# EQUIVALENCE OF CODES FOR COUNTABLE SETS OF REALS

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ABSTRACT. A set $U \subseteq \mathbb{R} \times \mathbb{R}$ is universal for countable subsets of $\mathbb{R}$ if and only for all $x \in \mathbb{R}$, the section $U_x = \{y \in \mathbb{R} : U(x, y)\}$ is countable and for all countable sets $A \subseteq \mathbb{R}$, there is an $x \in \mathbb{R}$ so that $U_x = A$. Define the equivalence relation $E_U$ on $\mathbb{R}$ by $x_0 \; E_U \; x_1$ if and only if $U_{x_0} = U_{x_1}$, which is the equivalence of codes for countable sets of reals according to $U$. The Friedman-Stanley jump, $=^+$, of the equality relation takes the form $E_{U^*}$ where $U^*$ is the most natural Borel set which is universal for countable sets.

For all $U$ which are Borel and universal for countable sets, $E_U$ is Borel bireducible to $=^+$.

If one assumes a particular instance of $\mathbf{\Sigma}^1_3$-generic absoluteness, then for all $U \subseteq \mathbb{R} \times \mathbb{R}$ which are $\mathbf{\Sigma}^1_1$ (continuous images of Borel sets) and universal for countable sets, there is a Borel reduction of $=^+$ into $E_U$.

## 1. Equivalence of Codes for Countable Sets of Reals

Let ${}^\omega 2$ be the collection of functions $f : \omega \to 2$. The elements of ${}^\omega 2$ are called reals. (Sometimes one will also denote ${}^\omega 2$ by $\mathbb{R}$ especially when typographically convenient.) Let $\mathsf{pair} : \omega \times \omega \to \omega$ be a recursive bijection. If $x \in {}^\omega 2$ and $n \in \omega$, let $\hat{x}_n \in {}^\omega 2$ be defined by $\hat{x}_n(k) = x(\mathsf{pair}(n, k))$. So a single real $x$ naturally gives a countable set of reals $\{\hat{x}_n : n \in \omega\}$

Suppose $U \subseteq {}^\omega 2 \times {}^\omega 2$. For $x \in {}^\omega 2$, let $U_x = \{y \in {}^\omega 2 : U(x, y)\}$. Define an equivalence relation $E_U$ on ${}^\omega 2$ by $x \; E_U \; y$ if and only $U_x = U_y$. $U \subseteq {}^\omega 2 \times {}^\omega 2$ is universal for countable sets if and only if for all $x \in {}^\omega 2$, $U_x$ is countable and for all countable $A \subseteq {}^\omega 2$, there exists an $x \in {}^\omega 2$ so that $U_x = A$. If $U_x = A$, then $x$ is a code for $A$ according to $U$. $E_U$ is essentially the equivalence relation stating two reals code the same countable set according to $U$.

Suppose $A \subseteq {}^\omega 2$ is a countable set. Let $\langle x_n : n \in \omega \rangle$ be an enumeration of $A$. Then $A = \bigcup_{n \in \omega} \{x_n\}$. Since singletons are $\mathbf{\Pi}^0_1$ subsets of ${}^\omega 2$, this shows that every countable subset of ${}^\omega 2$ is $\mathbf{\Sigma}^0_2$. Let $U^* \subseteq {}^\omega 2 \times {}^\omega 2$ be defined by $(x, y) \in U^*$ if and only if $(\exists n)(\hat{x}_n = y)$ if and only if $(\exists n)(\forall m)(x(\mathsf{pair}(n, m)) = y(m))$. Note that $U^*$ is $\Sigma^0_2$ and universal for countable sets of reals. $U^*$ is the most natural coding of countable sets. (The enumeration of a countable set is a code for that countable set.) Let $=^+$ be the equivalence relation on ${}^\omega 2$ defined to be $E_{U^*}$. Note that for any $x, y \in {}^\omega 2$, $x =^+ y$ if and only if $x \; E_{U^*} \; y$ if and only if $\{\hat{x}_n : n \in \omega\} = \{\hat{y}_n : n \in \omega\}$. The latter is the familiar definition of $=^+$ as the Friedman-Stanley jump of $=$.

Equivalence relations on ${}^\omega 2$ are compared by Borel reductions. That is, if $E$ and $F$ are two equivalence relations on ${}^\omega 2$, one writes $E \leq_{\mathbf{\Delta}^1_1} F$ if and only there is a Borel function $\Phi : {}^\omega 2 \to {}^\omega 2$ so that for all $x, y \in {}^\omega 2$, $x \; E \; y$ if and only if $\Phi(x) \; F \; \Phi(y)$. One writes $E \equiv_{\mathbf{\Delta}^1_1} F$ if and only if $E \leq_{\mathbf{\Delta}^1_1} F$ and $F \leq_{\mathbf{\Delta}^1_1} E$. Since $=^+$ is $E_{U^*}$ where $U^*$ is the most natural $\Sigma^0_2$ set universal for countable subsets of ${}^\omega 2$, a natural question ([2] Question 2.5) asked by Ding and Yu is whether for any Borel set $U \subseteq {}^\omega 2 \times {}^\omega 2$ which is universal for countable sets, is $E_U \equiv_{\mathbf{\Delta}^1_1} =^+$? They showed that if $U$ is Borel and universal for countable sets, then $E_U \leq_{\mathbf{\Delta}^1_1} =^+$. Thus the question becomes whether $=^+ \leq_{\mathbf{\Delta}^1_1} E_U$ when $U$ is Borel and universal for countable sets. They also asked if $=^+ \leq_{\mathbf{\Delta}^1_1} E_U$ when $U$ is $\mathbf{\Sigma}^1_1$ (a continuous image of a Borel set) and universal for countable sets.

This article will answer these questions. It will be shown that if $U \subseteq {}^\omega 2 \times {}^\omega 2$ is Borel and universal for countable subsets of ${}^\omega 2$, then $=^+$ is Borel bireducible to $E_U$. Intuitively, this means that every coding of countable sets via a Borel $U$ which is universal for countable sets is indistinguishable from the natural coding of countable sets given by $U^*$ via Borel procedures. The argument uses forcing ideas and absoluteness. Granting sufficient absoluteness of certain statements between the ground model and certain forcing extensions, the method in the Borel case can be extended to produce a Borel reduction from $=^+$ into $E_U$ when $U \subseteq {}^\omega 2 \times {}^\omega 2$ is a $\mathbf{\Sigma}^1_1$ set which is universal for countable sets. The end of the article has a broad overview of why forcing produces certain countable sets of reals for which one can easily search for the code

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for these countable sets according to the Borel set $U$ which is universal for countable sets. In general, the search for a code for a countable set of reals seems quite complex.

A tree $T$ on $2\times\omega$ (or 2 or $\omega$) is a subset of ${}^{<\omega}(2\times\omega)$ (or ${}^{<\omega}2$ or ${}^{<\omega}\omega$, respectively) which is $\subseteq$-downward closed (where here $\subseteq$ refers to string extension). Note that such trees are coded by reals. If $T$ is a tree on $2\times\omega$, then $[T]=\{f\in{}^{\omega}(2\times\omega):(\forall^{\omega}n)(f\restriction n\in T)\}$, where $f\restriction n$ refers to the length $n$ initial segment of $f$. Let $\pi_1:{}^{\omega}2\times{}^{\omega}\omega\to{}^{\omega}2$ be the projection onto the first coordinate. A set $B$ is $\Sigma^1_1(z)$ if and only if there is an $z$-recursive tree $T$ in $2\times\omega$ so that $B=\pi_1[[T]]$. It is important to note that whenever one writes $B$ in any universe of set theory containing $T$, it will always refer to the interpretation of $\pi_1[[T]]$. A set $B$ is $\Delta^1_1(z)$ if and only if there are $z$-recursive trees $T$ and $S$ so that $\pi_1[[T]]=B$ and $\pi_1[[S]]={}^{\omega}2\setminus B$. Note that the statement "$(\exists x)(T_x$ and $S_x$ are illfounded)" is $\Sigma^1_1(z)$. By Mostowski absoluteness, in any transitive set or class $M$ satisfying adequate amount of $\mathsf{ZF}$ with $\{z\}\cup\omega\subseteq M$, $M\models\pi_1[[T]]\cap\pi_1[[S]]=\emptyset$. Also $(\forall x)(T_x$ or $T_y$ is illfounded) is $\Pi^1_2(z)$. By Shoenfield absoluteness, in any transitive set or class $M$ satisfying adequate amount of $\mathsf{ZF}$ with $\{z\}\cup\omega_1\subseteq M$, $M\models\pi_1[[T]]={}^{\omega}2\setminus\pi_1[[S]]$. Thus if trees $T$ and $S$ define a $\Delta^1_1(z)$ subset of ${}^{\omega}2$, then in any transitive set or class model $M$ such that $\{z\}\cup\omega_1\subseteq M$, $T$ and $S$ will continue to represent a $\Delta^1_1(z)$ set. In this way, when one speaks of this $\Delta^1_1(z)$ set, one implicitly means $\pi_1[[T]]$ and its complement is $\pi_1[[S]]$.

**Fact 1.1.** *Suppose $U\subseteq{}^{\omega}2\times{}^{\omega}2$ is $\Sigma^1_1(z)$. Then the statement "$(\forall x)(U_x$ is countable)" is $\Pi^1_1(z)$.*

*Proof.* Since for all $x\in{}^{\omega}2$, $U_x$ is a countable $\Sigma^1_1(x,z)$ set, the effective perfect set theorem of Mansfield ([5] 4F.1) implies that $U_x$ consists only of $\Delta^1_1(x,z)$ reals. By [5] 4D.2, there is a $\Pi^1_1$ relation $H\subseteq\mathbb{R}\times\mathbb{R}$ such that $H(x,y)$ if and only if $y\in\Delta^1_1(x)$. Thus the statement "$(\forall x)(U_x$ is countable)" is equivalent to

$$(\forall x)(\forall y)(U(x,y)\Rightarrow H(x\oplus z,y))$$

where $(x\oplus z)\in{}^{\omega}2$ is the recursive join defined by $(x\oplus z)(2n)=x(n)$ and $(x\oplus z)(2n+1)=z(n)$ for all $n\in\omega$. The latter statement is $\Pi^1_1(z)$. $\square$

**Fact 1.2.** *(Ding-Yu, [2] Theorem 2.4) If $U$ is $\Delta^1_1(z)$ and universal for countable sets, then there is a $\Delta^1_1(z)$ reduction $\Phi:{}^{\omega}2\to{}^{\omega}2$ witnessing $E_U\leq_{\mathbf{\Delta}^1_1}=^+$. In particular, this implies that $E_U$ is a $\Delta^1_1(z)$ equivalence relation.*

*Proof.* First, one will show $\mathrm{dom}(U)=\{x\in{}^{\omega}2:(\exists y)U(x,y)\}$ is $\Delta^1_1(z)$. It is clearly $\Sigma^1_1(z)$. By [5] 4D.2, there is a $\Pi^1_1$-recursive partial function $\mathbf{d}:\omega\times\mathbb{R}\to\mathbb{R}$ so that $y\in\Delta^1_1(x)$ if and only if $(\exists^{\omega}n)((n,x)\in\mathrm{dom}(\mathbf{d})\wedge\mathbf{d}(n,x)=y)$. Since $U$ is countable, the effective perfect set theorem implies $U_x\subseteq\Delta^1_1(x\oplus z)$. Thus $x\in\mathrm{dom}(U)$ if and only $(\exists^{\omega}n)((n,x)\in\mathrm{dom}(\mathbf{d})\wedge U(x,\mathbf{d}(n,x)))$. By [5] 4D.1 (ii), the latter expression is $\Pi^1_1(z)$.

If $\sigma\in{}^{<\omega}2$, then let $N_\sigma=\{f\in{}^{\omega}2:\sigma\subseteq f\}$ be the basic neighborhood determined by $\sigma$. Let $\Psi:{}^{\omega}2\to N_{\langle 0\rangle}$ be a recursive bijection.

By the Lusin-Novikov theorem ([5] 4F.17), there is a $\Delta^1_1(z)$ relation $P\subseteq\omega\times{}^{\omega}2\times{}^{\omega}2$ so that $U(x,y)$ if and only if $(\exists^{\omega}n)P(n,x,y)$ and for each $n\in\omega$, $P_n=\{(x,y):P(n,x,y)\}$ uniformizes $U$.

Define $\Phi:{}^{\omega}2\to{}^{\omega}2$ by $\Phi(x)=w$ if and only if the disjunction of the following

- $x\in\mathrm{dom}(U)\wedge(\forall^{\omega}n)(\exists^{\mathbb{R}}y)[P(n,x,y)\wedge(\forall^{\omega}k)(w(\mathsf{pair}(n,k))=\Psi(y)(k))]$
- $x\notin\mathrm{dom}(U)\wedge(\forall^{\omega}n)(\forall^{\omega}k)(w(\mathsf{pair}(n,k))=1)$

if and only if the disjunction of the following

- $x\in\mathrm{dom}(U)\wedge(\forall^{\omega}n)(\forall^{\mathbb{R}}y)[P(n,x,y)\Rightarrow(\forall^{\omega}k)(w(\mathsf{pair}(n,k))=\Psi(y)(k))]$
- $x\notin\mathrm{dom}(U)\wedge(\forall^{\omega}n)(\forall^{\omega}k)(w(\mathsf{pair}(n,k))=1)$

By the properties of $P$ stated above, these two definitions are equivalent. Since the first definition is $\Sigma^1_1(z)$ and the second definition is $\Pi^1_1(z)$, $\Phi$ is $\Delta^1_1(z)$. Intuitively, if $U_x\neq\emptyset$, $\Phi(x)$ has the property that $\{\widehat{\Phi(x)}_n:n\in\omega\}=\{\Psi(y):U(x,y)\}$. If $U_x=\emptyset$, then $\Phi(x)$ has the property that $\{\widehat{\Phi(x)}_n:n\in\omega\}=\{\bar{1}\}$, where $\bar{1}$ is the constant function taking value 1. Since $\Psi:{}^{\omega}2\to N_{\langle 0\rangle}$, one has the $\Phi$ is a $\Delta^1_1(z)$ reduction of $E_U$ into $=^+$. $\square$

**Fact 1.3.** *Suppose $U\subseteq{}^{\omega}2\times{}^{\omega}2$ is $\Sigma^1_1(z)$, then the statement "$U$ is universal for countable sets" is $\Pi^1_3(z)$. If $U\subseteq{}^{\omega}2\times{}^{\omega}2$ is $\Delta^1_1(z)$, then the statement "$U$ is universal for countable sets" is $\Pi^1_2(z)$.*

*Proof.* Suppose $U$ is $\Sigma^1_1(z)$. Then $U$ is universal for countable set if and only if the conjunction of the following holds

- $(\forall^{\mathbb{R}} x)(U_x \text{ is countable})$
- $(\forall^{\mathbb{R}} z)(\exists^{\mathbb{R}} x)[(\forall^{\omega} n)U(x, \hat{z}_n) \wedge (\forall^{\mathbb{R}} y)(U(x,y) \Rightarrow (\exists^{\omega} n)(\hat{z}_n = y))]$

The first condition is $\Pi^1_1(z)$ by Fact 1.1. The second condition is $\Pi^1_3(z)$. The entire expression is $\Pi^1_3(z)$.

Now suppose that $U$ is $\Delta^1_1(z)$. If for all $x \in {}^{\omega}2$, $U_x$ is countable, then the Lusin-Novikov theorem ([5] 4F.17) states that there is a $\Delta^1_1(z)$ relation $P \subseteq \omega \times {}^{\omega}2 \times {}^{\omega}2$ so that $U(x,y)$ if and only $(\exists^{\omega} n)P(n,x,y)$ and for all $n \in \omega$, $P_n = \{(x,y) : P(n,x,y)\}$ is a uniformization for $U$. $U$ is universal for countable sets if and only if the conjunction of the following holds

- $(\forall^{\mathbb{R}} x)(U_x \text{ is countable})$
- $(\forall^{\mathbb{R}} z)(\exists^{\mathbb{R}} x)[(\forall^{\omega} n)U(x, \hat{z}_n) \wedge (\forall^{\omega} m)(\exists^{\mathbb{R}} y)(\exists^{\omega} n)(P(m,x,y) \wedge \hat{z}_n = y)]$

Note that the second condition of $\Pi^1_2(z)$. Thus the entire expression is $\Pi^1_2(z)$. □

**Definition 1.4.** Let $\mathbb{C}$ be the set of finite partial functions $p : \omega \to 2$. Let $\leq_{\mathbb{C}}$ be reverse inclusion. The largest condition is $1_{\mathbb{P}} = \emptyset$. $\mathbb{C} = (\mathbb{C}, \leq_{\mathbb{C}}, 1_{\mathbb{C}})$ is called Cohen forcing.

For any $\epsilon \in \mathrm{ON}$, let $\mathbb{C}_\epsilon = \prod_{\alpha < \epsilon} \mathbb{C}$ be the finite support product of $\mathbb{C}$. The conditions are $p : \epsilon \to \mathbb{C}$ so that $\mathsf{supp}(p) = \{\alpha < \epsilon : p(\alpha) \neq 1_{\mathbb{C}}\}$ is finite. If $p, q \in \mathbb{C}_\epsilon$, $p \leq_{\mathbb{C}_\epsilon} q$ if and only if for all $\alpha < \epsilon$, $p(\alpha) \leq_{\mathbb{C}} q(\alpha)$. $1_{\mathbb{C}_\epsilon}$ is the constant function on $\epsilon$ taking value $1_{\mathbb{C}}$.

Let $\mathrm{Coll}(\omega, \mathbb{R})$ be the forcing consisting of finite partial functions $p : \omega \to \mathbb{R}$. $\leq_{\mathrm{Coll}(\omega,\mathbb{R})}$ is reverse inclusion. $1_{\mathrm{Coll}(\omega,\mathbb{R})} = \emptyset$. Note that if $G \subseteq \mathsf{Coll}(\omega, \mathbb{R})$ is $\mathrm{Coll}(\omega,\mathbb{R})$-generic over the ground model, then the extension by $G$ adds a surjection $g$ from $\omega$ onto the reals of the ground model. Therefore, the set of ground model reals are countable in this forcing extension.

Throughout the article, one will need several effectiveness or uniformity observations concerning the forcing construction on countable models coded as reals. Some details will be provided without including too much burdensome coding notations.

**Definition 1.5.** If $x \in {}^{\omega}2$, let $\mathcal{R}_x(m,n)$ if and only if $x(\mathsf{pair}(m,n)) = 1$. $\mathcal{R}_x$ is the binary relation on $\omega$ coded by $x$.

WO is the collection of $x \in {}^{\omega}2$ so that $\mathcal{R}_x$ is a wellordering. If $x \in \mathrm{WO}$, then let $\mathrm{ot}(x)$ be the ordertype $(\omega, \mathcal{R}_x)$.

If $\mathcal{R}_x$ is a set-like, extensional, and wellfounded relation, then let the transitive set $(\mathcal{M}_x, \in)$ denote the Mostowski collapse of $(\omega, \mathcal{R}_x)$. Let $\mathsf{most}_x : (\omega, \mathcal{R}_x) \to (\mathcal{M}_x, \in)$ be the Mostowski collapse function.

Recall that the satisfaction relation $\mathsf{Sat}$ defined by $(x, \varphi, \langle i_1, ..., i_k\rangle) \in \mathsf{Sat}$ if and only if $(\omega, \mathcal{R}_x) \models \varphi(i_1, ..., i_k)$ is $\Delta^1_1$. (Formulas are coded by integers in some recursive manner.) The fact that $\mathsf{Sat}$ is $\Delta^1_1$ will often be implicitly used.

**Fact 1.6.** *Suppose $\mathfrak{m} \in {}^{\omega}2$ is such that $(\omega, \mathcal{R}_{\mathfrak{m}})$ is a wellfounded structure satisfying some adequate amount of $\mathsf{ZF} + \mathsf{AC}^{\mathbb{R}}_{\omega}$. Let $\mathsf{most}_{\mathfrak{m}} : (\omega, \mathcal{R}_{\mathfrak{m}}) \to (\mathcal{M}_{\mathfrak{m}}, \in)$ be the Mostowski collapse map. For notational simplicity, let $\mathcal{N} = \mathcal{M}_{\mathfrak{m}}$. Then there are $\boldsymbol{\Delta}^1_1$ functions $\mathsf{Gen}_0, \mathsf{GenMod}_0 : {}^{\omega}2 \to {}^{\omega}2$ so that*

1. *For all $x \in \mathbb{R}$, let $\mathcal{G}_x = \mathsf{most}_{\mathfrak{m}}[\{n \in \omega : \mathsf{Gen}_0(n) = 1\}]$. $\mathcal{G}_x$ is $\mathbb{C}$-generic over $\mathcal{N}$.*
2. *For any $k \in \omega$ and injective sequence $\ell : k \to {}^{\omega}2$, $\prod_{i<k} \mathcal{G}_{\ell(k)}$ is $\prod_{i<k} \mathbb{C}$-generic over $\mathcal{N}$.*
3. *$(\omega, \mathcal{R}_{\mathsf{GenMod}_0}(x))$ is a set-like, wellfounded, and extensional structure whose Mostowski collapse, $\mathcal{M}_{\mathsf{GenMod}_0(x)}$, is $\mathcal{N}[\mathcal{G}_x]$.*

*Proof.* Using the fact that the satisfaction relation is $\Delta^1_1$, one can obtain from $\mathfrak{m}$ in a $\Delta^1_1$ manner a function $\mathfrak{d} : \omega \times \omega \to \omega$ with the following properties: For $1 \leq k < \omega$ and $i \in \omega$, let $D^k_i = \mathsf{most}_{\mathfrak{m}}(\mathfrak{d}(i,k))$. For each $1 \leq k < \omega$, $\{D^k_i : i \in \omega\}$ enumerates all of the dense open subsets of $\prod_{j<k} \mathbb{C}$ in the countable transitive set $\mathcal{N}$.

Next, one will sketch the standard construction of a perfect set of mutually $\mathbb{C}$-generics filters over $\mathcal{N}$. One will build a perfect tree $\langle p_\sigma : \sigma \in {}^{<\omega}2\rangle$ of $\mathbb{C}$-conditions so that each path generates a $\mathbb{C}$-generic filter over $\mathcal{N}$.

Let $p_\emptyset = 1_{\mathbb{C}}$. Suppose for some $n \in \omega$, $p_\sigma$ has been defined for all $\sigma \in {}^{n}2$. For each $\sigma \in {}^{n}2$, let $n$ be least so that $n \notin \mathrm{dom}(p_\sigma)$. Let $q_{\sigma\hat{}i} = p_\sigma \cup \{(n,i)\}$ for $i \in \{0,1\}$. By repeatedly extending $q_\tau$ for all $\tau \in {}^{n+1}2$ as necessary to meet all the requisite dense open sets, one can find a collection $\{p_\tau : \tau \in {}^{n+1}2\}$ such that

- For all $\tau \in {}^{n+1}2$, $p_\tau \leq_{\mathbb{C}} q_\tau$.
- For all $k < 2^{n+1}$, for all injections $B : k \to {}^{n+1}2$, and any dense open set $D^k_i$ for $i \leq n$, $(p_{B(0)}, ..., p_{B(k-1)}) \in D^k_i$.

This completes the construction. For each $x \in {}^{\omega}2$, let $\mathcal{G}_x$ be the $\leq_{\mathbb{C}}$-upward closure of $\{p_{x \restriction n} : n \in \omega\}$. One can check that each $\mathcal{G}_x$ is $\mathbb{C}$-generic over $\mathcal{N}$ and any finite collection has the mutual genericity property.

The reader can check that by coding using $\mathfrak{m}$ and $\mathfrak{d}$ (which is obtained from $\mathfrak{m}$), one can find a $\Delta^1_1(\mathfrak{m})$ function $\mathsf{Gen}_0$ so that $\mathsf{Gen}_0(x)$ is a real which codes $\mathsf{most}_{\mathfrak{m}}^{-1}[\mathcal{G}_x]$. By the uniformity of the forcing construction, one can also find a $\Delta^1_1(\mathfrak{m})$ function $\mathsf{GenMod}_0$ so that for all $x \in {}^{\omega}2$, $\mathsf{GenMod}_0(x)$ codes a structure whose Mostowski collapse is $\mathcal{N}[\mathcal{G}_x]$. ☐

**Fact 1.7.** *Suppose $\epsilon < \omega_1$. Then the Cohen forcing $\mathbb{C}$ and the $\epsilon$-length finite support product of Cohen forcing $\mathbb{C}_\epsilon$ are isomorphic.*

*Proof.* Let $\epsilon < \omega_1$ and $B : \epsilon \times \omega \to \omega$ be a bijection. Let $\Phi : \mathbb{C} \to \mathbb{C}_\epsilon$ be defined by $\Phi(p)(\alpha)(n) = p(B(\alpha, n))$ whenever $B(\alpha, n)$ is in the domain of $p$. So for each $\alpha < \epsilon$, $\Phi(p)(\alpha) \in \mathbb{C}$ and for only finitely many $\alpha$, $\Phi(p)(\alpha) \neq 1_{\mathbb{C}} = \emptyset$. Recall that elements of $\mathbb{C}_\epsilon$ are functions from $\epsilon$ into $\mathbb{C}$ with finite support. Thus $\Phi$ is well defined and is an isomorphism. ☐

**Fact 1.8.** *Assume the notation of Fact 1.6. There is a uniform procedure which takes an injective sequence $\langle G_n : n \in \omega\rangle$ of $\mathbb{C}$-generic filters over $\mathcal{N}$ with the property that any finite collection is mutually generic to a $\mathbb{C}_{\omega_1^{\mathcal{M}}}$-generic filter $G^*$ over $\mathcal{N}$ so that $\mathbb{R}^{\mathcal{N}[G^*]} = \bigcup\{\mathbb{R}^{\mathcal{N}[\prod_{i<k} G_i]} : k \in \omega\}$.*

*Proof.* Recall that $\mathcal{N} = \mathcal{M}_{\mathfrak{m}}$. Using the fact that the satisfaction relation is $\Delta^1_1$, one can define, in a $\Delta^1_1$ manner using $\mathfrak{m}$, a sequence $\Xi : \omega \to \omega$ by induction as follows: $\Xi(0)$ is the least element $k$ of $\omega$ so that $\mathcal{N} \models \mathsf{most}_{\mathfrak{m}}(k) < \omega_1$. Suppose $\Xi(n)$ has been defined, let $\Xi(n+1)$ be the least integer $k > \Xi(n)$ so that $\mathcal{N} \models \mathsf{most}_{\mathfrak{m}}(k) < \omega_1$. Note that $\Xi[\omega] = \{n \in \omega : \mathcal{N} \models \mathsf{most}_{\mathfrak{m}}(n) \in \omega_1\}$; that is, $\Xi$ enumerates all the integers $n$ so that $(\omega, \mathcal{R}_{\mathfrak{m}}) \models$ "$n$ is a countable ordinal". Let $\rho(n) = \sup\{\mathsf{most}_{\mathfrak{m}}(\Xi(k)) : k \leq n\}$. Note that $\rho : \omega \to \omega_1^{\mathcal{N}}$ is a cofinal increasing sequence. Let $I_0 = \{\alpha \in \omega_1^{\mathcal{N}} : 0 \leq \alpha < \rho(0)\}$ and for $n > 0$, $I_n = \{\alpha \in \omega_1^{\mathcal{N}} : \rho(n-1) \leq \alpha < \rho(n)\}$. Note that for all $n \in \omega$, $I_n \in \mathcal{N}$ and $\mathcal{N} \models |I_n| \leq \aleph_0$. As before in a $\Delta^1_1$ manner from $\mathfrak{m}$, one can define $\Upsilon : \omega \to \omega$ by $\Upsilon(n)$ is the least integer $k$ so that $\mathcal{N} \models$ "$\mathsf{most}_{\mathfrak{m}}(k)$ is a bijection from $I_n \times \omega \to \omega$". Let $B_n = \mathsf{most}_{\mathfrak{m}}(\Upsilon(n))$. Thus for each $n \in \omega$, $B_n : I_n \times \omega \to \omega$ is a bijection and $B_n \in \mathcal{N}$ (however the entire sequence $\langle B_n : n \in \omega\rangle$ does not belong to $\mathcal{N}$).

Next, the idea is to create a $\mathbb{C}_{\omega_1^{\mathcal{N}}}$-generic filter by "transferring" each $G_n$ onto the interval $I_n$ via isomorphisms $\Phi_n : \mathbb{C} \to \prod_{I_n} \mathbb{C}$ created from $B_n$ as in the proof of Fact 1.7. More precisely, let $\Phi_n : \mathbb{C} \to \prod_{I_n} \mathbb{C}$ be defined by $\Phi_n(p)(\alpha)(k) = p(B_n(\alpha, k))$ whenever $\alpha \in I_n$ and $p(B_n(\alpha, k))$ is defined. Since $B_n \in \mathcal{N}$, $\Phi_n \in \mathcal{N}$ as well. For each $n \in \omega$, let $\Psi^*_n : \prod_{i \leq n} \mathbb{C} \to \mathbb{C}_{\rho(n)}$ by $\Psi^*_n(q)(\alpha)(k) = \Phi_j(q(j))(\alpha)(k)$ where $q \in \prod_{i \leq n} \mathbb{C}$ and $j \leq n$ is the unique $j$ so that $\alpha \in I_j$. Note that $\Psi^*_n \in \mathcal{N}$ for each $n \in \omega$ and $\Psi^*_n : \prod_{i \leq n} \mathbb{C} \to \mathbb{C}_{\rho(n)}$ is an isomorphism. For each $n \in \omega$, let $\mathfrak{I}_n : \mathbb{C}_{\rho(n)} \to \mathbb{C}_{\omega_1^{\mathcal{M}}}$ be the canonical order preserving injection defined by

$$\mathfrak{I}_n(p)(\alpha) = \begin{cases} p(\alpha) & \alpha < \rho(n) \\ 1_{\mathbb{C}} & \alpha \geq \rho(n) \end{cases}$$

Observe that for each $n \in \omega$, $\mathfrak{I}_n \in \mathcal{N}$. Let $\Psi_n : \prod_{i \leq n} \mathbb{C} \to \mathbb{C}_{\omega_1^{\mathcal{N}}}$ be defined by $\mathfrak{I}_n \circ \Psi^*_n$. Note also that for all $n \in \omega$, $\Psi_n \in \mathcal{N}$. Define $G^* = \bigcup\{\Psi_n[\prod_{i \leq n} G_i] : n \in \omega\}$. $G^* \subseteq \mathbb{C}_{\omega_1^{\mathcal{N}}}$ is a filter.

It remains to show that $G^*$ is $\mathbb{C}_{\omega_1^{\mathcal{N}}}$-generic over $\mathcal{N}$. Recall that $\mathcal{N} \models \mathbb{C}_{\omega_1}$ satisfies the $\omega_1$-chain condition. Let $A \in \mathcal{N}$ be such that $\mathcal{N}$ thinks $A$ is a maximal antichain of $\mathbb{C}_{\omega_1^{\mathcal{N}}}$. Since the $\omega_1$-chain condition holds, $\omega_1^{\mathcal{N}}$ is regular in $\mathcal{N} \models \mathsf{AC}^{\mathbb{R}}_{\omega}$, and each $\Psi^*_n$ is an isomorphism, one has that there is an $n \in \omega$ so that $\Psi_n^{-1}[A]$ is a maximal antichain of $\prod_{i \leq n} \mathbb{C}$. Since $\prod_{i \leq n} G_i$ is $\prod_{i \leq n} \mathbb{C}$-generic over $\mathcal{N}$, $\Psi_n^{-1}[A] \cap \prod_{i \leq n} G_i \neq \emptyset$. Hence $A \cap G^* \neq \emptyset$. This shows that $G^*$ is $\mathbb{C}_{\omega_1^{\mathcal{N}}}$-generic over $\mathcal{N}$.

Since all $\Psi_n \in \mathcal{N}$ and are isomorphisms, one has that $\mathcal{N}[\prod_{i \leq n} G_i] = \mathcal{N}[\Psi_n[\prod_{i \leq n} G_i]]$. Since $\mathcal{N}[\prod_{i \leq n} G_i] \subseteq \mathcal{N}[G^*]$, one has that $\bigcup\{\mathbb{R}^{\mathcal{N}[\prod_{i \leq n} G_i]} : n \in \omega\} \subseteq \mathbb{R}^{\mathcal{N}[G^*]}$. Now suppose that $x \in \mathbb{R}^{\mathcal{N}[G^*]}$. There is a nice name $\tau \in \mathcal{N}$ of the form $\tau = \bigcup_{n \in \omega}\{\check{n}\} \times A_n$ (where $A_n$ is an antichain of $\mathbb{C}_{\omega_1^{\mathcal{N}}}$) such that $\tau[G^*] = x$. Since $\mathcal{N}$ believes that $\mathbb{C}_{\omega_1^{\mathcal{N}}}$ has the $\omega_1^{\mathcal{N}}$-chain condition and $\omega_1^{\mathcal{N}}$ is regular, there is an $n \in \omega$ so that all conditions mentioned in the name $\tau$ occurs in $\mathbb{C}_{\rho(n)}$. Thus $x = \tau[G^*] = \tau[\Psi_n[\prod_{i \leq n} G_i]]$, where here one considers $\tau$ as a $\mathbb{C}_{\rho(n)}$-name in the natural way. Since $\mathcal{N}[\prod_{i\leq n} G_i] = \mathcal{N}[\Psi_n[\prod_{i\leq n} G_i]]$, $x \in \mathbb{R}^{\mathcal{N}[\prod_{i\leq n} G_i]}$. It has been shown that $\mathbb{R}^{\mathcal{N}[G^*]} \subseteq \bigcup\{\mathbb{R}^{\mathcal{N}[\prod_{i\leq n} G_i]} : n \in \omega\}$. Hence these two sets are equal.

It is important to note that there is an explicit and uniform method to obtain $G^*$ from $\langle G_n : n \in \omega\rangle$. One can check this procedure is $\Delta^1_1(\mathfrak{m})$ as a function in the codes in the sense of Fact 1.9. □

**Fact 1.9.** *Assume the setting from Fact 1.6. Then there are $\Delta^1_1(\mathfrak{m})$ function $\mathsf{Gen}_1, \mathsf{GenMod}_1 : {}^\omega 2 \to {}^\omega 2$ with the following properties. Let $\mathcal{H}_x = \mathsf{most}_{\mathfrak{m}}[\{n : \mathsf{Gen}_1(x)(n) = 1\}]$.*

1. *Suppose $\{\hat{x}_n : n \in \omega\}$ is finite. Let $E(x) : N \to {}^\omega 2$ be the enumeration of $\{\hat{x}_n : n \in \omega\}$ which removes the duplicates from $\langle \hat{x}_n : n \in \omega\rangle$ where $N \in \omega$. Then $\mathcal{H}_x$ is $\prod_{i<N} \mathbb{C}$-generic over $\mathcal{N}$ and $\mathcal{N}[\mathcal{H}_x] = \mathcal{N}[\prod_{i<N} \mathcal{G}_{E(x)(i)}]$.*
2. *Now suppose $x \in {}^\omega 2$ is such that $\{\hat{x}_n : n \in \omega\}$ is infinite. Let $E(x) : \omega \to {}^\omega 2$ be the enumeration of $\{\hat{x}_n : n \in \omega\}$ which removes the duplicate from $\langle \hat{x}_n : n \in \omega\rangle$. Then $\mathcal{H}_x$ is $\mathbb{C}_{\omega_1^{\mathcal{N}}}$-generic over $\mathcal{N}$ and $\mathbb{R}^{\mathcal{N}[\mathcal{H}_x]} = \bigcup\{\mathbb{R}^{\mathcal{N}[\prod_{i<n} \mathcal{G}_{E(x)(i)}]} : n \in \omega\}$.*
3. *$\mathcal{R}_{\mathsf{GenMod}_1(x)}$ is a set-like, wellfounded, and extensional relation on $\omega$ whose Mostowski collapse $\mathcal{M}_{\mathsf{GenMod}_1(x)}$ is equal to $\mathcal{N}[\mathcal{H}_x]$.*

*Proof.* In case (2), the existence of the $\Delta^1_1(\mathfrak{m})$ function $\mathsf{Gen}_1$ follows from the uniformity of the argument in the proof of Fact 1.8. Case (1) is similar and somewhat easier. $\mathsf{GenMod}_1$ again comes from the uniformity of the forcing construction. □

**Fact 1.10.** *Assume the setting of Fact 1.9. For all $x, y \in {}^\omega 2$, $x =^+ y$ if and only $\mathbb{R}^{\mathcal{N}[\mathcal{H}_x]} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_y]}$.*

*Proof.* Without loss of generality (since the arguments are similar), assume that $\{\hat{x}_n : n \in \omega\}$ and $\{\hat{y}_n : n \in \omega\}$ are infinite. Let $E(x)$ and $E(y)$ enumerate without repetition $\langle \hat{x}_n : n \in \omega\rangle$ and $\langle \hat{y}_n : n \in \omega\rangle$, respectively. Since $\mathbb{R}^{\mathcal{N}[\mathcal{H}_x]} = \bigcup\{\mathbb{R}^{\mathcal{N}[\prod_{i<n} \mathcal{G}_{E(x)(i)}]} : n \in \omega\}$ and $\mathbb{R}^{\mathcal{N}[\mathcal{H}_y]} = \bigcup\{\mathbb{R}^{\mathcal{N}[\prod_{i<n} \mathcal{G}_{E(y)(i)}]} : n \in \omega\}$, it is clear that if $x =^+ y$, then $\mathbb{R}^{\mathcal{N}[\mathcal{H}_x]} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_y]}$.

Now suppose that $\neg(x =^+ y)$. Without loss of generality, there is an $n^*$ so that $E(x)(n^*) \notin \{\hat{y}_n : n \in \omega\}$. Let $g \in {}^\omega 2$ denote the $\mathbb{C}$-generic real associated to the $\mathbb{C}$-generic filter $\mathcal{G}_{E(x)(n^*)}$. Suppose for the sake of contradiction that $\mathbb{R}^{\mathcal{N}[\mathcal{H}_x]} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_y]}$. Since $\mathcal{N} \models$ "$\mathbb{C}_{\omega_1^{\mathcal{N}}}$ has the $\omega_1^{\mathcal{N}}$-chain condition", there is some $m \in \omega$ so that $g \in \mathbb{R}^{\mathcal{N}[\prod_{i<m} \mathcal{G}_{E(y)(i)}]}$ as argued in the proof Fact 1.8. This is impossible since by Fact 1.6, $\{\mathcal{G}_{E(x)(n^*)}\} \cup \{\mathcal{G}_{E(y)(i)} : i < m\}$ is a collection of mutually $\mathbb{C}$-generic filters. □

**Fact 1.11.** *There are $\Delta^1_1(\mathfrak{m})$ functions $\mathsf{Gen}_2, \mathsf{GenMod}_2 : {}^\omega 2 \to {}^\omega 2$ with the following properties:*

1. *Let $\mathcal{K}_x = \mathsf{most}_{\mathsf{GenMod}_1(x)}[\{n : \mathsf{Gen}_2(x)(n) = 1\}]$. $\mathcal{K}_x$ is a $\mathsf{Coll}(\omega, \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]})$-generic filter over $\mathcal{N}[\mathcal{H}_x]$.*
2. *$\mathcal{R}_{\mathsf{GenMod}_2(x)}$ is a set-like, wellfounded, and extensional relation on $\omega$ whose Mostowski collapse $\mathcal{M}_{\mathsf{GenMod}_2(x)}$ is $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$.*

*Proof.* The main ideas are the following: Fix $x \in {}^\omega 2$. Using that the satisfaction relation is $\Delta^1_1$, one can obtain in a $\Delta^1_1$ manner from the real $\mathsf{GenMod}_1(x)$ an enumeration of all the $\mathsf{Coll}(\omega, \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]})$-dense open subsets that belong to $\mathcal{N}[\mathcal{H}_x]$. From this enumeration of dense open sets, one can construct the code $\mathsf{Gen}_2(x)$ for $\mathcal{K}_x$, a $\mathrm{Coll}(\omega, \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]})$-generic filter over $\mathcal{N}[\mathcal{H}_x]$ and the code $\mathsf{GenMod}_2(x)$ for a structure on $\omega$ whose Mostowski collapse is $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$. (The construction is a simplified version of the argument in Fact 1.6.) □

**Fact 1.12.** *Assume $\mathsf{ZF} + \mathsf{AC}^{\mathbb{R}}_\omega$. Let $U \subseteq {}^\omega 2 \times {}^\omega 2$ be a $\Delta^1_1(z)$ set which is universal for countable subsets of ${}^\omega 2$. Let $V$ denote the real world. Let $\mathfrak{m} \in {}^\omega 2$ be such that the Mostowski collapse $\mathcal{N} = \mathcal{M}_{\mathfrak{m}}$ of $(\omega, \mathcal{R}_{\mathfrak{m}})$ is an elementary substructure of $V_\kappa$ (for some cardinal $\kappa$) satisfying adequate amount of $\mathsf{ZF} + \mathsf{AC}^{\mathbb{R}}_\omega$ and $z \in \mathcal{N}$. Then there is a $\Delta^1_1(\mathfrak{m})$ function $\Phi : {}^\omega 2 \to {}^\omega 2$ so that $U_{\Phi(x)} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$.*

*Assume $\mathsf{ZF} + \mathsf{AC}^{\mathbb{R}}_\omega$ and $\Sigma^1_3(z)$-generic absoluteness holds (specifically for the two step iteration $\mathbb{C}_{\omega_1} * \mathrm{Coll}(\omega, \dot{\mathbb{R}})$). Let $U \subseteq {}^\omega 2 \times {}^\omega 2$ be a $\Sigma^1_1(z)$ set which is universal for countable subsets of ${}^\omega 2$. Let $\mathfrak{m} \in {}^\omega 2$ be such that Mostowski collapse $\mathcal{N} = \mathcal{M}_{\mathfrak{m}}$ of $(\omega, \mathcal{R}_{\mathfrak{m}})$ is an elementary substructure of $V_\kappa$ (for some cardinal $\kappa$) satisfying an adequate amount of $\mathsf{ZF} + \mathsf{AC}^{\mathbb{R}}_\omega + \Sigma^1_3(z)$-generic absoluteness and $z \in \mathcal{N}$. Then there is a $\Delta^1_1(\mathfrak{m})$ function $\Phi : {}^\omega 2 \to {}^\omega 2$ so that $U_{\Phi(x)} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$.*

*Proof.* Fix $x \in {}^\omega 2$. Since $\mathcal{K}_x$ is $\mathrm{Coll}(\omega, \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]})$-generic over $\mathcal{N}[\mathcal{H}_x]$, $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x] \models \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$ is countable.

If $U$ is $\Delta^1_1(z)$, then Fact 1.3 and the fact that $\mathcal{N}$ is an elementary substructure of $V_\kappa$ imply that "$U$ is universal for countable sets" holds in $\mathcal{N}$ and is a $\Pi^1_2(z)$ statement. By Schoenfield absoluteness, $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ continues to believe that $U$ is universal for countable sets. Thus there is an $e \in \mathbb{R} \cap \mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ so that $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x] \models U_e = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$.

If $U$ is $\Sigma^1_1(z)$, then Fact 1.3 and the fact that $\mathcal{N}$ is an elementary substructure of $V_\kappa$ imply that $\mathcal{N}$ believes that $U$ is universal for countable sets and that this statement is $\Sigma^1_3(z)$. Since $\mathcal{N}$ satisfies $\Sigma^1_3(z)$-generic absoluteness (for the forcing $\mathbb{C}_{\omega_1^{\mathcal{N}}} * \mathsf{Coll}(\omega, \dot{\mathbb{R}})$), $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ continues to believe that $U$ is universal for countable sets. Thus there is an $e \in \mathbb{R} \cap \mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ so that $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x] \models U_e = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$.

Thus in either the $\Delta^1_1(z)$ or $\Sigma^1_1(z)$ case, let $n^* \in \omega$ be least so that $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x] \models U_{\mathsf{most}_{\mathsf{GenMod}_2(x)}(n^*)} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]}$. Let $\Phi(x) = \mathsf{most}_{\mathsf{GenMod}_2(x)}(n^*)$. Since the satisfaction relation is $\Delta^1_1$, $\Phi$ is a $\Delta^1_1(\mathfrak{m})$ function.

It has only been shown that $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x] \models U_{\Phi(x)} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$. One needs to show that this holds in the real world $V$. Note that since $U$ is $\Delta^1_1(z)$ or $\Sigma^1_1(z)$, the effective perfect set theorem of Mansfield implies that $U_{\Phi(x)}$ consists only of $\Delta^1_1(z, \Phi(x))$ reals. Since the reals in $\Delta^1_1(z, \Phi(x))$ are exactly the reals which belong to every $z \oplus \Phi(x)$-admissible set (transitive model of Kripke-Platek set theory, $\mathsf{KP}$) and $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ is admissible (since admissibility is preserved by forcing and $\mathcal{N}$ is an elementary substructure of the admissible set $V_\kappa$), one has that $\Delta^1_1(z, \Phi(x)) \subseteq \mathbb{R} \cap \mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$. Thus by Mostowski absoluteness between $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ and $V$, one has that $\mathbb{R}^{\mathcal{N}[\mathcal{H}_x]} = (U_{\Phi(x)})^{\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]} = (U_{\Phi(x)})^V$. (Recall the Mostowski absoluteness states that $\boldsymbol{\Sigma}^1_1$ statements are absolute between two transitive models with the same $\omega$, and $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ and $V$ both have the same $\omega$ although they share very few other ordinals.) This completes the proof. $\square$

**Theorem 1.13.** *Assume $\mathsf{ZF} + \mathsf{AC}^{\mathbb{R}}_\omega$. Let $U \subseteq {}^\omega 2 \times {}^\omega 2$ be $\boldsymbol{\Delta}^1_1$ universal for countable sets. Then $=^+ \equiv_{\boldsymbol{\Delta}^1_1} E_U$.*

*Assume $\mathsf{ZF} + \mathsf{AC}^{\mathbb{R}}_\omega + \boldsymbol{\Sigma}^1_3$-generic absoluteness for the two step iteration $\mathbb{C}_{\omega_1} * \mathrm{Coll}(\omega, \dot{\mathbb{R}})$. Let $U \subseteq {}^\omega 2 \times {}^\omega 2$ be $\boldsymbol{\Sigma}^1_1$ universal for countable sets. Then $=^+ \leq_{\boldsymbol{\Delta}^1_1} E_U$.*

*Proof.* This follows from Fact 1.2, Fact 1.10, and Fact 1.12. $\square$

Finally, some comments on the arguments used in this article. Suppose $\Sigma : {}^\omega 2 \to \mathscr{P}_{\omega_1}({}^\omega 2)$ is a map from ${}^\omega 2$ to $\mathscr{P}_{\omega_1}({}^\omega 2)$, the collection of countable subsets of ${}^\omega 2$, with the property that $x =^+ y$ if and only $\Sigma(x) = \Sigma(y)$. Since $U$ is assumed to be universal for countable sets, for each $x \in {}^\omega 2$, there is an $e$ so that $U_e = \Sigma(x)$. Without any concrete knowledge of the definition of $U$, it seems that a function $\Phi : {}^\omega 2 \to {}^\omega 2$ so that $U_{\Phi(x)} = \Sigma(x)$ could be quite complex. Forcing and absoluteness allow for the simultaneous construction (for each for $x \in {}^\omega 2$) a countable set of reals $\Sigma(x)$ and another countable set of reals $\mathcal{C}_x$ so that one can successfully search within $\mathcal{C}_x$ to find an $e$ so that $U_e = \Sigma(x)$. Specifically in the above argument, $\Sigma(x) = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$ and $\mathcal{C}_x = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]}$. Since the search has been restricted to a countable set, one can produce a $\boldsymbol{\Delta}^1_1$ function $\Phi$ which essentially select the least $e \in \mathcal{C}_x$ so that $U_e = \Sigma(x)$.

The use of forcing to study $=^+$ is quite natural. The forcing $\mathrm{Coll}(\omega, \mathbb{R})$ and the canonical $\mathrm{Coll}(\omega, \mathbb{R})$-name for the generic surjection witness that $=^+$ is an unpinned equivalence relation in the sense of [3] or [4]. Similar forcing arguments are used in [4] to study the concept of pinnedness and the pinned cardinal. Moreover, the unpinnedness of $=^+$ is often used in the study of this equivalence relation. For instance the witness to the unpinnedness of $=^+$ is used in [1] Example 2.17 to show in $L(\mathbb{R}) \models \mathsf{AD}$ that $=^+$ has an OD equivalence class with no OD member. Under $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$, unpinnedness of $\boldsymbol{\Sigma}^1_1$ equivalence relations is in some sense the main obstacle to making definable selections from equivalence classes. (See [1] Corollary 2.14, Theorem 3.1, Example 3.5, and Example 3.6.)

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