

# MORE DEFINABLE COMBINATORICS AROUND THE FIRST AND SECOND UNCOUNTABLE CARDINAL

WILLIAM CHAN, STEPHEN JACKSON, AND NAM TRANG

ABSTRACT. Assume  $\mathbf{ZF} + \mathbf{AD}$ . The following two continuity results for functions on certain subsets of  $\mathcal{P}(\omega_1)$  and  $\mathcal{P}(\omega_2)$  will be shown:

For every  $\epsilon < \omega_1$  and function  $\Phi : [\omega_1]^\epsilon \rightarrow \omega_1$ , there is a club  $C \subseteq \omega_1$  and a  $\zeta < \epsilon$  so that for all  $f, g \in [C]^\zeta$ , if  $f \restriction \zeta = g \restriction \zeta$  and  $\sup(f) = \sup(g)$ , then  $\Phi(f) = \Phi(g)$ .

For every  $\epsilon < \omega_2$  and function  $\Phi : [\omega_2]^\epsilon \rightarrow \omega_2$ , there is an  $\omega$ -club  $C \subseteq \omega_2$  and a  $\zeta < \epsilon$  so that for all  $f, g \in [C]^\zeta$ , if  $f \restriction \zeta = g \restriction \zeta$  and  $\sup(f) = \sup(g)$ , then  $\Phi(f) = \Phi(g)$ .

The previous two continuity results will be used to distinguish cardinals below  $\mathcal{P}(\omega_2)$ :  $|\omega_1|^\omega < |\omega_1|^{<\omega_1}$ ,  $|\omega_2|^\omega < |\omega_2|^{<\omega_1} < |\omega_2|^{\omega_1} < |\omega_2|^{<\omega_2}$ .  $\neg(|\omega_1|^{<\omega_1} \leq |\omega_2|^\omega)$ .  $\neg(|\omega_1|^{\omega_1} \leq |\omega_2|^{<\omega_1})$ .

$[\omega_1]^\omega$  has the Jónsson property: That is, for every  $\Phi : {}^{<\omega}([\omega_1]^\omega) \rightarrow [\omega_1]^\omega$ , there is an  $X \subseteq [\omega_1]^\omega$  with  $|X| = |\omega_1|^\omega$  so that  $\Phi[{}^{<\omega}X] \neq [\omega_1]^\omega$ .

## 1. INTRODUCTION

The axiom of determinacy,  $\mathbf{AD}$ , asserts that every two player game where each player takes turns playing a natural number has a winning strategy for one of the two players. The axiom has proved to be a very robust general framework for extending results from classical descriptive set theory and for providing structure to sets which are surjective images of  $\mathbb{R}$ .

Let  $A$  and  $B$  be two sets. If there is an injection from  $A$  into  $B$  then one write  $|A| \leq |B|$ . Denote  $|A| < |B|$  if  $|A| \leq |B|$  but  $\neg(|B| \leq |A|)$ . If there is a bijection between  $A$  and  $B$ , then one writes  $|A| = |B|$ . By the Cantor-Schröder-Bernstein theorem (proved in  $\mathbf{ZF}$ ),  $|A| = |B|$  if and only  $|A| \leq |B|$  and  $|B| \leq |A|$ . In the absence of choice, the cardinality of  $A$ , refer to as  $|A|$ , is the equivalence class of  $A$  under the bijection relation.

One classical structural result of determinacy is that every subset of  $\mathbb{R}$  is either countable or contains a perfect set. Hence  $|\mathbb{R}|$  is the only uncountable cardinal less than or equal to  $|\mathbb{R}|$  assuming  $\mathbf{AD}$ .

Another classical descriptive set theoretic result assuming  $\mathbf{AD}$  is that every function  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  is continuous on a comeager set. As customary in descriptive set theory,  $\mathbb{R}$  is identified with  ${}^\omega\omega$ , which is the collection of functions from  $\omega$  into  $\omega$ . Continuity here can be interpreted using the following example:  $\Phi(f)(0)$ , the first bit of  $\Phi(f)$ , a priori could require global information about all of  $f$ . Continuity on a comeager set states that if  $f$  belongs to this comeager set, then  $\Phi(f)(0)$  only depends on a local behavior of  $f$ . That is, there is some  $n \in \omega$  so that for all  $g$  which belong to this appropriate comeager set, if  $g \restriction n = f \restriction n$ , then  $\Phi(g)(0) = \Phi(f)(0)$ .

This paper seeks to investigate the continuity phenomenon for functions defined on certain subsets of  $\mathcal{P}(\omega_1)$  and  $\mathcal{P}(\omega_2)$ . These continuity properties will then be used to distinguish several very concrete cardinals below  $|\mathcal{P}(\omega_1)|$  and  $|\mathcal{P}(\omega_2)|$  under  $\mathbf{AD}$ .

Specifically, this paper will be interested in the collection of subsets of  $\omega_1$  and  $\omega_2$  which have cardinality less than  $\omega_1$  and  $\omega_2$ , respectively. Identifying subsets of  $\omega_1$  or  $\omega_2$  by their increasing enumeration, one will prefer to work with the collection of increasing sequences through  $\omega_1$  and  $\omega_2$  (primarily because the partition properties are formulated for these sets). If  $X$  and  $Y$  are wellordered set, then  $[X]^Y$  is the collection of functions  $f : Y \rightarrow X$  which are increasing with respect to the two orderings. This paper will be particular interested in  $[\omega_1]^\omega$ ,  $[\omega_1]^{<\omega_1}$ ,  $[\omega_2]^\omega$ ,  $[\omega_2]^{\omega_1}$  and  $[\omega_1]^{<\omega_2}$ .

Thus this paper will study the short functions on  $\omega_1$  and  $\omega_2$ . The combinatorics of full functions on  $\omega_1$  is investigated in [4], and the techniques there are quite different than what is used here: for instance, the

---

August 30, 2019. The first author was supported by NSF grant DMS-1703708. The second author was supported by NSF grant DMS-1800323. The third author was supported by NSF grant DMS-1855757.

generic coding at  $\omega_1$ , category arguments, Moschovakis coding lemma, the good coding system for  ${}^{\omega_1}\omega_1$ , and projective uniformization are used. Chan and Jackson [4] showed that for every function  $\Phi : [\omega_1]^{\omega_1} \rightarrow \omega_1$ , there is a club  $C \subseteq \omega_1$  so that for all  $f \in [C]_*^{\omega_1}$ , there exists an  $\alpha < \omega_1$  so that for all  $g \in [C]_*^{\omega_1}$ , if  $g \restriction \alpha = f \restriction \alpha$ , then  $\Phi(f) = \Phi(g)$ . Chan, Jackson, and Trang showed an even stronger version that for every function  $\Phi : [\omega_1]^{\omega_1} \rightarrow {}^{\omega_1}\omega_1$ , there is a club  $C \subseteq \omega_1$  so that for all  $f \in [C]_*^{\omega_1}$  and  $\beta < \omega_1$ , there exists an  $\alpha < \omega_1$  so that for all  $g \in [C]_*^{\omega_1}$ , if  $g \restriction \alpha = f \restriction \beta$ ,  $\Phi(g) \restriction \beta = \Phi(f) \restriction \beta$ . Note that the second continuity property is simply the traditional notion of continuity where the domain and range space is given the topology generated by set of the form  $N_\sigma = \{f \in [\omega_1]^{\omega_1} : \sigma \subseteq f\}$  where  $\sigma \in [\omega_1]^{<\omega_1}$  (or  $N_\sigma = \{f \in {}^{\omega_1}\omega_1 : \sigma \subseteq f\}$  where  $\sigma \in {}^{<\omega_1}\omega_1$ ) as a basis.

As a consequence of Martin's result that  $\omega_1$  is a strong partition cardinal, the filter  $\mu_{\omega_1}$  on  $[\omega_1]_1^\omega$  defined by  $X \in \mu_{\omega_1}$  if and only if there exists a club  $C \subseteq \omega_1$  so that  $[C]_*^{\omega_1} \subseteq X$  is a countably complete measure on  $\omega_1$ . Thus in the above two continuity results, the notion of largeness given by comeagerness for classical continuity on  $\mathbb{R}$  is replaced with largeness on  $[\omega_1]^{\omega_1}$  given by the ultrafilter  $\mu_{\omega_1}$ . The above continuity property can be used to show that if  $\langle X_\alpha : \alpha < \omega_1 \rangle$  is a sequence of subsets of  $[\omega_1]^{\omega_1}$  so that  $[\omega_1]^{\omega_1} = \bigcup_{\alpha < \omega_1} X_\alpha$ , then there is an  $\alpha < \omega_1$  such that  $|X_\alpha| = |[\omega_1]^{\omega_1}|$ . This result can then be used to show that  $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$ . (It should be mentioned that there is a more set theoretic argument involving the measurability of  $\omega_1$  and a certain inner model of ZFC. See the argument in Fact 3.30.)

Returning to the subject of this article: One is concerned with continuity phenomenon for functions  $\Phi : [\omega_1]^\epsilon \rightarrow \omega_1$  where  $\epsilon < \omega_1$  is countable. Again the partition measure  $\mu_\epsilon$  on  $[\omega_1]^\epsilon$  will serve as the notion of largeness for subsets of  $[\omega_1]^\epsilon$ . However, one can not have continuity in the sense described above for the full functions,  $[\omega_1]^{\omega_1}$ . Recall that  $\text{AC}_\omega^\mathbb{R}$  and the Moschovakis coding lemma imply that  $\omega_1$  is regular. Consider the function  $\Psi : [\omega_1]^\omega \rightarrow \omega_1$  defined by  $\Psi(f) = \sup(f)$ . Clearly, there is no  $n \in \omega$  and no club  $C \subseteq \omega_1$  so that whenever  $f, g \in [C]_*^\omega$ , if  $f \restriction n = g \restriction n$ , then  $\Psi(f) = \Psi(g)$ . However,  $\Psi$  does satisfy a particular continuity phenomenon in the sense that  $\Psi(f)$  depend only on one piece of information, namely  $\sup(f)$ . That is, for any  $f, g \in [\omega_1]^\omega$ , if  $\sup(f) = \sup(g)$ , then  $\Psi(f) = \Psi(g)$ . The first main result is that this is general occurrence that holds for any function  $\Phi : [\omega_1]^\epsilon \rightarrow \omega_1$  when  $\epsilon < \omega_1$ . For each  $f \in [\omega_1]^\epsilon$  and  $\alpha \leq \epsilon$ , let  $\text{bound}(f, \alpha) = \sup\{f(\beta) : \beta < \alpha\}$ . Note that  $\text{bound}(f, 0) = 0$  and  $\text{bound}(f, \epsilon) = \sup(f)$ .

**Theorem 2.14.** *Assume ZF + AD. Let  $\epsilon < \omega_1$  and  $\Phi : [\omega_1]_*^\epsilon \rightarrow \omega_1$ . Then there is a decreasing sequence of ordinals which are less than or equal to  $\epsilon$ ,  $(\beta_i : i \leq n)$ , with  $\beta_n = 0$  and a club  $C \subseteq \omega_1$  so that for any  $f, g \in [C]_*^\epsilon$  with the property that  $\text{bound}(f, \beta_i) = \text{bound}(g, \beta_i)$  for all  $i \leq n$ , then  $\Phi(f) = \Phi(g)$ .*

This result is a continuity property which states that for any such function  $\Phi$ ,  $\Phi(f)$  depends only on local behaviors of  $f$  at certain finitely many places for  $\mu_\epsilon$ -almost all  $f$ . The following is a more coarse but useful formulation of the above result which states that for every function  $\Phi$ , there is a  $\delta < \epsilon$  so that  $\Phi(f)$  depends only on the  $\delta$ -length initial segment of  $f$  and  $\sup(f)$ .

**Theorem 2.15.** *Assume ZF + AD. Let  $\epsilon < \omega_1$  and  $\Phi : [\omega_1]_*^\epsilon \rightarrow \omega_1$ . Then there is a  $\delta < \epsilon$  and some club  $C \subseteq \omega_1$  so that for all  $f, g \in [C]_*^\epsilon$  with  $f \restriction \delta = g \restriction \delta$  and  $\sup(f) = \sup(g)$ ,  $\Phi(f) = \Phi(g)$ .*

There are two distinguished subsets of  $\mathcal{P}(\omega_1)$ , namely  $[\omega_1]^\omega$  and  $[\omega_1]^{<\omega_1}$ . One natural question is whether these two sets are different in terms of cardinality. Woodin [14] studied the cardinals below  $[\omega_1]^{<\omega_1}$  under ZF + AD $_{\mathbb{R}}$  + DC. From the dichotomy results in [14], it was known to Woodin that  $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$ . Moreover, Woodin isolated a subset of  $[\omega_1]^{<\omega_1}$  called  $S_1$  defined by  $S_1 = \{f \in [\omega_1]^{<\omega_1} : \sup(f) = \omega_1^{L[f]}\}$ . It is implicit in [14] that  $|S_1|$  is incomparable with  $[\omega_1]^\omega$  and hence one can conclude that  $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$ .

The proofs of some of the main properties of  $S_1$  (assuming ZF + AD + DC $_{\mathbb{R}}$  and all sets of reals have  $\infty$ -Borel codes) can be found [3]. For instance, assuming ZF + AD, one can show that  $|\mathbb{R}| \leq |S_1|$  and  $\neg(\omega_1 \leq |S_1|)$ . The main property of  $S_1$  shown in [3] is that there is no injection of  $S_1$  into  ${}^\omega\text{ON}$  assuming ZF + AD + DC $_{\mathbb{R}}$  and all sets of reals have  $\infty$ -Borel codes. From this, one can conclude that  $|\mathbb{R}| < |S_1|$  and  $\neg(|S_1| \leq |[\omega_1]^\omega|)$ . The argument for the main property of  $S_1$  in [3] goes roughly as follows: Suppose such an injection  $\Phi$  exists. Using  $\infty$ -Borel codes, one can find an inner model  $M$  of ZFC that “absorbs” some fragment of this injection in a suitable sense. (This is where  $\infty$ -Borel codes are needed.) Let  $\zeta < \omega_1^V$  be an inaccessible cardinal of  $M$ . Since  $\text{Coll}(<\omega, \zeta)$  is countable in the real world satisfying AD, one can find a  $G \subseteq \text{Coll}(\omega, <\zeta)$  which

is  $\text{Coll}(\omega, < \zeta)$ -generic over  $M$ . One can show that  $G$  adds a  $g \in S_1$  such that  $M[G] = M[g]$ . Since  $M$  “absorbs”  $\Phi$ ,  $\Phi(g) \in M[g]$ . (Intuitively, this is what “absorb” means.) Since  $\Phi$  is an injection, one can argue that  $M[g] = M[\Phi[g]]$ . However,  $\Phi(g)$  is an  $\omega$ -sequence of ordinals. By a crucial property of the Lévy collapse, there is a  $\xi < \zeta$  so that  $\Psi(g) \in M[G \restriction \xi]$ . Then one has that  $M[G] = M[g] = M[\Phi(g)] = M[G \restriction \xi]$ . This is impossible.

The authors know very little about the cardinal properties of  $S_1$  in the absence of  $\infty$ -Borel codes.  $S_1$  is a set whose definition is based upon the notion of constructibility. The two sets  $[\omega_1]^\omega$  and  $[\omega_1]^{<\omega_1}$  are very concrete combinatorial objects. There should be no need to employ  $\text{AD}^+$  concepts to distinguish these two cardinals. Using the continuity properties for short functions mentioned above, one can distinguish these two sets within  $\text{ZF} + \text{AD}$ .

**Theorem 2.16.** *Assume  $\text{ZF} + \text{AD}$ .  $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$ .*

Next, one will consider various subsets of  $\mathcal{P}(\omega_2)$ . Of particular interests are  $[\omega_2]^\omega$ ,  $[\omega_2]^{<\omega_1}$ ,  $[\omega_2]^{\omega_1}$ ,  $[\omega_2]^{<\omega_2}$ , and  $[\omega_2]^{\omega_2}$ . One would like to distinguish the cardinality of these sets from each other as well as from the cardinality of the subsets of  $\mathcal{P}(\omega_1)$  considered earlier such as  $[\omega_1]^\omega$ ,  $[\omega_1]^{<\omega_1}$ , and  $[\omega_1]^{\omega_1}$ .

Martin showed that  $\omega_2$  is a weak partition cardinal and hence measurable. Using the same technique mentioned above (for showing  $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$ ) involving using the measure and going into an appropriate inner model of ZFC, one can show  $|[\omega_2]^{<\omega_2}| < |[\omega_2]^{\omega_2}|$  under just  $\text{ZF} + \text{AD}$ .

In analogy with the study of  $\omega_1$ , one needs to establish the analogous continuity property for  $\omega_2$ .

**Theorem 3.21.** *Assume  $\text{ZF} + \text{AD}$ . Let  $\epsilon < \omega_2$  and  $\Phi : [\omega_2]_*^\epsilon \rightarrow \omega_2$ . Then there is a decreasing sequence of ordinals less than or equal to  $\epsilon$ ,  $(\beta_i : i \leq n)$ , with  $\beta_n = 0$  and an  $\omega$ -club  $B \subseteq \omega_2$  so that for any  $\mathcal{F}, \mathcal{G} \in [B]_*^\epsilon$  with the property that  $\text{bound}(\mathcal{F}, \beta_i) = \text{bound}(\mathcal{G}, \beta_i)$  for all  $i \leq n$ , then  $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$ .*

**Theorem 3.22.** *Assume  $\text{ZF} + \text{AD}$ . Let  $\epsilon < \omega_2$  and  $\Phi : [\omega_2]_*^\epsilon \rightarrow \omega_2$ . Then there is a  $\delta < \epsilon$  and an  $\omega$ -club  $B \subseteq \omega_2$  so that for all  $\mathcal{F}, \mathcal{G} \in [B]_*^\epsilon$  with  $\mathcal{F} \restriction \delta = \mathcal{G} \restriction \delta$  and  $\text{sup}(\mathcal{F}) = \text{sup}(\mathcal{G})$ ,  $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$ .*

Using these continuity results, one can establish the following cardinal relations:

**Theorem 3.23.** *Assume  $\text{ZF} + \text{AD}$ .  $|[\omega_2]^\omega| < |[\omega_2]^{<\omega_1}|$ .*

**Theorem 3.24.** *Assume  $\text{ZF} + \text{AD}$ .  $|[\omega_2]^{<\omega_1}| < |[\omega_2]^{\omega_1}|$ .*

**Theorem 3.26.** *Assume  $\text{ZF} + \text{AD}$ .  $|[\omega_2]^{\omega_1}| < |[\omega_2]^{<\omega_2}|$ .*

It should be mentioned that these results concerning  $\omega_2$  are proved in  $\text{ZF} + \text{AD}$  and the arguments provided here are the only proofs presently known to the authors. That is, the authors do not know of an  $\text{AD}^+$  style proof involving some analog  $S_1$ . In the proof that  $S_1$  does not inject into  ${}^\omega\text{ON}$  sketched above, one considered the forcing  $\text{Coll}(\omega, < \zeta)$  where  $\zeta < \omega_1^V$  is an inaccessible cardinal of an inner model  $M$  of ZFC. Above one was able to find, in the real world, a generic over  $M$  since the forcing is countable in the real world. One may attempt to make analogs of  $S_1$  to handle results at  $\omega_2$ . However, the naturally associated forcing appears to be uncountable even in the real world, and one can no longer be certain that generics for such forcings exist in the real world.

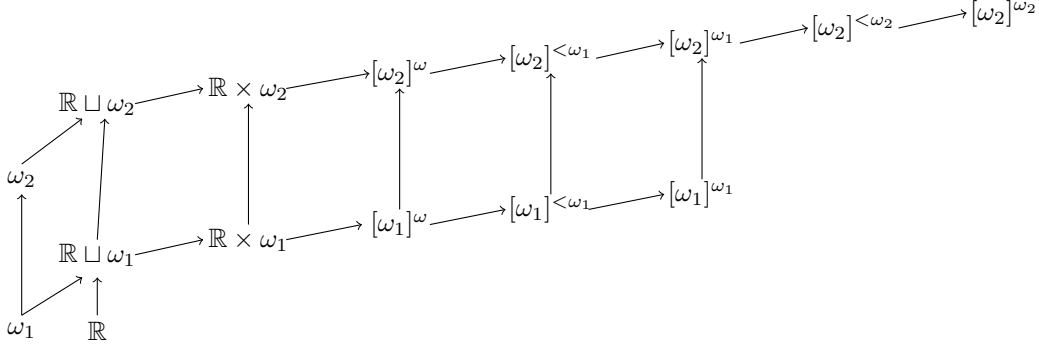
To give a more complete picture of the relations between cardinal, one also has the following result

**Theorem 3.29.** *Assume  $\text{ZF} + \text{AD}$ .  $\neg(|[\omega_1]^{<\omega_1}| \leq |[\omega_2]^\omega|)$ . Thus  $\neg(|[\omega_1]^{\omega_1}| \leq |[\omega_2]^\omega|)$ .*

Recall as mentioned above, that under  $\text{AD} + \text{DC}_{\mathbb{R}}$  and all sets of reals have  $\infty$ -Borel codes, one has that  $S_1 \subseteq [\omega_1]^{<\omega_1}$  and  $S_1$  does not inject into  ${}^\omega\text{ON}$ . Thus the previous theorem would follow from this observation under  $\text{AD}^+$ ; however, the above result is proved in just  $\text{AD}$ .

**Theorem 3.31.** *Assume  $\text{ZF} + \text{AD}$ . Then  $\neg(|[\omega_1]^{\omega_1}| \leq |[\omega_2]^{<\omega_1}|)$ .*

From the result mentioned throughout the paper, one has the following diagram depicting all the cardinal relationships between the uncountable cardinals below  $\mathcal{P}(\omega_2)$  which will be discussed in this paper. An arrow between  $A$  and  $B$  indicates  $|A| < |B|$ . All relations among these cardinals are those derivable by compositions of the arrows on the diagram. Of course, there are other cardinals below  $\mathcal{P}(\omega_2)$  which are not on the diagram, for instance  $[\omega_1]^{<\omega_1} \sqcup [\omega_2]^\omega$  and  $[\omega_1]^{\omega_1} \times [\omega_2]^{<\omega_1}$ . With additional determinacy assumptions such as  $\text{AD}^+$ , the set  $S_1$  can be proved to be distinct from all of these.



The main technique used in this paper involves Kunen functions for  $\omega_1$ . Let  $\mu$  be the club measure on  $\omega_1$ . Using the Kunen tree analysis, one can show that for any function  $f : \omega_1 \rightarrow \omega_1$ , there is a function  $\Xi : \omega_1 \times \omega_1 \rightarrow \omega_1$  so that for  $\mu$ -almost all  $\alpha$ ,  $f(\alpha) < \sup\{\Xi(\alpha, \beta) : \beta < \alpha\}$ . This  $\Xi$  will be called a Kunen function for  $f$ .  $\Xi$  allows for a uniform way of selecting a representative for any  $g <_\mu f$ , i.e. there is a  $\beta < \omega_1$  so that the function  $\Xi^\beta : \omega_1 \rightarrow \omega_1$  defined by  $\Xi^\beta(\alpha) = \Xi(\alpha, \beta)$  is  $\mu$ -almost equal to  $g$ . Using these Kunen functions and sliding arguments, Martin proved an ultrapower representation for  $\omega_2 = \prod_{\omega_1} \omega_1 / \mu$  and showed the weak partition property on  $\omega_2$ .

The basic facts about partition properties and Kunen functions can be found in [1]. These arguments are well known and due to Jackson, Kunen, and Martin. (See [10], [11], and [12].) However, the article will follow [1] which develop the minimal notation and theory necessary for the results in this paper.

The final section of this paper concerns a simple combinatorial property of sets called the Jónsson property. Let  $X$  be a set. Let  $[X]_\equiv^n = \{f \in {}^n X : (\forall i < j < n)(f(i) \neq f(j))\}$ . Let  $[X]_\equiv^{\leq \omega} = \bigcup_{n \in \omega} [X]_\equiv^n$ .  $X$  is  $n$ -Jónsson if and only if for every  $\Phi : [X]_\equiv^n \rightarrow X$ , there exists a  $Y \subseteq X$  with  $|Y| = |X|$  and  $\Phi[[Y]_\equiv^n] \neq X$ .  $X$  is Jónsson if and only if for every  $\Phi : [X]_\equiv^{\leq \omega} \rightarrow X$ , there is a  $Y \subseteq X$  with  $|Y| = |X|$  and  $\Phi[[Y]_\equiv^{\leq \omega}] \neq X$ .

Under  $\text{AD}$ , Kleinberg [13] showed that  $\omega_n$  is Jónsson for all  $n \in \omega$ . Jackson, Ketchersid, Schlutzenberg, and Woodin [9] showed that under  $\text{ZF} + \text{AD} + \text{V} = \text{L}(\mathbb{R})$  (and also  $\text{ZF} + \text{AD}^+$ ) that every cardinal  $\kappa < \Theta$  is Jónsson. Holshouser and Jackson showed that  $\mathbb{R}$  and  $\mathbb{R} \times \kappa$  of  $\kappa < \Theta$  are Jónsson. Chan [2] showed in fact that for all ordinals  $\kappa$ ,  $\mathbb{R} \times \kappa$  is Jónsson.

Define an equivalence relation  $E_0$  on  ${}^\omega 2$  by  $x E_0 y$  if and only if  $(\exists m)(\forall n \geq m)(x(n) = y(n))$ . The Hjorth  $E_0$ -dichotomy [8] (which generalize the  $E_0$ -dichotomy of Harrington, Kechris, and Louveau [7]) shows that  $E_0$  is the minimal cardinal (which is a surjective image of  $\mathbb{R}$ ) that does not inject into  $\mathcal{P}(\alpha)$  for any  $\alpha < \text{ON}$  under  $\text{AD}^+$ . Holshouser and Jackson showed that  ${}^\omega 2 / E_0$  is 2-Jónsson. Chan and Meehan [6] showed that  ${}^\omega 2 / E_0$  is not 3-Jónsson and hence not Jónsson.

Say an equivalence relation  $E$  on  $\mathbb{R}$  is smooth if and only if  $|\mathbb{R}/E| = |\mathbb{R}|$ . If  $\langle E_\alpha : \alpha < \kappa \rangle$  for some cardinal  $\kappa$  be a sequence of equivalence relations on  $\mathbb{R}$ , let  $\bigsqcup_{\alpha < \kappa} \mathbb{R}/E_\alpha$  be the disjoint union of the various quotients. Chan and Jackson [5] showed under  $\text{ZF} + \text{AD}^+ + \text{V} = \text{L}(\mathcal{P}(\mathbb{R}))$ , if  $\langle E_\alpha : \alpha < \kappa \rangle$  (for any cardinal  $\kappa$ ) is a sequence of smooth equivalence relations on  $\mathbb{R}$  with all classes countable, then  $\bigsqcup_{\alpha < \kappa} \mathbb{R}/E_\alpha$  must be in bijection with  $\mathbb{R} \times \kappa$  and therefore has the Jónsson property by the result of Chan [2] mentioned above.

The final result of this paper shows  $[\omega_1]^\omega$  has the Jónsson property:

**Theorem 4.12.** *Assume  $\text{ZF} + \text{AD}$ .  $[\omega_1]^\omega$  is Jónsson.*

## 2. CONTINUITY OF SHORT FUNCTIONS ON $\omega_1$

For the rest of the paper, assume  $\mathbf{ZF} + \mathbf{AD}$ . (Not even  $\mathbf{DC}_{\mathbb{R}}$  will implicitly assumed.)

If  $\epsilon \leq \kappa$  are ordinals, then  $[\kappa]^\epsilon$  is the collection of increasing functions  $f : \epsilon \rightarrow \kappa$ .

**Definition 2.1.** ([11]) Let  $\kappa$  be an ordinal and  $\epsilon \leq \kappa$ . A function  $f : \epsilon \rightarrow \kappa$  has uniform cofinality  $\omega$  if and only if there is a function  $g : \epsilon \times \omega \rightarrow \kappa$  with the following two properties:

(a) For all  $\alpha < \epsilon$  and  $n \in \omega$ ,  $g(\alpha, n) < g(\alpha, n+1)$ .

(b) For all  $\alpha < \epsilon$ ,  $f(\alpha) = \sup\{g(\alpha, n) : n \in \omega\}$ .

A function  $f : \epsilon \rightarrow \kappa$  is discontinuous at  $\alpha$  if and only if  $f(\alpha) > \sup\{f(\beta) : \beta < \alpha\}$ .

A function  $f : \epsilon \rightarrow \kappa$  is of the correct type if and only if  $f$  has uniform cofinality  $\omega$  and  $f$  is discontinuous everywhere.

Let  $A \subseteq \kappa$ ,  $[A]_*^\epsilon$  denote the collection of all increasing functions  $f : \epsilon \rightarrow A$  of the correct type.

The collection of increasing functions and the collection of increasing functions of the correct type have the same cardinality. In the following, one may use either sets for purpose of distinguishing cardinality.

**Fact 2.2.** Let  $\kappa$  be a cardinal. Let  $\epsilon \leq \kappa$ .  $[\kappa]^\epsilon \approx [\kappa]_*^\epsilon$ .

*Proof.* Let  $H : \kappa \rightarrow \kappa$  be any increasing function of the correct type. Define  $\Phi : [\kappa]^\epsilon \rightarrow [\kappa]_*^\epsilon$  by  $\Phi(f) = H \circ f$ . Then  $\Phi$  is an injection. The two sets are in bijection by the Cantor-Schröder-Bernstein theorem.  $\square$

**Definition 2.3.** Let  $\kappa$  be an ordinal and  $\epsilon \leq \kappa$ . One write  $\kappa \rightarrow_* (\kappa)_2^\epsilon$  to indicate that for every  $P : [\kappa]_*^\epsilon \rightarrow 2$ , there is some club  $C \subseteq \omega_1$  and an  $i \in 2$  so that for all  $f \in [C]_*^\epsilon$ ,  $\Phi(f) = i$ .

If  $\kappa \rightarrow_* (\kappa)_2^\epsilon$ , then one says that  $\kappa$  is a strong partition cardinal.

If  $\kappa \rightarrow_* (\kappa)_2^\epsilon$  for all  $\epsilon < \kappa$ , then  $\kappa$  is said to be a weak partition cardinal.

**Fact 2.4.** ([1] Section 2 and 4, [13] Chapter II, [12] Theorem 7.3 and 12.2.) (Solovay) The club measure  $\mu$  on  $\omega_1$  is a countably complete normal measure on  $\omega_1$ . (Martin)  $\omega_1$  is a strong partition cardinal.

**Definition 2.5.** Let  $\mu$  denote the club measure on  $\omega_1$ . For each  $\epsilon \leq \omega_1$ , let  $\mu^\epsilon$  be a filter on  $[\omega_1]_*^\epsilon$  defined by  $K \in \mu^\epsilon$  if and only if there is a club  $C \subseteq \omega_1$  so that  $[C]_*^\epsilon \subseteq K$ . Since  $\omega_1$  is a strong partition cardinal, one has that  $\mu^\epsilon$  is a countably complete ultrafilter for all  $\epsilon \leq \omega_1$ .

If  $\varphi$  is a formula, then one write  $(\forall_\epsilon^* f)\varphi(f)$  to indicate that the set  $\{f \in [\omega_1]_*^\epsilon : \varphi(f)\} \in \mu^\epsilon$ .

**Definition 2.6.** ([1] Section 5) Let  $\mu$  be a club measure on  $\omega_1$ .

Let  $\Xi : \omega_1 \times \omega_1 \rightarrow \omega_1$ . For each  $\alpha < \omega_1$ , let  $\delta_\alpha^\Xi = \sup\{\Xi(\alpha, \beta) : \beta < \alpha\}$ . Let  $\Xi_\alpha : \alpha \rightarrow \delta_\alpha^\Xi$  be defined by  $\Xi_\alpha(\beta) = \Xi(\alpha, \beta)$ .

$\Xi$  is a Kunen function for  $f$  with respect to  $\mu$  if and only if  $K_f^\Xi = \{\alpha < \omega_1 : f(\alpha) \leq \delta_\alpha^\Xi \wedge \Xi_\alpha \text{ is a surjection}\} \in \mu$ .  $K_f^\Xi$  is the set of  $\alpha$  on which  $\Xi$  provides a bounding for  $f$ .

For  $\beta < \omega_1$ , let  $\Xi^\beta : \omega_1 \rightarrow \omega_1$  be defined by  $\Xi^\beta(\alpha) = \Xi(\alpha, \beta)$  where  $\alpha > \beta$  and 0 otherwise.

**Fact 2.7.** ([1] Section 5, [11] Lemma 4.1) (Kunen) For every  $f : \omega_1 \rightarrow \omega_1$ , there is a Kunen function for  $f$  with respect to  $\mu$ .

**Definition 2.8.** Let  $\beta \leq \epsilon < \omega_1$  and  $f \in [\omega_1]_*^\epsilon$ . Let  $\text{bound}(f, \beta) = \sup\{f(\alpha) : \alpha < \beta\}$ , where  $\sup(\emptyset)$  is defined to be 0.

If  $A \subseteq \omega_1$  with  $|A| = \omega_1$ , then let  $\text{enum}_A : \omega_1 \rightarrow A$  denote the increasing enumeration of  $A$ .

Let  $C \subseteq \omega_1$  be a club. Let  $\text{next}_C^\omega(\alpha)$  denote  $\omega^{\text{th}}$  element of  $C$  above  $\alpha$ .

**Fact 2.9.** Let  $\epsilon < \omega_1$ . For all  $\Phi : [\omega]_*^\epsilon \rightarrow \omega_1$ , there exists a unique  $\mathfrak{b}_\Phi \leq \epsilon$  so that  $\mathfrak{b}_\Phi$  is the largest  $\beta \leq \epsilon$  so  $(\forall_\epsilon^* f)(\text{bound}(f, \beta) \leq \Phi(f))$ .

*Proof.* For each  $\beta \leq \epsilon < \omega_1$ , let  $A_\beta$  be the set of  $f$  so that  $\beta$  is the largest  $\gamma \leq \epsilon$  so that  $\Phi(f) \geq \text{bound}(f, \gamma)$ .  $[\omega_1]_*^\epsilon = \bigcup_{\beta \leq \epsilon} A_\beta$ . Since  $\mu^\epsilon$  is a countably complete ultrafilter on  $[\omega_1]_*^\epsilon$ , there is a  $\mathfrak{b}_\Phi$  so that  $A_{\mathfrak{b}_\Phi} \in \mu^\epsilon$ .  $\square$

**Lemma 2.10.** Let  $\epsilon < \omega_1$ . Let  $\Phi : [\omega_1]_*^\epsilon \rightarrow \omega_1$ . Then there are club subsets of  $\omega_1$ ,  $C$  and  $D$ , so that for all  $f \in [D]_*^\epsilon$ ,  $\Phi(f) < \text{next}_C^\omega(\text{bound}(f, \mathfrak{b}_\Phi))$ .

*Proof.* Let  $*$  be a new symbol. Define a linear ordering  $\mathcal{L}$  on  $\epsilon \cup \{*\}$  by  $x \prec y$  if and only if

- (a)  $x, y \in \epsilon$  and  $x < y$
- (b)  $x = *, y \in \epsilon$ , and  $y \geq \mathfrak{b}_\Phi$
- (c)  $x \in \epsilon, y = *$ , and  $x < \mathfrak{b}_\Phi$ .

Note that  $\mathcal{L}$  is a wellordering of ordertype less than  $\omega_1$ . If  $f : \mathcal{L} \rightarrow \omega_1$  is an increasing function, then let  $\text{main}(f) : \epsilon \rightarrow \omega_1$  be defined by  $\text{main}(f)(\alpha) = f(\alpha)$ . Let  $\text{extra}(f) \in \omega_1$  be defined by  $\text{extra}(f) = f(*)$ .

Define a partition  $P : [\omega_1]_*^\mathcal{L} \rightarrow 2$  by  $P(g) = 0 \Leftrightarrow \Phi(\text{main}(g)) < \text{extra}(g)$ . By the weak partition property of  $\omega_1$ , there is some  $C \subseteq \omega_1$  which is homogeneous for this partition. By intersecting with an appropriate club, one may assume that for all  $f \in [C]_*^\epsilon$ ,  $\mathfrak{b}_\Phi$  is the largest  $\gamma$  so that  $\Phi(f) \geq \text{bound}(f, \gamma)$ . Therefore if  $\mathfrak{b}_\Phi < \epsilon$ ,  $\Phi(f) < f(\mathfrak{b}_\Phi)$ .

The claim is that  $C$  is homogeneous for  $P$  taking value 0: Let  $D = \{\alpha \in C : \text{enum}_C(\alpha) = \alpha\}$  which is the club set of closure point of  $C$ . Let  $f \in D$ . In the case that  $\mathfrak{b}_\Phi < \epsilon$ , since  $\text{bound}(f, \mathfrak{b}_\Phi) \leq \Phi(f) < f(\mathfrak{b}_\Phi)$  and  $f(\mathfrak{b}_\Phi) \in D$ , the  $\omega^{\text{th}}$ -element of  $C$  above  $\Phi(f)$  is below  $f(\mathfrak{b}_\Phi)$ . In all cases, let  $g : \mathcal{L} \rightarrow C$  be defined by  $g(\alpha) = f(\alpha)$  for all  $\alpha \in \epsilon$  and  $g(*) = \text{next}_C^\omega(\Phi(f))$ . Using any function witnessing that  $f$  has uniform cofinality  $\omega$ , one can show that  $g$  has uniform cofinality  $\omega$ .  $g$  is discontinuous everywhere. So  $g \in [C]_*^\mathcal{L}$  and  $\Phi(\text{main}(g)) = \Phi(f) < \gamma = \text{extra}(g)$ . Thus  $P(g) = 0$  and hence  $C$  must have been homogeneous for  $P$  taking value 0. This establishes the claim.

Now suppose  $f \in [D]_*^\epsilon$ . In the case that  $\mathfrak{b}_\Phi < \epsilon$ , since  $\text{bound}(f, \mathfrak{b}_\Phi) \leq \Phi(f) < f(\mathfrak{b}_\Phi)$  and  $f(\mathfrak{b}_\Phi) \in D$ ,  $\text{next}_C^\omega(\text{bound}(f, \mathfrak{b}_\Phi)) < f(\mathfrak{b}_\Phi)$ . In all cases, let  $g : \mathcal{L} \rightarrow C$  be defined by  $g(\alpha) = f(\alpha)$  if  $\alpha < \epsilon$  and  $g(*) = \text{next}_C^\omega(\text{bound}(f, \mathfrak{b}_\Phi))$ . As before,  $g$  is a function of the correct type in  $[C]_*^\mathcal{L}$ .  $P(g) = 0$  implies that  $\Phi(f) = \Phi(\text{main}(g)) < \text{extra}(g) = \text{next}_C^\omega(\text{bound}(f, \mathfrak{b}_\Phi))$ . This completes the proof.  $\square$

**Lemma 2.11.** *Let  $\epsilon < \omega_1$  and  $\Phi : [\omega_1]_*^\epsilon \rightarrow \omega_1$  be such that  $\mathfrak{b}_\Phi \neq 0$ . Then there is some club  $D \subseteq \omega_1$ , some Kunen function  $\Xi : \omega_1 \times \omega_1 \rightarrow \omega_1$ , and some  $\Phi' : [\omega_1]_*^\epsilon \rightarrow \omega_1$  so that for all  $f \in [D]_2^{\omega_1}$ ,  $\Phi(f) = \Xi(\text{bound}(f, \mathfrak{b}_\Phi), \Phi'(f))$  where  $\mathfrak{b}_{\Phi'} < \mathfrak{b}_\Phi$ .*

*Proof.* By Lemma 2.10, there are clubs  $C$  and  $D_1$  so that for all  $f \in [D_1]_*^\epsilon$ ,  $\Phi(f) < \text{next}_C^\omega(\text{bound}(f, \mathfrak{b}_\Phi))$ . Let  $\Xi$  be a Kunen function for  $\text{next}_C^\omega : \omega_1 \rightarrow \omega_1$ . Since  $K_{\text{next}_C^\omega}^\Xi \in \mu$ , let  $D_2 \subseteq K_{\text{next}_C^\omega}^\Xi$  be a club subset of  $\omega_1$ . Let  $D_3 = D_1 \cap D_2$ . Thus for all  $f \in [D_3]_*^\epsilon$ ,  $\Phi(f) < \text{next}_C^\omega(\text{bound}(f, \mathfrak{b}_\Phi)) < \delta_{\text{bound}(f, \mathfrak{b}_\Phi)}^\Xi$ . Let  $\Phi' : [D_3]_*^\epsilon \rightarrow \omega_1$  be defined by  $\Phi'(f)$  is the least  $\gamma < \text{bound}(f, \mathfrak{b}_\Phi)$  so that  $\Phi(f) = \Xi(\text{bound}(f, \mathfrak{b}_\Phi), \gamma)$ . Thus one has that for all  $f \in [D_3]_*^\epsilon$ ,  $\Phi(f) = \Xi(\text{bound}(f, \mathfrak{b}_\Phi), \Phi'(f))$ . Also  $(\forall_\epsilon^* f)(\Phi'(f) < \text{bound}(f, \mathfrak{b}_\Phi))$  implies that  $\mathfrak{b}_{\Phi'} < \mathfrak{b}_\Phi$  as long as  $\mathfrak{b}_\Phi \neq 0$ .  $\square$

**Definition 2.12.** Let  $\epsilon < \omega_1$  and  $\Phi : [\omega_1]_*^\epsilon \rightarrow \omega_1$ .

A representation for  $\Phi$  is a tuple  $(\Xi_0, \dots, \Xi_{n-1}; \beta_0, \dots, \beta_n; \gamma)$  with the following properties

- (a)  $n \in \omega$ . If  $n = 0$ , then no  $\Xi$  appears.
- (b)  $\beta_0 > \beta_1 > \dots > \beta_{n-1} > \beta_n = 0$  is a sequence of strictly decreasing ordinals less than or equal to  $\epsilon$ .  $\gamma < \omega_1$ .
- (c) Each  $\Xi_i : \omega_1 \times \omega_1 \rightarrow \omega_1$  is a Kunen function (for some function with respect to  $\mu$ ).
- (c) Let  $\Phi_n(f) = \gamma$ . Suppose for  $0 < i \leq n$ ,  $\Phi_i$  has been defined, then let  $\Phi_{i-1}(f) = \Xi_{i-1}(\text{bound}(f, \beta_{i-1}), \Phi_i(f))$ . One has that  $(\forall_\epsilon^* f)(\Phi_0(f) = \Phi(f))$ .

**Theorem 2.13.** *Let  $\epsilon < \omega_1$ . Every  $\Phi : [\omega_1]_*^\epsilon \rightarrow \omega_1$  has a representation.*

*Proof.* Let  $T$  be the tree of decreasing sequences  $\sigma = (\beta_0, \dots, \beta_k)$  in  $\epsilon + 1$  ordered by reverse string extension with the property that there exists some Kunen functions  $\Xi_0, \dots, \Xi_{k-1}$  and functions  $\Phi_0, \dots, \Phi_k$  with the property that

- (i)  $\Phi_0 = \Phi$ .
- (ii)  $\beta_i = \mathfrak{b}_{\Phi_i}$ .
- (iii)  $(\forall_\epsilon^* f)(\Phi_i(f) = \Xi_i(\text{bound}(f, \beta_i), \Phi_{i+1}(f)))$  for all  $i < k$ .

The claim is that there is some  $\sigma = (\beta_0, \dots, \beta_n) \in T$  so that  $\beta_n = 0$ .

To see this: Suppose not. Let  $\sigma = (\beta_0, \dots, \beta_k) \in T$  with  $\beta_k \neq 0$ . Let  $\Xi_0, \dots, \Xi_{k-1}$  and  $\Phi_0, \dots, \Phi_k$  witness that  $\sigma \in T$ . (ii) implies that  $\mathfrak{b}_{\Phi_k} = \beta_k > 0$ . Lemma 2.11 implies that there is some  $\Xi_k$  and  $\Phi'$  so that  $(\forall_\epsilon^* f)(\Phi_k(f) = \Xi_k(\text{bound}(f, \beta_k), \Phi'(f)))$  with  $\mathfrak{b}_{\Phi'} < \mathfrak{b}_{\Phi_k} = \beta_k$ . Let  $\Phi_{k+1} = \Phi'$ . Let  $\beta_{k+1} = \mathfrak{b}_{\Phi'}$ . Let  $\sigma' = \sigma^\frown \beta_{k+1}$ . Then  $\Phi_0, \dots, \Phi_{k+1}$  and  $\Xi_0, \dots, \Xi_k$  witness that  $\sigma' \in T$ .

It has been shown that any  $\sigma \in T$  can be extended to some  $\sigma' \in T$ .  $T$  is a tree on  $\epsilon + 1$  with no dead branches. Since  $\epsilon$  is a wellordering,  $T$  must have an infinite branch. This is impossible since an infinite branch is an infinite descending sequence of ordinals.

The claim has been shown. So let  $\sigma = (\beta_0, \dots, \beta_n) \in T$  be such that  $\beta_n = 0$ . Let  $\Xi_0, \dots, \Xi_{n-1}$  and  $\Phi_0, \dots, \Phi_n$  be witnesses to  $\sigma \in T$ . Since  $\mathfrak{b}_{\Phi_n} = \beta_n = 0$ , one has that for  $\mu^\epsilon$ -almost all  $f$ ,  $\text{bound}(f, 0) = 0 \leq \Phi_n(f) < f(0)$ . So for  $\mu^\epsilon$ -almost all  $f$ ,  $\Phi_n(f)$  is a constant function taking value some  $\gamma \in \omega_1$ . This implies that  $(\Xi_0, \dots, \Xi_{n-1}; \beta_0, \dots, \beta_n; \gamma)$  is a representation of  $\Phi$ .  $\square$

The theorem implies a  $\mu^\epsilon$ -almost everywhere continuity result for function  $\Phi : [\omega_1]_*^\epsilon \rightarrow \omega_1$ .

**Theorem 2.14.** *Let  $\epsilon < \omega_1$  and  $\Phi : [\omega_1]_*^\epsilon \rightarrow \omega_1$ . Then there is a decreasing sequence of ordinals which are less than or equal to  $\epsilon$ ,  $(\beta_i : i \leq n)$ , with  $\beta_n = 0$  and a club  $C \subseteq \omega_1$  so that for any  $f, g \in [C]_*^\epsilon$  with the property that  $\text{bound}(f, \beta_i) = \text{bound}(g, \beta_i)$  for all  $i \leq n$ , then  $\Phi(f) = \Phi(g)$ .*

The following is an even coarser form of continuity:

**Theorem 2.15.** *Let  $\epsilon < \omega_1$  and  $\Phi : [\omega_1]_*^\epsilon \rightarrow \omega_1$ . Then there is a  $\delta < \epsilon$  and some club  $C \subseteq \omega_1$  so that for all  $f, g \in [C]_*^\epsilon$  with  $f \restriction \delta = g \restriction \delta$  and  $\sup(f) = \sup(g)$ ,  $\Phi(f) = \Phi(g)$ .*

*Proof.* If  $n = 0$ , then  $\Phi$  is a constant function so this is immediately true. If  $n = 1$ , then let  $\delta = \beta_0$  if  $\beta_0 < \epsilon$  and  $\delta = 0$  if  $\beta_0 = \epsilon$ . If  $n > 1$ , then let  $\delta = \beta_1$ .  $\square$

Woodin [14] has observed the conclusion of the next theorem at least under  $\text{ZF} + \text{DC} + \text{AD}_{\mathbb{R}}$  or  $\text{ZF} + \text{AD}^+$ .

**Theorem 2.16.**  $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$ .

*Proof.* Observe that  $[\omega_1]_*^\omega \approx [\omega_1]^\omega$  and  $[\omega_1]_*^{<\omega_1} \approx [\omega_1]^{<\omega_1}$ . So suppose there is an injection  $\Sigma : [\omega_1]_*^{<\omega_1} \rightarrow [\omega_1]_*^\omega$ .

For each  $\epsilon < \omega_1$  and  $n \in \omega$ , let  $\Sigma_n^\epsilon : [\omega_1]_*^\epsilon \rightarrow \omega_1$  be defined by  $\Sigma_n^\epsilon(f) = \Sigma(f)(n)$ . By Theorem 2.15, there is some  $\delta_n^\epsilon < \epsilon$  so that there is some  $C \subseteq \omega_1$  club with the property that for all  $f, g \in [C]_*^\epsilon$ ,  $\sup(f) = \sup(g)$  and  $f \restriction \delta_n^\epsilon = g \restriction \delta_n^\epsilon$  implies that  $\Sigma_n^\epsilon(f) = \Sigma_n^\epsilon(g)$ .

For each  $n \in \omega$ , define  $\Lambda_n : \omega_1 \rightarrow \omega_1$  by  $\Lambda_n(\epsilon) = \delta_n^\epsilon$ . Each  $\Lambda_n$  is a regressive function. Therefore, using  $\text{AC}_{\omega}^{\mathbb{R}}$ , let  $C_n$  and  $\delta_n$  be such that for all  $\epsilon \in C_n$ ,  $\Lambda_n(\epsilon) = \delta_n$ . Let  $C = \bigcap_{n \in \omega} C_n$ . Let  $\delta = \sup\{\delta_n : n \in \omega\}$ . Since  $\omega_1$  is regular,  $\delta < \omega_1$ .

Now let  $\epsilon > \delta$  be some limit ordinal with  $\epsilon \in C$ . Using  $\text{AC}_{\omega}^{\mathbb{R}}$ , let  $D_n \subseteq \omega_1$  be clubs so that for all  $f, g \in [D_n]_*^\epsilon$ ,  $\sup(f) = \sup(g)$  and  $f \restriction \delta_n^\epsilon = g \restriction \delta_n^\epsilon$  imply that  $\Sigma_n^\epsilon(f) = \Sigma_n^\epsilon(g)$ . Let  $D = \bigcap_{n \in \omega} D_n$ .

Now pick  $f, g \in [D]_*^\epsilon$  so that  $f \restriction \delta = g \restriction \delta$ ,  $\sup(f) = \sup(g)$ , and  $f \neq g$ . Since for all  $n \in \omega$ ,  $\delta \geq \delta_n = \delta_n^\epsilon$  and  $\epsilon \in C$ , one has that  $\Sigma(f) = \Sigma(g)$ . This contradicts  $\Sigma$  being an injection.  $\square$

### 3. CONTINUITY OF SHORT FUNCTIONS ON $\omega_2$

First, one will review the notations and basic tools needed to analyze  $\omega_2$  under  $\text{AD}$ . See [1] Section 5 and 6 for more details and the proofs of the following results.

Let  $\mu$  denote the club filter on  $\omega_1$ . An important application of the Kunen function for functions  $f : \omega_1 \rightarrow \omega_1$  is the existence of a uniform procedure to select representative of the ultrapower  $\prod_{\omega_1} \omega_1 / \mu$ .

**Fact 3.1.** *Let  $\mu$  be the club measure on  $\omega_1$ . Suppose  $f : \omega_1 \rightarrow \omega_1$  and possesses a Kunen function  $\Xi$  with respect to  $\mu$ . Suppose  $G \in \prod_{\alpha \in \omega_1} f(\alpha) / \mu$ . Then there is a  $\beta < \omega_1$  so that  $[\Xi^\beta]_\mu = G$*

As a consequence, one can show that  $\prod_{\omega_1} \omega_1 / \mu$  is wellfounded even without  $\text{DC}_{\mathbb{R}}$ .

**Fact 3.2.** *Let  $f : \omega_1 \rightarrow \omega_1$  and possesses a Kunen function  $\Xi$  with respect to  $\mu$ . Then  $\prod_{\alpha \in \omega_1} f(\alpha) / \mu$ , i.e. the initial segment of  $\prod_{\omega_1} \omega_1 / \mu$  determined by  $[f]_\mu$ , is a wellordering.*

$\prod_{\omega_1} \omega_1 / \mu$  is wellfounded.

For each  $F \in \prod_{\omega_1} \omega_1 / \mu$ ,  $F < \omega_2$ . Thus  $\prod_{\omega_1} \omega_1 / \mu \leq \omega_2$ .

**Fact 3.3.** (Martin) *Assume just  $\text{ZF}$ . Let  $\kappa$  be a strong partition cardinal.*

*If  $\nu$  is a  $\kappa$ -complete measure on  $\kappa$ , then  $\prod_{\kappa} \kappa / \nu$  is a cardinal.*

*If  $\nu$  is a normal  $\kappa$ -complete measure on  $\kappa$ , then  $\prod_{\kappa} \kappa / \nu$  is a regular cardinal.*

**Corollary 3.4.** (Martin) Let  $\mu$  be the club measure on  $\omega_1$ .  $\omega_2 = \prod_{\omega_1} \omega_1 / \mu$  and  $\omega_2$  is a regular cardinal.

**Definition 3.5.** Let  $\mu$  be the club measure on  $\omega_1$ . Let  $h : \omega_1 \rightarrow \omega_1$ . Suppose  $h$  possesses a Kunen function  $\Xi$  with respect to  $\mu$ . An ordinal  $\beta < \omega_1$  is a minimal code (relative to  $\Xi$ ) if and only if for all  $\gamma < \beta$ ,  $\neg(\Xi^\gamma =_\mu \Xi^\beta)$ . Let  $J$  be the collection of  $\beta$  which are minimal codes and  $\Xi^\beta <_\mu h$ . Define an ordering  $\prec$  on  $J$  by  $\alpha \prec \beta$  if and only if  $\Xi^\alpha <_\mu \Xi^\beta$ . By Fact 3.1, for every  $G < [h]_\mu$ , there is a unique  $\beta \in J$  so that  $\Xi^\beta \in G$ . In this way, one says that  $\beta$  is a minimal code for  $G$  or for any  $g \in G$  with respect to  $\Xi$ . Thus  $(J, \prec)$  has the same ordertype as  $[h]_\mu$ . By Fact 3.2,  $[h]_\mu$  is a wellordering. Let  $\epsilon \in \text{ON}$  denote the ordertype of  $([h]_\mu, <)$  which is equal to the ordertype of  $(J, \prec)$ . Let  $\pi : \epsilon \rightarrow (J, \prec)$  be the unique order-preserving isomorphism.

Note that the objects  $J$ ,  $\prec$ ,  $\epsilon$ , and  $\pi$  depend on  $\Xi$  and  $h$ . However, within this section, one will only work with a single  $\Xi$  and  $h$  at a given time. It should be clear in context that these object depend on this fixed  $\Xi$  and  $h$ .

**Definition 3.6.** Let  $\mu$  be the club measure on  $\omega_1$ . Let  $h : \omega_1 \rightarrow \omega_1$  be a function so that  $h(\alpha) > 0$   $\mu$ -almost everywhere. Let  $\Xi$  be a Kunen function for  $h$  with respect to  $\mu$ . Let  $\epsilon = [h]_\mu = \text{ot}(J, \prec)$  which are defined relative to  $\Xi$  and  $h$ .

Let  $T^h = \{(\alpha, \beta) \in \omega_1 \times \omega_1 : \beta < h(\alpha)\}$ . Let  $\mathcal{T}^h = (T^h, \sqsubset)$  where  $\sqsubset$  is the lexicographic ordering. Note that  $\text{ot}(\mathcal{T}^h) = \omega_1$ .

Suppose  $F : \mathcal{T}^h \rightarrow \omega_1$  is an order-preserving function. Let  $g \in \omega_1 \rightarrow \omega_1$  be such that  $g <_\mu h$ . Let  $A^g = \{\alpha : g(\alpha) < h(\alpha)\}$ . Let  $F^g : \omega_1 \rightarrow \omega_1$  be defined by

$$F^g(\alpha) = \begin{cases} F(\alpha, g(\alpha)) & \alpha \in A^g \\ F(\alpha, 0) & \text{otherwise} \end{cases}$$

Note that if  $g_1 <_\mu g_2 <_\mu h$ , then  $F^{g_1} <_\mu F^{g_2}$ .

If  $\beta \in \epsilon$ , then let  $F^{(\beta)} = F^{\Xi^{\pi(\beta)}}$ . Let  $\text{funct}(F) : \epsilon \rightarrow \text{ON}$  be defined by  $\text{funct}(F)(\alpha) = [F^{(\alpha)}]_\mu$ .

**Fact 3.7.** Let  $\mu$  be the club measure on  $\omega_1$ . Let  $h : \omega_1 \rightarrow \omega_1$  be a function possessing a Kunen function  $\Xi$  with respect to  $\mu$ . Suppose  $F_0, F_1 \in [\omega_1]^{\mathcal{T}^h}$  have the property that  $F_0^{(\beta)} =_\mu F_1^{(\beta)}$  for all  $\beta < \epsilon$ . Then for  $\mu$ -almost all  $\alpha$ ,  $F_0(\alpha, \beta) = F_1(\alpha, \beta)$  for all  $\beta < h(\alpha)$ .

Suppose  $\epsilon < \omega_2$  and  $\mathcal{F} : \epsilon \rightarrow \omega_2$ . Let  $h : \omega_1 \rightarrow \omega_1$  be such that  $[h]_\mu = \epsilon$ . Let  $\Xi$  be a Kunen function for  $h$ . Via a “sliding argument”, one can find an increasing function  $F : \mathcal{T}^h \rightarrow \omega_1$  so that for all  $\beta < \epsilon$ ,  $F^{(\beta)} = \mathcal{F}(\beta)$ . Hence one can study functions  $\mathcal{F} : \epsilon \rightarrow \omega_2$  by using the strong partition property of  $\omega_1$  on partitions of functions in  $[\omega_1]^{\mathcal{T}^h}$ . See [1] Section 5 on the statement of the sliding lemma and how it can be used to prove the following results:

**Theorem 3.8.** (Martin-Paris) Let  $\mu$  be the club measure on  $\omega_1$ . Then for all  $\alpha < \omega_2$ , the partition relation  $\omega_2 \rightarrow (\omega_2)_2^\alpha$  holds. That is,  $\omega_2$  is a weak partition cardinal.

As a consequence of the weak partition property on  $\omega_1$ , one can completely characterize the normal measures on  $\omega_2$ .

**Corollary 3.9.** Let  $W_\omega^{\omega_2}$  and  $W_2^{\omega_2}$  denote the  $\omega$ -club and  $\omega_1$ -club filter, respectively.

$W_\omega^{\omega_2}$  and  $W_2^{\omega_2}$  are the only two  $\omega_2$ -complete normal ultrafilter on  $\omega_2$ .

The next two results show that club subsets and  $\omega$ -club subsets of  $\omega_2$  are lift (in a certain sense) of some club subsets of  $\omega_1$ .

**Fact 3.10.** Let  $\mu$  be the club measure on  $\omega_1$ . If  $C \subseteq \omega_1$  is a club subset of  $\omega_1$ , then  $[C]^{\omega_1} / \mu$  is a club subset of  $\omega_2$ .

If  $D \subseteq \omega_2$  is club, then there is a club  $C \subseteq \omega_1$  so that  $[C]^{\omega_1} / \mu \subseteq D$ .

**Fact 3.11.** Let  $\mu$  be the club measure on  $\omega_1$ . Let  $C \subseteq \omega_1$  be a club. Then  $[C]_*^{\omega_1} / \mu$  is an  $\omega$ -club subset. Moreover, for every  $\omega$ -club  $D \subseteq \omega_2$ , there is a club  $C \subseteq \omega_1$  so that  $[C]_*^{\omega_1} / \mu \subseteq D$ .

**Fact 3.12.** Let  $\mu$  denote the club measure on  $\omega_1$ . Let  $C \subseteq \omega_1$  be club. Let  $B = [C]_*^{\omega_1} / \mu$  which is an  $\omega$ -club subset of  $\omega_2$ .

Let  $\epsilon < \omega_2$ . Let  $h : \omega_1 \rightarrow \omega_1$  with  $h(\alpha) > 0$  for all  $\alpha < \omega_1$  and  $[h]_\mu = \epsilon$ . Let  $\Xi$  be a Kunen function for  $h$ .

Let  $\mathcal{F} \in [B]_*^\epsilon$  (be of correct type). Then there is an  $F \in [C]_*^{\mathcal{T}^h}$  so that for all  $\alpha < \epsilon$ ,  $[F^{(\alpha)}]_\mu = \mathcal{F}(\alpha)$ .



**Definition 3.13.** Let  $\mu$  denote the club measure on  $\omega_1$ . Let  $\nu = W_\omega^{\omega_2}$  denote the  $\omega$ -club measure on  $\omega_2$ . Let  $\epsilon < \omega_2$ . Define  $\nu^\epsilon$  as follows: for all  $A \subseteq [\omega_2]_*^\epsilon$ ,  $A \in \nu^\epsilon$  if and only if there is a  $\omega$ -club  $B \subseteq \omega_2$  so that  $[B]^\epsilon \subseteq A$ .  $\nu^\epsilon$  is an  $\omega_2$ -complete measure on  $[\omega_2]_*^\epsilon$  by the weak partition property of  $\omega_2$ .

Let  $\mathcal{F} \in [\omega_2]_*^\epsilon$ . For  $\beta \leq \epsilon$ , let  $\text{bound}(\mathcal{F}, \beta) = \sup\{\mathcal{F}(\alpha) : \alpha < \beta\}$ .

Let  $\Phi : [\omega_2]_*^\epsilon \rightarrow \omega_2$ . Let  $\mathfrak{b}_\Phi$  be defined so that for  $\nu^\epsilon$ -almost all  $\mathcal{F} \in [\omega_2]_*^\epsilon$ ,  $\mathfrak{b}_\Phi$  is the largest  $\gamma \leq \epsilon$  so that  $\Phi(\mathcal{F}) \geq \text{bound}(\mathcal{F}, \gamma)$ .

Let  $h \in \omega_1 \rightarrow \omega_1$  with  $h(\alpha) > 0$  be such that  $[h]_\mu = \epsilon$ . Let  $\Xi$  be a Kunen function for  $h$  with respect to  $\mu$ .

Suppose  $F \in [\omega_1]_*^{\mathcal{T}^h}$  and  $\beta \leq \epsilon$ . Define  $\text{Bound}_\beta(F)(\gamma) = \sup\{F^{(\alpha)}(\gamma) : \alpha < \beta\}$ . Note that although  $\beta$  may be uncountable, for each  $\gamma$ , this is a supremum of a set containing at most  $|h(\gamma)| = \aleph_0$  many elements.

For the next several results, assume the setting of Definition 3.13.

The next results states that if  $\mathcal{F} \in [\omega_2]_*^\epsilon$  and  $F \in [\omega_1]_*^{\mathcal{T}^h}$  is a lifted representation of  $\mathcal{F}$ , then  $\text{Bound}_\beta(F)$  is a lifted representation of  $\text{bound}(\mathcal{F}, \beta)$ .

**Fact 3.14.** Let  $\beta \leq \epsilon$ . Let  $\mathcal{F} \in [\omega_2]_*^\epsilon$ . Let  $F \in [\omega_1]_*^{\mathcal{T}^h}$  be such that for all  $\alpha < \epsilon$ ,  $[F^{(\alpha)}]_\mu = \mathcal{F}(\alpha)$ . Then  $\text{bound}(\mathcal{F}, \beta) = [\text{Bound}_\beta(F)]_\mu$ .

*Proof.* First observe that for any  $\mathcal{F}$ , there is an  $F$  with the above property by Fact 3.12.

Let  $\delta < \text{bound}(\mathcal{F}, \beta)$ . Then there is some  $\gamma < \beta$  so that  $\delta < \mathcal{F}(\gamma)$ . So  $\delta < [F^{(\gamma)}]_\mu$ . Hence  $\delta < [\text{Bound}_\beta(F)]_\mu$ .

Now suppose that  $\delta < [\text{Bound}_\beta(F)]_\mu$ . Let  $\ell : \omega_1 \rightarrow \omega_1$  be such that  $[\ell]_\mu = \delta$ . Then for  $\mu$ -almost all  $\gamma$ ,  $\ell(\gamma) < \sup\{F^{(\alpha)}(\gamma) : \alpha < \beta\}$ . Therefore, for  $\mu$ -almost all  $\gamma$ , there is a  $\zeta < h(\gamma)$  and, in fact, if  $\beta < \epsilon$ , there is a  $\zeta < \Xi^{\pi(\beta)}(\gamma)$  so that  $\ell(\gamma) < F(\gamma, \zeta)$ . Let  $\iota : \omega_1 \rightarrow \omega_1$  be defined so that for the set of  $\mu$ -almost all  $\gamma$  with the previous property,  $\iota(\gamma)$  is the least such  $\zeta$  with  $\ell(\gamma) < F(\gamma, \zeta)$ . There is some  $\rho < \beta$  so that  $\iota =_\mu \Xi^{\pi(\rho)}$ . Thus  $\ell <_\mu F^\iota =_\mu F^{\Xi^{\pi(\rho)}} = F^{(\rho)}$ . Hence  $\delta < \mathcal{F}(\rho)$  where  $\rho < \beta$ . This shows that  $[\text{Bound}_\beta(F)]_\mu < \text{bound}(\mathcal{F}, \beta)$ .  $\square$

**Definition 3.15.** Let  $\beta \leq \epsilon$ . Let  $C \subseteq \omega_1$  be a club subset of  $\omega_1$ .

For each  $F \in [\omega_1]_*^{\mathcal{T}^h}$ , define  $\text{Fnext}_{\beta, C}(F)(\alpha) = \text{next}_C^\omega(\text{Bound}_\beta(F)(\alpha))$ .

Using either Fact 3.7 or Fact 3.14, if  $F_0, F_1 \in [\omega_1]_*^{\mathcal{T}^h}$  have the property that for all  $\beta \leq \epsilon$ ,  $F_0^{(\beta)} =_\mu F_1^{(\beta)}$ , then  $\text{Fnext}_{\beta, C}(F_0) =_\mu \text{Fnext}_{\beta, C}(F_1)$ .

Therefore the following is well defined: if  $\mathcal{F} \in [\omega_2]_*^\epsilon$ , let  $\text{fnext}_{\beta, C}(\mathcal{F}) = [\text{Fnext}_{\beta, C}(F)]_\mu$ , for any  $F \in [\omega_1]_*^{\mathcal{T}^h}$  such that for all  $\alpha < \epsilon$ ,  $[F^{(\alpha)}]_\mu = \mathcal{F}(\alpha)$ .

**Lemma 3.16.** Assume the setting of Definition 3.13. There is a club  $C \subseteq \omega_1$  and an  $\omega$ -club  $B \subseteq \omega_2$  so that for all  $\mathcal{F} \in [B]_*^\epsilon$ ,  $\Phi(\mathcal{F}) < \text{fnext}_{\mathfrak{b}_\Phi, C}(\mathcal{F})$ .

*Proof.* For each  $\alpha < \omega_1$ , one will define a wellordering  $\mathcal{L}_\alpha$ : Let  $*_\alpha$  be a distinct new object. The underlying domain of  $\mathcal{L}_\alpha$  is  $h(\alpha) \cup \{*_\alpha\}$ .

First assume  $\mathfrak{b}_\Phi < \epsilon$ . Define the linear ordering  $\prec_\alpha$  by  $x \prec_\alpha y$  if and only if

- (a)  $x, y \in h(\alpha)$  and  $x < y$ .
- (b)  $x = *_\alpha$  and  $y \in h(\alpha)$ , and  $y \geq \Xi^{\pi(\mathfrak{b}_\Phi)}(\alpha)$ .
- (c)  $x \in h(\alpha)$ ,  $y = *_\alpha$ , and  $x < \Xi^{\pi(\mathfrak{b}_\Phi)}(\alpha)$ .

If  $\mathfrak{b}_\Phi = \epsilon$ , then define  $x \prec_\alpha y$  if and only if

- (a)  $x, y \in h(\alpha) \wedge x < y$ .
- (b)  $x \in h(\alpha)$  and  $y = *_\alpha$ .

Let  $\mathcal{L} = (L, \prec)$  be a linear ordering on  $L = \{(\alpha, x) : \alpha \in \omega_1 \wedge x \in \mathcal{L}_\alpha\}$  where  $\prec$  is the lexicographic ordering on  $L$  with  $\prec_\alpha$  on the  $\alpha^{\text{th}}$ -coordinate. Note that  $\mathcal{L}$  has ordertype  $\omega_1$ .

In the case the  $\mathfrak{b}_\Phi = \epsilon$ , let  $\tilde{h} = h(\alpha) + 1$ . By initially choosing  $\Xi$  large enough, one may assume that  $\Xi$  is also a Kunen function  $\tilde{h}$  with respect to  $\mu$ . Note that  $\mathcal{L}$  is order isomorphic to  $\mathcal{T}^{\tilde{h}}$ .

Suppose  $K \in [\omega_1]_*^{\mathcal{L}}$ . Define  $\text{main}(K) : [\omega_1]_*^{\mathcal{T}^h} \rightarrow \omega_1$  by  $\text{main}(K)(\alpha, \beta) = K(\alpha, \beta)$ . Define  $\text{extra}(K) : \omega_1 \rightarrow \omega_1$  by  $\text{extra}(K)(\alpha) = K(\alpha, *_\alpha)$ .

Let  $P : [\omega_1]_*^{\mathcal{L}} \rightarrow 2$  be defined by  $P(K) = 0 \Leftrightarrow \Phi(\text{fnext}(\text{main}(K))) < [\text{extra}(K)]_\mu$ . By  $\omega_1 \rightarrow_* (\omega_1)_2^{\omega_1}$ , there is a club  $C \subseteq \omega_1$  which is homogeneous for  $P$ .

Claim 1:  $C$  is homogeneous for  $P$  taking value 0.

By definition of  $\mathfrak{b}_\Phi$ , there is an  $\omega$ -club  $B' \subseteq \omega_2$  so that all  $\mathcal{F} \in [B]_*^\epsilon$ ,  $\mathfrak{b}_\Phi$  is the largest  $\gamma \leq \epsilon$  so that  $\Phi(\mathcal{F}) \geq \text{bound}(\mathcal{F}, \gamma)$ . By Fact 3.14, there is a club  $C'$  so that  $[C']_*^{\omega_1}/\mu \subseteq B$ . By intersecting with  $C'$ , assume that  $C \subseteq C'$ .

(Case I)  $\mathfrak{b}_\Phi < \epsilon$ .

Let  $D = \{\alpha \in C : \text{enum}_C(\alpha) = \alpha\}$  be the closure points of  $C$ . Let  $B = [D]_*^{\omega_1}$ . Pick any  $\mathcal{F} \in [B]_*^\epsilon$ . By Fact 3.12, there is some  $F \in [D]_*^{\mathcal{T}^h}$  so that for all  $\alpha < \epsilon$ ,  $[F^{(\alpha)}]_\mu = \mathcal{F}(\alpha)$ . Let  $f : \omega_1 \rightarrow \omega_1$  be such that  $[f]_\mu = \Phi(\mathcal{F})$ . By Fact 3.14,  $\text{bound}(\mathcal{F}, \mathfrak{b}_\Phi) = [\text{Bound}_{\mathfrak{b}_\Phi}(F)]_\mu$ . Since  $\mathfrak{b}_\Phi$  is the least  $\gamma$  so that  $\Phi(\mathcal{F}) \geq \text{bound}(\mathcal{F}, \gamma)$ , one has that the set  $A$  of  $\alpha$ 's so that  $\text{Bound}_{\mathfrak{b}_\Phi}(F)(\alpha) \leq f(\alpha) < F^{(\mathfrak{b}_\Phi)}(\alpha)$  belongs to  $\mu$ . Define  $K \in [C]_*^\mathcal{L}$  by

$$K(\alpha, z) = \begin{cases} F(\alpha, z) & z \in h(\alpha) \\ \text{next}_C^\omega(f(\alpha)) & \alpha \in A \wedge z = *_\alpha \\ \text{next}_C^\omega(\text{Bound}_{\mathfrak{b}_\Phi}(F)(\alpha)) & \alpha \notin A \wedge z = *_\alpha \end{cases}$$

Note that since  $F(\alpha, \Xi^{\pi(\mathfrak{b}_\Phi)}) \in D$ ,  $K(\alpha, *_\alpha) < K(\alpha, \Xi^{\pi(\mathfrak{b}_\Phi)})$  for all  $\alpha$ . Thus  $K : \mathcal{L} \rightarrow C$  is indeed an increasing function. Since  $F$  is a function of the correct type, one can check that  $K$  is also of the correct type.

Note that  $\text{main}(K) = F$  and for  $\mu$ -almost all  $\alpha$ ,  $\text{extra}(K)(\alpha) = \text{next}_C^\omega(f(\alpha)) > f(\alpha)$ . Thus  $\Phi(\text{funct}(\text{main}(K))) = \Phi(\mathcal{F}) = [f]_\mu < [\text{extra}(K)]_\mu$ . Thus  $P(K) = 0$ . However since  $C$  is homogeneous for  $P$  and  $K \in [C]_*^\mathcal{L}$ , one has that  $C$  is homogeneous for  $P$  taking value 0.

(Case II)  $\mathfrak{b}_\Phi = \epsilon$ .

Let  $B = [C]_*^{\omega_1}$ . Pick any  $\mathcal{F} \in [B]_*^\epsilon$ . Let  $f : \omega_1 \rightarrow \omega_1$  be such that  $[f]_\mu = \Phi(\mathcal{F})$ . Let  $g(\alpha) = \text{next}_C^\omega(f(\alpha))$ . Let  $\mathcal{G} \in [B]_*^{\epsilon+1}$  be defined by

$$\mathcal{G}(\alpha) = \begin{cases} \mathcal{F}(\alpha) & \alpha < \epsilon \\ [g]_\mu & \alpha = \epsilon \end{cases}$$

By Fact 3.12, there is some  $K \in [C]_*^{\mathcal{T}^h} = [C]_*^\mathcal{L}$  so that for all  $\alpha < \epsilon + 1$ ,  $K^{(\alpha)} = \mathcal{G}(\alpha)$ .

Then one has that  $\Phi(\text{funct}(\text{main}(K))) = \Phi(\mathcal{F}) = [f]_\mu < [g]_\mu = [\text{extra}(K)]_\mu$ . Thus  $P(K) = 0$ . Since  $K \in [C]_*^\mathcal{L}$ ,  $C$  is homogeneous for  $P$  taking value 0.

The claim has now been established.

Let  $D = \{\alpha \in C : \text{enum}_C(\alpha) = \alpha\}$ . Let  $B = [D]_*^{\omega_1}$ . Now suppose  $\mathcal{F} \in [B]_*^\epsilon$ . By Fact 3.12, pick any  $F \in [D]_*^{\mathcal{T}^h}$  so that for all  $\alpha < \epsilon$ ,  $[F^{(\alpha)}]_\mu = \mathcal{F}(\alpha)$ . Now define  $K \in [C]_*^\mathcal{L}$  by

$$K(\alpha, z) = \begin{cases} F(\alpha, z) & z \in h(\alpha) \\ \text{next}_C^\omega(\text{Bound}_{\mathfrak{b}_\Phi}(F)(\alpha)) & z = *_\alpha \end{cases}$$

Since  $C$  is homogeneous for  $P$  taking value 0, one has  $P(K) = 0$ . This implies  $\Phi(\mathcal{F}) = \Phi(\text{funct}(\text{main}(K))) < [\text{extra}(K)]_\mu = [\text{fnext}_{\mathfrak{b}_\Phi, C}(F)]_\mu = \text{fnext}_{\mathfrak{b}_\Phi, C}(\mathcal{F})$ . This completes the proof.  $\square$

**Definition 3.17.** Suppose  $\Sigma : \omega_1 \times \omega_1 \rightarrow \omega_1$ .

Suppose  $f_0 : \omega_1 \rightarrow \omega_1$  and  $f_1 : \omega_1 \rightarrow \omega_1$ . Let  $v_{f_0, f_1} : \omega_1 \rightarrow \omega_1$  be defined by  $v_{f_0, f_1}(\alpha) = \Sigma(f_0(\alpha), f_1(\alpha))$ . Note that if  $f'_0 =_\mu f_0$  and  $f'_1 =_\mu f_1$ , then  $v_{f_0, f_1} =_\mu v_{f'_0, f'_1}$ .

Therefore, define  $\hat{\Sigma} : \omega_2 \times \omega_2 \rightarrow \omega_2$  by  $\hat{\Sigma}(\alpha, \beta) = [v_{f_\alpha, f_\beta}]_\mu$ , where  $f_\alpha, f_\beta : \omega_1 \rightarrow \omega_1$  are such that  $[f_\alpha]_\mu = \alpha$  and  $[f_\beta]_\mu = \beta$ .

**Lemma 3.18.** Suppose  $\mathfrak{b}_\Phi > 0$ . Then there is a Kunen function  $\Sigma : \omega_1 \times \omega_1 \rightarrow \omega_1$  and a function  $\Phi' : [\omega_2]_*^\epsilon \rightarrow \omega_2$  so that for  $\nu^\epsilon$ -almost all  $\mathcal{F}$ ,  $\Phi(\mathcal{F}) = \hat{\Sigma}(\text{bound}(\mathcal{F}, \mathfrak{b}_\Phi), \Phi'(\mathcal{F}))$  where  $\mathfrak{b}_{\Phi'} < \mathfrak{b}_\Phi$ .

*Proof.* Let  $B \subseteq \omega_2$  be the  $\omega$ -club and  $C \subseteq \omega_1$  be the club from Lemma 3.16.

Pick any  $\mathcal{F} \in [B]_*^\epsilon$ . Let  $F \in [\omega_1]_*^{\mathcal{T}^h}$  be so that for all  $\alpha < \omega_1$ ,  $[F^{(\alpha)}]_\mu = \mathcal{F}(\alpha)$ . Let  $f : \omega_1 \rightarrow \omega_1$  be such that  $[f]_\mu = \Phi(\mathcal{F})$ . By Lemma 3.16, for  $\mu$ -almost all  $\alpha$ ,  $f(\alpha) < \text{next}_C^\omega(\text{Bound}_{\mathfrak{b}_\Phi}(F)(\alpha))$ . Let  $\Sigma : \omega_1 \times \omega_1 \rightarrow \omega_1$  be a Kunen function for  $\text{next}_C^\omega$ . For  $\mu$ -almost all  $\alpha$ , let  $v_{f, F}(\alpha)$  be the least  $\gamma < \text{Bound}_{\mathfrak{b}_\Phi}(F)(\alpha)$  so that  $f(\alpha) = \Sigma(\text{Bound}_{\mathfrak{b}_\Phi}(F)(\alpha), \gamma)$ . Observe that if  $g =_\mu f$  and  $G \in [\omega_1]_*^{\mathcal{T}^h}$  is such that  $G^{(\alpha)} =_\mu F^{(\alpha)}$  for all  $\alpha < \epsilon$ , then  $v_{f, F} =_\mu v_{g, G}$ . Therefore, define  $\Phi'(\mathcal{F}) = [v_{f, F}]_\mu$ . Note by construction,  $\Phi(\mathcal{F}) = [f]_\mu = \hat{\Sigma}(\text{bound}(\mathcal{F}, \mathfrak{b}_\Phi), [v_{f, F}]_\mu) = \hat{\Sigma}(\text{bound}(\mathcal{F}, \mathfrak{b}_\Phi), \Phi'(\mathcal{F}))$ . Since  $\Phi'(\mathcal{F}) < \text{bound}(\mathcal{F}, \mathfrak{b}_\Phi)$ , one has that  $\mathfrak{b}_{\Phi'} < \mathfrak{b}_\Phi$  if  $\mathfrak{b}_\Phi > 0$ .  $\square$

**Definition 3.19.** Let  $\epsilon < \omega_2$  and  $\Phi : [\omega_2]_*^\epsilon \rightarrow \omega_2$ .

A representation for  $\Phi$  is a tuple  $(\Xi_0, \dots, \Xi_{n-1}; \beta_0, \dots, \beta_n; \gamma)$  with the following properties

- (a)  $n \in \omega$ . If  $n = 0$ , then no  $\Xi$  appears.
- (b)  $\beta_0 > \beta_1 > \dots > \beta_{n-1} > \beta_n = 0$  is a sequence of strictly decreasing ordinals less than or equal to  $\epsilon$ .  $\gamma < \omega_2$ .
- (c) Each  $\Xi_i : \omega_1 \times \omega_1 \rightarrow \omega_1$ .
- (d) Let  $\Phi_n(\mathcal{F}) = \gamma$ . Suppose for  $0 < i \leq n$ ,  $\Phi_i$  has been defined, then let  $\Phi_{i-1}(\mathcal{F}) = \hat{\Xi}_i(\text{bound}(\mathcal{F}, \beta_{i-1}), \Phi_i(\mathcal{F}))$ . One has that for  $\nu^\epsilon$ -almost all  $\mathcal{F}$ ,  $\Phi_0(\mathcal{F}) = \Phi(\mathcal{F})$ .

**Theorem 3.20.** Let  $\epsilon < \omega_2$ . Every  $\Phi : [\omega_2]_*^\epsilon \rightarrow \omega_2$  has a representation.

*Proof.* The proof is analogous to the proof of Theorem 2.13 using the  $\omega_2$  version of the analogous lemmas.  $\square$

Now one has the analogous continuity result for functions  $\Phi : [\omega_2]_*^\epsilon \rightarrow \omega_2$  where  $\epsilon < \omega_2$ .

**Theorem 3.21.** Let  $\epsilon < \omega_2$  and  $\Phi : [\omega_2]_*^\epsilon \rightarrow \omega_2$ . Then there is a decreasing sequence of ordinals less than or equal to  $\epsilon$ ,  $(\beta_i : i \leq n)$ , with  $\beta_n = 0$  and an  $\omega$ -club  $B \subseteq \omega_2$  so that for any  $\mathcal{F}, \mathcal{G} \in [B]_*^\epsilon$  with the property that  $\text{bound}(\mathcal{F}, \beta_i) = \text{bound}(\mathcal{G}, \beta_i)$  for all  $i \leq n$ , then  $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$ .

**Theorem 3.22.** Let  $\epsilon < \omega_2$  and  $\Phi : [\omega_2]_*^\epsilon \rightarrow \omega_2$ . Then there is a  $\delta < \epsilon$  and an  $\omega$ -club  $B \subseteq \omega_2$  so that for all  $\mathcal{F}, \mathcal{G} \in [B]_*^\epsilon$  with  $\mathcal{F} \upharpoonright \delta = \mathcal{G} \upharpoonright \delta$  and  $\text{sup}(\mathcal{F}) = \text{sup}(\mathcal{G})$ ,  $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$ .

Now one has some new cardinality results:

**Theorem 3.23.**  $|[\omega_2]^\omega| < |[\omega_2]^{<\omega_1}|$ .

*Proof.* Suppose  $\Phi : [\omega_2]_*^{<\omega_1} \rightarrow [\omega_2]_*^\omega$  is a function. For each  $\epsilon < \omega_1$  and each  $n \in \omega$ , let  $\Phi_n^\epsilon : [\omega_2]_*^\epsilon \rightarrow \omega_2$  be defined by  $\Phi_n^\epsilon(\mathcal{F}) = \Phi(\mathcal{F})(n)$ . By Theorem 3.22, there is some  $\delta < \epsilon$  so that  $\Phi_n^\epsilon(\mathcal{F}) = \Phi_n^\epsilon(\mathcal{G})$  for  $\nu^\epsilon$ -almost all  $\mathcal{F}$  and  $\mathcal{G}$  so that  $\mathcal{F} \upharpoonright \delta = \mathcal{G} \upharpoonright \delta$  and  $\text{sup}(\mathcal{F}) = \text{sup}(\mathcal{G})$ . Let  $\delta_n^\epsilon$  be the least such  $\delta$ . The function  $\Lambda_n : \omega_1 \rightarrow \omega_1$  defined by  $\Lambda_n(\epsilon) = \delta_n^\epsilon$  is a regressive function. Using  $\text{AC}_\omega^\mathbb{R}$ , there is a  $\delta_n < \omega_1$  and  $A_n \in \mu$  so that for all  $\epsilon \in A_n$ ,  $\Lambda_n(\epsilon) = \delta_n$ . Let  $A = \bigcap_{n \in \omega} A_n \in \mu$  and  $\delta = \sup_{n \in \omega} \delta_n < \omega_1$ . Pick a limit ordinal  $\epsilon \in A$  with  $\epsilon > \delta$ . By  $\text{AC}_\omega^\mathbb{R}$ , let  $B_n$  be an  $\omega$ -club subset of  $\omega_2$  so that for all  $\mathcal{F}, \mathcal{G} \in [B_n]_*^\epsilon$ , if  $\text{sup}(\mathcal{F}) = \text{sup}(\mathcal{G})$  and  $\mathcal{F} \upharpoonright \delta_n = \mathcal{G} \upharpoonright \delta_n$ , then  $\Phi_n^\epsilon(\mathcal{F}) = \Phi_n^\epsilon(\mathcal{G})$ . Since  $\nu$  is  $\omega_2$ -complete,  $B = \bigcap_{n \in \omega} B_n \in \nu$ . Thus pick some  $\mathcal{F}, \mathcal{G} \in [B]_*^\epsilon$  with  $\mathcal{F} \neq \mathcal{G}$ ,  $\text{sup}(\mathcal{F}) = \text{sup}(\mathcal{G})$ , and  $\mathcal{F} \upharpoonright \delta = \mathcal{G} \upharpoonright \delta$ . Then for all  $n \in \omega$ ,  $\Phi_n^\epsilon(\mathcal{F}) = \Phi_n^\epsilon(\mathcal{G})$ . So  $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$ .  $\Phi$  can not be an injection.  $\square$

**Theorem 3.24.**  $|[\omega_2]^{<\omega_1}| < |[\omega_2]^{\omega_1}|$ .

*Proof.* Let  $\Phi : [\omega_2]_*^{\omega_1} \rightarrow [\omega_2]_*^{<\omega_1}$  be a function. Let  $\Psi : [\omega_2]_*^{\omega_1} \rightarrow \omega_1$  be  $\text{length} \circ \Phi$ , where  $\text{length}(\mathcal{F}) = \epsilon$  if  $\mathcal{F} : \epsilon \rightarrow \omega_2$ . Since  $\nu$  is  $\omega_2$ -complete, there is a  $B \in \nu$  and an  $\epsilon < \omega_1$  so that for all  $\mathcal{F} \in [B]_*^{\omega_1}$ ,  $\Psi(\mathcal{F}) = \epsilon$ . In other words, for all  $\mathcal{F} \in [B]_*^{\omega_1}$ ,  $\Phi(\mathcal{F}) \in [\omega_2]_*^\epsilon$ .

Let  $\alpha < \epsilon$ . Let  $\Phi_\alpha(\mathcal{F}) = \Phi(\mathcal{F})(\alpha)$ . By Theorem 3.22 and  $\text{AC}_\omega^\mathbb{R}$ , there are  $\delta_\alpha < \omega_1$  and  $\omega$ -club  $B_\alpha \subseteq \omega_2$  so that for all  $\mathcal{F}, \mathcal{G} \in [B_\alpha]_*^{\omega_1}$ , if  $\mathcal{F} \upharpoonright \delta_\alpha = \mathcal{G} \upharpoonright \delta_\alpha$  and  $\text{sup}(\mathcal{F}) = \text{sup}(\mathcal{G})$ , then  $\Phi_\alpha(\mathcal{F}) = \Phi_\alpha(\mathcal{G})$ .

Now let  $U = \bigcap_{\alpha < \epsilon} B_\alpha \in \nu$  since  $\nu$  is  $\omega_2$ -complete. Let  $\delta = \sup\{\delta_\alpha : \alpha < \epsilon\}$ . Note that  $\delta < \omega_1$  since  $\omega_1$  is regular. Pick  $\mathcal{F}, \mathcal{G} \in [U]_*^{\omega_1}$  with  $\mathcal{F} \neq \mathcal{G}$ ,  $\mathcal{F} \upharpoonright \delta = \mathcal{G} \upharpoonright \delta$ ,  $\text{sup}(\mathcal{F}) = \text{sup}(\mathcal{G})$ . Since  $\mathcal{F}, \mathcal{G} \in [B]_*^{\omega_1}$ ,  $\Phi(\mathcal{F})$  and  $\Phi(\mathcal{G})$  both have length  $\epsilon$ . By choice,  $\Phi(\mathcal{F})(\alpha) = \Phi_\alpha(\mathcal{F}) = \Phi_\alpha(\mathcal{G}) = \Phi(\mathcal{G})(\alpha)$  for all  $\alpha < \epsilon$ . So  $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$ .  $\Phi$  is not an injection.  $\square$

Previously, one only needed  $\text{AC}_\omega^\mathbb{R}$  to make a countable selection of subsets of  $\omega_1$  or  $\omega_2$ . For the next theorem, one will need to make an  $\omega_1$ -length selection of club subset of  $\omega_1$ . The following fact ensures this can be done.

**Fact 3.25.** ([1] Section 4) Let  $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$  be such that each  $\mathcal{A}_\alpha$  is a nonempty  $\subseteq$ -downward closed collection of club subsets of  $\omega_1$ . Then there is a sequence  $\langle C_\alpha : \alpha < \omega_1 \rangle$  with each  $C_\alpha \subseteq \omega_1$  a club subset of  $\omega_1$  and  $C_\alpha \in \mathcal{A}_\alpha$ .

**Theorem 3.26.**  $|[\omega_2]^{\omega_1}| < |[\omega_2]^{<\omega_2}|$ .

*Proof.* Let  $\Phi : [\omega_2]_*^{<\omega_2} \rightarrow [\omega_2]_*^{\omega_1}$  be a function. For each  $\epsilon < \omega_2$  and  $\alpha < \omega_1$ , let  $\Phi_\alpha^\epsilon : [\omega_2]_*^\epsilon \rightarrow \omega_2$  be defined by  $\Phi_\alpha^\epsilon(\mathcal{F}) = \Phi(\mathcal{F})(\alpha)$ . By Theorem 3.22, there is a minimal  $\delta_\alpha^\epsilon < \epsilon$  so that for  $\nu^\epsilon$ -almost all  $\mathcal{F}, \mathcal{G} \in [\tilde{B}]_*^\epsilon$ , if  $\mathcal{F} \upharpoonright \delta_\alpha^\epsilon = \mathcal{G} \upharpoonright \delta_\alpha^\epsilon$  and  $\sup(\mathcal{F}) = \sup(\mathcal{G})$ , then  $\Phi_\alpha^\epsilon(\mathcal{F}) = \Phi_\alpha^\epsilon(\mathcal{G})$ .

For each  $\alpha < \omega_1$ , let  $\Lambda_\alpha : \omega_2 \rightarrow \omega_2$  be defined by  $\Lambda_\alpha(\epsilon) = \delta_\alpha^\epsilon$ . Since  $\nu$  is a normal measure on  $\omega_2$  and  $\Lambda_\alpha$  is a regressive function, there is a minimal  $\delta_\alpha < \omega_2$  so that for  $\nu$ -almost all  $\epsilon$ ,  $\Lambda_\alpha(\epsilon) = \delta_\alpha$ . By Fact 3.11, for every  $B \in \nu$ , there is a  $C \subseteq \omega_1$  club so that  $[C]_*^{\omega_1}/\mu \subseteq B$ . Let  $\mathcal{A}_\alpha$  be the collection of all club  $C \subseteq \omega_1$  so that for all  $\epsilon \in [C]_*^{\omega_1}/\mu$ ,  $\Lambda_\alpha(\epsilon) = \delta_\alpha$ .  $\mathcal{A}_\alpha$  is clearly  $\subseteq$ -downward closed. Apply Fact 3.25 to obtain a sequence  $\langle C_\alpha : \alpha < \omega \rangle$  so that  $C_\alpha \in \mathcal{A}_\alpha$ . Let  $B = \bigcap_{\alpha < \omega_1} [C_\alpha]_*^{\omega_1}/\mu$  which belong to  $\nu$  as  $\nu$  is  $\omega_2$ -complete. Let  $\delta = \sup\{\delta_\alpha : \alpha < \omega_1\} < \omega_2$  since  $\omega_2$  is regular. Now pick a limit ordinal  $\epsilon > \delta$  with  $\epsilon \in B$ .

For  $\alpha < \omega_1$ , let  $\mathcal{A}'_\alpha$  be the collection of club  $C \subseteq \omega_1$  so that if  $D = [C]_*^{\omega_1}/\mu$ , then  $D$  has the property that for all  $\mathcal{F}, \mathcal{G} \in [D]_*^\epsilon$ , if  $\mathcal{F} \upharpoonright \delta_\alpha = \mathcal{G} \upharpoonright \delta_\alpha$  and  $\sup(\mathcal{F}) = \sup(\mathcal{G})$ , then  $\Phi_\alpha^\epsilon(\mathcal{F}) = \Phi_\alpha^\epsilon(\mathcal{G})$ .  $\mathcal{A}'_\alpha$  is a  $\subseteq$ -downward closed nonempty collection of club subsets of  $\omega_1$ . Apply Fact 3.25 to obtain a collection  $\langle C'_\alpha : \alpha < \omega_1 \rangle$  of club subsets of  $\omega_1$  with the property that for all  $\alpha < \omega_1$ ,  $C'_\alpha \in \mathcal{A}'_\alpha$ . Let  $B' = \bigcap_{\alpha < \omega_1} [C'_\alpha]_*^\omega/\mu$  which belongs to  $\nu$  since  $\nu$  is  $\omega_2$ -complete. Now pick  $\mathcal{F}, \mathcal{G} \in [B']_*^\epsilon$  with  $\mathcal{F} \upharpoonright \delta = \mathcal{G} \upharpoonright \delta$ ,  $\sup(\mathcal{F}) = \sup(\mathcal{G})$ , and  $\mathcal{F} \neq \mathcal{G}$ . Note that for all  $\alpha < \omega_1$ ,  $\Phi(\mathcal{F})(\alpha) = \Phi_\alpha^\epsilon(\mathcal{F}) = \Phi_\alpha^\epsilon(\mathcal{G})_\alpha = \Phi(\mathcal{G})(\alpha)$ . Thus  $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$ .  $\Phi$  is not an injection.  $\square$

**Theorem 3.27.**  $||[\omega_2]^\omega| < ||[\omega_2]^{<\omega_1}| < ||[\omega_2]^{\omega_1}| < ||[\omega_2]^{<\omega_2}|$ .

*Proof.* Given the previous theorems, one needs only to show that the appropriate injections exists. The only one that is not immediately clear is the injection from  $[\omega_2]^{<\omega_1}$  into  $[\omega_2]^{\omega_1}$ .

Let  $\text{add} : \omega_2 \times [\omega_2]^{<\omega_1}$  be defined by if  $\mathcal{F} \in [\omega_2]^\epsilon$  for some  $\epsilon < \omega_1$ , then  $\text{add}(\lambda, \mathcal{F}) \in [\omega_2]^\epsilon$  is defined by  $\text{add}(\lambda, \mathcal{F})(\alpha) = \lambda + \mathcal{F}(\alpha)$ .

If  $\mathcal{F} \in [\omega_2]^{<\omega_1}$ , then let  $\text{fill}(\mathcal{F}) \in [\omega_2]^{\omega_1}$  is defined by appending onto  $\mathcal{F}$  the next  $\omega_1$ -many ordinals after  $\sup(\mathcal{F})$ .

Let  $\Phi : [\omega_2]^{<\omega_1} \rightarrow [\omega_2]^{\omega_1}$  be defined by  $\Phi(\mathcal{F}) = \text{fill}(\text{length}(\mathcal{F}) \hat{\ } \text{add}(\text{length}(\mathcal{F}), \mathcal{F}))$ . In words,  $\Phi(\mathcal{F})$  starts with  $\text{length}(\mathcal{F})$ , then shifts up all the values of  $\mathcal{F}$  by  $\text{length}(\mathcal{F})$ , and fill in the rest with successive ordinals until one reaches length  $\omega_1$ . One can check that  $\Phi$  is an injection.  $\square$

**Fact 3.28.**  $\omega_2$  does not inject into  $[\omega_1]^{\omega_1}$ . Thus  $[\omega_2]^\omega$  does not inject into  $[\omega_1]^{\omega_1}$ .

*Proof.* This is a consequence of the measurability of  $\omega_2$  in the same way the fact that there are no uncountable wellordered sequences of reals follows from the measurability of  $\omega_1$ . The details follow:

Let  $\nu$  be an  $\omega_2$ -complete measure on  $\omega_2$ . Suppose  $\langle f_\alpha : \alpha < \omega_2 \rangle$  is an injection of  $\omega_2$  into  $[\omega_1]^{\omega_1}$ . Let  $F_\alpha = \text{rang}(f_\alpha)$ . Then  $\langle F_\alpha : \alpha < \omega_2 \rangle$  is an  $\omega_2$ -sequence of distinct subsets of  $\omega_1$ .

For each  $\beta < \omega_1$ , let  $A_\beta^0 = \{\alpha < \omega_2 : \beta \notin F_\alpha\}$  and  $A_\beta^1 = \{\alpha < \omega_2 : \beta \in F_\alpha\}$ . Since  $\mu$  is a measure, there is some  $i_\beta \in 2$  so that  $A_\beta^{i_\beta} \in \nu$ .

By the  $\omega_2$ -completeness of  $\nu$ ,  $\bigcap_{\beta \in \omega_1} A_\beta^{i_\beta} \in \nu$ . Let  $\alpha_0, \alpha_1 \in \bigcap_{\beta \in \omega_1} A_\beta^{i_\beta}$ . Let  $F \subseteq \omega_1$  be defined  $\beta \in F \Leftrightarrow i_\beta = 1$ . Then  $F_{\alpha_0} = F_{\alpha_1} = F$ . This contradicts the fact that  $\langle F_\alpha : \alpha < \omega_2 \rangle$  is a sequence of distinct subsets of  $\omega_1$ .  $\square$

Like the original argument for the cardinal relation  $||[\omega_1]^\omega| < ||[\omega_1]^{<\omega_1}|$ , the argument that  $[\omega_1]^{<\omega_1}$  does not inject into  $[\omega_2]^\omega$  passes through the set  $S_1$  using  $\infty$ -Borel code and forcing arguments. This originally was proved under  $\text{ZF} + \text{AD}^+$ . The following gives a purely descriptive set theoretic proof using just  $\text{AD}$ .

**Theorem 3.29.**  $\neg(||[\omega_1]^{<\omega_1}| \leq ||[\omega_2]^\omega|)$ . Thus  $\neg(||[\omega_1]^{\omega_1}| \leq ||[\omega_2]^\omega|)$ .

*Proof.* Suppose  $\Phi : [\omega_1]^{<\omega_1} \rightarrow [\omega_2]^\omega$  is an injection.

For each  $\epsilon < \omega_1$ , let  $\text{tail}(f, \epsilon) \in [\omega_1]^{\omega_1}$  be defined by  $\text{tail}(f, \epsilon)(\beta) = f(\epsilon + \beta)$ . Note that for all  $\epsilon < \omega_1$  and  $f \in [\omega_1]^{\omega_1}$ ,  $f = (f \upharpoonright \epsilon) \hat{\ } \text{tail}(f, \epsilon)$ . Let  $\mu$  denote the club measure on  $\omega_1$ .

For each  $\epsilon < \omega_1$ , let  $P_\epsilon : [\omega_1]_*^{\omega_1} \rightarrow 2$  be defined by  $P_\epsilon(f) = 0$  if and only if  $\sup(\Phi(f \upharpoonright \epsilon)) < [\text{tail}(f, \epsilon)]_\mu$ . (Recall that  $\prod_{\omega_1} \omega_1/\mu = \omega_2$ .)

Let  $C \subseteq \omega_1$  be a club which is homogeneous for  $P_\epsilon$ . The claim is that  $C$  is homogeneous for  $P_\epsilon$  taking value 0. Suppose otherwise, then pick any  $\sigma \in [C]_*^\epsilon$ . For any  $g \in [C]_*^{\omega_1}$  with  $\min(g) > \sup(\sigma)$ , define  $f^g \in [C]_*^{\omega_1}$  by  $\sigma \hat{\ } g$ . Then  $P(f^g) = 1$  implies that  $[g]_\mu = \text{tail}(f^g, \epsilon) \leq \sup(\Phi(f^g \upharpoonright \epsilon)) = \sup(\Phi(\sigma))$ . This impossible since  $\sigma$  is fixed,  $[C]_*^{\omega_1}/\mu = \omega_2$ , and  $g$  can be any member of  $[C]_*^{\omega_1}$  with  $\min(g) > \sup(\sigma)$ .

It has been shown that  $C$  is homogeneous for  $P_\epsilon$  taking value 0. Let  $\ell \in [C]_*^{\omega_1}$  and let  $\beta = [\ell]_\mu$ . Note that for all  $\epsilon < \omega_1$ ,  $\ell =_\mu \text{tail}(\ell, \epsilon)$ . Let  $\sigma \in [C]_*^\epsilon$ . Let  $\gamma_\sigma$  be the least  $\gamma$  so that  $\ell(\gamma) > \sup(\sigma)$ . Define  $f_\sigma = \sigma \hat{\text{tail}}(f, \gamma_\sigma)$ . Note that  $f_\sigma \in [C]_*^{\omega_1}$ . Thus  $P_\epsilon(f_\sigma) = 0$  implies that  $\sup(\Phi(\sigma)) = \sup(\Phi(f_\sigma \upharpoonright \epsilon)) < [\text{tail}(f_\sigma, \epsilon)]_\mu = [\text{tail}(\ell, \gamma_\sigma)]_\mu = [\ell]_\mu = \beta$ . That is,  $\Phi$  maps  $[C]_*^\epsilon$  into  $[\beta]^\omega$ .

For each  $\epsilon < \omega_1$ , let  $\beta_\epsilon$  be the least  $\beta < \omega_2$  so that there exists a club  $C \subseteq \omega_1$  with the property that for all  $\sigma \in [C]_*^\epsilon$ ,  $\sup(\Phi(\sigma)) < \beta$ . This defines a sequence  $\langle \beta_\epsilon : \epsilon < \omega_1 \rangle$ . Let  $\delta = \sup\{\beta_\epsilon : \epsilon < \omega_1\}$ . Since  $\omega_2$  is regular,  $\delta < \omega_2$ .

For  $\epsilon < \omega_1$ , let  $\mathcal{A}_\epsilon$  be the collection of clubs  $C \subseteq \omega_1$  so that for all  $\sigma \in [C]_*^\epsilon$ ,  $\sup(\Phi(\sigma)) < \beta_\epsilon$ . This defines a sequence  $\langle \mathcal{A}_\epsilon : \epsilon < \omega_1 \rangle$ . Note that for all  $\epsilon < \omega_1$ ,  $\mathcal{A}_\epsilon$  is a nonempty  $\subseteq$ -downward closed collection of club subsets of  $\omega_1$ . By Fact 3.25, let  $\langle C_\epsilon : \epsilon < \omega_1 \rangle$  be a sequence so that  $C_\epsilon \in \mathcal{A}_\epsilon$  for all  $\epsilon \in \omega_1$ . So for any  $\epsilon < \omega_1$ , if  $\sigma \in [C_\epsilon]_*^\epsilon$ , then  $\sup(\Phi(\sigma)) < \delta$ .

Note that  $\bigcup_{\epsilon < \omega_1} [C_\epsilon]_*^\epsilon \approx [\omega_1]^{<\omega_1}$ . Observe that

$$\Phi \left[ \bigcup_{\epsilon < \omega_1} [C_\epsilon]_*^\epsilon \right] \subseteq [\delta]^\omega.$$

Hence  $\Phi$  induces an injection of  $[\omega_1]^{<\omega_1}$  into  $[\delta]^\omega \approx [\omega_1]^\omega$  since  $\delta < \omega_2$ . By Theorem 2.16, this is impossible.  $\square$

**Fact 3.30.**  $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$ .

*Proof.* There is a purely descriptive set theoretic proof of this result in the flavor of the continuity argument used throughout this paper in [4]. However, the requisite continuity property is more challenging to establish than the analogous continuity properties in this paper. However, there is a very simple set theoretic proof of this result:

Suppose there was an injection  $\Phi : [\omega_1]^{\omega_1} \rightarrow [\omega_1]^{<\omega_1}$ . Let  $L[\Phi] \models \text{ZFC}$  be the Gödel constructible universe built relative to  $\Phi$  as a predicate.

Note that  $\omega_1^V$  is inaccessible in  $L[\Phi]$ : Suppose  $\delta < \omega_1^V$  and  $|\mathcal{P}(\delta)^{L[\Phi]}|^{L[\Phi]} \geq \omega_1^V$ . Since  $L[\Phi] \models \text{AC}$ ,  $\mathcal{P}(\delta)^{L[\Phi]}$  is a wellorderable collection of subsets of  $\delta$  of cardinality  $\omega_1^V$ . In the real world  $V$ ,  $\delta$  is a countable ordinal and hence there is a bijection of  $\delta$  with  $\omega$ . Using this bijection, one can obtain an  $\omega_1^V$ -length sequence of distinct reals from  $\mathcal{P}(\delta)^{L[\Phi]}$ . This is impossible under  $\text{AD}$  by a simple form of the argument in Fact 3.28. Thus  $|\mathcal{P}(\delta)^{L[\Phi]}|^{L[\Phi]} < \omega_1^V$ . This implies  $\omega_1^V$  is inaccessible.

Since  $L[\Phi] \models \text{ZFC}$ , Cantor's theorem assert that  $L[\Phi] \models |[\omega_1^V]^{\omega_1^V}| = |2^{\omega_1^V}| \geq (\omega_1^V)^+$ . Also since  $L[\Phi] \models \text{ZFC}$  and  $\omega_1^V$  is inaccessible in  $L[\Phi]$ ,  $L[\Phi] \models |[\omega_1^V]^{<\omega_1^V}| = |2^{<\omega_1^V}| = \omega_1^V$ . By absoluteness,  $L[\Phi] \models \Phi$  is an injection. It is impossible that  $L[\Phi]$  thinks that  $\Phi$  is an injection of  $2^{\omega_1^V}$  into  $\omega_1^V$ .  $\square$

**Theorem 3.31.** *Then  $\neg(|[\omega_1]^{\omega_1}| \leq |[\omega_2]^{<\omega_1}|)$ .*

*Proof.* Let  $\mathcal{T} = (\omega_1 \times 2, <)$  where  $<$  is the lexicographic ordering. (Note that  $\text{ot}(\mathcal{T}) = \omega_1$ .) If  $F \in [\omega_1]_*^{\omega_1}$  and  $i \in 2$ , let  $F_i \in [\omega_1]_*^{\omega_1}$  be defined by  $F_i(\alpha) = F(\alpha, i)$ .

Now suppose  $\Phi : [\omega_1]_*^{\omega_1} \rightarrow [\omega_2]^{<\omega_1}$  is an injection. Define a partition  $P : [\omega_1]_*^{\omega_1} \rightarrow 2$  by  $P(F) = 0$  if and only if  $\sup(\Phi(F_0)) \leq \sup(\Phi(F_1))$ . Let  $C \subseteq \omega_1$  be a club homogeneous subset for  $P$ . The claim is  $C$  is homogeneous for  $P$  taking value 0.

Suppose  $C$  was homogeneous for  $P$  taking value 1. Let  $g_0(0) = \text{next}_C^\omega(0)$ . Suppose  $g_k(\alpha)$  has been defined, then let  $g_{k+1}(\alpha) = \text{next}_C^\omega(g_k(\alpha))$ . Suppose  $g_n(\beta)$  has been defined for all  $n \in \omega$  and  $\beta < \alpha$ . Then let  $g_0(\alpha) = \text{next}_C^\omega(\sup\{g_n(\beta) : n \in \omega \wedge \beta < \alpha\})$ .

For each  $n \in \omega$ ,  $g_n \in [C]_*^{\omega_1}$ . Define for  $\alpha < \omega_1$  and  $i \in 2$ ,  $G^n(\alpha, i) = g_{n+i}(\alpha)$ . By the construction of  $\langle g_n : n \in \omega \rangle$ , one has that  $G^n \in [C]_*^T$ .

Thus one has that  $P(G^n) = 1$  for all  $n \in \omega$ . This implies for all  $n \in \omega$ .

$$\sup(\Phi(g_{n+1})) = \sup(\Phi(G_1^n)) < \sup(\Phi(G_0^n)) = \sup(\Phi(g_n)).$$

It has been shown that  $\langle \sup(\Phi(g_n)) : n \in \omega \rangle$  is an infinite decreasing sequence of ordinals. This contradicts the wellfoundedness of the ordinals.

One must have that  $C$  is homogeneous for  $P$  taking value 0. For the next part, take  $g_0, g_1$ , and  $g_2$  from the sequence  $\langle g_n : n \in \omega \rangle$  constructed above. The important observation from above is that  $g_0(\alpha) < g_1(\alpha) < g_2(\alpha) < g_0(\alpha + 1)$  for all  $\alpha$ .

For each  $A \in {}^{\omega_1}2$ , let  $h_A \in [C]_*^{\omega_1}$  be defined by  $h_A(\alpha) = g_{A(\alpha)}(\alpha)$ . Let  $H^A \in [C]_*^T$  be defined by

$$H^A(\alpha, i) = \begin{cases} h_A(\alpha) & i = 0 \\ g_2(\alpha) & i = 1 \end{cases}.$$

Note that  $H_0^A = h_A$  and  $H_1^A = g_2$ .  $P(H^A) = 0$  implies that  $\sup(\Phi(h_A)) = \sup(\Phi(H_0^A)) \leq \sup(\Phi(H_1^A)) = \sup(\Phi(g_2))$ . Let  $\zeta = \sup(\Phi(g_n))$  which is some ordinal less than  $\omega_2$ .

Define  $\Psi : {}^{\omega_1}2 \rightarrow [\omega_2]^{<\omega_1}$  by  $\Psi(A) = \Phi(h_A)$ . Note that  $\Psi$  is an injection. By the above,  $\Psi : {}^{\omega_1}2 \rightarrow [\zeta]^{<\omega_1}$ . Since  ${}^{\omega_1}2 \approx \mathcal{P}(\omega_1) \approx [\omega_1]^{\omega_1}$ , one has shown that there is an injection of  $[\omega_1]^{\omega_1}$  into  $[\zeta]^{<\omega_1} \approx [\omega_1]^{<\omega_1}$ . This is not possible by Fact 3.30.  $\square$

For the sake of completeness, one sketches the remaining well-known cardinal relations among the sets considered in this paper:

**Fact 3.32.**  $\neg(\omega_1 \leq |\mathbb{R}|)$  and  $\neg(|\mathbb{R}| \leq \omega_1)$ .

*Proof.* By a simple form of the argument in the proof of Fact 3.28, there are no uncountable wellordered sequences of distinct reals.  $\omega_1$  can not inject into  $\mathbb{R}$ .

Under AD,  $\mathbb{R}$  can not be wellordered. (For instance, a category argument can be used to show that a wellordered union of meager sets is meager under AD.) Hence  $\mathbb{R}$  can not inject into  $\omega_1$ .  $\square$

**Fact 3.33.** Let  $\kappa$  be an ordinal.  $\neg(|[\omega_1]^\omega| \leq \kappa)$ ,  $\neg(|[\omega_1]^\omega| \leq \mathbb{R})$ , and  $\neg(|[\omega_1]^\omega| \leq |\mathbb{R} \sqcup \kappa|)$ .  $\neg(|[\omega_1]^\omega| \leq |\mathbb{R} \times \kappa|)$ .

Similarly,  $\neg(|[\omega_2]^\omega| \leq \kappa)$ ,  $\neg(|[\omega_2]^\omega| \leq \mathbb{R})$ , and  $\neg(|[\omega_2]^\omega| \leq |\mathbb{R} \sqcup \kappa|)$ .  $\neg(|[\omega_2]^\omega| \leq |\mathbb{R} \times \kappa|)$ .

*Proof.* Since  $\mathbb{R}$  injects into  $[\omega_1]^\omega$  and  $\mathbb{R}$  is not wellorderable,  $[\omega_1]^\omega$  is not wellorderable. So  $[\omega_1]^\omega$  can not inject into any ordinal  $\kappa$ .

Let  $\Phi : [\omega_1]^\omega \rightarrow {}^\omega 2$ . For each  $n \in \omega$ , define  $P_n : [\omega_1]^\omega \rightarrow 2$  by  $P_n(f) = f(n)$ . By  $\text{AC}_\omega^\mathbb{R}$ , let  $C_n \subseteq \omega_1$  be club homogeneous for  $P_n$  taking some value  $i_n \in 2$ . Let  $C = \bigcap_{n \in \omega} C_n$ . Let  $r \in {}^\omega 2$  by  $r(n) = i_n$ . Note that for all  $\Phi[[C]_*^\omega] = \{r\}$ . Thus  $\Phi$  is not an injection.

Now suppose  $\Phi : [\omega_1]^\omega \rightarrow \kappa \sqcup \mathbb{R}$ . Define  $Q : [\omega_1]^\omega \rightarrow 2$  by

$$Q(f) = \begin{cases} 0 & \Phi(f) \in \kappa \\ 1 & \Phi(f) \in \mathbb{R} \end{cases}$$

Let  $C \subseteq \omega_1$  be club homogeneous for  $Q$ . If  $C$  is homogeneous for  $Q$  taking value 0, then  $\Phi$  maps  $[C]_*^\omega$  into  $\kappa$ . By the earlier argument,  $\Phi$  can not be an injection. If  $C$  is homogeneous for  $Q$  taking value 1, the  $\Phi$  maps  $[C]_*^\omega$  into  $\mathbb{R}$ . Again by the earlier argument,  $\Phi$  can not be an injection.

Suppose  $\Phi : [\omega_1]^\omega \rightarrow \mathbb{R} \times \omega_1$ . Let  $\pi_1 : \mathbb{R} \times \omega_1 \rightarrow \mathbb{R}$  be the projection onto the first coordinate. Then  $\pi_1 \circ \Phi : [\omega_1]^\omega \rightarrow \mathbb{R}$ . By the argument above, there is a club  $C \subseteq \omega_1$  and an  $r \in \mathbb{R}$  so that  $(\pi_1 \circ \Phi)[[C]_*^\omega] = \{r\}$ . Then  $\Phi : [C]_*^\omega \rightarrow \{r\} \times \omega_1$ . Since  $\{r\} \times \omega_1$  is in bijection with  $\omega_1$ ,  $\Phi$  can not be an injection by the earlier part of this proof.

The result for  $[\omega_2]^\omega$  follows by the same argument using the weak partition property for  $\omega_2$ .  $\square$

The cardinal relations displayed in the diagram at the beginning should follow from the work so far.

#### 4. $[\omega_1]^\omega$ IS JÓNSSON

**Definition 4.1.** Let  $X$  be a set. Define  $[X]_\equiv^n = \{f \in {}^n X : (\forall i < j < n)(f(i) \neq f(j))\}$ . Let  $[X]_\equiv^{\leq \omega} = \bigcup_{n \in \omega} [X]_\equiv^n$ .

For  $n < \omega$ ,  $X$  is  $n$ -Jónsson if and only if for every  $\Phi : [X]_\equiv^n \rightarrow X$ , there is some  $Z \subseteq X$  with  $Z \approx X$  so that  $\Phi[[Z]_\equiv^n] \neq X$ .

$X$  is Jónsson if and only if for all  $\Phi : [X]_\equiv^{\leq \omega} \rightarrow X$ , there is some  $Z \subseteq X$  with  $Z \approx X$  so that  $\Phi[[X]_\equiv^{\leq \omega}] \neq X$ .

**Definition 4.2.** Let  $\bar{f} \in {}^{<\omega}([\omega_1]^\omega)$ . The tuple-type of  $\bar{f}$ , denoted  $\text{type}(\bar{f})$ , is a 4-tuple  $(n, m, G, D)$  with the following properties:

(1)  $n$  is the length of the tuple  $\bar{f}$ .

Let  $S = \{\sup(f_i) : i < n\}$ . Let  $\text{rang}(\bar{f}) = \bigcup_{i < n} \text{rang}(f_i)$ . Note that  $m$  also has the property that  $\text{ot}(\text{rang}(\bar{f})) = \omega \cdot m$ . Let  $\langle a_0, \dots, a_{m-1} \rangle$  be the increasing enumeration of  $S$ . Let  $F : \omega \cdot m \rightarrow \text{rang}(\bar{f})$  be the increasing enumeration of  $\text{rang}(\bar{f})$ .

(2) Then  $m = |S|$ .

(3)  $G : m \rightarrow \mathcal{P}(n)$  is defined by  $G(i) = \{k \in n : \sup(f_k) = a_i\}$ .

(4) Let  $D : \omega \cdot m \rightarrow \mathcal{P}(n)$  be defined by  $D(\alpha) = \{i \in n : F(\alpha) \in \text{rang}(f_i)\}$ .

If  $Z \subseteq [\omega_1]^{\omega_1}$ , then let  $\text{type}(Z) = \{\bar{f} \in {}^{<\omega}([Z]^\omega) : \text{type}(\bar{f})\}$ .

**Example 4.3.** Consider  $f_0, f_1, f_2 \in [\omega_1]^\omega$  defined by

$$f_0(x) = \begin{cases} 0 & x = 0 \\ x + 1 & x \geq 1 \end{cases}, \quad f_1(x) = \begin{cases} x & x = 0, 1 \\ \omega + 2(x - 1) & x \geq 2 \end{cases}$$

$$f_2(x) = \begin{cases} x & x = 0, 1 \\ \omega + (x - 2) & x = 2, 3 \\ \omega + 2(x - 3) + 1 & x \geq 4 \end{cases}$$

The first several values of  $f_0, f_1$ , and  $f_2$  are the following:

$$f_0 = \langle 0, 2, 3, 4, 5, 6, 7, \dots \rangle \quad f_1 = \langle 0, 1, \omega + 2, \omega + 4, \omega + 6, \omega + 8, \omega + 10, \dots \rangle$$

$$f_2 = \langle 0, 1, \omega, \omega + 1, \omega + 3, \omega + 5, \omega + 7, \omega + 9, \omega + 11, \dots \rangle.$$

The picture looks as follows: There are  $\omega \cdot 2$  many columns. Row 0, 1, and 2 indicate which values among  $\omega \cdot 2$  are taken by  $f_0, f_1$ , and  $f_2$ , respectively.

$$\begin{array}{cccccccc|cccccccc} 0 & & 0 & 0 & 0 & 0 & \dots & | & & & & & & & & & \\ 1 & 1 & & & & & & | & & 1 & & 1 & & 1 & & 1 & \dots \\ 2 & 2 & & & & & & | & 2 & 2 & & 2 & & 2 & & 2 & \dots \end{array}$$

Then  $\text{type}((f_0, f_1, f_2)) = (3, 2, G, D)$  where  $G$  and  $D$  are defined as follows:  $G : 2 \rightarrow \mathcal{P}(3)$  is defined by  $G(0) = \{0\}$  and  $G(1) = \{1, 2\}$ . The function  $D : \omega \cdot 2 \rightarrow \mathcal{P}(3)$  can be read off the diagram above by

$$D(\alpha) = \begin{cases} \{0, 1, 2\} & \alpha = 0 \\ \{1, 2\} & \alpha = 1 \\ \{0\} & 2 \leq \alpha < \omega \\ \{2\} & \alpha = \omega, \omega + 1 \\ \{1\} & (\exists k \in \omega)[\alpha = \omega + 2(k + 1)] \\ \{2\} & (\exists k \in \omega)[\alpha = \omega + 2(k + 1) + 1] \end{cases}$$

With Definition 4.2 as the motivation, one makes the following abstract definition of a tuple-type:

**Definition 4.4.** A tuple-type  $t$  is a 4-tuple  $(n, m, G, D)$  with the following properties:

(1)  $n \in \omega$  and  $n > 0$  which is called the length of tuple type.

(2)  $1 \leq m \leq n$  which is called the arrangement number of the tuple type.

(3)  $G : m \rightarrow \mathcal{P}(n)$  with the property that for all  $i < m$ ,  $G(i) \neq \emptyset$  and for all  $i < j < m$ ,  $G(i) \cap G(j) = \emptyset$ .  $G$  is called the grouping order of the tuple-type.

(4)  $D : \omega \cdot m \rightarrow \mathcal{P}(n)$ , which is called the distribution of the type, is a function with the following properties:

(a) For each  $i < m$  and  $l \in \omega$ ,

$$D(\omega \cdot i + l) \cap \left( \bigcup_{j < i} G(j) \right) = \emptyset.$$

(b) For each  $k < n$ , if  $k \in G(i)$ , then  $\{l \in \omega : k \in D(\omega \cdot i + l)\}$  is infinite.

(c) For each  $k < n$ , if  $k \in G(i)$ , then for each  $j < i$ ,  $\{l \in \omega : k \in D(\omega \cdot j + l)\}$  is finite.

Observe that if  $\bar{f} \in {}^{<\omega}([\omega_1]^\omega)$ , then the tuple-type of  $\bar{f}$ ,  $\text{type}(\bar{f})$ , is a tuple-type as defined in Definition 4.4.

**Definition 4.5.** Let  $t = (n, m, G, D)$  be a tuple-type. Let  $h : [\omega_1]^{\omega \cdot m} \rightarrow \omega_1$ . For  $i < n$ , let  $f_i^{t,h}$  be defined to be the increasing enumeration of  $\{h(\alpha) : i \in D(\alpha)\}$ . Note that the properties of the distribution imply that  $f_i^{t,h} \in [\omega_1]^\omega$ .

Define  $\text{extract}(t, h) = (f_0^{t,h}, \dots, f_{n-1}^{t,h})$ . This is the  $n$ -tuple extracted from  $h$  of tuple-type  $t$ . Note that  $\text{type}(\text{extract}(t, h)) = t$ .

**Definition 4.6.** Let  $X$  be any set and  $P : \omega \rightarrow X$ .  $P$  is eventually periodic if and only if there exists  $k, p \in \omega$  and  $x_0, \dots, x_{p-1} \in X$  so that for all  $n > k$ ,  $P(n) = x_i$  where  $i < p$  is such that  $n - k$  is congruent to  $i \bmod p$ .

A tuple-type  $t = (n, m, G, D)$  is an eventually periodic tuple-type if and only if for each  $i < m$ , the function  $P_i : \omega \rightarrow \mathcal{P}(n)$  defined by  $P_i(k) = D(\omega \cdot i + k)$  is eventually periodic.

Note that there are only countably many eventually periodic tuple-types.

**Definition 4.7.** Let  $L$  be the collection of finite tuples  $(\alpha, n, \beta_0, \dots, \beta_n)$  where  $\alpha < \omega_1$ ,  $n \in \omega$ ,  $\beta_0 < \beta_1 < \dots < \beta_n < \alpha$ . Let  $\prec$  be the lexicographic ordering on  $L$ . Let  $\mathcal{L} = (L, \prec)$ . Note that  $\text{ot}(\mathcal{L}) = \omega_1$ .

Let  $H \in [\omega_1]^\mathcal{L}$ , that is an order-preserving function of  $\mathcal{L}$  into  $\omega_1$ .

Define  $\Lambda^H : [\omega_1]^\omega \rightarrow [\omega_1]^\omega$  by  $\Lambda(f)(k) = H(\sup(f), k, f(0), \dots, f(k))$ .

**Lemma 4.8.**  $\Lambda^H$  is an injection and  $\text{type}(\Lambda^H[[\omega_1]^\omega])$  consists only of eventually periodic tuple-types.

*Proof.* Suppose  $f, g \in [\omega_1]^\omega$  with  $f \neq g$ .

(Case I) Suppose  $\sup(f) \neq \sup(g)$ . Without loss of generality, suppose  $\sup(f) < \sup(g)$ . Then  $\Lambda^H(f)(0) = H(\sup(f), 0, f(0)) < H(\sup(g), 0, g(0)) = \Lambda^H(g)(0)$ . Therefore,  $\Lambda^H(f) \neq \Lambda^H(g)$ .

(Case II) Suppose  $\sup(f) = \sup(g)$ .  $f \neq g$  implies that there is a least  $k$  so that  $f(k) \neq g(k)$ . Without loss of generality, suppose  $f(k) < g(k)$ . Then  $\Lambda^H(f)(k) = H(\sup(f), k, f(0), \dots, f(k)) < H(\sup(g), k, g(0), \dots, g(k)) = \Lambda^H(g)(k)$ . So  $\Lambda^H(f) \neq \Lambda^H(g)$ .

It has been shown that  $\Lambda^H$  is an injection.

Now suppose  $\bar{f} = (f_0, \dots, f_{n-1}) \in {}^{<\omega}([\omega_1]^\omega)$ . Let  $\Lambda^H(\bar{f}) = (\Lambda^H(f_0), \dots, \Lambda^H(f_{n-1}))$ . Let  $\text{type}(\bar{f}) = (n, m, G, D)$ . Suppose  $\text{type}(\Lambda^H(\bar{f})) = (n', m', G', D')$ .

For  $i < j < n$ , if  $\sup(f_i) < \sup(f_j)$ , then

$$\Lambda^H(f_i)(a) = H(\sup(f_i), a, f_i(0), \dots, f_i(a)) < H(\sup(f_j), b, f_j(0), \dots, f_j(b)) = \Lambda^H(f_j)(b)$$

for any  $a, b \in \omega$ . This implies that if  $\sup(f_i) < \sup(f_j)$ , then  $\sup(\Lambda^H(f_i)) < \sup(\Lambda^H(f_j))$ . This shows that  $m' = m$  and  $G' = G$ .

Pick any  $i < m$ . Let  $P_i(k) = D'(\omega \cdot i + k)$ . Pick a  $\ell \in \omega$  large enough so that for all  $a, b \in G(i)$ , if  $f_a \neq f_b$ , then there is some  $\iota < \ell$  so that  $f_a(\iota) \neq f_b(\iota)$ .

Define an preordering  $\sqsubseteq$  on  $G(i)$  by  $a \sqsubseteq b$  if and only if  $f_a \upharpoonright \ell = f_b \upharpoonright \ell$  or  $f_a \upharpoonright \ell$  is lexicographically less than  $f_b \upharpoonright \ell$ . The  $\sqsubseteq$ -preordering classes of  $G(i)$  are naturally linearly ordered. Note that  $P_i$  is eventually periodic by repeating the  $\sqsubseteq$ -preordering classes of  $G(i)$  in this natural order.

It has been established that  $\text{type}(\Lambda^H(\bar{f}))$  is an eventually periodic tuple-type.  $\square$

**Example 4.9.** Let  $f_0, f_1$ , and  $f_2$  be the functions from Example 4.3. Let  $H : \mathcal{L} \rightarrow \omega_1$  be any order-preserving function of the correct type. Let  $\text{type}((f_0, f_1, f_2)) = (3, 2, G, D)$ . Let  $\Lambda^H$  be the associated function as defined above. Let  $\text{type}((\Lambda^H(f_0), \Lambda^H(f_1), \Lambda^H(f_2))) = (3, 2, G, D')$ , where  $D'$  is defined below:

Observe that in  $\mathcal{L} = (L, \preceq)$ , the following objects are arranged as follows:

$$\begin{aligned} &(\omega, 0, 0) \prec (\omega, 1, 0, 2) \prec (\omega, 2, 0, 2, 3) \prec (\omega, 3, 0, 2, 3, 4) \prec \dots \prec (\omega \cdot 2, 0, 0) \prec (\omega \cdot 2, 1, 0, 1) \\ &\prec (\omega \cdot 2, 2, 0, 1, \omega) \prec (\omega \cdot 2, 2, 0, 1, \omega + 2) \prec (\omega \cdot 2, 3, 0, 1, \omega + 1) \prec (\omega \cdot 2, 3, 0, 1, \omega + 2, \omega + 4) \prec (\omega \cdot 2, 4, 0, 1, \omega, \omega + 1, \omega + 3) \\ &\prec (\omega \cdot 2, 4, 0, 1, \omega + 2, \omega + 4, \omega + 6) \prec (\omega \cdot 2, 5, 0, 1, \omega, \omega + 1, \omega + 3, \omega + 5) \prec \dots \end{aligned}$$

This implies that

$$\begin{aligned} &\Lambda^H(f_0)(0) < \Lambda^H(f_0)(1) < \Lambda^H(f_0)(2) < \Lambda^H(f_0)(3) < \Lambda^H(f_0)(4) < \dots \\ &< \Lambda^H(f_1)(0) = \Lambda^H(f_2)(0) < \Lambda^H(f_2)(1) = \Lambda^H(f_1)(1) < \Lambda^H(f_2)(2) \end{aligned}$$



$$< \Lambda^H(f_1)(2) < \Lambda^H(f_2)(3) < \Lambda^H(f_1)(3) < \Lambda^H(f_2)(4) < \Lambda^H(f_1)(4) < \Lambda^H(f_2)(5)$$

From the example above, the diagram for  $D'$  is given below. In his diagram,  $\hat{0}$ ,  $\hat{1}$ , and  $\hat{2}$  represent  $\Lambda^H(f_0)$ ,  $\Lambda^H(f_1)$ , and  $\Lambda^H(f_2)$ :

$$\begin{array}{cccccccc|cccccccc} \hat{0} & \hat{0} & \hat{0} & \hat{0} & \hat{0} & \hat{0} & \cdots & | & & & & & & & & & \\ & & & & & & & | & \hat{1} & \hat{1} & \hat{1} & \hat{1} & \hat{1} & \hat{1} & \cdots & & \\ & & & & & & & | & \hat{2} & \hat{2} & \hat{2} & \hat{2} & \hat{2} & \hat{2} & \hat{2} & \cdots \end{array}$$

Explicitly,  $D' : \omega \cdot 2 \rightarrow \mathcal{P}(3)$  is

$$D'(\alpha) = \begin{cases} \{0\} & \alpha < \omega \\ \{1, 2\} & \alpha = \omega, \omega + 1 \\ \{2\} & (\exists k \in \omega)[\alpha = \omega + 2(k + 1)] \\ \{1\} & (\exists k \in \omega)[\alpha = \omega + 2(k + 1) + 1] \end{cases}$$

Note that  $P_0(k) = D'(k)$  is eventually periodic by repeating  $\{0\}$  and  $P_1(k) = D'(\omega + k)$  is eventually periodic by eventually alternating between  $\{1\}$  and  $\{2\}$ .

**Fact 4.10.** Let  $\Phi : {}^{<\omega}([\omega_1]^\omega) \rightarrow [\omega_1]^\omega$  be a function. Let  $t = (n, m, G, D)$  be a tuple-type. Let  $\mu$  denote the club measure on  $\omega_1$ . Let  $\Phi^{t,k} : [\omega_1]_{*}^{\omega \cdot m} \rightarrow \omega_1$  be defined  $\Phi^{t,k}(h) = \Phi(\text{extract}(t, h))(k)$ .

If for  $\mu^{\omega \cdot m}$ -almost all  $h$ ,  $\Phi^{t,k}(h) < h(0)$ , then for  $\mu^{\omega \cdot m}$ -almost all  $h$ ,  $\Phi^{t,k}(h)$  take a constant value  $c_k^{\Phi,t}$ .

*Proof.* This follows from the countable additivity of  $\mu^{\omega \cdot m}$ .  $\square$

**Definition 4.11.** Assume the setting of fact 4.10. Let  $d^{\Phi,t}$  be the least  $k$  if it exists so that  $\Phi^{t,k}(h) \geq h(0)$  for  $\mu^{\omega \cdot m}$ -almost all  $h$ . Otherwise, let  $d^{\Phi,t} = \omega$ .

Let  $\text{stem}^{\Phi,t} : d^{\Phi,t} \rightarrow \omega_1$  be defined by  $\text{stem}^{\Phi,t}(j) = c_j^{\Phi,t}$ , where  $j < d^{\Phi,t}$ .

Thus for  $\mu^{\omega \cdot m}$ -almost all  $h$ ,  $\text{stem}^{\Phi,t} \subseteq \Phi(\text{extract}(t, h))$  and if  $d^{\Phi,t} < \omega$ , then  $\Phi(\text{extract}(\Phi, t))(d^{\Phi,t}) \geq h(0)$ .

**Theorem 4.12.**  $[\omega_1]^\omega$  is Jónsson.

*Proof.* One will prove a slightly stronger version of the Jónsson property:

Let  $\Phi : {}^{<\omega}([\omega_1]^\omega) \rightarrow [\omega_1]^\omega$ .

Using  $\text{AC}_\omega^\mathbb{R}$  and the discussion in Definition 4.11, for each (of the countably many) eventually periodic tuple-type  $t$ , let  $C_t \subseteq \omega_1$  be a club so that for all  $h \in [C_t]_{*}^\omega$ ,  $\text{stem}^{\Phi,t} \subseteq \Phi(\text{extract}(t, h))$  and if  $d^{\Phi,t} < \omega$ , then  $\Phi(\text{extract}(\Phi, t))(d^{\Phi,t}) \geq h(0)$ .

Let  $\zeta$  be the supremum of  $\sup(\text{stem}^{\Phi,t})$  as  $t$  ranges over the countable set of eventually periodic tuple-types. As  $\omega_1$  is regular,  $\zeta < \omega_1$ . Let  $C$  be the intersection of all  $C_t$  as  $t$  ranges over all eventually periodic tuple-type. By removing an initial segment of  $C$ , one may assume that  $\zeta < \min(C) + 1$ .

Let  $H : \mathcal{L} \rightarrow C$  be any order-preserving function of the correct type. Note that  $\Lambda^H(f) \in [\omega_1]_{*}^\omega$ , i.e. is also a function of the correct type for any  $f \in [\omega_1]^\omega$ .

Let  $Z = \Lambda^H[[\omega_1]^\omega]$ . Since  $\Lambda^H$  is an injection by Lemma 4.8,  $Z \approx [\omega_1]^\omega$ .

Now suppose  $\bar{f} = (f_0, \dots, f_{n-1}) \in {}^{<\omega}Z$ . By Lemma 4.8,  $t = \text{type}(\bar{f}) = (n, m, G, D)$  is an eventually periodic tuple-type. There is a unique  $h \in [C]_{*}^{\omega \cdot m}$  so that  $\text{extract}(t, h) = \bar{f}$ . In particular, since  $h \in [C_t]_{*}^{\omega \cdot m}$ ,  $\text{stem}^{\Phi,t} \subseteq \Phi(\bar{f})$  and if  $d^{\Phi,t} < \omega$ ,  $\Phi(\bar{f})(d^{\Phi,t}) \geq h(0) \geq \min(C) > \zeta$ . This and the definition of  $\zeta$  imply that  $\zeta \notin \text{rang}(\Phi(\bar{f}))$ .

It has been shown that for all  $\bar{f} \in {}^{<\omega}Z$ ,  $\zeta \notin \text{rang}(\Phi(\bar{f}))$ . In particular,  $\Phi[{}^{<\omega}Z] \neq [\omega_1]^\omega$ .

As  $\Phi$  was arbitrary, this implies that  $[\omega_1]^\omega$  is Jónsson.  $\square$

## REFERENCES

1. William Chan, *An introduction to combinatorics of determinacy*, In preparation.
2. William Chan, *Ordinal definability and combinatorics of equivalence relations*, Journal of Mathematical Logic **0** (0), no. 0, 1950009.
3. William Chan and Stephen Jackson, *Applications of infinity-borel codes to definability and definable cardinals*, In Preparation.
4. William Chan and Stephen Jackson, *Definable combinatorics at the first uncountable cardinal*, In preparation.

5. William Chan and Stephen Jackson, *Cardinality of Wellordered Disjoint Unions of Quotients of Smooth Equivalence Relations*, arXiv e-prints (2019), arXiv:1903.03902.
6. William Chan and Connor Meehan, *Definable Combinatorics of Some Borel Equivalence Relations*, ArXiv e-prints (2017).
7. L. A. Harrington, A. S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), no. 4, 903–928. MR 1057041
8. Greg Hjorth, *A dichotomy for the definable universe*, J. Symbolic Logic **60** (1995), no. 4, 1199–1207. MR 1367205
9. S. Jackson, R. Ketchersid, F. Schlutzenberg, and W. H. Woodin, *Determinacy and Jónsson cardinals in  $L(\mathbb{R})$* , J. Symb. Log. **79** (2014), no. 4, 1184–1198. MR 3343535
10. Steve Jackson, *A computation of  $\delta_5^1$* , Mem. Amer. Math. Soc. **140** (1999), no. 670, viii+94. MR 1606035
11. ———, *Structural consequences of AD*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 1753–1876. MR 2768700
12. Alexander S. Kechris, *AD and projective ordinals*, Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77), Lecture Notes in Math., vol. 689, Springer, Berlin, 1978, pp. 91–132. MR 526915
13. Eugene M. Kleinberg, *Infinitary combinatorics and the axiom of determinateness*, Lecture Notes in Mathematics, Vol. 612, Springer-Verlag, Berlin-New York, 1977. MR 0479903
14. W. Hugh Woodin, *The cardinals below  $|\omega_1|^{<\omega_1}$* , Ann. Pure Appl. Logic **140** (2006), no. 1-3, 161–232. MR 2224057

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON, TX 76203  
*Email address:* William.Chan@unt.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON, TX 76203  
*Email address:* Stephen.Jackson@unt.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON, TX 76203  
*Email address:* Nam.Trang@unt.edu