T}^h \to C$ which is order preserving so that $[F^{(\gamma)}]_\mu = \nu_\gamma$.
   Let $\ell : \omega_1 \to \omega_1$ be defined by $\ell(\alpha) = \sup\{F(\alpha, \beta) : \beta < h(\alpha)\}$. Since $\mathrm{rang}(F) \subseteq C$, one has that $\ell(\alpha) \in C$ for each $\alpha \in \omega_1$.
   Suppose $j : \omega_1 \to \omega_1$ is such that $j <_\mu \ell$. Let $B = \{\alpha \in \omega : j(\alpha) < \ell(\alpha)\} \in \mu$. For $\alpha \in B$, let $p(\alpha)$ be the least $\beta < h(\alpha)$ so that $j(\alpha) < F(\alpha, \beta)$. For $\alpha \notin B$, let $p(\alpha) = 0$. Since $p <_\mu h$, Fact 5.6 implies there $\gamma < \alpha$ so that $p =_\mu \Xi^{\pi_h^\Xi(\gamma)}$. Then $j <_\mu F^{(\gamma)}$. Thus $[j]_\mu < \nu_\gamma$. This establishes that $[\ell]_\mu$ is the limit of $\langle \nu_\gamma : \gamma < \epsilon \rangle$. Since $\ell \in [C]^{\omega_1}$, this shows the supremum belongs to $[C]^{\omega_1}/\mu$. This shows that $[C]^{\omega_1}/\mu$ is closed. It is easy to see that $[C]^{\omega_1}/\mu$ is unbounded.
   Now suppose $D \subseteq \omega_2$ is club. Let $\mathcal{T}^2 = (\omega_1 \times 2, \sqsubset)$ where $\sqsubset$ is the lexicographic ordering. If $F : \mathcal{T}^2 \to \omega_1$ is an increasing function, then let $F_0, F_1 \in [\omega_1]^{\omega_1}$ be defined by $F_i(\alpha) = F(\alpha, i)$. Note that $F_0 <_\mu F_1$.
   Define $P : [\omega_1]^{\mathcal{T}^2} \to 2$ by $P(F) = 0$ if and only if $(\exists \alpha \in D)([F_0]_\mu < \alpha < [F_1]_\mu)$. By $\omega_1 \to_* (\omega_1)_2^{\omega_1}$ (the correct type strong partition property), there is a club $C \subseteq \omega_1$ which is homogeneous for P (for all functions $F \in [C]_*^{\mathcal{T}^2}$ of correct type).
   Suppose C is homogeneous for P taking value 1. Fix some $\alpha_0 \in [C]_*^{\omega_1}/\mu$. Let $f_0 \in [C]_*^{\omega_1}$ with $[f_0]_\mu = \alpha_0$. Pick $\beta < \omega_2$ with $\alpha_0 < \beta$. Since $[C]_*^{\omega_1}/\mu$ is cofinal through ω_2, pick some $\alpha_1 \in [C]_*^{\omega_1}/\mu$ with $\beta < \alpha_1$. Let $f_1 \in [C]_*^{\omega_1}$ be such that $[f_1]_\mu = \alpha_1$. By an argument as in Fact 5.3 with some additional care to maintain the correct type, there is a function $F \in [C]_*^{\mathcal{T}^2}$ so that $[F_0]_\mu = \alpha_0$ and $[F_1]_\mu = \alpha_1 > \beta$. Since $P(F) = 1$, there is no element of D between α_0 and α_1. In particular, there are no elements of D between α_0 and β. However since β is arbitrary, this implies $D \subseteq \alpha_0$. This contradicts that D is unbounded.
   This shows that C is homogeneous for P taking value 0. Let $\tilde{C} = \{\alpha \in C : \mathsf{enum}_C(\alpha) = \alpha\}$ which is a club subset of ω_1. Let $f \in [\tilde{C}]^{\omega_1}$. Suppose $h <_\mu f$. For each $\alpha < \omega_1$, let γ_α be least so $h(\alpha) < \mathsf{enum}_C(\gamma_\alpha)$. Let $f_0(\alpha) = \mathsf{enum}_C(\gamma_\alpha + \omega)$ and $f_1 = \mathsf{enum}_C(\gamma_\alpha + \omega + \omega)$. Since $f(\alpha) \in \tilde{C}$, $f_0(\alpha) < f_1(\alpha) < f(\alpha)$. Let $g_0(\alpha, n) = \mathsf{enum}_C(\gamma_\alpha + n)$ and $g_1(\alpha, n) = \mathsf{enum}_C(\gamma_\alpha + \omega + n)$. Note that $h <_\mu f_0 <_\mu f_1 <_\mu f$, $f_0, f_1 \in [C]_*^{\omega_1}$, and g_0, g_1 witnesses that f_0, f_1 are functions of the correct type. As above, one can find some $F \in [C]_*^{\mathcal{T}^2}$ so that $F_0 =_\mu f_0$ and $F_1 =_\mu f_1$. Since $P(F) = 0$, there is some $\alpha \in D$ so that $[h]_\mu < [f_0]_\mu < \alpha < [f_1]_\mu < [f]_\mu$. Since D is a club and $h <_\mu f$ was arbitrary, this shows that $[f]_\mu \in D$.
   It has been shown that for all $D \subseteq \omega_1$, there is some club $\tilde{C} \subseteq \omega_1$ so that $[\tilde{C}]^{\omega_1}/\mu \subseteq D$. $\qquad$ □

6. Failure of the Strong Partition Property at ω_2

Fact 6.1. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* μ *be the club measure on* ω_1. *Suppose* $f \in {}^{\omega_1}\omega_1$ *is a function of uniform cofinality* ω. *Then* $\mathrm{cof}([f]_\mu) = \omega$.

Proof. Suppose f has uniform cofinality ω. Let $g : \omega_1 \times \omega \to \omega_1$ be such that for all $\alpha < \omega_1$, $f(\alpha) = \sup\{g(\alpha, n) : n \in \omega\}$. Let $f_n : \omega_1 \to \omega_1$ be defined by $f_n(\alpha) = g(\alpha, n)$. Then $m < n$ implies $f_m <_\mu f_n$. Suppose $h <_\mu f$. Let $A = \{\alpha : h(\alpha) < f(\alpha)\}$. Define $k(\alpha)$ to be the least $n \in \omega$ so that $h(\alpha) < f_n(\alpha)$ whenever $\alpha \in A$. By the countable additivity of μ, there is some n^* so that $k(\alpha) = n^*$ for μ-almost all $\alpha \in \kappa$. Then $h <_\mu f_{n^*}$. Thus $[f]_\mu = \sup\{[f_n]_\mu : n \in \omega\}$. So $\mathrm{cof}([f]_\mu) = \omega$. $\qquad\qquad\qquad$ □

Fact 6.2. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* μ *be the club measure on* ω_1. *Let* $C \subseteq \omega_1$ *be a club. Then* $[C]_*^{\omega_1}/\mu$ *is an* ω-*club subset of* ω_2. *Moreover, for every* ω-*club* $D \subseteq \omega_2$, *there is a club* $C \subseteq \omega_1$ *so that* $[C]_*^{\omega_1}/\mu \subseteq D$.

Proof. It is clear that $[C]_*^{\omega_1}/\mu$ is unbounded.

Suppose $\langle \nu_n : n \in \omega \rangle$ is an increasing ω-sequence in $[C]_*^{\omega_1}/\mu$. By $\mathsf{AC}_\mathbb{R}^\omega$, let $\langle g_n : n \in \omega \rangle$ be such that $\nu_n = [g_n]_\mu$ and each g_n is of the correct type. Using $\mathsf{AC}_\omega^\mathbb{R}$, one can also select a sequence $\langle k_n : n \in \omega \rangle$ so that k_n witnesses that g_n has uniform cofinality ω. By Fact 5.1, one may assume $g_n : \omega_1 \to C$. Using Lemma 5.16, let $F : \mathcal{T}^\omega \to C$ be an order preserving map of the correct type so that for all $n \in \omega$, $F^n =_\mu g_n$. Let $g : \omega_1 \to C$ be defined by $g(\alpha) = \sup\{F^n(\alpha) : n \in \omega\}$. Note that g is of the correct type. As before, one can check that $[g]_\mu = \sup\{\nu_n : n \in \omega\}$ and $[g]_\nu \in [C]_*^{\omega_1}/\mu$. This shows that $[C]_*^{\omega_1}/\mu$ is an ω-club subset of ω_2.

Let $D \subseteq \omega_2$ be an ω-club subset of ω_2. Define $P : [\omega_1]_*^{\omega_1} \to 2$ by

$$P(f) = \begin{cases} 0 & [f]_\mu \notin D \\ 1 & [f]_\mu \in D \end{cases}$$

By the correct type partition property $\omega_1 \to_* (\omega_1)_*^{\omega_1}$, there is a club $C \subseteq \omega_1$ which is homogeneous for P (in the correct type sense). Since it was shown that $[C]_*^{\omega_1}/\mu$ is an ω-club, $([C]_*^{\omega_1}/\mu) \cap D \neq \emptyset$. Thus C must be homogeneous for D taking value 1. It has been shown that every ω-club $D \subseteq \omega_2$ contains an ω-club of the form $[C]_*^{\omega_1}/\mu$. $\qquad\square$

Fact 6.3. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* μ *denote the club measure on* ω_1. *Let* $C \subseteq \omega_1$ *be club. Let* $B = [C]_*^{\omega_1}/\mu$ *which is an* ω-*club subset of* ω_2.

Let $\epsilon < \omega_2$. *Let* $h : \omega_1 \to \omega_1$ *with* $h(\alpha) > 0$ *for all* $\alpha < \omega_1$ *and* $[h]_\mu = \epsilon$. *Let* Ξ *be a Kunen function for* h.
Let $\mathcal{F} \in [B]_*^\epsilon$ *(be of correct type). Let* $\mathcal{G} : [\omega_2]^\epsilon \times \omega$ *be a function witnessing that* $\mathcal{F}$ *has uniform cofinality* ω *with the property that for all* $\alpha_0 < \alpha_1 < \epsilon$ *and* $m, n \in \omega$, $\mathcal{G}(\alpha_0, m) < \mathcal{G}(\alpha_1, n)$.

Then there is a sequence $\langle G_n : n \in \omega \rangle$ *with each* $G_n : \mathcal{T}^h \to \omega_1$ *so that* $\mathcal{G}(\alpha, n) = [G_n^{(\alpha)}]_\mu$.
There is an $F \in [C]_*^{\mathcal{T}^h}$ *so that for all* $\alpha < \epsilon$, $[F^{(\alpha)}]_\mu = \mathcal{F}(\alpha)$.

Proof. For each n, let $\mathcal{G}_n : \epsilon \to \omega_1$ be defined by $\mathcal{G}_n(\alpha) = \mathcal{G}(\alpha, n)$. Using Fact 5.18 on h, Ξ, and $\mathcal{G}_n$, one obtains G_n.

Fix $\gamma < \epsilon$. Define $g'_{\gamma,n} : \omega_1 \to \omega_1$ by defined by

$$g'_{\gamma,n}(\alpha) = \begin{cases} G_n^{(\gamma)}(\alpha) & \Xi^{\pi_h^\Xi(\gamma)}(\alpha) < h(\alpha) \\ 0 & \text{otherwise} \end{cases}$$

Let $f'_\gamma : \omega_1 \to \omega_1$ be defined by $f'_\gamma(\alpha) = \sup\{g'_{\gamma,n}(\alpha) : n \in \omega\}$. Note that $\mathcal{F}(\gamma) = [f'_\gamma]_\mu$.

Claim 1: The set D_γ of $\alpha < \omega_1$ so that f'_γ is discontinuous at α belongs to μ.

Suppose $\omega_1 \setminus D_\gamma \in \mu$ and hence contains a club E_γ. Since $\mathcal{F}(\gamma) \in [C]_*^{\omega_1}/\mu$, there is some $\bar{f}_\gamma \in [C]_*^{\omega_1}$ (of the correct type) so that $f'_\gamma =_\mu \bar{f}_\gamma$. Let $J_\gamma = \{\alpha : \bar{f}_\gamma(\alpha) = f'_\gamma(\alpha)\} \in \mu$. Let $K_\gamma \subseteq J_\gamma$ be a club. Then $E_\gamma \cap K_\gamma$ is a club subset of ω_1. Let $\langle \lambda_n : n \in \omega \rangle$ be an increasing sequence in $E_\gamma \cap K_\gamma$. Then $\lambda = \sup\{\lambda_n : n \in \omega\} \in E_\gamma \cap K_\gamma$ since $E_\gamma \cap K_\gamma$ is a club. Since λ_n and λ belong to $K_\gamma \subseteq J_\gamma$, $\bar{f}_\gamma(\lambda_n) = f'_\gamma(\lambda_n)$ and $\bar{f}_\gamma(\lambda) = f'_\gamma(\lambda)$. Since $\lambda \in E_\gamma$, one has that f'_γ is continuous at λ. However, the previous statement implies that $\bar{f}_\gamma$ is also continuous at λ. But $\bar{f}_\gamma$ is discontinuous everywhere since it is of the correct type. This proves the claim.

Since $[f'_\gamma]_\mu = \mathcal{F}(\gamma) \in [C]_*^{\omega_1}/\mu$, there is some $\bar{f}_\gamma \in [C]_*^{\omega_1}$ so that $f'_\gamma =_\mu \bar{f}_\gamma$. Thus $P_\gamma = \{\alpha \in \omega_1 : f'_\gamma(\alpha) \in C\} \in \mu$. Then $Q_\gamma = P_\gamma \cap D_\gamma \in \mu$. Let $f_\gamma = f'_\gamma \circ \mathsf{enum}_{Q_\gamma}$. By Fact 5.1, $f'_\gamma =_\mu f_\gamma$. Thus $[f_\gamma]_\mu = \mathcal{F}(\gamma)$. Note that f_γ is discontinuous everywhere. Define $g_\gamma : \omega_1 \times \omega \to \omega_1$ by $g_\gamma(\alpha, n) = g'_{\gamma,n}(\mathsf{enum}_{Q_\gamma}(\alpha))$. g_γ witnesses that f_γ has uniform cofinality ω. Thus f_γ is a function of the correct type.

By the uniformity of the construction, one has now produced a sequence $\langle f_\gamma : \gamma < \epsilon \rangle$ of functions of the correct type and $\langle g_\gamma : \gamma < \epsilon \rangle$ so that $[f_\gamma]_\mu = \mathcal{F}(\gamma)$ and g_γ witnesses the uniform cofinality of f_γ. Now apply Lemma 5.16 to h, Ξ, $\langle f_\gamma : \gamma < \epsilon \rangle$, and $\langle g_\gamma : \gamma < \epsilon \rangle$ to obtained the desired function $F \in [C]_*^{\mathcal{T}^h}$. $\qquad\square$

Fact 6.4. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* μ *be the club measure on* ω_1. *Suppose* f *has uniform cofinality* id. *Then* $[f]_\mu$ *(as an ordinal in* ω_2*) has cofinality* ω_1.

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Proof. Let $T^{\mathsf{id}} = \{(\alpha, \beta) : \beta < \alpha\}$. Let $g : T^{\mathsf{id}} \to \omega_1$ witness that f has uniform cofinality id. For each $\beta < \omega_1$, let

$$f_\beta(\gamma) = \begin{cases} g(\gamma, \beta) & \gamma > \beta \\ 0 & \text{otherwise} \end{cases}$$

One can show that $\langle [f_\beta]_\mu : \beta < \omega_1 \rangle$ is an increasing sequence cofinal below $[f]_\mu$. $\square$

Fact 6.5. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let μ denote the club measure on ω_1. Let $f : \omega_1 \to \omega_1$ be a function so that $f(\alpha)$ is a limit ordinal for μ-almost all α. Then either f has uniform cofinality ω or uniform cofinality* id *μ-almost everywhere but not both.*

Proof. Let Ξ be a Kunen function for f. Let $A = \{\alpha : f(\alpha) \in \mathrm{Lim}\} \in \mu$. By Definition 5.5, for all $\alpha \in K_f^\Xi$, $f(\alpha) < \delta_\alpha^\Xi = \sup\{\Xi(\alpha, \beta) : \beta < \alpha\}$.

Fix $\alpha \in K_f^\Xi \cap A$, define ϵ_0^α be the least $\epsilon < \alpha$ so that $\Xi(\alpha, \epsilon) < f(\alpha)$. Suppose $\beta < \alpha$ and $\epsilon_{\beta'}^\alpha$ has been defined for all $\beta' < \beta < \alpha$. If $\sup\{\Xi_\alpha(\epsilon_{\beta'}^\alpha) : \beta' < \beta\} = f(\alpha)$, then let α^* denote this β and declare the construction to have terminated. Otherwise, let ϵ_β^α be the least $\epsilon < \alpha$ so that $\sup\{\Xi_\alpha(\epsilon_{\beta'}^\alpha) : \beta' < \beta\} < \Xi_\alpha(\epsilon) < f(\alpha)$. This defines a sequence $\langle \epsilon_\beta^\alpha : \beta < \alpha^* \rangle$ where $\alpha^* \leq \alpha$. Note that $\langle \epsilon_\beta^\alpha : \beta < \alpha^* \rangle$ is an increasing sequence and $\sup\{\Xi_\alpha(\epsilon_\beta^\alpha) : \beta < \alpha^*\} = f(\alpha)$.

Consider the function $\Phi : K_f^\Xi \cap A \to \omega_1$ by $\Phi(\alpha) = \alpha^*$. Let $B = \{\alpha \in A \cap K_f^\Xi : \Phi(\alpha) = \alpha^* < \alpha\}$.

(Case I) $B \in \mu$.

Then Φ is a regressive function. Since μ is normal, there is a $C \subseteq B$ and $\delta < \omega_1$ so that $C \in \mu$ and $\Phi(\alpha) = \delta$ for all $\alpha \in C$. Let $\phi : \omega \to \delta$ be a cofinal sequence. Let $g : \omega_1 \times \omega \to \omega_1$ be defined by

$$g(\alpha, n) = \begin{cases} n & \alpha \notin C \\ \Xi_\alpha(\epsilon_{\phi(n)}^\alpha) & \alpha \in C \end{cases}$$

The function g witnesses that f is μ-almost everywhere a function of uniform cofinality ω.

(Case II) $B \notin \mu$.

Let $C = (K_f^\Xi \cap A) \setminus B$. For all $\alpha \in C$, $\Phi(\alpha) = \alpha^* = \alpha$. Let $T^{\mathsf{id}} = \{(\alpha, \beta) : \alpha < \omega_1 \wedge \beta < \alpha\}$. Define $g : T^{\mathsf{id}} \to \omega_1$ by

$$g(\alpha, \beta) = \begin{cases} \beta & \alpha \notin C \\ \Xi_\alpha(\epsilon_\beta^\alpha) & \alpha \in C \end{cases}$$

Then g witnesses that f has uniform cofinality id μ-almost everywhere.

Fact 6.1 implies that if f has uniform cofinality ω, then $[f]_\mu$ has cofinality ω. Fact 6.4 implies that if f has uniform cofinality id, then $[f]_\mu$ has cofinality ω_1. Since ω_1 is regular under AD, f cannot have both uniform cofinality. $\square$

In the previous result, it is shown that f cannot have uniform cofinality ω and id by showing these uniform cofinalities correspond to different cofinalities of $[f]_\mu$. This is however not the case in general. The following argument avoids considering the function in the ultrapower. This argument can be generalized to show functions from $f : [\omega_1]^n \to \omega_1$ can only have one uniform cofinality.

Fact 6.6. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Every function $f : \omega_1 \to \omega_1$ has uniform cofinality ω or* id *μ-almost everywhere but not both.*

Proof. This is already proved in Fact 6.5, so one will just give an argument that no function can have both uniform cofinality which does not involve the ultrapower.

Suppose that $f : \omega_1 \to \omega_1$ μ-almost everywhere has both uniform cofinality ω and id. Let $g_\omega : \omega_1 \times \omega \to \omega_1$ witness that f has uniform cofinality ω μ-almost everywhere. Let $g_{\mathsf{id}} : \omega_1 \times \omega_1 \to \omega_1$ witness that f has uniform cofinality id μ-almost everywhere.

Let $A_\omega \in \mu$ be the set of $\alpha < \omega_1$ so that $f(\alpha) = \sup\{g_\omega(\alpha, n) : n \in \omega\}$. Let $A_{\mathsf{id}} \in \mu$ be the set of $\alpha < \omega_1$ so that $f(\alpha) = \sup\{g_{\mathsf{id}}(\alpha, \beta) : \beta < \alpha\}$. Let $A = A_\omega \cap A_{\mathsf{id}}$ which also belongs to μ.

For each $\beta \in A$ and $\alpha < \beta$, let $n_{\alpha,\beta}$ be least $n \in \omega$ so that $g_\omega(\beta, n) > g_{\mathsf{id}}(\beta, \alpha)$ which exists since $\beta \in A$. Consider the function $\Phi : [A]^2 \to \omega$ defined by $\Phi(\alpha, \beta) = n_{\alpha,\beta}$. By the weak partition property and the countable additivity of μ, one has that there is some $B \subseteq A$ with $B \in \mu$ and $n \in \omega$ so that for all $(\alpha, \beta) \in [B]^2$, $n_{\alpha,\beta} = n$.

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Now let $\beta \in B$ be a limit point of B, that is $\sup B \cap \beta = \beta$. For each $\alpha < \beta$ with $\alpha \in B$, one has that $n_{\alpha,\beta} = n$. Then $f(\beta) = \sup\{g_{\mathsf{id}}(\beta,\alpha) : \alpha \in \beta\} \leq g_\omega(\beta, n) < f(\beta)$. Contradiction. $\qquad\square$

Next one will consider the ultrapower, $\mathrm{ult}(V, \mu)$, where μ is the club measure on ω_1. One needs some care when working with this structure as one may not have Łoś's theorem without AC. It will be shown later that Łoś's theorem fails for this ultrapower and in fact $\mathrm{ult}(V, \mu)$ is not a model ZF.

Fact 6.7. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* μ *denote the club measure on* ω_1. *Suppose* $f : \omega_1 \to V$ *is such that* $\mathrm{ult}(V, \mu) \models [f]_\mu \subseteq \omega_2$, *then there is an* $f' : \omega_1 \to \mathscr{P}(\omega_1)$ *so that* $[f']_\mu = [f]_\mu$.

Proof. By Corollary 5.13, one knows that $\omega_2 = \prod_{\omega_1} \omega_1 / \mu$. Thus in $\mathrm{ult}(V, \mu)$, each $\zeta < \omega_2$ is represented by some $h : \omega_1 \to \omega_1$.

Now let $f'(\alpha) = f(\alpha) \cap \omega_1$. The claim is that $[f']_\mu = [f]_\mu$.

Suppose $h : \omega_1 \to V$ is a function so that $\mathrm{ult}(V, \mu) \models [h]_\mu \in [f]_\mu$. Since $\mathrm{ult}(V, \mu) \models [f]_\mu \subseteq \omega_2$, one must have that $A = \{\alpha \in \omega_1 : h(\alpha) \in f(\alpha) \cap \omega_1\} \in \mu$. Let $h' : \omega_1 \to \omega_1$ be defined by $h'(\alpha) = h(\alpha)$ if $\alpha \in A$ and $h'(\alpha) = \emptyset$ otherwise. Note that $[h]_\mu = [h']_\mu$ and $[h']_\mu \in [f']_\mu$. This shows that $[f]_\mu \subseteq [f']_\mu$. It is clear that $[f']_\mu \subseteq [f]_\mu$. Thus $[f]_\mu = [f']_\mu$. $\qquad\square$

Fact 6.8. *Assume* $\mathsf{ZF} + \mathsf{AD}$ *and* μ *is the club measure on* ω_1. *Suppose* $h : \omega_1 \to \mathscr{P}(\omega_1)$ *has the property that for all* $\alpha < \omega_1$, $|h(\alpha)| = \omega_1$ *and* $\mathsf{enum}_{h(\alpha)}$ *is a function of correct type. Suppose there is a function* G *so that for each* $\alpha < \omega_1$, $G(\alpha) : \omega_1 \times \omega \to \omega_1$ *is a witness to* $\mathsf{enum}_{h(\alpha)}$ *having uniform cofinality* ω, *then* $\mathsf{enum}_{[h]_\mu} : \omega_2 \to \omega_2$ *is a function of correct type.*

Proof. For each $\xi < \omega_1$, let $g_\xi : \omega_1 \times \omega \to \omega_1$ be defined by $g_\xi = G(\xi)$. Hence g_ξ is a witnesses to $\mathsf{enum}_{h(\alpha)}$ having the correct type.

For each $\xi < \omega_2$, let $p_\xi : \omega_1 \to \omega_1$ have the property that for all $\alpha < \omega_1$, $p_\xi(\alpha) \in h(\alpha)$ and $[p_\xi]_\mu$ represents the ξth-element of $[h]_\mu$. (Note that one does not have a uniform procedure for finding p_ξ as ξ ranges over ordinals below ω_2.) Define $k_{\xi,n} : \omega_1 \to \omega_1$ by $k_{\xi,n}(\alpha) = g_\xi(\mathsf{enum}_{h(\xi)}^{-1}(p_\xi(\alpha)), n)$. Let $\delta_{\xi,n} = [k_{\xi,n}]_\mu$. Note that if $p'_\xi =_\mu p_\xi$ and $k'_{\xi,n}$ was defined in the same manner as k using p'_ξ instead of p_ξ, then $k'_{\xi,n} =_\mu k_{\xi,n}$. This shows that $\delta_{\xi,n}$ is well defined independent of the choice of p_ξ.

Define $r : \omega_2 \times \omega \to \omega_2$ by $r(\xi, n) = \delta_{\xi,n}$. One can check that r witnesses that $\mathsf{enum}_{[h]_\mu}$ has uniform cofinality ω.

Let $\zeta \in [h]_\mu$. Let $\ell : \omega_1 \to \omega_1$ be such that $[\ell]_\mu = \zeta$. One may assume that for all α, $\ell(\alpha) \in h(\alpha)$. Let $\iota : \omega_1 \to \omega_1$ be defined by $\iota(\alpha) = \sup(h(\alpha) \cap \ell(\alpha))$. Note that $\iota(\alpha) < \ell(\alpha)$ since $\mathsf{enum}_{h(\alpha)}$ was assumed to be discontinuous everywhere. One can check that every element of $[h]_\mu$ which is below $[\ell]_\mu$ is below $[\iota]_\mu < [\ell]_\mu$. Thus $\mathsf{enum}_{[h]_\mu}$ is discontinuous everywhere. $\qquad\square$

Fact 6.9. *Suppose* $h : \omega_1 \to \mathscr{P}(\omega_1)$ *with* $|h(\alpha)| = \omega_1$ *for all* $\alpha < \omega_1$. *Let* $\xi < \omega_2$ *and* $\ell : \omega_1 \to \omega_1$ *be such that* $[\ell]_\mu = \xi$. $\mathsf{enum}_{[h]_\mu}(\xi)$ *is represented by the function* $g : \omega_1 \to \omega_1$ *defined by* $g(\alpha) = \mathsf{enum}_{h(\alpha)}(\ell(\alpha))$.

Proof. For each $\ell : \omega_1 \to \omega_1$, let $g_\ell : \omega_1 \to \omega_1$ be defined by $g_\ell(\alpha) = \mathsf{enum}_{h(\alpha)}(\ell(\alpha))$. Note that if $\ell =_\mu \ell'$, then $g_\ell =_\mu g_{\ell'}$. Note also that $[g_\ell]_\mu \in [h]_\mu$.

For each $\xi < \omega_1$, let $\gamma_\xi \in \omega_2$ be defined by $\gamma_\xi = [g_\ell]_\mu$ where ℓ is any function so that $[\ell]_\mu = \xi$. This is well defined by the previous paragraph. $\langle \gamma_\xi : \xi < \omega_2 \rangle$ is an increasing sequence through $[h]_\mu$.

Let $k : \omega_1 \to \omega_1$ be such that $[k]_\mu \in [h]_\mu$. Note $A = \{\alpha \in \omega_1 : k(\alpha) \in h(\alpha)\} \in \mu$. By modifying k off A, one will assume without loss of generality that $k(\alpha) \in h(\alpha)$ for all $\alpha < \omega_1$. Let $\ell(\alpha) = \mathsf{enum}_{h(\alpha)}^{-1}(k(\alpha))$. Then $k =_\mu g_\ell$.

This shows that $\mathsf{enum}_{[h]_\mu}(\xi) = \gamma_\xi$. This completes the proof. $\qquad\square$

Fact 6.10. *Assume* $\mathsf{ZF} + \mathsf{AD}$ *and let* μ *be the club measure on* ω_1. *Suppose* $E \subseteq \omega_2$ *is such that* $|E| = \omega_1$. *Then* $E \in \mathrm{ult}(V, \mu)$.

Proof. Let $\zeta = \mathrm{ot}(E)$. Let $h : \omega_1 \to \omega_1$ be such that $[h]_\mu = \zeta$ and $h(\alpha) > 0$ for all $\alpha < \omega_1$. Let Ξ be a Kunen function for h. Using Fact 5.18, there is an $F : \mathcal{T}^h \to \omega_1$ which is increasing so that for all $\xi < \zeta$, $[F^{(\xi)}]_\mu = \mathsf{enum}_E(\xi)$.

Let $g : \omega_1 \to \mathscr{P}(\omega_1)$ be defined by $g(\alpha) = \{F(\alpha, \beta) : \beta < h(\alpha)\}$. The claim is that $[g]_\mu = E$.

Suppose $p : \omega_1 \to \omega_1$ is such that $[p]_\mu \in [g]_\mu$. Note $A = \{\alpha : p(\alpha) \in g(\alpha)\} \in \mu$. By modifying p off A, one may assume that for all α, $p(\alpha) \in g(\alpha)$. Let $f(\alpha)$ be the $\beta < h(\alpha)$ so that $F(\alpha, \beta) = p(\alpha)$. Then $p = F^f = F^{(\xi)} = \mathsf{enum}_E(\xi)$ for some $\xi < \zeta$. Then $[g]_\mu \subseteq E$. It is straightforward to see that $E \subseteq [g]_\mu$. $\qquad \square$

Fact 6.11. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Suppose* $D \subseteq \omega_2$ *is such that there is some* $g : \omega_1 \to \mathscr{P}(\omega_1)$ *so that* $[g]_\mu = D$. *Suppose* $E \subseteq \omega_2$ *is such that* $\sup E < \min D$, *then* $D \cup E \in \mathrm{ult}(V, \mu)$.

Proof. By Fact 6.10, $E \in \mathrm{ult}(V, \mu)$. Thus there is a $g' : \omega_1 \to \mathscr{P}(\omega_1)$ so that $[g']_\mu = E$. Let $g'' : \omega_1 \to \mathscr{P}(\omega_1)$ be defined by $g''(\alpha) = g(\alpha) \cup g'(\alpha)$. One can check that $[g'']_\mu = D \cup E$. $\qquad \square$

Fact 6.12. *Assume* $\mathsf{ZF} + \mathsf{AD}$ *and let* μ *be the club measure on* ω_1. *Let* $D = \{\alpha < \omega_2 : \mathrm{cof}(\alpha) = \omega_1\}$. *Then* $D \notin \mathrm{ult}(V, \mu)$.

Proof. Suppose $D \in \mathrm{ult}(V, \mu)$. By Fact 6.7, there is some $h : \omega_1 \to \mathscr{P}(\omega_1)$ so that $D = [h]_\mu$.
(Case I) $A = \{\alpha < \omega_1 : h(\alpha) \in \mu\} \in \mu$.
By Fact 4.8, there is a sequence $\langle C_\alpha : \alpha \in A \rangle$ of club subsets of ω_1 so that $C_\alpha \subseteq h(\alpha)$ for all $\alpha \in A$.
Define $\ell : \omega_1 \to \omega_1$ by $\ell(\alpha) = \mathsf{enum}_{C_{\mathsf{enum}_A(\alpha)}}(\omega)$. Let $\ell' : \omega_1 \to \omega_1$ be defined by

$$\ell'(\alpha) = \begin{cases} \mathsf{enum}_{C_\alpha}(\omega) & \alpha \in A \\ \min(h(\alpha)) & \alpha \notin A \end{cases}$$

Clearly $[\ell']_\mu \in [h]_\mu$. Note that $\ell = \ell' \circ \mathsf{enum}_A$. By Fact 5.1, $\ell =_\mu \ell'$. Define $g : \omega_1 \times \omega \to \omega$ by $g(\alpha, n) = \mathsf{enum}_{C_{\mathsf{enum}_A(\alpha)}}(n)$. The function g witnesses that ℓ has uniform cofinality ω. Thus $\mathrm{cof}([\ell]_\mu) = \omega$ by Fact 6.1. Since $[\ell]_\mu \in D = \{\alpha < \omega_2 : \mathrm{cof}(\alpha) = \omega_1\}$, one has a contradiction.
(Case II) $A = \{\alpha < \omega_1 : h(\alpha) \in \mu\} \notin \mu$.
Then $B = \omega_1 \setminus A = \{\alpha < \omega_1 : \omega_1 \setminus h(\alpha) \in \mu\} \in \mu$. By Fact 4.8, there is a sequence $\langle C_\alpha : \alpha \in B \rangle$ of club subsets of ω_1 so that $C_\alpha \subseteq \omega_1 \setminus h(\alpha)$. Define $\ell : \omega_1 \to \omega_1$ by $\ell(\alpha) = \mathsf{enum}_{C_{\mathsf{enum}_B(\alpha)}}(\alpha)$. Let $\ell' : \omega_1 \to \omega_1$ be defined by

$$\ell'(\alpha) = \begin{cases} \mathsf{enum}_{C_\alpha}(\alpha) & \alpha \in B \\ \min h(\alpha) & \alpha \notin B \end{cases}$$

Note that $[\ell']_\mu \notin [h]_\mu$. Observe $\ell = \ell \circ \mathsf{enum}_B$. By Fact 5.1, $\ell =_\mu \ell'$. Let $g : \mathcal{T}^{\mathsf{id}} \to \omega_1$ be defined by $g(\alpha, \beta) = \mathsf{enum}_{C_{\mathsf{enum}_B(\alpha)}}(\beta)$ if $\beta < \alpha$. Then g witnesses that ℓ has uniform cofinality id. Therefore, $\mathrm{cof}([\ell]_\mu) = \omega_1$ by Fact 6.4. However, $[\ell]_\mu \notin D = \{\alpha < \omega_2 : \mathrm{cof}(\alpha) = \omega_1\}$ yields a contradiction.
The proof is complete. $\qquad \square$

One can now show that μ does not satisfy Łoś's theorem and in fact the ultrapower does not satisfy the ZF axioms.

Fact 6.13. *Assume* $\mathsf{ZF} + \mathsf{AD} + V = L(\mathbb{R})$. *Let* μ *denote the club measure on* ω_1. *Then* $\mathrm{ult}(L(\mathbb{R}), \mu)$ *is not a model of* ZF. *Thus Łoś's theorem fails for* μ.

Proof. Note that $L(\mathbb{R}) \models \mathsf{AD}$ implies $L(\mathbb{R}) \models \mathsf{DC}_\mathbb{R}$ and hence $L(\mathbb{R}) \models \mathsf{DC}$ by a result of Kechris [15]. Thus $\mathrm{ult}(L(\mathbb{R}), \mu)$ may be considered a transitive inner model of $L(\mathbb{R})$. One can check that $\mathbb{R} \subseteq \mathrm{ult}(L(\mathbb{R}), \mu)$. If $\mathrm{ult}(L(\mathbb{R}), \mu)$ is an inner model of ZF containing all the reals of $L(\mathbb{R})$, then one must have that $\mathrm{ult}(L(\mathbb{R}), \mu) = L(\mathbb{R})$. This is impossible since Fact 6.12 asserts that $\mathrm{ult}(L(\mathbb{R}), \mu)$ is missing a subset of ω_2 which belongs to $L(\mathbb{R})$. $\qquad \square$

Theorem 6.14. *(Jackson) Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* μ *denote the club measure on* ω_1. *Define a partition* $P : [\omega_2]_*^{\omega_2} \to 2$ *by*

$$P(f) = 0 \Leftrightarrow \mathrm{rang}(f) \in \mathrm{ult}(V, \mu).$$

Then there is no club $D \subseteq \omega_2$ *and no* $i \in 2$ *so that* $P(f) = i$ *for all* $f \in [D]_*^{\omega_2}$.

Proof. (Case I) Suppose there is a club $D \subseteq \omega_2$ so that $P(f) = 1$ for all $f \in [D]_*^{\omega_2}$. By Fact 5.21, there is a club $C \subseteq \omega_1$ so that $[C]^{\omega_1}/\mu \subseteq D$.
Let $A = \{\alpha : (\exists \gamma)(\alpha = \mathsf{enum}_C(\gamma + \omega)\}$. Note that $\mathsf{enum}_A : \omega_1 \to \omega_1$ is a function of the correct type. Let $g : \omega_1 \times \omega \to \omega$ witness that enum_A has uniform cofinality ω. For each $\xi < \omega_1$, let $A_\xi = \{\alpha \in A : \alpha \geq \xi\}$. Let $g_\xi : \omega_1 \times \omega \to \omega$ be defined by $g_\xi(\alpha, n) = g(\mathsf{enum}_A^{-1}(\mathsf{enum}_{A_\xi}(\alpha)), n)$. Then for each $\xi < \omega_1$, g_ξ

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witnesses that enum_{A_ξ} has uniform cofinality ω. Let $h : \omega_1 \to \mathscr{P}(\omega_1)$ be defined by $h(\xi) = A_\xi$. By Fact 6.8, $\mathsf{enum}_{[h]_\mu} : \omega_2 \to \omega_2$ is a function of the correct type. Let $f = \mathsf{enum}_{[h]_\mu}$. One can check that $\mathrm{rang}(f) = [h]_\mu \subseteq [C]^{\omega_1}/\mu$. Thus $f \in [D]^{\omega_2}_*$. However since $\mathrm{rang}(f) = [h]_\mu \in \mathrm{ult}(V, \mu)$, one must have that $P(f) = 0$. Contradiction.

(Case II) Suppose there is a club $D \subseteq \omega_2$ so that $P(f) = 0$ for all $f \in [D]^{\omega_2}_*$.

Fix any $f \in [D]^{\omega_2}_*$. Since $P(f) = 0$, there is some $h : \omega_1 \to \mathscr{P}(\omega_1)$ so that $\mathrm{rang}(f) = [h]_\mu$ by Fact 6.7.

Now let $A \subseteq \omega_2$ be arbitrary. $\mathsf{enum}_{f[A]} : \omega_2 \to D$ is a function of the correct type since $f : \omega_2 \to D$ is a function of the correct type. Then $P(\mathsf{enum}_{f[A]}) = 0$. Thus $f[A] \in \mathrm{ult}(V, \mu)$. There is some $g : \omega_1 \to \mathscr{P}(\omega_1)$ so that $[g]_\mu = f[A]$. Let $k : \omega_1 \to \mathscr{P}(\omega_1)$ be defined by $k(\alpha) = \mathsf{enum}^{-1}_{h(\alpha)}[g(\alpha)]$.

The claim is that $[k]_\mu = A$.

To show $[k]_\mu \subseteq A$: Suppose $\xi \in [k]_\mu$. There is a representative $\ell : \omega_1 \to \omega_1$ of ξ so that for all μ-almost all α, $\ell(\alpha) \in k(\alpha)$. There is a function $p : \omega_1 \to \omega_1$ so that for μ-almost all α, $p(\alpha) \in g(\alpha)$ and $\ell(\alpha) = \mathsf{enum}^{-1}_{h(\alpha)}(p(\alpha))$. In particular $[p]_\mu \in [g]_\mu = f[A]$. By Fact 6.9, the ξth element of $[h]_\mu$ is represented by the function $q : \omega_1 \to \omega_1$ defined by $q(\alpha) = \mathsf{enum}_{h(\alpha)}(\ell(\alpha))$. However μ-almost everywhere, $q(\alpha) = \mathsf{enum}_{h(\alpha)}(\mathsf{enum}^{-1}_{h(\alpha)}(p(\alpha))) = p(\alpha)$. Thus $q =_\mu p$. It has been shown that the ξth element of $[h]_\mu = \mathrm{rang}(f)$ belongs to $[g]_\mu = f[A]$. Thus $\xi \in A$.

To show that $A \subseteq [k]_\mu$: Let $\xi \in A$. Let $\ell : \omega_1 \to \omega_1$ be such that $[\ell]_\mu = \xi$. By Fact 6.9, the ξth-element of $[h]$ is represented by the function $q(\alpha) = \mathsf{enum}_{h(\alpha)}(\ell(\alpha))$. Thus $[q]_\mu \in f[A] = [g]_\mu$. So μ-almost everywhere, $q(\alpha) = \mathsf{enum}_{h(\alpha)}(\ell(\alpha)) \in g(\alpha)$. Thus for μ-almost all α, $\ell(\alpha) \in \mathsf{enum}^{-1}_{h(\alpha)}[g(\alpha)] = k(\alpha)$. It has been shown that $\xi = [\ell]_\mu \in [k]_\mu$.

The claim has been shown. Since A was arbitrary, this implies that everywhere subset of ω_2 belongs to $\mathrm{ult}(V, \mu)$. This contradicts Fact 6.12.

It has been shown that the partition P has no club set which is homogeneous. $\qquad\square$

Corollary 6.15. *(Martin and Paris) Assume* $\mathsf{ZF} + \mathsf{AD}$. *The partition relation* $\omega_2 \to (\omega_2)^{\omega_2}_2$ *does not hold. Thus* ω_2 *is a weak partition cardinal which is not a strong partition cardinal.*

Remark 6.16. The example of Jackson from Theorem 6.14 gives an explicit example of a partition $P : [\omega_2]^{\omega_2}_* \to 2$ which has no homogeneous club subset.

Martin and Paris original argument roughly shows that if $\omega_2 \to (\omega_2)^{\omega_2}_2$ holds, then ω_3 would satisfy $\omega_3 \to (\omega_3)^\alpha_2$ for all $\alpha < \omega_1$. Fact 2.7 implies that ω_3 must be regular. However, it can be shown that ω_3 is a singular cardinal of cofinality ω_2. For more information on the result of Martin and Paris, see [20] and specifically Lemma 5.19. Also see Section 13 of [14].

Corollary 6.17. *Let* $\sigma \in [\omega_2]^{<\omega_2}_*$. *Define* $P_\sigma : [\omega_2 \setminus (\sup(\sigma) + \omega)]^{\omega_2}_* \to 2$ *by* $P(\sigma ^\frown f)$, *where* P *is the partition from Theorem 6.14.* P_σ *also does not have have a club homogeneous set.*

Proof. Essentially the same argument as Theorem 6.14 with the assistance of Fact 6.11. $\qquad\square$

7. $L(\mathbb{R})$ as a Symmetric Collapse Extension of HOD

Definition 7.1. Let $S \subseteq \mathrm{ON}$ be a set of ordinals. Let φ be a formula of set theory. The pair (S, φ) is called an ∞-Borel code. Let $n \in \omega$. Define $\mathfrak{B}^n_{(S,\varphi)} = \{x \in \mathbb{R}^n : L[S, x] \models \varphi(S, x)\}$.

Let $A \subseteq \mathbb{R}^n$. (S, φ) is said to be an ∞-Borel code for A if and only if $A = \mathfrak{B}^n_{(S,\varphi)}$.

If a set $A \subseteq \mathbb{R}^n$ has an ∞-Borel code, then A has a very absolute definition. That is, in ordered to determine membership of x in A, one needs only to ask whether $\varphi(S, x)$ holds in $L[S, x]$, which is the minimal model of ZFC containing x and the code set S.

Definition 7.2. Let P be a set. Recall a set A is ordinal definable from P if and only if there is a formula φ of set theory, a finite tuple of ordinals $\bar\alpha$, and a finite tuple $\bar p$ of elements of P so that $A = \{x : \varphi(x, \bar\alpha, \bar p)\}$.

Using the reflection theorem, one can show that a set A is ordinal definable if and only if there is some $\xi \in \mathrm{ON}$, tuple of ordinals $\bar\alpha$, tuple $\bar p$ from P, and formula φ so that all these objects belong to V_ξ and $A = \{x \in V_\xi : V_\xi \models \varphi(x, \bar\alpha, \bar p)\}$.

This shows the collection OD_P of all sets which are ordinal definable from P forms a first order class. If P has an OD_P wellordering, then OD_P has a wellordering which is definable with parameter from ordinals and P. Thus there is a bijection of OD_P with ON.

Let HOD_P denote the subclass of OD_P which is hereditarily OD_P. That is, HOD_P consists of those $x \in \mathrm{OD}_P$ such that $\mathrm{tc}(\{x\}) \subseteq \mathrm{OD}_P$, where tc refers to the transitive closure.

As a matter of convention, if $S \subseteq \mathrm{ON}$ is a set of ordinals, one will often write OD_S and HOD_S for $\mathrm{OD}_{\{S\}}$ and $\mathrm{HOD}_{\{S\}}$.

Definition 7.3. Let $n \in \omega$ and $S \subseteq \mathrm{ON}$ be a set of ordinals. Let $_n\mathbb{O}_S$ denote the forcing of nonempty ordinal definable in S subsets of $\mathbb{R}^n$. Let the ordering be $\leq_{n\mathbb{O}_S} = \subseteq$. The largest element is $1_{n\mathbb{O}_S} = \mathbb{R}^n$.

Since there is a definable (in S) bijection of the class OD_S with the ordinal ON, one can identify $_n\mathbb{O}_S$ as a set of ordinals in HOD_S. In this way, $_n\mathbb{O}_S \in \mathrm{HOD}_S$. $_n\mathbb{O}_S$ is called the n-dimensional S-Vopěnka forcing. If $n = 1$, $_1\mathbb{O}_S$ will be denoted simply $\mathbb{O}_S$.

Definition 7.4. Let $n \in \omega$ and $S \subseteq \mathrm{ON}$ be a set of ordinals. Let $_n\mathbb{A}_S$ denote the forcing of nonempty subsets of $\mathbb{R}^n$ which possess OD_S ∞-Borel codes. $_n\mathbb{A}_S$ is ordered by $\leq_{n\mathbb{A}_S} = \subseteq$. It has a largest element $1_{n\mathbb{O}_S} = \mathbb{R}^n$.

Since $_n\mathbb{A}_S \subseteq {}_n\mathbb{O}_S$, one can consider $_n\mathbb{A}_S$ to be a forcing of HOD_S. $_n\mathbb{A}_S$ will be called the n-dimensional OD_S ∞-Borel code forcing.

Remark 7.5. One can be more specific about how $_n\mathbb{A}_S$ is coded as a set of ordinals. One can identify $_n\mathbb{A}_S$ with a (set sized) collection of pairs of (S', φ), where S' is a OD_S set of ordinals and φ is a formula. Using the canonical global wellordering of HOD_S, let $\langle (S_\alpha, \varphi_\alpha) : \alpha < \delta \rangle$, for some ordinal δ, be an enumeration of ∞-Borel codes that include at least one code for each element of $_n\mathbb{A}_S$. Fix $\langle \phi_n : n \in \omega \rangle$ to be a coding of formulas of set theory by natural numbers. Using the Gödel pairing function, let $K = \{(\alpha, \beta, n) : \beta \in S_\alpha \wedge \phi_n = \varphi_\alpha\}$. One will often identify $_n\mathbb{A}_S$ with this set of ordinals K. If this is done, then from $_n\mathbb{A}_S$, one can obtain uniformly ∞-Borel codes for each condition in $_n\mathbb{A}_S$.

In nearly every regard, the Vopěnka forcing is a more practical forcing than $\mathbb{A}_S$. It will shown that in $\mathsf{ZF} + \mathsf{AD} + V = L(\mathbb{R})$, $\mathbb{O}_S$ and $\mathbb{A}_S$ are identical. To establish this, one will prove a structural theorem about $L(\mathbb{R})$ due to Woodin that involves the forcing $\mathbb{A}_S$. The presentation of Woodin's result that $L(\mathbb{R})$ is a symmetric collapse extension of its HOD follows closely [21]. Of particular importance to the study of cardinals and combinatorics in $L(\mathbb{R}) \models \mathsf{AD}$ will be existence of an ultimate ∞-Borel code which follows from the proof.

For simplicity, $S = \emptyset$ in the following results. The result can be appropriately relativized.

The main benefit of $\mathbb{A}$ over $\mathbb{O}$ is the following result:

Fact 7.6. *Let $x \in \mathbb{R}^n$. Then there is a generic filter $G_x \subseteq {}_n\mathbb{A}$ which is $_n\mathbb{A}$-generic over HOD so that $\mathrm{HOD}[G_x] = \mathrm{HOD}[x]$, where $\mathrm{HOD}[x]$ refers to the smallest transitive model of ZF extending HOD and containing x.*

Proof. For simplicity, let $n = 1$. Let $x \in \mathbb{R}$. Let $G_x = \{p \in \mathbb{A} : x \in p\}$.

Using the convention of Remark 7.5, from $\mathbb{A} \in \mathrm{HOD}$, one can obtain an enumeration, $\langle (S_p, \varphi_p) : p \in \mathbb{A} \rangle$ so that (S_p, φ_p) is an OD ∞-Borel for the condition p.

First to show that G_x is an $\mathbb{A}$-generic filter over HOD: Let $A \subseteq \mathbb{A}$ be a maximal antichain which belongs to HOD and is hence OD. Considering $\mathbb{A}$ as the set K defined in Remark 7.5 and using the fact that A is OD, one can find a formula φ so that $(\mathbb{A}, \varphi)$ is an OD-Borel code for $\bigcup A$. Therefore, $\bigcup A = \mathbb{R}$ since otherwise $\mathbb{R} \setminus \bigcup A$ would be a nonempty set with an OD ∞-Borel code. Then $\mathbb{R} \setminus \bigcup A$ is a condition of $\mathbb{A}$ which is incompatible with every element of A. This contradicts A being a maximal antichain. Thus $x \in \bigcup A$. There is some $a \in A$ so that $x \in a$. Thus $a \in G_x$. It has been shown that $G_x \cap A \neq \emptyset$. G_x is $\mathbb{A}$-generic over HOD.

Thinking of $\mathbb{R} = {}^\omega 2$, let $b_n = \{x \in \mathbb{R} : x(n) = 1\}$. Note that $b_n \neq \emptyset$ and b_n clearly has an OD ∞-Borel code. Thus $b_n \in \mathbb{A}$. Note that $b_n \in G_x$ if and only if $x(n) = 1$. Thus $x \in \mathrm{HOD}[G_x]$.

Note that $p \in G_x$ if and only if $x \in p$ if and only if $L[S_p, x] \models \varphi_p(S_p, x)$. Note that $V \models L[S_p, x] \models \varphi_p(S_p, x)$ if and only if $\mathrm{HOD}[x] \models L[S_p, x] \models \varphi_p(S_p, x)$. (This is an application of the important absoluteness property of ∞-Borel codes.) Thus G_x can be defined in $\mathrm{HOD}[x]$ as the set of p such that $L[S_p, x] \models \varphi_p(S_p, x)$. This shows $G_x \in \mathrm{HOD}[x]$.

It has been shown that $\mathrm{HOD}[x] = \mathrm{HOD}[G_x]$. $\qquad \square$

Definition 7.7. Suppose $n \in \omega$. For each $s \in {}^n\omega$, let $b_s = \{x \in \mathbb{R}^n : x(s) = 1\}$.

Let $\dot{x}_{\text{gen}}^n = \{(\check{s}, b_s) : s \in {}^n\omega\}$. In light of the argument of Fact 7.6, if $x \in \mathbb{R}^n$, then $\dot{x}_{\text{gen}}^n[G_x] = x$.

Definition 7.8. Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcings. A surjective map $\pi : \mathbb{Q} \to \mathbb{P}$ is a forcing projection if and only if the following holds:

(1) For all $q_0, q_1 \in \mathbb{Q}$, $q_0 \leq_{\mathbb{Q}} q_1$ implies $\pi(q_0) \leq_{\mathbb{P}} \pi(q_1)$ and $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$.

(2) For all $q \in \mathbb{Q}$ and $p' \in \mathbb{P}$ such that $p' \leq_{\mathbb{P}} \pi(q)$, there exists some $q' \leq_{\mathbb{Q}} q$ so that $\pi(q') = p'$.

Let $\pi : \mathbb{Q} \to \mathbb{P}$ be a projection. If $D \subseteq \mathbb{P}$ is dense, then $\pi^{-1}[D]$ is dense in $\mathbb{Q}$. This implies that if $G \subseteq \mathbb{Q}$ is a $\mathbb{Q}$-generic over V, then $\pi[G]$ is a $\mathbb{P}$-generic over V.

Let $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V, then let $\mathbb{Q}/G = \{q \in \mathbb{Q} : \pi(q) \in G\}$. Let $\leq_{\mathbb{Q}/G} = \leq_{\mathbb{Q}} \restriction \mathbb{Q}/G$.

Let $\dot{G} \in V^{\mathbb{P}}$ denote the $\mathbb{P}$-name for the generic filter. Let $\mathbb{Q}/\dot{G}$ denote the $\mathbb{P}$-name so that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \mathbb{Q}/\dot{G} = \{p \in \check{\mathbb{Q}} : \check{\pi}(p) \in \dot{G}\}$.

It can be checked that $\mathbb{Q}$ embeds densely into the iteration $\mathbb{P} * (\mathbb{Q}/\dot{G})$. Therefore, these two forcings are equivalent as forcings. Also if G is $\mathbb{P}$-generic over V then $(\mathbb{Q}/\dot{G})[G] = \mathbb{Q}/G$.

Suppose H is $\mathbb{Q}$-generic over V. Let $G = \pi[H]$ be the associated $\mathbb{P}$-generic filter. One can check that H is $(\mathbb{Q}/G)$-generic over $V[G]$, $G * H$ is $(\mathbb{P} * \mathbb{Q}/\dot{G})$-generic over V and $V[G][H] = V[G * H] = V[H]$.

Definition 7.9. For the moment, consider $\mathbb{R} = {}^\omega 2 = \mathscr{P}(\omega)$. If $x, y \in \mathbb{R}$, then one writes $x \leq_T y$ if and only if there is a Turing machine (taking oracle input) so that x can be computed from this Turing machine when given y as its oracle. Since Turing programs can be coded by natural numbers, for any $x \in \mathbb{R}$, there are only countably many $y \in \mathbb{R}$ so that $y \leq_T x$. A Turing program is also absolute between models of ZF with the same ω.

Define $x =_T y$ if and only if $x \leq_T y$ and $y \leq_T x$.

Let $\mathcal{D} = \mathscr{P}(\omega)/ =_T$ denote the collection of $=_T$ equivalence classes. An element of $\mathcal{D}$ is called a Turing degree. If $x \in \mathbb{R}$, then one uses the notation $[x]_T$ rather than $[x]_{=_T}$ to denote the Turing degree of x. As observed above, each Turing degree contains only countable many reals.

If $X, Y \in \mathcal{D}$, one defines $X \leq Y$ if and only if there exist $x \in X$ and $y \in Y$ so that $x \leq_T y$. (One can check this is a well defined relation.)

A Turing cone of reals with base x is the collection $C_x = \{y \in \mathbb{R} : x \leq_T y\}$. A Turing cone of degrees with base X is the collection $C_X = \{Y \in \mathcal{D} : X \leq Y\}$.

Let $\mu_{\mathcal{D}} \subseteq \mathscr{P}(\mathcal{D})$ consists of those subsets of $\mathcal{D}$ which contain a Turing cone of degrees. $\mu_{\mathcal{D}}$ is a filter.

Fact 7.10. *(Martin) Assume* ZF + AD. *$\mu_{\mathcal{D}}$ is a countably complete ultrafilter.*

Proof. Let $K \subseteq \mathcal{D}$. Let $\tilde{K} = \{x \in \mathbb{R} : x \in K\}$. Note that $\tilde{K}$ is a $=_T$-invariant subset of $\mathbb{R}$.

Consider the usual game $G_{\tilde{K}}$

$$
\begin{array}{ccccccc}
 & \text{I} & a(0) & & a(1) & & a(2) & & a(3) & & a \\
G_{\tilde{K}} & & & & & & & & & & \\
 & \text{II} & & b(0) & & b(1) & & b(2) & & b(3) & & b
\end{array}
$$

where Player 1 wins if and only if $a \oplus b \in \tilde{K}$, where $a \oplus b$ is defined by $a \oplus b(2n) = a(n)$ and $a \oplus b(2n+1) = b(n)$.

By AD, one of the two players has a winning strategy. Suppose Player 1 has a winning strategy σ.

Let $Z = [\sigma]_T$ be the Turing degree of σ. The claim is that $C_Z \subseteq K$:

Let $Y \in \mathcal{D}$ be such that $Z \leq_T Y$. Pick any $y \in Y$. Thus $\sigma \leq_T y$. Let $\sigma * y$ denote the result of the play where Player 1 uses σ and Player 2 plays the bits of y each turn. Since σ is a Player 1 winnings strategy, $\sigma * y \in \tilde{K}$. Since $\sigma \leq_T y$, $\sigma * y \leq_T y$ and clearly $y \leq_T \sigma * y$. Thus $\sigma * y =_T y$. Since $\tilde{K}$ is $=_T$-invariant, $y \in \tilde{K}$. Thus $[y]_T = Y \in K$.

This shows that if Player 1 has a winning strategy in $G_{\tilde{K}}$, then $K \in \mu_{\mathcal{D}}$. A similar argument shows that if Player 2 has a winning strategy then $\mathcal{D} \setminus K \in \mu_{\mathcal{D}}$. Thus $\mu_{\mathcal{D}}$ is an ultrafilter.

Next to show countable completeness: Suppose $\langle K_n : n \in \omega \rangle$ is a sequence in $\mu_{\mathcal{D}}$. By $\mathsf{AC}_\omega^{\mathbb{R}}$, let x_n be such that the cone above $X_n = [x_n]_T$ is contained in K_n. Let $x = \bigoplus x_n = \{(n, m) : m \in x_n\}$ and $X = [x]_T$. Then the cone above X is contained inside of $\bigcap_{n \in \omega} K_n$. Thus $\bigcap_{n \in \omega} K_n \in \mu_{\mathcal{D}}$. □

Remark 7.11. Let S be a set. Note that if $x =_T y$, then $L[S, x] = L[S, y]$. Therefore, if $X \in \mathcal{D}$, one will often write $L[S, X]$ to denote $L[S, x]$ for any $x \in X$.

The canonical constructibility wellordering is based on the hierarchy $\{L_\alpha[S,x] : \alpha \in \mathrm{ON}\}$. Even if $x =_T y$, the levels $L_\alpha[S,x]$ and $L_\alpha[S,y]$ can differ. Thus, the canonical constructibility wellordering on $L[S,x]$ is not invariant under $=_T$.

If V is a model of ZF (in the language $\dot{\in}$), then the canonical wellordering $<^{\mathrm{HOD}_S^V}$ of HOD_S^V depends only on V. Since $x =_T y$ implies that $L[S,x] = L[S,y]$, one has that $\mathrm{HOD}_S^{L[S,x]} = \mathrm{HOD}_S^{L[S,y]}$ and $<^{\mathrm{HOD}_S^{L[S,x]}} = <^{\mathrm{HOD}_S^{L[S,y]}}$. (Note that although the constructibility hierarchy of $L[S,x]$ is naturally formulated in the language $\{\dot{\in}, \dot{E}_0, \dot{E}_1\}$, where $\dot{E}_0$ and $\dot{E}_1$ are unary predicate symbols meant to interpret S and x, when constructing $\mathrm{HOD}_S^{L[S,x]}$, $L[S,x]$ is considered as merely a $\{\dot{\in}\}$-structure.) Thus if X is a Turing degree, one will often write $\mathrm{HOD}_S^{L[S,X]}$ and $\leq^{\mathrm{HOD}_S^{L[S,X]}}$ to refer to $\mathrm{HOD}_S^{L[S,x]}$ and $<^{\mathrm{HOD}_S^{L[S,x]}}$ for any $x \in X$.

The invariance of the canonical wellordering of HOD allows one to take ultrapowers of local HOD's (models of the form $\mathrm{HOD}_S^{L[S,X]}$) by the Martin's measure. This is a very powerful technique as indicated by the following results.

The existence of the canonical wellorderings of the local $\mathrm{HOD}_S^{L[S,X]}$ shows that the resulting ultraproduct satisfies Łoś's theorem:

Fact 7.12. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* $S \subseteq \mathrm{ON}$ *be a set of ordinals.* $\prod_{X \in \mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu_\mathcal{D}$ *satisfies the Łoś's theorem: Let* φ *be a formula. Let* $f_0, ..., f_{n-1}$ *be functions on* $\mathcal{D}$ *with the property that for all* $X \in \mathcal{D}$, $f_i(X) \in \mathrm{HOD}_S^{L[S,X]}$. *Then*

$$\prod_{X \in \mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu_\mathcal{D} \models \varphi([f_0]_{\mu_\mathcal{D}}, ..., [f_{n-1}]_{\mu_\mathcal{D}}) \Leftrightarrow \{X \in \mathcal{D} : \mathrm{HOD}_S^{L[S,X]} \models \varphi(f_0(X), ..., f_{n-1}(X))\} \in \mu_\mathcal{D}.$$

Proof. Only the existential quantification case requires a choice-like principle. One will give a sketch:

Let $\mathcal{M}$ denote this ultraproduct. Let φ be a formula and assume inductively one has already shown the result for φ. Suppose

$$K = \{X \in \mathcal{D} : \mathrm{HOD}_S^{L[S,X]} \models (\exists v)\varphi(v, f_0(X), ..., f_{n-1}(X))\} \in \mu_\mathcal{D}.$$

Let g be defined on K by letting $g(X)$ be the $\mathrm{HOD}_S^{L[S,X]}$-least v so that $\mathrm{HOD}_S^{L[S,X]} \models \varphi(v, f_0(X), ..., f_{n-1}(X))$. Then using the induction hypothesis, one can show that $\mathcal{M} \models \varphi([g]_{\mu_\mathcal{D}}, [f]_{\mu_\mathcal{D}}, ..., [f_{n-1}]_{\mu_\mathcal{D}})$. Thus $\mathcal{M} \models (\exists v)\varphi(v, [f]_{\mu_\mathcal{D}}, ..., [f_{n-1}]_{\mu_\mathcal{D}})$. $\qquad\square$

There is no claim that $\prod_{X \in \mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu_\mathcal{D}$ is a wellfounded model. This is true assuming DC. It is open whether AD implies this.

Fact 7.13. *(Woodin) Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$. *Suppose* $A \subseteq \mathbb{R}^2$ *has an* ∞*-Borel code* (S, φ), *then* $B(x) = (\exists^\mathbb{R} y)A(x,y)$ *has an* ∞*-Borel code which is* OD_S.

Proof. Note that $L(S, \mathbb{R}) \models \mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$. Work in $L(S, \mathbb{R})$.

Let $\mathcal{M} = \prod_{X \in \mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu_\mathcal{D}$. $\mathcal{M}$ is an S-definable class which is wellfounded by DC. Implicitly, it will be assumed that all objects in this ultrapower have been Mostowski collapsed. By Fact 7.12, $\mathcal{M}$ satisfies Łoś's theorem. Define $\Phi_{\mathbb{A}_S^\infty}$ on $\mathcal{D}$ by $\Phi_{\mathbb{A}_S^\infty}(X) = \mathbb{A}_S^{L[S,X]}$. (Recall Remark 7.5 concerning the convention on $\mathbb{A}_S$.) Let $\mathbb{A}_S^\infty = [\Phi_{\mathbb{A}_S^\infty}]_{\mu_\mathcal{D}}$. By Łoś's theorem, $\mathbb{A}_S^\infty$ is a forcing poset. Let $\lambda_X = |\mathbb{A}_S|^{\mathrm{HOD}_S^{L[S,X]}}$. Let Φ_λ be a function on $\mathcal{D}$ defined by $\Phi_\lambda(X) = \lambda_X$. Let $\lambda = [\Phi_\lambda]_{\mu_\mathcal{D}}$. By Łoś's theorem, $\mathcal{M} \models \lambda = |\mathbb{A}_S^\infty|$. Let Φ_{S^∞} be a function of $\mathcal{D}$ defined by $\Phi_{S^\infty}(X) = S$. Let $S^\infty = [\Phi_{S^\infty}]_{\mu_\mathcal{D}}$.

Claim: For all $a \in \mathbb{R}$,

$$a \in B \Leftrightarrow L[S^\infty, \mathbb{A}_S^\infty, a] \models 1_{\mathrm{Coll}(\omega, \lambda)} \Vdash (\exists b) L[S^\infty, a, b] \models \varphi(S^\infty, a, b).$$

To prove the claim: First observe that for all $X \in \mathcal{D}$, λ_X is countable in $L(S, \mathbb{R})$ which is a model of AD. To see this: Note that $\mathbb{R}^{L[S,X]}$ is countable since it is a wellorderable collection of reals; thus, there is a bijection of ω with $\mathbb{R}^{L[S,X]}$. $\mathbb{A}_S^{L[S,X]}$ is a collection of subsets of $\mathbb{R}^{L[S,X]}$ in $L[S,X]$. Identifying $\mathbb{R}^{L[S,X]}$ with ω, this collection $\mathbb{A}_S^{L[S,X]}$ can be identified as a wellorderable collection of $\mathbb{R}$, as well. Thus $\lambda_X = |\mathbb{A}_S|^{L[S,X]}$ is countable in $L(S, \mathbb{R})$. By the same reasoning, $(2^{\lambda_X})^{L[S,X]}$ is countable in $L(S, \mathbb{R})$.

($\Leftarrow$) For all $X \in \mathcal{D}$ so that $a \in X$, one can define $\mathrm{HOD}_S^{L[S,X]}[a]$ as in Fact 7.6. Let

$$\mathcal{M}[a] = \prod_{X \in \mathcal{D}} \mathrm{HOD}_S^{L[S,X]}[a]/\mu_{\mathcal{D}}.$$

Assume that

$$V \models L[S^\infty, \mathbb{A}_S^\infty, a] \models 1_{\mathrm{Coll}(\omega,\lambda)} \Vdash (\exists b) L[S^\infty, a, b] \models \varphi(S^\infty, a, b).$$

Thus

$$\mathcal{M}[a] \models L[S^\infty, \mathbb{A}_S^\infty, a] \models 1_{\mathrm{Coll}(\omega,\lambda)} \Vdash (\exists b) L[S^\infty, a, b] \models \varphi(S^\infty, a, b).$$

By the idea of the proof of Łoś's theorem (Fact 7.12), for $\mu_{\mathcal{D}}$-almost all $X \in \mathcal{D}$ with $a \in X$,

$$L[S, \mathbb{A}_S^{L[S,X]}, a] \models 1_{\mathrm{Coll}(\omega,\lambda_X)} \Vdash (\exists b) L[S, a, b] \models \varphi(S, a, b).$$

Fix such an X. Since $(2^{\lambda_X})^{L[S,X]}$ is countable in $L(S, \mathbb{R})$, there is a $g \in L(S, \mathbb{R})$ so that $g \subseteq \mathrm{Coll}(\omega, \lambda_X)$ is $\mathrm{Coll}(\omega, \lambda_X)$-generic over $\mathrm{HOD}_S^{L[S,X]}[a]$. Then g is also generic over $L[S, \mathbb{A}_S^{L[S,X]}, a]$. By the forcing theorem, $L[S, \mathbb{A}_S^{L[S,X]}, a][g] \models (\exists b) L[S, a, b] \models \varphi(S, a, b)$. Pick some $b \in L[S, \mathbb{A}_S^{L[S,X]}, a][g]$ witnessing the existential. Then one has in particular that $L[S, a, b] \models \varphi(S, a, b)$. Since (S, φ) is the ∞-Borel code for A, one has that $(a, b) \in A$. Thus $a \in B$.

($\Rightarrow$) Suppose that $a \in B$. Let $b \in \mathbb{R}$ be such that $(a, b) \in A$. Let X be a Turing degree such that $[a \oplus b] \leq X$. By Fact 7.6, let $G_{a \oplus b}$ be the $\mathbb{A}_S^{L[S,X]}$-generic over $\mathrm{HOD}_S^{L[S,X]}$-filter derived from $a \oplus b$. Note that $G_{a \oplus b}$ is also $\mathbb{A}_S^{L[S,X]}$-generic over $L[S, \mathbb{A}_S^{L[S,X]}]$. Using the convention from Remark 7.5 and an argument similar to Fact 7.6, one can recover $a \oplus b$ from $G_{a,b}$ in $L[S, \mathbb{A}_S^{L[S,X]}][G_{a,b}]$. Thus

$$L[S, \mathbb{A}_S^{L[S,X]}][G_{a \oplus b}] \models (\exists y) L[S, a, y] \models \varphi(S, a, y).$$

Since $|\mathbb{A}_S^{L[S,X]}|$ is a forcing of size λ_X, one has that $\mathbb{A}_S^{L[S,X]}$ regularly embeds into $\mathrm{Coll}(\omega, \lambda_X)$ by [13] Corollary 26.8. There is some $g \subseteq \mathrm{Coll}(\omega, \lambda_X)$ which is generic over $L[S, \mathbb{A}_S^{L[S,X]}]$ so that $G_{a \oplus b} \in L[S, \mathbb{A}_S^{L[S,X]}][g]$. Thus

$$L[S, \mathbb{A}_S^{L[S,X]}][g] \models (\exists y) L[S, a, y] \models \varphi(S, a, y).$$

Note that $a \in L[S, \mathbb{A}_S^{L[S,X]}][g]$. By one of the main properties of the collapse forcing ([13] Corollary 26.10), there is some $h \subseteq \mathrm{Coll}(\omega, \lambda_X)$ which is $\mathrm{Coll}(\omega, \lambda_X)$-generic over $L[S, \mathbb{A}_S^{L[S,X]}][a]$ so that $L[S, \mathbb{A}_S^{L[S,X]}][a][h] = L[S, \mathbb{A}_S^{L[S,X]}][g]$. Thus

$$L[S, \mathbb{A}_S^{L[S,X]}][a][h] \models (\exists y) L[S, a, y] \models \varphi(S, a, y).$$

There is some $p \in \mathrm{Coll}(\omega, \lambda_X)$ which forces the inner statement. By the homogeneity of $\mathrm{Coll}(\omega, \lambda_X)$, $1_{\mathrm{Coll}(\omega,\lambda_X)}$ forces this statement:

$$L[S, \mathbb{A}_S^{L[S,X]}][a] \models 1_{\mathrm{Coll}(\omega,\lambda_X)} \Vdash (\exists y) L[S, a, y] \models \varphi(S, a, y).$$

In particular,

$$\mathrm{HOD}_S^{L[S,X]}[a] \models L[S, \mathbb{A}_S^{L[S,X]}, a] \models 1_{\mathrm{Coll}(\omega,\lambda_X)} \Vdash (\exists y) L[S, a, y] \models \varphi(S, a, y).$$

By Łoś's theorem (Fact 7.12), one has

$$\mathcal{M}[a] \models L[S^\infty, \mathbb{A}_S^\infty, a] \models 1_{\mathrm{Coll}(\omega,\lambda)} \Vdash (\exists y) L[S^\infty, a, y] \models \varphi(S^\infty, a, y).$$

So in particular,

$$L[S^\infty, \mathbb{A}_S^\infty, a] \models 1_{\mathrm{Coll}(\omega,\lambda)} \Vdash (\exists y) L[S^\infty, a, y] \models \varphi(S^\infty, a, y).$$

This completes the proof of the claim.

Let J be a set of ordinals that codes S^∞ and $\mathbb{A}_S^\infty$ in some fixed way. Let $\psi(J, a)$ be the formula that asserts

$$1_{\mathrm{Coll}(\omega,\lambda)} \Vdash (\exists y) L[S^\infty, a, y] \models \varphi(S^\infty, a, y).$$

(J, ψ) is an ∞-Borel code for B. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Fact 7.14. *(Woodin) Assume* ZF + AD + DC. *Let* $1 \leq m \leq n$. *Let* $\pi_{n,m} : \mathbb{R}^n \to \mathbb{R}^m$ *be defined by* $\pi_{n,m}(s) = s \upharpoonright m$. *Define* $\pi_{n,m} : {}_n\mathbb{A} \to {}_m\mathbb{A}$ *by* $\pi_{n,m}(b) = \pi_{n,m}[b]$, *where the latter* $\pi_{n,m}$ *refers to the earlier function* $\pi_{n,m} : \mathbb{R}^n \to \mathbb{R}^m$. *Then* $\pi_{n,m}$ *is a forcing projection.*

Proof. By Fact 7.13, one can show that $\pi_{n,m}(p) \in {}_m\mathbb{A}$ for all $p \in {}_n\mathbb{A}$.

Now suppose that $q \in {}_n\mathbb{A}$. Suppose $p' \in {}_m\mathbb{A}$ is such that $p' \leq_{m\mathbb{A}} \pi_{n,m}(q)$. Let $q' = \{x \in \mathbb{R}^n : x \upharpoonright n \in \pi_{n,m}(q)\}$. Since $\pi_{n,m}(q) \in {}_m\mathbb{A}$ (that is, it has an OD ∞-Borel code), q' has an OD ∞-Borel code and hence belongs to ${}_n\mathbb{A}$. Clearly, $\pi_{n,m}(q') = p'$. This establishes that $\pi_{n,m}$ is a forcing projection. $\square$

Now one will work in $L(\mathbb{R}) \models \mathsf{AD}$. Kechris ([15]) showed that if AD holds, then $L(\mathbb{R}) \models \mathsf{DC}_\mathbb{R}$. Thus, for the following results, the background theory $\mathsf{ZF} + \mathsf{AD}$ is sufficient.

Definition 7.15. Using the projections from Fact 7.14, let ${}_\omega\mathbb{A}$ be the finite support direct limit of $\langle {}_n\mathbb{A} : n \in \omega \setminus \{0\}\rangle$. That is, ${}_\omega\mathbb{A}$ is the collection of $p : (\omega \setminus \{0\}) \to \bigcup_{n\in\omega} {}_n\mathbb{A}$ so that for all $m \leq n$, $\pi_{n,m}(p(n)) = p(m)$ and there exists a $N \in \omega$ so that for all $k \geq N$, $p(k) = p(N) \times \mathbb{R}^{k-N}$. The least such N is denoted $\dim(p)$, the dimension of p. For $n \in \omega$ and $p \in {}_\omega\mathbb{A}$, let $\pi_{\omega,n}(p) = p(n)$. Each $\pi_{\omega,n} : {}_\omega\mathbb{A} \to {}_n\mathbb{A}$ is a forcing projection.

Since each ${}_n\mathbb{A} \in \mathrm{HOD}$ is identified as a set of ordinals (see Remark 7.5) and the projection maps are in HOD, ${}_\omega\mathbb{A}$ belongs to HOD and may be identified as a set of ordinals having the property expressed in Remark 7.5.

Let $m \leq n$, let τ_m be a ${}_n\mathbb{A}$-name for the last real of the $\dot{x}_{\mathrm{gen}}^m$, which is a name for the generic m-tuple coming from a ${}_n\mathbb{A}$-generic filter. (Technically, τ_m is different for each n, but the projection can be used to interpret it in suitable n's.)

Let $\dot{\mathbb{R}}_{\mathrm{sym}}$ be a ${}_\omega\mathbb{A}$-name so that $1_{{}_\omega\mathbb{A}} \Vdash \dot{\mathbb{R}}_{\mathrm{sym}} = \{\tau_n : n \in \omega \setminus \{0\}\}$.

Observe that from Fact 7.6, every $z \in \mathbb{R}^n$ induces a ${}_n\mathbb{A}$-generic filter G_z over HOD so that $\mathrm{HOD}[z] = \mathrm{HOD}[G_z]$. Note that $\dot{x}_{\mathrm{gen}}[G_z] = z$ and $\tau_m[G_z] = z(m)$ for all $m < n$.

Theorem 7.16. *(Woodin) Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = L(\mathbb{R})$. Suppose $g \subseteq \mathrm{Coll}(\omega, \mathbb{R})$-generic over $L(\mathbb{R})$. From g, one can derive a ${}_\omega\mathbb{A}$-generic over $\mathrm{HOD}^{L(\mathbb{R})}$ filter G_g.*

Proof. Let $g \subseteq \mathrm{Coll}(\omega, \mathbb{R})$ be a generic over $\mathrm{HOD}^{L(\mathbb{R})}$. Let $G_g \subseteq {}_\omega\mathbb{A}$ be the collection of condition $p \in {}_\omega\mathbb{A}$ so that $g \upharpoonright \dim(p) \in p(\dim(p))$.

The claim is that G_g is ${}_\omega\mathbb{A}$-generic over $\mathrm{HOD}^{L(\mathbb{R})}$.

Suppose $D \subseteq {}_\omega\mathbb{A}$ belongs to $\mathrm{HOD}^{L(\mathbb{R})}$ and is dense. Let $\tilde{D} \subseteq \mathrm{Coll}(\omega, \mathbb{R})$ be the collection of $s \in \mathrm{Coll}(\omega, \mathbb{R})$ so that there is some $p \in {}_\omega\mathbb{A}$ with $\dim(p) = |s|$, $s \in p(|s|)$, and $p \in D$. (Note that s as a condition of $\mathrm{Coll}(\omega, \mathbb{R})$ is a finite tuple of reals.)

One will show that $\tilde{D}$ is dense in $\mathrm{Coll}(\omega, \mathbb{R})$. Let $s \in \mathrm{Coll}(\omega, \mathbb{R})$. Let $n = |s|$. Let

$$E = \{p \in {}_n\mathbb{A} : (\exists q \in {}_\omega\mathbb{A})(\dim(q) \geq n \land q \in D \land \pi_{\omega,n}(q) = p)\}.$$

First, one will show E is dense in ${}_n\mathbb{A}$. Let $r \in {}_n\mathbb{A}$. Since D is dense in ${}_\omega\mathbb{A}$, there is some $q \in {}_\omega\mathbb{A}$ with $\dim(q) \geq n$ so that $q \leq_{\omega\mathbb{A}} r$ and $q \in D$. Then $p = \pi_{\omega,n}(q)$ belongs to E and $p \leq_{n\mathbb{A}} r$. Thus E is dense in ${}_n\mathbb{A}$.

Let G_s^n be the ${}_n\mathbb{A}$-generic over $\mathrm{HOD}^{L(\mathbb{R})}$ filter derived from s. By genericity, pick some $p \in G_s^n \cap E$. Thus there is some $q \in D$ so that $\pi_{\omega,n}(q) = p$. Let $s' \supseteq s$ be such that $s' \in q$. This means that $s' \leq_{\mathrm{Coll}(\omega,\mathbb{R})} s$ and $s' \in \tilde{D}$. It has been shown that $\tilde{D}$ is dense in $\mathrm{Coll}(\omega, \mathbb{R})$.

Now since $g \subseteq \mathrm{Coll}(\omega, \mathbb{R})$ is $\mathrm{Coll}(\omega, \mathbb{R})$-generic over $L(\mathbb{R})$, $g \cap \tilde{D} \neq \emptyset$. There is some $n \in \omega$ so that $g \upharpoonright n \in \tilde{D}$. By definition, there is some $p \in D$ so that $g \upharpoonright n \in p(n)$. Thus $p \in G_g \cap D$.

This shows that G_g is ${}_\omega\mathbb{A}$-generic over HOD. $\square$

Fact 7.17. *(Woodin) Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = L(\mathbb{R})$. $\mathrm{HOD}^{L(\mathbb{R})} \models 1_{{}_\omega\mathbb{A}} \Vdash$ "the reals of $L(\dot{\mathbb{R}}_{\mathrm{sym}})$ is $\dot{\mathbb{R}}_{\mathrm{sym}}$".*

Proof. Let $p \in {}_\omega\mathbb{A}$ be some condition and let $n = \dim(p)$. Let $s \in \mathbb{R}^n$ with $s \in p(n)$. Consider s as a condition of $\mathrm{Coll}(\omega, \mathbb{R})$. Let $g \subseteq \mathrm{Coll}(\omega)$ be $\mathrm{Coll}(\omega, \mathbb{R})$-generic over $L(\mathbb{R})$ containing s. An easy density argument shows that if g is considered as a function from ω to $\mathbb{R}$, g must be a surjection onto $\mathbb{R}^{L(\mathbb{R})}$. Let G_g be the ${}_\omega\mathbb{A}$-generic filter over $\mathrm{HOD}^{L(\mathbb{R})}$ derived from g. Then $\mathrm{HOD}^{L(\mathbb{R})}[G_g] \models$ "the reals of $L(\dot{\mathbb{R}}_{\mathrm{sym}}[G_g])$ is $\dot{\mathbb{R}}_{\mathrm{sym}}[G_g]$" since $L(\dot{\mathbb{R}}_{\mathrm{sym}}[G_g]) = L(\mathbb{R})$. Thus there is some $q \leq_{\omega\mathbb{A}} p$ so that $\mathrm{HOD}^{L(\mathbb{R})} \models q \Vdash_{\omega\mathbb{A}}$ "the reals of $L(\dot{\mathbb{R}}_{\mathrm{sym}})$ is $\dot{\mathbb{R}}_{\mathrm{sym}}$". Since p was an arbitrary condition, one has that $1_{{}_\omega\mathbb{A}}$ forces this same statement. $\square$

Theorem 7.18. *(Woodin) Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} + L(\mathbb{R})$. Let $s \in \mathbb{R}^n$. Suppose G_s^n is the ${}_n\mathbb{A}$-generic filter over $\mathrm{HOD}^{L(\mathbb{R})}$ derived from s. Let φ be a formula. Suppose $z \in \mathrm{HOD}^{L(\mathbb{R})}[G_s^n]$. Then*

$$\mathrm{HOD}^{L(\mathbb{R})}[G_s^n] \models 1_{{}_\omega\mathbb{A}/G_s^n} \Vdash_{{}_\omega\mathbb{A}/G_s^n} L(\dot{\mathbb{R}}_{\mathrm{sym}}) \models \varphi(\check{z}, \dot{x}_{\mathrm{gen}}^n)$$

or

$$\mathrm{HOD}^{L(\mathbb{R})}[G_s^n] \models 1_{{}_{\omega}\mathbb{A}/G_s^n} \Vdash_{{}_{\omega}\mathbb{A}/G_s^n} L(\dot{\mathbb{R}}_{\mathrm{sym}}) \models \neg\varphi(\check{z}, \dot{x}_{\mathrm{gen}}^n)$$

Proof. In $L(\mathbb{R})$, one either has $L(\mathbb{R}) \models \varphi(z,s)$ or $L(\mathbb{R}) \models \neg\varphi(z,s)$. The claim is that which ever case occurs, this is the side that is forced over $\mathrm{HOD}^{L(\mathbb{R})}[G_s^n]$.

So without loss generality, suppose $L(\mathbb{R}) \models \varphi(z,s)$.

Take any $q \in {}_{\omega}\mathbb{A}/G_s^n$ with $\dim(q) \geq n$. Let $m = \dim(q)$. This means that $\pi_{\omega,n}(q) \in G_s^n$. Let $r \in q$ be a $\mathbb{R}^m$ sequence extending s. Let $g \subseteq \mathrm{Coll}(\omega, \mathbb{R})$ be a $\mathrm{Coll}(\omega, \mathbb{R})$-generic over $L(\mathbb{R})$ filter so that $r \subseteq g$. Let G_g be the ${}_{\omega}\mathbb{A}$-generic over $\mathrm{HOD}^{L(\mathbb{R})}$ filter derived from g. Note that G_g is a ${}_{\omega}\mathbb{A}/G_s^n$-generic filter over $\mathrm{HOD}^{L(\mathbb{R})}[G_s^n]$ (see the basic properties of projection from Definition 7.8) and that $q \in G_g$. Using Fact 7.17 and the fact that $\mathrm{HOD}[G_s^n][G_g] = \mathrm{HOD}[G_g]$ (also see Definition 7.8),

$$\mathrm{HOD}[G_s^n][G_g] \models L(\dot{\mathbb{R}}_{\mathrm{sym}}[G_g]) \models \varphi(z, \dot{x}_{\mathrm{gen}}^n[G_s^n])$$

since $L(\mathbb{R}) \models \varphi(z,s)$. Hence there is some $q' \leq_{{}_{\omega}\mathbb{A}/G_s^n} q$ so that

$$\mathrm{HOD}[G_s^n] \models q' \Vdash_{{}_{\omega}\mathbb{A}/G_s^n} L(\dot{\mathbb{R}}_{\mathrm{sym}}) \models \varphi(\check{z}, \dot{x}_{\mathrm{gen}}^n).$$

Since $q \in {}_{\omega}\mathbb{A}/G_s^n$ was arbitrary, $1_{{}_{\omega}\mathbb{A}/G_s^n}$ forces this statement. This completes the proof. □

Theorem 7.19. *(Woodin) Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$. *Let* $s \in \mathbb{R}^n$, $z \in L[{}_{\omega}\mathbb{A}, s]$, *and* φ *be a formula. Then* $L(\mathbb{R}) \models \varphi(s, z)$ *if and only if*

$$L[{}_{\omega}\mathbb{A}, s] \models 1_{{}_{\omega}\mathbb{A}/G_s^n} \Vdash_{{}_{\omega}\mathbb{A}/G_s^n} L(\dot{\mathbb{R}}_{\mathrm{sym}}) \models \varphi(\dot{x}_{\mathrm{gen}}^n, \check{z})$$

Proof. Recall ${}_{\omega}\mathbb{A} \in \mathrm{HOD}^{L(\mathbb{R})}$. Using the convention from Remark 7.5, one may assume that from a $p \in {}_{\omega}\mathbb{A}$, one can obtain an ∞-Borel code for this condition. From this, one can see that given s and ${}_{\omega}\mathbb{A}$, one can reconstruct G_s^n. Also the names $\dot{x}_{\mathrm{gen}}^n$ and $\dot{\mathbb{R}}_{\mathrm{sym}}$ belong to $L[{}_{\omega}\mathbb{A}]$.

($\Rightarrow$) Using the idea from Fact 7.6, one can show that $L[{}_{\omega}\mathbb{A}, s] = L[{}_{\omega}\mathbb{A}][G_s^n]$. Take any $p \in {}_{\omega}\mathbb{A}/G_s^n$. In particular, $p \in {}_{\omega}\mathbb{A}$. Pick some $r \in \mathrm{Coll}(\omega, \mathbb{R})$ with $r \supseteq s$ and $r \in p(|r|)$. Let $g \subseteq \mathrm{Coll}(\omega, \mathbb{R})$ be $\mathrm{Coll}(\omega, \mathbb{R})$-generic over $L(\mathbb{R})$ and $r \subseteq g$. Using ${}_{\omega}\mathbb{A}$, one can reconstruct G_g which is a ${}_{\omega}\mathbb{A}$-generic filter over $\mathrm{HOD}^{L(\mathbb{R})}$. Thus G_g is $({}_{\omega}\mathbb{A}/G_s^n)$-generic over $\mathrm{HOD}^{L(\mathbb{R})}[G_s^n]$ and hence generic over $L[{}_{\omega}\mathbb{A}, s]$. Since $L(\mathbb{R}) \models \varphi(s, z)$, $L[{}_{\omega}\mathbb{A}, s][G_g] \models L(\dot{\mathbb{R}}_{\mathrm{sym}}[G_g]) \models \varphi(s, \check{z})$ because $L(\dot{\mathbb{R}}_{\mathrm{sym}}[G_g]) = L(\mathbb{R})$. Pick a $q \leq_{{}_{\omega}\mathbb{A}} p$ so that

$$L[{}_{\omega}\mathbb{A}, s] \models q \Vdash_{{}_{\omega}\mathbb{A}/G_s^n} L(\dot{\mathbb{R}}_{\mathrm{sym}}) \models \varphi(s, \check{z}).$$

Since p was arbitrary, $1_{{}_{\omega}\mathbb{A}/G_s^n}$ forces this statement.

($\Leftarrow$) Let $g \supseteq s$ be a $\mathrm{Coll}(\omega, \mathbb{R})$-generic filter over $L(\mathbb{R})$. Let G_g be the ${}_{\omega}\mathbb{A}$-generic filter of $\mathrm{HOD}^{L(\mathbb{R})}$ derived from g. It is $({}_{\omega}\mathbb{A}/G_s^n)$-generic over $\mathrm{HOD}^{L(\mathbb{R})}[G_s^n]$ and hence generic over $L[{}_{\omega}\mathbb{A}, s]$. Then one has

$$L[{}_{\omega}\mathbb{A}, s][G_g] \models L(\dot{\mathbb{R}}_{\mathrm{sym}}[G_g]) \models \varphi(s, z).$$

However, $L(\dot{\mathbb{R}}_{\mathrm{sym}}[G_g]) = L(\mathbb{R})$. Thus $L(\mathbb{R}) \models \varphi(s, z)$. □

Corollary 7.20. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$. *Every set of reals has an* ∞-*Borel code. Moreover, if* A *is* OD_s *where* s *is a finite tuple of reals, then* A *has an* ∞-*Borel code in* $L[{}_{\omega}\mathbb{A}, s] \subseteq \mathrm{HOD}_s$.

Proof. Suppose $A \subseteq \mathbb{R}$. In $L(\mathbb{R})$, every set is definable from ordinals and some reals. Let $s \in \mathbb{R}^n$, $\bar{\alpha}$ be a tuple of ordinals, and φ be a formula so that $r \in A \Leftrightarrow \varphi(s, r, \bar{\alpha})$. By Theorem 7.19, one has that $r \in A$ if and only if

$$L(\mathbb{R}) \models L[{}_{\omega}\mathbb{A}, s, r] \models 1_{{}_{\omega}\mathbb{A}/G_{s^\frown r}^{n+1}} \Vdash L(\dot{\mathbb{R}}_{\mathrm{sym}}) \models \varphi(s, r, \bar{\alpha}).$$

Let S be a set of ordinals coding $({}_{\omega}\mathbb{A}, s, \bar{\alpha})$. Let ψ be the formula expressing the inner forcing statement. One has that (S, ψ) is an ∞-Borel code for A. □

Note that ${}_{\omega}\mathbb{A}$ can be considered an ultimate ∞-Borel code in the sense that for any $A \subseteq \mathbb{R}$, there is some tuple of reals, tuple of ordinals, and a formula so that together with ${}_{\omega}\mathbb{A}$ they form an ∞-Borel code for A. This is very useful in diagonalization arguments.

Corollary 7.21. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$. *Then* $\mathrm{HOD}^{L(\mathbb{R})} = L[{}_{\omega}\mathbb{A}]$.

Proof. Since ${}_{\omega}\mathbb{A} \in \mathrm{HOD}^{L(\mathbb{R})}$, $L[{}_{\omega}\mathbb{A}] \subseteq \mathrm{HOD}^{L(\mathbb{R})}$. $\mathrm{HOD}^{L(\mathbb{R})} \subseteq L[{}_{\omega}\mathbb{A}]$ by Theorem 7.19. □

Corollary 7.22. *Assume* ZF + AD + V = L($\mathbb{R}$). $\mathbb{A} = \mathbb{O}$, *that is the forcing with nonempty subsets of* $\mathbb{R}$ *possessing* OD ∞-*Borel codes is the same as the Vopénka forcing.*

Proof. This follows from Corollary 7.20. $\hfill\square$

Therefore, in the following, one will use the more practical Vopénka forcing $\mathbb{O}$ rather than $\mathbb{A}$.

8. THE VOPÉNKA FORCING

Fact 8.1. *([9] Theorem 2.4.) Let M be a transitive inner model of* ZF. *Let $S \in M$ be a set of ordinals. Suppose K is an* OD_S^M *set of ordinals and φ is a formula. Let N be some transitive inner model with* $\mathrm{HOD}_S^M \subseteq N$. *Suppose $p = \{x \in \mathbb{R}^M : L[K,x] \models \varphi(K,x)\}$ is a condition of $\mathbb{O}_S^M$ (i.e. is nonempty). Then* $N \models p \Vdash_{\mathbb{O}_S^M} L[\check{K}, \dot{x}_{\mathrm{gen}}] \models \varphi(\check{K}, \dot{x}_{\mathrm{gen}})$.

Proof. Suppose not. Then there is some $q' \leq_{\mathbb{O}_S^M} p$ so that $N \models q' \Vdash_{\mathbb{O}_S^M} L[\check{K}, \dot{x}_{\mathrm{gen}}] \models \neg\varphi(\check{K}, \dot{x}_{\mathrm{gen}})$. Because every $\mathbb{O}_S^M$-generic filter over N is also generic over HOD_S^M, one can find some $q \leq_{\mathbb{O}_S^M} q'$ so that $\mathrm{HOD}_S^M \models q \Vdash_{\mathbb{O}_S^M} L[\check{K}, \dot{x}_{\mathrm{gen}}] \models \neg\varphi(\check{K}, \dot{x}_{\mathrm{gen}})$. Since $q \neq \emptyset$, let $y \in q$. Let G_y be the $\mathbb{O}_S^M$-generic filter over HOD_S^M derived from y. Note that $q \in G_y$. Thus $\mathrm{HOD}_S^M[G_y] \models L[K,y] \models \neg\varphi(K,y)$. Hence $L[K,y] \models \neg\varphi(K,y)$. This contradicts $q \subseteq p$. $\hfill\square$

Fact 8.2. *Let M be an inner model of* ZF. *Let $S \in M$ be a set of ordinals. Let N be an inner model of* ZF *such that $N \supseteq \mathrm{HOD}_S^M$. Let $n \geq 1$ be a natural number. Suppose $(g_0, ..., g_{n-1})$ is an $_n\mathbb{O}_S^M$-generic reals over N. Then each $g_0,...,g_{n-1}$ is $\mathbb{O}_S^M$-generic over N.*

Proof. Here g is a $\mathbb{O}_S^M$-generic real over N if and only if there is a filter G which $\mathbb{O}_S^M$-generic over N so that $g = \dot{x}_{\mathrm{gen}}[G]$. For simplicity assume $n = 2$.

Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first coordinate. For each $p \in {}_2\mathbb{O}_S^M$, $\pi_1[p] \in \mathbb{O}_S^M$.

Let (g_0, g_1) be $_2\mathbb{O}_S^M$-generic over N. Let $G_{(g_0,g_1)}$ be the $_2\mathbb{O}_S^M$-generic filter over N which adds (g_0, g_1). Let $G = \{\pi_1(p) : p \in G_{(g_0,g_1)}\}$. G is a filter on $\mathbb{O}_S^M$.

Suppose $D \subseteq \mathbb{O}_S^M$ belongs to N and is dense open. Let $D' = \{p \in {}_2\mathbb{O}_S^M : \pi_1(p) \in D\}$. Let $r \in {}_2\mathbb{O}_S^M$. Since D is dense, there is some $r' \leq_{\mathbb{O}_S^M} \pi_1(r)$ with $r' \in D$. Let $s = (r' \times \mathbb{R}) \cap r$. Note $s \in {}_2\mathbb{O}_S^M$, $\pi_1(s) = r' \in D$, and $s \leq_{{}_2\mathbb{O}_S^M} r$. Hence $s \in D'$. This shows that D' is dense in $_2\mathbb{O}_S^M$.

By genericity, there is some $r \in D' \cap G_{(g_0,g_1)}$. Then $\pi_1(r) \in D \cap G$. This shows that G is $\mathbb{O}_S^M$-generic over N. Since g_0 is the real added by G, g_0 is a $\mathbb{O}_S^M$-generic real over N. $\hfill\square$

First, one will give a proof of Woodin's countable section uniformization. This follows a presentation of this result in [19]. It is not known whether countable section uniformization is provable in AD alone.

Lemma 8.3. *Assume* ZF + AD. *Let $R \subseteq \mathbb{R} \times \mathbb{R}$ be a relation. Suppose there is a set of ordinals S with the property that for all $x \in \mathbb{R}$ so that $R_x = \{y \in \mathbb{R} : R(x,y)\} \neq \emptyset$ (i.e. $x \in \mathrm{dom}(R)$), for $\mu_\mathcal{D}$-almost all Turing degrees Z, $R_x \cap \mathrm{HOD}_{S,x}^{L[S,x,Z]} \neq \emptyset$. Then R has a uniformization.*

Proof. Think of $\mathbb{R} = {}^\omega 2$. Let $x \in \mathrm{dom}(R)$. Let $E^x = \{Z \in \mathcal{D} : R_x \cap \mathrm{HOD}_{S,x}^{L[S,x,Z]} \neq \emptyset\}$. By the assumption, $E^x \in \mu_\mathcal{D}$.

For each $n \in \omega$ and $i \in 2$, let $E_{n,i}^x$ be the collection of $Z \in E^x$ so that $z(n) = i$, where z is the $\mathrm{HOD}_{S,x}^{L[S,x,Z]}$-least element of $R_x \cap \mathrm{HOD}_{S,x}^{L[S,x,Z]}$ according to the canonical wellordering of $\mathrm{HOD}_{S,x}^{L[S,x,Z]}$. Note that $E^x = E_{n,0}^x \cup E_{n,1}^x$. Since $\mu_\mathcal{D}$ is an ultrafilter and $E^x \in \mu_\mathcal{D}$, for each $n \in \omega$, there is a unique $a_n^x \in 2$ so that $E_{n,a_n^x}^x \in \mu_\mathcal{D}$.

Define a function $\Phi : \mathrm{dom}(R) \to {}^\omega 2$ by $\Phi(x)(n) = a_n^x$. The claim is that Φ is a uniformization for R.

Suppose $x \in \mathrm{dom}(R)$. Since $\mu_\mathcal{D}$ is a countably complete ultrafilter on $\mathcal{D}$, $\bigcap_{n \in \omega} E_{n,a_n^x}^x \in \mu_\mathcal{D}$. Pick any $Z \in \bigcap_{n \in \omega} E_{n,a_n^x}^x$. Then $\Phi(x)$ is the $\mathrm{HOD}_{S,x}^{L[S,x,Z]}$-least element of $R_x \cap \mathrm{HOD}_{S,x}^{L[S,x,Z]}$. In particular, $\Phi(x) \in R_x$. Φ is a uniformization. $\hfill\square$

Theorem 8.4. *(Woodin's countable section uniformization) Assume* ZF + AD *and that all sets of reals have an ∞-Borel code. Let $R \subseteq \mathbb{R} \times \mathbb{R}$ be such that for all $x \in \mathrm{dom}(R)$, $R_x = \{y \in \mathbb{R} : R(x,y)\}$ is countable.*

Then R has a uniformization function, that is some function $F : \mathrm{dom}(R) \to \mathbb{R}$ so that for all $x \in \mathrm{dom}(R)$, $R(x, F(x))$.

In particular, countable section uniformization holds in $L(\mathbb{R}) \models \mathsf{AD}$.

Proof. Let (S, φ) be an ∞-Borel code for R, that is $\mathfrak{B}^2_{(S,\varphi)} = R$.

Let $x \in \mathrm{dom}(R)$. Thus there is some y^* such that $R(x, y^*)$. Let $Z \in \mathcal{D}$ be such that $Z \geq [y^*]_T$. In the model $L[S, x, Z]$, define

$$ p = \{y \in \mathbb{R} : L[S, x, y] \models \varphi(S, x, y)\}. $$

This condition is clearly $\mathrm{OD}^{L[S,x,Z]}_{S,x}$. Note $[y^*]_T \leq Z$ implies that $y^* \in L[S, x, Z]$. Since in the real world V, $R(x, y^*)$ holds, one has that $V \models L[S, x, y^*] \models \varphi(S, x, y^*)$. Thus $L[S, x, Z] \models L[S, x, y^*] \models \varphi(S, x, y^*)$. So $y^* \in p$. Hence $p \neq \emptyset$. It has been shown that $p \in \mathbb{O}^{L[S,x,Z]}_{S,x}$.

Claim 1: There is a dense below p set of conditions in $\mathbb{O}^{L[S,x,Z]}_{S,x}$ which forces a value for $\dot{x}_{\mathrm{gen}}$.

To prove Claim 1: Suppose otherwise. As argued in the proof of Fact 7.13, since $\mathrm{HOD}^{L[S,x,Z]}_{S,x} \models \mathsf{AC}$, $\mathscr{P}(\mathbb{O}^{L[S,x,Z]}_{S,x})^{\mathrm{HOD}^{L[S,x,Z]}_{S,x}}$ is countable in the real world V. Let $\langle D_n : n \in \omega \rangle$ enumerate (in the real world) all the dense open subsets of $\mathbb{O}^{L[S,x,Z]}_{S,x}$ that belong to $\mathrm{HOD}^{L[S,x,Z]}_{S,x}$.

Let $p_\emptyset$ be any condition below p that meets D_0. Let $m_\emptyset = 0$. Suppose for some $\sigma \in {}^{<\omega}2$, p_σ and m_σ have been defined. First find some $p' \leq p_\sigma$ so that $p' \in D_{|\sigma|+1}$. Since p' can not determine $\dot{x}_{\mathrm{gen}}$, there is some $N \geq m_\sigma$ and two conditions q_0 and q_1 below p' so that
(1) q_0 and q_1 determine $\dot{x}_{\mathrm{gen}} \upharpoonright N + 1$ and
(2) For $i \in 2$, $q_i \Vdash \dot{x}_{\mathrm{gen}}(\check{N}) = \check{i}$. (That is, q_i forces the generic real to take value i at N.)

Let q_0 and q_1 be the $\mathrm{HOD}^{L[S,x,Z]}_{S,x}$-least pair with the above property. Now let $p_{\sigma ^\frown i} = q_i$ and $m_{\sigma ^\frown i} = N+1$. Observe that $p_{\sigma ^\frown i} \in D_{n+1}$ (as D_{n+1} is dense open). This completes the construction of a sequence $\langle p_\sigma : \sigma \in {}^{<\omega}2 \rangle$ of conditions in $\mathbb{O}^{L[S,x,Z]}_{S,x}$.

For each $f \in {}^\omega 2$, let G_f be the $\leq_{\mathbb{O}^{L[S,x,Z]}_{S,x}}$ upward closure of the set $\{p_{f\upharpoonright n} : n \in \omega\}$. By construction, G_f is $\mathbb{O}^{L[S,x,Z]}_{S,x}$-generic over $\mathrm{HOD}^{L[S,x,Z]}_{S,x}$. Also by construction, if $f \neq g$, then $\dot{x}_{\mathrm{gen}}[G_f] \neq \dot{x}_{\mathrm{gen}}[G_g]$.

Hence $A = \{\dot{x}_{\mathrm{gen}}[G_f] : f \in {}^\omega 2\}$ is an uncountable set of reals.

Subclaim 1.1: $A \subseteq R_x$.

To see Subclaim 1.1: Note that for all $f \in {}^\omega 2$, $p \in G_f$. Note that the condition p takes the form specified in Fact 8.1. Hence the fact asserts that

$$ \mathrm{HOD}^{L[S,x,Z]}_{S,x} \models p \Vdash_{\mathbb{O}^{L[S,x,Z]}_{S,x}} L[S, x, \dot{x}_{\mathrm{gen}}] \models \varphi(\check{S}, \check{x}, \dot{x}_{\mathrm{gen}}). $$

Therefore,

$$ \mathrm{HOD}^{L[S,x,Z]}_{S,x}[G_f] \models L[S, x, \dot{x}_{\mathrm{gen}}[G_f]] \models \varphi(S, x, \dot{x}_{\mathrm{gen}}[G_f]). $$

So in particular,

$$ V \models L[S, x, \dot{x}_{\mathrm{gen}}[G_f]] \models \varphi(S, x, \dot{x}_{\mathrm{gen}}[G_f]). $$

Since $R = \mathfrak{B}^2_{(S,\varphi)}$, $R(x, \dot{x}_{\mathrm{gen}}[G_f])$ for each $f \in {}^\omega 2$. This shows that $A \subseteq R_x$ which completes the proof of the subclaim.

However, A is uncountable and R_x was assumed to be countable. Contradiction. This completes the proof of Claim 1.

Let $D \subseteq \mathbb{O}^{L[S,x,Z]}_{S,x}$ be the dense below p set that witnesses Claim 1. Take any $q \in D$ with $q \leq p$. Define $t \in {}^\omega 2$ by $t(n) = i \Leftrightarrow q \Vdash \dot{x}_{\mathrm{gen}}(\check{n}) = \check{i})$. It is clear that $t \in \mathrm{HOD}^{L[S,x,Z]}_{S,x}$. Again since $\mathscr{P}(\mathbb{O}^{L[S,x,Z]}_{S,x})^{\mathrm{HOD}^{L[S,x,Z]}_{S,x}}$ is countable in V, let G be any $\mathbb{O}^{L[S,x,Z]}_{S,x}$-generic filter over $\mathrm{HOD}^{L[S,x,Z]}_{S,x}$ with $q \in G$ (and hence $p \in G$). By the forcing theorem, $\dot{x}_{\mathrm{gen}}[G] = t$. Using Fact 8.1 as above, one has $\mathrm{HOD}^{L[S,x,Z]}_{S,x}[G] \models L[S, x, t] \models \varphi(S, x, t)$. Hence $V \models L[S, x, t] \models \varphi(S, x, t)$. Thus $R(x, t)$. So $t \in R_x \cap \mathrm{HOD}^{L[S,x,Z]}_{S,x}$.

It has been shown that $R_x \cap \mathrm{HOD}^{L[S,x,Z]}_{S,x} \neq \emptyset$ for any $Z \in \mathcal{D}$ with $Z \geq_T [y^*]_T$. Lemma 8.3 implies that R has a uniformization. $\qquad\square$

In ZF: The Silver's dichotomy [22] states that for every $\mathbf{\Pi}_1^1$ equivalence E on $\mathbb{R}$, $\mathbb{R}/E$ injects into ω or $\mathbb{R}$ injects into $\mathbb{R}/E$. Burgess [1] showed that if E is a $\mathbf{\Sigma}_1^1$ equivalence relation, $\mathbb{R}/E$ injects into ω, $\mathbb{R}/E$ is in bijection with ω_1, or $\mathbb{R}$ inject into $\mathbb{R}/E$. Note that both results imply that either $\mathbb{R}/E$ is wellorderable or $\mathbb{R}$ injects into $\mathbb{R}/E$.

Woodin's perfect set dichotomy is an extension of this property to all equivalence relations on $\mathbb{R}$ assuming AD, all sets of reals have ∞-Borel codes, and the ultrapower, $\prod_{X \in D} \omega_1/\mu_\mathcal{D}$, is wellfounded. The wellfoundedness of the ultrapower is certainly a consequence of DC.

The proof of Woodin's perfect set dichotomy presented below follows the outline of Hjorth's [9] generalization of the [8] E_0-dichotomy in $L(\mathbb{R}) \models$ AD and Harrington's proof of the Silver's dichotomy. Harrington's proof uses the Gandy-Harrington forcing of nonempty Σ_1^1 definable subsets of $\mathbb{R}$. The Vopénka forcing is simply the OD version of the Gandy-Harrington forcing.

The argument presented below appears in [6] where the uniformity of this proof is needed to make further conclusions about wellordered disjoint unions of quotients of "smooth" equivalence relations with countable classes.

Theorem 8.5. (*Woodin's perfect set dichotomy*) *Assume* ZF + AD + V = L($\mathbb{R}$). *Let E be an equivalence relation on $\mathbb{R}$. Then either*
  (i) $\mathbb{R}/E$ *is wellorderable.*
  (ii) $\mathbb{R}$ *inject into* $\mathbb{R}/E$.

Proof. Let E be an equivalence relation on $\mathbb{R}$. An E-component is a nonempty set A so that for all $x, y \in A$, $x \mathrel{E} y$. That is, an E-component is simply a nonempty subset of an E-class.

By Corollary 7.20, every set of reals has an ∞-Borel code. Let (S, φ) be an ∞-Borel code for E; that is, $E = \mathfrak{B}_{(S,\varphi)}^2$. Throughout this argument, E will always be considered as the set defined by the ∞-Borel code (S, φ).

(Case I) For all $X \in \mathcal{D}$, for all $a \in \mathbb{R}^{L[S,X]}$, there is some $\mathrm{OD}_S^{L[S,X]}$ E-component containing a.

In other words, for all degrees $X \in \mathcal{D}$, in the local model $L[S, X]$, every real belongs to an OD_S E-component.

For each $\alpha \in \prod_{X \in \mathcal{D}} \omega_1/\mu_\mathcal{D}$, let $f : \mathcal{D} \to \omega_1$ be such that f is a representative for α. Define $A_\alpha \subseteq \mathbb{R}$ as follows: $a \in A_\alpha$ if and only if for $\mu_\mathcal{D}$-almost all X, a belongs to the $f(X)^{\mathrm{th}}$ $\mathrm{OD}_S^{L[S,X]}$ E-component according to the canonical wellordering of $\mathrm{HOD}_S^{L[S,X]}$. (If there is no $F(X)^{\mathrm{th}}$ $\mathrm{OD}_S^{L[S,X]}$ E-component, then one just let this set be $\emptyset$.) One can check that A_α is well defined, independent of the choice of f representing α.

A_α is an E-component: To see this, let $a, b \in A_\alpha$. Thus there is a set $K \in \mu_\mathcal{D}$ so that for all $X \in K$, $[a \oplus b]_T \leq X$ and a and b both belong to the $f(X)^{\mathrm{th}}$ OD_S E-component of $\mathrm{HOD}_S^{L[S,X]}$. Thus, $L[S, X] \models a \mathrel{E} b$. Since E is always defined by it ∞-Borel code, one has, $L[S, X] \models L[S, a, b] \models \varphi(S, a, b)$. Thus $V \models L[S, a, b] \models \varphi(S, a, b)$. Therefore, in V, $a \mathrel{E} b$. It has been shown that A_α is an E-component.

For every $a \in \mathbb{R}$, there is some $\alpha \in \prod_{X \in \mathcal{D}} \omega_1/\mu_\mathcal{D}$ such that $a \in A_\alpha$: To see this. Define $f : \mathcal{D} \to \omega_1$ by letting $f(X)$ be the least β so that a belongs to the βth $\mathrm{OD}_S^{L[S,X]}$ E-component of $L[S, X]$ according to the canonical wellordering of $\mathrm{HOD}_S^{L[S,X]}$. This β exists due to the Case I assumption. Let $\alpha = [f]_{\mu_\mathcal{D}}$. Then $a \in A_\alpha$.

Since $L(\mathbb{R}) \models$ DC, $\prod_{X \in \mathcal{D}} \omega_1/\mu_\mathcal{D}$ is a wellordered set. Thus $\langle A_\alpha : \alpha \in \prod_{X \in \mathcal{D}} \omega_1/\mu_\mathcal{D} \rangle$ is a wellordered sequence of E-components with the property that every real belongs to some A_α. One can now wellorder $\mathbb{R}/E$ as follows: For two E-classes $u, v \in \mathbb{R}/E$, $u \prec v$ if and only if the least α so that $A_\alpha \subseteq u$ is less than the least α so that $A_\alpha \subseteq v$. Thus $\prec$ wellorders $\mathbb{R}/E$.

(Case II) There exists an $X \in \mathcal{D}$ and an $a \in \mathbb{R}^{L[S,X]}$ so that there is no $\mathrm{OD}_S^{L[S,X]}$ E-component containing a.

In other word, there is a particular local model $L[S, X]$ so that within this model, there is a real a which does not belong to any OD_S E-component.

Fix such a degree $X \in \mathcal{D}$. One will always work in this local model $L[S, X]$.

In $L[S, X]$, define u be the collection of reals of $L[S, X]$ that do not belong to any $\mathrm{OD}_S^{L[S,X]}$ E-component. Note that $u \neq \emptyset$ by the Case II assumption and u is $\mathrm{OD}_S^{L[S,X]}$. Thus u is condition of $\mathbb{O}_S^{L[S,X]}$.

Let $\dot{x}_{\text{left}}$ and $\dot{x}_{\text{right}}$ be $\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}$-names so that $\dot{x}_{\text{left}}$ and $\dot{x}_{\text{right}}$ is the evaluation of the $\mathbb{O}_S^{L[S,X]}$-name $\dot{x}_{\text{gen}}$ according to the left and right $\mathbb{O}_S^{L[S,X]}$-generic filter, respectively, coming from an $\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}$-generic filter.

Claim 1: $\text{HOD}_S^{L[S,X]} \models (u,u) \Vdash_{\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}} \neg(\dot{x}_{\text{left}} \ E \ \dot{x}_{\text{right}})$.

To see Claim 1: Suppose not. Then there is some $(v,w) \leq_{\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}} (u,u)$ so that $\text{HOD}_S^{L[S,X]} \models (v,w) \Vdash_{\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}} \dot{x}_{\text{left}} \ E \ \dot{x}_{\text{right}}$.

Subclaim 1.1: Suppose G_0 and G_1 are $\mathbb{O}_S^{L[S,X]}$-generic filter over $\text{HOD}_S^{L[S,X]}$ which belong to V and contain the condition v, then $\dot{x}_{\text{gen}}[G_0] \ E \ \dot{x}_{\text{gen}}[G_1]$. (Note that G_0 and G_1 are not necessarily mutually generic.)

To see Subclaim 1.1: As before, since $\text{HOD}_S^{L[S,X]} \models \mathsf{AC}$, $\mathscr{P}(\mathbb{O}_S^{L[S,X]})^{\text{HOD}_S^{L[S,X]}}$ is countable in the real world which is a model of AD. Therefore, one can find in the real world V, an $H \subseteq \mathbb{O}_S^{L[S,X]}$ which is $\mathbb{O}_S^{L[S,X]}$-generic over both $\text{HOD}_S^{L[S,X]}[G_0]$ and $\text{HOD}_S^{L[S,X]}[G_1]$ and $w \in H$. Applying the forcing theorem, one has that $\text{HOD}_S^{L[S,X]}[G_0][H] \models \dot{x}_{\text{gen}}[G_0] \ E \ \dot{x}_{\text{gen}}[H]$ and $\text{HOD}_S^{L[S,X]}[G_1][H] \models \dot{x}_{\text{gen}}[G_1] \ E \ \dot{x}_{\text{gen}}[H]$. As E is defined by its ∞-Borel code (S,φ), one has

$$\text{HOD}_S^{L[S,X]}[G_0][H] \models L[S, \dot{x}_{\text{gen}}[G_0], \dot{x}_{\text{gen}}[H]] \models \varphi(S, \dot{x}_{\text{gen}}[G_0], \dot{x}_{\text{gen}}[H])$$

$$\text{HOD}_S^{L[S,X]}[G_1][H] \models L[S, \dot{x}_{\text{gen}}[G_1], \dot{x}_{\text{gen}}[H]] \models \varphi(S, \dot{x}_{\text{gen}}[G_1], \dot{x}_{\text{gen}}[H])$$

In particular

$$V \models L[S, \dot{x}_{\text{gen}}[G_0], \dot{x}_{\text{gen}}[H]] \models \varphi(S, \dot{x}_{\text{gen}}[G_0], \dot{x}_{\text{gen}}[H])$$

$$V \models L[S, \dot{x}_{\text{gen}}[G_1], \dot{x}_{\text{gen}}[H]] \models \varphi(S, \dot{x}_{\text{gen}}[G_1], \dot{x}_{\text{gen}}[H])$$

Thus in the real world, $\dot{x}_{\text{gen}}[G_0] \ E \ \dot{x}_{\text{gen}}[H]$ and $\dot{x}_{\text{gen}}[G_1] \ E \ \dot{x}_{\text{gen}}[H]$. By the transitivity of the equivalence relation E, $\dot{x}_{\text{gen}}[G_0] \ E \ \dot{x}_{\text{gen}}[G_1]$. This proves Subclaim 1.1.

Observe that there must be some $a, b \in v$ so that $\neg(a \ E \ b)$ since otherwise v would be an $\text{OD}_S^{L[S,X]}$ E-component containing a. This is impossible since $v \leq_{\mathbb{O}_S^{L[S,X]}} u$ implies that $v \subseteq u$ and u consists of those $a \in \mathbb{R}$ which do not belong to any $\text{OD}_S^{L[S,X]}$ E-component. Thus $p = v \times v \setminus E = \{(a,b) \in v \times v : \neg(a \ E \ b)\}$ is a nonempty $\text{OD}_S^{L[S,X]}$ subset of $\mathbb{R}^2$. Thus $p \in {}_2\mathbb{O}_S^{L[S,X]}$.

Let $\dot{x}_{\text{gen}}^2$ denote the generic element of $\mathbb{R}^2$ added by ${}_2\mathbb{O}_S^{L[S,X]}$. Let τ_0 and τ_1 be the ${}_2\mathbb{O}_S^{L[S,X]}$-names for the first and second coordinate of $\dot{x}_{\text{gen}}^2$. Using the ∞-Borel code (S,φ), the condition $q = \{(a,b) : \neg(a \ E \ b)\} = \{(a,b) : L[S,a,b] \models \neg\varphi(S,a,b)\}$ takes the form specified in Fact 8.1. Therefore, Fact 8.1 implies that $q \Vdash_{{}_2\mathbb{O}_S^{L[S,X]}} L[S, \tau_0, \tau_1] \models \neg\varphi(S, \tau_0, \tau_1)$.

As before, in the real world, find some $G \subseteq {}_2\mathbb{O}_S^{L[S,X]}$ containing p which is ${}_2\mathbb{O}_S^{L[S,X]}$-generic over $\text{HOD}_S^{L[S,X]}$. Since $p \leq_{{}_2\mathbb{O}_S^{L[S,X]}} q$, one has that $q \in G$. Hence

$$\text{HOD}_S^{L[S,X]}[G] \models L[S, \tau_0[G], \tau_1[G]] \models \neg\varphi(S, \tau_0[G], \tau_1[G]).$$

In particular,

$$V \models L[S, \tau_0[G], \tau_1[G]] \models \neg\varphi(S, \tau_0[G], \tau_1[G]).$$

Since (S,φ) is the ∞-Borel code for E, one has that $\neg(\tau_0[G] \ E \ \tau_1[G])$. However, $\tau_0[G]$ and $\tau_1[G]$ are just the two coordinates of $\dot{x}_{\text{gen}}^2[G]$, which is the generic element of $\mathbb{R}^2$ added by G. Fact 8.2 implies that $\tau_0[G]$ and $\tau_1[G]$ are individually $\mathbb{O}_S^{L[S,X]}$-generic filters over $\text{HOD}_S^{L[S,X]}$. Then Subclaim 1.1 implies that $\tau_0[G] \ E \ \tau_1[G]$. Contradiction. This shows Claim 1.

As before, $\mathscr{P}(\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]})^{\text{HOD}_S^{L[S,X]}}$ is countable in the real world. Let $\langle D_n : n \in \omega \rangle$ enumerate all the dense open subsets of $\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}$ which belong to $\text{HOD}_S^{L[S,X]}$. By intersecting, one may assume that $D_{n+1} \subseteq D_n$ for all $n \in \omega$. Now in the usual manner, one will build a perfect tree of conditions whose distinct branches correspond to mutually generic filters. The details follow:

Let $p_\emptyset$ be any condition below u. Suppose for some $n \in \omega$, p_σ has been defined for all $\sigma \in {}^n 2$. For each $\sigma \in {}^n 2$, $(p_\sigma, p_\sigma) \leq_{\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}} (u,u)$ and Claim 1 implies that there must be some $q_0 \leq_{\mathbb{O}_S^{L[S,X]}} p_\sigma$

and $q_1 \leq_{\mathbb{O}_S^{L[S,X]}} p_\sigma$ which are incompatible conditions. Let $(\rho_{\sigma^\frown 0}, \rho_{\sigma^\frown 1})$ denote the $\mathrm{HOD}_S^{L[S,X]}$-least such pair (q_0, q_1).

For $\sigma \in {}^{n+1}2$, let $\rho_\sigma^{-1} = \rho_\sigma$. Let $\langle \sigma_i : i < 2^{n+1} \rangle$ enumerate ${}^{n+1}2$.

Now suppose that ρ_σ^j has been defined for all $\sigma \in {}^{n+1}2$ and $-1 \leq j < 2^{n+1}$. By considering all possible pairs and extending by density, find a collection $\{\rho_{\sigma_i}^{j+1} : 0 \leq i < 2^{n+1}\}$ with the property that

(1) $\rho_\sigma^{j+1} \leq_{\mathbb{O}_s^{L[S,X]}} \rho_\sigma^j$

(2) $(\rho_{\sigma_{j+1}}^{j+1}, \rho_{\sigma_\ell}^{j+1}) \in D_n$ and $(\rho_{\sigma_\ell}^{j+1}, \rho_{\sigma_{j+1}}^{j+1}) \in D_n$ whenever $\ell \neq j+1$.

For $\sigma \in {}^{n+1}2$, let $p_\sigma = \rho_\sigma^{2^{n+1}-1}$.

This defines a sequence $\langle p_\sigma : \sigma \in {}^{<\omega}2 \rangle$. For each $f \in {}^\omega 2$, let G_f be the $\leq_{\mathbb{O}_S^{L[S,X]}}$-upward closure of $\{p_{f\upharpoonright n} : n \in \omega\}$. By construction, if $f \neq g$, then $G_f \times G_g$ is an $\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}$-generic filter over $\mathrm{HOD}_S^{L[S,X]}$ containing the condition (u, u). Therefore, by Claim 1, for any $f, g \in {}^\omega 2$ with $f \neq g$, $\neg(\dot{x}_{\mathrm{gen}}[G_f] \ E \ \dot{x}_{\mathrm{gen}}[G_g])$. Thus $A = \{\dot{x}_{\mathrm{gen}}[G_f] : f \in {}^\omega 2\}$ is a perfect set of E-inequivalent reals. Thus $\Phi : \mathbb{R} \to \mathbb{R}/E$ defined by $\Phi(f) = [\dot{x}_{\mathrm{gen}}[G_f]]_E$ is an injection. $\square$

Corollary 8.6. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$. *If X is a surjective image of $\mathbb{R}$ (equivalently $X \in L_\Theta(\mathbb{R})$), then X is either wellorderable or $\mathbb{R}$ injects into X.*

In fact, as observed by [2], in $L(\mathbb{R})$, this dichotomy actually holds for all sets X in $L(\mathbb{R})$ not just $X \in L_\Theta(\mathbb{R})$.

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