DEFINABLE COMBINATORICS AT THE FIRST UNCOUNTABLE CARDINAL

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Abstract. Assume ZF and the axiom of determinacy, AD.

Almost everywhere $[\omega_1]^{<\omega_1}$-club uniformization holds: Let club_{ω_1} denote the collection of club subsets of ω_1. Suppose $R \subseteq [\omega_1]^{<\omega_1} \times \mathsf{club}_{\omega_1}$ is $\subseteq$-downward closed in the sense that for all $\sigma \in [\omega_1]^{<\omega_1}$, for all clubs $C \subseteq D$, $R(\sigma, D)$ implies $R(\sigma, C)$. Then there is a club $C \subseteq \omega_1$ and a function $F : \mathrm{dom}(R) \cap [C]^{<\omega_1}_* \to \mathsf{club}_{\omega_1}$ so that for all $\sigma \in \mathrm{dom}(R) \cap [C]^{<\omega_1}_*$, $R(\sigma, F(\sigma))$.

For every function $\Phi : [\omega_1]^{\omega_1} \to \omega_1$, there is a club $C \subseteq \omega_1$ so that $\Phi \restriction [C]^{\omega_1}_*$ is a continuous function.

The following cardinal relation holds: $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$.

If $\langle X_\alpha : \alpha < \omega_1 \rangle$ is a collection of subsets of $[\omega_1]^{\omega_1}$ with the property that $\bigcup_{\alpha<\omega_1} X_\alpha = [\omega_1]^{\omega_1}$, then there is an $\alpha < \omega_1$ so that X_α and $[\omega_1]^{\omega_1}$ are in bijection.

1. Introduction

The setting throughout this article will be $\mathsf{ZF} + \mathsf{AD}$. AD is the axiom of determinacy which asserts that every integer game of a certain form, one of the two players must have a winning strategy. AD and its various extensions have been shown to be a fruitful and general framework for extending properties of simple subsets of $\mathbb{R}$ to a much more general class. Within this setting, sets which are surjective images of $\mathbb{R}$ have a very interesting structure.

The definable properties of $\mathbb{R}$ and it subsets have long been studied within descriptive set theory. Under determinacy, the first uncountable cardinal, ω_1, is a minimal uncountable set much like $\mathbb{R}$. AD can distinguish ω_1 and $\mathbb{R}$ via bijections: ω_1 and $\mathbb{R}$ are incomparable cardinals in the sense that neither can inject into the other. Moreover, under a strengthening of AD called AD^+, Woodin's perfect set dichotomy implies that every uncountable set X which is a surjective image of $\mathbb{R}$ must contain a copy of $\mathbb{R}$ or ω_1. (See [2] Section 8 or [4].) More generally, [1] showed that in $L(\mathbb{R}) \models \mathsf{AD}$, every uncountable set X must contain a copy of $\mathbb{R}$ or ω_1. Like its companion $\mathbb{R}$, ω_1 and its subsets deserves a definable analysis.

Note that $\mathbb{R}$, $\mathscr{P}(\omega)$, and $[\omega]^\omega$ (where $[\omega]^\omega$ is the collection of increasing functions from ω into ω) are all in bijection. Let $[\omega_1]^{\omega_1}$ denote the collection of increasing functions from ω_1 to ω_1. $[\omega_1]^{\omega_1}$ is in bijection with $\mathscr{P}(\omega_1)$. Under AD, the cardinal structure below $|\mathbb{R}| = |\mathscr{P}(\omega)| = |[\omega]^\omega|$ is fully understood. One motivation for this article was to explore the definable cardinals around $|\mathscr{P}(\omega_1)| = |[\omega_1]^{\omega_1}|$ under AD. A continuity phenomenon for functions of the form $\Phi : [\omega_1]^{\omega_1} \to \omega_1$ will be a useful tool for studying the cardinals below $\mathscr{P}(\omega_1)$. The continuity phenomenon will be shown to be a consequence of a choice principle for club subsets of ω_1 which is fundamentally useful for studying definable combinatorics on $|[\omega_1]^{\omega_1}| = |\mathscr{P}(\omega_1)|$ under AD.

The continuity phenomenon in a general sense asserts that a local property of the output of a function can be determined by a local behavior of the input. Philosophically, this is motivated by a question of whether it is possible for one to truly use all of a function $f \in [\omega_1]^{\omega_1}$ in order to assign to f a countable ordinal?

As motivation, consider the classical case of a function $\Phi : \mathbb{R} \to \mathbb{R}$. As customary in descriptive set theory, $\mathbb{R}$ denote ${}^\omega\omega$ which is the collection of functions from ω into ω. A priori, Φ may need all of $f \in \mathbb{R}$ even to determine the first bit $\Phi(f)(0)$ of $\Phi(f)$. That is, if g differs from f at any place, $\Phi(f)(0)$ could potentially be different from $\Phi(g)(0)$. However, if Φ is continuous, then there is a $j \in \omega$ so that if $f \restriction j = g \restriction j$, then $\Phi(f)(0) = \Phi(g)(0)$. Thus one can determine the value of $\Phi(f)(0)$ forever by freezing an appropriate local behavior of the input f. Certainly not all functions $\Phi : \mathbb{R} \to \mathbb{R}$ are continuous. However, under AD, every function is continuous almost everywhere in the sense that there is a comeager set $C \subseteq \mathbb{R}$ so that $\Phi \restriction C$ is a continuous function.

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Now consider a function $\Phi : [\omega_1]^{\omega_1} \to \omega_1$. First, one needs an appropriate notion of "almost-everywhere". Let μ be the collection of subsets of ω_1 which contain a club subset of ω_1. Solovay showed that μ is a normal countably complete measure on ω_1 under AD. It has the distinction of being the unique normal measure on ω_1. Let μ_{ω_1} be the filter on $[\omega_1]^{\omega_1}_*$ defined by $X \in \mu_{\omega_1}$ if and only if there is a club $C \subseteq \omega_1$ so that $[C]^{\omega_1}_* \subseteq X$. (If $A \subseteq \omega_1$, $[A]^{\omega_1}_*$ is the collection of increasing functions from ω_1 into A which are of the correct type. See Definition 2.1.) Using the (correct type) strong partition property $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ of Martin, one can show that μ_{ω_1} is a countably complete measure on $[\omega_1]^{\omega_1}_*$. Using μ_{ω_1} as the notion of almost-everywhere is both natural and robust since it allows the strong partition property as a powerful tool in analyzing the continuity phenomenon. (The use of correct type is needed to obtain club homogeneous set for partitions. One can show $[\omega_1]^{\omega_1}$ and $\mathscr{P}(\omega_1)$ are in bijection with $[\omega_1]^{\omega_1}_*$. For this reason, this article will prefer $[\omega_1]^{\omega_1}_*$ over $\mathscr{P}(\omega_1)$.)

So the question becomes: For every $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$, is Φ continuous μ_{ω_1}-almost everywhere? Precisely, is there a club $C \subseteq \omega_1$ so that for all $f \in [C]^{\omega_1}_*$, there is an $\alpha < \omega_1$ so that for all $g \in [C]^{\omega_1}_*$ with $f \restriction \alpha = g \restriction \alpha$, $\Phi(f) = \Phi(g)$.

There is a great deal of empirical evidence that the continuity property holds. Any function $\Phi : [\omega_1]^{\omega_1} \to \omega_1$ which is of bounded dependence μ_{ω_1}-almost everywhere is continuous μ_{ω_1}-almost everywhere. (This means that there is an $\epsilon < \omega_1$ and a function $\Psi : [\omega_1]^{\epsilon}_* \to \omega_1$ so that for μ_{ω_1} almost all f, $\Phi(f) = \Psi(f \restriction \epsilon)$.) The function $\Phi : [\omega_1]^{\omega_1} \to \omega_1$ defined by $\Phi(f) = \sup_{\alpha < f(0)} f(\alpha)$ does not have bounded dependence, but it is continuous.

One can even attempt to use definability notions to construct a function that ostensibly seems to use the entire sequence to define an output: For instance, let $\Phi(f) = \omega_1^{L[f]}$. This example is discussed in Example 4.2 where it is shown that μ_{ω_1}-almost everywhere this function is constant. Thus for μ_{ω_1}-almost all f, Φ actually use no information about f to determine the output $\Phi(f)$.

This article will show that that continuity phenomenon holds for every function $\Phi : [\omega_1]^{\omega_1} \to \omega_1$:

Theorem 4.5. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Every function* $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ *is continuous* μ_{ω_1}*-almost everywhere.*

The continuity property, in its various forms, has interesting mathematical consequences for definable combinatorics under determinacy. The continuity property for function $f : \mathbb{R} \to \mathbb{R}$ is an important tool for the study of the Mycielski and Jónsson property for quotient of E_0 in [6] and [3]. Furthermore in [5], a form of the continuity property is established for functions $\Phi : [\omega_1]^{\epsilon}_* \to \omega_1$ where $\epsilon < \omega_1$ and for functions $\Phi : [\omega_2]^{\epsilon}_* \to \omega_2$ where $\epsilon < \omega_2$ in order to give a purely descriptive set theoretic proof under AD that $|[\omega_1]^{\omega}| < |[\omega_1]^{<\omega_1}|$ and $|[\omega_2]^{\omega}| < |[\omega_2]^{<\omega_1}| < |[\omega_2]^{\omega_1}| < |[\omega_2]^{<\omega_2}|$.

Using the continuity property at ω_1, one can give a purely descriptive set theoretic proof of the following cardinality computation:

Theorem 4.7. *Assume* $\mathsf{ZF} + \mathsf{AD}$. $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$.

Zapletal also asked the first author the following basic combinatorial question: Assume AD. If one partitions $[\omega_1]^{\omega_1}$ (or equivalently $\mathscr{P}(\omega_1)$) into ω_1 many pieces, $\langle X_\alpha : \alpha < \omega_1 \rangle$, so that $X_\alpha \subseteq [\omega_1]^{\omega_1}$ and $\bigcup_{\alpha<\omega_1} X_\alpha = [\omega_1]^{\omega_1}$, then must there be a piece X_α so that $X_\alpha \approx [\omega_1]^{\omega_1}$, meaning X_α is in bijection with $[\omega_1]^{\omega_1}$? The consequence of the continuity property gives a positive answer:

Theorem 4.6. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Suppose* $\langle X_\alpha : \alpha < \omega_1 \rangle$ *is a sequence of subsets of* $[\omega_1]^{\omega_1}$ *so that* $\bigcup_{\alpha<\omega_1} X_\alpha = [\omega_1]^{\omega_1}$. *Then there is an* $\alpha < \omega_1$ *so that* $X_\alpha \approx [\omega_1]^{\omega_1}$.

A natural question extending Theorem 4.5 is to ask whether every function $\Phi : [\omega_1]^{\omega_1}_* \to {}^{\omega_1}\omega_1$ is continuous μ_{ω_1}-almost everywhere. (Here ${}^{\omega_1}\omega_1$ refers to the set of all functions $f : \omega_1 \to \omega_1$.) Given such a function Φ, one can define $\Phi_\beta : [\omega_1]^{\omega_1}_* \to \omega_1$ by $\Phi_\beta(f) = \Phi(f)(\beta)$. By applying Theorem 4.5 to Φ_β, there is a club C so that $\Phi_\beta \restriction [C]^{\omega_1}_*$ is continuous. Although it is possible to show there is a sequence $\langle C_\beta : \beta < \omega_1 \rangle$ so that for all $\beta < \omega_1$, $\Phi_\beta \restriction [C_\beta]^{\omega_1}_*$ is continuous (see [2] Section 4), it is not clear how to use this sequence to obtain one single club C which witnesses that the original function $\Phi : [\omega_1]^{\omega_1} \to {}^{\omega_1}\omega_1$ is continuous on $[C]^{\omega_1}_*$ since an intersection of ω_1-many club subsets of ω_1 may not be a club. With Trang, using ideas similar to the proof of Theorem 4.5 but with more elaborate partitions, one can establish the following almost everywhere continuity result:

Theorem 5.3 *(With Trang) Assume* $\mathsf{ZF} + \mathsf{AD}$*. Every function* $\Phi : [\omega_1]^{\omega_1} \to {}^{\omega_1}\omega_1$ *is continuous* μ_{ω_1}*-almost everywhere.*

The strong partition property for ω_1 is crucial in the arguments for establishing the continuity property for functions $\Phi : [\omega_1]^{\omega_1} \to \omega_1$. The second uncountable cardinal ω_2 fails to have the strong partition property but by a result of Martin and Paris, it does have the weak partition property, that is, $\omega_2 \to (\omega_2)^\epsilon_2$ for each $\epsilon < \omega_2$. Using an explicit failure of the strong partition property for ω_2, Section 6 shows that there is a function $\Phi : [\omega_2]^{\omega_2} \to 2$ so that there is no club $C \subseteq \omega_2$ so that $\Phi \restriction [C]^{\omega_2}_*$ is continuous.

The main argument in establishing Theorem 4.5 that every function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ is continuous μ_{ω_1}-almost everywhere is to show that a certain natural partition $P : [\omega_1]^{\omega_1}_* \to 2$ has a club homogeneous set for the desired side of the partition. As described in the proof of Theorem 4.5, one needs to make choices of club subsets of ω_1 which is dependent on previous choices of clubs. The axiom of determinacy is incompatible with many consequences of the axiom of choice. A selection principle for subsets of ω_1 is generally not possible in AD. To perform the construction mentioned above, one would need to prove a club uniformization result.

Let club_{ω_1} denote the club subsets of ω_1. In the applications of this paper, one has a relation $R \subseteq [\omega_1]^{<\omega_1} \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed in the sense that for all $C \subseteq D$ which are club subsets of ω_1 and for all σ, if $R(\sigma, D)$ holds, then $R(\sigma, C)$ holds. $[\omega_1]^{<\omega_1}$-club uniformization is the statement that there is a function $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_{\omega_1}$ so that for all $\sigma \in [\omega_1]^{<\omega_1}$, $R(\sigma, \Lambda(\sigma))$.

For any $R \subseteq [\omega_1]^{<\omega_1} \times \mathsf{club}_{\omega_1}$ as above, there is a coded version $\tilde{R} \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ of R. Theorem 3.7 shows that if $\tilde{R}$ has a uniformization, then one can use the simple ω_1-version of the Kechris-Woodin generic coding function (see [9]) and a category argument to establish R has an (everywhere) uniformization. Thus under $\mathsf{AD}_\mathbb{R}$, (everywhere) $[\omega_1]^{<\omega_1}$-club uniformization holds:

Theorem 3.7. *Assuming* $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$*,* $[\omega_1]^{<\omega_1}$*-club uniformization holds.*

Under $\mathsf{AD}_\mathbb{R}$, every relation $S \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized. AD cannot prove this full uniformization since $L(\mathbb{R}) \models \mathsf{AD}$ has a relation on $\mathbb{R} \times \mathbb{R}$ that cannot be uniformized. However, there is an almost everywhere uniformization result that does hold in AD: for any relation $S \subseteq \mathbb{R} \times \mathbb{R}$, there is a comeager $C \subseteq \mathbb{R}$ and a function $F : C \to \mathbb{R}$ which uniformizes S on C.

Similarly, AD cannot prove (everywhere) $[\omega_1]^{<\omega_1}$-club uniformization since Fact 3.9 shows that it fails in $L(\mathbb{R}) \models \mathsf{AD}$. One says that almost-everywhere $[\omega_1]^{<\omega_1}$-club uniformization holds if and only if for every relation $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed, there is a club $C \subseteq \omega_1$ so that $R \cap ([C]^{<\omega_1}_* \times \mathsf{club}_{\omega_1})$ has a uniformization. By combining the generic coding function, category arguments, the Moschovakis coding lemma, and a fundamental idea of Martin (used in the study of the partition properties on ω_1) where the player with the winning strategy determines the property of the output but the losing player determines the identity of the output, one can prove a main result of this paper:

Theorem 3.10. *Assume* $\mathsf{ZF} + \mathsf{AD}$*. Almost everywhere* $[\omega_1]^{<\omega_1}$*-club uniformization holds.*

Almost everywhere $[\omega_1]^{<\omega_1}$-club uniformization is used to verify the partition used in the proof of the continuity property (Theorem 4.5) has a club homogeneous set which is homogeneous for the desired side. However, Theorem 3.10 is a powerful general technique for constructing functions $h \in [\omega_1]^{\omega_1}_*$ which verify that partitions of a certain form are homogeneous for the desired side. The following template illustrates a very typical and simple use of Theorem 3.10:

Suppose $P : [\omega_1]^{\omega_1}_* \to 2$ is a partition defined by $P(f) = 0$ if and only if f does not have any "errors". An error is a property of f so that if f has an error it must be witnessed at a $\gamma < \omega_1$. An example of an error property could be that $L[f] \models \neg\mathsf{GCH}$, i.e. the generalized continuum hypothesis fails in $L[f]$. For this example, if f has an error, then by a condensation argument, there is a $\gamma < \omega_1$ which witnesses this error in the sense that $L[f] \models 2^\gamma > \gamma^+$. By the Martin's partition relation, there is a club $D_0 \subseteq \omega_1$ which is homogeneous for P. Suppose one could show that for all $\sigma \in [D_0]^{<\omega_1}$, there is a club $C \subseteq \omega_1$ so that for all $g \in [C]^{\omega_1}_*$ such that $\sup(\sigma) < g(0)$, $\sigma\hat{\ }g$ does not have an error at any γ such that $\sup(\sigma) \leq \gamma < g(0)$. Define a relation $R \subset [D_0]^{<\omega_1} \times \mathsf{club}_{\omega_1}$ by $R(\sigma, C)$ if and only if C has the above property with respect to σ. R is a relation which is $\subseteq$-downward closed in the club_{ω_1}-coordinate. By Theorem 3.10, let $D_1 \subseteq D_0$ and $\Lambda : \mathrm{dom}(R) \cap [D_1]^{<\omega_1} \to \mathsf{club}_{\omega_1}$ be such that for all $\sigma \in \mathrm{dom}(R) \cap [D_1]^{<\omega_1}_*$, $R(\sigma, \Lambda(\sigma))$. Now construct a function $h \in [D_1]^{\omega_1}_*$ by recursion as follows: Let $F_0 = D_1 \cap \Lambda(\emptyset)$. Let $h(0)$ the ω^{th} element of F_0. Suppose for some α, $h \restriction \alpha$ and F_β, for all $\beta < \alpha$, have been defined. Let $F_\alpha = \Lambda(h \restriction \alpha)$ and let $h(\alpha)$ be the ω^{th} element of F_α larger than $\sup(h \restriction \alpha)$. This completes the construction. Note that h belongs to $[D_1]^{\omega_1}_*$. For each $\alpha < \omega_1$, let $\mathsf{drop}(h, \alpha) \in [D_1]^{\omega_1}_*$ be defined by $\mathsf{drop}(h, \alpha)(\gamma) = h(\alpha + \gamma)$. For all $\alpha < \omega_1$, by construction, one has $\mathsf{drop}(h, \alpha) \in [F_\alpha]^{\omega_1}_* \subseteq [\Lambda(h \restriction \alpha)]^{\omega_1}_*$ and therefore h does not have an error at any γ with $\sup(h \restriction \alpha) \leq \gamma < \mathsf{drop}(h, \alpha)(0) = h(\alpha)$. Thus h has no errors at any $\gamma < \omega_1$. Because errors must be witnessed at some $\gamma < \omega_1$, h has no error. $P(h) = 0$ and therefore D_0 is homogeneous for P taking value 0.

The almost everywhere $[\omega_1]^{<\omega_1}$-club uniformization of Theorem 3.10 is particularly important for studying the stable theory of the partition measure μ_{ω_1}: Each $f \in [\omega_1]^{\omega_1}$, $L[f]$ is naturally an $\mathscr{L} = \{\dot{\in}, \dot{E}\}$ structure. Since μ_{ω_1} is an ultrafilter, for any $\mathscr{L}$-sentence, either (1) for μ_{ω_1}-almost all f, $L[f] \models \varphi$ or (2) for μ_{ω_1}-almost all f, $L[f] \models \neg\varphi$. The ω_1-stable theory is $\mathfrak{T}^{\omega_1}$ which is defined to be the collection of $\mathscr{L}$-sentences φ so that for μ_{ω_1}-almost all f, $L[f] \models \varphi$. One can ask which natural statements of set theory, such as GCH, belong to $\mathfrak{T}^{\omega_1}$. For instance, in forthcoming work of Chan, Jackson, and Trang, one can show that for any $\sigma \in [\omega_1]^{<\omega_1}$, there is a club $C \subseteq \omega_1$ so that for all $g \in [C]^{\omega_1}_*$, for all κ with $\sup(\sigma) \leq \kappa < g(0)$, $L[\sigma\hat{\ }g] \models 2^\kappa = \kappa^+$. Thus using the outline above, one has that $\mathsf{GCH} \in \mathfrak{T}^{\omega_1}$. Using Theorem 3.10, one can also show that for μ_{ω_1}-almost all f, $L[f] \models (\forall \alpha < \omega_1)(f(\alpha)$ is a strongly inaccessible cardinal) and $L[f]$ satisfies $\boldsymbol{\Sigma}^1_1$-determinacy. One can also show that for μ_{ω_1}-almost all f, $L[f]$ has a canonical inner models $L[\bar{\nu}_f]$ where $\bar{\nu}_f$ is an ω_1-length sequence of normal measure with discontinuous increasing sequence of critical points $\bar{\kappa}$ so that f is generic over $L[\bar{\nu}_f]$ for a generalized Prikry forcing $\bar{\mathbb{P}}_{\bar{\nu}_f}$, considered by Fuchs [7]. This can be used to show that for μ_{ω_1}-almost all f, $\boldsymbol{\Delta}^1_2$-determinacy fails in $L[f]$. Welch [10] has investigated similar questions in a different setting.

2. Basics

Throughout the entire paper, assume $\mathsf{ZF}+\mathsf{AD}$ (but not necessarily $\mathsf{DC}_{\mathbb{R}}$) unless otherwise explicitly stated.

Except for Theorem 2.16 and Theorem 2.17 proved by the authors for this paper, the results of this section are well known and due to Martin and Solovay. This section will introduce the necessary notation and results. Although the proofs use a simple and fundamentally important idea of Martin that appears in his arguments for the partition properties, the exposition is quite tedious. A careful presentation is given in [2]. Specifically, see [2] Section 2, 3, and 4 for more details.

Definition 2.1. Let $[\omega_1]^{\omega_1}$ denote the collection of strictly increasing functions $f : \omega_1 \to \omega_1$.

A function $f \in [\omega_1]^{\omega_1}$ has uniform cofinality ω if and only if there is a function $F : \omega_1 \times \omega \to \omega_1$ so that for all $\alpha < \omega_1$, for all $n \in \omega$, $F(\alpha, n) < F(\alpha, n+1)$ and $f(\alpha) = \sup\{F(\alpha, n) : n \in \omega\}$.

A function $f \in [\omega_1]^{\omega_1}$ has correct type if and only if f has uniform cofinality ω and for all $\alpha < \omega_1$, $f(\alpha) > \sup\{f(\beta) : \beta < \alpha\}$, that is, f is discontinuous everywhere.

Let $[\omega_1]^{\omega_1}_*$ denote the subset of $[\omega_1]^{\omega_1}$ consisting of the functions of correct type.

Fact 2.2. $[\omega_1]^{\omega_1} \approx [\omega_1]^{\omega_1}_*$.

Proof. Let $A = \{\omega\cdot(\alpha+1) : \alpha \in \omega_1\}$. Suppose $f \in [A]^{\omega_1}$. Let $F'(\alpha)$ be the unique β so that $f(\alpha) = \omega\cdot(\beta+1)$. Let $F : \omega_1 \times \omega \to \omega_1$ be defined by $F(\alpha, n) = F'(\alpha) + n$. Note that for all α, $f(\alpha) = \sup\{F(\alpha, n) : n \in \omega\}$. Thus f has uniformly cofinality ω. For any α, for any $\beta < \alpha$, $f(\beta) = \omega \cdot (F'(\beta) + 1) \leq \omega \cdot F'(\alpha) < \omega \cdot (F'(\alpha) + 1) = f(\alpha)$. This shows that every $f \in [A]^{\omega_1}$ is of the correct type. Clearly, $[\omega_1]^{\omega_1} \approx [A]^{\omega_1}$. Thus one has an injection of $[\omega_1]^{\omega_1}$ into $[\omega_1]^{\omega_1}_*$. The inclusion map in an injection of $[\omega_1]^{\omega_1}_*$ into $[\omega_1]^{\omega_1}$. ☐

Definition 2.3. Let $\epsilon \leq \omega_1$. Write $\omega_1 \to_* (\omega_1)^\epsilon_2$ to indicate that for all $P : [\omega_1]^\epsilon_* \to 2$, there is an $i \in 2$ and a club $C \subseteq \omega_1$ so that for all $f \in [\omega_1]^\epsilon_*$, $P(f) = i$. In this case, one says that C is homogeneous for P taking value i.

Fact 2.4. *(Martin) For all $\epsilon \leq \omega_1$, $\omega_1 \to_* (\omega_1)^\epsilon_2$.*

Proof. See [8] Theorem 12.2 or [2] Section 4. ☐

Definition 2.5. For $\epsilon \leq \omega_1$, let μ_ϵ denote the collection of $X \subseteq [\omega_1]^\epsilon_*$ so that there exists a club $C \subseteq \omega_1$ so that $[C]^\epsilon_* \subseteq X$.

(Martin) As a consequence of $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, one has that for all $\epsilon \leq \omega_1$, μ_ϵ is a countably complete ultrafilter on $[\omega_1]^\epsilon_*$.

Definition 2.6. Let $\pi : \omega \times \omega \to \omega$. If $R \subseteq \omega \times \omega$, then $x \in {}^\omega\omega$ codes R if and only if $(a,b) \in R \Leftrightarrow x(\pi(a,b)) = 0$. This gives a coding of binary relations on ω by elements of $\mathbb{R}$.

For each $x \in \mathbb{R}$, $\text{field}(x)$ is the set of n so that there exists some m, such that $x(\pi(m,n)) = 0$ or $x(\pi(n,m)) = 0$.

Let WO denote the set of reals coding wellordering on subsets of ω. If $w \in \text{WO}$, then $<_w$ refers to the wellordering on $\text{field}(w)$ coded by w.

For each $w \in \text{WO}$ and $\alpha < \text{ot}(w)$, let n^w_α be the element of $\text{field}(w)$ which has rank α according to $<_w$.

For each $\alpha < \omega_1$, let $\text{WO}_\alpha = \{w \in \text{WO} : \text{ot}(w) = \alpha\}$. Similarly, one can define $\text{WO}_{<\alpha}$, $\text{WO}_{\leq\alpha}$, $\text{WO}_{>\alpha}$, and $\text{WO}_{\leq\alpha}$.

Note that WO is Π^1_1 and for each $\alpha < \omega_1$, $\text{WO}_{>\alpha}$ and $\text{WO}_{\geq\alpha}$ are $\boldsymbol{\Pi}^1_1$; and WO_α, $\text{WO}_{<\alpha}$, and $\text{WO}_{\leq\alpha}$ are $\boldsymbol{\Delta}^1_1$.

Fact 2.7. *([2] Fact 4.3) Suppose τ is a Player 2 strategy with the property that for all $x \in \text{WO}$, $\tau(x) \in \text{WO}$ and $\text{ot}(\tau(x)) > \text{ot}(x)$. Let $C_\tau = \{\eta : (\forall w)(w \in \text{WO}_{<\eta} \Rightarrow \tau(w) \in \text{WO}_{<\eta})\}$. Then C_τ is a club.*

Definition 2.8. Let $\mathsf{clubcode}_{\omega_1}$ denote the collection $\tau \in \mathbb{R}$ so that τ is a Player 2 winning strategy with the property that for all $w \in \text{WO}$, $\tau(w) \in \text{WO}$ and $\text{ot}(\tau(w)) > \text{ot}(w)$. Note that $\mathsf{clubcode}_{\omega_1}$ is a Π^1_2 set.

Fact 2.9. *(Solovay, [2] Fact 4.6) Suppose C is a club. There is a $\tau \in \mathsf{clubcode}_{\omega_1}$ so that $C_\tau \subseteq C$.*

Fact 2.10. *([2] Fact 4.7) Suppose $A \subseteq \mathsf{clubcode}_{\omega_1}$ is $\boldsymbol{\Sigma}^1_1$. Then one can find a club C uniformly in A (as a set; e.g. not depending on any $\boldsymbol{\Sigma}^1_1$ representation of A) so that for all $\tau \in A$, $C \subseteq C_\tau$.*

Definition 2.11. Suppose $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ is continuous if and only if for all $f \in [\omega_1]^{\omega_1}_*$, there is some $\alpha < \omega_1$ so that for all $g \in [\omega_1]^{\omega_1}_*$, if $g \restriction \alpha = f \restriction \alpha$, then $\Phi(g) = \Phi(f)$.

If one gives $[\omega_1]^{\omega_1}_*$ the topology generated by $N_s = \{f \in [\omega_1]^{\omega_1}_* : s \subset f\}$ for each $s \in [\omega_1]^{<\omega_1}_*$, then Φ is continuous in the above sense if and only if it is continuous in the topological sense with ω_1 given the discrete topology.

$\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ is continuous almost everywhere if and only if there is a $C \subseteq \omega_1$ club so that Φ is continuous on $[C]^{\omega_1}_*$; that is, for all $f \in [C]^{\omega_1}_*$, there exists an α so that for all $g \in [C]^{\omega_1}_*$ with $g \restriction \alpha = f \restriction \alpha$, $\Phi(g) = \Phi(f)$.

Definition 2.12. Let BS denote the collection of $(x, y) \in \mathbb{R}$ so that
(i) $x \in \text{WO}$.
(ii) For all $n \in \text{field}(x)$, $y_n \in \text{WO}$.
(iii) For all $m, n \in \text{field}(x)$, $m <_x n$ if and only if $\text{ot}(y_m) < \text{ot}(y_n)$.

Note that BS is Π^1_1.

For each $(x,y) \in \mathsf{BS}$, let $\sigma_{(x,y)} : \text{ot}(x) \to \omega_1$ be defined by $\sigma_{(x,y)}(\alpha) = \text{ot}(y_{n^x_\alpha})$. Observe that for every $\sigma \in [\omega_1]^{<\omega_1}$, there is some $(x,y) \in \mathsf{BS}$ so that $\sigma_{(x,y)} = \sigma$.

Definition 2.13. Let κ be a regular cardinal and $\lambda \leq \kappa$ be an ordinal. A good coding system for ${}^\lambda\kappa$ consists of Γ, decode, and $\mathsf{GC}_{\beta,\gamma}$ for each $\beta < \lambda$ and $\gamma < \kappa$ with the following properties:
(1) Γ is a pointclass closed under continuous substitution and $\exists^\mathbb{R}$. Let $\check{\Gamma}$ denote the dual pointclass. Let $\Delta = \Gamma \cap \check{\Gamma}$.
(2) $\mathsf{decode} : \mathbb{R} \to \mathscr{P}(\lambda \times \kappa)$. For all $f \in {}^\lambda\kappa$, there is some $x \in \mathbb{R}$ so that $\mathsf{decode}(x) = f$.
(3) For all $\beta < \lambda$ and $\gamma < \kappa$, $\mathsf{GC}_{\beta,\gamma} \subseteq \mathbb{R}$, $\mathsf{GC}_{\beta,\gamma} \in \Delta$, and $\mathsf{GC}_{\beta,\gamma}$ has the property that $x \in \mathsf{GC}_{\beta,\gamma}$ if and only if

$$\mathsf{decode}(x)(\beta,\gamma) \wedge (\forall \gamma' < \kappa)(\mathsf{decode}(x)(\beta,\gamma') \Rightarrow \gamma = \gamma').$$

For each $\beta < \lambda$, let $\mathsf{GC}_\beta = \bigcup_{\gamma<\kappa} \mathsf{GC}_{\beta,\gamma}$.
(4) (Boundedness property) Suppose $A \in \exists^\mathbb{R}\Delta$ and $A \subseteq \mathsf{GC}_\beta = \bigcup_{\gamma<\kappa} \mathsf{GC}_{\beta,\gamma}$, then there exists some $\delta < \kappa$ so that $A \subseteq \bigcup_{\gamma<\delta} \mathsf{GC}_{\beta,\gamma}$.

(5) Δ is closed under less than κ wellordered unions.

Suppose $x \in \mathbb{R}$, let $\mathsf{fail}(x)$ be the least $\beta < \lambda$ so that $x \notin \mathsf{GC}_\beta$ if it exists. Otherwise, let $\mathsf{fail}(x) = \infty$.

Let $\mathsf{GC} = \bigcap_{\beta<\lambda} \mathsf{GC}_\beta$. Note that if $x \in \mathsf{GC}$, then $\mathsf{decode}(x)$ is the graph of a function in ${}^\lambda\kappa$. If $x \in \mathsf{GC}$, then one will use function notations such as $\mathsf{decode}(x)(\beta) = \gamma$ to indicate $(\beta, \gamma) \in \mathsf{decode}(x)$.

Definition 2.14. Suppose κ is a regular cardinal and λ is such that $\omega \cdot \lambda < \kappa$. Suppose $f \in {}^{\omega\cdot\lambda}\kappa$. Let $\mathsf{block} : {}^{\omega\cdot\lambda}\kappa \to {}^\lambda\kappa$ be defined by $\mathsf{block}(f)(\alpha) = \sup\{f(\omega \cdot \alpha + k) : k \in \omega\}$.

Suppose $f, g \in {}^{\omega\cdot\lambda}\kappa$. Let $\mathsf{joint} : {}^{\omega\cdot\lambda}\kappa \times {}^{\omega\cdot\lambda}\kappa \to {}^\lambda\kappa$ be defined by

$$\mathsf{joint}(f, g)(\alpha) = \sup\{f(\omega \cdot \alpha + k), g(\omega \cdot \alpha + k) : k \in \omega\} = \max\{\mathsf{block}(f)(\alpha), \mathsf{block}(g)(\alpha)\}.$$

Theorem 2.15. *(Martin, [2] Theorem 3.7) Suppose λ, κ are ordinals such that $\omega \cdot \lambda \leq \kappa$. Suppose there is a good coding system $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta \in \omega \cdot \lambda, \gamma < \kappa)$ for ${}^{\omega\cdot\lambda}\kappa$. Then $\kappa \to_* (\kappa)^\lambda_2$ holds.*

Theorem 2.16. *([2] Theorem 3.8) (Almost everywhere uniformization on good codes) Let κ be a regular cardinal and $\lambda \leq \kappa$. Let $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \omega \cdot \lambda, \gamma < \kappa)$ be a good coding system for ${}^{\omega\cdot\lambda}\kappa$. Let $R \subseteq [\kappa]^\lambda_* \times \mathbb{R}$ be a relation.*

There is a club $C \subseteq \kappa$ and a Lipschitz continuous function $F : \mathbb{R} \to \mathbb{R}$ so that for all $x \in \mathsf{GC}$ with $\mathsf{decode}(x) \in [C]^{\omega\cdot\lambda}$ and $\mathsf{block}(\mathsf{decode}(x)) \in [C]^\lambda_* \cap \mathrm{dom}(R)$, $R(\mathsf{block}(\mathsf{decode}(x)), F(x))$.

Theorem 2.17. *([2] Theorem 3.9) Let κ be a regular cardinal and $\lambda < \kappa$. Suppose $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \lambda, \gamma < \kappa)$ is a good coding system for ${}^\lambda\kappa$. Let $M \models \mathsf{AD}$ be an inner model containing all the reals and within M, $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \lambda, \gamma < \kappa)$ is a good coding system.*

Then for any $\Phi : [\kappa]^\lambda \to \kappa$, there is a club D, necessarily in M by the coding lemma, so that $\Phi \restriction [D]^\lambda_* \in M$.

Definition 2.18. Let κ be a regular cardinal and $\lambda \leq \kappa$. κ is λ-reasonable if and only if there is a good coding system for ${}^\lambda\kappa$.

Theorem 2.19. *(Martin, [2] Fact 4.9, Theorem 4.18, Theorem 4.26, Corollary 4.27) For any $\lambda \leq \omega_1$, ω_1 is λ-reasonable.*

Remark 2.20. One can check that for $\lambda < \omega_1$, one can produce a good coding system so that for any $f \in [\omega_1]^\lambda_*$, the collection of $x \in \mathsf{GC}$ so that $\mathsf{decode}(x) = f$ is $\mathbf{\Delta}^1_1$. See [2] Section 4.

3. Club Uniformization

Definition 3.1. Let club_{ω_1} denote the collection of club subsets of ω_1.

Let $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$. If $\sigma \in [\omega_1]^{<\omega_1}_*$, then let $R_\sigma = \{C \in \mathsf{club}_{\omega_1} : R(\sigma, C)\}$. Let $\mathrm{dom}(R) = \{\sigma \in [\omega_1]^{<\omega_1}_* : R_\sigma \neq \emptyset\}$.

Suppose $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ is a relation. A function $F : \mathsf{dom}(R) \to \mathsf{club}_{\omega_1}$ is a uniformization for R if and only if for all $\sigma \in \mathrm{dom}(R)$, $R(\sigma, F(\sigma))$.

R is $\subseteq$-downward closed if and only if for all $\sigma \in [\omega_1]^{<\omega_1}_*$ and for all $C \subseteq D$ with $C, D \in \mathsf{club}_{\omega_1}$, $R(\sigma, D)$ implies $R(\sigma, C)$.

$[\omega_1]^{<\omega_1}_*$-club uniformization is the statement that every $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed has a uniformization.

Almost everywhere $[\omega_1]^{<\omega_1}_*$-club uniformization is the statement that for every $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed, there is a club $C \subseteq \omega_1$ so that the relation $R \cap ([C]^{<\omega_1}_* \times \mathsf{club}_{\omega_1})$ has a uniformization.

The primary purpose of this section is to establish almost everywhere $[\omega_1]^{<\omega_1}_*$-club uniformization which will be applied in the next section to establish continuity results.

As a warmup, the following is a simple form of club uniformization.

Definition 3.2. Let $\alpha < \omega_1$. Let $R \subseteq [\omega_1]^\alpha_* \times \mathsf{club}_{\omega_1}$ be a $\subseteq$-downward closed relation. A uniformization for R is a function $F : \mathrm{dom}(R) \to \mathsf{club}_{\omega_1}$ so that for all $\sigma \in \mathrm{dom}(R)$, $R(\sigma, F(\sigma))$.

$[\omega_1]^\alpha_*$-club uniformization is the statement that every $R \subseteq [\omega_1]^\alpha_* \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed has a uniformization.

Almost everywhere $[\omega_1]^\alpha_*$-club uniformization is the statement that for every $R \subseteq [\omega_1]^\alpha_* \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed, there is a club $C \subseteq \omega_1$ so that $R \cap ([C]^\alpha_* \times \mathsf{club}_{\omega_1})$ has a uniformization.

Theorem 3.3. *Let $\alpha < \omega_1$. Almost everywhere $[\omega_1]^\alpha_*$-club uniformization holds.*

Proof. Let $(\mathbf{\Sigma}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \alpha, \gamma < \omega_1)$ be a good coding system for ${}^{\omega\cdot\alpha}\omega_1$.

Fix $R \subseteq [\omega_1]^\alpha_* \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed. Let $S \subseteq [\omega_1]^\alpha_* \times \mathsf{clubcode}_{\omega_1}$ be defined by $S(f,z)$ if and only if $R(f, C_z)$.

By Theorem 2.16, there is a Lipschitz continuous function $F : \mathbb{R} \to \mathbb{R}$ and a club C so that if one lets D be the set of limit points of C, then for all $x \in \mathsf{GC}$ so that $\mathsf{decode}(x) \in [C]^{\omega\cdot\alpha}_*$ and $\mathsf{block}(\mathsf{decode}(x)) \in \mathrm{dom}(S) \cap [D]^\alpha_*$, $S(\mathsf{block}(\mathsf{decode}(x)), F(x))$.

One can check that for each $f \in \mathrm{dom}(R) \cap [D]^\alpha_*$, $K_f = \{x \in \mathbb{R} : \mathsf{decode}(x) \in [C]^{\omega\cdot\alpha}_* \wedge \mathsf{block}(\mathsf{decode}(x)) = f\}$ is a $\mathbf{\Delta}^1_1$ set. (See Remark 2.20.) Thus $\tau[K_f]$ is a $\mathbf{\Sigma}^1_1$ subset of $\mathsf{clubcode}_{\omega_1}$. By Fact 2.10, there is a club C_f obtained uniformly from K_f (and hence f) so that $C_f \subseteq C_z$ for all $z \in K_f$. Since for any $z \in K_f$, $R(f, C_z)$ and R is $\subseteq$-downward closed, $R(f, C_f)$.

Thus the function mapping f to C_f defined by the procedure above is a uniformization for R. $\square$

Note that Theorem 3.3 uses only a boundedness principle. It does not use uniformization or any other consequences of scales. This is in contrast to the argument for almost everywhere $[\omega_1]^{<\omega_1}_*$-club uniformization which seems to require the relevant sets to be within scales.

The following is the simple generic coding function for ω_1.

Fact 3.4. *There is a continuous function $G : {}^\omega\omega_1 \to \mathrm{WO}$ so that for all $f \in {}^\omega\omega_1$, $G(f) \in \mathrm{WO}$ and for all $f \in {}^\omega\omega_1$ such that $f(0) = \{f(n+1) : n \in \omega\}$, $\mathrm{ot}(G(f)) = f(0)$.*

Let $\mathrm{cut} : {}^\omega\omega_1 \to {}^\omega\omega_1$ be defined by $\mathrm{cut}(f)(n) = f(n+1)$.

In other words, G has the property that for all $f \in {}^\omega\omega_1$, $G(f) \in \mathrm{WO}$ and if $\mathrm{cut}(f)$ is a surjection of ω onto $f(0)$, then $\mathrm{ot}(G(f)) = f(0)$.

Proof. For all such $f \in {}^\omega\omega_1$, let $A_f = \{n \in \omega \setminus \{0\} : (\forall m)(f(n) = f(m) \Rightarrow n \leq m)\}$. Let $G(f)$ code a binary relation with domain A_f by letting $m <_{G(f)} n$ if and only if $f(m) < f(n)$. It is clear that $G(f) \in \mathrm{WO}$ and $\mathrm{ot}(G(f)) = f(0)$ if $\mathrm{cut}(f)$ is a surjection of ω onto $f(0)$. $\square$

Definition 3.5. Let $\alpha < \omega_1$, let $s \in {}^{<\omega}\alpha$. Let $N^\alpha_s = \{f \in {}^\omega\alpha : s \subset f\}$. ${}^\omega\alpha$ is given the topology generated by N^α_s, and ${}^\omega\alpha$ is homeomorphic to ${}^\omega\omega$. The concepts of meagerness, comeagerness, and nonmeagerness can be defined as usual.

Note that the set $\mathrm{surj}_\alpha = \{f \in {}^\omega\alpha : f : \omega \to \alpha \text{ is a surjection}\}$ is a comeager subset of ${}^\omega\alpha$.

Under AD, a wellordered intersection of comeager subsets of ${}^\omega\alpha$ is a comeager subset of ${}^\omega\alpha$.

Fact 3.6. *There is a function $H : [\omega_1]^{<\omega_1} \times \mathrm{WO} \to \mathsf{BS}$ with the property that for all $\sigma \in [\omega_1]^{<\omega_1}$ and for all $w \in \mathrm{WO}$ so that $\mathrm{ot}(w) = \sup(\sigma) + 2$, $H(\sigma, w) \in \mathsf{BS}$ and $H(\sigma, w)$ codes σ, that is, $\sigma_{H(\sigma,w)} = \sigma$.*

Proof. Fix $\sigma \in [\omega_1]^{<\omega_1}$. If $w \in \mathrm{WO}$ and $\mathrm{ot}(w) \neq \sup(\sigma) + 2$, then let $H(\sigma, w)$ be some fixed element of BS as this case is insignificant.

Observe that $\mathrm{length}(\sigma) < \sup(\sigma) + 2$. Suppose $w \in \mathrm{WO}_{\sup(\sigma)+2}$. Canonically from w and σ, one will produce $(x, y) \in \mathrm{BS}$ as follows: Let $x \in \mathrm{WO}$ (which is produced canonically from w and σ) code a relation on ω whose field is $\{n \in \mathrm{field}(w) : n <_w n^w_{\mathrm{length}(\sigma)}\}$ and for $m, n \in \mathrm{field}(x)$, $m <_x n$ if and only if $m <_w n$. Then $\mathrm{ot}(x) = \mathrm{length}(\sigma)$.

Similarly, produce y canonically from w and σ as follows: Fix a $k \in \omega$. If $k \notin \mathrm{field}(x)$, then let $y_k = \bar{0}$, the constant 0 sequence. If $k \in \mathrm{field}(x)$, then let $\alpha < \mathrm{length}(\sigma)$ so that $k = n^w_\alpha$. Let y_k be the unique real coding a binary relation such that $\mathrm{field}(y_k) = \{n \in \mathrm{field}(w) : n <_w n^w_{\sigma(\alpha)}\}$ and for all $m, n \in \mathrm{field}(y_k)$, $m <_{y_k} n \Leftrightarrow m <_w n$. Then $y_k \in \mathrm{WO}$ and $\mathrm{ot}(y_k) = \sigma(\alpha)$. Let $y \in \mathbb{R}$ be such that k^{th} section of y is y_k.

Thus $(x, y) \in \mathrm{BS}$ and $\sigma_{(x,y)} = \sigma$. Let $H(\sigma, w) = (x, y)$. $\square$

Theorem 3.7. *Assume all sets of reals have the Baire property. Let $R \subseteq [\omega_1]^{<\omega_1} \times \mathsf{club}_{\omega_1}$ be a $\subseteq$-downward closed relation. Define $\tilde{R} \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by*

$$\tilde{R}(x, y, z) \Leftrightarrow (x, y) \in \mathrm{BS} \wedge z \in \mathsf{clubcode}_{\omega_1} \wedge R(\sigma_{(x,y)}, C_z).$$

Consider $\tilde{R}$ as a relation on $\mathsf{BS} \times \mathsf{clubcode}_{\omega_1}$. Suppose there is a $J : \mathrm{dom}(\tilde{R}) \to \mathsf{clubcode}_{\omega_1}$ which is a uniformization for $\tilde{R}$. Then there is an $F : \mathrm{dom}(R) \to \mathsf{club}_{\omega_1}$ which is a uniformization for R.

Thus $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$ proves $[\omega_1]^{<\omega_1}$-club uniformization.

Proof. Let add : $\omega_1 \times {}^\omega\omega_1 \to {}^\omega\omega_1$ be defined by

$$\mathrm{add}(\alpha, f)(n) = \begin{cases} \alpha & n = 0 \\ f(n-1) & n > 0 \end{cases}$$

Fix $\sigma \in \mathrm{dom}(R)$. One will describe how to define $F(\sigma)$:

Let $A = \mathrm{surj}_{\sup(\sigma)+2}$. As observed earlier, A is comeager as a subset of ${}^\omega(\sup(\sigma) + 2)$. Let $O : A \to \mathsf{BS}$ be defined by

$$O(f) = J(H(\sigma, G(\mathrm{add}(\sup(\sigma) + 2, f)))).$$

Note that $G(\mathrm{add}(\sup(\sigma) + 2, f)) \in \mathrm{WO}_{\sup(\sigma)+2}$ for all $f \in A$. Therefore for all $f \in A$,

$$H(\sigma, G(\mathrm{add}(\sup(\sigma) + 2, f))) \in \mathrm{BS}$$

and codes σ. Thus

$$\tilde{R}(H(\sigma, G(\mathrm{add}(\sup(\sigma) + 2, f))), O(f))$$

holds for all $f \in A$. Let $B = \{O(f) : f \in A\}$. Thus it has been shown that any $z \in B$ belongs to $\mathsf{clubcode}_{\omega_1}$ and $R(\sigma, C_z)$.

For any club C, let $\mathsf{enum}_C : \omega_1 \to C$ be the increasing enumeration of C. For each $\gamma < \omega_1$, let $K(\gamma)$ be the least $\delta < \omega_1$ so that for comeagerly many $f \in A$ (in the topological space ${}^\omega(\sup(\sigma) + 2)$), $\mathsf{enum}_{C_{O(f)}}(\gamma) < \delta$.

Claim 1: K is a well defined function.

To see this: Fix γ. One will show that $K(\gamma)$ is defined.

For each $\epsilon < \omega_1$, let $T_\epsilon = \{f \in A : \mathsf{enum}_{C_{O(f)}}(\gamma) = \epsilon\}$. Note that $A = \bigcup_{\epsilon<\omega_1} T_\epsilon$. Since A is comeager and wellordered unions of meager sets are meager, there is a least ϵ_0 so that T_{ϵ_0} is nonmeager.

Suppose α is a limit ordinal. Suppose $\langle \epsilon_\beta : \beta < \alpha\rangle$ has been defined with the property that for all $\beta < \alpha$, $\bigcup_{\nu<\epsilon_\beta} T_\nu$ is not comeager. Let $\mu = \sup\{\epsilon_\nu : \nu < \alpha\}$. If $\bigcup_{\nu<\mu} T_\nu$ is comeager, then say the construction has stopped at stage α. If it is not comeager, then $A \setminus \bigcup_{\nu<\mu} T_\nu = \bigcup_{\nu\geq\mu} T_\nu$ is nonmeager. Again since wellordered unions of meager sets are meager, there must be a least $\epsilon_\alpha \geq \mu$ so that T_{ϵ_α} is nonmeager.

Suppose $\langle \epsilon_\beta : \beta \leq \alpha\}$ has been defined. If $\bigcup_{\nu\leq\epsilon_\alpha} T_\nu$ is comeager, then say the construction ended at stage α. If not, then $A \setminus \bigcup_{\nu\leq\epsilon_\alpha} T_\mu = \bigcup_{\nu>\epsilon_\alpha} T_\nu$ is nonmeager. Since wellordered union of meager sets is meager, there must be a least $\epsilon_{\alpha+1} < \omega_1$ with $\epsilon_{\alpha+1} > \epsilon_\alpha$ and $T_{\epsilon_{\alpha+1}}$ is nonmeager.

In this way, one constructed a sequence $\langle \epsilon_\nu : \nu < \rho\rangle$ where $\rho \leq \omega_1$ is the stage by which is the construction stops. Since for each $\nu \neq \nu'$, $T_{\epsilon_\nu} \cap T_{\epsilon_{\nu'}} = \emptyset$ and each T_ν is nonmeager, one must have that $\rho < \omega_1$ by the fact that all sets of reals have the Baire property and the countable chain condition for the meager ideal. Let $\delta = \sup\{\epsilon_\nu : \nu < \rho\}$. Let $T = \bigcup_{\nu<\rho} T_{\epsilon_\nu}$. $T \subseteq A$ is a comeager set. For all $f \in T$, $\mathsf{enum}_{C_{O(f)}}(\gamma) < \delta$. So $K(\gamma)$ exists. This completes the proof of Claim 1.

Let $D = \{\eta : (\forall\nu < \eta)(K(\nu) < \eta)\}$. Note that since for any club $C \subseteq \omega_1$, $\mathsf{enum}_C(\gamma) \geq \gamma$, one can conclude that $K(\gamma) > \gamma$. Also if $\gamma \leq \gamma'$, $K(\gamma) \leq K(\gamma')$. Let $\epsilon < \omega_1$. Let $\alpha_0 = \epsilon$. Let $\alpha_{n+1} = K(\alpha_n)$. Hence $\alpha_{n+1} > \alpha_n$. Let $\alpha = \sup\{\alpha_n : n \in \omega\}$. Note that for all $\nu < \alpha$, then $\nu < K(\alpha_n)$ for some n. Then one has $K(\nu) < K(\alpha_n) = \alpha_{n+1} < \alpha$. Thus $\alpha \in D$ and $\epsilon < \alpha$. This shows that D is unbounded. D is clearly closed.

Claim 2: $R(\sigma, D)$.

To see this: Let $\eta \in D$. For each $\beta < \eta$, let $F^\eta_\beta = \{f \in A : \mathsf{enum}_{C_{O(f)}}(\beta) < \eta\}$. Since $\eta \in D$, for all $\beta < \eta$, $K(\beta) < \eta$. So the set of $f \in A$ so that $\mathsf{enum}_{C_{O(f)}}(\beta) < K(\beta) < \eta$ is comeager, i.e. F^η_β is comeager. Then $Y^\eta = \bigcap_{\beta<\eta} F^\eta_\beta$ is a comeager set. For all $f \in Y^\eta$, for all $\beta < \eta$, $\beta \leq \mathsf{enum}_{C_{O(f)}}(\beta) < \eta$. Since $C_{O(f)}$ is a club, $\eta \in C_{O(f)}$. It has been shown that if $\eta \in D$, then Y^η has the property that for all $f \in Y^\eta$, $\eta \in C_{O(f)}$. Let $Y = \bigcap_{\eta\in D} Y^\eta$. Since wellordered intersection of comeager sets are comeager, Y is comeager. Pick an $f \in Y$. For any $\eta \in D$, $f \in Y^\eta$. So $\eta \in C_{O(f)}$. Thus $D \subseteq C_{O(f)}$. Since $R(\sigma, C_{O(f)})$ holds and R is $\subseteq$-downward closed, $R(\sigma, D)$ holds.

Note that D was produced uniformly from σ by the procedure above. So finally, let $F(\sigma) = D$. This defines $F : [\omega_1]^{<\omega_1} \to \mathsf{club}_{\omega_1}$. F is a uniformization for R.

Assume $\mathsf{AD}_\mathbb{R}$. Every set of reals has the Baire property. Moreover, the uniformization J for $\tilde{R}$ exists since $\mathsf{AD}_\mathbb{R}$ proves uniformization for all relations on $\mathbb{R} \times \mathbb{R}$. Club uniformization follows from the first part of the theorem. ☐

For $\alpha < \omega_1$, let BS_α be the subset of BS coding elements of $[\omega_1]^\alpha$.

Corollary 3.8. *Assume all sets of reals have the Baire property. Let $\alpha < \omega_1$. Let $R \subseteq [\omega_1]^\alpha_* \times \mathsf{club}_{\omega_1}$ be a $\subseteq$-closed relation. Define $\tilde{R} \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by*

$$\tilde{R}(x,y,z) \Leftrightarrow (x,y) \in \mathsf{BS}_\alpha \wedge z \in \mathsf{clubcode}_{\omega_1} \wedge R(\sigma_{(x,y)}, C_z).$$

Consider $\tilde{R}$ as a relation on $\mathsf{BS}_\alpha \times \mathsf{clubcode}_{\omega_1}$. Suppose there is a $J : \mathrm{dom}(\tilde{R}) \to \mathsf{clubcode}_{\omega_1}$ which is a uniformization for $\tilde{R}$. Then there is a $F : \mathrm{dom}(R) \to \mathsf{club}_{\omega_1}$ which is a uniformization for R.

Thus $\mathsf{AD}_\mathbb{R}$ proves $[\omega_1]^\alpha$-club uniformization, for all $\alpha < \omega_1$.

Fact 3.9. *Assume $\mathsf{ZF} + \mathsf{AD}$. Then $L(\mathbb{R}) \models \mathsf{AD}$ (and even AD^+) and $L(\mathbb{R})$ does not satisfy $[\omega_1]^\alpha$-club uniformization (when $\omega \leq \alpha < \omega_1$) or $[\omega_1]^{<\omega_1}$-club uniformization.*

Proof. Work in $L(\mathbb{R})$. Consider the relation $S \subseteq \mathbb{R} \times \mathsf{club}_{\omega_1}$ defined by $S(x,C)$ if and only if for all club $D \subseteq C$, $D \notin \mathrm{HOD}_x$.

Fix an $x \in \mathbb{R}$. Since ω_2 is measurable in V, every wellordered sequence of elements of $\mathscr{P}(\omega_1)$ has length less than ω_2. Thus $(\mathscr{P}(\omega_1))^{\mathrm{HOD}_x}$ has cardinality less than ω_2 in $L(\mathbb{R})$. Let $\langle C_\alpha : \alpha < \omega_1\rangle$ be an enumeration of all club subsets of ω_1 which belong to HOD_x. (This enumeration does not belong to HOD_x.)

One will construct a club $E \subseteq \omega_1$ as follows:

Let $\alpha_0 = \min C_0$. Let $E_0 = \{\alpha_0 + 1\}$. Note that $\alpha_0 \notin E_0$.

If γ is a limit ordinal, let E_γ be the closure of $\bigcup_{\nu<\gamma} E_\nu$.

Now suppose γ is an ordinal so that $\langle \alpha_\nu : \nu < \gamma\rangle$ and the closed set E_γ have been defined with $\alpha_\nu \notin E_\gamma$. Let α_γ be least element of C_γ greater than $\sup E_\gamma$. Let $E_{\gamma+1} = E_\gamma \cup \{\alpha_{\gamma+1} + 1\}$.

In the end, one has constructed a sequence $\langle \alpha_\gamma : \gamma < \omega_1\rangle$ and a sequence $\langle E_\gamma : \gamma < \omega_1\rangle$ so that each $\alpha_\gamma \in C_\gamma$. Let $E = \bigcup_{\gamma<\omega_1} E_\gamma$. One can check that E is club and $\alpha_\nu \notin E$ for any $\nu < \omega_1$.

Now suppose $D \subseteq E$, D is a club subset of ω_1, and $D \in \mathrm{HOD}_x$. Then there is some $\gamma < \omega_1$ so that $D = C_\gamma$. But $\alpha_\gamma \notin D$ since $\alpha_\gamma \notin E$. But $\alpha_\gamma \in C_\gamma$. Hence $D \neq C_\gamma$. This shows that $E \subseteq \omega_1$ is a club subset with the property that E has no club subsets that belong to HOD_x.

It has been shown that for all $x \in \mathbb{R}$, there is some C so that $S(x,C)$.

Observe that S is $\subseteq$-downward closed in the sense that for all $x \in \mathbb{R}$, $S(x,C)$ and $D \subseteq C$, then $S(x,D)$.

Suppose there is a function $F : \mathbb{R} \to \mathsf{club}_{\omega_1}$ so that F uniformizes S. In $L(\mathbb{R})$, F is OD_z for some $z \in \mathbb{R}$. Since F is a uniformization, $S(z, F(z))$. Therefore $F(z)$ is a club subset of ω_1 which is OD_z and thus $F(z) \in \mathrm{HOD}_z$. This contradicts the definition of S.

Considering $\mathbb{R}$ as increasing sequences in ω, define $R \subseteq [\omega_1]^\omega \times \mathsf{club}_{\omega_1}$ by

$$R(x,C) \Leftrightarrow (x \in \mathbb{R} \wedge S(x,C)) \vee (x \notin [\omega]^\omega).$$

R can not be uniformized or else S could be uniformized. This shows $[\omega_1]^\omega$-club uniformization fails. Similar examples give the failure of $[\omega_1]^\alpha$-club uniformization for all $\omega \leq \alpha < \omega_1$ and a failure of $[\omega_1]^{<\omega_1}$-club uniformization. ☐

Thus almost everywhere $[\omega_1]^{<\omega_1}$-club uniformization is the best one can expect in AD alone. This is verified by the following result.

Theorem 3.10. *Almost everywhere $[\omega_1]^{<\omega_1}$-club uniformization holds: That is, let $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed. There is a club $D \subseteq \omega_1$ so that $R \cap ([D]^{<\omega_1}_* \times \mathsf{club}_{\omega_1})$ has a uniformization.*

Proof. Suppose $D \subseteq \omega_1$ is a club. Let BS^D denote the subset of BS which code elements of $[D]^{<\omega_1}_*$.

If one can find a club $D \subset \omega_1$ so that $\tilde{R} \cap (\mathsf{BS}^D \times \mathsf{clubcode}_{\omega_1})$ has a uniformization, then Theorem 3.7 would give the conclusion of this theorem.

Fix $U \subseteq \mathbb{R}^3$, a universal set for Σ^1_2 subsets of $\mathbb{R}^2$. Take any $f \in [\omega_1]^{\omega_1}$. Let $T_f \subseteq \mathrm{WO} \times \mathsf{clubcode}_{\omega_1}$ be defined by $T_f(w,z)$ if and only if

$$f \restriction \mathrm{ot}(w) \in \mathrm{dom}(R) \wedge z \in \mathsf{clubcode}_{\omega_1} \wedge R(f \restriction \mathrm{ot}(w), C_z).$$

By the coding lemma applied to the pointclass $\mathbf{\Sigma}^1_2$ and the usual prewellordering on WO, there is some e so that

(1) $U_e \subseteq T_f$.

(2) For all $w \in \mathrm{WO}$, $(T_f)_w \neq \emptyset$ if and only if $U_{e,w} \neq \emptyset$.

(Note that $(T_f)_w = \{c \in \mathbb{R} : T_f(w,c)\}$. Recall that $U \subseteq \mathbb{R}^3$ and $U_{a,b} = \{c \in \mathbb{R} : U(a,b,c)\}$.)

Say that $e \in \mathbb{R}$ is an f-selector if and only if (1) and (2) holds for e and f.

Fix a good coding system $(\mathbf{\Sigma}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \omega_1, \gamma < \omega_1)$ for ${}^{\omega\cdot\omega_1}\omega_1$. Consider the relation, $S \subseteq [\omega_1]^{\omega_1}_* \times \mathbb{R}$ defined by $S(f,e)$ if and only if e is an f-selector. Let F be a Lipschitz function and $E \subseteq \omega_1$ be a club witnessing the properties given by Theorem 2.16 for the relation S. By Fact 2.9, let $z^* \in \mathsf{clubcode}_{\omega_1}$ be such that $C_{z^*} \subseteq E$. C_{z^*} will also be a club satisfying the conclusion of Theorem 2.16. Let D be the limit points of C_{z^*}.

Now consider the relation $K \subseteq \mathsf{BS} \times \mathbb{R}$ by $K((x,y),r)$ if and only if the conjunction of the two holds
(1) $\sigma_{(x,y)} \in [D]^{<\omega_1}_*$. (That is, $(x,y) \in \mathsf{BS}^D$.)
(2) $r \in \mathsf{GC}$, $\mathsf{decode}(r) \in [C_{z^*}]^{\omega\cdot\omega_1}_*$, and $\sigma_{(x,y)} \subseteq \mathsf{block}(\mathsf{decode}(r))$.

Roughly, $K((x,y),r)$ holds if (x,y) is a code for a function of length less than ω_1 of the correct type through D (which is the set of limit points of C_{z^*}) and r is a code (according to the good coding system) for a full $\omega_1 = \omega \cdot \omega_1$ length function with the property that $\sigma_{(x,y)}$ is an initial segment of $\mathsf{block}(\mathsf{decode}(r))$.

One can check that K is projective using z^* as a parameter. Hence let $G : \mathbb{R} \to \mathbb{R}$ be a projective uniformization for this relation. Thus if $(x,y) \in \mathsf{BS}^D$ is such that $\sigma_{(x,y)}$ is a bounded function of the correct type, then $\mathsf{decode}(G(x,y)) \in [C_{z^*}]^{\omega_1}_*$, and $\mathsf{block}(\mathsf{decode}(G(x,y)))$ is an extension of $\sigma_{(x,y)}$ to a full sequence.

Define $\tilde{Y} \subseteq \mathsf{BS}^D \times \mathsf{clubcode}_{\omega_1}$ by

$$\tilde{Y}((x,y),v) \Leftrightarrow (x,y) \in \mathsf{BS}^D \wedge v \in U_{F(G(x,y)),x}$$

Note that $\tilde{Y}$ is projective since D is the limit points of C_{z^*}, U is Σ^1_2, F is a Lipschitz function, and G is a projective function. Whenever $(x,y) \in \mathsf{BS}^D$ and $\sigma_{(x,y)}$ codes a sequence of the correct type of length less than ω_1 through D, $G(x,y) \in \mathsf{GC}$ is a code for a full function passing through C_{z^*} so that $\mathsf{block}(\mathsf{decode}(G(x,y)))$ extends $\sigma_{(x,y)}$. By the property of F, $F(G(x,y))$ is then a $\mathsf{block}(\mathsf{decode}(G(x,y)))$-selector. Recall that x is the length of $\sigma_{(x,y)}$. So for all such (x,y), $U_{F(G(x,y)),x} \subseteq \tilde{R}_{(x,y)}$ and $U_{F(G(x,y)),x} \neq \emptyset$ if and only if $\tilde{R}_{(x,y)} \neq \emptyset$. Hence $\tilde{Y} \subseteq \tilde{R} \cap (\mathsf{BS}^D \times \mathsf{clubcode}_{\omega_1})$ and any uniformization for $\tilde{Y}$ is a uniformization for $\tilde{R} \cap (\mathsf{BS}^D \times \mathsf{clubcode}_{\omega_1})$.

However, $\tilde{Y}$ does have a uniformization since it is projective. Thus $\tilde{R} \cap (\mathsf{BS}^D \times \mathsf{clubcode}_{\omega_1})$ has a uniformization. By the remarks at the beginning of this proof, this suffices to complete the proof. ☐

Theorem 3.11. *Assume all sets of reals have the Baire property. Let $R \subseteq \omega_1 \times [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ be a $\subseteq$-downward closed relation (on the club_{ω_1}-coordinate). Define $\tilde{R} \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by*

$$\tilde{R}(w,x,y,z) \Leftrightarrow w \in \mathrm{WO} \wedge (x,y) \in \mathrm{BS} \wedge z \in \mathsf{clubcode}_{\omega_1} \wedge R(\mathrm{ot}(w), \sigma_{(x,y)}, C_z).$$

Consider $\tilde{R}$ as a relation on $(\mathrm{WO} \times \mathsf{BS}) \times \mathsf{clubcode}_{\omega_1}$. Suppose there is a $J : \mathrm{dom}(\tilde{R}) \to \mathsf{clubcode}_{\omega_1}$ which is a uniformization for $\tilde{R}$. Then there is an $F : \mathrm{dom}(R) \to \mathsf{club}_{\omega_1}$ which is a uniformization for R.

Thus under $\mathsf{AD}_{\mathbb{R}}$, such relations have a uniformization.

Let $R \subseteq \omega_1 \times [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ be a $\subseteq$-downward closed relation as above. Then there is a club $D \subseteq \omega_1$ so that $R \cap (\omega_1 \times [D]^{<\omega_1}_* \times \mathsf{club}_{\omega_1})$ has a uniformization.

Proof. This requires some small modifications in the arguments for Theorem 3.7 and Theorem 3.10.

By an argument similar to Fact 3.6, there is a function $H : \omega_1 \times [\omega_1]^{<\omega_1} \times \mathrm{WO} \to \mathrm{WO} \times \mathsf{BS}$ with the property that for all $\alpha < \omega_1$, $\sigma \in [\omega_1]^{<\omega_1}$, and all $w \in \mathrm{WO}$ so that $\mathrm{ot}(w) = \max\{\sup(\sigma) + 2, \alpha + 1\}$, one has that $H(\alpha, \sigma, w) \in \mathrm{WO} \times \mathsf{BS}$ with the property that $\mathrm{ot}(\pi_1(H(\alpha,\sigma,w))) = \alpha$ and $\pi_2(H(\alpha,\sigma,w))$ codes σ, where $\pi_1, \pi_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are the projection maps onto the first and second coordinate, respectively. Using H, one can prove the first part by making the necessary modifications to the argument of Theorem 3.7.

For the second part, let $\rho : \omega_1 \to \omega_1 \times \omega_1$ be a bijection. Let $\varpi_1, \varpi_2 : \omega_1 \times \omega_1 \to \omega_1$ be the projection onto the first and second coordinate, respectively. Define a relation $T_f \subseteq \mathrm{WO} \times \mathsf{clubcode}_{\omega_1}$ if and only

$$(\varpi_1(\rho(\mathrm{ot}(w))), f \restriction \varpi_2(\rho(\mathrm{ot}(w)))) \in \mathrm{dom}(R) \wedge z \in \mathsf{clubcode}_{\omega_1} \wedge R(\varpi_1(\rho(\mathrm{ot}(w))), f \restriction \varpi_2(\rho(\mathrm{ot}(w))), z).$$

With this version of the relation T_f, one can prove the second statement with a modification of the argument in Theorem 3.10. ☐

4. Continuity of Functions $[\omega_1]^{\omega_1} \to \omega_1$

Lemma 4.1. *Suppose $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ has the property that there is a club $C \subseteq \omega_1$ so that for all $f \in [C]^{\omega_1}_*$, $\Phi(f) < f(0)$. Then there is a club $D \subseteq \omega_1$ and a $\zeta < \omega_1$ so that for all $f \in [D]^{\omega_1}_*$, $\Phi(f) = \zeta$.*

Proof. Define a partition $P : [\omega_1]^{\omega_1}_* \to 2$ by $P(\alpha\hat{\ } f) = 0$ if and only if $\Phi(f) < \alpha$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is a club $E \subseteq \omega_1$ which is homogeneous for P. Let $\tilde{E} = \{\alpha \in E : \mathsf{enum}_E(\alpha) = \alpha\}$ where $\mathsf{enum}_E : \omega_1 \to E$ is the increasing enumeration of E. $\tilde{E} \subseteq E$ is also a club subset of ω_1. Let $f \in [\tilde{E} \cap C]^{\omega_1}_*$. Then $\Phi(f) < f(0)$ by the assumption on C. Since f is a function of the correct type and $f(0) \in \tilde{E}$, one can find an $\alpha \in E$ with $\Phi(f) < \alpha < f(0)$. Then $\alpha\hat{\ } f \in [E]^{\omega_1}_*$ and $P(\alpha\hat{\ } f) = 0$. Since E is homogeneous for P, one must have that E is homogeneous for P taking value 0. Let $E_0 = E \setminus (\min E + 1)$. For all $f \in [E_0]^{\omega_1}_*$, one has that $\Phi(f) < \min E$ since $(\min E)\hat{\ } f \in [E]^{\omega_1}_*$ and $P(\min(E)\hat{\ } f) = 0$. By the countable completeness of the strong partition measure on ω_1, there is a club $D \subseteq E_0$ and a $\zeta < \min E$ so that for all $f \in [D]^{\omega_1}_*$, $\Phi(f) = \zeta$. ☐

Example 4.2. The existence of a function $\Phi : [\omega_1]^{\omega_1} \to \omega_1$ which is not continuous μ_{ω_1}-almost everywhere intuitively amount to asking whether there is a way a define a map that truly uses all information about f and not merely an initial segment of f, for μ_{ω_1}-almost all $f \in [\omega_1]^{\omega_1}_*$.

One function that at first glance may appear to use the whole function $f \in [\omega_1]^{\omega_1}$ is $\Phi(f) = \omega_1^{L[f]}$. However, almost everywhere Φ uses no information about f. It is μ_{ω_1}-almost everywhere a constant function.

To see this: Let $f \in [\omega_1]^{\omega_1}_*$. For each $\alpha < \omega_1$, let $f_\alpha \in [\omega_1]^{\omega_1}_*$ be defined by $f_\alpha(\beta) = f(\alpha + \beta)$. Note that for all $\alpha < \beta < \omega_1$, $f_\beta \in L[f_\alpha]$. So $\omega_1^{L[f_\beta]} \leq \omega_1^{L[f_\alpha]}$. The sequence $\langle \omega_1^{L[f_\alpha]} : \alpha < \omega_1\rangle$ is a nonincreasing sequence of ordinals. It must be eventually constant else one would have an infinite decreasing sequence of ordinals. Let ϵ_f be the eventual constant value of this sequence.

Let $Q : [\omega_1]^{\omega_1}_* \to 2$ be defined by $Q(f) = 0 \Leftrightarrow \omega_1^{L[f]} = \epsilon_f$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is some $D \subseteq \omega_1$ club which is homogeneous for Q. Let $f \in [D]^{\omega_1}_*$. Let α be minimal so that $\omega_1^{L[f_\alpha]} = \epsilon_f$. Note that $\omega_1^{L[f_\alpha]} = \epsilon_{f_\alpha}$ and $f_\alpha \in [D]^{\omega_1}_*$. Thus $Q(f_\alpha) = 0$. Thus D is homogeneous for Q taking value 0. It has been shown that for all $f \in [D]^{\omega_1}_*$, $\omega_1^{L[f]} = \omega_1^{L[f_\alpha]}$ for all $\alpha < \omega_1$.

Consider $P : [D]^{\omega_1}_* \to 2$ defined by $P(f) = 0 \Leftrightarrow \Phi(f) = \omega_1^{L[f]} < f(0)$.

By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, let $C \subseteq D$ be a club such that C is homogeneous for P. Take any $f \in [C]^{\omega_1}_*$. Note that for all $\alpha < \omega_1$, $\Phi(f_\alpha) = \omega_1^{L[f_\alpha]} = \omega_1^{L[f]} = \Phi(f)$ since $f \in [D]^{\omega_1}_*$. Pick δ so that $f(\delta) > \omega_1^{L[f]}$. Then $\Phi(f_\delta) = \Phi(f) < f(\delta) = f_\delta(0)$. Since $f_\delta \in [C]^{\omega_1}_*$, one has that C is homogeneous for P taking value 0.

So for all $f \in [C]^{\omega_1}_*$, $\Phi(f) < f(0)$. By Lemma 4.1, Φ is μ_{ω_1}-almost everywhere a constant function.

With Trang, it has been shown that the constant almost everywhere value of Φ is quite large. It is in particular not ω_1^L.

Claim 1: There is some $\epsilon < \omega_1$ and some club $C_0 \subseteq \omega_1$ so that for all $f \in [C_0]^{\omega_1}_*$, the canonical $L[f]$ wellordering of $\mathbb{R}^{L[f]}$ has ordertype ϵ.

Suppose not. For all $\alpha < \omega_1$, there is a club $C \subseteq \omega_1$ so that the canonical wellordering of $\mathbb{R}^{L[f]}$ has length greater than α. By the countable additivity of the strong partition measure, for each $\alpha < \omega_1$, there is a real r_α so that for almost all $f \in [\omega_1]^{\omega_1}_*$, r_α is the α^{th} real in the canonical wellordering.

Suppose $\alpha \neq \beta$. There is a club C_α and C_β so that for all $f \in [C_\alpha]^{\omega_1}_*$ and $g \in [C_\beta]^{\omega_1}_*$, the α^{th} real of $L[f]$ is r_α and the β^{th} real of $L[g]$ is r_β. If $f \in [C_\alpha \cap C_\beta]^{\omega_1}_*$, then the r_α and r_β are the α^{th} and β^{th} real of $L[f]$. Hence $r_\alpha \neq r_\beta$.

Then $\langle r_\alpha : \alpha < \omega_1\rangle$ is an ω_1 sequence of distinct reals. This is impossible under AD. Claim 1 has been shown.

By a countable additivity argument, one can show that there is a club C_1 so that for all $f, g \in [C_1]^{\omega_1}_*$, $\mathbb{R}^{L[f]} = \mathbb{R}^{L[g]}$. Let $\mathbb{R}^*$ denote this common set of reals.

Suppose $x \in \mathbb{R}^*$. AD implies that $x^\sharp$ exists. Let C_x be the club set of Silver indiscernible for $L[x]$. Let $f \in [C_1 \cap C_x]^{\omega_1}_*$. Note that $f \restriction \omega \in L[f]$ and $x \in L[f]$. Thus $x^\sharp$ can be defined within $L[f]$ as the collection of formulas true in $L[x]$ using $\{f(n) : n \in \omega\}$ as indiscernibles. So $x^\sharp \in \mathbb{R}^*$.

$\mathbb{R}^*$ is closed under sharps. Then certainly for $f \in [C_1]^{\omega_1}_*$, $\omega_1^{L[f]} > \omega_1^L$.

This example motivates the further study of the stable theory of the partition measures μ_ϵ for $\epsilon \leq \omega_1$: Note that for each sentence φ in the language of set theory, since μ_ϵ is an ultrafilter, one has exactly one of the following holds: (i) for μ_ϵ-almost all f, $L[f] \models \varphi$ or (ii) for μ_ϵ-almost all f, $L[f] \models \neg\varphi$.

In work with Trang, the authors study which natural statements belong to the stable theory of the partition measure μ_ϵ. For example, for all $\epsilon \leq \omega_1$, for μ_ϵ-almost all f, $L[f] \models \mathsf{GCH}$. The most difficult case is $\epsilon = \omega_1$ where the $[\omega_1]^{<\omega_1}_*$-almost everywhere club uniformization plays as essential role as a construction principle. This result will appear elsewhere.

As a warmup to showing every function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ is continuous, one will show that elements of $\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu$ which have representatives that are continuous form an initial segment of the ultraproduct.

Fact 4.3. *Suppose $\Psi, \Phi : [\omega_1]^{\omega_1}_* \to \omega_1$. Suppose Φ is continuous μ_{ω_1}-almost everywhere and $\Psi <_{\mu_{\omega_1}} \Phi$, which means $\{f \in [\omega_1]^{\omega_1}_* : \Psi(f) < \Phi(f)\} \in \mu_{\omega_1}$. Then Ψ is continuous μ_{ω_1}-almost everywhere.*

Proof. Let $C_0 \subseteq \omega_1$ be a club so that Φ is continuous on $[C_0]^{\omega_1}_*$ and $\Psi(f) < \Phi(f)$ for all $f \in [\omega_1]^{\omega_1}_*$. Let $K \subseteq [C_0]^{<\omega_1}_*$ be the collection of σ so that for all $f, g \in [C_0]^{\omega_1}$ with $f \restriction |\sigma| = g \restriction |\sigma| = \sigma$, $\Phi(f) = \Phi(g)$. Since Φ is continuous on $[C_0]^{\omega_1}_*$, K is dense in $[C_0]^{\omega_1}_*$ in the senes that for all $f \in [C_0]^{\omega_1}_*$, there exists an $\alpha < \omega_1$ so that $f \restriction \alpha \in K$. For each $\sigma \in K$, let $d_\sigma = \Phi(f)$ for any $f \in [C_0]^{\omega_1}_*$ such that $f \restriction |\sigma| = \sigma$, which is well defined by the definition of K.

For each $\sigma \in K$, define $\Gamma_\sigma : [C_0 \backslash \sup(\sigma)+1]^{\omega_1}_* \to \omega_1$ by $\Gamma_\sigma(g) = \Psi(\sigma\hat{\ }g)$. Thus for all $g \in [C_0 \backslash \sup(\sigma)+1]^{\omega_1}_*$, $\Gamma_\sigma(g) < d_\sigma$. By countable additivity of μ_{ω_1}, for μ_{ω_1}-almost all g, $\Gamma_\sigma(g)$ takes a constant value denoted c_σ.

Define $\Psi' : [C_0]^{\omega_1}_* \to \omega_1$ as follows: For each $f \in [C_0]^{\omega_1}_*$, find the least α so that $f \restriction \alpha \in K$ (which exists by the density of K), and let $\Psi'(f) = c_{f \restriction \alpha}$.

The claim is that $\Psi =_\mu \Psi'$:

Define $P : [C_0]^{\omega_1}_* \to 2$ by $P(f) = 0$ if and only if $\Psi(f) = \Psi'(f)$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, let $C_1 \subseteq C_0$ be a club on which P is homogeneous. Let $f \in [C_1]^{\omega_1}_*$. Find the least α so that $f \restriction \alpha \in K$. There is some club D so that $\Gamma_{f\restriction\alpha}$ takes constant value $c_{f\restriction\alpha}$ on $[D]^{\omega_1}_*$. Let $D' = (C_1 \setminus \sup(\sigma) + 1) \cap D$. Let $g \in [D']^{\omega_1}_*$. Then $\Psi(f \restriction \alpha\hat{\ }g) = \Gamma_{f\restriction\alpha}(g) = c_{f\restriction\alpha} = \Psi'(f \restriction \alpha\hat{\ }g)$. So $P(f \restriction \alpha\hat{\ }g) = 0$ and $f \restriction \alpha\hat{\ }g \in [C_1]^{\omega_1}_*$. Hence P is homogeneous taking value 0. This implies $\Psi = \Psi'$ on $[C_1]^{\omega_1}_*$, so Ψ' is continuous. ☐

The following is a useful notation:

Definition 4.4. Define $\mathsf{drop} : [\omega_1]^{\omega_1} \times \omega_1 \to [\omega_1]^{\omega_1}$ by $\mathsf{drop}(f, \delta)(\alpha) = f(\delta + \alpha)$. Thus $\mathsf{drop}(f, \delta)$ is merely f with its δ^{th}-initial segment, $f \restriction \delta$, removed.

Let $A \subseteq \omega_1$ be an unbounded subsets of ω_1. Let $\mathsf{next}_A : \omega_1 \to A$ be defined by $\mathsf{next}_A(\alpha)$ is the smallest element of A strictly larger than α. Let $\mathsf{next}^\omega_A : \omega_1 \to A$ be defined by $\mathsf{next}^\omega_A(\alpha)$ is the ω^{th}-element of A larger than α.

Theorem 4.5. *Every function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ is continuous almost everywhere.*

Proof. Let $P : [\omega_1]^{\omega_1}_* \to 2$ be defined by $P(f) = 0$ if and only if there exists $\alpha < \omega_1$ so that for all club $C \subseteq \omega_1$, there exists $g \in [C]^{\omega_1}_*$ so that $f \restriction \alpha\hat{\ }g \in [\omega_1]^{\omega_1}_*$ (i.e. is strictly increasing) and $\Phi(f \restriction \alpha\hat{\ }g) < g(0)$.

By $\omega_1 \to_* (\omega_1)^{\omega_1}_*$, let $D \subseteq \omega_1$ be homogeneous for P.

Claim 1: D is homogeneous for P taking value 0.

To prove this: Suppose that it is homogeneous for P taking value 1. This means that for all $f \in [D]^{\omega_1}_*$, for all $\alpha < \omega_1$, there exists a club $C \subseteq [\omega_1]^{\omega_1}$ so that for all $g \in [C]^{\omega_1}_*$ such that $f \restriction \alpha\hat{\ }g \in [\omega_1]^{\omega_1}_*$, $\Phi(f \restriction \alpha\hat{\ }g) \geq g(0)$.

Define $R \subseteq [D]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ by $R(\sigma, C)$ if and only if for all $g \in [C]^{\omega_1}_*$ with $\sigma\hat{\ }g \in [\omega_1]^{\omega_1}_*$, $\Phi(\sigma\hat{\ }g) \geq g(0)$. R is $\subseteq$-downward closed. Each section is nonempty since D is homogeneous for P taking value 1. By Theorem 3.10, there is a club $E \subseteq D$ so that $R \cap ([E]^{<\omega_1}_* \times \mathsf{club}_{\omega_1})$ has a uniformization. Let $\Lambda : [E]^{<\omega_1}_* \to \mathsf{club}_{\omega_1}$ be such a uniformization function for R on $[E]^{<\omega_1}_*$.

First one will construct an $h \in [E]^{\omega_1}_*$ by recursion as follows: Let $F_0 = E \cap \Lambda(\emptyset)$. Let $h(0) = \mathsf{next}^\omega_{F_0}(0)$.

Suppose $h \restriction \alpha$ and clubs F_β for all $\beta < \alpha$ have been defined. Let $F_\alpha = \Lambda(h \restriction \alpha) \cap \bigcap_{\beta<\alpha} F_\beta$. Let $h(\alpha) = \mathsf{next}^\omega_{F_\alpha}(\sup(h \restriction \beta))$.

This completes the construction of $h \in [E]^{\omega_1}$ and sequence of clubs $\langle F_\beta : \beta < \omega_1 \rangle$.

Since one has the sequence $\langle F_\beta : \beta < \omega_1 \rangle$, one also can define the sequence of functions $\langle \mathsf{next}_{F_\beta} : \beta < \omega_1 \rangle$. Define $H : \omega_1 \times \omega \to \omega_1$ by recursions as follows: $H(0, 0) = \mathsf{next}_{F_0}(0)$. $H(0, n+1) = \mathsf{next}_{F_0}(H(0, n))$. If for some α, $H(\beta, n)$ has been defined for all $\beta < \alpha$ and $n \in \omega$, then let $\mu = \sup\{H(\beta, n) : \beta < \alpha \wedge n \in \omega\}$. Let $H(\alpha, 0) = \mathsf{next}_{F_\alpha}(\mu)$. Let $H(\alpha, n+1) = \mathsf{next}_{F_\alpha}(H(\alpha, n))$. Now H witnesses h has uniform cofinality ω.

By the construction, it is clear that h is increasing and discontinuous everywhere. Thus $h \in [E]^{\omega_1}_*$, i.e. is increasing and has correct type.

Now pick any $\alpha < \omega_1$. Since $\mathsf{drop}(h, \alpha) \in [F_\alpha]^{\omega_1}_* \subseteq [\Lambda(h \restriction \alpha)]^{\omega_1}_*$, the definition of Λ being a uniformization for R implies that $\Phi(h) = \Phi(h \restriction \alpha\hat{}\mathsf{drop}(h, \alpha)) \geq \mathsf{drop}(h, \alpha)(0) = h(\alpha)$. Since $\alpha < \omega_1$ was arbitrary, this shows that for all $\alpha < \omega_1$, $\Phi(h) \geq h(\alpha)$. Since $h \in [E]^{\omega_1}_*$ is a strictly increasing function, $\Phi(h) \geq \omega_1$. This is impossible since $\Phi : [\omega_1]^{\omega_1} \to \omega_1$ is a function which takes values among the countable ordinals. This establishes Claim 1.

Thus D is homogeneous for P taking value 0. Let $K \subseteq [D]^{<\omega_1}_*$ be the collection of σ such that for all club $C \subseteq \omega_1$, there is some $g \in [C]^{\omega_1}_*$ with $\sigma\hat{}g \in [\omega_1]^{\omega_1}_*$ and $\Phi(\sigma\hat{}g) < g(0)$. Note that K is dense in $[D]^{\omega_1}_*$ since D is homogeneous for P taking value 0.

Fix $\sigma \in K$. Let $Q_\sigma : [D \setminus \sup\sigma + \omega]^{\omega_1}_* \to 2$ be defined by $Q_\sigma(g) = 0$ if and only if $\Phi(\sigma\hat{}g) < g(0)$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is some $E_\sigma \subseteq D$ club which is homogeneous for Q_σ. By the property of $\sigma \in K$, there is some $g \in [E_\sigma]^{\omega_1}$ so that $\sigma\hat{}g \in [\omega_1]^{\omega_1}_*$ and $\Phi(\sigma\hat{}g) < g(0)$. Thus one has $Q_\sigma(g) = 0$. This shows that E_σ is homogeneous for Q_σ taking value 0.

Now define $V_\sigma : [E_\sigma \setminus \sup\sigma + \omega]^{\omega_1}_* \to \omega_1$ by $V_\sigma(g) = \Phi(\sigma\hat{}g)$. For all $g \in [E_\sigma \setminus \sup\sigma + \omega]^{\omega_1}_*$, $V_\sigma(g) < g(0)$. By Lemma 4.1, there is an $E'_\sigma \subseteq E_\sigma$ club so that V_σ is constant on $[E'_\sigma]^{\omega_1}_*$ taking value c_σ. Note that c_σ does not depend on the choice of E_σ or E'_σ in the sense that for any club E so that V_σ is constant on $[E]^{\omega_1}_*$, the constant value must be c_σ.

Define $\Psi : [D]^{\omega_1}_* \to \omega$ as follows: For $f \in [D]^{\omega_1}_*$, find the least α so that $f \restriction \alpha \in K$ and let $\Psi(f) = c_{f\restriction\alpha}$. Such an α exists by the density of K. Ψ is continuous on $[D]^{\omega_1}_*$. This is because for any $f \in [D]^{\omega_1}_*$, let α be the least ordinal so that $f \restriction \alpha \in K$. For any $g \in [D]^{\omega_1}_*$ with $g \restriction \alpha = f \restriction \alpha$, one has that α is also the least ordinal so that $g \restriction \alpha \in K$. Thus $\Psi(g) = c_{g\restriction\alpha} = c_{f\restriction\alpha} = \Psi(f)$.

Claim 2 : For μ_{ω_1}-almost all f, $\Phi(f) = \Psi(f)$.

To see this: Define $Y : [D]^{\omega_1}_* \to 2$ by $Y(f) = 0$ if and only if $\Phi(f) = \Psi(f)$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_*$, there is some club $F \subseteq D$ which is homogeneous for Y. Let $f \in [F]^{\omega_1}_*$. Let α be least so that $f \restriction \alpha \in K$. Let $\sigma = f \restriction \alpha$. There is some $F' \subseteq F$ club so that V_σ takes constant value c_σ on $[F']^{\omega_1}_*$. Let $g \in [F']^{\omega_1}_*$ be such that $\sigma\hat{}g \in [\omega_1]^{\omega_1}$. Let $f' = \sigma\hat{}g$. Then $\Phi(f') = \Phi(\sigma\hat{}g) = V_\sigma(g) = c_\sigma$. As noted above, the least α so that $f' \restriction \alpha \in K$ is the same as the least α so that $f \restriction \alpha \in K$. So $c_\sigma = \Psi(f')$. Thus $Y(f') = 0$. Since $f' \in [F]^{\omega_1}_*$, F must be homogeneous for Y taking value 0.

It has been shown that for all $f \in [F]^{\omega_1}_*$, $\Phi(f) = \Psi(f)$. Since Ψ is a continuous function, Φ is μ_{ω_1}-almost equal to a continuous function. □

Zapletal asked the first author whether every partition of $[\omega_1]^{\omega_1}$ into ω_1 many pieces must have at least one piece of cardinality $[\omega_1]^{\omega_1}$. The following gives a positive answer.

Theorem 4.6. *Suppose $\langle X_\alpha : \alpha < \omega_1\rangle$ is a sequence of subsets of $[\omega_1]^{\omega_1}$ so that $\bigcup_{\alpha<\omega_1} X_\alpha = [\omega_1]^{\omega_1}$. Then there is an $\alpha < \omega_1$ so that $X_\alpha \approx [\omega_1]^{\omega_1}$.*

Proof. Define $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ by letting $\Phi(f)$ be the least α such that $f \in X_\alpha$.

By Theorem 4.5, there is some club $C \subseteq \omega_1$ so that Φ is continuous on $[C]^{\omega_1}_*$. Pick any $f \in [C]^{\omega_1}_*$. Let $\delta = \Phi(f)$. By continuity, there is some α so that for all $g \in [C]^{\omega_1}_*$ with $g \restriction \alpha = f \restriction \alpha$, $\Phi(g) = \Phi(f) = \delta$.

Using Fact 2.2, let $\Delta : [\omega_1]^{\omega_1} \to [C \setminus f(\alpha)]^{\omega_1}_*$ be a bijection. Define $\Gamma : [\omega_1]^{\omega_1} \to X_\delta$ by $\Gamma(g) = (f \restriction \alpha)\hat{}\Delta(g)$. Then Γ is an injection. Thus $|X_\delta| = |[\omega_1]^{\omega_1}|$. □

Theorem 4.7. $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$.

Proof. For $\alpha, \beta < \omega_1$, let $X_{\alpha,\beta} = [\beta]^\alpha$. Note that for all $\alpha, \beta < \omega_1$, $|X_{\alpha,\beta}| \leq |\mathbb{R}|$. Observe that $[\omega_1]^{<\omega_1} = \bigcup_{\alpha,\beta<\omega_1} X_{\alpha,\beta}$. By using the Gödel pairing function, one can recognize this union as an ω_1-length union of subsets of $[\omega_1]^{<\omega_1}$ with cardinality less than or equal to $\mathbb{R}$ (but non-uniformly). Therefore $|[\omega_1]^{<\omega_1}| = |[\omega_1]^{\omega_1}|$ is impossible since it would violate Theorem 4.6. □

5. Continuity of Functions $[\omega_1]^{\omega_1} \to {}^{\omega_1}\omega_1$

Recall that ${}^{\omega_1}\omega_1$ is the collection of all functions $f : \omega_1 \to \omega_1$.

Definition 5.1. A function $\Phi : [\omega_1]^{\omega_1} \to {}^{\omega_1}\omega_1$ is continuous if and only if for all $f \in [\omega_1]^{\omega_1}$, for all $\epsilon < \omega_1$, there exists a $\delta < \omega_1$ so that for all $g \in [\omega_1]^{\omega_1}$, if $f \restriction \delta = g \restriction \delta$, then $\Phi(f) \restriction \epsilon = \Phi(g) \restriction \epsilon$.

If one gives $[\omega_1]^{\omega_1}$ and ${}^{\omega_1}\omega_1$ the topology indicated in Definition 2.11, then $\Phi : [\omega_1]^{\omega_1} \to {}^{\omega_1}\omega_1$ is continuous if and only if Φ is continuous in the topological sense.

$\Phi : [\omega_1]^{\omega_1} \to {}^{\omega_1}\omega_1$ is continuous almost everywhere if and only if there club $C \subseteq \omega_1$ so that Φ is continuous on $[C]^{\omega_1}_*$.

Lemma 5.2. *There is no club $D \subseteq \omega_1$ and no function $\Lambda : [D]^{\omega_1}_* \to \omega_1$ with the property that for all $f \in [D]^{\omega_1}_*$, for all $\alpha < \omega_1$, there exists a club $C \subseteq \omega_1$ so that for all $g \in [C]^{\omega_1}$, if $(f \upharpoonright \alpha)\hat{}g \in [\omega_1]^{\omega_1}_*$, then $\Lambda((f \upharpoonright \alpha)\hat{}g) \geq g(0)$.*

Proof. The proof of Claim 1 in Theorem 4.5 is precisely this lemma. As there, one can prove this by using the almost everywhere $[\omega_1]^{<\omega_1}_*$-club uniformization (Theorem 3.10). However having already established the continuity property in Theorem 4.5, this lemma can be derived easily as follows:

Suppose such a club $D \subseteq \omega_1$ and function Λ exist. By Theorem 4.5, there is a $D_0 \subseteq D$ so that $\Lambda \upharpoonright [D_0]^{\omega_1}_*$ is continuous. Take any $f \in [D_0]^{\omega_1}_*$. Let $\zeta = \Lambda(f)$. By continuity, there is an $\alpha < \omega_1$ so that for all $h \in [D_0]^{\omega_1}_*$, if $f \upharpoonright \alpha = h \upharpoonright \alpha$, then $\Lambda(h) = \Lambda(f) = \zeta$.

By the hypothesis applied to this $f \in [D_0]^{\omega_1}_*$ and α, there exists some club $C \subseteq \omega_1$ so that for all $g \in [C]^{\omega_1}_*$, if $(f \upharpoonright \alpha)\hat{}g \in [\omega_1]^{\omega_1}_*$, then $\Lambda((f \upharpoonright \alpha)\hat{}g) \geq g(0)$. Pick $g \in [C \cap D_0]^{\omega_1}$ such that $(f \upharpoonright \alpha)\hat{}g \in [\omega_1]^{\omega_1}_*$ and $g(0) > \zeta$. Then $\Lambda((f \upharpoonright \alpha)\hat{}g) \geq g(0) > \zeta$ by choice of C. However $(f \upharpoonright \alpha)\hat{}g \in [D_0]^{\omega_1}_*$ and $((f \upharpoonright \alpha)\hat{}g) \upharpoonright \alpha = f \upharpoonright \alpha$. Since $f \upharpoonright \alpha$ is a point of continuity for Λ on $[D_0]^{\omega_1}_*$ for taking value ζ, one must have $\Lambda((f \upharpoonright \alpha)\hat{}g) = \zeta$. Contradiction. ☐

Theorem 5.3. *(With Trang) Every function $\Phi : [\omega_1]^{\omega_1} \to {}^{\omega_1}\omega_1$ is continuous almost everywhere.*

Proof. Define a partition $P_0 : [\omega_1]^{\omega_1}_* \to 2$ by $P_0(f) = 0$ if and only if for all $\beta < \omega_1$, for all $\gamma < \omega_1$, there exists an $\alpha > \gamma$ so that for all club $C \subseteq \omega_1$, there exists a $g \in [C]^{\omega_1}_*$ so that $(f \upharpoonright \alpha)\hat{}g \in [\omega_1]^{\omega_1}_*$ and $\Phi((f \upharpoonright \alpha)\hat{}g)(\beta) < g(0)$.

By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is a club $D_0 \subseteq \omega_1$ which is homogeneous for P_0.

Claim 1: D_0 is homogeneous for P_0 taking value 0.

To see this, suppose D_0 is homogeneous for P_0 taking value 1.

Negating the definition of P_0, one see that $P_0(f) = 1$ if and only if there exists $\beta < \omega_1$, there exists $\gamma < \omega_1$ so that for all $\alpha > \gamma$, there exists a club $C \subseteq \omega_1$ so that for all $g \in [C]^{\omega_1}_*$, if $(f \upharpoonright \alpha)\hat{}g \in [\omega_1]^{\omega_1}_*$, then $\Phi((f \upharpoonright \alpha)\hat{}g)(\beta) \geq g(0)$.

Let $\Psi_0 : [D_0]^{\omega_1}_* \to \omega_1$ be defined by letting $\Psi_0(f)$ be the least β witnessing the first existential quantifier in the definition of $P_0(f) = 1$. By Theorem 4.5, there is a club $D_0' \subseteq D_0$ so that $\Psi_0 \upharpoonright [D_0']^{\omega_1}_*$ is continuous.

Define $\Psi_1 : [D_0']^{\omega_1}_* \to \omega_1$ by $\Psi_1(f)$ is the least γ witnessing the second existential quantifier in the definition of $P_0(f) = 1$ for $\beta = \Psi_0(f)$. Again by Theorem 4.5, there is a club $D_1 \subseteq D_0'$ so that $\Psi_1 \upharpoonright [D_1]^{\omega_1}_*$ is continuous. (Note that $\Psi_0 \upharpoonright [D_1]^{\omega_1}_*$ is also continuous.)

Now take an $h \in [D_1]^{\omega_1}_*$. Let $\hat{\beta} = \Psi_0(h)$ and $\hat{\gamma} = \Psi_1(h)$. By the continuity of $\Psi_0 \upharpoonright [D_1]^{\omega_1}_*$ and $\Psi_1 \upharpoonright [D_1]^{\omega_1}_*$, there is $\zeta < \omega_1$ so that for all $g \in [D_1]^{\omega_1}_*$, if $g \upharpoonright \zeta = h \upharpoonright \zeta$, then $\Psi_0(g) = \Psi_0(h) = \hat{\beta}$ and $\Psi_1(g) = \Psi_1(h) = \hat{\gamma}$.

Let $\xi = \max\{\zeta, \hat{\gamma}\}$. Let $\sigma = h \upharpoonright \xi$. Note that for all $g \in [D_1]^{\omega_1}_*$ so that $\sigma\hat{}g \in [D_1]^{\omega_1}_*$, one has that $\Psi_0(\sigma\hat{}g) = \hat{\beta}$ and $\Psi_1(\sigma\hat{}g) = \hat{\gamma}$.

Define $\Lambda : [D_1 \setminus (\sup\sigma + 1)]^{\omega_1}_* \to \omega_1$ by $\Lambda(f) = \Phi(\sigma\hat{}f)(\hat{\beta})$. Observe that Λ has the property that for all α and $f \in [D_1 \setminus (\sup\sigma + 1)]^{\omega_1}_*$, there is a club $C \subseteq \omega_1$ so that for all $g \in [C]^{\omega_1}_*$, if $(f \upharpoonright \alpha)\hat{}g \in [\omega_1]^{\omega_1}_*$, then $\Lambda((f \upharpoonright \alpha)\hat{}g) \geq g(0)$. Such a function can not exist by Lemma 5.2. Claim 1 has been shown.

Thus D_0 is homogeneous for P_0 taking value 0.

For each $\beta < \omega_1$, let K_β be the collection of $\sigma \in [D_0]^{<\omega_1}_*$ so that for all club $C \subseteq \omega_1$, there exists a $g \in [C]^{\omega_1}_*$ so that $\sigma\hat{}g \in [\omega_1]^{\omega_1}_*$ and $\Phi(\sigma\hat{}g)(\beta) < g(0)$.

Note that for all $\beta < \omega_1$, K_β is dense in $[D_0]^{\omega_1}_*$, which means that for all $f \in [D_0]^{\omega_1}_*$, for all $\gamma < \omega_1$, there exists an $\alpha > \gamma$ with $f \upharpoonright \alpha \in K_\beta$. To see this: for any $f \in [D_0]^{\omega_1}_*$ and $\gamma < \omega_1$, $P_0(f) = 0$ implies there exists some $\alpha > \gamma$ so that for all club $C \subseteq \omega_1$, there exists a $g \in [C]^{\omega_1}_*$ with $f \upharpoonright \alpha\hat{}g \in [\omega_1]^{\omega_1}_*$ and $\Phi((f \upharpoonright \alpha)\hat{}g)(\beta) < g(0)$. This α would suffice.

For each $\beta < \omega_1$ and $\sigma \in K_\beta$, define the partition $Q^\beta_\sigma : [\omega_1 \setminus (\sup\sigma + 1)]^{\omega_1}_* \to 2$ by $Q^\beta_\sigma(g) = 0$ if and only if $\Phi(\sigma\hat{}g)(\beta) < g(0)$. By $\omega_1 \to (\omega_1)^{\omega_1}_*$, there is a club $E^\beta_\sigma \subseteq \omega_1$ which is homogeneous for Q^β_σ. By definition of $\sigma \in K_\beta$, there is a $g \in [E^\beta_\sigma]$ so that $\sigma\hat{}g \in [\omega_1]^{\omega_1}_*$ and $\Phi(\sigma\hat{}g) < g(0)$. Thus E^β_σ is homogeneous for Q^β_σ taking value 0. Define $\Phi^\beta_\sigma : [E^\beta_\sigma \setminus (\sup\sigma + 1)]^{\omega_1}_* \to \omega_1$ by $\Phi^\beta_\sigma(g) = \Phi(\sigma\hat{}g)(\beta)$. Since E^β_σ is homogeneous for

Q^β_σ taking value 0, one has that for all $g \in [E^\beta_\sigma \setminus (\sup(\sigma)+1)]^{\omega_1}_*$, $\Phi^\beta_\sigma(g) < g(0)$. By Lemma 4.1, there is a club $F^\beta_\sigma \subseteq E^\beta_\sigma$ and a $c^\beta_\sigma < \omega_1$ so that for all $g \in [F^\beta_\sigma]^{\omega_1}_*$, $\Phi^\beta_\sigma(g) = c^\beta_\sigma$. (Note that c^β_σ does not depend on the choice of clubs E^β_σ or F^β_σ.)

For each $f \in [D_0]^{\omega_1}_*$, define a strictly increasing sequence $\langle \alpha^\beta_f : \beta < \omega_1\rangle$ by recursion as follows: Let α^0_f be the least α so that $f \restriction \alpha \in K_0$, which exists by the density of K_0 in $[D_0]^{\omega_1}_*$. Suppose $\beta < \omega_1$ and for all $\gamma < \beta$, α^γ_f has been defined with the property that $f \restriction \alpha^\gamma_f \in K_\gamma$. Let $\xi = \sup\{\alpha^\gamma_f : \gamma < \beta\}$. Let α^β_f be the least $\alpha > \xi$ so that $f \restriction \alpha \in K_\beta$, which exists by the density of K_β in $[D_0]^{\omega_1}_*$.

Note that the map $f \mapsto \langle \alpha^\beta_f : \beta < \omega_1\rangle$ is continuous in the sense that for any $f \in [D_0]^{\omega_1}_*$, any $\gamma < \omega_1$, and for all $g \in [D_0]^{\omega_1}_*$, if $g \restriction \alpha^\gamma_f = f \restriction \alpha^\gamma_f$, then $\alpha^\beta_f = \alpha^\beta_g$ for all $\beta \leq \gamma$.

Define $\Gamma : [D_0]^{\omega_1}_* \to {}^{\omega_1}\omega_1$ by $\Gamma(f)(\beta) = c^\beta_{f \restriction \alpha^f_\beta}$. Pick any $f \in [D_0]^{\omega_1}_*$ and $\gamma < \omega_1$. As observed above, for all g so that $g \restriction \alpha^\gamma_f = f \restriction \alpha^\gamma_f$, one has that $\langle \alpha^\beta_g : \beta \leq \gamma\rangle = \langle \alpha^\beta_f : \beta \leq \gamma\rangle$. Hence $\Gamma(f) \restriction \gamma+1 = \Gamma(g) \restriction \gamma+1$ for all g so that $g \restriction \alpha^\gamma_f = f \restriction \alpha^\gamma_f$. This shows that $\Gamma : [D_0]^{\omega_1}_* \to {}^{\omega_1}\omega_1$ is continuous.

Claim 2 : There is a club $D_2 \subseteq \omega_1$ so that $\Phi \restriction [D_2]^{\omega_1}_* = \Gamma \restriction [D_2]^{\omega_1}_*$.

To see Claim 2: Define a partition $P_1 : [D_0]^{\omega_1}_* \to 2$ by $P_1(f) = 0$ if and only if $\Gamma(f) = \Phi(f)$.

By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there exists a club $D_2 \subseteq D_0$ which is homogeneous for P_1.

Define a relation $R \subseteq \omega_1 \times [D_2]^{\omega_1}_* \times \mathsf{club}_{\omega_1}$ by $R(\beta, \sigma, C)$ if and only $\sigma \in K_\beta$ and Φ^β_σ is constant on $[C]^{\omega_1}_*$ taking value c^β_σ. Note that the domain of R is $Y = \{(\beta,\sigma) : \sigma \in K_\beta\}$. R is $\subseteq$-closed in the club_{ω_1}-coordinate. By Theorem 3.11, there is a club $D_3 \subseteq D_2$ and a uniformization function $\Sigma : Z \to \mathrm{club}_{\omega_1}$ so that for all $(\beta,\sigma) \in Z$, $R(\beta,\sigma,\Sigma(\beta,\sigma))$, where $Z = \{(\beta,\sigma) : \beta \in \omega_1 \wedge \sigma \in K_\beta \cap [D_3]^{<\omega_1}_*\}$.

If $C \subseteq \omega_1$ is a club, then let $p_C \in [C]^{\omega_1}_*$ be defined by $\rho_C(\alpha) = \mathsf{enum}_C(\omega \cdot (\alpha+1))$. p_C can be regarded as the canonical correct type function passing through the club C.

One will construct by recursion a function $h \in [D_3]^{\omega_1}_*$, an increasing sequence of ordinals $\langle \gamma_\delta : \delta < \omega_1\rangle$, and a sequence of clubs $\langle F_\delta : \delta < \omega_1\rangle$.

Let $g_0 = p_{D_3}$. Note that $g_0 \in [D_3]^{\omega_1}_*$. Let $\gamma_0 = \alpha^0_{g_0}$. Define $h \restriction \gamma_0 = g_0 \restriction \gamma_0$. Note that $h \restriction \gamma_0 \in K_0$ and therefore $(0, h \restriction \gamma_0) \in Z$. Let $F_0 = \Sigma(0, h \restriction \gamma_0)$. Note that for any $h' \supseteq h \restriction \gamma_0$, one has that $\alpha^0_{h'} = \gamma_0$.

Suppose γ_β, $h \restriction \gamma_\beta$, and F_β have been defined for all $\beta < \delta$. Suppose it has also been shown that for all $\beta < \delta$, for all $h' \in [D_3]^{\omega_1}_*$ such that $h' \supseteq h \restriction \gamma_\beta$, one has that $\alpha^\beta_{h'} = \gamma_\beta$. Let $\xi_\delta = \sup\{\gamma_\beta : \beta < \delta\}$. Let $E_\delta = \bigcap_{\beta<\delta} F_\beta$. Let $G_\delta = E_\delta \setminus (\sup(h \restriction \xi_\delta)+1)$. Let $g_\delta = h \restriction \xi_\delta \hat{\ } p_{G_\delta}$. Note that $g_\delta \in [D_3]^{\omega_1}_*$. Let $\gamma_\delta = \alpha^\delta_{g_\delta}$. Let $h \restriction \gamma_\delta = g_\delta \restriction \gamma_\delta$. Since $h \restriction \gamma_\beta \subseteq g_\delta$ for all $\beta < \delta$, one has that $\gamma_\delta > \gamma_\beta$ for all $\beta < \delta$. Note that for all $h' \supseteq h \restriction \gamma_\delta$, $\alpha^\delta_{h'} = \gamma_\delta$. Also $h \restriction \gamma_\delta \in K_\delta$ and therefore $(\delta, h \restriction \gamma_\delta) \in Z$. Let $F_\delta = \Sigma(\delta, h \restriction \gamma_\delta) \cap \bigcap_{\beta<\delta} F_\beta$.

This completes the construction. Note that $h \in [D_3]^{\omega_1}_*$. By construction, $\langle \gamma_\delta : \delta < \omega_1\rangle = \langle \alpha^\delta_h : \delta < \omega_1\rangle$. Fix any $\delta < \omega_1$. Also due to the construction, $\mathsf{drop}(h, \alpha^\delta_h) \in [F_\delta]^{\omega_1}_* \subseteq [\Sigma(\delta, h \restriction \alpha^\delta_h)]^{\omega_1}_*$. Since Σ is a uniformization for R, one has that $\Phi(h)(\delta) = \Phi(h \restriction \alpha^\delta_h \hat{\ } \mathsf{drop}(h, \alpha^\delta_h))(\delta) = \Phi^\delta_{h \restriction \alpha^\delta_h}(\mathsf{drop}(h, \alpha^\delta_h)) = c^\delta_{h \restriction \alpha^\delta_h} = \Gamma(h)(\delta)$. Since δ was arbitrary, $\Phi(h) = \Gamma(h)$. Thus $P_1(h) = 0$. Since D_3 was homogeneous for P_1 and $h \in [D_3]^{\omega_1}_*$, D_3 is homogeneous for P_1 taking value 0. Thus $\Phi \restriction [D_3]^{\omega_1}_* = \Gamma \restriction [D_3]^{\omega_1}_*$. Φ is continuous on $[D_3]^{\omega_1}_*$. This completes the proof. ☐

6. Failure of Continuity Property at ω_2

A natural question would be whether the continuity phenomenon occurs at ω_2. That is, for every function $\Phi : [\omega_2]^{\omega_2}_* \to \omega_2$, is there is a club $C \subseteq \omega_2$ so that $\Phi \restriction [C]^{\omega_2}_*$ is continuous?

Fact 6.1. *(Martin) For all $\alpha < \omega_2$, $\omega_2 \to_* (\omega_2)^\alpha_2$. That is, ω_2 is a weak partition cardinal.*

(Martin and Paris) The partition relation $\omega_2 \to_* (\omega_2)^{\omega_2}_2$ fails.

The strong partition property for ω_1 played an essential role in establishing the continuity property for functions $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$. The failure of the strong partition property at ω_2 seems to suggest that one should use a counterexample to the strong partition property as a counterexample to the continuity property. However, it is not clear if the fact that a function $P : [\omega_2]^{\omega_2} \to 2$ has no club homogeneous set alone can imply the failure of the continuity property. For this, one needs to analyze explicit counterexamples to the strong partition property at ω_2.

The proof of the Martin-Paris theorem roughly shows that if the partition relation $\omega_2 \to (\omega_2)^{\omega_2}_2$ holds, then $\omega_3 \to (\omega_3)^{\alpha}_2$ holds for each $\alpha < \omega_1$. The partition relation $\omega_3 \to (\omega_3)^2_2$ already implies that ω_3 is regular. However, AD proves that $\mathrm{cof}(\omega_3) = \omega_2$. See [8] Section 13.

The second author produced an explicit example of the failure of the strong partition property at ω_2. Its proof gives some additional property that will show this function also fails to have the continuity property. The proof of the following theorem requires an analysis of the ultrapower $\prod_{\omega_1} \omega_1/\mu$, where μ is the club measure on ω_1, the Kunen tree, and Kunen functions for functions of the form $f : \omega_1 \to \omega_1$.

Let $\mathrm{ult}(V, \mu)$ denote the sets of the form $[h]_\mu$ where $h : \omega_1 \to V$. It can be shown that if $[h']_\mu$ represents a subset of ω_2, then $[h']_\mu = [h]_\mu$ where $h : \omega_1 \to \mathscr{P}(\omega_1)$. Thus to say that $A \subseteq \omega_2$ belongs to $\mathrm{ult}(V, \mu)$ means there is some $h : \omega_1 \to \mathscr{P}(\omega_1)$ so that $A = [h]_\mu$.

One can also show that $\mathrm{ult}(V, \mu)$ is necessarily missing a subset of ω_2. For instance, $\{\alpha \in \omega_2 : \mathrm{cof}(\alpha) = \omega_1\}$ does not belong to $\mathrm{ult}(V, \mu)$. This fact can be used to show that this ultrapower may not satisfy Łoś's theorem and in fact may not be a model of ZF. These results are essential ingredients of the proof of the follow theorem. The background material and proof of the results mentioned above and the following theorem are given in [2] Section 6.

Theorem 6.2. *(Jackson) Let μ denote the club measure on ω_1. Let $P : [\omega_2]^{\omega_2}_* \to 2$ be defined by $P(f) = 0$ if and only if $\mathrm{rang}(f) \in \mathrm{ult}(V, \mu)$.*

Then there are no club $C \subseteq \omega_2$ and no $i \in 2$ so that for all $f \in [C]^{\omega_2}_*$, $P(f) = i$.

As a corollary of the proof of this result, one also has

Corollary 6.3. *Let P denote the function from Theorem 6.2. Let $\sigma \in [\omega_2]^{<\omega_2}_*$. Define $P_\sigma : [\omega_2 \setminus (\sup(\sigma) + \omega)]^{\omega_2}_* \to 2$ by $P_\sigma(f) = 0$ if and only if $P(\sigma\hat{\ } f) = 0$.*

Then there are no club $C \subseteq \omega_2$ and no $i \in 2$ so that for all $f \in [C]^{\omega_2}_*$, $P_\sigma(f) = i$.

With these results, one can show that P is not continuous on $[C]^{\omega_2}_*$ for any club $C \subseteq \omega_2$.

Theorem 6.4. *Let $P : [\omega_2]^{\omega_2}_* \to 2$ be the function from Theorem 6.2. Then there is no club $C \subseteq \omega_2$ so that $P \restriction [C]^{\omega_2}_*$ is continuous.*

Proof. Suppose there was a club C so that $P \restriction [C]^{\omega_2}_*$ is continuous. Take any $f \in [C]^{\omega_2}_*$. Without loss of generality, say that $P(f) = 0$. By continuity, there is some $\zeta < \omega_2$ so that for all $g \in [C]^{\omega_2}_*$ with $g \restriction \zeta = f \restriction \zeta$, $P(f) = P(g)$. Let $\sigma = f \restriction \zeta$. This would means that $C \setminus (\sup(\sigma) + \omega)$ would be a club homogeneous for P_σ. This contradicts Corollary 6.3. ☐

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