MORE DEFINABLE COMBINATORICS AROUND THE FIRST AND SECOND UNCOUNTABLE CARDINAL

WILLIAM CHAN, STEPHEN JACKSON, AND NAM TRANG

Abstract. Assume $\mathsf{ZF} + \mathsf{AD}$. If ϵ is an ordinal and X is a set of ordinals, then $[X]^\epsilon_*$ is the collection of order-preserving functions $f : \epsilon \to X$ which have uniform cofinality ω and discontinuous everywhere. The weak partition properties on ω_1 and ω_2 yield partition measures on $[\omega_1]^\epsilon_*$ when $\epsilon < \omega_1$ and $[\omega_2]^\epsilon_*$ when $\epsilon < \omega_2$. The following almost everywhere continuity properties for functions on partition spaces with respect to these partition measures will be shown.

For every $\epsilon < \omega_1$ and function $\Phi : [\omega_1]^\epsilon \to \omega_1$, there is a club $C \subseteq \omega_1$ and a $\zeta < \epsilon$ so that for all $f, g \in [C]^\epsilon_*$, if $f \restriction \zeta = g \restriction \zeta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.

For every $\epsilon < \omega_2$ and function $\Phi : [\omega_2]^\epsilon \to \omega_2$, there is an ω-club $C \subseteq \omega_2$ and a $\zeta < \epsilon$ so that for all $f, g \in [C]^\epsilon_*$, if $f \restriction \zeta = g \restriction \zeta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.

The previous two continuity results will be used to distinguish the cardinalities of some important subsets of $\mathscr{P}(\omega_2)$: $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$. $|[\omega_2]^\omega| < |[\omega_2]^{<\omega_1}| < |[\omega_2]^{\omega_1}| < |[\omega_2]^{<\omega_2}|$. $\neg(|[\omega_1]^{<\omega_1}| \leq |[\omega_2]^\omega|)$. $\neg(|[\omega_1]^{\omega_1}| \leq |[\omega_2]^{<\omega_1}|)$.

It will also be shown that $[\omega_1]^\omega$ has the Jónsson property: For every $\Phi : {}^{<\omega}([\omega_1]^\omega) \to [\omega_1]^\omega$, there is an $X \subseteq [\omega_1]^\omega$ with $|X| = |[\omega_1]^\omega|$ so that $\Phi[{}^{<\omega}X] \neq [\omega_1]^\omega$.

1. Introduction

Under the axiom of determinacy, AD, the cardinalities of sets have a very rich and non-linear structure. The cardinalities of wellorderable sets are called cardinals. ω_1 and ω_2 refer to the first and second uncountable cardinals, respectively. This article will distinguish the cardinalities of some important subsets of $\mathscr{P}(\omega_1)$ (the power set of ω_1) and $\mathscr{P}(\omega_2)$ (the power set of ω_2) under AD. Since cardinalities are compared through injections, a deep understanding of the behavior of functions between the relevant sets will be necessary. This will be obtained through a complete analysis of the continuity properties of functions of the form $\Phi : [\omega_1]^\epsilon \to \omega_1$ when $\epsilon < \omega_1$ and functions of the form $\Phi : [\omega_2]^\epsilon \to \omega_2$ when $\epsilon < \omega_2$. The arguments in this article are entirely combinatorial and should be accessible with minimal knowledge of determinacy. The necessary combinatorial consequences of determinacy such as the partition relations on ω_1 and ω_2, the ultrapower representation of ω_2, and some combinatorial tools to handle this ultrapower such as Kunen functions and sliding arguments will be reviewed.

Descriptive set theorists have recently studied the definable cardinalities of quotients of equivalence relations on Polish spaces through definable reductions. If E is an equivalence relation on $\mathbb{R}$, then let R/E denote the set of equivalence classes of E. If E and F are two equivalence relations on $\mathbb{R}$, then a reduction between E and F is a function $\Lambda : \mathbb{R} \to \mathbb{R}$ so that for all $x, y \in \mathbb{R}$, $x \ E \ y$ if and only if $\Lambda(x) \ F \ \Lambda(y)$. The reduction Λ between E and F induces an injection $\Sigma : \mathbb{R}/E \to \mathbb{R}/F$. Motivated by this, an injection $\Sigma : \mathbb{R}/E \to \mathbb{R}/F$ is said to be a Borel definable injection if and only if Σ is induced by a Borel reduction $\Lambda : \mathbb{R} \to \mathbb{R}$ between E and F.

There are several important dichotomy results of descriptive set theory which elucidate the structure of the quotients of Borel equivalence relations under Borel definable injections. Silver ([17]) showed that if E is a Borel (or even coanalytic) equivalence relation, then either

- E has countably many classes or
- there is a Borel reduction of the equality equivalence relation $=$ on $\mathbb{R}$ into E.

Thus the quotient of a Borel equivalence relation E is either countable or there is a Borel definable injection of $\mathbb{R}$ into $\mathbb{R}/E$. Let E_0 be the equivalence relation on ${}^\omega 2$ of eventual equality defined by $x \ E_0 \ y$ if and only

October 31, 2022. The first author was supported by NSF grant DMS-1703708. The second author was supported by NSF grant DMS-1800323. The third author was supported by NSF grant DMS-1855757 and DMS-1945592.

if $(\exists m)(\forall n \geq m)(x(n) = y(n))$. Harrington, Kechris, and Louveau [9] showed that for any Borel equivalence relation E, either

- there is a Borel reduction of E into the equality relation $=$ or
- there is a Borel reduction of E_0 into E.

Thus for any Borel equivalence relation E, either there is a Borel definable injection of $\mathbb{R}/E$ into $\mathbb{R}$ (which is in bijection with $\mathscr{P}(\omega)$) or there is a Borel definable injection of $\mathbb{R}/E_0$ into $\mathbb{R}/E$.

With the axiom of choice, this nice structure for the definable cardinalities under definable injections collapses since all these quotients are in bijection with $\mathbb{R}$. In the spirit of descriptive set theory, this paper will be interested in definable cardinalities studied using definable maps which can either be interpreted by restricting functions to certain classes (like the class of Borel functions, as in the classical examples above) or by working within models of determinacy, which will be the approach taken here. The axiom of determinacy, AD, asserts that every two player game where each player takes turns playing a natural number has a winning strategy for one of the two players. Determinacy axioms allow the structure of the definable cardinalities of sets (which are surjective images of $\mathbb{R}$) to possess a structure that resembles the structure of Borel definable cardinalities and this structure is established through techniques that have a descriptive set theoretic flavor.

The two dichotomy results for Borel reductions mentioned above are proved by using the Gandy-Harrington forcing of lightface Σ^1_1 subsets of $\mathbb{R}$ developed in [10]. In an extension of AD called AD^+, highly absolute definitions for equivalence relations called ∞-Borel codes exist. The Vopěnka forcing of ordinal definable (relative to the ∞-Borel code) subsets of $\mathbb{R}$ can be used to extend Silver's dichotomy and the E_0-dichotomy into cardinality dichotomies in AD^+. Generalizing Silver's dichotomy, Woodin's perfect set dichotomy ([3], [1]) states that if E is an equivalence relation on $\mathbb{R}$, then either

- $\mathbb{R}/E$ is wellorderable (that is, injects into an ordinal) or
- $\mathbb{R}$ injects into $\mathbb{R}/E$.

Since all sets which are surjective images of $\mathbb{R}$ are in bijection with a quotient of an equivalence relation on $\mathbb{R}$, this can be reformulated to say that for all sets X which are surjective images of $\mathbb{R}$, either X is wellorderable or $\mathbb{R}$ injects into X. In $L(\mathbb{R}) \models \mathsf{AD}$, Caicedo and Ketchersid [1] extended these results further by showing every set $X \in L(\mathbb{R})$ is either wellorderable or $\mathbb{R}$ injects into X. Generalizing the E_0-dichotomy, Hjorth's E_0-dichtomy ([11]) states that if E is an equivalence relation on $\mathbb{R}$, then either

- $\mathbb{R}/E$ injects into $\mathscr{P}(\delta)$ for some ordinal δ or
- $\mathbb{R}/E_0$ injects into $\mathbb{R}/E$.

The first two authors have recently obtained additional new cardinality results for quotients of equivalence relations on $\mathbb{R}$ in $L(\mathbb{R}) \models \mathsf{AD}$. Borrowing a term from classical descriptive set theory, an equivalence relation E on $\mathbb{R}$ is strongly smooth if and only if $\mathbb{R}/E$ is in bijection with $\mathbb{R}$. In $L(\mathbb{R}) \models \mathsf{AD}$, many subsets of $\mathscr{P}(\omega_1)$ are in bijection with an ω_1-length disjoint union of quotients of strongly smooth equivalence relations on $\mathbb{R}$; however, only one cardinality can be represented in this way if each equivalence relation has only countable equivalence classes: Combining ideas from the Woodin perfect set dichotomy and Hjorth's E_0-dichotomy, [5] Theorem 5.8 showed that in $L(\mathbb{R}) \models \mathsf{AD}$, if $\langle E_\alpha : \alpha < \omega_1 \rangle$ is a sequence of strongly smooth equivalence relations on $\mathbb{R}$ so that each E_α has all countable equivalence classes, then the disjoint union $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ is in bijection with $\mathbb{R} \times \omega_1$.

Another classical cardinality result under AD is the perfect set property which asserts that every subset of $\mathbb{R}$ is either countable or contains a perfect subset (a nonempty closed set with no isolated points). Since $\mathbb{R}$ is in bijection with $\mathscr{P}(\omega)$, this result completely characterizes the cardinalities of sets below $\mathscr{P}(\omega)$ by establishing a suitable form of the continuum hypothesis: All subsets of $\mathscr{P}(\omega)$ are either countable or in bijection with $\mathscr{P}(\omega)$. This article and other recent work of the authors seek to understand the structure of the cardinalities below $\mathscr{P}(\omega_1)$ and $\mathscr{P}(\omega_2)$.

By the Moschovakis coding lemma, $\mathbb{R}$ surjects onto $\mathscr{P}(\omega_1)$ and $\mathscr{P}(\omega_2)$. Thus every subset of $\mathscr{P}(\omega_1)$ and $\mathscr{P}(\omega_2)$ is in bijection with a quotient of an equivalence relation on $\mathbb{R}$. Rather than viewing these sets as quotients of equivalence relations, the approach of this paper will be to consider these sets as sets of increasing sequences of ordinals and use an important consequence of determinacy known as the partition relations on ω_1 and ω_2. Both the descriptive set theoretic and the combinatorial approaches seem useful and necessary for studying cardinalities under determinacy. The following will summarize the results of this paper and its context within determinacy.

Let A and B be two sets. If there is an injection from A into B, then write $|A| \leq |B|$. Denote $|A| < |B|$ if $|A| \leq |B|$ but $\neg(|B| \leq |A|)$. If there is a bijection between A and B, then one writes $|A| = |B|$. By the Cantor-Schröder-Bernstein theorem (proved in ZF), $|A| = |B|$ if and only $|A| \leq |B|$ and $|B| \leq |A|$. In the absence of choice, the cardinality of A, referred to as $|A|$, is the equivalence class of A under the bijection relation.

To understand cardinalities and injections, one will need to study functions between sets under determinacy. One such classical result concerns continuity for functions from $\mathbb{R}$ to $\mathbb{R}$. Assuming AD, every function $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous on a comeager subset of $\mathbb{R}$. As customary in descriptive set theory, thinking of $\mathbb{R}$ as ${}^\omega\omega$ (the collection of functions from ω into ω), continuity can be understood using the following example: $\Phi(f)(0)$, the first bit of $\Phi(f)$, a priori could require global information about all of f. Continuity on a comeager set implies that if f belongs to this comeager set, then $\Phi(f)(0)$ only depends on a local behavior of f. That is, there is some $n \in \omega$ so that for all g which belong to this appropriate comeager set, if $g \upharpoonright n = f \upharpoonright n$, then $\Phi(g)(0) = \Phi(f)(0)$. Continuity of Φ on this comeager set means this property holds for the k^{th} bit of $\Phi(f)$ for each $k \in \omega$ and f belonging to the suitable comeager set.

Identifying subsets of ω_1 or ω_2 by their increasing enumeration, one will prefer to work with the collection of increasing sequences through ω_1 and ω_2 (primarily because the partition properties are formulated for these sets). If $\epsilon \leq \delta$ are two ordinals, then $[\delta]^\epsilon$ is the collection of increasing functions $f : \epsilon \to \delta$. Let $[\delta]^{<\epsilon} = \bigcup_{\gamma<\epsilon}[\delta]^\gamma$. This paper will be particularly interested in $[\omega_1]^\omega$, $[\omega_1]^{<\omega_1}$, $[\omega_2]^\omega$, $[\omega_2]^{\omega_1}$, and $[\omega_1]^{<\omega_2}$.

This article will study the short functions on ω_1 and ω_2 (i.e. functions $\Phi : [\omega_1]^\epsilon \to \omega_1$ when $\epsilon < \omega_1$ or $\Phi : [\omega_2]^\epsilon \to \omega_2$ when $\epsilon < \omega_2$). The continuity phenomenon for full functions on ω_1 (i.e. $\Phi : [\omega_1]^{\omega_1} \to \omega_1$) is investigated in [6], and the techniques there are quite different than what is used here. The first two authors [6] showed that for every function $\Phi : [\omega_1]^{\omega_1} \to \omega_1$, there is a club $C \subseteq \omega_1$ with the property that for all $f \in [C]^{\omega_1}_*$, there exists an $\alpha < \omega_1$ so that for all $g \in [C]^{\omega_1}_*$, if $g \upharpoonright \alpha = f \upharpoonright \alpha$, then $\Phi(f) = \Phi(g)$. ($[C]^{\omega_1}_*$ is the collection of increasing functions from ω_1 into C of the correct type, which will be defined in Definition 2.1.) The authors [6] also showed an even stronger version that for every function $\Phi : [\omega_1]^{\omega_1} \to {}^{\omega_1}\omega_1$, there is a club $C \subseteq \omega_1$ so that for all $f \in [C]^{\omega_1}_*$ and $\beta < \omega_1$, there exists an $\alpha < \omega_1$ so that for all $g \in [C]^{\omega_1}_*$, if $g \upharpoonright \alpha = f \upharpoonright \alpha$, then $\Phi(g) \upharpoonright \beta = \Phi(f) \upharpoonright \beta$. Note that this latter continuity property is just the standard notion of continuity where the domain and range spaces are given the topology generated by sets of the form $N_\sigma = \{f \in [\omega_1]^{\omega_1} : \sigma \subseteq f\}$ where $\sigma \in [\omega_1]^{<\omega_1}$ (or $N_\sigma = \{f \in {}^{\omega_1}\omega_1 : \sigma \subseteq f\}$ where $\sigma \in {}^{<\omega_1}\omega_1$) as a basis.

As a consequence of Martin's result that ω_1 is a strong partition cardinal, the filter μ^{ω_1} on $[\omega_1]^{\omega_1}$ defined by $X \in \mu^{\omega_1}$ if and only if there exists a club $C \subseteq \omega_1$ so that $[C]^{\omega_1}_* \subseteq X$ is a countably complete measure on ω_1. Thus in the above two continuity results, the notion of largeness given by comeagerness for classical continuity on $\mathbb{R}$ is replaced with largeness on $[\omega_1]^{\omega_1}$ given by the ultrafilter μ^{ω_1}. The continuity property for functions mentioned in the previous paragraph can be used to show that $|\mathscr{P}(\omega_1)| = |[\omega_1]^{\omega_1}|$ is "regular cardinality" with respect to wellordered decompositions: if $\langle X_\alpha : \alpha < \omega_1\rangle$ is a sequence of subsets of $[\omega_1]^{\omega_1}$ so that $[\omega_1]^{\omega_1} = \bigcup_{\alpha<\omega_1} X_\alpha$, then there is an $\alpha < \omega_1$ such that $|X_\alpha| = |[\omega_1]^{\omega_1}|$. This result can then be used to show that $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$. (See Fact 3.30 for a different argument using measures and certain inner models of ZFC.)

This article will be concerned with continuity phenomena for functions $\Phi : [\omega_1]^\epsilon \to \omega_1$ where $\epsilon < \omega_1$. The partition measure μ^ϵ on $[\omega_1]^\epsilon$ will serve as the notion of largeness for subsets of $[\omega_1]^\epsilon$. However, continuity in the sense described above is impossible for functions from $[\omega_1]^\omega$ into ω_1 by the following example. Consider the function $\Psi : [\omega_1]^\omega \to \omega_1$ defined by $\Psi(f) = \sup(f)$. There is no club $C \subseteq \omega_1$ so that for all $f \in [C]^\omega_*$, there is an $n < \omega$ such that whenever $g \in [C]^\omega_*$ and $f \upharpoonright n = g \upharpoonright n$, $\Psi(f) = \Psi(g)$. However, Ψ does satisfy a particular continuity phenomenon in the sense that $\Psi(f)$ depends only on one piece of information, namely $\sup(f)$. That is (by definition of Ψ), for any $f, g \in [\omega_1]^\omega$, if $\sup(f) = \sup(g)$, then $\Psi(f) = \Psi(g)$. The first main result is that this is a general occurrence that holds for any function $\Phi : [\omega_1]^\epsilon \to \omega_1$ when $\epsilon < \omega_1$. For each $f \in [\omega_1]^\epsilon$ and $\alpha \leq \epsilon$, let $\mathsf{bound}(f, \alpha) = \sup\{f(\beta) : \beta < \alpha\}$. Note that $\mathsf{bound}(f, 0) = 0$ and $\mathsf{bound}(f, \epsilon) = \sup(f)$.

Theorem 2.14. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* $\epsilon < \omega_1$ *and* $\Phi : [\omega_1]^\epsilon_* \to \omega_1$. *Then there is a decreasing sequence of ordinals which are less than or equal to* ϵ, $(\beta_i : i \leq n)$, *with* $\beta_n = 0$ *and a club* $C \subseteq \omega_1$ *so that if* $f, g \in [C]^\epsilon_*$ *has the property that* $\mathsf{bound}(f, \beta_i) = \mathsf{bound}(g, \beta_i)$ *for all* $i \leq n$, *then* $\Phi(f) = \Phi(g)$.

This result is a continuity property which states that for any such function Φ, $\Phi(f)$ depends only on local behaviors of f at certain finitely many places for μ^ϵ-almost all f. The following is a more coarse but useful consequence of the above result which states that for every function Φ, there is a $\delta < \epsilon$ so that $\Phi(f)$ depends only on the δ-length initial segment of f and $\sup(f)$.

Theorem 2.15. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* $\epsilon < \omega_1$ *and* $\Phi : [\omega_1]^\epsilon_* \to \omega_1$. *Then there is a* $\delta < \epsilon$ *and some club* $C \subseteq \omega_1$ *so that for all* $f, g \in [C]^\epsilon_*$ *with* $f \restriction \delta = g \restriction \delta$ *and* $\sup(f) = \sup(g)$, $\Phi(f) = \Phi(g)$.

$[\omega_1]^\omega$ and $[\omega_1]^{<\omega_1}$ are two distinguished subsets of $\mathscr{P}(\omega_1)$. One natural question is whether these two sets are different in terms of cardinality. Woodin [18] studied the cardinalities below $[\omega_1]^{<\omega_1}$ under $\mathsf{ZF}+\mathsf{AD}_\mathbb{R}+\mathsf{DC}$. From the dichotomy results in [18], it was known to Woodin that $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$. Moreover, Woodin isolated a subset of $[\omega_1]^{<\omega_1}$ called S_1 defined by $S_1 = \{f \in [\omega_1]^{<\omega_1} : \sup(f) = \omega_1^{L[f]}\}$. It is implicit in [18] that $|S_1|$ is incomparable with $[\omega_1]^\omega$ and hence one can conclude that $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$.

The proofs of some of the main properties of S_1 (assuming $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$ and all sets of reals have ∞-Borel codes) can be found [4] and [5]. Assuming just $\mathsf{ZF}+\mathsf{AD}$, one can show that $|\mathbb{R}| \leq |S_1|$ and $\neg(\omega_1 \leq |S_1|)$ (see [5] Fact 6.3). The main property of S_1 shown in [4] is that there is no injection of S_1 into ${}^\omega\mathrm{ON}$ assuming $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$ and all sets of reals have ∞-Borel codes. From this, one can conclude that $|\mathbb{R}| < |S_1|$ and $\neg(|S_1| \leq |[\omega_1]^\omega|)$. The argument for the main property of S_1 in [4] goes roughly as follows: Suppose such an injection Φ exists. Using ∞-Borel codes, one can find an inner model M of ZFC that "absorbs" some fragment of this injection in a suitable sense. Let $\zeta < \omega_1^V$ be an inaccessible cardinal of M. Since $\mathrm{Coll}(\omega, <\zeta)$ is countable in the real world satisfying AD, one can find a $G \subseteq \mathrm{Coll}(\omega, <\zeta)$ which is $\mathrm{Coll}(\omega, <\zeta)$-generic over M. One can show that G adds a $g \in S_1$ such that $M[G] = M[g]$. Since M "absorbs" Φ, $\Phi(g) \in M[g]$. Since Φ is an injection, one can argue that $M[g] = M[\Phi[g]]$. However, $\Phi(g)$ is an ω-sequence of ordinals. By a crucial property of the Lévy collapse, there is a $\xi < \zeta$ so that $\Psi(g) \in M[G \restriction \xi]$. Then one has that $M[G] = M[g] = M[\Phi(g)] = M[G \restriction \xi]$. This is impossible.

The authors know very little about the cardinality properties of S_1 in the absence of ∞-Borel codes. S_1 is a set whose definition is based upon the notion of constructibility. The two sets $[\omega_1]^\omega$ and $[\omega_1]^{<\omega_1}$ are very concrete combinatorial objects. There should be no need to employ AD^+ concepts to distinguish these two cardinalities. Using the continuity properties for short functions mentioned above, one can distinguish these two sets within $\mathsf{ZF} + \mathsf{AD}$ using combinatorial arguments.

Theorem 2.16. *Assume* $\mathsf{ZF} + \mathsf{AD}$. $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$.

Recently, the authors have used Theorem 2.16 as a backbone for more general results concerning injections of $[\omega_1]^{<\omega_1}$. For example, [7] showed under just $\mathsf{ZF} + \mathsf{AD}$ that there is no injection of $[\omega_1]^{<\omega_1}$ into ${}^\omega(\omega_\omega)$, the set of ω-sequences into ω_ω. Moreover with the addition of $\mathsf{DC}_\mathbb{R}$, [7] proved in $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_\mathbb{R}$ that there is no injection of $[\omega_1]^{<\omega_1}$ into ${}^\omega\mathrm{ON}$, the class of ω-sequences of ordinals. These results use a variety of combinatorial and descriptive set theoretic consequences of determinacy to reduce back to Theorem 2.16.

Next, one will consider various subsets of $\mathscr{P}(\omega_2)$. Of particular interests are $[\omega_2]^\omega$, $[\omega_2]^{<\omega_1}$, $[\omega_2]^{\omega_1}$, $[\omega_2]^{<\omega_2}$, and $[\omega_2]^{\omega_2}$. One would like to distinguish the cardinality of these sets from each other as well as from the cardinality of the subsets of $\mathscr{P}(\omega_1)$ considered earlier such as $[\omega_1]^\omega$, $[\omega_1]^{<\omega_1}$, and $[\omega_1]^{\omega_1}$.

Martin showed that ω_2 is a weak partition cardinal and hence measurable. Using the same technique mentioned above (for showing $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$) which involved using a measure and going into an appropriate inner model of ZFC, one can show $|[\omega_2]^{<\omega_2}| < |[\omega_2]^{\omega_2}|$ under just $\mathsf{ZF} + \mathsf{AD}$.

Similar to the study of ω_1, one needs to establish the analogous continuity property for ω_2.

Theorem 3.21. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* $\epsilon < \omega_2$ *and* $\Phi : [\omega_2]^\epsilon_* \to \omega_2$. *Then there is a decreasing sequence of ordinals less than or equal to* ϵ, $(\beta_i : i \leq n)$, *with* $\beta_n = 0$ *and an* ω*-club* $B \subseteq \omega_2$ *so that if* $\mathcal{F}, \mathcal{G} \in [B]^\epsilon_*$ *has the property that* $\mathsf{bound}(\mathcal{F}, \beta_i) = \mathsf{bound}(\mathcal{G}, \beta_i)$ *for all* $i \leq n$, *then* $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$.

Theorem 3.22. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* $\epsilon < \omega_2$ *and* $\Phi : [\omega_2]^\epsilon_* \to \omega_2$. *Then there is a* $\delta < \epsilon$ *and an* ω*-club* $B \subseteq \omega_2$ *so that for all* $\mathcal{F}, \mathcal{G} \in [B]^\epsilon_*$ *with* $\mathcal{F} \restriction \delta = \mathcal{G} \restriction \delta$ *and* $\sup(\mathcal{F}) = \sup(\mathcal{G})$, $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$.

Using these continuity results, one can establish the following cardinality relations:

Theorem 3.23. *Assume* $\mathsf{ZF} + \mathsf{AD}$. $|[\omega_2]^\omega| < |[\omega_2]^{<\omega_1}|$.

Theorem 3.24. *Assume* $\mathsf{ZF} + \mathsf{AD}$. $|[\omega_2]^{<\omega_1}| < |[\omega_2]^{\omega_1}|$.

Theorem 3.26. *Assume* $\mathsf{ZF} + \mathsf{AD}$. $|[\omega_2]^{\omega_1}| < |[\omega_2]^{<\omega_2}|$.

It should be mentioned that these results concerning ω_2 are proved in $\mathsf{ZF} + \mathsf{AD}$ and the arguments provided here are the only proofs presently known to the authors. That is, the authors do not know of an AD^+ style proof involving some analog of S_1. In the proof that S_1 does not inject into ${}^\omega\mathrm{ON}$ sketched above, one considered the forcing $\mathrm{Coll}(\omega, <\zeta)$ where $\zeta < \omega_1^V$ is an inaccessible cardinal of an inner model M of ZFC. In that case, one was able to find, in the real world, a generic over M since the forcing is countable in the real world. One may attempt to make analogs of S_1 to handle results at ω_2. However, the naturally associated forcing appears to be uncountable even in the real world, and one can no longer be certain that generics for such forcings exist in the real world.

To give a more complete picture of the relations between cardinalities, one also has the following results.

Theorem 3.29. *Assume* $\mathsf{ZF} + \mathsf{AD}$. $\neg(|[\omega_1]^{<\omega_1}| \leq |[\omega_2]^\omega|)$. *Thus* $\neg(|[\omega_1]^{\omega_1}| \leq [\omega_2]^\omega)$.

Theorem 3.31. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Then* $\neg(|[\omega_1]^{\omega_1}| \leq |[\omega_2]^{<\omega_1}|)$.

From the results mentioned throughout the paper, one has the following diagram depicting the relationships between the uncountable cardinalities below $\mathscr{P}(\omega_2)$ which will be discussed in this paper. An arrow between A and B indicates $|A| < |B|$. All relations among these cardinals are those derivable by compositions of the arrows on the diagram. Of course, there are other cardinals below $\mathscr{P}(\omega_2)$ which are not on the diagram, for instance $[\omega_1]^{<\omega_1} \sqcup [\omega_2]^\omega$ and $[\omega_1]^{\omega_1} \times [\omega_2]^{<\omega_1}$. With additional determinacy assumptions such as AD^+, the set S_1 can be proved to be distinct from all of these.

$$\begin{tikzcd}
 & \mathbb{R} \sqcup \omega_2 \arrow[r] & \mathbb{R} \times \omega_2 \arrow[r] & {[\omega_2]^\omega} \arrow[r] & {[\omega_2]^{<\omega_1}} \arrow[r] & {[\omega_2]^{\omega_1}} \arrow[r] & {[\omega_2]^{<\omega_2}} \arrow[r] & {[\omega_2]^{\omega_2}} \\
\omega_2 \arrow[ur] & & & & & & & \\
 & \mathbb{R} \sqcup \omega_1 \arrow[uu] \arrow[r] & \mathbb{R} \times \omega_1 \arrow[uu] \arrow[r] & {[\omega_1]^\omega} \arrow[uu] \arrow[r] & {[\omega_1]^{<\omega_1}} \arrow[uu] \arrow[r] & {[\omega_1]^{\omega_1}} \arrow[uu] & & \\
\omega_1 \arrow[uu] \arrow[ur] & \mathbb{R} \arrow[u] & & & & & &
\end{tikzcd}$$

The main technique used in this paper involves Kunen functions for ω_1. Let μ be the club measure on ω_1. Using the Kunen tree analysis, one can show that for any function $f : \omega_1 \to \omega_1$, there is a function $\Xi : \omega_1 \times \omega_1 \to \omega_1$ so that for μ-almost all α, $f(\alpha) < \sup\{\Xi(\alpha, \beta) : \beta < \alpha\}$ and $\{\Xi(\alpha, \beta) : \beta < \alpha\}$ is an ordinal (not just a set of ordinals). This function Ξ will be called a Kunen function for f. Ξ allows for a uniform way of selecting a representative for any $g <_\mu f$, i.e. there is a $\beta < \omega_1$ so that the function $\Xi^\beta : \omega_1 \to \omega_1$ defined by $\Xi^\beta(\alpha) = \Xi(\alpha, \beta)$ is μ-almost equal to g. Using these Kunen functions and sliding arguments, Martin proved an ultrapower representation for $\omega_2 = \prod_{\omega_1} \omega_1/\mu$ and showed the weak partition property on ω_2.

The ultrapower representation is important for studying the continuity property at ω_2 in this paper. In fact, these continuity properties for functions $\Phi : [\omega_2]^\epsilon \to \omega_2$ expressed in Theorem 3.21 and Theorem 3.22 when $\epsilon < \omega_2$ and has uncountable cofinality are exceptionally remarkable and unique to ω_2. For instance, one can show under AD that the ultrapower of ω_2 by the club measure μ on ω_1, $\prod_{\omega_1} \omega_2/\mu$, is ω_3. Define

$\Psi : [\omega_2]^{\omega_1} \to \omega_3$ by $\Psi(f) = [f]_\mu$ (where $[f]_\mu$ is the element of this ultrapower represented by f). There is no ω-club B, ordinal $\delta < \epsilon$ so that if $f, g \in [B]^{\omega_1}_*$ with $f \upharpoonright \delta = g \upharpoonright \delta$ and $\sup(f) = \sup(g)$, then $\Psi(f) = \Psi(g)$. This example shows that the continuity property expressed in Theorem 3.22 fails if one considers functions whose range is larger than ω_2. For partition cardinals greater than ω_2, the failure of the continuity property at ϵ of uncountable cofinality is even worse. $\omega_{\omega+1}$ is the next strong partition cardinal after ω_1 under AD. The ultrapower of $\omega_{\omega+1}$ by the club measure μ on ω_1, $\prod_{\omega_1} \omega_{\omega+1}/\mu$, is $\omega_{\omega+1}$. Define $\Psi : [\omega_{\omega+1}]^{\omega_1} \to \omega_{\omega+1}$ by $\Psi(f) = [f]_\mu$. For the same reason as before, the continuity property expressed in Theorem 3.22 fails. These continuity results at ω_2 are largely possible due to the combinatorial tool available from the ultrapower representation of ω_2.

The basic facts about partition properties and Kunen functions can be found in [3]. These arguments are well known and due to Jackson, Kunen, and Martin. (See [13], [14], and [15].) However, the article will follow [3] which develops the minimal notation and theory necessary for the results in this paper.

The final section of this paper will study functions on tuples in $[\omega_1]^\omega$ using partition properties to establish a basic combinatorial property called the Jónsson property for $[\omega_1]^\omega$. Let X be a set. Let $[X]^n_= = \{f \in {}^nX : (\forall i < j < n)(f(i) \neq f(j))\}$. Let $[X]^{<\omega}_= = \bigcup_{n\in\omega} [X]^n_=$. X is n-Jónsson if and only if for every $\Phi : [X]^n_= \to X$, there exists a $Y \subseteq X$ with $|Y| = |X|$ and $\Phi[[Y]^n_=] \neq X$. X is Jónsson if and only if for every $\Phi : [X]^{<\omega}_= \to X$, there is a $Y \subseteq X$ with $|Y| = |X|$ and $\Phi[[Y]^{<\omega}_=] \neq X$.

Under AD, Kleinberg [16] showed that ω_n is Jónsson for all $n \in \omega$. Jackson, Ketchersid, Schlutzenberg, and Woodin [12] showed that under $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$ (and also $\mathsf{ZF} + \mathsf{AD}^+$) that every cardinal $\kappa < \Theta$ is Jónsson. Holshouser and Jackson showed that $\mathbb{R}$ and $\mathbb{R} \times \kappa$ for $\kappa < \Theta$ are Jónsson. The first author [2] showed in fact that for all ordinals κ, $\mathbb{R} \times \kappa$ is Jónsson. Holshouser and Jackson showed that ${}^\omega2/E_0$ is 2-Jónsson. The first author and Meehan [8] showed that ${}^\omega2/E_0$ is not 3-Jónsson and hence not Jónsson. The final result of this paper shows $[\omega_1]^\omega$ has the Jónsson property:

Theorem 4.12. *Assume* $\mathsf{ZF} + \mathsf{AD}$*.* $[\omega_1]^\omega$ *is Jónsson.*

2. Continuity of Short Functions on ω_1

For the rest of the paper, assume $\mathsf{ZF} + \mathsf{AD}$. (Not even $\mathsf{DC}_\mathbb{R}$ will be implicitly assumed.)

If $\epsilon \leq \kappa$ are ordinals, then $[\kappa]^\epsilon$ is the collection of increasing functions $f : \epsilon \to \kappa$.

Definition 2.1. ([14]) Let κ be an ordinal and $\epsilon \leq \kappa$. A function $f : \epsilon \to \kappa$ has uniform cofinality ω if and only if there is a function $g : \epsilon \times \omega \to \kappa$ with the following two properties:
(a) For all $\alpha < \epsilon$ and $n \in \omega$, $g(\alpha, n) < g(\alpha, n + 1)$.
(b) For all $\alpha < \epsilon$, $f(\alpha) = \sup\{g(\alpha, n) : n \in \omega\}$.

A function $f : \epsilon \to \kappa$ is discontinuous at α if and only if $f(\alpha) > \sup\{f(\beta) : \beta < \alpha\}$.

A function $f : \epsilon \to \kappa$ is of the correct type if and only if f has uniform cofinality ω and f is discontinuous everywhere.

Let $A \subseteq \kappa$, $[A]^\epsilon_*$ denote the collection of all increasing functions $f : \epsilon \to A$ of the correct type.

The collection of increasing functions and the collection of increasing functions of the correct type have the same cardinality. In the following, one may use either sets for purpose of distinguishing cardinality.

Fact 2.2. *Let* κ *be a cardinal. Let* $\epsilon \leq \kappa$*.* $[\kappa]^\epsilon \approx [\kappa]^\epsilon_*$*.*

Proof. Let $H : \kappa \to \kappa$ be any increasing function of the correct type. Define $\Phi : [\kappa]^\epsilon \to [\kappa]^\epsilon_*$ by $\Phi(f) = H \circ f$. Then Φ is an injection. The two sets are in bijection by the Cantor-Schröder-Bernstein theorem. □

Definition 2.3. Let κ be an ordinal and $\epsilon \leq \kappa$. One write $\kappa \to_* (\kappa)^\epsilon_2$ to indicate that for every $P : [\kappa]^\epsilon_* \to 2$, there is some club $C \subseteq \omega_1$ and an $i \in 2$ so that for all $f \in [C]^\epsilon_*$, $\Phi(f) = i$.

If $\kappa \to_* (\kappa)^\kappa_2$, then one says that κ is a strong partition cardinal.

If $\kappa \to_* (\kappa)^\epsilon_2$ for all $\epsilon < \kappa$, then κ is said to be a weak partition cardinal.

Fact 2.4. *(*[3] *Section 2 and 4,* [16] *Chapter II,* [15] *Theorem 7.3 and 12.2.) (Solovay) The club measure* μ *on* ω_1 *is a countably complete normal measure on* ω_1*. (Martin)* ω_1 *is a strong partition cardinal.*

Definition 2.5. Let μ denote the club measure on ω_1. For each $\epsilon \leq \omega_1$, let μ^ϵ be a filter on $[\omega_1]^\epsilon_*$ defined by $K \in \mu^\epsilon$ if and only if there is a club $C \subseteq \omega_1$ so that $[C]^\epsilon_* \subseteq K$. Since ω_1 is a strong partition cardinal, one has that μ^ϵ is a countably complete ultrafilter for all $\epsilon \leq \omega_1$.

If φ is a formula, then one writes $(\forall^*_\epsilon f)\varphi(f)$ to indicate that the set $\{f \in [\omega_1]^\epsilon_* : \varphi(f)\} \in \mu^\epsilon$.

Definition 2.6. ([3] Section 5) Let μ be the club measure on ω_1.

Let $\Xi : \omega_1 \times \omega_1 \to \omega_1$. For each $\alpha < \omega_1$, let $\delta^\Xi_\alpha = \sup\{\Xi(\alpha, \beta) : \beta < \alpha\}$. Let $\Xi_\alpha : \alpha \to \delta^\Xi_\alpha$ be defined by $\Xi_\alpha(\beta) = \Xi(\alpha, \beta)$.

Ξ is a Kunen function for f with respect to μ if and only if $K^\Xi_f = \{\alpha < \omega_1 : f(\alpha) \leq \delta^\Xi_\alpha \wedge \Xi_\alpha \text{ is a surjection}\} \in \mu$. K^Ξ_f is the set of α on which Ξ provides a bounding for f.

For $\beta < \omega_1$, let $\Xi^\beta : \omega_1 \to \omega_1$ be defined by $\Xi^\beta(\alpha) = \Xi(\alpha, \beta)$ where $\alpha > \beta$ and 0 otherwise.

Fact 2.7. *([3] Section 5, [14] Lemma 4.1) (Kunen) For every $f : \omega_1 \to \omega_1$, there is a Kunen function for f with respect to μ.*

Definition 2.8. Let $\beta \leq \epsilon < \omega_1$ and $f \in [\omega_1]^\epsilon_*$. Let $\mathsf{bound}(f, \beta) = \sup\{f(\alpha) : \alpha < \beta\}$, where $\sup(\emptyset)$ is defined to be 0.

If $A \subseteq \omega_1$ with $|A| = \omega_1$, then let $\mathsf{enum}_A : \omega_1 \to A$ denote the increasing enumeration of A.

Let $C \subseteq \omega_1$ be a club. Let $\mathsf{next}^\omega_C(\alpha)$ denote the ω^{th} element of C above α.

Fact 2.9. *Let $\epsilon < \omega_1$. For all $\Phi : [\omega_1]^\epsilon_* \to \omega_1$, there exists a unique $\mathfrak{b}_\Phi \leq \epsilon$ so that $\mathfrak{b}_\Phi$ is the largest $\beta \leq \epsilon$ so $(\forall^*_\epsilon f)(\mathsf{bound}(f, \beta) \leq \Phi(f))$.*

Proof. For each $\beta \leq \epsilon < \omega_1$, let A_β be the set of f so that β is the largest $\gamma \leq \epsilon$ so that $\Phi(f) \geq \mathsf{bound}(f, \gamma)$. $[\omega_1]^\epsilon_* = \bigcup_{\beta \leq \epsilon} A_\beta$. Since μ^ϵ is a countably complete ultrafilter on $[\omega_1]^\epsilon_*$, there is a $\mathfrak{b}_\Phi$ so that $A_{\mathfrak{b}_\Phi} \in \mu^\epsilon$. ☐

Lemma 2.10. *Let $\epsilon < \omega_1$. Let $\Phi : [\omega_1]^\epsilon_* \to \omega_1$. Then there are club subsets of ω_1, C and D, so that for all $f \in [D]^\epsilon_*$, $\Phi(f) < \mathsf{next}^\omega_C(\mathsf{bound}(f, \mathfrak{b}_\Phi))$.*

Proof. Let $*$ be a new symbol. Define a linear ordering $\mathcal{L}$ on $\epsilon \cup \{*\}$ by $x \prec y$ if and only if
(a) $x, y \in \epsilon$ and $x < y$
(b) $x = *$, $y \in \epsilon$, and $y \geq \mathfrak{b}_\Phi$
(c) $x \in \epsilon$, $y = *$, and $x < \mathfrak{b}_\Phi$.

Note that $\mathcal{L}$ is a wellordering of ordertype less than ω_1. If $f : \mathcal{L} \to \omega_1$ is an increasing function, then let $\mathsf{main}(f) : \epsilon \to \omega_1$ be defined by $\mathsf{main}(f)(\alpha) = f(\alpha)$. Let $\mathsf{extra}(f) \in \omega_1$ be defined by $\mathsf{extra}(f) = f(*)$.

Define a partition $P : [\omega_1]^\mathcal{L}_* \to 2$ by $P(g) = 0 \Leftrightarrow \Phi(\mathsf{main}(g)) < \mathsf{extra}(g)$. By the weak partition property of ω_1, there is some $C \subseteq \omega_1$ which is homogeneous for this partition. By intersecting with an appropriate club, one may assume that for all $f \in [C]^\epsilon_*$, $\mathfrak{b}_\Phi$ is the largest γ so that $\Phi(f) \geq \mathsf{bound}(f, \gamma)$. Therefore if $\mathfrak{b}_\Phi < \epsilon$, $\Phi(f) < f(\mathfrak{b}_\Phi)$.

The claim is that C is homogeneous for P taking value 0: Let $D = \{\alpha \in C : \mathsf{enum}_C(\alpha) = \alpha\}$ which is the club set of closure points of C. Let $f \in D$. In the case that $\mathfrak{b}_\Phi < \epsilon$, since $\mathsf{bound}(f, \mathfrak{b}_\Phi) \leq \Phi(f) < f(\mathfrak{b}_\Phi)$ and $f(\mathfrak{b}_\Phi) \in D$, the ω^{th}-element of C above $\Phi(f)$ is below $f(\mathfrak{b}_\Phi)$. In all cases, let $g : \mathcal{L} \to C$ be defined by $g(\alpha) = f(\alpha)$ for all $\alpha \in \epsilon$ and $g(*) = \mathsf{next}^\omega_C(\Phi(f))$. Using any function witnessing that f has uniform cofinality ω, one can show that g has uniform cofinality ω. g is discontinuous everywhere. So $g \in [C]^\mathcal{L}_*$ and $\Phi(\mathsf{main}(g)) = \Phi(f) < \mathsf{next}^\omega_C(\Phi(f)) = \mathsf{extra}(g)$. Thus $P(g) = 0$ and hence C must have been homogeneous for P taking value 0. The establishes the claim.

Now suppose $f \in [D]^\epsilon_*$. In the case that $\mathfrak{b}_\Phi < \epsilon$, since $\mathsf{bound}(f, \mathfrak{b}_\Phi) \leq \Phi(f) < f(\mathfrak{b}_\Phi)$ and $f(\mathfrak{b}_\Phi) \in D$, $\mathsf{next}^\omega_C(\mathsf{bound}(f, \mathfrak{b}_\Phi)) < f(\mathfrak{b}_\Phi)$. In all cases, let $g : \mathcal{L} \to C$ be defined by $g(\alpha) = f(\alpha)$ if $\alpha < \epsilon$ and $g(*) = \mathsf{next}^\omega_C(\mathsf{bound}(f, \mathfrak{b}_\Phi))$. As before, g is a function of the correct type in $[C]^\mathcal{L}_*$. $P(g) = 0$ implies that $\Phi(f) = \Phi(\mathsf{main}(g)) < \mathsf{extra}(g) = \mathsf{next}^\omega_C(\mathsf{bound}(f, \mathfrak{b}_\Phi))$. This completes the proof. ☐

Lemma 2.11. *Let $\epsilon < \omega_1$ and $\Phi : [\omega_1]^\epsilon_* \to \omega_1$ be such that $\mathfrak{b}_\Phi \neq 0$. Then there is some club $D \subseteq \omega_1$, some Kunen function $\Xi : \omega_1 \times \omega_1 \to \omega_1$, and some $\Phi' : [\omega_1]^\epsilon_* \to \omega_1$ so that for all $f \in [D]^{\omega_1}_*$, $\Phi(f) = \Xi(\mathsf{bound}(f, \mathfrak{b}_\Phi), \Phi'(f))$ where $\mathfrak{b}_{\Phi'} < \mathfrak{b}_\Phi$.*

Proof. By Lemma 2.10, there are clubs C and D_1 so that for all $f \in [D_1]^\epsilon_*$, $\Phi(f) < \mathsf{next}^\omega_C(\mathsf{bound}(f, \mathfrak{b}_\Phi))$. Let Ξ be a Kunen function for $\mathsf{next}^\omega_C : \omega_1 \to \omega_1$. Since $K^\Xi_{\mathsf{next}^\omega_C} \in \mu$, let $D_2 \subseteq K^\Xi_{\mathsf{next}^\omega_C}$ be a club subset of ω_1. Let

$D_3 = D_1 \cap D_2$. Thus for all $f \in [D_3]^\epsilon_*$, $\Phi(f) < \mathsf{next}^\omega_C(\mathsf{bound}(f, \mathfrak{b}_\Phi)) \leq \delta^\Xi_{\mathsf{bound}(f,\mathfrak{b}_\Phi)}$. Let $\Phi' : [D_3]^\epsilon_* \to \omega_1$ be defined by $\Phi'(f)$ is the least $\gamma < \mathsf{bound}(f, \mathfrak{b}_\Phi)$ so that $\Phi(f) = \Xi(\mathsf{bound}(f, \mathfrak{b}_\Phi), \gamma)$. Thus one has that for all $f \in [D_3]^\epsilon_*$, $\Phi(f) = \Xi(\mathsf{bound}(f, \mathfrak{b}_\Phi), \Phi'(f))$. Also $(\forall^*_\epsilon f)(\Phi'(f) < \mathsf{bound}(f, \mathfrak{b}_\Phi))$ implies that $\mathfrak{b}_{\Phi'} < \mathfrak{b}_\Phi$ as long as $\mathfrak{b}_\Phi \neq 0$. ☐

Definition 2.12. Let $\epsilon < \omega_1$ and $\Phi : [\omega_1]^\epsilon_* \to \omega_1$.

A representation for Φ is a tuple $(\Xi_0, ..., \Xi_{n-1}; \beta_0, ..., \beta_n; \gamma)$ with the following properties

(a) $n \in \omega$. If $n = 0$, then no Ξ appears.

(b) $\beta_0 > \beta_1 > ... > \beta_{n-1} > \beta_n = 0$ is a sequence of strictly decreasing ordinals less than or equal to ϵ. $\gamma < \omega_1$.

(c) Each $\Xi_i : \omega_1 \times \omega_1 \to \omega_1$ is a Kunen function (for some function with respect to μ).

(c) Let $\Phi_n(f) = \gamma$. Suppose for $0 < i \leq n$, Φ_i has been defined, then let $\Phi_{i-1}(f) = \Xi_{i-1}(\mathsf{bound}(f, \beta_{i-1}), \Phi_i(f))$. One has that $(\forall^*_\epsilon f)(\Phi_0(f) = \Phi(f))$.

Theorem 2.13. *Let $\epsilon < \omega_1$. Every $\Phi : [\omega_1]^\epsilon_* \to \omega_1$ has a representation.*

Proof. Let T be the tree of decreasing sequences $\sigma = (\beta_0, ..., \beta_k)$ in $\epsilon + 1$ ordered by reverse string extension with the property that there exists some Kunen functions $\Xi_0, ..., \Xi_{k-1}$ and functions $\Phi_0, ..., \Phi_k$ with the property that

(i) $\Phi_0 = \Phi$.

(ii) $\beta_i = \mathfrak{b}_{\Phi_i}$.

(iii) $(\forall^*_\epsilon f)(\Phi_i(f) = \Xi_i(\mathsf{bound}(f, \beta_i), \Phi_{i+1}(f)))$ for all $i < k$.

The claim is that there there is some $\sigma = (\beta_0, ..., \beta_n) \in T$ so that $\beta_n = 0$.

To see this: Suppose not. Let $\sigma = (\beta_0, ..., \beta_k) \in T$ with $\beta_k \neq 0$. Let $\Xi_0, ..., \Xi_{k-1}$ and $\Phi_0, ..., \Phi_k$ witness that $\sigma \in T$. (ii) implies that $\mathfrak{b}_{\Phi_k} = \beta_k > 0$. Lemma 2.11 implies that there is some Ξ_k and Φ' so that $(\forall^*_\epsilon f)(\Phi_k(f) = \Xi_k(\mathsf{bound}(f, \mathfrak{b}_{\Phi_k}), \Phi'(f)))$ with $\mathfrak{b}_{\Phi'} < \mathfrak{b}_{\Phi_k} = \beta_k$. Let $\Phi_{k+1} = \Phi'$. Let $\beta_{k+1} = \mathfrak{b}_{\Phi'}$. Let $\sigma' = \sigma \hat{\ } \beta_{k+1}$. Then $\Phi_0, ..., \Phi_{k+1}$ and $\Xi_0, ..., \Xi_k$ witness that $\sigma' \in T$.

It has been shown that any $\sigma \in T$ can be extended to some $\sigma' \in T$. T is a tree on $\epsilon + 1$ with no terminal nodes. Since ϵ is a wellordering, T must have an infinite branch. This is impossible since an infinite branch is an infinite descending sequence of ordinals.

The claim has been shown. So let $\sigma = (\beta_0, ..., \beta_n) \in T$ be such that $\beta_n = 0$. Let $\Xi_0, ..., \Xi_{n-1}$ and $\Phi_0, ..., \Phi_n$ be witnesses to $\sigma \in T$. Since $\mathfrak{b}_{\Phi_n} = \beta_n = 0$, one has that for μ^ϵ-almost all f, $\mathsf{bound}(f, 0) = 0 \leq \Phi_n(f) < f(0)$. This implies that $\mathfrak{b}_{\Phi_n} = 0$. By Lemma 2.10, there is a club $C \subseteq \omega_1$ so that $\Phi_n(f) < \mathsf{next}^\omega_C(\mathsf{bound}(f, \mathfrak{b}_{\Phi_n})) = \mathsf{next}^\omega_C(\mathsf{bound}(f, 0)) = \mathsf{next}^\omega_C(0)$. Since μ^ϵ is countably complete and $\mathsf{next}^\omega_C(0) < \omega_1$, Φ_n is μ^ϵ-almost everywhere a constant function taking value some $\gamma \in \mathsf{next}^\omega_C(0)$. This implies that $(\Xi_0, ..., \Xi_{n-1}; \beta_0, ..., \beta_n; \gamma)$ is a representation of Φ. ☐

The theorem implies a μ^ϵ-almost everywhere continuity result for functions $\Phi : [\omega_1]^\epsilon_* \to \omega_1$.

Theorem 2.14. *Let $\epsilon < \omega_1$ and $\Phi : [\omega_1]^\epsilon_* \to \omega_1$. Then there is a decreasing sequence of ordinals which are less than or equal to ϵ, $(\beta_i : i \leq n)$, with $\beta_n = 0$ and a club $C \subseteq \omega_1$ so that if $f, g \in [C]^\epsilon_*$ has the property that $\mathsf{bound}(f, \beta_i) = \mathsf{bound}(g, \beta_i)$ for all $i \leq n$, then $\Phi(f) = \Phi(g)$.*

The following is an even coarser form of continuity:

Theorem 2.15. *Let $\epsilon < \omega_1$ and $\Phi : [\omega_1]^\epsilon_* \to \omega_1$. Then there is a $\delta < \epsilon$ and some club $C \subseteq \omega_1$ so that for all $f, g \in [C]^\epsilon_*$ with $f \upharpoonright \delta = g \upharpoonright \delta$ and $\sup(f) = \sup(g)$, $\Phi(f) = \Phi(g)$.*

Proof. If $n = 0$, then Φ is a constant function so this immediately true. If $n = 1$, then let $\delta = \beta_0$ if $\beta_0 < \epsilon$ and $\delta = 0$ if $\beta_0 = \epsilon$. If $n > 1$, then let $\delta = \beta_1$. ☐

Woodin [18] has observed the conclusion of the next theorem at least under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_\mathbb{R}$ or $\mathsf{ZF} + \mathsf{AD}^+$. The following gives a combinatorial proof in AD.

Theorem 2.16. $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$.

Proof. Observe that $[\omega_1]^\omega_* \approx [\omega_1]^\omega$ and $[\omega_1]^{<\omega_1}_* \approx [\omega_1]^{<\omega_1}$. So suppose there is an injection $\Sigma : [\omega_1]^{<\omega_1}_* \to [\omega_1]^\omega_*$.

For each $\epsilon < \omega_1$ and $n \in \omega$, let $\Sigma_n^\epsilon : [\omega_1]^\epsilon_* \to \omega_1$ be defined by $\Sigma_n^\epsilon(f) = \Sigma(f)(n)$. By Theorem 2.15, there is some $\delta_n^\epsilon < \epsilon$ so that there is some $C \subseteq \omega_1$ club with the property that for all $f, g \in [C]^\epsilon_*$, $\sup(f) = \sup(g)$ and $f \restriction \delta_n^\epsilon = g \restriction \delta_n^\epsilon$ implies that $\Sigma_n^\epsilon(f) = \Sigma_n^\epsilon(g)$.

For each $n \in \omega$, define $\Lambda_n : \omega_1 \to \omega_1$ by $\Lambda_n(\epsilon) = \delta_n^\epsilon$. Each Λ_n is a regressive function. For each $n \in \omega$, there is a club C so that Λ_n is constant on C. Define a relation $S \subseteq \omega \times \mathscr{P}(\omega_1)$ by $S(n, C)$ if and only if C is a club subset of ω_1 and Λ_n is constant on C. By the Moschovakis coding lemma, there is a surjection $\pi : \mathbb{R} \to \mathscr{P}(\omega_1)$. Define $R \subseteq \omega \times \mathbb{R}$ by $R(n, r)$ if and only if $R(n, \pi(r))$. Therefore, using $\mathsf{AC}_\omega^\mathbb{R}$, there is a function $\Gamma : \omega \to \mathbb{R}$ so that for all $n \in \omega$, $R(n, \Gamma(n))$. Let $C_n = \pi(\Gamma(n))$ which is a club subset of ω_1. Thus Λ_n is constant on C_n. Let δ_n be such that for all $\epsilon \in C_n$, $\Lambda_n(\epsilon) = \delta_n$. Let $C_\omega = \bigcap_{n\in\omega} C_n$. Let $\delta = \sup\{\delta_n : n \in \omega\}$. Since ω_1 is regular, $\delta < \omega_1$.

Now fix an $\epsilon > \delta$ be some limit ordinal with $\epsilon \in C_\omega$. Since $\epsilon \in C_\omega$, for all n, $\delta_n^\epsilon = \delta_n$. As observed above and since $\delta_n^\epsilon = \delta_n \leq \delta$, there is a club $C \subseteq \omega_1$ so that for all $f, g \in [C]^\epsilon_*$, if $\sup(f) = \sup(g)$ and $f \restriction \delta = g \restriction \delta$, then $\Sigma_n^\epsilon(f) = \Sigma_n^\epsilon(g)$. Let $T \subseteq \omega \times \mathscr{P}(\omega_1)$ be defined by $T(n, C)$ if and only if C is a club subset of ω_1 and for all $f, g \in [C]^\epsilon_*$, if $\sup(f) = \sup(g)$ and $f \restriction \delta = g \restriction \delta$, then $\Sigma_n^\epsilon(f) = \Sigma_n^\epsilon(g)$. By an argument as above using $\mathsf{AC}_\omega^\mathbb{R}$ and the Moschovakis coding lemma, there is a sequence $\langle D_n : n \in \omega \rangle$ so that for all $n \in \omega$, D_n is a club subset of ω_1 and for all $f, g \in [D_n]^\epsilon_*$, if $\sup(f) = \sup(g)$ and $f \restriction \delta = g \restriction \delta$, then $\Sigma_n^\epsilon(f) = \Sigma_n^\epsilon(g)$. Let $D = \bigcap_{n\in\omega} D_n$.

Now pick $f, g \in [D]^\epsilon_*$ so that $f \restriction \delta = g \restriction \delta$, $\sup(f) = \sup(g)$, and $f \neq g$. Since for all $n \in \omega$, $\delta \geq \delta_n = \delta_n^\epsilon$, one has that $\Sigma(f) = \Sigma(g)$. This contradicts Σ being an injection. ☐

3. Continuity of Short Functions on ω_2

First, one will review the notations and basic tools needed to analyze ω_2 under AD. See [3] Section 5 and 6 for more details and the proofs of the following results.

Let μ denote the club filter on ω_1. An important application of the Kunen function for functions $f : \omega_1 \to \omega_1$ is the existence of a uniform procedure to select representatives of the ultrapower $\prod_{\omega_1} \omega_1/\mu$.

Fact 3.1. *Let μ be the club measure on ω_1. Suppose $f : \omega_1 \to \omega_1$ and possesses a Kunen function Ξ with respect to μ. Suppose $G \in \prod_{\alpha\in\omega_1} f(\alpha)/\mu$. Then there is a $\beta < \omega_1$ so that $[\Xi^\beta]_\mu = G$*

As a consequence, one can show that $\prod_{\omega_1} \omega_1/\mu$ is wellfounded even without $\mathsf{DC}_\mathbb{R}$.

Fact 3.2. *Let $f : \omega_1 \to \omega_1$ and possesses a Kunen function Ξ with respect to μ. Then $\prod_{\alpha\in\omega_1} f(\alpha)/\mu$, i.e. the initial segment of $\prod_{\omega_1} \omega_1/\mu$ determined by $[f]_\mu$, is a wellordering.*

$\prod_{\omega_1} \omega_1/\mu$ is wellfounded.

For each $F \in \prod_{\omega_1} \omega_1/\mu$, $F < \omega_2$. Thus $\prod_{\omega_1} \omega_1/\mu \leq \omega_2$.

Fact 3.3. *(Martin) Assume just ZF. Let κ be a strong partition cardinal.*

If ν is a measure on κ, then $\prod_\kappa \kappa/\nu$ is a cardinal.

If ν is a normal κ-complete measure on κ, then $\prod_\kappa \kappa/\nu$ is a regular cardinal.

Corollary 3.4. *(Martin) Let μ be the club measure on ω_1. $\omega_2 = \prod_{\omega_1} \omega_1/\mu$ and ω_2 is a regular cardinal.*

Definition 3.5. Let μ be the club measure on ω_1. Let $h : \omega_1 \to \omega_1$. Suppose h possesses a Kunen function Ξ with respect to μ. An ordinal $\beta < \omega_1$ is a minimal code (relative to Ξ) if and only if for all $\gamma < \beta$, $\neg(\Xi^\gamma =_\mu \Xi^\beta)$. Let J be the collection of β which are minimal codes and satisfy $\Xi^\beta <_\mu h$. Define an ordering $\prec$ on J by $\alpha \prec \beta$ if and only if $\Xi^\alpha <_\mu \Xi^\beta$. By Fact 3.1, for every $G < [h]_\mu$, there is a unique $\beta \in J$ so that $\Xi^\beta \in G$ (i.e. $[\Xi^\beta]_\mu = G$). In this way, one says that β is a minimal code for G or for any $g \in G$ with respect to Ξ. Thus $(J, \prec)$ has the same ordertype as $[h]_\mu$. By Fact 3.2, $[h]_\mu$ is a wellordering. Let $\epsilon \in \mathrm{ON}$ denote the ordertype of $([h]_\mu, <)$ which is equal to the ordertype of $(J, \prec)$. Let $\pi : \epsilon \to (J, \prec)$ be the unique order-preserving isomorphism.

Note that the objects J, $\prec$, ϵ, and π depend on Ξ and h. However, within this section, one will only work with a single Ξ and h at a given time. It should be clear in context that these objects depend on this fixed Ξ and h.

Definition 3.6. Let μ be the club measure on ω_1. Let $h : \omega_1 \to \omega_1$ be a function so that $h(\alpha) > 0$ μ-almost everywhere. Let Ξ be a Kunen function for h with respect to μ. Let $\epsilon = [h]_\mu = \mathrm{ot}(J, \prec)$ which are defined relative to Ξ and h.

Let $T^h = \{(\alpha,\beta) \in \omega_1 \times \omega_1 : \beta < h(\alpha)\}$. Let $\mathcal{T}^h = (T^h, \sqsubset)$ where $\sqsubset$ is the lexicographic ordering. Note that $\text{ot}(\mathcal{T}^h) = \omega_1$.

Suppose $F : \mathcal{T}^h \to \omega_1$ is an order-preserving function. Let $g \in \omega_1 \to \omega_1$ be such that $g <_\mu h$. Let $A^g = \{\alpha : g(\alpha) < h(\alpha)\}$. Let $F^g : \omega_1 \to \omega_1$ be defined by

$$F^g(\alpha) = \begin{cases} F(\alpha, g(\alpha)) & \alpha \in A^g \\ F(\alpha, 0) & \text{otherwise} \end{cases}$$

Note that if $g_1 <_\mu g_2 <_\mu h$, then $F^{g_1} <_\mu F^{g_2}$.

If $\beta \in \epsilon$, then let $F^{(\beta)} = F^{\Xi^{\pi(\beta)}}$. Let $\mathsf{funct}(F) : \epsilon \to \text{ON}$ be defined by $\mathsf{funct}(F)(\alpha) = [F^{(\alpha)}]_\mu$.

If $X \subseteq \omega_1$, then let $[X]^{\mathcal{T}^h}$ be the collection of order-preserving functions $f : \mathcal{T}^h \to X$. Let $[X]^{\mathcal{T}^h}_*$ be the collection of correct type order-preserving functions $f : \mathcal{T}^h \to \omega_1$ (since $\mathcal{T}^h$ is order-isomorphic to ω_1, this is equivalent to the earlier notion of $f : \omega_1 \to X$ having the correct type).

Fact 3.7. *Let μ be the club measure on ω_1. Let $h : \omega_1 \to \omega_1$ be a function possessing a Kunen function Ξ with respect to μ. Suppose $F_0, F_1 \in [\omega_1]^{\mathcal{T}^h}$ have the property that $F_0^{(\beta)} =_\mu F_1^{(\beta)}$ for all $\beta < \epsilon$. Then for μ-almost all α, $F_0(\alpha,\beta) = F_1(\alpha,\beta)$ for all $\beta < h(\alpha)$.*

Suppose $\epsilon < \omega_2$ and $\mathcal{F} : \epsilon \to \omega_2$. Let $h : \omega_1 \to \omega_1$ be such that $[h]_\mu = \epsilon$. Let Ξ be a Kunen function for h. Via a "sliding argument", one can find an increasing function $F : \mathcal{T}^h \to \omega_1$ so that for all $\beta < \epsilon$, $[F^{(\beta)}]_\mu = \mathcal{F}(\beta)$. Hence one can study functions $\mathcal{F} : \epsilon \to \omega_2$ by using the strong partition property of ω_1 on partitions of functions in $[\omega_1]^{\mathcal{T}^h}_*$. See [3] Section 5 on the statement of the sliding lemma and how it can be used to prove the following results:

Theorem 3.8. *(Martin-Paris) Let μ be the club measure on ω_1. Then for all $\alpha < \omega_2$, the partition relation $\omega_2 \to (\omega_2)^\alpha_2$ holds. That is, ω_2 is a weak partition cardinal.*

As a consequence of the weak partition property on ω_2, one can completely characterize the normal measures on ω_2.

Corollary 3.9. *Let $W^{\omega_2}_\omega$ and $W^{\omega_2}_{\omega_1}$ denote the ω-club and ω_1-club filter, respectively.*

$W^{\omega_2}_\omega$ and $W^{\omega_2}_{\omega_1}$ are the only two ω_2-complete normal ultrafilters on ω_2.

The next two results show that club subsets and ω-club subsets of ω_2 are lifts (in a certain sense) of some club subsets of ω_1.

Fact 3.10. *Let μ be the club measure on ω_1. If $C \subseteq \omega_1$ is a club subset of ω_1, then $[C]^{\omega_1}/\mu$ is a club subset of ω_2.*

If $D \subseteq \omega_2$ is club, then there is a club $C \subseteq \omega_1$ so that $[C]^{\omega_1}/\mu \subseteq D$.

Fact 3.11. *Let μ be the club measure on ω_1. Let $C \subseteq \omega_1$ be a club. Then $[C]^{\omega_1}_*/\mu$ is an ω-club subset. Moreover, for every ω-club $D \subseteq \omega_2$, there is a club $C \subseteq \omega_1$ so that $[C]^{\omega_1}_*/\mu \subseteq D$.*

Fact 3.12. *Let μ denote the club measure on ω_1. Let $C \subseteq \omega_1$ be club. Let $B = [C]^{\omega_1}_*/\mu$ which is an ω-club subset of ω_2.*

Let $\epsilon < \omega_2$. Let $h : \omega_1 \to \omega_1$ with $h(\alpha) > 0$ for all $\alpha < \omega_1$ and $[h]_\mu = \epsilon$. Let Ξ be a Kunen function for h.

Let $\mathcal{F} \in [B]^\epsilon_*$ (be of correct type). Then there is an $F \in [C]^{\mathcal{T}^h}_*$ so that for all $\alpha < \epsilon$, $[F^{(\alpha)}]_\mu = \mathcal{F}(\alpha)$.

Definition 3.13. Let μ denote the club measure on ω_1. Let $\nu = W^{\omega_2}_\omega$ denote the ω-club measure on ω_2. Let $\epsilon < \omega_2$. Define ν^ϵ as follows: for all $A \subseteq [\omega_2]^\epsilon_*$, $A \in \nu^\epsilon$ if and only if there is a ω-club $B \subseteq \omega_2$ so that $[B]^\epsilon_* \subseteq A$. ν^ϵ is an ω_2-complete measure on $[\omega_2]^\epsilon_*$ by the weak partition property of ω_2.

Let $\mathcal{F} \in [\omega_2]^\epsilon_*$. For $\beta \leq \epsilon$, let $\mathsf{bound}(\mathcal{F},\beta) = \sup\{\mathcal{F}(\alpha) : \alpha < \beta\}$.

Let $\Phi : [\omega_2]^\epsilon_* \to \omega_2$. Let $\mathfrak{b}_\Phi$ be defined so that for ν^ϵ-almost all $\mathcal{F} \in [\omega_2]^\epsilon_*$, $\mathfrak{b}_\Phi$ is the largest $\gamma \leq \epsilon$ so that $\Phi(\mathcal{F}) \geq \mathsf{bound}(\mathcal{F},\gamma)$.

Let $h \in \omega_1 \to \omega_1$ with $h(\alpha) > 0$ be such that $[h]_\mu = \epsilon$. Let Ξ be a Kunen function for h with respect to μ.

Suppose $F \in [\omega_1]^{\mathcal{T}^h}_*$ and $\beta \leq \epsilon$. Define $\mathsf{Bound}_\beta : \omega_1 \to \omega_1$ by $\mathsf{Bound}_\beta(F)(\gamma) = \sup\{F^{(\alpha)}(\gamma) : \alpha < \beta\}$. Note that although β may be uncountable, for each γ, this is a supremum of a set containing at most $|h(\gamma)| = \aleph_0$ many elements.

For the next several results, assume the setting of Definition 3.13.

The next result states that if $\mathcal{F} \in [\omega_2]^\epsilon_*$ and $F \in [\omega_1]^{\mathcal{T}^h}_*$ is a lifted representation of $\mathcal{F}$, then $\mathsf{Bound}_\beta(F)$ is a lifted representation of $\mathsf{bound}(\mathcal{F}, \beta)$.

Fact 3.14. *Let $\beta \leq \epsilon$. Let $\mathcal{F} \in [\omega_2]^\epsilon_*$. Let $F \in [\omega_1]^{\mathcal{T}^h}_*$ be such that for all $\alpha < \epsilon$, $[F^{(\alpha)}]_\mu = \mathcal{F}(\alpha)$. Then $\mathsf{bound}(\mathcal{F}, \beta) = [\mathsf{Bound}_\beta(F)]_\mu$.*

Proof. First observe that for any $\mathcal{F}$, there is an F with the above property by Fact 3.12.

Let $\delta < \mathsf{bound}(\mathcal{F}, \beta)$. Then there is some $\gamma < \beta$ so that $\delta < \mathcal{F}(\gamma)$. So $\delta < [F^{(\gamma)}]_\mu$. Hence $\delta < [\mathsf{Bound}_\beta(F)]_\mu$.

Now suppose that $\delta < [\mathsf{Bound}_\beta]_\mu$. Let $\ell : \omega_1 \to \omega_1$ be such that $[\ell]_\mu = \delta$. Then for μ-almost all γ, $\ell(\gamma) < \sup\{F^{(\alpha)}(\gamma) : \alpha < \beta\}$. Therefore, for μ-almost all γ, there is a $\zeta < h(\gamma)$ and, in fact, if $\beta < \epsilon$, there is a $\zeta < \Xi^{\pi(\beta)}(\gamma)$ so that $\ell(\gamma) < F(\gamma, \zeta)$. Let $\iota : \omega_1 \to \omega_1$ be defined so that for the set of μ-almost all γ with the previous property, $\iota(\gamma)$ is the least such ζ with $\ell(\gamma) < F(\gamma, \zeta)$. There is some $\rho < \beta$ so that $\iota =_\mu \Xi^{\pi(\rho)}$. Thus $\ell <_\mu F^\iota =_\mu F^{\Xi^{\pi(\rho)}} = F^{(\rho)}$. Hence $\delta < \mathcal{F}(\rho)$ where $\rho < \beta$. This shows that $[\mathsf{Bound}_\beta]_\mu < \mathsf{bound}(\mathcal{F}, \beta)$. ☐

Definition 3.15. Let $\beta \leq \epsilon$. Let $C \subseteq \omega_1$ be a club subset of ω_1.

For each $F \in [\omega_1]^{\mathcal{T}^h}_*$, define $\mathsf{Fnext}_{\beta,C}(F) : \omega_1 \to \omega_1$ by $\mathsf{Fnext}_{\beta,C}(F)(\alpha) = \mathsf{next}^\omega_C(\mathsf{Bound}_\beta(F)(\alpha))$.

Using either Fact 3.7 or Fact 3.14, if $F_0, F_1 \in [\omega_1]^{\mathcal{T}^h}_*$ have the property that for all $\beta \leq \epsilon$, $F_0^{(\beta)} =_\mu F_1^{(\beta)}$, then $\mathsf{Fnext}_{\beta,C}(F_0) =_\mu \mathsf{Fnext}_{\beta,C}(F_1)$.

Therefore the following is well defined: if $\mathcal{F} \in [\omega_2]^\epsilon_*$, let $\mathsf{fnext}_{\beta,C}(\mathcal{F}) = [\mathsf{Fnext}_{\beta,C}(F)]_\mu$, for any $F \in [\omega_1]^{\mathcal{T}^h}_*$ such that for all $\alpha < \epsilon$, $[F^{(\alpha)}]_\mu = \mathcal{F}(\alpha)$.

The following gives an intuitive summary of the previous notations.

If $\mathcal{F} : \epsilon \to \omega_2$ and $\beta \leq \epsilon$, $\mathsf{bound}(F, \beta) = \sup\{\mathcal{F}(\alpha) : \alpha < \beta\}$ which is an ordinal less than ω_2. If $F \in [\omega_1]^{\mathcal{T}^h}_*$ represents $\mathcal{F}$ in the sense stated in Fact 3.14, then $\mathsf{Bound}_\beta(F) : \omega_1 \to \omega_1$ is a representative for the ordinal $\mathsf{bound}(\mathcal{F}, \beta)$ in the ultrapower of ω_1 by the club measure on ω_1.

Now suppose $C \subseteq \omega_1$ is a club, $\beta \leq \epsilon$, and $\mathcal{F} : \epsilon \to \omega_2$ which is represented by $F : \mathcal{T}^h \to \omega_1$. $\mathsf{Fnext}_{\beta,C} : \omega_1 \to \omega_1$ is defined using F and C to represent $\mathsf{fnext}_{\beta,C}(\mathcal{F})$ which is roughly the next element of the ω-club on ω_2 induced by the club $C \subseteq \omega_1$ after the ordinal $\mathsf{bound}_\beta(\mathcal{F})$.

Lemma 3.16. *Assume the setting of Definition 3.13. There is a club $C \subseteq \omega_1$ and an ω-club $B \subseteq \omega_2$ so that for all $\mathcal{F} \in [B]^\epsilon_*$, $\Phi(\mathcal{F}) < \mathsf{fnext}_{\mathfrak{b}_\Phi,C}(\mathcal{F})$.*

Proof. For each $\alpha < \omega_1$, one will define a wellordering $\mathcal{L}_\alpha$: Let $*_\alpha$ be a distinct new object. The underlying domain of $\mathcal{L}_\alpha$ is $h(\alpha) \cup \{*_\alpha\}$.

First assume $\mathfrak{b}_\Phi < \epsilon$. Define the linear ordering $\prec_\alpha$ by $x \prec_\alpha y$ if and only if

(a) $x, y \in h(\alpha)$ and $x < y$.

(b) $x = *_\alpha$ and $y \in h(\alpha)$, and $y \geq \Xi^{\pi(\mathfrak{b}_\Phi)}(\alpha)$.

(c) $x \in h(\alpha)$, $y = *_\alpha$, and $x < \Xi^{\pi(\mathfrak{b}_\Phi)}(\alpha)$.

If $\mathfrak{b}_\Phi = \epsilon$, then define $x \prec_\alpha y$ if and only if

(a) $x, y \in h(\alpha) \wedge x < y$.

(b) $x \in h(\alpha)$ and $y = *_\alpha$.

Let $\mathcal{L} = (L, \prec)$ be a linear ordering on $L = \{(\alpha, x) : \alpha \in \omega_1 \wedge x \in \mathcal{L}_\alpha\}$ where $\prec$ is the lexicographic ordering on L with $\prec_\alpha$ on the α^{th}-coordinate. Note that $\mathscr{L}$ has ordertype ω_1.

In the case that $\mathfrak{b}_\Phi = \epsilon$, let $\tilde{h}(\alpha) = h(\alpha) + 1$. By initially choosing Ξ large enough, one may assume that Ξ is also a Kunen function for $\tilde{h}$ with respect to μ. Note that $\mathcal{L}$ is order isomorphic to $\mathcal{T}^{\tilde{h}}$.

Suppose $K \in [\omega_1]^{\mathcal{L}}_*$. Define $\mathsf{main}(K) : [\omega_1]^{\mathcal{T}^h} \to \omega_1$ by $\mathsf{main}(K)(\alpha, \beta) = K(\alpha, \beta)$. Define $\mathsf{extra}(K) : \omega_1 \to \omega_1$ by $\mathsf{extra}(K)(\alpha) = K(\alpha, *_\alpha)$.

Let $P : [\omega_1]^{\mathcal{L}} \to 2$ be defined by $P(K) = 0 \Leftrightarrow \Phi(\mathsf{funct}(\mathsf{main}(K))) < [\mathsf{extra}(K)]_\mu$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is a club $C \subseteq \omega_1$ which is homogeneous for P.

Claim 1: C is homogeneous for P taking value 0.

By definition of $\mathfrak{b}_\Phi$, there is an ω-club $B' \subseteq \omega_2$ so that all $\mathcal{F} \in [B]^\epsilon_*$, $\mathfrak{b}_\Phi$ is the largest $\gamma \leq \epsilon$ so that $\Phi(\mathcal{F}) \geq \mathsf{bound}(\mathcal{F}, \gamma)$. By Fact 3.14, there is a club C' so that $[C']^{\omega_1}_*/\mu \subseteq B$. By intersecting with C', assume that $C \subseteq C'$.

(Case I) $\mathfrak{b}_\Phi < \epsilon$.

Let $D = \{\alpha \in C : \mathsf{enum}_C(\alpha) = \alpha\}$ be the closure points of C. Let $B = [D]^{\omega_1}_*$. Pick any $\mathcal{F} \in [B]^\epsilon_*$. By Fact 3.12, there is some $F \in [D]^{\mathcal{T}^h}_*$ so that for all $\alpha < \epsilon$, $[F^{(\alpha)}]_\mu = \mathcal{F}(\alpha)$. Let $f : \omega_1 \to \omega_1$ be such that $[f]_\mu = \Phi(\mathcal{F})$. By Fact 3.14, $\mathsf{bound}(\mathcal{F}, \mathfrak{b}_\Phi) = [\mathsf{Bound}_{\mathfrak{b}_\Phi}(F)]_\mu$. Since $\mathfrak{b}_\Phi$ is the least γ so that $\Phi(\mathcal{F}) \geq \mathsf{bound}(\mathcal{F}, \gamma)$, one has that the set A of α's so that $\mathsf{Bound}_{\mathfrak{b}_\Phi}(F)(\alpha) \leq f(\alpha) < F^{(\mathfrak{b}_\Phi)}(\alpha)$ belongs to μ. Define $K \in [C]^{\mathcal{L}}_*$ by

$$K(\alpha, z) = \begin{cases} F(\alpha, z) & z \in h(\alpha) \\ \mathsf{next}^\omega_C(f(\alpha)) & \alpha \in A \wedge z = *_\alpha \\ \mathsf{next}^\omega_C(\mathsf{Bound}_{\mathfrak{b}_\Phi}(F)(\alpha)) & \alpha \notin A \wedge z = *_\alpha \end{cases}.$$

Note that since $F(\alpha, \Xi^{\pi(\mathfrak{b}_\Phi)}) \in D$, $K(\alpha, *_\alpha) < K(\alpha, \Xi^{\pi(\mathfrak{b}_\Phi)})$ for all α. Thus $K : \mathcal{L} \to C$ is indeed an increasing function. Since F is a function of the correct type, one can check that K is also of the correct type.

Note that $\mathsf{main}(K) = F$ and for μ-almost all α, $\mathsf{extra}(K)(\alpha) = \mathsf{next}^\omega_C(f(\alpha)) > f(\alpha)$. Thus $\Phi(\mathsf{funct}(\mathsf{main}(K))) = \Phi(\mathcal{F}) = [f]_\mu < [\mathsf{extra}(K)]_\mu$. Thus $P(K) = 0$. However since C is homogeneous for P and $K \in [C]^{\mathcal{L}}_*$, one has that C is homogeneous for P taking value 0.

(Case II) $\mathfrak{b}_\Phi = \epsilon$.

Let $B = [C]^{\omega_1}_*$. Pick any $\mathcal{F} \in [B]^\epsilon_*$. Let $f : \omega_1 \to \omega_1$ be such that $[f]_\mu = \Phi(\mathcal{F})$. Let $g(\alpha) = \mathsf{next}^\omega_C(f(\alpha))$. Let $\mathcal{G} \in [B]^{\epsilon+1}_*$ be defined by

$$\mathcal{G}(\alpha) = \begin{cases} \mathcal{F}(\alpha) & \alpha < \epsilon \\ [g]_\mu & \alpha = \epsilon \end{cases}$$

By Fact 3.12, there is some $K \in [C]^{\mathcal{T}^{\hat{h}}}_* = [C]^{\mathcal{L}}_*$ so that for all $\alpha < \epsilon + 1$, $K^{(\alpha)} = \mathcal{G}(\alpha)$.

Then one has that $\Phi(\mathsf{funct}(\mathsf{main}(K))) = \Phi(\mathcal{F}) = [f]_\mu < [g]_\mu = [\mathsf{extra}(K)]_\mu$. Thus $P(K) = 0$. Since $K \in [C]^{\mathcal{L}}_*$, C is homogeneous for P taking value 0.

The claim has now been established.

Let $D = \{\alpha \in C : \mathsf{enum}_C(\alpha) = \alpha\}$. Let $B = [D]^{\omega_1}_*$. Now suppose $\mathcal{F} \in [B]^\epsilon_*$. By Fact 3.12, pick any $F \in [D]^{\mathcal{T}^h}_*$ so that for all $\alpha < \epsilon$, $[F^{(\alpha)}]_\mu = \mathcal{F}(\alpha)$. Now define $K \in [C]^{\mathcal{L}}_*$ by

$$K(\alpha, z) = \begin{cases} F(\alpha, z) & z \in h(\alpha) \\ \mathsf{next}^\omega_C(\mathsf{Bound}_{\mathfrak{b}_\Phi}(F)(\alpha)) & z = *_\alpha \end{cases}.$$

Since C is homogeneous for P taking value 0, one has $P(K) = 0$. This implies $\Phi(\mathcal{F}) = \Phi(\mathsf{funct}(\mathsf{main}(K))) < [\mathsf{extra}(K)]_\mu = [\mathsf{Fnext}_{\mathfrak{b}_\Phi, C}(F)]_\mu = \mathsf{fnext}_{\mathfrak{b}_\Phi, C}(\mathcal{F})$. This completes the proof. □

Definition 3.17. Suppose $\Sigma : \omega_1 \times \omega_1 \to \omega_1$.

Suppose $f_0 : \omega_1 \to \omega_1$ and $f_1 : \omega_1 \to \omega_1$. Let $v_{f_0, f_1} : \omega_1 \to \omega_1$ be defined by $v_{f_0, f_1}(\alpha) = \Sigma(f_0(\alpha), f_1(\alpha))$. Note that if $f_0' =_\mu f_0$ and $f_1' =_\mu f_1$, then $v_{f_0, f_1} =_\mu v_{f_0', f_1'}$.

Therefore, define $\hat{\Sigma} : \omega_2 \times \omega_2 \to \omega_2$ by $\hat{\Sigma}(\alpha, \beta) = [v_{f_\alpha, f_\beta}]_\mu$, where $f_\alpha, f_\beta : \omega_1 \to \omega_1$ are such that $[f_\alpha]_\mu = \alpha$ and $[f_\beta]_\mu = \beta$.

Lemma 3.18. *Suppose $\mathfrak{b}_\Phi > 0$. Then there is a Kunen function $\Sigma : \omega_1 \times \omega_1 \to \omega_1$ and a function $\Phi' : [\omega_2]^\epsilon_* \to \omega_2$ so that for ν^ϵ-almost all $\mathcal{F}$, $\Phi(\mathcal{F}) = \hat{\Sigma}(\mathsf{bound}(\mathcal{F}, \mathfrak{b}_\Phi), \Phi'(\mathcal{F}))$ where $\mathfrak{b}_{\Phi'} < \mathfrak{b}_\Phi$.*

Proof. Let $B \subseteq \omega_2$ be the ω-club and $C \subseteq \omega_1$ be the club from Lemma 3.16.

Pick any $\mathcal{F} \in [B]^\epsilon_*$. Let $F \in [\omega_1]^{\mathcal{T}^h}_*$ be so that for all $\alpha < \omega_1$, $[F^{(\alpha)}]_\mu = \mathcal{F}(\alpha)$. Let $f : \omega_1 \to \omega_1$ be such that $[f]_\mu = \Phi(\mathcal{F})$. By Lemma 3.16, for μ-almost all α, $f(\alpha) < \mathsf{next}^\omega_C(\mathsf{Bound}_{\mathfrak{b}_\Phi}(F)(\alpha))$. Let $\Sigma : \omega_1 \times \omega_1 \to \omega_1$ be a Kunen function for next^ω_C. For μ-almost all α, let $v_{f,F}(\alpha)$ be the least $\gamma < \mathsf{Bound}_{\mathfrak{b}_\Phi}(F)(\alpha)$ so that $f(\alpha) = \Sigma(\mathsf{Bound}_{\mathfrak{b}_\Phi}(F)(\alpha), \gamma)$. Observe that if $g =_\mu f$ and $G \in [\omega_1]^{\mathcal{T}^h}$ is such that $G^{(\alpha)} =_\mu F^{(\alpha)}$ for all $\alpha < \epsilon$, then $v_{f,F} =_\mu v_{g,F}$. Therefore, define $\Phi'(\mathcal{F}) = [v_{f,F}]_\mu$. Note by construction, $\Phi(\mathcal{F}) = [f]_\mu = \hat{\Sigma}(\mathsf{bound}(\mathcal{F}, \mathfrak{b}_\Phi), [v_{f,F}]_\mu) = \hat{\Sigma}(\mathsf{bound}(\mathcal{F}, \mathfrak{b}_\Phi), \Phi'(\mathcal{F}))$. Since $\Phi'(\mathcal{F}) < \mathsf{bound}(\mathcal{F}, \mathfrak{b}_\Phi)$, one has that $\mathfrak{b}_{\Phi'} < \mathfrak{b}_\Phi$ if $\mathfrak{b}_\Phi > 0$. □

Definition 3.19. Let $\epsilon < \omega_2$ and $\Phi : [\omega_2]^\epsilon_* \to \omega_2$.

A representation for Φ is a tuple $(\Xi_0, ..., \Xi_{n-1}; \beta_0, ..., \beta_n; \gamma)$ with the following properties

(a) $n \in \omega$. If $n = 0$, then no Ξ appears.
(b) $\beta_0 > \beta_1 > ... > \beta_{n-1} > \beta_n = 0$ is a sequence of strictly decreasing ordinals less than or equal to ϵ. $\gamma < \omega_2$.
(c) Each $\Xi_i : \omega_1 \times \omega_1 \to \omega_1$.
(d) Let $\Phi_n(\mathcal{F}) = \gamma$. Suppose for $0 < i \leq n$, Φ_i has been defined, then let $\Phi_{i-1}(\mathcal{F}) = \hat{\Xi}_i(\mathsf{bound}(\mathcal{F}, \beta_{i-1}), \Phi_i(\mathcal{F}))$. One has that for ν^ϵ-almost all $\mathcal{F}$, $\Phi_0(\mathcal{F}) = \Phi(\mathcal{F})$.

Theorem 3.20. *Let $\epsilon < \omega_2$. Every $\Phi : [\omega_2]^\epsilon_* \to \omega_2$ has a representation.*

Proof. The proof is analogous to the proof of Theorem 2.13 using the ω_2 version of the analogous lemmas. ☐

Now one has the analogous continuity result for functions $\Phi : [\omega_2]^\epsilon_* \to \omega_2$ where $\epsilon < \omega_2$.

Theorem 3.21. *Let $\epsilon < \omega_2$ and $\Phi : [\omega_2]^\epsilon_* \to \omega_2$. Then there is a decreasing sequence of ordinals less than or equal to ϵ, $(\beta_i : i \leq n)$, with $\beta_n = 0$ and an ω-club $B \subseteq \omega_2$ so that if $\mathcal{F}, \mathcal{G} \in [B]^\epsilon_*$ has the property that $\mathsf{bound}(\mathcal{F}, \beta_i) = \mathsf{bound}(\mathcal{G}, \beta_i)$ for all $i \leq n$, then $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$.*

Theorem 3.22. *Let $\epsilon < \omega_2$ and $\Phi : [\omega_2]^\epsilon_* \to \omega_2$. Then there is a $\delta < \epsilon$ and an ω-club $B \subseteq \omega_2$ so that for all $\mathcal{F}, \mathcal{G} \in [B]^\epsilon_*$ with $\mathcal{F} \restriction \delta = \mathcal{G} \restriction \delta$ and $\sup(\mathcal{F}) = \sup(\mathcal{G})$, $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$.*

Now one has some new cardinality results:

Theorem 3.23. $|[\omega_2]^\omega| < |[\omega_2]^{<\omega_1}|$.

Proof. Suppose $\Phi : [\omega_2]^{<\omega_1}_* \to [\omega_2]^\omega_*$ is a function. For each $\epsilon < \omega_1$ and each $n \in \omega$, let $\Phi^\epsilon_n : [\omega_2]^\epsilon_* \to \omega_2$ be defined by $\Phi^\epsilon_n(\mathcal{F}) = \Phi(\mathcal{F})(n)$. By Theorem 3.22, there is some $\delta < \epsilon$ so that $\Phi^\epsilon_n(\mathcal{F}) = \Phi^\epsilon_n(\mathcal{G})$ for ν^ϵ-almost all $\mathcal{F}$ and $\mathcal{G}$ so that $\mathcal{F} \restriction \delta = \mathcal{G} \restriction \delta$ and $\sup(\mathcal{F}) = \sup(\mathcal{G})$. Let δ^ϵ_n be the least such δ. The function $\Lambda_n : \omega_1 \to \omega_1$ defined by $\Lambda_n(\epsilon) = \delta^\epsilon_n$ is a regressive function. Using $\mathsf{AC}^{\mathbb{R}}_\omega$, there is a $\delta_n < \omega_1$ and $A_n \in \mu$ so that for all $\epsilon \in A_n$, $\Lambda_n(\epsilon) = \delta_n$. Let $A = \bigcap_{n \in \omega} A_n \in \mu$ and $\delta = \sup_{n \in \omega} \delta_n < \omega_1$. Pick a limit ordinal $\epsilon \in A$ with $\epsilon > \delta$. By $\mathsf{AC}^{\mathbb{R}}_\omega$, let B_n be an ω-club subset of ω_2 so that for all $\mathcal{F}, \mathcal{G} \in [B_n]^\epsilon_*$, if $\sup(\mathcal{F}) = \sup(\mathcal{G})$ and $\mathcal{F} \restriction \delta_n = \mathcal{G} \restriction \delta_n$, then $\Phi^\epsilon_n(\mathcal{F}) = \Phi^\epsilon_n(\mathcal{G})$. Since ν is ω_2-complete, $B = \bigcap_{n \in \omega} B_n \in \nu$. Thus pick some $\mathcal{F}, \mathcal{G} \in [B]^\epsilon_*$ with $\mathcal{F} \neq \mathcal{G}$, $\sup(\mathcal{F}) = \sup(\mathcal{G})$, and $\mathcal{F} \restriction \delta = \mathcal{G} \restriction \delta$. Then for all $n \in \omega$, $\Phi^\epsilon_n(\mathcal{F}) = \Phi^\epsilon_n(\mathcal{G})$. So $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$. Φ can not be an injection. ☐

Theorem 3.24. $|[\omega_2]^{<\omega_1}| < |[\omega_2]^{\omega_1}|$.

Proof. First, it will be shown that there is an injection of $[\omega_2]^{<\omega_1}$ into $[\omega_2]^{\omega_1}$. Let $\mathsf{add} : \omega_2 \times [\omega_2]^{<\omega_1} \to [\omega_2]^{<\omega_1}$ be defined as follows: if $\mathcal{F} \in [\omega_2]^\epsilon$ for some $\epsilon < \omega_1$, then define $\mathsf{add}(\lambda, \mathcal{F}) \in [\omega_2]^\epsilon$ by $\mathsf{add}(\lambda, \mathcal{F})(\alpha) = \lambda + \mathcal{F}(\alpha)$.

If $\mathcal{F} \in [\omega_2]^{<\omega_1}$, then let $\mathsf{fill}(\mathcal{F}) \in [\omega_2]^{\omega_1}$ be defined by appending onto $\mathcal{F}$ the next ω_1-many ordinals after $\sup(\mathcal{F})$.

Let $\Phi : [\omega_2]^{<\omega_1} \to [\omega_2]^{\omega_1}$ be defined by $\Phi(\mathcal{F}) = \mathsf{fill}(\mathsf{length}(\mathcal{F})\hat{\ }\mathsf{add}(\mathsf{length}(\mathcal{F}), \mathcal{F}))$. In words, $\Phi(\mathcal{F})$ starts with $\mathsf{length}(\mathcal{F})$, then shifts up all the values of $\mathcal{F}$ by $\mathsf{length}(\mathcal{F})$, and fill in the rest with successive ordinals until one reaches length ω_1. One can check that Φ is an injection.

Next to show that $[\omega_2]^{\omega_1}$ cannot inject into $[\omega_2]^{<\omega_1}$. Let $\Phi : [\omega_2]^{\omega_1}_* \to [\omega_2]^{<\omega_1}_*$ be a function. Let $\Psi : [\omega_2]^{\omega_1}_* \to \omega_1$ be $\mathsf{length} \circ \Phi$, where $\mathsf{length}(\mathcal{F}) = \epsilon$ if $\mathcal{F} : \epsilon \to \omega_2$. Since ν is ω_2-complete, there is a $B \in \nu$ and an $\epsilon < \omega_1$ so that for all $\mathcal{F} \in [B]^{\omega_1}_*$, $\Psi(\mathcal{F}) = \epsilon$. In other words, for all $\mathcal{F} \in [B]^{\omega_1}_*$, $\Phi(\mathcal{F}) \in [\omega_2]^\epsilon_*$.

Let $\alpha < \epsilon$. Let $\Phi_\alpha(\mathcal{F}) = \Phi(\mathcal{F})(\alpha)$. By Theorem 3.22 and $\mathsf{AC}^{\mathbb{R}}_\omega$, there are $\delta_\alpha < \omega_1$ and ω-club $B_\alpha \subseteq \omega_2$ so that for all $\mathcal{F}, \mathcal{G} \in [B_\alpha]^{\omega_1}_*$, if $\mathcal{F} \restriction \delta_\alpha = \mathcal{G} \restriction \delta_\alpha$ and $\sup(\mathcal{F}) = \sup(\mathcal{G})$, then $\Phi_\alpha(\mathcal{F}) = \Phi_\alpha(\mathcal{G})$.

Now let $U = \bigcap_{\alpha < \epsilon} B_\alpha \in \nu$ since ν is ω_2-complete. Let $\delta = \sup\{\delta_\alpha : \alpha < \epsilon\}$. Note that $\delta < \omega_1$ since ω_1 is regular. Pick $\mathcal{F}, \mathcal{G} \in [U]^{\omega_1}_*$ with $\mathcal{F} \neq \mathcal{G}$, $\mathcal{F} \restriction \delta = \mathcal{G} \restriction \delta$, $\sup(\mathcal{F}) = \sup(\mathcal{G})$. Since $\mathcal{F}, \mathcal{G} \in [B]^{\omega_1}_*$, $\Phi(\mathcal{F})$ and $\Phi(\mathcal{G})$ both have length ϵ. By choice, $\Phi(\mathcal{F})(\alpha) = \Phi_\alpha(\mathcal{F}) = \Phi_\alpha(\mathcal{G}) = \Phi(\mathcal{G})(\alpha)$ for all $\alpha < \epsilon$. So $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$. Φ is not an injection. ☐

Previously, one only needed $\mathsf{AC}^{\mathbb{R}}_\omega$ to make a countable selection of subsets of ω_1 or ω_2. For the next theorem, one will need to make an ω_1-length selection of club subsets of ω_1. The following fact ensures this can be done.

Fact 3.25. *([3] Section 4) Let $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ be such that each $\mathcal{A}_\alpha$ is a nonempty $\subseteq$-downward closed collection of clubs subsets of ω_1. Then there is a sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ with each $C_\alpha \subseteq \omega_1$ a club subset of ω_1 and $C_\alpha \in \mathcal{A}_\alpha$.*

Theorem 3.26. $|[\omega_2]^{\omega_1}| < |[\omega_2]^{<\omega_2}|$.

Proof. Let $\Phi : [\omega_2]^{<\omega_2}_* \to [\omega_2]^{\omega_1}_*$ be a function. For each $\epsilon < \omega_2$ and $\alpha < \omega_1$, let $\Phi^\epsilon_\alpha : [\omega_2]^\epsilon_* \to \omega_2$ be defined by $\Phi^\epsilon_\alpha(\mathcal{F}) = \Phi(\mathcal{F})(\alpha)$. By Theorem 3.22, there is a minimal $\delta^\epsilon_\alpha < \epsilon$ so that for ν^ϵ-almost all $\mathcal{F}, \mathcal{G} \in [\omega_2]^\epsilon_*$, if $\mathcal{F} \upharpoonright \delta^\epsilon_\alpha = \mathcal{G} \upharpoonright \delta^\epsilon_\alpha$ and $\sup(\mathcal{F}) = \sup(\mathcal{G})$, then $\Phi^\epsilon_\alpha(\mathcal{F}) = \Phi^\epsilon_\alpha(\mathcal{G})$.

For each $\alpha < \omega_1$, let $\Lambda_\alpha : \omega_2 \to \omega_2$ be defined by $\Lambda_\alpha(\epsilon) = \delta^\epsilon_\alpha$. Since ν is a normal measure on ω_2 and Λ_α is a regressive function, there is a minimal $\delta_\alpha < \omega_2$ so that for ν-almost all ϵ, $\Lambda_\alpha(\epsilon) = \delta_\alpha$. By Fact 3.11, for every $B \in \nu$, there is a $C \subseteq \omega_1$ club so that $[C]^{\omega_1}_*/\mu \subseteq B$. Let $\mathcal{A}_\alpha$ be the collection of all club $C \subseteq \omega_1$ so that for all $\epsilon \in [C]^{\omega_1}_*/\mu$, $\Lambda_\alpha(\epsilon) = \delta_\alpha$. $\mathcal{A}_\alpha$ is clearly $\subseteq$-downward closed. Apply Fact 3.25 to obtain a sequence $\langle C_\alpha : \alpha < \omega_1\rangle$ so that $C_\alpha \in \mathcal{A}_\alpha$. Let $B = \bigcap_{\alpha<\omega_1}[C_\alpha]^{\omega_1}_*/\mu$ which belongs to ν as ν is ω_2-complete. Let $\delta = \sup\{\delta_\alpha : \alpha < \omega_1\} < \omega_2$ since ω_2 is regular. Now pick a limit ordinal $\epsilon > \delta$ with $\epsilon \in B$.

For $\alpha < \omega_1$, let $\mathcal{A}'_\alpha$ be the collection of club $C \subseteq \omega_1$ so that if $D = [C]^{\omega_1}_*/\mu$, then D has the property that for all $\mathcal{F}, \mathcal{G} \in [D]^\epsilon_*$, if $\mathcal{F} \upharpoonright \delta_\alpha = \mathcal{G} \upharpoonright \delta_\alpha$ and $\sup(\mathcal{F}) = \sup(\mathcal{G})$, then $\Phi^\epsilon_\alpha(\mathcal{F}) = \Phi^\epsilon_\alpha(\mathcal{G})$. $\mathcal{A}'_\alpha$ is a $\subseteq$-downward closed nonempty collection of club subsets of ω_1. Apply Fact 3.25 to obtain a collection $\langle C'_\alpha : \alpha < \omega_1\rangle$ of club subsets of ω_1 with the property that for all $\alpha < \omega_1$, $C'_\alpha \in \mathcal{A}'_\alpha$. Let $B' = \bigcap_{\alpha<\omega_1}[C'_\alpha]^{\omega_1}_*/\mu$ which belongs to ν since ν is ω_2-complete. Now pick $\mathcal{F}, \mathcal{G} \in [B']^\epsilon_*$ with $\mathcal{F} \upharpoonright \delta = \mathcal{G} \upharpoonright \delta$, $\sup(\mathcal{F}) = \sup(\mathcal{G})$, and $\mathcal{F} \neq \mathcal{G}$. Note that for all $\alpha < \omega_1$, $\Phi(\mathcal{F})(\alpha) = \Phi^\epsilon_\alpha(\mathcal{F}) = \Phi^\epsilon_\alpha(\mathcal{G}) = \Phi(\mathcal{G})(\alpha)$. Thus $\Phi(\mathcal{F}) = \Phi(\mathcal{G})$. Φ is not an injection. ☐

Theorem 3.27. $|[\omega_2]^\omega| < |[\omega_2]^{<\omega_1}| < |[\omega_2]^{\omega_1}| < |[\omega_2]^{<\omega_2}|$.

Proof. These follow from Theorem 3.23, Theorem 3.24, and Theorem 3.26. ☐

Fact 3.28. *ω_2 does not inject into $[\omega_1]^{\omega_1}$. Thus $[\omega_2]^\omega$ does not inject into $[\omega_1]^{\omega_1}$.*

Proof. This is a consequence of the measurability of ω_2 in the same way the fact that there are no uncountable wellordered sequences of reals follows from the measurability of ω_1. The details follow:

Let ν be an ω_2-complete measure on ω_2. Suppose $\langle f_\alpha : \alpha < \omega_2\rangle$ is an injection of ω_2 into $[\omega_1]^{\omega_1}$. Let $F_\alpha = \text{rang}(f_\alpha)$. Then $\langle F_\alpha : \alpha < \omega_2\rangle$ is an ω_2-sequence of distinct subsets of ω_1.

For each $\beta < \omega_1$, let $A^0_\beta = \{\alpha < \omega_2 : \beta \notin F_\alpha\}$ and $A^1_\beta = \{\alpha < \omega_2 : \beta \in F_\alpha\}$. Since μ is a measure, there is some $i_\beta \in 2$ so that $A^{i_\beta}_\beta \in \nu$.

By the ω_2-completeness of ν, $\bigcap_{\beta\in\omega_1} A^{i_\beta}_\beta \in \nu$. Let $\alpha_0, \alpha_1 \in \bigcap_{\beta\in\omega_1} A^{i_\beta}_\beta$. Let $F \subseteq \omega_1$ be defined by $\beta \in F \Leftrightarrow i_\beta = 1$. Then $F_\alpha = F_\beta = F$. This contradicts the fact that $\langle F_\alpha : \alpha < \omega_2\rangle$ is a sequence of distinct subsets of ω_1. ☐

Like the original argument for the cardinal relation $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$, the argument that $[\omega_1]^{<\omega_1}$ does not inject into $[\omega_2]^\omega$ passes through the set S_1 using ∞-Borel codes and forcing arguments. This originally was proved under $\mathsf{ZF} + \mathsf{AD}^+$. The following gives a purely descriptive set theoretic proof using just AD.

Theorem 3.29. $\neg(|[\omega_1]^{<\omega_1}| \leq |[\omega_2]^\omega|)$. *Thus* $\neg(|[\omega_1]^{\omega_1}| \leq [\omega_2]^\omega)$.

Proof. Suppose $\Phi : [\omega_1]^{<\omega_1} \to [\omega_2]^\omega$ is an injection.

For each $\epsilon < \omega_1$ and $f \in [\omega_1]^{\omega_1}$, let $\mathsf{tail}(f, \epsilon) \in [\omega_1]^{\omega_1}$ be defined by $\mathsf{tail}(f, \epsilon)(\beta) = f(\epsilon + \beta)$. Note that for all $\epsilon < \omega_1$ and $f \in [\omega_1]^{\omega_1}$, $f = (f \upharpoonright \epsilon)\hat{\ }\mathsf{tail}(f, \epsilon)$. Let μ denote the club measure on ω_1.

For each $\epsilon < \omega_1$, let $P_\epsilon : [\omega_1]^{\omega_1}_* \to 2$ by defined by $P_\epsilon(f) = 0$ if and only if $\sup(\Phi(f \upharpoonright \epsilon)) < [\mathsf{tail}(f, \epsilon)]_\mu$. (Recall that $\prod_{\omega_1} \omega_1/\mu = \omega_2$.)

Let $C \subseteq \omega_1$ be a club which is homogeneous for P_ϵ. The claim is that C is homogeneous for P_ϵ taking value 0. Suppose otherwise, then pick any $\sigma \in [C]^\epsilon_*$. For any $g \in [C]^{\omega_1}_*$ with $\min(g) > \sup(\sigma)$, define $\sigma^g \in [C]^{\omega_1}_*$ by $\sigma\hat{\ }g$. Then $P(\sigma^g) = 1$ implies that $[g]_\mu = \mathsf{tail}(\sigma^g, \epsilon) \leq \sup(\Phi(\sigma^g \upharpoonright \epsilon)) = \sup(\Phi(\sigma))$. This impossible since σ is fixed, $[C]^{\omega_1}/\mu = \omega_2$, and g can be any member of $[C]^{\omega_1}_*$ with $\min(g) > \sup(\sigma)$.

It has been shown that C is homogeneous for P_ϵ taking value 0. Let $\ell \in [C]^{\omega_1}_*$ and let $\beta = [\ell]_\mu$. Note that for all $\epsilon < \omega_1$, $\ell =_\mu \mathsf{tail}(\ell, \epsilon)$. Let $\sigma \in [C]^\epsilon_*$. Let γ_σ be the least γ so that $\ell(\gamma) > \sup(\sigma)$. Define $f_\sigma = \sigma\hat{\ }\mathsf{tail}(\ell, \gamma_\sigma)$. Note that $f_\sigma \in [C]^{\omega_1}_*$. Thus $P_\epsilon(f_\sigma) = 0$ implies that $\sup(\Phi(\sigma)) = \sup(\Phi(f_\sigma \upharpoonright \epsilon)) < [\mathsf{tail}(f_\sigma, \epsilon)]_\mu = [\mathsf{tail}(\ell, \gamma_\sigma)]_\mu = [\ell]_\mu = \beta$. That is, Φ maps $[C]^\epsilon_*$ into $[\beta]^\omega$.

For each $\epsilon < \omega_1$, let β_ϵ be the least $\beta < \omega_2$ so that there exists a club $C \subseteq \omega_1$ with the property that for all $\sigma \in [C]^\epsilon_*$, $\sup(\Phi(\sigma)) < \beta$. This defines a sequence $\langle \beta_\epsilon : \epsilon < \omega_1\rangle$. Let $\delta = \sup\{\beta_\epsilon : \epsilon < \omega_1\}$. Since ω_2 is regular, $\delta < \omega_2$.

For $\epsilon < \omega_1$, let $\mathcal{A}_\epsilon$ be the collection of clubs $C \subseteq \omega_1$ so that for all $\sigma \in [C]^\epsilon_*$, $\sup(\Phi(\sigma)) < \beta_\epsilon$. This defines a sequence $\langle \mathcal{A}_\epsilon : \epsilon < \omega_1\rangle$. Note that for all $\epsilon < \omega_1$, $\mathcal{A}_\epsilon$ is a nonempty $\subseteq$-downward closed collection of club subsets of ω_1. By Fact 3.25, let $\langle C_\epsilon : \epsilon < \omega_1\rangle$ be a sequence so that $C_\epsilon \in \mathcal{A}_\epsilon$ for all $\epsilon \in \omega_1$. So for any $\epsilon < \omega_1$, if $\sigma \in [C_\epsilon]^\epsilon_*$, then $\sup(\Phi(\sigma)) < \delta$.

Note that $\bigcup_{\epsilon<\omega_1} [C_\epsilon]^\epsilon_* \approx [\omega_1]^{<\omega_1}$. Observe that

$$\Phi\left[\bigsqcup_{\epsilon<\omega_1} [C_\epsilon]^\epsilon_*\right] \subseteq [\delta]^\omega.$$

Hence Φ induces an injection of $[\omega_1]^{<\omega_1}$ into $[\delta]^\omega \approx [\omega_1]^\omega$ since $\delta < \omega_2$. By Theorem 2.16, this is impossible. $\square$

Fact 3.30. $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$.

Proof. There is a purely descriptive set theoretic proof of this result in the flavor of the continuity argument used throughout this paper in [6]. However, the requisite continuity property is more challenging to establish than the analogous continuity properties in this paper. However, there is a very simple set theoretic proof of this result:

Suppose there was an injection $\Phi : [\omega_1]^{\omega_1} \to [\omega_1]^{<\omega_1}$. Let $\tilde{\Phi} = \{(f,\beta) : f \in [\omega_1]^{\omega_1} \wedge \beta \in \Phi(f)\}$, where $\Phi(x) \in [\omega_1]^{<\omega_1}$ is considered as a countable subset of ω_1. Let $L[\tilde{\Phi}] \models \mathsf{ZFC}$ be the Gödel constructible universe built relative to $\tilde{\Phi}$ as a predicate. Note that if $f \in [\omega_1]^{\omega_1} \cap L[\tilde{\Phi}]$, then $\Phi(f) \in L[\tilde{\Phi}]$.

Note that ω_1^V is inaccessible in $L[\tilde{\Phi}]$: Suppose $\delta < \omega_1^V$ and $|\mathscr{P}(\delta)^{L[\tilde{\Phi}]}|^{L[\tilde{\Phi}]} \geq \omega_1^V$. Since $L[\tilde{\Phi}] \models \mathsf{AC}$, $\mathscr{P}(\delta)^{L[\tilde{\Phi}]}$ is a wellorderable collection of subsets of δ of cardinality ω_1^V. In the real world V, δ is a countable ordinal and hence there is a bijection of δ with ω. Using this bijection, one can obtain an ω_1^V-length sequence of distinct reals from $\mathscr{P}(\delta)^{L[\tilde{\Phi}]}$. This is impossible under AD by a simple form of the argument in Fact 3.28. Thus $|\mathscr{P}(\delta)^{L[\tilde{\Phi}]}|^{L[\tilde{\Phi}]} < \omega_1^V$. This implies ω_1^V is inaccessible in $L[\tilde{\Phi}]$.

Since $L[\tilde{\Phi}] \models \mathsf{ZFC}$, Cantor's theorem assert that $L[\tilde{\Phi}] \models |[\omega_1^V]^{\omega_1^V}| = |2^{\omega_1^V}| \geq (\omega_1^V)^+$. Also since $L[\tilde{\Phi}] \models \mathsf{ZFC}$ and ω_1^V is inaccessible in $L[\tilde{\Phi}]$, $L[\tilde{\Phi}] \models |[\omega_1^V]^{<\omega_1^V}| = |2^{<\omega_1^V}| = \omega_1^V$. By absoluteness, $L[\tilde{\Phi}] \models \Phi$ is an injection. It is impossible that $L[\tilde{\Phi}]$ thinks that Φ is an injection of $2^{\omega_1^V}$ into ω_1^V. $\square$

A very similar argument can be used to show that $|[\omega_2]^{<\omega_2}| < |[\omega_2]^{\omega_2}|$. See [4] Section 4.

Theorem 3.31. $\neg(|[\omega_1]^{\omega_1}| \leq |[\omega_2]^{<\omega_1}|)$.

Proof. Let $\mathcal{T} = (\omega_1 \times 2, \prec)$ where $\prec$ is the lexicographic ordering. (Note that $\mathrm{ot}(\mathcal{T}) = \omega_1$.) If $F \in [\omega_1]^{\mathcal{T}}_*$ and $i \in 2$, let $F_i \in [\omega_1]^{\omega_1}$ be defined by $F_i(\alpha) = F(\alpha, i)$.

Now suppose $\Phi : [\omega_1]^{\omega_1} \to [\omega_2]^{<\omega_1}$ is an injection. Define a partition $P : [\omega_1]^{\mathcal{T}} \to 2$ by $P(F) = 0$ if and only if $\sup(\Phi(F_0)) \leq \sup(\Phi(F_1))$. Let $C \subseteq \omega_1$ be a club homogeneous subset for P. The claim is C is homogeneous for P taking value 0.

Suppose C was homogeneous for P taking value 1. Let $g_0(0) = \mathsf{next}^\omega_C(0)$. Suppose $g_k(\alpha)$ has been defined, then let $g_{k+1}(\alpha) = \mathsf{next}^\omega_C(g_k(\alpha))$. Suppose $g_n(\beta)$ has been defined for all $n \in \omega$ and $\beta < \alpha$. Then let $g_0(\alpha) = \mathsf{next}^\omega_C(\sup\{g_n(\beta) : n \in \omega \wedge \beta < \alpha\})$.

For each $n \in \omega$, $g_n \in [C]^{\omega_1}_*$. Define for $\alpha < \omega_1$ and $i \in 2$, $G^n(\alpha, i) = g_{n+i}(\alpha)$. By the construction of $\langle g_n : n \in \omega\rangle$, one has that $G^n \in [C]^{\mathcal{T}}_*$.

Thus one has that $P(G^n) = 1$ for all $n \in \omega$. This implies for all $n \in \omega$.

$$\sup(\Phi(g_{n+1})) = \sup(\Phi(G^n_1)) < \sup(\Phi(G^n_0)) = \sup(\Phi(g_n)).$$

It has been shown that $\langle \sup(\Phi(g_n)) : n \in \omega\rangle$ is an infinite decreasing sequence of ordinals. This contradicts the wellfoundedness of the ordinals.

One must have that C is homogeneous for P taking value 0. For the next part, take g_0, g_1, and g_2 from the sequence $\langle g_n : n \in \omega\rangle$ constructed above. The important observation from above is that $g_0(\alpha) < g_1(\alpha) < g_2(\alpha) < g_0(\alpha + 1)$ for all α.

For each $A \in {}^{\omega_1}2$, let $h_A \in [C]^{\omega_1}_*$ be defined by $h_A(\alpha) = g_{A(\alpha)}(\alpha)$. Let $H^A \in [C]^{\mathcal{T}}_*$ be defined by

$$H^A(\alpha, i) = \begin{cases} h_A(\alpha) & i = 0 \\ g_2(\alpha) & i = 1 \end{cases}.$$

Note that $H^A_0 = h_A$ and $H^A_1 = g_2$. $P(H^A) = 0$ implies that $\sup(\Phi(h_A)) = \sup(\Phi(H^A_0)) \leq \sup(\Phi(H^A_1)) = \sup(\Phi(g_2))$. Let $\zeta = \sup(\Phi(g_2))$ which is an ordinal less than ω_2.

Define $\Psi : {}^{\omega_1}2 \to [\omega_2]^{<\omega_1}$ by $\Psi(A) = \Phi(h_A)$. Note that Ψ is a injection. By the above, $\Psi : {}^{\omega_1}2 \to [\zeta]^{<\omega_1}$. Since ${}^{\omega_1}2 \approx \mathscr{P}(\omega_1) \approx [\omega_1]^{\omega_1}$, one has shown that there is an injection of $[\omega_1]^{\omega_1}$ into $[\zeta]^{<\omega_1} \approx [\omega_1]^{<\omega_1}$. This is not possible by Fact 3.30. $\square$

For the sake of completeness, one sketches the remaining well-known cardinal relations among the sets considered in this paper:

Fact 3.32. $\neg(\omega_1 \leq |\mathbb{R}|)$ *and* $\neg(|\mathbb{R}| \leq \omega_1)$.

Proof. By a simple form of the argument in the proof of Fact 3.28, there are no uncountable wellordered sequences of distinct reals. That is, ω_1 can not inject into $\mathbb{R}$.

Under AD, $\mathbb{R}$ can not be wellordered. Hence $\mathbb{R}$ can not inject into ω_1. $\square$

Fact 3.33. *Let κ be an ordinal.* $\neg(|[\omega_1]^\omega| \leq \kappa)$, $\neg(|[\omega_1]^\omega| \leq \mathbb{R})$, $\neg(|[\omega_1]^\omega| \leq |\mathbb{R} \sqcup \kappa|)$, *and* $\neg(|[\omega_1]^\omega| \leq |\mathbb{R} \times \kappa|)$.

Similarly, $\neg(|[\omega_2]^\omega| \leq \kappa)$, $\neg(|[\omega_2]^\omega| \leq \mathbb{R})$, $\neg(|[\omega_2]^\omega| \leq |\mathbb{R} \sqcup \kappa|)$, *and* $\neg(|[\omega_2]^\omega| \leq |\mathbb{R} \times \kappa|)$.

Proof. Since $\mathbb{R}$ injects into $[\omega_1]^\omega$ and $\mathbb{R}$ is not wellorderable, $[\omega_1]^\omega$ is not wellorderable. So $[\omega_1]^\omega$ can not inject into any ordinal κ.

Let $\Phi : [\omega_1]^\omega \to {}^\omega 2$. For each $n \in \omega$, define $P_n : [\omega_1]^\omega \to 2$ by $P_n(f) = f(n)$. By $\mathsf{AC}^{\mathbb{R}}_\omega$, let $C_n \subseteq \omega_1$ be club homogeneous for P_n taking some value $i_n \in 2$. Let $C = \bigcap_{n\in\omega} C_n$. Let $r \in {}^\omega 2$ be defined by $r(n) = i_n$. Note that $\Phi[[C]^\omega_*] = \{r\}$. Thus Φ is not an injection.

Now suppose $\Phi : [\omega_1]^\omega \to \kappa \sqcup \mathbb{R}$. Define $Q : [\omega_1]^\omega \to 2$ by

$$Q(f) = \begin{cases} 0 & \Phi(f) \in \kappa \\ 1 & \Phi(f) \in \mathbb{R} \end{cases}$$

Let $C \subseteq \omega_1$ be club homogeneous for Q. If C is homogeneous for Q taking value 0, then Φ maps $[C]^\omega_*$ into κ. By the earlier argument, Φ can not be an injection. If C is homogeneous for Q taking value 1, the Φ maps $[C]^\omega_*$ into $\mathbb{R}$. Again by the earlier argument, Φ can not be an injection.

Suppose $\Phi : [\omega_1]^\omega \to \mathbb{R} \times \omega_1$. Let $\pi_1 : \mathbb{R} \times \omega_1 \to \mathbb{R}$ be the projection onto the first coordinate. Then $\pi_1 \circ \Phi : [\omega_1]^{\omega_1} \to \mathbb{R}$. By the argument above, there is a club $C \subseteq \omega_1$ and an $r \in \mathbb{R}$ so that $(\pi \circ \Phi)[[C]^\omega_*] = \{r\}$. Then $\Phi : [C]^\omega_* \to \{r\} \times \omega_1$. Since $\{r\} \times \omega_1$ is in bijection with ω_1, Φ can not be an injection by the earlier part of this proof.

The result for $[\omega_2]^\omega$ follows by the same argument using the weak partition property for ω_2. $\square$

The cardinal relations displayed in the diagram from the introduction follow from the work so far.

4. $[\omega_1]^\omega$ is Jónsson

Definition 4.1. Let X be a set. Define $[X]^n_= = \{f \in {}^n X : (\forall i < j < n)(f(i) \neq f(j))\}$. Let $[X]^{<\omega}_= = \bigcup_{n\in\omega} [X]^n_=$.

For $n < \omega$, X is n-Jónsson if and only if for every $\Phi : [X]^n_= \to X$, there is some $Z \subseteq X$ with $Z \approx X$ so that $\Phi[[Z]^n_=] \neq X$.

X is Jónsson if and only if for all $\Phi : [X]^{<\omega}_= \to X$, there is some $Z \subseteq X$ with $Z \approx X$ so that $\Phi[[X]^{<\omega}_=] \neq X$.

Definition 4.2. Let $\bar{f} \in {}^{<\omega}([\omega_1]^\omega)$. The tuple-type of $\bar{f}$, denoted $\mathsf{type}(\bar{f})$, is a 4-tuple (n, m, G, D) with the following properties:

(1) n is the length of the tuple $\bar{f}$.

(2) Let $S = \{\sup(f_i) : i < n\}$. Then $m = |S|$.

Let $\text{rang}(\bar{f}) = \bigcup_{i<n} \text{rang}(f_i)$. Note that m also has the property that $\text{ot}(\text{rang}(\bar{f})) = \omega \cdot m$. Let $\langle a_0, ..., a_{m-1}\rangle$ be the increasing enumeration of S. Let $F : \omega \cdot m \to \text{rang}(\bar{f})$ be the increasing enumeration of $\text{rang}(\bar{f})$.

(3) $G : m \to \mathscr{P}(n)$ is defined by $G(i) = \{k \in n : \sup(f_k) = a_i\}$.
(4) Let $D : \omega \cdot m \to \mathscr{P}(n)$ be defined by $D(\alpha) = \{i \in n : F(\alpha) \in \operatorname{rang}(f_i)\}$.

If $Z \subseteq [\omega_1]^\omega$, then let $\mathsf{type}(Z) = \{\mathsf{type}(\bar{f}) : \bar{f} \in {}^{<\omega}Z\}$.

Example 4.3. Consider $f_0, f_1, f_2 \in [\omega_1]^\omega$ defined by

$$f_0(x) = \begin{cases} 0 & x = 0 \\ x+1 & x \geq 1 \end{cases}, \quad f_1(x) = \begin{cases} x & x = 0,1 \\ \omega + 2(x-1) & x \geq 2 \end{cases}$$

$$f_2(x) = \begin{cases} x & x = 0,1 \\ \omega + (x-2) & x = 2,3 \\ \omega + 2(x-3)+1 & x \geq 4 \end{cases}$$

The first several values of f_0, f_1, and f_2 are the following:

$$f_0 = \langle 0, 2, 3, 4, 5, 6, 7, ... \rangle \quad f_1 = \langle 0, 1, \omega+2, \omega+4, \omega+6, \omega+8, \omega+10, ... \rangle$$
$$f_2 = \langle 0, 1, \omega, \omega+1, \omega+3, \omega+5, \omega+7, \omega+9, \omega+11, ... \rangle.$$

The picture looks as follows: There are $\omega \cdot 2$ many columns. Row 0, 1, and 2 indicate which values among $\omega \cdot 2$ are taken by f_0, f_1, and f_2, respectively.

$$\begin{array}{ccccccc|cccccccccccc}
0 & & 0 & 0 & 0 & 0 & \cdots & & & & & & & & & & & & \\
1 & 1 & & & & & & & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
2 & 2 & & & & & & 2 & 2 & & 2 & & 2 & & 2 & & 2 & & \cdots
\end{array}$$

Then $\mathsf{type}((f_0, f_1, f_2)) = (3, 2, G, D)$ where G and D are defined as follows: $G : 2 \to \mathscr{P}(3)$ is defined by $G(0) = \{0\}$ and $G(1) = \{1, 2\}$. The function $D : \omega \cdot 2 \to \mathscr{P}(3)$ can be read off the diagram above by

$$D(\alpha) = \begin{cases} \{0,1,2\} & \alpha = 0 \\ \{1,2\} & \alpha = 1 \\ \{0\} & 2 \leq \alpha < \omega \\ \{2\} & \alpha = \omega, \omega+1 \\ \{1\} & (\exists k \in \omega)[\alpha = \omega + 2(k+1)] \\ \{2\} & (\exists k \in \omega)[\alpha = \omega + 2(k+1)+1] \end{cases}$$

With Definition 4.2 as the motivation, one makes the following abstract definition of a tuple-type:

Definition 4.4. A tuple-type t is a 4-tuple (n, m, G, D) with the following properties:
(1) $n \in \omega$ and $n > 0$ which is called the length of tuple type.
(2) $1 \leq m \leq n$ which is called the arrangement number of the tuple type.
(3) $G : m \to \mathscr{P}(n)$ with the property that for all $i < m$, $G(i) \neq \emptyset$, $\bigcup_{i \in m} G(i) = n$, and for all $i < j < m$, $G(i) \cap G(j) = \emptyset$. G is called the grouping order of the tuple-type.
(4) $D : \omega \cdot m \to \mathscr{P}(n)$, which is called the distribution of the type, is a function with the following properties:

(a) For each $i < m$ and $l \in \omega$,

$$D(\omega \cdot i + l) \cap \left(\bigcup_{j<i} G(j) \right) = \emptyset.$$

(b) For each $k < n$, if $k \in G(i)$, then $\{l \in \omega : k \in D(\omega \cdot i + l)\}$ is infinite.
(c) For each $k < n$, if $k \in G(i)$, then for each $j < i$, $\{l \in \omega : k \in D(\omega \cdot j + l)\}$ is finite.

Observe that if $\bar{f} \in {}^{<\omega}([\omega_1]^\omega)$, then the tuple-type of $\bar{f}$, $\mathsf{type}(\bar{f})$, is a tuple-type as defined in Definition 4.4.

Definition 4.5. Let $t = (n, m, G, D)$ be a tuple-type. Let $h \in [\omega_1]^{\omega \cdot m}$. For $i < n$, let $f_i^{t,h}$ be defined to be the increasing enumeration of $\{h(\alpha) : \alpha < \omega \cdot m \wedge i \in D(\alpha)\}$. Note that the properties of the distibution imply that $f_i^{t,h} \in [\omega_1]^\omega$.

Define $\mathsf{extract}(t, h) = (f_0^{t,h}, ..., f_{n-1}^{t,h})$. This is the n-tuple extracted from h of tuple-type t. Note that $\mathsf{type}(\mathsf{extract}(t, h)) = t$.

Definition 4.6. Let X be any set and $P : \omega \to X$. P is eventually periodic if and only if there exists $k, p \in \omega$ and $x_0, ..., x_{p-1} \in X$ so that for all $n > k$, $P(n) = x_i$ where $i < p$ is such that $n - k$ is congruent to $i \bmod p$.

A tuple-type $t = (n, m, G, D)$ is an eventually periodic tuple-type if and only if for each $i < m$, the function $P_i : \omega \to \mathscr{P}(n)$ defined by $P_i(k) = D(\omega \cdot i + k)$ is eventually periodic.

Note that there are only countably many eventually periodic tuple-types.

Definition 4.7. Let L be the collection of finite tuples $(\alpha, n, \beta_0, ..., \beta_n)$ where $\alpha < \omega_1$, $n \in \omega$, $\beta_0 < \beta_1 < ... < \beta_n < \alpha$. Let $\prec$ be the lexicographic ordering on L. Let $\mathcal{L} = (L, \prec)$. Note that $\mathrm{ot}(\mathcal{L}) = \omega_1$.

Let $H \in [\omega_1]^{\mathcal{L}}$, that is an order-preserving function of $\mathcal{L}$ into ω_1.

Define $\Lambda^H : [\omega_1]^\omega \to [\omega_1]^\omega$ by $\Lambda(f)(k) = H(\sup(f), k, f(0), ..., f(k))$.

Lemma 4.8. *Λ^H is an injection and $\mathsf{type}(\Lambda^H[[\omega_1]^\omega])$ consists only of eventually periodic tuple-types.*

Proof. Suppose $f, g \in [\omega_1]^\omega$ with $f \neq g$.

(Case I) Suppose $\sup(f) \neq \sup(g)$. Without loss of generality, suppose $\sup(f) < \sup(g)$. Then $\Lambda^H(f)(0) = H(\sup(f), 0, f(0)) < H(\sup(g), 0, g(0)) = \Lambda^H(g)(0)$. Therefore, $\Lambda^H(f) \neq \Lambda^H(g)$.

(Case II) Suppose $\sup(f) = \sup(g)$. $f \neq g$ implies that there is a least k so that $f(k) \neq g(k)$. Without loss of generality, suppose $f(k) < g(k)$. Then $\Lambda^H(f)(k) = H(\sup(f), k, f(0), ..., f(k)) < H(\sup(g), k, g(0), ..., g(k)) = \Lambda^H(g)(k)$. So $\Lambda^H(f) \neq \Lambda^H(g)$.

It has been shown that Λ^H is an injection.

Now suppose $\bar{f} = (f_0, ..., f_{n-1}) \in {}^{<\omega}([\omega_1]^\omega)$. Let $\Lambda^H(\bar{f}) = (\Lambda^H(f_0), ..., \Lambda^H(f_{n-1})$. Let $\mathsf{type}(\bar{f}) = (n, m, G, D)$. Suppose $\mathsf{type}(\Lambda^H(\bar{f})) = (n', m', G', D')$.

For $i < j < n$, if $\sup(f_i) < \sup(f_j)$, then

$$\Lambda^H(f_i)(a) = H(\sup(f_i), a, f_i(0), ..., f_i(a)) < H(\sup(f_j), b, f_j(0), ..., f_j(b)) = \Lambda^H(f_j)(b)$$

for any $a, b \in \omega$. This implies that if $\sup(f_i) < \sup(f_j)$, then $\sup(\Lambda^H(f_i)) < \sup(\Lambda^H(f_j))$. This shows that $m' = m$ and $G' = G$.

Pick any $i < m$. Let $P_i(k) = D'(\omega \cdot i + k)$. Pick a $\ell \in \omega$ large enough so that for all $a, b \in G(i)$, if $f_a \neq f_b$, then there is some $\iota < \ell$ so that $f_a(\iota) \neq f_b(\iota)$.

Define an preordering $\sqsubseteq$ on $G(i)$ by $a \sqsubseteq b$ if and only if $f_a \upharpoonright \ell = f_b \upharpoonright \ell$ or $f_a \upharpoonright \ell$ is lexicographically less than $f_b \upharpoonright \ell$. The $\sqsubseteq$-preordering classes of $G(i)$ are naturally linearly ordered. Note that P_i is eventually periodic by repeating the $\sqsubseteq$-preordering classes of $G(i)$ in this natural order.

It has been established that $\mathsf{type}(\Lambda^H(\bar{f}))$ is an eventually periodic tuple-type. $\square$

Example 4.9. Let f_0, f_1, and f_2 be the functions from Example 4.3. Let $H : \mathcal{L} \to \omega_1$ be any order-preserving function of the correct type. Let $\mathsf{type}((f_0, f_1, f_2)) = (3, 2, G, D)$. Let Λ^H be the associated function as defined above. Let $\mathsf{type}((\Lambda^H(f_0), \Lambda^H(f_1), \Lambda^H(f_2))) = (3, 2, G, D')$, where D' is defined below:

Observe that in $\mathcal{L} = (L, \preceq)$, the following objects are arranged as follows:

$$(\omega, 0, 0) \prec (\omega, 1, 0, 2) \prec (\omega, 2, 0, 2, 3) \prec (\omega, 3, 0, 2, 3, 4) \prec ... \prec (\omega \cdot 2, 0, 0) \prec (\omega \cdot 2, 1, 0, 1)$$

$$\prec (\omega{\cdot}2, 2, 0, 1, \omega) \prec (\omega{\cdot}2, 2, 0, 1, \omega{+}2) \prec (\omega{\cdot}2, 3, 0, 1, \omega, \omega{+}1) \prec (\omega{\cdot}2, 3, 0, 1, \omega{+}2, \omega{+}4) \prec (\omega{\cdot}2, 4, 0, 1, \omega, \omega{+}1, \omega{+}3)$$

$$\prec (\omega \cdot 2, 4, 0, 1, \omega + 2, \omega + 4, \omega + 6) \prec (\omega \cdot 2, 5, 0, 1, \omega, \omega + 1, \omega + 3, \omega + 5) \prec ...$$

This implies that

$$\Lambda^H(f_0)(0) < \Lambda^H(f_0)(1) < \Lambda^H(f_0)(2) < \Lambda^H(f_0)(3) < \Lambda^H(f_0)(4) < ...$$

$$< \Lambda^H(f_1)(0) = \Lambda^H(f_2)(0) < \Lambda^H(f_2)(1) = \Lambda^H(f_1)(1) < \Lambda^H(f_2)(2)$$

$$< \Lambda^H(f_1)(2) < \Lambda^H(f_2)(3) < \Lambda^H(f_1)(3) < \Lambda^H(f_2)(4) < \Lambda^H(f_1)(4) < \Lambda^H(f_2)(5)$$

From the example above, the diagram for D' is given below. In his diagram, $\hat{0}$, $\hat{1}$, and $\hat{2}$ represent $\Lambda^H(f_0)$, $\Lambda^H(f_1)$, and $\Lambda^H(f_2)$:

$$\begin{array}{ccccccc|cccccccccccc}
\hat{0} & \hat{0} & \hat{0} & \hat{0} & \hat{0} & \hat{0} & \cdots & & & & & & & & & & & & \\
 & & & & & & & \hat{1} & \hat{1} & & \hat{1} & & \hat{1} & & \hat{1} & & \hat{1} & & \cdots \\
 & & & & & & & \hat{2} & \hat{2} & \hat{2} & & \hat{2} & & \hat{2} & & \hat{2} & & \hat{2} & \cdots
\end{array}$$

Explicitly, $D' : \omega \cdot 2 \to \mathscr{P}(3)$ is

$$D'(\alpha) = \begin{cases} \{0\} & \alpha < \omega \\ \{1,2\} & \alpha = \omega, \omega+1 \\ \{2\} & (\exists k \in \omega)[\alpha = \omega + 2(k+1)] \\ \{1\} & (\exists k \in \omega)[\alpha = \omega + 2(k+1)+1] \end{cases}$$

Note that $P_0(k) = D'(k)$ is eventually periodic by repeating $\{0\}$ and $P_1(k) = D'(\omega+k)$ is eventually periodic by eventually alternating between $\{1\}$ and $\{2\}$.

Fact 4.10. *Let $\Phi : {}^{<\omega}([\omega_1]^\omega) \to [\omega_1]^\omega$ be a function. Let $t = (n, m, G, D)$ be a tuple-type. Let μ denote the club measure on ω_1. Let $\Phi^{t,k} : [\omega_1]^{\omega \cdot m}_* \to \omega_1$ be defined by $\Phi^{t,k}(h) = \Phi(\mathsf{extract}(t, h))(k)$.*

If for $\mu^{\omega \cdot m}$-almost all h, $\Phi^{t,k}(h) < h(0)$, then for $\mu^{\omega \cdot m}$-almost all h, $\Phi^{t,k}(h)$ takes a constant value $c_k^{\Phi,t}$.

Proof. This follows from the countable additivity of $\mu^{\omega \cdot m}$. □

Definition 4.11. Assume the setting of fact 4.10. Let $d^{\Phi,t}$ be the least k if it exists so that $\Phi^{t,k}(h) \geq h(0)$ for $\mu^{\omega \cdot m}$-almost all h. Otherwise, let $d^{\Phi,t} = \omega$.

Let $\mathsf{stem}^{\Phi,t} : d^{\Phi,t} \to \omega_1$ be defined by $\mathsf{stem}^{\Phi,t}(j) = c_j^{\Phi,t}$, where $j < d^{\Phi,t}$

Thus for $\mu^{\omega \cdot m}$-almost all h, $\mathsf{stem}^{\Phi,t} \subseteq \Phi(\mathsf{extract}(t, h))$ and if $d^{\Phi,t} < \omega$, then $\Phi(\mathsf{extract}(\Phi, t))(d^{\Phi,t}) \geq h(0)$.

Theorem 4.12. *$[\omega_1]^\omega$ is Jónsson.*

Proof. A slightly stronger version of the Jónsson property will be shown: Let $\Phi : {}^{<\omega}([\omega_1]^\omega) \to [\omega_1]^\omega$. A $Z \subseteq [\omega_1]^\omega$ with $|Z| = |[\omega_1]^\omega$ will be found so that $\Phi[{}^{<\omega}Z] \neq [\omega_1]^\omega$. (The Jónsson property merely asks that $\Phi[[Z]^{<\omega}_=] \neq [\omega_1]^\omega$.)

Using $\mathsf{AC}^{\mathbb{R}}_\omega$ and the discussion in Definition 4.11, for each (of the countably many) eventually periodic tuple-type t, let $C_t \subseteq \omega_1$ be a club so that for all $h \in [C_t]^\omega_*$, $\mathsf{stem}^{\Phi,t} \subseteq \Phi(\mathsf{extract}(t, h))$ and if $d^{\Phi,t} < \omega$, then $\Phi(\mathsf{extract}(\Phi, t))(d^{\Phi,t}) \geq h(0)$.

Let ζ be the supremum of $\sup(\mathsf{stem}^{\Phi,t})$ as t ranges over the countable set of eventually periodic tuple-types. As ω_1 is regular, $\zeta < \omega_1$. Let C be the intersection of all C_t as t ranges over all eventually periodic tuple-types. By removing an initial segment of C, one may assume that $\zeta < \min(C) + 1$.

Let $H : \mathcal{L} \to C$ be any order-preserving function of the correct type. Note that $\Lambda^H(f) \in [\omega_1]^\omega_*$, i.e. it is also a function of the correct type for any $f \in [\omega_1]^\omega$.

Let $Z = \Lambda^H[[\omega_1]^\omega]$. Since Λ^H is an injection by Lemma 4.8, $Z \approx [\omega_1]^\omega$.

Now suppose $\bar{f} = (f_0, ..., f_{n-1}) \in {}^{<\omega}Z$. By Lemma 4.8, $t = \mathsf{type}(\bar{f}) = (n, m, G, D)$ is an eventually periodic tuple-type. There is a unique $h \in [C]^{\omega \cdot m}_*$ so that $\mathsf{extract}(t, h) = \bar{f}$. In particular, since $h \in [C_t]^{\omega \cdot m}_*$, $\mathsf{stem}^{\Phi,t} \subseteq \Phi(\bar{f})$ and if $d^{\Phi,t} < \omega$, $\Phi(\bar{f})(d^{\Phi,t}) \geq h(0) \geq \min(C) > \zeta$. This and the definition of ζ imply that $\zeta \notin \mathrm{rang}(\Phi(\bar{f}))$.

It has been shown that for all $\bar{f} \in {}^{<\omega}Z$, $\zeta \notin \mathrm{rang}(\Phi(\bar{f}))$. In particular, $\Phi[{}^{<\omega}Z] \neq [\omega_1]^\omega$.

As Φ was arbitrary, this implies that $[\omega_1]^\omega$ is Jónsson. □

References

1. Andrés Eduardo Caicedo and Richard Ketchersid, *A trichotomy theorem in natural models of* AD^+, Set theory and its applications, Contemp. Math., vol. 533, Amer. Math. Soc., Providence, RI, 2011, pp. 227–258. MR 2777751
2. William Chan, *Ordinal definability and combinatorics of equivalence relations*, J. Math. Log. **19** (2019), no. 2, 1950009, 24. MR 4014889
3. ______, *An introduction to combinatorics of determinacy*, Trends in Set Theory, Contemp. Math., vol. 752, Amer. Math. Soc., Providence, RI, 2020, pp. 21–75. MR 4132099
4. William Chan and Stephen Jackson, *Applications of infinity-borel codes to definability and definable cardinals*, Submitted.
5. William Chan and Stephen Jackson, *Cardinality of wellordered disjoint unions of quotients of smooth equivalence relations*, Ann. Pure Appl. Logic **172** (2021), no. 8, 102988. MR 4256846
6. ______, *Definable combinatorics at the first uncountable cardinal*, Trans. Amer. Math. Soc. **374** (2021), no. 3, 2035–2056. MR 4216731
7. William Chan, Stephen Jackson, and Nam Trang, *The size of the class of countable sequences of ordinals*, Trans. Amer. Math. Soc. **375** (2022), no. 3, 1725–1743. MR 4378077
8. William Chan and Connor Meehan, *Definable Combinatorics of Some Borel Equivalence Relations*, ArXiv e-prints (2017).

9. L. A. Harrington, A. S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), no. 4, 903–928. MR 1057041
10. Leo Harrington, *A powerless proof of a theorem of Silver*, Unpublished.
11. Greg Hjorth, *A dichotomy for the definable universe*, J. Symbolic Logic **60** (1995), no. 4, 1199–1207. MR 1367205
12. S. Jackson, R. Ketchersid, F. Schlutzenberg, and W. H. Woodin, *Determinacy and Jónsson cardinals in* $L(\mathbb{R})$, J. Symb. Log. **79** (2014), no. 4, 1184–1198. MR 3343535
13. Steve Jackson, *A computation of* δ^1_5, Mem. Amer. Math. Soc. **140** (1999), no. 670, viii+94. MR 1606035
14. ———, *Structural consequences of* AD, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 1753–1876. MR 2768700
15. Alexander S. Kechris, AD *and projective ordinals*, Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77), Lecture Notes in Math., vol. 689, Springer, Berlin, 1978, pp. 91–132. MR 526915
16. Eugene M. Kleinberg, *Infinitary combinatorics and the axiom of determinateness*, Lecture Notes in Mathematics, Vol. 612, Springer-Verlag, Berlin-New York, 1977. MR 0479903
17. Jack H. Silver, *Counting the number of equivalence classes of Borel and coanalytic equivalence relations*, Ann. Math. Logic **18** (1980), no. 1, 1–28. MR 568914
18. W. Hugh Woodin, *The cardinals below* $|[\omega_1]^{<\omega_1}|$, Ann. Pure Appl. Logic **140** (2006), no. 1-3, 161–232. MR 2224057

Department of Mathematics, Carnegie Mellon University, Pittsburgh, PA 15213
Email address: wchan3@andrew.cmu.edu

Department of Mathematics, University of North Texas, Denton, TX 76203
Email address: Stephen.Jackson@unt.edu

Department of Mathematics, University of North Texas, Denton, TX 76203
Email address: Nam.Trang@unt.edu