Combinatorics of the Partition Measures on the First Uncountable Cardinal. Part 1

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We work in ZF + AD. Recall if $A \subseteq {}^{\omega}\omega$ $(= \mathbb{R})$, one has a game

$$G_A \quad \begin{array}{llllll} \mathrm{I} & a(0) & & a(2) & & \cdots \\ \mathrm{II} & & a(1) & & a(3) & \cdots \end{array} \quad a$$

Player I wins G_A if and only if $a \in A$.
The Axiom of Determinacy states that for all $A \subseteq {}^{\omega}\omega$, one of the two players has a winning strategy.

Let $\epsilon \leq \omega_1$. A function $f : \epsilon \to \omega_1$ has uniform cofinality ω if and only if there is a function $F : \omega_1 \times \omega \to \omega_1$ such that

1) For all $\alpha < \omega_1$, for all $n \in \omega$, $F(\alpha, n) < F(\alpha, n+1)$.
2) For all $\alpha < \omega_1$, $f(\alpha) = \sup\{F(\alpha, n) : n \in \omega\}$.

If $\epsilon < \omega_1$, then every function $f : \epsilon \to \omega_1$ taking limit ordinal values has uniform cofinality ω since AD proves $AC_\omega^{\mathbb{R}}$. However, the function $f : \omega_1 \to \omega_1$ which is the increasing enumeration of all countable limit ordinals does not have uniform cofinality ω if ω_1 has a countably complete normal measure.

Let $\alpha \leq \epsilon$. Let $\text{bound}(f, \alpha) = \sup\{f(\beta) : \beta < \alpha\}$.
f is discontinuous everywhere if and only if for all $\alpha < \epsilon$, $\text{bound}(f, \alpha) < f(\alpha)$.

f has the correct type iff f has uniform cofinality ω and is discontinuous everywhere.

If $X \subseteq \omega_1$, then $[X]^\epsilon_*$ is the set of increasing $f : \epsilon \to X$ which have the correct type.

Definition: (Correct Type Partition Relation) Let $\epsilon \leq \omega_1$. $\omega_1 \to_* (\omega_1)^\epsilon_2$ asserts that all $P : [\omega_1]^\epsilon_* \to 2$, there is an $i \in 2$ and a club $C \subseteq \omega_1$ so that for all $f \in [C]^\epsilon_*$, $P(f) = i$.

The main theorem we will be proving in the first part of this series of talk will be the following.

Thm: (Martin) (ZF+AD) For all $\epsilon \leq \omega_1$, $\omega_1 \to_* (\omega_1)^\epsilon_2$.

ω_1 is called a strong partition cardinal since $\omega_1 \to_* (\omega_1)^{\omega_1}_2$.

Definition: Let $\epsilon \leq \omega_1$. Define a filter μ_ϵ on $[\omega_1]^\epsilon_*$ by $X \in \mu_\epsilon$ iff there is a club $C \subseteq \omega_1$ so that $[C]^\epsilon_* \subseteq X$.

Fact: $\omega_1 \to (\omega_1)^{\varepsilon}$ implies that μ_ε is an ultrafilter.

Exercise: Use the partition relations to show μ_ε is countably complete. (Actually AD proves all ultrafilters are countably complete.)
Show μ_1 is a normal measure on ω_1.

Definition: Let $\varepsilon \le \omega_1$. A good coding system for ${}^{\varepsilon}\omega_1$ consists of $(\underset{\sim}{\Pi}^1_1, \text{decode}, GC_{\beta,\gamma} : \beta < \varepsilon, \gamma < \omega_1)$ with the following properties

(1) ω_1 is regular by $AC^{\mathbb{R}}_{\omega}$.

(2) $\underset{\sim}{\Pi}^1_1$ is closed under $\forall^{\mathbb{R}}$. $\underset{\sim}{\Delta}^1_1 = \underset{\sim}{\Pi}^1_1 \cap \check{\underset{\sim}{\Pi}}^1_1 = \underset{\sim}{\Pi}^1_1 \cap \underset{\sim}{\Sigma}^1_1$.

(3) $\text{decode} : \mathbb{R} \to \mathcal{P}(\varepsilon \times \omega_1)$ such that for all $f : \varepsilon \to \omega_1$, there is an $x \in \mathbb{R}$ with $\text{decode}(x) = f$.

(4) For all $\beta < \varepsilon$, $\gamma < \omega_1$, $GC_{\beta,\gamma} \in \underset{\sim}{\Delta}^1_1$. Also $x \in GC_{\beta,\gamma}$ iff $\text{decode}(x)(\beta,\gamma) \wedge (\forall \xi < \omega_1)(\text{decode}(x)(\beta,\xi) \Rightarrow \xi = \gamma)$.

(5) For $\beta < \varepsilon$, let $GC_\beta = \bigcup_{\gamma<\omega_1} GC_{\beta,\gamma}$. Suppose $A \in \exists^{\mathbb{R}} \underset{\sim}{\Delta}^1_1 = \underset{\sim}{\Sigma}^1_1$ and $A \subseteq GC_\beta$, then there is a $\delta < \omega_1$ so that $A \subseteq \bigcup_{\gamma<\delta} GC_{\beta,\gamma}$.

(6) $\underset{\sim}{\Delta}^1_1$ is closed under $<\omega_1$ unions.

ω_1 is said to be ε-reasonable iff there is a good coding system for ${}^{\varepsilon}\omega_1$.

Example: ω_1 is ω-reasonable.

Proof: Let $\pi : \omega \times \omega \to \omega$ be a bijection. Using π, one can code relations on ω by reals. Let WO be those reals which code well orderings. WO is a $\underset{\sim}{\Pi}^1_1$ set. If $\alpha < \omega_1$, then WO_α are the reals which code the ordinal α. WO_α is $\underset{\sim}{\Delta}^1_1$.

If $x \in {}^{\omega}\omega$ and $n \in \omega$, $\hat{x}_n \in {}^{\omega}\omega$ is defined by $\hat{x}_n(k) = x(\pi(n,k))$.

Now we code $f : \omega \to \omega_1$ in the obvious way. That is x will code f iff for all n, $\hat{x}_n \in WO_{f(n)}$.

Define $GC_{n,\gamma}$ to set the set of x so that $\hat{x}_n \in WO_\gamma$. $GC_{n,\gamma}$ is $\underset{\sim}{\Delta}^1_1$.

Let decode$(x)(n,\gamma)$ iff $x \in GC_{n,\gamma}$.

Suppose A is $\underset{\sim}{\Sigma}^1_1$ and $A \subseteq GC_n = \bigcup_{\gamma<\omega_1} GC_{n,\gamma}$.

Let $B = \{z : (\exists x)(x \in A \wedge z = \hat{x}_n)\}$ is a $\underset{\sim}{\Sigma}^1_1$ subset of WO. By $\underset{\sim}{\Sigma}^1_1$ boundedness principle, $B \subseteq WO_{<\delta}$ for some δ. Thus

$$A \subseteq \bigcup_{\gamma<\delta} GC_{n,\gamma}.$$

We have shown this defines a good coding system for ${}^{\omega}\omega_1$. ▨

By using a bijection of $\varepsilon^{<\omega_1}$ with ω, this example can be modified to show

Example: ω_1 is ε-reasonable for all $\varepsilon < \omega_1$.

Thm: (Martin) Let $\varepsilon \leq \omega_1$. If ω_1 is $\omega \cdot \varepsilon$-reasonable, then $\omega_1 \rightarrow_* (\omega_1)^{\varepsilon}_2$ holds.

Proof: We will illustrate this by proving $\omega_1 \rightarrow_* (\omega_1)^1_2$.

Fix the good coding system $\langle \underset{\sim}{\Pi}^1_1, \text{decode}, GC_{n,\gamma} : n<\omega, \gamma<\omega_1 \rangle$ for ${}^{\omega}\omega_1$ from the example.

For $x \in \mathbb{R}$, let fail(x) be the least $n \in \omega$ so that $x \notin GC_n$, if such an n exists. Otherwise, let fail$(x) = \infty$.

Let σ is a Player 1 strategy and $x \in \mathbb{R}$, let $\sigma(x)$ be moves of PI using σ against Player 2 playing the bits of x. Similarly, define $\tau(x)$ is τ is a Player 2 strategy.

Let S_1 be the collection of PI strategies σ st

$$\forall y \,[\, \text{fail}(y) \leq \text{fail}(\sigma(y)) \wedge \text{fail}(y) < \infty \Rightarrow \text{fail}(\sigma(y)) > \text{fail}(y) \,]$$

Let S_2 be the collection of PII strategy τ st

$$\forall x \,(\, \text{fail}(x) \leq \text{fail}(\tau(x)) \,)$$

Lemma: If σ is in S_1, then there is a club $C \subseteq \omega_1$ so that for all $\delta \in C$, for all $n \in \omega$ and $\gamma < \delta$

$$\sigma\Big[\bigcap_{n'<n} \bigcup_{\gamma'<\gamma} GC_{n',\gamma'}\Big] \subseteq \bigcap_{n' \leq n} \bigcup_{\gamma'<\delta} GC_{n',\gamma'}$$

If $\tau \in S_2$, then there is a club $C \subseteq \omega_1$ so that for all $\delta \in C$, for all $n \in \omega$ and $\gamma < \delta$

$$\tau\Big[\bigcap_{n' \leq n} \bigcup_{\gamma'<\gamma} GC_{n',\gamma'}\Big] \subseteq \bigcap_{n' \leq n} \bigcup_{\gamma'<\delta} GC_{n',\gamma'}$$

<u>Proof</u>: Fix $\sigma \in S_1$.

For each $r < \omega_1$, let $R_{n,r} = \bigcap_{n' < n} \bigcup_{r' < r} GC_{n',r'}$.

R_r is $\underset{\sim}{\Delta}^1_1$. For each $y \in R_{n,r}$, $\mathrm{fail}(y) \geq n$. Since $\sigma \in S_1$, $\mathrm{fail}(\sigma(y)) > n$. This shows that

$$\sigma[R_{n,r}] \subseteq GC_n$$ and is $\underset{\sim}{\Sigma}^1_1$. By boundedness condition (5) in the definition of the good coding system, there is a least ordinal δ^r_n so that

$$\sigma[R_{n,r}] \subseteq \bigcup_{r' < \delta^r_n} GC_{n,r'}.$$

Let $C = \{\eta < \omega_1 : \forall n \in \omega\ \forall r < \eta\ \ \delta^r_n < \eta\}$.

C is clearly closed. Let $\alpha < \omega_1$. Let $a_0 = \alpha$.
If a_n has been defined, let $a_{n+1} = \sup\{\delta^r_n : r < a_n \wedge n \in \omega\}$
Let $a_\omega = \sup\{a_n : n \in \omega\}$. One can see that $a_\omega \in C$.
It has been shown that C is a club.

From the definition of C, it is clear that this club satisfies the statement for σ.

The argument for $\tau \in S_2$ is similar $\boxtimes$

If $f : \omega \cdot 1 \to \omega_1$, then let $\mathrm{block}(f) : 1 \to \omega_1$ be defined by $\mathrm{block}(f)(0) = \sup\{\omega \cdot 0 + n : n \in \omega\} = \sup(f)$.
If $f, g : \omega \cdot 1 \to \omega_1$, then let $\mathrm{joint}(f,g) : 1 \to \omega_1$ be defined by
$\mathrm{joint}(f,g)(0) = \sup\{f(\omega \cdot 0 + n), g(\omega \cdot 0 + n) : n \in \omega\} = \sup\{f(n), g(n) : n \in \omega\}$

Now we start the proof of $\omega_1 \to_* (\omega_1)^1_2$.

Let $P : [\omega_1]^1 \to 2$. Consider the following game

$$G \quad \begin{array}{llllll} \mathrm{I} & x_0 & x_1 & x_2 & \cdots & x \\ \mathrm{II} & y_0 & y_1 & y_2 & \cdots & y \end{array}$$

Player I wins iff the conjunction of the following 2 statement holds,

(1) $\mathrm{fail}(x) > \mathrm{fail}(y) \ \vee\ \mathrm{fail}(x) = \mathrm{fail}(y) = \infty$

(2) $(\mathrm{fail}(x) = \mathrm{fail}(y) = \infty) \Rightarrow P(\mathrm{joint}(\mathrm{decode}(x), \mathrm{decode}(y))) = 0$.

$$\begin{array}{lllllllll} \mathrm{I} & x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \mathrm{II} & y & \cdot & \cdot & \cdot & \cdot & \cdots & \cdots & \end{array}$$

$f_x = \text{decode}(x)$

$\Big\rangle\ \text{joint}(f_x, f_y)$

$f_y = \text{decode}(y)$

Ideally $x, y \in GC$ and code $f_x, f_y : \omega \to \omega_1$. The relevant ordinal is $\text{joint}(f_x, f_y) = \sup\{ f_x(n), f_y(n) \}$.

Player I wins if $P(\text{joint}(f_x, f_y)) = 0$.

By AD, one of the two players has a winning strategy.

<u>Case I</u>: P I has a winning strategy σ.

Condition (1) of the payoff implies that $\sigma \in S_1$. Using the lemma Let $C \subseteq \omega_1$ be a club so that for all $\delta \in C$, $n \in \omega$, $\gamma < \delta$

$$\sigma\Big[\bigcap_{n' < n} \bigcup_{\gamma' < \gamma} GC_{n', \gamma'} \Big] \subseteq \bigcap_{n' \leq n} \bigcup_{\gamma' < \delta} GC_{n', \gamma'}.$$

Let D be the club of limit points of C.

Now suppose $\lambda \in D$. Pick any $g : \omega \to C$ with $\sup(g) = \lambda$. Let $y \in GC$ so that $\text{decode}(y) = g$.

Since $\sigma \in S_1$ and $y \in GC$, $\text{fail}(y) = \infty$ implies $\text{fail}(\sigma(y)) = \infty$ so $\sigma(y) \in GC$.

$$\begin{array}{llllll} \mathrm{I} & \sigma(y) & f_{\sigma(y)} & \cdot & & \cdot \\ \mathrm{II} & y & g = f_y & & \cdot & & \cdot \\ & & & & g(0) & & g(1) \end{array}$$

From here, we know

$$y \in \bigcap_{n' < 1} \bigcup_{\gamma' < g(0)+1} GC_{n', \gamma'}$$

Since $g(0)+1 < g(1)$,

$$\sigma(y) \in \bigcap_{n' \leq 1} \bigcup_{\gamma' < g(1)} GC_{n', \gamma'}$$

so $f_{\sigma(y)}(1)$ is less than $g(1)$.

In this way, $f_{\sigma(y)}(n) < g(n)$ for all $n \in \omega$.

So $\text{joint}(\text{decode}(\sigma(y)), \text{decode}(y)) = \text{joint}(f_{\sigma(y)}, g) = \sup(g) = \lambda$.

Since σ is winning for P I, $P(\lambda) = 0$. This shows D is homogeneous for P taking value 0.

<u>Case II</u> : P II has winning strategy τ.

Then $\tau \in S_2$. Using the lemma this case will yield a club $D \subseteq \omega_1$ which is homogeneous for P taking value 1.

$\boxtimes$

Next we will prove that ω_1 is ω_1-reasonable using an argument due to Kechris.

(Solovay)

<u>Fact</u>: Assume ZF + AD. Let $f : \omega_1 \to \omega_1$. There is a Player 2 strategy τ so that for all $v \in WO$,

$$\{ (\alpha, \beta) \in \omega_1 \times \omega_1 : \exists n \ (\widehat{\tau(v)}_n)_0 \in WO_\alpha \wedge (\widehat{\tau(v)}_n)_1 \in WO_\beta \} = f \restriction \gamma$$

for some $\gamma > ot(v)$.

<u>Pf</u>: Consider the game

$$S_f \quad \begin{array}{lcccccc} I & v(0) & & v(1) & & \cdots & v \\ II & & r(0) & & r(1) & & r \end{array}$$

P II wins iff

$$v \in WO \Rightarrow \left[\begin{array}{l} \{ (\alpha, \beta) : \exists n \ (\hat{r}_n)_0 \in WO_\alpha \wedge (\hat{r}_n)_1 \in WO_\beta \} = f \restriction \gamma \\ \text{for some } \gamma > ot(v) \end{array} \right]$$

AD implies one of the two players has a winning strategy. The claim is that P II has a winning strategy τ

Suppose P I has a winning strategy σ. By the payoff condition, for any $r \in \mathbb{R}$, $\sigma(r) \in WO$ else P II immediately wins.

So $\sigma[\mathbb{R}]$ is a Σ^1_1 subset of WO. By the Σ^1_1 boundedness principle there is a $\delta < \omega_1$ so that for all $v \in \sigma[\mathbb{R}]$, $ot(v) < \delta$.

Fix a bijection $B : \omega \to \delta$. By $AC^{\mathbb{R}}_\omega$, pick for each $n \in \omega$, a real S_n so that

$$(\hat{S_n})_0 \in WO_{B(n)} \text{ and } (\hat{S_n})_1 \in WO_{f(B(n))}.$$

Let $r \in \mathbb{R}$ so that for all $n \in \omega$, $\hat{r}_n = S_n$.

Let P II play r against σ. $ot(\sigma(r)) < \delta$ but

$$\{ (\alpha, \beta) : \exists n \ (\hat{r}_n)_0 \in WO_\alpha \wedge (\hat{r}_n)_1 \in WO_\beta \} = f \restriction \delta.$$

Thus P I loses using α. Contradiction.

So P II has a winning strategy τ, as desired $\boxed{\text{IV}}$

Let $A \subseteq {}^\omega\omega$ and $B \subseteq {}^\omega\omega \times {}^\omega\omega$ be such that $A \subseteq \pi_1[B]$.

$$G^*_{A,B} \quad \begin{array}{lllllll} \text{I} & s_0 & & s_2 & & \cdots & x = s_0{}^\frown s_1{}^\frown \cdots \\ \text{II} & & s_1 & & s_1 & & \\ & & y(0) & & y(1) & & y \end{array}$$

All $s_i \in {}^{<\omega}\omega$. $y(i) \in \omega$.

Player II wins $G^*_{A,B}$ iff $x \in A$ and $(x,y) \in B$.

$G^*_{A,B}$ is the unfolded Banach-Mazur game.

<u>Thm</u>: (ZF) $A \subseteq {}^\omega\omega$ $\mathbf{\Sigma}^1_1$, $B \subseteq {}^\omega\omega$ $\mathbf{\Pi}^0_1$ be such that $A = \pi_1[B]$.

Then A is comeager iff P II has a winning strategy in $G^*_{A,B}$.

Let $\forall^*_\omega y\, \varphi(y)$ iff $\{y \in {}^\omega\omega : \varphi(y)\}$ is comeager.

<u>Thm</u>: (ZF) Let $A \subseteq {}^\omega\omega \times {}^\omega\omega$. be $\mathbf{\Sigma}^1_1$. Then

$A_0(r) \iff \forall^*_\omega x\, A(r,x)$ is $\mathbf{\Sigma}^1_1$

$A_1(r) \iff \forall^*_\omega x\, \neg A(r,x)$ is $\mathbf{\Pi}^1_1$.

<u>Pf</u>: Let $B \subseteq {}^\omega\omega \times {}^\omega\omega \times {}^\omega\omega$ $\mathbf{\Pi}^0_1$ so that $A(r,x) = \exists_y B(r,x,y)$.

$A_0(r) \iff A_r$ is comeager

$\iff$ P II has a w.s. in $G^*_{A_r, B_r}$

$$\iff \underbrace{\exists \tau\, \underbrace{\forall \sigma \quad B(r, x_{\sigma * \tau}, y_{\sigma * \tau})}_{\text{This } \Pi^0_1 \text{ since } B \text{ is } \mathbf{\Pi}^0_1.}}_{\text{This entire expression is } \mathbf{\Sigma}^1_1.}$$

The other statement is similar after using the Baire property. $\boxed{\text{III}}$

Fact: AD implies the meager ideal has full wellordered additivity. That is if $\kappa \in On$, $\langle A_\alpha : \alpha < \kappa \rangle$ is a sequence of meager subset of $\mathbb{R}$, then $\bigcup_{\alpha < \kappa} A_\alpha$ is meager

For each $\alpha < \omega_1$, one can give ${}^\omega\alpha$ the topology generated by $N_s = \{ f \in {}^\omega\alpha : s \subseteq f \}$ where $s \in {}^{<\omega}\alpha$. ${}^\omega\alpha$ is homeomorphic to ${}^\omega\omega$.

Thus we can formulate the familiar category notions on ${}^\omega\alpha$.

Note $\mathrm{surj}_\alpha = \{ f \in {}^\omega\alpha : f[\omega] = \alpha \}$ is comeager in ${}^\omega\alpha$

Fact: (ZF) There is a continuous function $\mathscr{D} : {}^\omega\omega_1 \to WO$ so that for $\alpha < \omega_1$, if $f \in \mathrm{surj}_\alpha$, $\mathscr{D}(f) \in WO_\alpha$.

Proof: Let $A_f = \{ n \in \omega : (\forall m < n)\ (f(m) \neq f(n)) \}$.

Let $\mathscr{D}(f)$ be code a relation on A_f by

$m <_{\mathscr{D}(f)} n \iff f(m) < f(n)$.

It is clear that $\mathscr{D}(f) \in WO$ and if $f \in \mathrm{surj}_\alpha$, then $\mathscr{D}(f) \in WO_\alpha$

$\square$

Let $A \subseteq {}^\omega\omega$ and $\kappa \in On$. A is κ-Suslin if and only if there is a tree T on $\omega \times \kappa$ so that
$A = \{ r \in {}^\omega\omega : \exists f \in {}^\omega\kappa\ \forall n\ (r \upharpoonright n, f \upharpoonright n) \in T \}$

Note that $\boldsymbol{\Sigma}^1_1$ sets are ω-Suslin.

Fact (Kunen-Martin Thm) (ZF) If $\prec$ is a κ-Suslin well founded relation, then the $\mathrm{length}(\prec) < \kappa^+$.

Thm (Martin) ω_1 is ω_1-reasonable.

Pf: We will present Kechris' proof.

We need to define a good coding system for ${}^{\omega_1}\omega$.

ω_1 is regular by $AC_\omega^{\mathbb{R}}$. The pointclass is $\underset{\sim}{\Pi}^1_1$ which is closed under $\forall^{\mathbb{R}}$

$\underset{\sim}{\Delta}^1_1 = \underset{\sim}{\Pi}^1_1 \cap \underset{\sim}{\Sigma}^1_1 = \underset{\sim}{\Pi}^1_1 \cap \check{\underset{\sim}{\Pi}}^1_1$ is closed under countable union.

If $x \in \mathbb{R}$, let T_x be the P II strategy coded by the real x.

For each $\beta, \gamma < \omega_1$, define $\varphi^\beta_\gamma(x, v)$ iff

$$\exists n \left[\left(\widehat{T_x(v)_n}\right)_0 \in WO_\beta \wedge \left(\widehat{T_x(v)_n}\right)_1 \in WO_\gamma \wedge (\forall m < n) \left(\widehat{T_x(v)_m}\right)_0 \notin WO_\beta \right]$$

Note that φ^β_γ defines a $\underset{\sim}{\Delta}^1_1$ subset of $\mathbb{R}^2$

Let $x \in GC_{\beta,\gamma} \iff (\forall^*_\beta f \in {}^\omega\beta_1)\ \varphi^\beta_\gamma(x, \mathcal{D}(f))$

Since surj_α is comeager in ${}^\omega\alpha$, for comeagerly many $f \in {}^\omega\alpha$ $\mathcal{D}(f) \in WO_\alpha$.

Pick a $B : \omega \to \alpha$. Let $\tilde{\mathcal{D}} : {}^\omega\omega \to WO$ be defined by $\tilde{\mathcal{D}}(r) = \mathcal{D}(B \circ r)$. $\tilde{\mathcal{D}}$ is continuous since $\mathcal{D}$ is continuous.

Observe that

$$x \in GC_{\beta,\gamma} \iff (\forall^*_\omega r \in {}^\omega\omega)\ \varphi^\beta_\gamma(x, \tilde{\mathcal{D}}(r))$$

By closure under the category quantifiers, $GC_{\beta,\gamma}$ is $\underset{\sim}{\Delta}^1_1$.

Define $\mathrm{decode}(x)(\beta,\gamma) \iff x \in GC_{\beta,\gamma}$.

One can check that $x \in GC_{\beta,\gamma} \iff \mathrm{decode}(x)(\beta,\gamma) \wedge \forall \xi\, (\mathrm{decode}(x)(\beta,\xi) \Rightarrow \xi = \gamma)$

from the Solovay game

For each $f: \omega_1 \to \omega_1$, there is a P II strategy τ_x (coded by the real x) so that for all $v \in WO$

$$\{ (\beta, \gamma) : \exists n \, ((\widehat{\widehat{\tau_x(v)}_n})_0 \in WO_\beta \wedge (\widehat{\widehat{\tau_x(v)}_n})_1 \in WO_\gamma \} = f \restriction \delta$$

for some $\delta > ot(v)$.

Thus $\mathrm{decode}(x) = f$.

Fix $\beta < \omega_1$. Define $\psi(x, y, v)$ by the conjunction of the following.

(1) $\exists n \; (\widehat{\widehat{\tau_x(v)}_n})_0 \in WO_\beta$

(2) $\exists n \; (\widehat{\widehat{\tau_y(v)}_n})_0 \in WO_\beta$

(3) There exists $w_0, w_1 \in \mathbb{R}$ $n_0, n_1 \in \omega$

(3a) $(\widehat{\widehat{\tau_x(v)}_{n_0}})_0 \in WO_\beta \wedge (\widehat{\widehat{\tau_x(v)}_{n_0}})_1 = w_0 \wedge$
$\forall m < n_0 \; (\widehat{\widehat{\tau_x(v)}_m})_0 \notin WO_\beta$

(3b) $(\widehat{\widehat{\tau_y(v)}_{n_1}})_0 \in WO_\beta \wedge (\widehat{\widehat{\tau_y(v)}_{n_1}})_1 = w_1 \wedge$
$\forall m < n_1 \; (\widehat{\widehat{\tau_y(v)}_m})_0 \notin WO_\beta$

(3c) $w_0 <_{\Sigma_1} w_1$ ($<_{\Sigma_1}$ is Σ^1_1 relation witnessing ot is Π^1_1-norm.)

Note that ψ defines a $\underset{\sim}{\Sigma}^1_1$ subset of $\mathbb{R}^3$.

Now let $A \subseteq GC_\beta$ be $\underset{\sim}{\Sigma}^1_1$.

Define a relation $<$ on A by

$$x < y \iff x \in A \wedge y \in A \wedge \forall^*_\beta f \in {}^\omega\beta \; \psi(x, y, \mathcal{D}(f))$$

Again by closure under category quantifiers, $<$ is $\underset{\sim}{\Sigma}^1_1$.
(and identify ω and β by bijection as before)

Fix $x \in GC_\beta$. For each $v \in WO_\beta$, let $\Lambda(x,v) = ot(\widehat{T_x(v)_n})$, where n is least so that $(\widehat{T_x(v)})_0 \in WO_\beta$. (if possible)

Let $B_\gamma = \{ f \in \mathrm{surj}_\beta : \Lambda(x, \mathfrak{D}(f)) = \gamma \}$.

Since $x \in GC_\beta$, there is some $\gamma < \omega_1$ so that B_γ is comeager.

Let $\Sigma(x)$ be this γ.

Then note that $x \prec y \iff \Sigma(x) < \Sigma(y)$.

Since Σ is $\prec$-preserving map into ω_1, $\prec$ is well founded.

Since $\prec$ is $\underset{\sim}{\Sigma}^1_1$ (ω-Suslin) and well founded, the Kunen-Martin Theorem implies the length of $\prec$ is less than ω_1.

Thus $\Sigma[A]$ is a countable set. Let $\delta \in \omega_1$ bound $\Sigma[A]$.

This implies $A \subseteq \bigcup_{\gamma < \delta} GC_{\beta,\gamma}$.

ω_1 is ω_1-reasonable $\blacksquare$