APPLICATIONS OF INFINITY-BOREL CODES TO DEFINABILITY AND DEFINABLE CARDINALS

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Abstract. Woodin introduced an extension of the axiom of determinacy, AD, called AD^+ which includes an assertion that all sets of reals have an ∞-Borel-code. An ∞-Borel code is a pair (φ, S) where φ is a formula and S is a set of ordinals which provides a highly absolute definition for a set of reals. This paper will use AD^+ and ∞-Borel codes to establish a property of ordinal definability analogous to a property for Σ^1_1 shown by Harrington-Shore-Slaman in [9]. Under AD^+, the paper will also use ∞-Borel codes to explore the cardinality of sets below $\mathscr{P}(\omega_1)$ which Woodin began investigating in [20] under $\mathsf{AD}_{\mathbb{R}}$ and DC. The following summarizes the main results.

Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. If $H \subseteq \mathbb{R}$ has the property that there is a nonempty OD set of reals K so that H is OD_z for any $z \in K$, then H is OD.

Assume $\mathsf{ZF} + \mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}} + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Then there is a cardinal strictly between $|[\omega_1]^{<\omega_1}|$ and $|[\omega_1]^{\omega_1}| = |\mathscr{P}(\omega_1)|$.

Assume $\mathsf{ZF} + \mathsf{AD}^+$. $S_1 = \{f \in [\omega_1]^{<\omega_1} : \sup(f) = \omega_1^{L[f]}\}$ does not inject into ${}^{\omega}\mathrm{ON}$, the class of ω-sequences of ordinals. This implies $|\mathbb{R}| < |S_1|$ and $|[\omega_1]^{\omega}| < |[\omega_1]^{<\omega_1}|$.

Assuming $\mathsf{ZF} + \mathsf{AD}^+$. Let X be a surjective image of $\mathbb{R}$ and let $\mathscr{P}_{\omega_1}(X) = \{A \subseteq X : |A| < \omega_1\}$. If $\omega_1 \leq |\mathscr{P}_{\omega_1}(X)|$, then $\omega_1 \leq |X|$. If $|\mathscr{P}(\omega_1)| = |[\omega_1]^{\omega_1}| \leq |\mathscr{P}_{\omega_1}(X)|$, then $|\mathbb{R} \sqcup \omega_1| \leq |X|$.

$\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ implies that the uncountable cardinals below $|\mathbb{R} \times \omega_1|$ are ω_1, $|\mathbb{R}|$, $|\mathbb{R} \sqcup \omega_1|$, and $|\mathbb{R} \times \omega_1|$. An elaborate structure of cardinals below $|\mathbb{R} \times \omega_1|$ will be described under the assumption of $\mathsf{ZF} + \mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}} + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$.

1. Introduction

An ∞-Borel code is simply a pair (S, φ) where S is a set of ordinals and φ is a formula of set theory. The set of reals defined by (S, φ) is $\mathfrak{B}^1_{(S,\varphi)} = \{x \in \mathbb{R} : L[S, x] \models \varphi(S, x)\}$. If A is a set of reals, then one says that (S, φ) is an ∞-Borel code for A if and only if $\mathfrak{B}^1_{(S,\varphi)} = A$. An ∞-Borel code for A is a highly absolute definition for A in the sense that to determine membership of $x \in A$, one simply needs to go into $L[S, x]$, which is the minimal model of ZFC containing the code S and x, and ask whether $L[S, x] \models \varphi(S, x)$. Note that for any inner model $M \models \mathsf{ZF}$ with $S \in M$, $(\mathfrak{B}^1_{(S,\varphi)})^M = \mathfrak{B}^1_{(S,\varphi)} \cap M$.

The axiom of determinacy, AD, states that certain two player games have a winning strategy for one of the two players. Mathematics under AD is often regarded as being more effective, uniform, or definable. Woodin [21] isolated an extension of AD called AD^+ which includes $\mathsf{DC}_{\mathbb{R}}$, a technical statement called ordinal determinacy, and the statement that all sets of reals have an ∞-Borel code. The existence of ∞-Borel codes strengthens the claim that AD^+ captures definable combinatorics.

It is not known if AD can prove any of the three statements in AD^+. Kechris [12] and Woodin showed that if $L(\mathbb{R}) \models \mathsf{AD}$, then $L(\mathbb{R}) \models \mathsf{AD}^+$. Moreover, Woodin showed that in natural models of AD^+, i.e. those models which satisfy $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$, more is known about the structure of ∞-Borel codes. In particular, in models of the form $L(J, \mathbb{R}) \models \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ where J is a set of ordinals, Woodin's result that $L(J, \mathbb{R})$ is a symmetric collapse extension of $\mathrm{HOD}_J^{L(J,\mathbb{R})}$ outlines a procedure to obtain ∞-Borel codes from definitions witnessing ordinal definability.

Under AD^+, the Vopěnka forcing of nonempty OD subsets of $\mathbb{R}$ ordered by $\subseteq$ becomes a very powerful tool. In the presence of strongly absolute definitions provided by the ∞-Borel codes, the method of the Vopěnka forcing in local models of the form $\mathrm{HOD}_S^{L[S,X]}$, where S is a fixed set of ordinals and X varies over

January 21, 2024. The first author was supported by NSF grant DMS-1703708. The second author was supported by NSF grant DMS-1800323.

the Turing degrees, combined with the ultraproduct $\prod_{X\in\mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu$ where μ is the Martin measure on Turing degrees is especially useful for combinatorics under AD^+.

For instance, similar techniques were used by Woodin to prove the perfect set dichotomy (see [2]) which generalized Silver's $\mathbf{\Pi}^1_1$ equivalence relation dichotomy ([18]) and by Hjorth [10] to prove the more general E_0-dichotomy which generalizes the E_0-dichotomy of Harrington-Kechris-Louveau [8]. It is also used in Woodin's result that countable section uniformization for relations on $\mathbb{R}\times\mathbb{R}$ holds under AD^+ (see [15] or [2]). Such techniques are also used in [3] to answer a question of Foreman that there are Suslin lines in $L(\mathbb{R})\models\mathsf{AD}$. In [4], the ∞-Borel codes, Vopěnka forcing, and the ultraproduct is used to show that if $\langle E_\alpha : \alpha<\omega_1\rangle$ is a sequence of equivalence relations on $\mathbb{R}$ with all classes countable such that $|\mathbb{R}/E_\alpha| = |\mathbb{R}|$, then the disjoint union $\bigsqcup_{\alpha<\omega_1}\mathbb{R}/E_\alpha$ is in bijection with $\mathbb{R}\times\omega_1$.

This article provides some new applications of ∞-Borel codes and the Vopěnka forcing to questions about ordinal definability and definable cardinals assuming AD^+ or specifically in natural models of AD^+.

Harrington, Shore, and Slaman [9] showed that if $H\subseteq\mathbb{R}$ has the property that there is a nonempty Σ^1_1 $K\subseteq\mathbb{R}$ so that H is $\Sigma^1_1(z)$ for any $z\in K$, then H is Σ^1_1. In other words, if H is Σ^1_1 in any parameter z from a nonempty Σ^1_1 set K, then H is actually Σ^1_1 with no parameters.

One can ask if a similar phenomenon holds for other notions of lightface definability. Ordinal definability is a strong notion of definability which is closed under nearly any operation which does not introduce non-ordinal parameters. One can ask if $H\subseteq\mathbb{R}$ is OD_z in any parameter z from a nonempty OD set of reals K, then is H ordinal definable with no parameters.

The answer is generally not positive under ZF since Fact 3.2 shows that in the Sacks generic extension of the constructible universe L, the Sacks generic real is OD_z from any nonconstructible z but the Sacks generic real is not OD. However, in natural models of AD^+ is answer is positive:

Theorem 3.1 *Assume $\mathsf{ZF}+\mathsf{AD}^++\mathsf{V}=\mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let J be a set of ordinals. Let $H\subseteq\mathbb{R}$. Let $K\subseteq\mathbb{R}$ be nonempty and OD_J. If H is $\mathrm{OD}_{J,z}$ for all $z\in K$, then H is OD_J.*

Using the arguments of Woodin in the proof that $L(J,\mathbb{R})\models\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_\mathbb{R}$ is a symmetric collapse extension of $\mathrm{HOD}_J^{L(J,\mathbb{R})}$, one can show that in $L(J,\mathbb{R})$, there is a set of ordinals $\mathbb{X}$ which "absorbs" functions of various types. As an example, this means that for any function $\Phi:[\omega_1]^{\omega_1}\to[\omega_1]^{<\omega_1}$ (or $\Phi:\mathbb{R}\times\omega_1\to\mathbb{R}\times\omega_1$), there is a real e so that for all z with $e\leq_\mathbb{X} z$, and $f\in[\omega_1]^{\omega_1}\cap L[\mathbb{X},z]$, $\Phi(f)\in L[\mathbb{X},z]$ and $\Phi\cap L[\mathbb{X},z]\in L[\mathbb{X},z]$. This function absorption idea is especially useful for studying definable cardinality under AD^+ and for producing intermediate cardinalities in natural models of AD^+.

[5] shows that $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}| = |\mathscr{P}(\omega_1)|$ by establishing an almost everywhere continuity phenomenon for functions of the form $\Phi:[\omega_1]^{\omega_1}\to\omega_1$. Section 4 gives a more set theoretic argument as well as other conditions on cardinals κ which implies that $|[\kappa]^{<\kappa}|<|[\kappa]^\kappa|$. This section also shows that in models of the form $L(J,\mathbb{R})$, where J is a set of ordinals, there is a cardinal intermediate between $|[\omega_1]^{<\omega_1}|$ and $|[\omega_1]^{\omega_1}|$:

Theorem 4.10 *Assume $\mathsf{ZF}+\mathsf{AD}^+$. Let $J\subseteq\mathrm{ON}$ be a set of ordinals so that $V=L(J,\mathbb{R})$. Let $\mathbb{X}=(J,{}_\omega\mathbb{O}_J)$ (see Section 2 for more details). Define N_1^J by*

$$N_1^J=\bigsqcup_{r\in\mathbb{R}}((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]}=\{(r,\alpha):\alpha<((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]}\}.$$

One has the following cardinal relations: $\neg(|N_1^J|\leq[\omega_1]^{<\omega_1})$, $|\mathbb{R}\times\omega_1|<|N_1^J|<|\mathbb{R}\times\omega_2|$, $|N_1^J|<|[\omega_1]^{\omega_1}|$, $\neg(|[\omega_1]^\omega|\leq|N_1^J|)$, and $|[\omega_1]^{<\omega_1}|<|[\omega_1]^{<\omega_1}\sqcup N_1^J|<|[\omega_1]^{\omega_1}|$.

Intuitively, $[\omega_1]^\omega$ and $[\omega_1]^{<\omega_1}$ appear to be distinct subsets of $\mathscr{P}(\omega_1)$ in terms of cardinality. It is implicit in [20] that under $\mathsf{ZF}+\mathsf{AD}_\mathbb{R}+\mathsf{DC}$, $|[\omega_1]^\omega|<|[\omega_1]^{<\omega_1}|$. It appears that these cardinal distinctions are obtained through an analysis of the set $S_1=\{f\in[\omega_1]^{<\omega_1}:\sup(f)=\omega_1^{L[f]}\}$, defined by Woodin. Section 5 will study S_1 using ∞-Borel codes and the function absorption idea under AD^+.

In just AD, one can show that $|\mathbb{R}|\leq|S_1|$ and $\neg(\omega_1\leq|S_1|)$. However, all other interesting properties of S_1 appear to be only known under the existence of ∞-Borel codes. The main property of S_1 is that it does not inject into the class of ω-sequences of ordinals.

Theorem 5.7 *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and all sets of reals have ∞-Borel codes. Then there is no injection of S_1 into ${}^{\omega}\mathrm{ON}$, the class of ω-sequences of ordinals.*

This result can then be used to give the following cardinality computation under AD^+:

Theorem 5.8 *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and all sets of reals have ∞-Borel codes. Then $|\mathbb{R}| < |S_1|$ and $|[\omega_1]^{\omega}| < |[\omega_1]^{<\omega_1}|$.*

The proof of Theorem 5.7 involves finding a filter which is generic over a model of ZFC for a forcing in this model which is countable in the real world satisfying AD. If one would like to imitate this argument to establish similar results on ω_2, then the naturally associated forcing in a model of ZFC would be uncountable in even the real world and hence one may not have generics for this forcing. Thus the AD^+ methods in Theorem 5.7 are not suitable for generalization to ω_2.

S_1 by its definition involves notions of constructibility which makes ∞-Borel definition quite useful for studying properties of its cardinality. However $[\omega_1]^{\omega}$ and $[\omega_1]^{<\omega_1}$ are purely combinatorial objects whose cardinal distinctions should be obtainable under AD alone. By establishing an almost everywhere continuity result for functions of the form $\Phi : [\omega_1]^{\epsilon} \to \omega_1$, where $\epsilon < \omega_1$, [7] shows in just AD that $|[\omega_1]^{\omega}| < |[\omega_1]^{<\omega_1}|$. This argument provides the suitable template for studying combinatorics on ω_2. By establishing an almost everywhere continuity result for functions of the form $\Phi : [\omega_2]^{\epsilon} \to \omega_2$, where $\epsilon < \omega_2$, [7] shows in AD that $|[\omega_2]^{\omega}| < |[\omega_2]^{<\omega_1}| < |[\omega_2]^{\omega_1}| < |[\omega_2]^{<\omega_2}|$. More recently, [6] established these almost everywhere continuity properties purely from combinatorial principles. Thus [6] showed that if κ is a weak partition cardinal (satisfies $\kappa \to (\kappa)^{<\kappa}_2$), then for all $\chi < \kappa$, $[\kappa]^{<\kappa}$ does not inject into ${}^{\chi}\mathrm{ON}$, the class of χ-length sequences of ordinals, and hence $|[\kappa]^{\chi}| < |[\kappa]^{<\kappa}|$. Hence these cardinality results apply to all the familiar weak and strong partition cardinals of determinacy such as $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ and $\boldsymbol{\delta}^2_1$.

Using the properties of S_1, one can answer an interesting question of Zapletal: If X is a set, let $\mathscr{P}_{\omega_1}(X) = \{A \subseteq X : |A| < \omega_1\}$ and let $\mathscr{P}_{\mathrm{WO}}(X)$ be the collection of $A \subseteq X$ which are wellorderable. Zapletal asked that if $\mathscr{P}_{\omega_1}(X)$ has certain cardinality properties, then what can be said about the cardinality properties of X. A concrete question would be if ω_1 injects into $\mathscr{P}_{\omega_1}(X)$, then does ω_1 already inject into X? The following gives a positive answer:

Theorem 6.6 *Assuming* $\mathsf{ZF} + \mathsf{AD}^+$, *for all cardinals $\kappa < \Theta$ and all sets X which are surjective images of $\mathbb{R}$, $\kappa \leq |\mathscr{P}_{\mathrm{WO}}(X)|$ implies $\kappa \leq |X|$. In particular, $\kappa \leq |\mathscr{P}_{\omega_1}(X)|$ implies $\kappa \leq |X|$.*

Corollary 6.7 *Assume* $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \mathsf{AD}$ *and all sets of reals have ∞-Borel codes. Let X be a set which is a surjective image of $\mathbb{R}$. Then $\omega_1 \leq |\mathscr{P}_{\mathrm{WO}}(X)|$ implies $\omega_1 \leq |X|$. In particular, $\omega_1 \leq |\mathscr{P}_{\omega_1}(X)|$ implies $\omega_1 \leq |X|$.*

One can ask what other sets Y has the property that if Y injects into $\mathscr{P}_{\omega_1}(X)$, then X already has a copy of Y. Note that $\mathscr{P}_{\omega_1}(\omega_1) = [\omega_1]^{<\omega_1}$. Thus for any uncountable $Y \subseteq [\omega_1]^{<\omega_1}$ such that $|Y| \neq \omega_1$, Y injects into $\mathscr{P}_{\omega_1}(\omega_1)$, but Y does not inject into ω_1. This reflection property fails for any $Y \subseteq [\omega_1]^{<\omega_1}$ such that $|Y| \neq \omega_1$. Naturally, one can ask if $[\omega_1]^{\omega_1}$ injects into $\mathscr{P}_{\omega_1}(X)$, then what can be said about the cardinality of X. The following results shows that X must contain a copy of ω_1 and $\mathbb{R}$:

Theorem 6.10 *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and all sets of reals have an ∞-Borel code. Let X be a set which is a surjective image of $\mathbb{R}$. If $|[\omega_1]^{\omega_1}| \leq |\mathscr{P}_{\omega_1}(X)|$, then $|\mathbb{R} \sqcup \omega_1| \leq |X|$.*

A natural conjecture would be that if $[\omega_1]^{\omega_1}$ injects into $\mathscr{P}_{\omega_1}(X)$, then $[\omega_1]^{\omega_1}$ already injects into X. However, an easier question may be if $[\omega_1]^{\omega_1}$ injects into $\mathscr{P}_{\omega_1}(X)$, then does $\mathbb{R} \times \omega_1$ inject into X?

Woodin [20] showed using elaborate AD^+ techniques that under $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$, the uncountable cardinals below $[\omega_1]^{\omega}$ are ω_1, $|\mathbb{R}|$, $|\mathbb{R} \sqcup \omega_1|$, $|\mathbb{R} \times \omega_1|$, and $[\omega_1]^{\omega}$. Using a simple uniformization argument, Corollary 7.6

shows that under $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$, the uncountable cardinals below $|\mathbb{R} \times \omega_1|$ are ω_1, $|\mathbb{R}|$, $|\mathbb{R} \sqcup \omega_1|$, and $|\mathbb{R} \times \omega_1|$. Woodin showed that if $\mathsf{AD}_{\mathbb{R}}$ fails, then there may be other cardinalities below $|\mathbb{R} \times \omega_1|$.

The final section studies the uncountable cardinalities below $|\mathbb{R} \times \omega_1|$ in natural models of $\mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}}$ such as $L(J, \mathbb{R})$ where J is a set of ordinals which "absorbs" all functions from $\mathbb{R} \times \omega_1$ into $\mathbb{R} \times \omega_1$. Let $\mathfrak{V}$ denote all the cardinals $\mathcal{X}$ below $|\mathbb{R} \times \omega_1|$ such that $\neg(\omega_1 \leq \mathcal{X})$. Fact 7.4 shows that every cardinal $\mathcal{Z} \leq |\mathbb{R} \times \omega_1|$ is either in $\mathfrak{V}$ or is the disjoint union of ω_1 with some cardinality in $\mathfrak{V}$. Thus a complete understanding of $\mathfrak{V}$ would elucidate the structure of the cardinals below $|\mathbb{R} \times \omega_1|$.

Let $\mathcal{D}_J$ and μ_J denote the J-constructible degrees and the Martin measures on J-degrees, respectively. For any $F : \mathbb{R} \to \omega_1$ which is J-invariant, let $W^J_F = \bigsqcup_{r \in \mathbb{R}} \omega^{L[J,r]}_{F(r)}$. For any $\mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J$, there exists an everywhere increasing J-invariant $F : \mathbb{R} \to \omega_1$ which represents $\mathcal{F}$. Let $Y^J_{\mathcal{F}} = |W^J_F|$ for any everywhere increasing J-invariant $F : \mathbb{R} \to \omega_1$ which represents $\mathcal{F}$. (It can be shown that $Y^J_{\mathcal{F}}$ is independent of the choice of F.)

Woodin showed that $\prod_{X \in \mathcal{D}_J} \omega_1^{L[J,X]}/\mu_J = \omega_1$ for any set of ordinals J and $\prod_{X \in \mathcal{D}_J} \omega_2^{L[J,X]} = \Theta$ if J is an "ultimate ∞-Borel code" in $V = L(J, \mathbb{R})$. For $\alpha < \omega_1$, let $F^\alpha : \mathbb{R} \to \omega_1$ be the constant function taking value α. It can be shown that F^α represents the ordinal α in $\prod_{\mathcal{D}_J} \omega_1/\mu_J$. Thus $Y^J_\alpha = |W^J_{F^\alpha}|$ for each $\alpha < \omega_1$.

Let $\mathfrak{Y} = \{Y^J_{\mathcal{F}} : \mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J\}$. $\mathfrak{Y} \subseteq \mathfrak{V}$. It can be shown that $Y^J_0 = Y^J_1 = |\mathbb{R}|$. If $\mathcal{F}_1 < \mathcal{F}_2$ in the ultrapower ordering, then $Y^J_{\mathcal{F}_1} < Y^J_{\mathcal{F}_2}$. Also for any $\mathcal{Y} \in \mathfrak{V}$, there is some $\mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J$ so that $\mathcal{Y} \leq Y^J_{\mathcal{F}}$. By analyzing the behavior of $\mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J$ which are succesor ordinals and limit ordinals of cofinality ω, one can show that $\langle Y^J_\alpha : \alpha < \omega_1\rangle$ is the ω_1-length initial segment of $\mathfrak{V}$. The following summarizes the results of Section 7.

Theorem: Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ and $V = L(J, \mathbb{R})$ where J is a set of ordinals which absorbs function from $\mathbb{R} \times \omega_1$ to $\mathbb{R} \times \omega_1$.

$\langle Y^J_{\mathcal{F}} : \mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J \setminus \{0\}\rangle$ is an order preserving injection of the ultraproduct ordering into $\mathfrak{Y}$ with the injection ordering.

$\mathfrak{Y}$ is cofinal in $\mathfrak{V}$: For all $\mathcal{X} \in \mathfrak{V}$, there is an $\mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J \setminus \{0\}$ so that $\mathcal{X} \leq Y^J_{\mathcal{F}}$.

For any $\mathcal{X} \in \mathfrak{V}$ and $F \in \prod_{\mathcal{D}_J} \omega_1/\mu_J \setminus \{0\}$, either $\mathcal{X} \leq Y^J_{\mathcal{F}}$ or $Y^J_{\mathcal{F}} \leq \mathcal{X}$.

$\{Y^J_\alpha : \alpha < \omega_1\}$ is the ω_1-length initial segment of $\mathfrak{V}$: for any cardinality $\mathcal{X}$ below $|\mathbb{R} \times \omega_1|$ so that $\neg(\omega_1 \leq \mathcal{X})$, either there exists an $\alpha < \omega_1$ so that $\mathcal{X} = Y^J_\alpha$ or for all $\alpha < \omega_1$, $Y^J_\alpha \leq \mathcal{X}$.

A very appealing conjecture given these results is that $\mathfrak{V} = \mathfrak{Y}$. Let $F^{\omega_1} : \mathbb{R} \to \omega_1$ be defined by $F^{\omega_1}(x) = \omega_1^{L[J,x]}$. It can be shown that F^{ω_1} represents ω_1 in $\prod_{\mathcal{D}_J} \omega_1/\mu_J$. Is $Y^J_{\omega_1} = |W^J_{F^{\omega_1}}|$ the $\omega_1{}^{\text{th}}$ cardinality in $\mathfrak{V}$ in the sense that for all $\mathcal{X} \in \mathfrak{V}$ such that $\mathcal{X} \leq Y^J_{\omega_1}$, there is an $\alpha \leq \omega_1$ so that $\mathcal{X} = Y^J_\alpha$? The difficulty is that the behavior of cardinalities below $Y^J_{\mathcal{F}}$ when $\mathcal{F}$ has uncountable cofinality is not well understood.

2. Basics

This section summarizes some properties about ∞-Borel codes, Vopénka forcing, and the Martin measure that will be needed throughout the paper. The reader can refer to [2] for a detailed exposition of these ideas at least in the $L(\mathbb{R}) \models \mathsf{AD}$ setting.

Definition 2.1. Let $S \subseteq \mathrm{ON}$ be a set of ordinals and φ be a formula of set theory. The pair (S, φ) is called an ∞-Borel code. For any $n \in \omega$, define $\mathfrak{B}^n_{(S,\varphi)} = \{x \in \mathbb{R}^n : L[S, x] \models \varphi(S, x)\}$.

If $A \subseteq \mathbb{R}^n$, then (S, φ) is an ∞-Borel code for A if and only if $\mathfrak{B}^n_{(S,\varphi)} = A$.

A set $A \subseteq \mathbb{R}^n$ is said to be ∞-Borel if and only if it has an ∞-Borel code.

Note that an ∞-Borel Borel set of reals has a very absolute definition in the following sense: If $A \subseteq \mathbb{R}$ is ∞-Borel with ∞-Borel code (S, φ), then $x \in A$ is determined completely by whether $\varphi(S, x)$ holds in the minimal model of ZFC, $L[S, x]$, containing the code (S, φ) and the real x.

Definition 2.2. Let $n > 0$ and $S \subseteq \mathrm{ON}$ be a set of ordinals. Let ${}_n\mathbb{O}_S$ denote the forcing of nonempty OD_S subsets of $\mathbb{R}^n$ ordered by $\subseteq$ with largest element $1_{{}_n\mathbb{O}_S} = \mathbb{R}^n$. One will write $\mathbb{O}_S$ for ${}_1\mathbb{O}_S$.

Since there is an S-definable bijection of OD_S with ON, one can transfer ${}_n\mathbb{O}_S$ onto the ordinals. In this way, ${}_n\mathbb{O}_S$ is a forcing in HOD_S.

Definition 2.3. Let S be a set of ordinals. For each $k \in \omega$, let $b_k = \{x \in \mathbb{R} : x(k) = 1\}$. Note that $b_k \in \mathbb{O}_S$. Let $\dot{x}_{\mathrm{gen}} = \{(\check{k}, b_k) : k \in \omega\}$. Note that $\dot{x}_{\mathrm{gen}}$ is an $\mathbb{O}_S$-name which adds a real.

One can formulate the analogous ${}_n\mathbb{O}_S$-name $\dot{x}^n_{\mathrm{gen}}$ for adding an element of $\mathbb{R}^n$ for all $n \geq 1$.

Fact 2.4. *Let S be a set of ordinals. For each $x \in \mathbb{R}^n$, $G^n_x = \{p \in {}_n\mathbb{O}_S : x \in p\}$ is a HOD_S-generic filter so that $\dot{x}^n_{\mathrm{gen}}[G^n_x] = x$ and $\mathrm{HOD}_S[G^n_x] = \mathrm{HOD}_S[x]$.*

Fact 2.5. *([10] Theorem 2.4, [2] Fact 8.1) Let M be a transitive inner model of ZF. Let $S \in M$ be a set of ordinals. Suppose $K \in \mathrm{HOD}^M_S$ is a set of ordinals and φ is a formula. Let N be a transitive inner model with $\mathrm{HOD}^M_S \subseteq N$. Let $p = \{x \in \mathbb{R} : L[K, x] \models \varphi(K, x)\}$, so p is a condition of $\mathbb{O}^M_S$. Then $N \models p \Vdash_{\mathbb{O}^M_S} L[\check{K}, \dot{x}_{\mathrm{gen}}] \models \varphi(\check{K}, \dot{x}_{\mathrm{gen}})$.*

Definition 2.6. (Woodin, [21] Section 9.1) AD^+ consists of the following:
(1) $\mathsf{DC}_{\mathbb{R}}$.
(2) Every $A \subseteq \mathbb{R}$ is ∞-Borel.
(3) For all $\lambda < \Theta$, $A \subseteq \mathbb{R}$, and continuous $\pi : {}^\omega\lambda \to \mathbb{R}$, $\pi^{-1}[A]$ is determined.

Models satisfying $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$ are called natural models of AD^+. Woodin showed that these either are models of $\mathsf{AD}_{\mathbb{R}}$ or take the form $V = L(J, \mathbb{R})$ for a set of ordinals J:

Fact 2.7. *(Woodin, [1] Corollary 3.2) Assume $\mathsf{ZF} + \mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}} + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Then there is a set of ordinals J so that $V = L(J, \mathbb{R})$.*

Many results about $L(\mathbb{R})$ proved by Vopénka forcing can be relativized to an analogous statement about models of the form $L(J, \mathbb{R})$.

Fact 2.8. *(Woodin, [1] Theorem 3.4) Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let J be a set of ordinals and $A \subseteq \mathbb{R}$. If A is OD_J, then A has an OD_J ∞-Borel code.*

Fact 2.9. *(Woodin, [1] Theorem 2.8) Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let J be a set of ordinals. There is some set of ordinals $\mathbb{X}$ so that $\mathrm{HOD}_J = L[\mathbb{X}]$.*

Proof. See [2] Corollary 7.21 for a proof of this under $\mathsf{AD} + V = L(\mathbb{R})$. □

Woodin's works showing that $L(J, \mathbb{R}) \models \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ is a symmetric collapse extension of $\mathrm{HOD}^{L(J,\mathbb{R})}_J$ gives additional information about ∞-Borel codes in such models. In particular, it shows the existence of an ultimate ∞-Borel code mentioned above which will be particularly useful in this article for "absorbing fragments of functions" in relevant models of ZFC.

Assume $V = L(J, \mathbb{R}) \models \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. For each $m \leq n < \omega$, let $\pi_{n,m} : \mathbb{R}^n \to \mathbb{R}^m$ be the projection of $\mathbb{R}^n$ onto $\mathbb{R}^m$. One can induce map $\pi_{n,m} : {}_n\mathbb{O}_J \to {}_m\mathbb{O}_J$ by $\pi_{n,m}(p) = \pi_{n,m}[p]$, where the latter $\pi_{n,m} : \mathbb{R}^n \to \mathbb{R}^m$ is the projection map. These maps $\pi_{n,m} : {}_n\mathbb{O}_J \to {}_m\mathbb{O}_J$ are forcing projections. Let ${}_\omega\mathbb{O}_J$ denote the finite support direct limit induced by $\langle {}_n\mathbb{O}_J, \pi_{n,m} : 1 \leq m \leq n < \omega\rangle$. Let $\pi_{\omega,n} : {}_\omega\mathbb{O}_J \to {}_n\mathbb{O}_J$ be the natural associated projection map.

Each $s \in \mathbb{R}^n$ induces canonically an ${}_n\mathbb{O}_J$-generic filter over $\mathrm{HOD}^{L(J,\mathbb{R})}_J$ denoted by G^n_s. Using $\pi_{\omega,n}$, let ${}_\omega\mathbb{O}_J / G^n_s$ refer to the associated remainder forcing. Moreover, every $G \subseteq {}_n\mathbb{O}_J$ which is ${}_n\mathbb{O}_J$-generic over HOD_J adds a generic element of $\mathbb{R}^n$. For each n, let τ_n be the ${}_\omega\mathbb{O}_J$-name for the real in the last coordinate of the generic n-tuple of reals added by the ${}_n\mathbb{O}_J$-generic filter induced from an ${}_\omega\mathbb{O}_J$-generic filter. Let $\dot{\mathbb{R}}_{\mathrm{sym}}$ be the ${}_\omega\mathbb{O}_J$-name for the set $\{\tau_n : n \in \omega\}$. Let $\dot{x}^n_{\mathrm{gen}}$ be a name denoting $\langle \tau_i : i < n\rangle$.

Fact 2.10. *(Woodin) Suppose $L(J, \mathbb{R}) \models \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. Let $s \in \mathbb{R}^n$, $z \in L[J, {}_\omega\mathbb{O}_J, s]$, and φ is a formula. Then $L(J, \mathbb{R}) \models \varphi(J, s, z)$ if and only if*

$$L[J, {}_\omega\mathbb{O}_J, s] \models 1_{{}_\omega\mathbb{O}_J / G^n_s} \Vdash_{{}_\omega\mathbb{O}_J / G^n_s} L(\check{J}, \dot{\mathbb{R}}_{\mathrm{sym}}) \models \varphi(\check{J}, \dot{x}^n_{\mathrm{gen}}, \check{z}).$$

Fact 2.10 can be used to show that in $L(J, \mathbb{R}) \models \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$, for any $A \subseteq \mathbb{R}$, there is an $r \in \mathbb{R}$ and a formula φ so that $(J \oplus {}_\omega\mathbb{O}_J \oplus r, \varphi)$ forms an $\mathrm{OD}_{J,s}$ ∞-Borel code for A, where $J \oplus {}_\omega\mathbb{O}_J \oplus r$ is a set of ordinals that codes these three objects in some fixed way. It also gives following result.

Fact 2.11. *(Woodin) Assume* $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ *and there is a set* $J \subseteq \mathrm{ON}$ *so that* $V = L(J,\mathbb{R})$*. For each* $x \in \mathbb{R}$*,* $\mathrm{HOD}_{J,x}^{L(J,\mathbb{R})} = L[J, {}_{\omega}\mathbb{O}_J, x]$*.*

A more detailed exposition of these above results can be found in [2] in the $L(\mathbb{R})$ case. It should be noted that here these results are stated for the Vopěnka forcing $\mathbb{O}$. These results are initially proved using $\mathbb{A}$ which is the forcing of nonempty sets of reals with OD ∞-Borel codes. It is then shown that $\mathbb{O}$ and $\mathbb{A}$ are the same.

Definition 2.12. Let $x \leq_{\mathsf{Turing}} y$ indicate that x is Turing reducible to y. Let $x \equiv_{\mathsf{Turing}} y$ indicate $x \leq_{\mathsf{Turing}} y$ and $y \leq_{\mathsf{Turing}} y$. Let $\mathcal{D} = \mathbb{R}/\equiv_{\mathsf{Turing}}$ denote the collection of Turing degrees. For $X, Y \in \mathcal{D}$, let $X \leq Y$ indicate that there is some $x \in X$ and $y \in Y$ so that $x \leq_{\mathrm{Turing}} y$. If $X \in \mathcal{D}$, then the Turing cone above X is the set $\{Y \in \mathcal{D} : X \leq Y\}$. The Martin's measure μ on $\mathcal{D}$ is the collection of subsets of $\mathcal{D}$ which contain a Turing cone.

If $J \subseteq \mathrm{ON}$ is a set of ordinals. On $\mathbb{R}$, define $x \leq_J y$ if and only if $x \in L[J,y]$. Let $x \equiv_J y$ if and only if $x \leq_J y$ and $y \leq_J x$. Let $\mathcal{D}_J = \mathbb{R}/\equiv_J$ denote the collection of J-constructibility degrees. If $X, Y \in \mathcal{D}_J$, then let $X \leq Y$ indicates that there exist $x \in X$ and $y \in Y$ so that $x \leq_J y$. If $X \in \mathcal{D}_J$, then the J-cone above X is the set $\{Y \in \mathcal{D}_J : X \leq Y\}$. Let μ_J be collection of subsets of $\mathcal{D}_J$ which contain a J-cone.

Fact 2.13. *(Martin) Assume* $\mathsf{ZF}+\mathsf{AD}$*.* μ *is a countably complete ultrafilter. For any* $J \subseteq \mathrm{ON}$*,* μ_J *is a countably complete ultrafilter.*

Fact 2.14. *(Woodin, [1] Section 2.2) Assume* $\mathsf{ZF}+\mathsf{AD}^+$*.* $\prod_{X\in\mathcal{D}} \mathrm{ON}/\mu$*, the ultrapower of the ordinals by Martin's Turing cone measure, is a wellordering.* $\prod_{X\in\mathcal{D}_J} \mathrm{ON}/\mu_J$*, the ultrapower of the ordinals by the* J*-constructibility cone measure, is a wellordering.*

Corollary 2.15. *Assume* $\mathsf{ZF}+\mathsf{AD}^+$*. Let* $S \subseteq \mathrm{ON}$ *be a set of ordinals.* $\prod_{X\in\mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu$ *is wellfounded.*

Proof. Suppose $F \in \prod_{X\in\mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu$. Let f be a function on $\mathcal{D}$ such that $[f]_\mu = F$. Define $\phi(f)$ by $\phi(f)(X) = \mathrm{rk}^{\mathrm{HOD}_S^{L[S,X]}}(f(X))$. Let $\Phi : \prod_{X\in\mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu \to \prod_{X\in\mathcal{D}} \mathrm{ON}/\mu$ be defined by $\Phi([f]_\mu) = [\phi(f)]_\mu$. Φ is a well defined function. Moreover, it has the property that if $F \in G$, then $\Phi(F) < \Phi(G)$. Fact 2.14 implies that $\prod_{X\in\mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu$ is wellfounded. $\square$

3. OD in OD is OD

One will write $\mathbb{R}$ for ${}^{\omega}2$, which is the collection of functions $f : \omega \to 2$.

Theorem 3.1. *Assume* $\mathsf{ZF}+\mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$*. Let* J *be a set of ordinals. Let* $H \subseteq \mathbb{R}$*. Let* $K \subseteq \mathbb{R}$ *be nonempty and* OD_J*. If* H *is* $\mathrm{OD}_{J,z}$ *for all* $z \in K$*, then* H *is* OD_J*.*

Proof. For simplicity, assume $J = \emptyset$. By Fact 2.9, let $\mathbb{X} \in \mathrm{HOD}^V$ be such that $\mathrm{HOD}^V = L[\mathbb{X}]$. Using the constructibility ordering of $L[\mathbb{X}]$, let $\langle (S_\alpha, \varphi_\alpha) : \alpha \in \mathrm{ON}\rangle$ enumerate all the ∞-Borel codes in $L[\mathbb{X}] = \mathrm{HOD}^V$. (This is merely the canonical constructibility enumeration of all pairs (S,φ) in $\mathrm{HOD}^V = L[\mathbb{X}]$ where S is a set of ordinals and φ is a formula.) The main observation is that for any $X \in \mathcal{D}$, $\mathrm{HOD}^V = L[\mathbb{X}] \subseteq \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$ and therefore the sequence $\langle (S_\alpha, \varphi_\alpha) : \alpha \in \mathrm{ON}\rangle$ is definable in $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$ uniformly (by the same formula using $\mathbb{X}$ as a parameter for all $X \in \mathcal{D}$). In particular, every OD^V ∞-Borel code belongs to $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$.

<u>Claim 1</u>: For any $R \subseteq \mathbb{R}$, R is OD_z^V for some $z \in \mathbb{R}$ if and only if there is some OD^V ∞-Borel code (S,φ) so that

$$R = (\mathfrak{B}^2_{(S,\varphi)})_z = \{x \in \mathbb{R} : (z,x) \in \mathfrak{B}^2_{(S,\varphi)}\} = \{x \in \mathbb{R} : L[S,z,x] \models \varphi(S,z,x)\}.$$

Proof. ($\Rightarrow$) Suppose R is OD_z^V. There is some formula ς so that $x \in R \Leftrightarrow V \models \varsigma(z,x,\bar{\alpha})$ where $\bar{\alpha}$ is a tuple of ordinals. Let $R' = \{(a,b) \in \mathbb{R}^2 : \varsigma(a,b,\bar{\alpha})\}$. R' is an OD^V subset of $\mathbb{R}^2$. By Fact 2.8, there is some $(S,\varphi) \in \mathrm{HOD}^V$ so that $\mathfrak{B}^2_{(S,\varphi)} = R'$. Then $R = (\mathfrak{B}^2_{(S,\varphi)})_z$. ($\Leftarrow$) is clear. $\square$

Since $K \subseteq \mathbb{R}$ is OD^V, K has an ∞-Borel code $(U,\psi) \in \mathrm{HOD}^V$ by Fact 2.8. Since $K \neq \emptyset$, let $z^* \in K$. Let $Z^* = [z^*]_{\equiv_{\mathsf{Turing}}}$. Throughout this entire argument, one will only work on the Turing cone above Z^*.

For all $X \in \mathcal{D}$, since $(U, \psi) \in \mathrm{HOD}^V = L[\mathbb{X}] \subseteq L[\mathbb{X}, X]$, $(U, \psi) \in \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. For any $X \geq Z^*$, let $q^X = \{x \in \mathbb{R}^{L[\mathbb{X},X]} : L[U, x] \models \psi(U, x)\}$. Note that q^X is $\mathrm{OD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. Since $z^* \in \mathbb{R}^{L[\mathbb{X},X]}$, $z^* \in K$, and (U, ψ) is the ∞-Borel code for K, one has $V \models L[U, z^*] \models \psi(U, z^*)$. Thus $L[\mathbb{X}, X] \models L[U, z^*] \models \psi(U, z^*)$. Thus $z^* \in q^X$ and $q^X \neq \emptyset$. It has been shown that $q^X \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}$.

(Case I) There is a cone of $X \in \mathcal{D}$ so that there are no atoms in

$$\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \upharpoonright q^X = \{p \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} : p \leq_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} q^X\}.$$

Let $Z^{**} \in \mathcal{D}$ with $Z^{**} \geq Z^*$ be a base of a cone for which the Case I assumption holds. Now suppose $X \in \mathcal{D}$ with $X \geq Z^{**}$.

Claim 2: There is a sequence $\mathcal{J} = \langle J_n : n \in \omega\rangle$ of dense open subsets of $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \upharpoonright q^X$ and a sequence of ordinals $\langle \epsilon_n : n \in \omega\rangle$ so that for all $h \in \mathbb{R}$ which are $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \upharpoonright q^X$-generic with respect to $\mathcal{J}$, the following holds:
(1) $h \in K$.
(2) h is $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \upharpoonright q^X$-generic over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$.
(3) There is some $m \in \omega$ so that $H = (\mathfrak{B}^2_{(S_{\epsilon_m}, \varphi_{\epsilon_m})})_h$.

Proof. Since $L[\mathbb{X}, X] \models \mathsf{ZFC}$ and $V \models \mathsf{AD}$, ω_1^V is inaccessible in $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. This can be used to show that $\mathbb{O}_{X}^{L[\mathbb{X},X]} \upharpoonright q^X$ is a countable atomless forcing. In V, fix a forcing isomorphism $\Phi : \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \upharpoonright q^X \to \mathbb{C}$, where $\mathbb{C}$ is the Cohen forcing. (Specifically $\mathbb{C} = ({}^{<\omega}2, \leq_{\mathbb{C}})$ is the forcing of finite binary strings ordered by $\leq_{\mathbb{C}}$ which is reverse string inclusion. Note there is generally no way to uniformly choose Φ depending on the degree X.) Let $\mathcal{E}$ be the collection of all dense open subsets of $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \upharpoonright q^X$ which belongs to $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for some $y \in \mathbb{R}^{L[\mathbb{X},X]}$. Since $V \models \mathsf{AD}$, $L[\mathbb{X}, X] \models \mathsf{ZFC}$, and $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y] \models \mathsf{ZFC}$ for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$, one has that $\mathcal{E}$ is countable in V. Let $\mathcal{F} = \{\Phi[D] : D \in \mathcal{E}\}$. Then $\mathcal{F}$ is a countable collection of dense open subsets of Cohen forcing $\mathbb{C}$.

For each $g \in \mathbb{R}$, let $G_g^{\mathbb{C}} \subseteq \mathbb{C}$ be the derived $\mathbb{C}$-filter defined by $G_g^{\mathbb{C}} = \{g \upharpoonright n : n \in \omega\}$. One say that g is $\mathbb{C}$-generic with respect to $\mathcal{F}$ if and only if $G_g^{\mathbb{C}}$ intersects each dense open set in $\mathcal{F}$. Similarly if $\mathcal{J}$ is a collection of dense open subsets of $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \upharpoonright q^X$, one says that a real $x \in \mathbb{R}$ is $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \upharpoonright q^X$-generic with respect to $\mathcal{J}$ if and only if there is an $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \upharpoonright q^X$-generic filter $G \subseteq \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \upharpoonright q^X$ so that G meets each dense open set in $\mathcal{J}$ and $\dot{x}_{\mathrm{gen}}[G] = x$.

Since $\mathcal{F}$ is countable in V, let $C \subseteq \mathbb{R}$ be the comeager set of reals which are $\mathbb{C}$-generic with respect to $\mathcal{F}$. Let B be the collection of reals which are $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \upharpoonright q^X$-generic over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$. By the definition of Φ, $\mathcal{E}$, and $\mathcal{F}$, the forcing isomorphism Φ induces a bijection $\tilde{\Phi} : B \to C$.

For each $g \in C$, let $G_{\tilde{\Phi}^{-1}(g)} = \Phi^{-1}[G_g^{\mathbb{C}}]$. $G_{\tilde{\Phi}^{-1}(g)}$ is an $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \upharpoonright q^X$-generic filter over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$. Note that $\dot{x}_{\mathrm{gen}}[G_{\tilde{\Phi}^{-1}(g)}] = \tilde{\Phi}^{-1}(g)$. Since $q^X \in G_{\tilde{\Phi}^{-1}(g)}$ and q^X is a condition of the form mentioned in Fact 2.5,

$$\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[G_{\tilde{\Phi}^{-1}(g)}] \models L[U, \tilde{\Phi}^{-1}(g)] \models \psi(U, \tilde{\Phi}^{-1}(g)).$$

Thus

$$V \models L[U, \tilde{\Phi}^{-1}(g)] \models \psi(U, \tilde{\Phi}^{-1}(g)).$$

Since (U, ψ) is the ∞-Borel code for K, $\tilde{\Phi}^{-1}(g) \in K$. It has been shown that whenever $g \in C$, $\tilde{\Phi}^{-1}(g) \in K$.

By assumption, H is OD_x for all $x \in K$. In particular, for each $g \in C$, H is $\mathrm{OD}_{\tilde{\Phi}^{-1}(g)}$. By Claim 1, there is some $\epsilon \in \mathrm{ON}$ so that $H = (\mathfrak{B}^2_{(S_\epsilon, \varphi_\epsilon)})_{\tilde{\Phi}^{-1}(g)}$. Define $\Psi : C \to \mathrm{ON}$ by $\Psi(g)$ is the least such ϵ.

Under AD, a wellordered union of meager sets is meager, therefore, there must be some $\epsilon \in \mathrm{ON}$ so that $\Psi^{-1}[\{\epsilon\}]$ is nonmeager. Let $\delta_0 \in \mathrm{ON}$ be the least ordinal so that $\Psi^{-1}[\{\delta_0\}]$ is nonmeager. Suppose $\delta_\xi \in \mathrm{ON}$ has been defined. If $\bigcup_{\alpha \geq \delta_\xi} \Psi^{-1}[\{\alpha\}]$ is meager, then declare the construction to have finished. Otherwise, again using the fact that wellordered unions of meager sets are meager under AD, there is a least ordinal $\delta_{\xi+1}$ greater than δ_ξ so that $\Phi^{-1}[\{\delta_{\xi+1}\}]$ is nonmeager. If ξ is a limit ordinal and δ_ζ has been defined for all $\zeta < \xi$, then let $\delta_\xi = \sup\{\delta_\zeta : \zeta < \xi\}$. Since all sets of reals have the Baire property under AD and the topology on $\mathbb{R}$ has the countable chain condition, there must be a countable $\lambda \in \mathrm{ON}$ so that the construction is finished at stage λ.

As λ is countable, one can enumerate $\langle \delta_\xi : \xi < \lambda \rangle$ by $\langle \epsilon_n : n \in \omega \rangle$. Let $D = \bigcup_{n\in\omega} \Psi^{-1}[\{\epsilon_n\}]$ which is comeager by definition of λ being the ordinal by which the construction finished.

Since D is comeager, there is a sequence $\langle I_n : n \in \omega\rangle$ of topologically dense open subsets of $\mathbb{R}$ so that $\bigcap_{n\in\omega} I_n \subseteq D$. Let $J_n = \{\Phi^{-1}(\sigma) : \sigma \in \mathbb{C} \wedge N_\sigma \subseteq I_n\}$, where $N_\sigma = \{f \in \mathbb{R} : \sigma \subseteq f\}$ is the basic open subset of $\mathbb{R}$ determined by σ and recall that $\mathbb{C} = {}^{<\omega}2$. Define $\mathcal{J} = \langle J_n : n \in \omega\rangle$ which is a sequence of dense open subsets of $\mathbb{O}_{\mathbb{S}}^{L[\mathbb{X},X]} \restriction q^X$. Note that if x is $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$-generic with respect to $\mathcal{J} = \langle J_n : n \in \omega\rangle$, then $\tilde{\Phi}(x) \in D$. Since $D \subseteq C$ and by the observation above, $G_x = G_{\tilde{\Phi}^{-1}(\tilde{\Phi}(x))}$ is $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$-generic over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$ and $x = \dot{x}_{\mathrm{gen}}[G_x]$. This completes the proof of Claim 2. $\square$

One will construct a sequence of conditions in $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$ for as long as possible:

Let $p_{-1} = q^X$.

Suppose one has succeeded to construct p_k.

(Subcase i) There is some $y \in \mathbb{R}^{L[\mathbb{X},X]}$ and some $u \leq_{\mathbb{O}_X^{L[\mathbb{X},X]}\restriction q^X} p_k$ so that

$$y \notin H \wedge \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y] \models u \Vdash_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} L[\check{S}_{\epsilon_{k+1}}, \dot{x}_{\mathrm{gen}}, \check{y}] \models \varphi_{\epsilon_{k+1}}(\check{S}_{\epsilon_{k+1}}, \dot{x}_{\mathrm{gen}}, \check{y})$$

or

$$y \in H \wedge \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y] \models u \Vdash_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} L[\check{S}_{\epsilon_{k+1}}, \dot{x}_{\mathrm{gen}}, \check{y}] \models \neg\varphi_{\epsilon_{k+1}}(\check{S}_{\epsilon_{k+1}}, \dot{x}_{\mathrm{gen}}, \check{y})$$

In this case, let $p_{k+1} \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}$ be the least $u \in J_{k+1}$ according to the canonical wellordering of $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$.

(Subcase ii) Subcase i fails. Declare that the construction has failed at stage $k+1$.

<u>Claim 3</u>: The construction must fail at some stage.

Proof. Suppose the construction never failed. Then one would have successfully produced a sequence $\langle p_k : k \in \omega\rangle$ in $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$ with the properties specified above. Let $\hat{G}$ be the $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$-generic filter produced by $\leq_{\mathbb{Q}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X}$-upward closing $\{p_k : k \in \omega\}$. By construction, $p_k \in J_k$. Hence $\hat{G}$ is $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$-generic filter with respect to $\mathcal{J}$. Let $h = \dot{x}_{\mathrm{gen}}[\hat{G}]$ be the associated $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$-generic real. By Claim 2, $h \in K$, h is $\mathbb{O}_X^{L[\mathbb{X},X]} \restriction q^X$-generic over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$, and there is some $m \in \omega$ so that $H = (\mathfrak{B}^2_{(S_{\epsilon_m},\varphi_{\epsilon_m})})_h$. However, the construction did not fail at stage m. Without loss of generality (the other case being similar), p_m was found with the property that there is some $y \in \mathbb{R}^{L[\mathbb{X},X]}$ so that

$$y \notin H \wedge \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y] \models p_m \Vdash_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} L[\check{S}_{\epsilon_m}, \dot{x}_{\mathrm{gen}}, \check{y}] \models \varphi_{\epsilon_m}(\check{S}_{\epsilon_m}, \dot{x}_{\mathrm{gen}}, \check{y})$$

Thus

$$\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y][h] \models L[S_{\epsilon_m}, h, y] \models \varphi_{\epsilon_m}(S_{\epsilon_m}, h, y).$$

Thus

$$V \models L[S_{\epsilon_m}, h, y] \models \varphi_{\epsilon_m}(S_{\epsilon_k}, h, y).$$

Since $H = (\mathfrak{B}^2_{(S_{\epsilon_m},\varphi_{\epsilon_m})})_h$, this implies that $y \in H$. However, it was also assumed that $y \notin H$. Contradiction. This completes the proof of Claim 3. $\square$

<u>Claim 4</u>: For all $X \geq Z^{**}$, there is some $p \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}$ and some ordinal ϵ so that for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$, $y \in H$ if and only if $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y] \models p \Vdash_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} L[\check{S}_\epsilon, \dot{x}_{\mathrm{gen}}, \check{y}] \models \varphi(\check{S}_\epsilon, \dot{x}_{\mathrm{gen}}, \check{y})$.

Proof. By Claim 3, the construction described above must fail at some stage k. This means that the forcing relation written above in $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for p_{k-1} and the ∞-Borel code $(S_{\epsilon_k}, \varphi_{\epsilon_k})$ can be used to determine membership of $y \in H$ for any $y \in \mathbb{R}^{L[\mathbb{X},X]}$. This completes the proof of Claim 4.

As mentioned in the proof of Claim 2, one non-uniformly selected a forcing isomorphism Φ. The choice of Φ is irrelevant, however, since one will only need the existence of any condition p with the above property in Claim 4. $\square$

For $X \geq Z^{**}$, using the canonical wellordering of $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$, let $\langle p_\alpha^X : \alpha < \delta^X \rangle$, where $\delta^X \in \mathrm{ON}$, be the canonical enumeration of $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$.

In summary, it has been established that for any $y \in \mathbb{R}$, if one drops into a local model $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$, where X is a sufficiently strong Turing degree (i.e. $X \geq Z^{**} \oplus [y]_{\mathrm{Turing}}$), then one can determine membership of y in H by merely two pieces of information: a condition $p \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}$ and an ordinal ϵ. Note that p is coded by an ordinal since one can identify p with the least ordinal $\alpha < \delta^X$ so that $p = p_\alpha^X$. Next one will show that roughly all this local information can be captured by just two ordinals by taking an ultrapower by μ.

Using Claim 4, let $\Sigma_{\alpha^*} : \mathcal{D} \to \mathrm{ON}$ be defined by $\Phi_{\alpha^*}(X)$ is the least α so that p_α^X satisfies Claim 4 for some ϵ whenever $X \geq Z^{**}$. For other $X \in \mathcal{D}$, let $\Sigma_{\alpha^*}(X) = 0$. Define $\Sigma_{\epsilon^*} : \mathcal{D} \to \mathrm{ON}$ by $\Sigma_{\epsilon^*}(X)$ is the least ϵ satisfying Claim 4 with respect to $p_{\Sigma_{\alpha^*}(X)}$ whenever $X \geq Z^{**}$. For other $X \in \mathcal{D}$, let $\Sigma_{\epsilon^*}(X) = 0$.

$[\Sigma_{\alpha^*}]_\mu$ and $[\Sigma_{\epsilon^*}]_\mu$ are ordinals since $\prod_{X \in \mathcal{D}} \mathrm{ON}/\mu$ is a wellordering by Fact 2.14. Let $\alpha^* = [\Sigma_{\alpha^*}]_\mu$ and $\epsilon^* = [\Sigma_{\epsilon^*}]_\mu$.

Claim 5: H is OD.

Proof. Note that for $y \in \mathbb{R}$, $y \in H$ if and only if for any $\Sigma_0, \Sigma_1 : \mathcal{D} \to \mathrm{ON}$ so that $[\Sigma_0]_\mu = \alpha^*$ and $[\Sigma_1]_\mu = \epsilon^*$, for a cone of $X \in \mathcal{D}$,

$$\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y] \models p_{\Sigma_0(X)}^X \Vdash_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} L[\check{S}_{\Sigma_1(X)}, \dot{x}_{\mathrm{gen}}, \check{y}] \models \varphi_{\Sigma_1(X)}(\check{S}_{\Sigma_1(X)}, \dot{x}_{\mathrm{gen}}, \check{y}).$$

The latter is ordinal definable (using the two ordinals α^* and ϵ^*). The expression successfully defines H by the definition of $\alpha^* = [\Sigma_{\alpha^*}]_\mu$ and $\epsilon^* = [\Sigma_{\epsilon^*}]_\mu$ as well as Claim 4. □

The theorem is complete in the setting of Case I.

(Case II) There is a cone of $X \in \mathcal{D}$ so that there is an atom in $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$.
Let $Z^{**} \geq Z^*$ be the base of a cone satisfying the Case II assumption.
Fix an $X \geq Z^{**}$. Let $p \leq_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X} q^X$ be an atom.

Claim 6: There is some $r \in K \cap \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. Note that $r \in K$ implies there is an ordinal ϵ so that $H = (\mathfrak{B}^2_{(S_\epsilon, \varphi_\epsilon)})_r$.

Proof. Since $p \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}$, one has that $p \neq \emptyset$. Let $r \in p$. Let $G_r^1 = \{p \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} : r \in p\}$. By Fact 2.4, G_r^1 is an $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}$-generic filter over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$ and $\dot{x}_{\mathrm{gen}}[G_r^1] = r$. Also $p \in G_r^1$. Therefore thinking of reals as subsets of ω, for each $n \in \omega$, $n \in r$ if and only if $p \Vdash_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} \check{n} \in \dot{x}_{\mathrm{gen}}$ since p was assumed to be an atom and hence has no nontrivial extensions. The latter is $\mathrm{OD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. This shows that $r \in \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. (Since $r \in p$ was arbitrary, this argument actually shows that $p = \{r\}$.) Since $p \leq_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} q^X$ and $p \in G_r^1$, one has that $r \in q^X$. By definition of q^X, one has that $L[U, r] \models \Psi(U, r)$. Since (U, ψ) is the ∞-Borel code for K, $V \models r \in K$. It has been shown that $r \in K \cap \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. □

Let $\langle r_\alpha^X : \alpha < \delta^X \rangle$, where $\delta^X \in \mathrm{ON}$, be the enumeration of $\mathbb{R}^{\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}}$ according to the canonical wellordering of $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. Define $\Sigma_{\alpha^*} : \mathcal{D} \to \mathrm{ON}$ by $\Sigma_{\alpha^*}(X)$ is the least ordinal α so that r_α^X satisfies Claim 6 whenever $X \geq Z^{**}$. Otherwise, let $\Sigma_{\alpha^*}(X) = 0$. Let $\Sigma_{\epsilon^*} : \mathcal{D} \to \mathrm{ON}$ be defined by $\Sigma_{\epsilon^*}(X)$ is the least $\epsilon \in \mathrm{ON}$ so that $H = (\mathfrak{B}^2_{(S_\epsilon, \varphi_\epsilon)})_{r^X_{\Sigma_{\alpha^*}(X)}}$ whenever $X \geq Z^{**}$. Otherwise, let $\Sigma_{\epsilon^*}(X) = 0$.

Again since $\prod_{X \in \mathcal{D}} \mathrm{ON}/\mu$ is a wellordering by Fact 2.14, $[\Sigma_{\alpha^*}]_\mu$ and $[\Sigma_{\epsilon^*}]_\mu$ are ordinals. Let $\alpha^* = [\Sigma_{\alpha^*}]_\mu$ and $\epsilon^* = [\Sigma_{\epsilon^*}]_\mu$.

Claim 7: H is OD.

Proof. Note that for all $y \in \mathbb{R}$, $y \in H$ if and only for all $\Sigma_0, \Sigma_1 : \mathcal{D} \to \mathrm{ON}$ so that $[\Sigma_0]_\mu = \alpha^*$ and $[\Sigma_1]_\mu = \epsilon^*$, for a cone of $X \in \mathcal{D}$,

$$L[S_{\Sigma_1(X)}, r^X_{\Sigma_0(X)}, y] \models \varphi_{\Sigma_1(X)}(S_{\Sigma_1(X)}, r^X_{\Sigma_0(X)}, y).$$

This equivalence is true by Claim 6 and the definitions of Σ_{α^*} and Σ_{ϵ^*}. The latter is ordinal definable (using the ordinals α^* and ϵ^*). $\square$

The theorem has been shown in Case II as well. The entire argument is complete. $\square$

Some assumptions beyond ZF or ZFC are necessary to prove the conclusion of Theorem 3.1. The next result shows that in a Sacks forcing extension of the constructible universe L, there is a nonempty OD set K and a real g so that g is OD_z for any $z \in K$ but g is not OD.

Fact 3.2. *Let $\mathbb{S}$ denote the Sacks forcing of perfect trees. Let $G \subseteq \mathbb{S}$ be an $\mathbb{S}$-generic filter over L.*

In $L[G]$: Let $K = \mathbb{R}^{L[G]} \setminus \mathbb{R}^L$ be the collection of nonconstructible reals. K is an OD *set of reals. Let $g \in \mathbb{R}^{L[G]}$ be the $\mathbb{S}$-generic real over L derived from G. Then g is OD_z for any $z \in K$, but g is not* OD.

Proof. A perfect tree is a subset p of ${}^{<\omega}2$ with the property that for all $\sigma, \tau \in {}^{<\omega}2$, if $\sigma \subseteq \tau$ and $\tau \in p$, then $\sigma \in p$ and for all $\sigma \in p$, there exists a $\tau \supseteq \sigma$ so that $\tau\hat{\ }0, \tau\hat{\ }1 \in p$. Let $\mathbb{S}$ consists of the collection of perfect trees. Define $p \leq_{\mathbb{S}} q$ if and only if $p \subseteq q$. The largest element is $1_{\mathbb{S}} = {}^{<\omega}2$. Sacks forcing is $\mathbb{S} = (\mathbb{S}, \leq_{\mathbb{S}}, 1_{\mathbb{S}})$. If $p \in \mathbb{S}$, then define $[p] = \{f \in {}^{\omega}2 : (\forall n)(f \upharpoonright n \in p)\}$. If $r \in \mathbb{R}$, then let $G_r^{\mathbb{S}} = \{p \in \mathbb{S} : r \in [p]\}$. If $G_r^{\mathbb{S}}$ is an $\mathbb{S}$-generic filter over L, then one says that r is an $\mathbb{S}$-generic real over L. See [11] Chapter 15 for the basic facts about the Sacks forcing $\mathbb{S}$.

Fix $G \subseteq \mathbb{S}$ a Sacks generic filter over L. Work in $L[G]$. Let g be the Sacks generic real derived from G, i.e. $\{g\} = \bigcap_{p \in G} [p]$.

Let $K = \mathbb{R}^{L[G]} \setminus \mathbb{R}^L$ be the collection of nonconstructible reals. This set K is OD. Using a fusion argument, one can reconstruct g from any nonconstructible real z (that is $z \in K$) using only parameters in L. (This is the argument used in [11] Theorem 15.34 to show that g is a real of minimal constructibility degree. It also shows that every element of K is itself an $\mathbb{S}$-generic real for some $\mathbb{S}$-generic filter over L.) So g is OD_z for any $z \in K$.

However g is not OD. Suppose otherwise that g was OD. Then there must be some formula φ and some ordinal ϵ so that g is the unique solution $v \in L[G]$ to $L[G] \models \varphi(v, \epsilon)$. Therefore, there is some $q \in G$ so that $L \models q \Vdash_{\mathbb{S}} \varphi(\dot{x}_{\mathrm{gen}}, \check{\epsilon})$ where $\dot{x}_{\mathrm{gen}}$ is the canonical $\mathbb{S}$-name for the generic real added by an $\mathbb{S}$-generic filter. Since q is still a perfect tree in $L[G]$, $[q]^{L[G]}$ must contain a nonconstructible real h with $h \neq g$. As mentioned above, by the fusion argument of [11] Theorem 15.34, h is also $\mathbb{S}$-generic over L. Let $G_h^{\mathbb{S}} = \{p \in \mathbb{S} : h \in [p]\}$ be the $\mathbb{S}$-generic filter over L derived from h so that $\dot{x}_{\mathrm{gen}}[G_h^{\mathbb{S}}] = h$. Note that $G_h^{\mathbb{S}} \in L[G]$ and $q \in G_h^{\mathbb{S}}$. Thus $L[G_h^{\mathbb{S}}] \models \varphi(h, \epsilon)$. Since [11] Theorem 15.34 implies every nonconstructible real in $L[G]$ has minimal constructibility degree, $L[G] = L[G_h^{\mathbb{S}}]$. Hence $L[G] \models \varphi(h, \epsilon)$ and $h \neq g$. This contradicts g being the unique solution in $L[G]$ to $\varphi(v, \epsilon)$. $\square$

4. Cardinals Below $[\omega_1]^{\omega_1}$

This section will show under AD^+ that $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}| = |\mathscr{P}(\omega_1)|$. In $L(\mathbb{R})$, a cardinality intermediate between $|[\omega_1]^{<\omega_1}|$ and $|[\omega_1]^{\omega_1}|$ will be isolated.

The argument for Theorem 4.5 showing that $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$ presented below using Fact 4.1 was suggested by Neeman and is simpler than the original argument. The original argument will be presented later and is required in other settings involved absorbing a fragment of an arbitrary injection into a suitable ZFC model. This idea is a powerful technique for studying cardinalities under AD^+ and especially for producing intermediate cardinality under $\mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}}$.

Fact 4.1. *Assume* ZF*. Suppose κ is a cardinal which is inaccessible in any inner model of* ZFC*. Then $|[\kappa]^{<\kappa}| < |[\kappa]^{\kappa}|$.*

Proof. Suppose there was an injection $\Phi : [\kappa]^\kappa \to [\kappa]^{<\kappa}$. Consider $\hat{\Phi} \subseteq [\kappa]^\kappa \times \kappa$ defined by $(f, \alpha) \in \hat{\Phi} \Leftrightarrow \alpha \in \Phi(f)$. (Here $[\kappa]^{<\kappa}$ is identified as a subset of κ.) Note that if $f \in L[\hat{\Phi}]$, then $\Phi(f) \in L[\hat{\Phi}]$.

Identify the predicate Φ with $\hat{\Phi}$. Then $L[\Phi] \models \mathsf{ZFC}$ and $L[\Phi] \models$ "Φ is an injection". By Cantor's theorem, $L[\Phi] \models |[\kappa]^\kappa| = 2^\kappa > \kappa$. However, since $L[\Phi]$ thinks κ is inaccessible, $L[\Phi] \models |[\kappa]^{<\kappa}| = |2^{<\kappa}| = \kappa$. Then within $L[\Phi]$, Φ induces an injection of 2^κ into κ which is not possible. $\square$

Fact 4.2. *Assume* ZF*. Suppose κ is a cardinal such that there is a κ-complete nonprincipal ultrafilter on κ. Let M be any inner model of* ZFC*. Then κ is inaccessible in M.*

Proof. Let μ be a κ-complete measure on κ. It is clear that κ is regular in M.

Suppose κ is not a strong limit cardinal in M. Then there is a $\delta < \kappa$ so that $M \models |\mathscr{P}(\delta))| \geq \kappa$. Since $M \models \mathsf{ZFC}$, one can find a κ-length sequence of distinct subsets of δ, $\langle A_\alpha : \alpha < \kappa\rangle$.

For each $\beta < \delta$, let $C^0_\beta = \{\alpha < \kappa : \beta \notin A_\alpha\}$ and $C^1_\beta = \{\alpha < \kappa : \beta \in A_\alpha\}$. Since $C^0_\beta \cup C^1_\beta = \kappa$ and μ is a measure, there is some $i_\beta \in 2$ so that $C^{i_\beta}_\beta \in \mu$. Let $A = \{\beta : i_\beta = 1\}$. Since μ is κ-complete and $\delta < \kappa$, $C = \bigcap_{\beta<\delta} C^{i_\beta}_\beta \in \mu$. Since μ is nonprincipal, let $\alpha_0, \alpha_1 \in C$ with $\alpha_0 \neq \alpha_1$. Then $A_{\alpha_0} = A_{\alpha_1} = A$. This contradicts $\langle A_\alpha : \alpha < \kappa\rangle$ being a sequence of distinct subsets of δ. ☐

Fact 4.3. *([16] Theorem 3.2) Assume* ZF*. Let κ be a cardinal. Let $\eta < \kappa$ be a limit ordinal. The partition relation $\kappa \rightarrow (\kappa)^{\eta+\eta}_2$ implies that the η-club filter on κ, W^η_κ, is a normal κ-complete ultrafilter on κ.*

Fact 4.4. *Assume* $\mathsf{ZF} + \mathsf{AD}$*.*

(Solovay) $\omega_1 \rightarrow (\omega_1)^{\omega_1}_2$ and therefore ω_1 is measurable. (Martin) $\omega_2 \rightarrow (\omega_2)^{\alpha}_2$, for each $\alpha < \omega_2$, and therefore ω_2 is measurable.

([13]) Suppose $A \subseteq \mathbb{R}$. Let δ_A be the least ordinal so that $L_\delta(A,\mathbb{R}) \prec_1 L(A,\mathbb{R})$. $\delta_A \rightarrow (\delta_A)^{\delta_A}_2$ and hence δ_A is measurable.

Theorem 4.5. *Assume* $\mathsf{ZF} + \mathsf{AD}$*.*

- $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$.
- $|[\omega_2]^{<\omega_2}| < |[\omega_2]^{\omega_2}|$.
- *For any set $A \subseteq \mathbb{R}$, $|[\delta_A]^{<\delta_A}| < |[\delta_A]^{\delta_A}|$.*
- *More generally, for any cardinal κ satisfying the partition relation $\kappa \rightarrow (\kappa)^{\omega+\omega}_2$, one has $|[\kappa]^{<\kappa}| < |[\kappa]^{\kappa}|$.*

Proof. Under AD, ω_1, ω_2, and δ_A for any $A \subseteq \mathbb{R}$ satisfies the $\omega+\omega$ exponent partition relation by Fact 4.4 and are thus measurable cardinals by Fact 4.3. Each result now follows from Fact 4.2 and Fact 4.1. ☐

Fact 4.6. *Assume $V = L(J,\mathbb{R}) \models \mathsf{AD} + \mathsf{DC}_\mathbb{R}$ where J is a set of ordinals. Suppose $\Phi : [\kappa]^\kappa \rightarrow [\kappa]^{<\kappa}$. Then there is a $e \in \mathbb{R}$ so that for all $x \in \mathbb{R}$ with $e \leq_{J,{}_\omega\mathbb{O}_J} x$ (which refers to the $(J, {}_\omega\mathbb{O}_J)$-constructibility reduction), one has the following properties:*
(i) For all $f \in [\kappa]^\kappa \cap L[J, {}_\omega\mathbb{O}_J, x]$, $\Phi(f) \in L[J, {}_\omega\mathbb{O}_J, x]$.
(ii) $\Phi \cap L[J, {}_\omega\mathbb{O}_J, x] \in L[J, {}_\omega\mathbb{O}_J, x]$.

(i) and (ii) together imply that $\Phi \cap L[J, {}_\omega\mathbb{O}_J, x]$ is a function which is even a set in $L[J, {}_\omega\mathbb{O}_J, x]$.

Proof. In $L(J,\mathbb{R})$, every set is $\mathrm{OD}_{J,e}$ for some real e. Let φ be a formula and let $\bar\alpha$ be a tuple of ordinals so that

$$(f,\sigma) \in \Phi \Leftrightarrow L(J,\mathbb{R}) \models \varphi(J, e, \bar\alpha, f, \sigma)$$

Now fix $x \in \mathbb{R}$ so that $e \in L[J, {}_\omega\mathbb{O}_J, x]$. By Fact 2.10 and the above, one has that for all $(f,\sigma) \in ([\kappa]^\kappa \times [\kappa]^{<\kappa}) \cap L[J, {}_\omega\mathbb{O}_J, x]$

$$(f,\sigma) \in \Phi \Leftrightarrow L[J, {}_\omega\mathbb{O}_J, x] \models 1_{{}_\omega\mathbb{O}_J/G^n_x} \Vdash_{{}_\omega\mathbb{O}_J/G^1_x} L(\check J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models \varphi(J, e, \bar\alpha, f, \sigma).$$

By comprehension in $L[J, {}_\omega\mathbb{O}_J, x]$, one see that (ii) follows.

Note that for each $f \in [\kappa]^\kappa$ and $\beta \in \kappa$, one has that

$$\beta \in \Phi(f) \Leftrightarrow L(J,\mathbb{R}) \models (\exists\sigma)(\varphi(J, e, \bar\alpha, f, \sigma) \wedge \beta \in \sigma).$$

(Here $\sigma \in [\kappa]^{<\kappa}$ is construed as a subset of κ.)

So for each $x \in \mathbb{R}$ so that $e \in L[J, {}_\omega\mathbb{O}_J, x]$, if $f \in L[J, {}_\omega\mathbb{O}_J, x]$, one has

$$\beta \in \Phi(f) \Leftrightarrow L[J, {}_\omega\mathbb{O}_J, x] \models 1_{{}_\omega\mathbb{O}_J/G^1_x} \Vdash_{{}_\omega\mathbb{O}_J/G^1_x} L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists\sigma)(\varphi(J, e, \bar\alpha, f, \sigma) \wedge \beta \in \sigma).$$

Again by comprehension in $L[J, {}_\omega\mathbb{O}_J, x]$, one has that $\Phi(f) \in L[J, {}_\omega\mathbb{O}_J, x]$ and thus (i). ☐

The following result due to Steel is proved by inner model theoretic techniques:

Fact 4.7. *(Steel, [19] Theorem 8.27) Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$. If κ is regular, then for all $x \in \mathbb{R}$, $\mathrm{HOD}_x \models$ "κ is measurable".*

Theorem 4.8. *Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} + \mathsf{L}(\mathbb{R})$. Suppose $\kappa < \Theta$ is regular. Then $|[\kappa]^{<\kappa}| < |[\kappa]^\kappa|$.*

Proof. If $\kappa < \Theta$ is regular, then Fact 4.7 implies that $\mathrm{HOD}_x^{L(\mathbb{R})} \models$ "κ is measurable" for any $x \in \mathbb{R}$. Let $\mathbb{X} = {}_\omega\mathbb{O}$. By Fact 2.11, $\mathrm{HOD}_x^{L(\mathbb{R})} = L[\mathbb{X}, x]$.

Now suppose that there is an injection $\Phi : [\kappa]^\kappa \to [\kappa]^{<\kappa}$. By Fact 4.6, there is an $e \in \mathbb{R}$ so that $\Phi \cap L[\mathbb{X}, e] \in L[\mathbb{X}, e]$ and this set is a function in $L[\mathbb{X}, e]$. Let $\Psi = \Phi \cap L[\mathbb{X}, e]$. By absoluteness, $L[\mathbb{X}, e] \models$ "$\Psi : [\kappa]^\kappa \to [\kappa]^{<\kappa}$ is an injection". However, since κ is measurable in $\mathrm{HOD}_e = L[\mathbb{X}, e]$, one has that $L[\mathbb{X}, e] \models |[\kappa]^{<\kappa}| = \kappa$. By Cantor's theorem applied in $L[\mathbb{X}, e]$, it is impossible such an injection exists. $\square$

By Theorem 4.5, $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$. A natural question at this point would be whether it is possible under $\mathsf{ZF} + \mathsf{AD}$ that there exists a set K such that $|[\omega_1]^{<\omega_1}| < |K| < |[\omega_1]^{\omega_1}|$. Next, it will be shown that such a set exists under $\mathsf{ZF} + \mathsf{AD}^+ + \neg\mathsf{AD}_\mathbb{R} + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Recall under this assumption, there is a set of ordinal J so that $V = L(J, \mathbb{R})$.

Definition 4.9. Assume $\mathsf{ZF} + \mathsf{AD}^+$. Let $J \subseteq \mathrm{ON}$ be a set of ordinals so that $V = L(J, \mathbb{R})$. Let $\mathbb{X} = (J, {}_\omega\mathbb{O}_J)$.
$$N_1^J = \bigsqcup_{r \in \mathbb{R}} ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]} = \{(r, \alpha) : \alpha < ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]}\}.$$
In other words, this is a disjoint union over $r \in \mathbb{R}$ of the successor of $\omega_1^{L(J,\mathbb{R})}$ as computed in $L[\mathbb{X}, r]$.

Theorem 4.10. *Assume $\mathsf{ZF} + \mathsf{AD}^+$ and there is a set of ordinals $J \subseteq \mathrm{ON}$ so that $V = L(J, \mathbb{R})$.*
(1) $\neg(|N_1^J| \leq [\omega_1]^{<\omega_1})$.
(2) $|\mathbb{R} \times \omega_1| < |N_1^J| < |\mathbb{R} \times \omega_2|$.
(3) $|N_1^J| < |[\omega_1]^{\omega_1}|$.
(4) $\neg(|[\omega_1]^\omega| \leq |N_1^J|)$.
(5) $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{<\omega_1} \sqcup N_1^J| < |[\omega_1]^{\omega_1}|$.

Proof. Let $\mathbb{X} = (J, {}_\omega\mathbb{O}_J)$.

Suppose there is an injection $\Phi : N_1^J \to [\omega_1]^{<\omega_1}$. By the idea of Fact 4.6, there is an $e \in \mathbb{R}$ so that $\Phi \cap L[\mathbb{X}, e] \in L[\mathbb{X}, e]$ and $L[\mathbb{X}, e]$ thinks that $\tilde{\Phi} = \Phi \cap L[\mathbb{X}, e]$ is an injective function with domain $N_1^J \cap L[\mathbb{X}, e]$. Thus with the model $L[\mathbb{X}, e]$, the restriction of $\tilde{\Phi}$ to $\{e\} \times ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},e]}$ is an injection into $([\omega_1^{L(J,\mathbb{R})}]^{<\omega_1^{L(J,\mathbb{R})}}) \cap L[\mathbb{X}, e]$. This is impossible since the inaccessibility of $\omega_1^{L(J,\mathbb{R})}$ in the model $L[\mathbb{X}, e]$ implies that $L[\mathbb{X}, e] \models |[\omega_1^{L(J,\mathbb{R})}]^{<\omega_1^{L(J,\mathbb{R})}}| = \omega_1^{L(J,\mathbb{R})}$. This shows that $\neg(N_1^J \leq [\omega_1]^{<\omega_1})$. This also implies $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{<\omega_1} \sqcup N_1^J|$.

Suppose there is an injection $\Phi : N_1^J \to \mathbb{R} \times \omega_1$. Using the same idea as the proof of Fact 4.6, there is an e so that $\Phi \cap L[\mathbb{X}, e] \in L[\mathbb{X}, e]$ and $L[\mathbb{X}, e]$ thinks that $\Phi \cap L[\mathbb{X}, e]$ is an injective function with domain $N_1^J \cap L[\mathbb{X}, e]$. Let $\tilde{\Phi} = \Phi \cap L[\mathbb{X}, e]$. Then $L[\mathbb{X}, e] \models$ "$\tilde{\Phi}$ restricted to $\{e\} \times ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},e]} = \{e\} \times (\omega_1^{L(J,\mathbb{R})})^+$ is an injection of $\{e\} \times (\omega_1^{L(J,\mathbb{R})})^+$ into $\mathbb{R} \times \omega_1^{L(J,\mathbb{R})}$". Note that $L[\mathbb{X}, e] \models |\mathbb{R}| < \omega_1^{L(J,\mathbb{R})}$ since $\omega_1^{L(J,\mathbb{R})}$ is inaccessible in $L[\mathbb{X}, e]$. Thus $L[\mathbb{X}, e] \models |\mathbb{R} \times \omega_1^{L(J,\mathbb{R})}| = \omega_1^{L(J,\mathbb{R})}$. It is impossible that $L[\mathbb{X}, e]$ has an injection of the successor $\omega_1^{L(J,\mathbb{R})}$ (as computed in $L[\mathbb{X}, e]$) into $\omega_1^{L(J,\mathbb{R})}$. This establishes $\neg(|N_1^J| \leq |\mathbb{R} \times \omega_1|)$.

Suppose there is an injection $\Phi : \mathbb{R} \times \omega_2 \to N_1^J$. Again using the idea for Fact 4.6, there is an e so that $\Phi \cap L[\mathbb{X}, e] \in L[\mathbb{X}, e]$ and $L[\mathbb{X}, e]$ thinks that $\tilde{\Phi} = \Phi \cap L[\mathbb{X}, e]$ is a function with domain $(\mathbb{R} \times \omega_2^{L(J,\mathbb{R})}) \cap L[\mathbb{X}, e]$. Since $L[\mathbb{X}, e] \models \mathsf{AC}$ and there are no uncountable wellordered sequences of distinct reals, $L[\mathbb{X}, e] \models |\mathbb{R}| < \omega_1^{L(J,\mathbb{R})}$. Since AD implies that ω_1 and ω_2 are measurable, the argument of Fact 4.2 implies that there are no uncountable wellordered sequence of distinct reals and no ω_2 length sequence of distinct subsets of ω_1. Thus $\mathbb{R}^{L[\mathbb{X},e]}$ is countable and for each $r \in \mathbb{R}$, $((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]} < \omega_2^{L(J,\mathbb{R})}$. Hence $L[\mathbb{X}, e] \models |\bigsqcup_{r \in \mathbb{R}} ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]}| < \omega_2^{L(J,\mathbb{R})}$. Thus it is impossible that $L[\mathbb{X}, e]$ thinks that $\tilde{\Phi}$ restricted to $\{e\} \times \omega_2^{L(J,\mathbb{R})}$ is an injection of $\{e\} \times \omega_2^{L(J,\mathbb{R})}$ into $L[\mathbb{X}, e] \cap N_1^J = \bigsqcup_{r \in \mathbb{R}^{L[\mathbb{X},e]}} ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]}$. This establishes that $\neg(|\mathbb{R} \times \omega_2| \leq |N_1^J|)$.

As observed above, for each $r \in \mathbb{R}$, $((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]} < \omega_2^{L(J,\mathbb{R})}$. Thus it is clear that N_1^J is a subset of $\mathbb{R} \times \omega_2$. Thus $|\mathbb{R} \times \omega_1| < |N_1^J| < |\mathbb{R} \times \omega_2|$.

For each $r \in \mathbb{R}$, define in $L[\mathbb{X}, r]$, $A_r = \{f \in [\omega_1^{L(J,\mathbb{R})}]^{\omega_1^{L(J,\mathbb{R})}} : \min(f) \geq \omega\}$. Observe that $L[\mathbb{X}, r] \models |A_r| = |[\omega_1^{L(J,\mathbb{R})}]^{\omega_1^{L(J,\mathbb{R})}}| = |2^{\omega_1^{L(J,\mathbb{R})}}| \geq (\omega_1^{L(J,\mathbb{R})})^+$. Let $\Psi_r : ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]} \to A_r$ be the least injection from $((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]}$ into A_r according to the constructibility order on $L[\mathbb{X}, r]$. (Note that $\langle \Psi_r : r \in \mathbb{R} \rangle$ does exists as a set in $L(J,\mathbb{R})$.) Out in $L(J,\mathbb{R})$, define an injection $\Gamma : N_1^J \to [\omega_1]^{\omega_1}$ by $\Gamma(r,\alpha) = r\hat{\ }\Psi_r(\alpha)$, which is well defined if one considers $\mathbb{R}$ as $[\omega]^\omega$, the collection of strictly increasing ω-sequence in ω, and the fact that $\min \Psi_r(\alpha) \geq \omega$ since $\Psi_r(\alpha) \in A_r$. Γ witnesses that $|N_1^J| \leq |[\omega_1]^{\omega_1}|$.

Let $\mathrm{add} : \omega_1 \times [\omega_1]^{<\omega_1} \to [\omega_1]^{<\omega_1}$ be defined by $\mathrm{add}(\alpha, f)(\beta) = \alpha + f(\beta)$, whenever $\beta < \mathrm{dom}(f)$. If $B \subseteq \omega_1$ is unbounded in ω_1, then let $\mathrm{enum}_B : \omega_1 \to \omega_1$ denote the increasing enumeration of B. Let

$$\Lambda(f) = \langle \sup(f) \rangle\hat{\ }\mathrm{add}(\sup(f), f)\hat{\ }\mathrm{enum}_{\omega_1 \setminus \mathrm{rang}(\mathrm{add}(\sup(f),f))}.$$

In words, $\Lambda(f)$ first outputs $\sup(f)$, then put down the values $\sup(f) + f(\beta)$ for each $\beta < \mathrm{dom}(f)$, and then fills up the rest with an increasing enumerating of the remaining countable ordinals. Λ is an injection of $[\omega_1]^{<\omega_1}$ into $[\omega_1]^{\omega_1}$.

Let $A = \{f \in [\omega_1]^{<\omega_1} : \min(f) \geq \omega\}$. Observe that $|A| = |[\omega_1]^{<\omega_1}|$. Note that $\Lambda[A]$ and $\Gamma[N_1^J]$ are disjoint subsets of $[\omega_1]^{\omega_1}$ since for any $f \in \Lambda[A]$, $\min(f) \geq \omega$ but for all $f \in \Gamma[N_1^J]$, $\min(f) < \omega$. Thus one can merge these two injections together to obtain an injection of $[\omega_1]^{<\omega_1} \sqcup N_1^J$ into $[\omega_1]^{\omega_1}$. This shows that $|[\omega_1]^{<\omega_1} \sqcup N_1^J| \leq |[\omega_1]^{\omega_1}|$.

Now suppose $\Phi : [\omega_1]^\omega \to N_1^J$ is an injection. Let $\pi : \mathbb{R} \times \omega_2 \to \mathbb{R}$ denote the projection onto the first coordinate. Thinking of $N_1^J \subseteq \mathbb{R} \times \omega_2$, $\pi \circ \Phi : [\omega_1]^\omega \to \mathbb{R}$. Thinking of $\mathbb{R}$ as ${}^\omega 2$, let $\sigma_n : \mathbb{R} \to 2$ be defined to be the projection onto the n^{th}-coordinate, that is, $\sigma_n(r) = r(n)$. Thus for each $n \in \omega$, $\sigma_n \circ \pi \circ \Phi : [\omega_1]^\omega \to 2$. By the correct-type partition relation, $\omega_1 \to_* (\omega_1)^\omega_2$, there is a club C_n and $i_n \in 2$ so that for all $f \in [C_n]^\omega_*$, $\sigma_n(\pi(\Phi(f))) = i_n$, where $[C_n]^\omega_*$ is the collection of all $f \in [C_n]^\omega$ which are of the correct type. (See [2] Section 2 for the definition of functions of correct type, the correct-type partition relation, and its equivalence with the usual partition property.) By $\mathsf{AC}^{\mathbb{R}}_\omega$, let $\langle C_n : n \in \omega \rangle$ be such that C_n is a club subset of ω_1 which is homogeneous for $\sigma_n \circ \pi \circ \Phi$ in the sense above for each $n \in \omega$. Let $s \in \mathbb{R}$ be defined by $s(n) = i_n$. Let $C = \bigcap_{n \in \omega} C_n$. Then for all $f \in [C]^\omega_*$, $\pi(\Phi(f)) = s$. Thus Φ restricted to $[C]^\omega_*$ is an injection of $[C]^\omega_*$ into $\{s\} \times ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},e]}$. This is impossible since $[C]^\omega_*$ is not wellorderable under AD. This shows $\neg(|[\omega_1]^\omega| \leq |N_1^J|)$.

Now suppose $\Phi : [\omega_1]^{\omega_1} \to [\omega_1]^{<\omega_1} \sqcup N_1^J$. Define $P : [\omega_1]^{\omega_1} \to 2$ by

$$P(f) = \begin{cases} 0 & \Phi(f) \in [\omega_1]^{<\omega_1} \\ 1 & \Phi(f) \in N_1^J \end{cases}$$

By $\omega_1 \to (\omega_1)^{\omega_1}_2$, let $C \subseteq \omega_1$ with $|C| = \omega_1$ and homogeneous for P. If C is homogeneous for 0, then Φ gives an injection of $[C]^{\omega_1}$ (which is in bijection with $[\omega_1]^{\omega_1}$) into $[\omega_1]^{<\omega_1}$. This contradicts Theorem 4.5. Suppose C was homogeneous for P taking value 1. Then Φ is an injection of $[C]^{\omega_1}$ into N_1^J. From this, one obtains an injection of $[\omega_1]^\omega$ into N_1^J. But it was shown above that $\neg(|[\omega_1]^\omega| \leq |N_1^J|)$.

This completes the proof of the theorem. ☐

Note that the failure of $\mathsf{AD}_{\mathbb{R}}$ is important. With $\mathsf{AD}_{\mathbb{R}}$, one cannot have a set $\mathbb{X}$ that absorbs fragments of functions as in Fact 4.6. Moreover, the natural analog of the N_1^J sets under $\mathsf{AD}_{\mathbb{R}}$ are simply in bijection with $\mathbb{R} \times \omega_1$.

Fact 4.11. *Assume* $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$. *Let* $S \subseteq \mathrm{ON}$ *be a set of ordinals. Let* $N = \bigsqcup_{r \in \mathbb{R}} ((\omega_1^V)^+)^{L[S,r]}$. *Then* $|N| = |\mathbb{R} \times \omega_1|$.

Proof. Using a prewellordering on $\mathbb{R}$ of length ω_1, one can code subsets of ω_1 (and also subsets of $\omega_1 \times \omega_1$) by reals using the Moschovakis coding lemma. Define a relation $R \subseteq \mathbb{R} \times \mathbb{R}$ by $R(x,y)$ if and only if y codes a subset of $\omega_1 \times \omega_1$ which is a wellordering of ω_1 of ordertype $((\omega_1^V)^+)^{L[S,x]}$. By $\mathsf{AD}_{\mathbb{R}}$, let $F : \mathbb{R} \to \mathbb{R}$ be a uniformizing function for R. For each $x \in \mathbb{R}$, let $\Psi_x : \omega_1^V \to ((\omega_1^V)^+)^{L[S,x]}$ be the bijection induced by the wellordering on ω_1 coded by $F(x)$ according to the fixed prewellordering of length ω_1.

Define $\Phi : \mathbb{R} \times \omega_1 \to N$ by $\Phi(x, \alpha) = (x, \Psi_x(\alpha))$. Φ is a bijection. ☐

A natural question, under $\mathsf{AD}_{\mathbb{R}}$, is whether there an intermediate cardinal between $|[\omega_1]^{<\omega_1}|$ and $|[\omega_1]^{\omega_1}|$?

5. Cardinality of S_1

Recall the definition of S_1 from the introduction.

Definition 5.1. (Woodin) Let $S_1 = \{f \in [\omega_1]^{<\omega_1} : \sup(f) = \omega_1^{L[f]}\}$.

This section will establish several properties of the cardinality of S_1 under AD, $\mathsf{DC}_{\mathbb{R}}$, the statement that all sets of reals have ∞-Borel codes. It will be shown that S_1 does not inject into ${}^{\omega}\mathrm{ON}$, the class of ω-sequences of ordinals, which implies that $|[\omega_1]^{\omega}| < |[\omega_1]^{<\omega_1}|$.

Woodin [20] defines the set S_1 and establishes a very elaborate dichotomy which asserts that S_1 has a very special position among uncountable subsets of $[\omega_1]^{<\omega_1}$.

Fact 5.2. *([20] Theorem 19) (Woodin's S_1 dichotomy) Assume $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$. If $X \subseteq [\omega_1]^{<\omega_1}$ is uncountable, then either $|X| \leq |[\omega_1]^{\omega}|$ or $|S_1| \leq |X|$.*

The proof of the Woodin's S_1 dichotomy is very elaborate. This section will present some elementary arguments to establish several of the basic cardinal properties of S_1 under AD^+.

The next result shows that S_1 contains a copy of $\mathbb{R}$ but has no uncountable wellorderable subsets. These properties are mentioned in [20] without a proof, but for completeness, the brief arguments given in [4] will be reproduced below.

Fact 5.3. *(Woodin) Assume ZF. $|\mathbb{R}| \leq |S_1|$.*

Assume ZF and there are no uncountable wellorderable sets of reals. Then $\neg(\omega_1 \leq |S_1|)$.

Proof. For this proof, consider $\mathbb{R}$ as the collection of infinite subsets of ω. For each $r \in \mathbb{R}$, let $A_r = r \cup \{\alpha : \omega \leq \alpha < \omega_1^{L[r]}\}$. Let $f_r \in [\omega_1]^{<\omega_1}$ be the increasing enumeration of A_r. Note that $\omega_1^{L[f_r]} = \omega_1^{L[r]} = \sup(f_r)$. Thus $f_r \in S_1$. The function $\Phi : \mathbb{R} \to S_1$ defined by $\Phi(r) = f_r$ is an injection.

Suppose $\Phi : \omega_1 \to S_1$ is an injection.

Claim: $\sup\{\omega_1^{L[\Phi(\alpha)]} : \alpha < \omega_1\} = \omega_1$:

To see this, suppose not. Let $\epsilon = \sup\{\sup(\Phi(\alpha)) : \alpha < \omega_1\}$ and $\epsilon < \omega_1$. Since Φ maps into S_1, one has that $\sup\{\omega_1^{L[\Phi(\alpha)]} : \alpha < \omega_1\} = \sup\{\sup(\Phi(\alpha)) : \alpha < \omega_1\} = \epsilon < \omega_1$. Then Φ would be an injection into $[\epsilon + 1]^{<\epsilon+1}$ which is in bijection with $\mathbb{R}$. This is impossible since there are no uncountable wellorderable set of reals.

Let $\varpi : \omega_1 \times \omega_1 \to \omega_1$ be a constructible bijection, for instance the Gödel pairing function. Think of $S_1 \subseteq [\omega_1]^{<\omega_1}$ as subsets of ω_1. Then let $\tilde{\Phi} = \{\varpi(\alpha, \beta) : \beta \in \Phi(\alpha)\}$. Note that $\tilde{\Phi}$ is a subset of ω_1 which codes the function Φ. That is, $\Phi \in L[\tilde{\Phi}]$. Therefore, one has that $\Phi \in L[\Phi] \models \mathsf{ZFC}$.

Since there are no uncountable wellordered sets of reals, one has that $\omega_1^{L[\Phi]} < \omega_1$. By the claim, there is some $\alpha < \omega_1$ so that $\omega_1^{L[\Phi(\alpha)]} > \omega_1^{L[\Phi]}$. However, since $\Phi \in L[\Phi]$, $\Phi(\alpha) \in L[\Phi]$. Thus one has $\omega_1^{L[\Phi(\alpha)]} \leq \omega_1^{L[\Phi]}$. Contradiction. $\square$

Woodin's S_1-dichotomy (Fact 5.2) and Fact 5.3 are not sufficient to distinguish $|S_1|$ from $|\mathbb{R}|$, or $|[\omega_1]^{\omega}|$ from $|[\omega_1]^{<\omega_1}|$. Next, Theorem 5.7 will be shown in order to make these distinctions. (These cardinality distinctions seem to be implicit in [20].) The most interesting properties of S_1 require at least some of the properties of AD^+.

First one will fix a simple coding for elements of ${}^{<\omega_1}\omega_1$ by reals.

Definition 5.4. Let $\rho : \omega \times \omega \to \omega$ denote a fixed recursive and bijective pairing function. Thinking of $\mathbb{R}$ as ${}^{\omega}2$, one can code relations on ω by reals. That is, for each $x \in X$, let $R_x(n, m) \Leftrightarrow x(\rho(n, n)) = 1$. Recall WO is the collection of x so that R_x is a wellordering on ω.

For each $x \in \mathbb{R}$, let $x_n \in \mathbb{R}$ be defined by $x_n(k) = x(\rho(n, k))$.

Say that $x \in \mathsf{BS}$ if and only if $x_0 \in \mathrm{WO}$ and for all $n \in \omega$, $(x_1)_n \in \mathrm{WO}$. For each $x \in \mathsf{BS}$, let $\sigma_x : \mathrm{ot}(x_0) \to \omega_1$ defined by $\sigma(\alpha) = \beta$ if and only if for the unique $n \in \omega$ with rank α according to the wellordering R_{x_0}, $\mathrm{ot}((x_1)_n) = \beta$.

In this way, every $\sigma \in {}^{<\omega_1}\omega_1$ has a code $x \in \mathsf{BS}$ so that $\sigma_x = \sigma$.

Fact 5.5. *Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$, and all sets of reals have ∞-Borel codes. Suppose $R \subseteq {}^{<\omega_1}\omega_1 \times \kappa$, where $\kappa < \Theta$. Then there is a set of ordinals $S \subseteq \mathrm{ON}$ and a formula ϑ so that for all $\sigma \in {}^{<\omega_1}\omega_1$ and $\beta < \kappa$*

$$R(\sigma, \beta) \Leftrightarrow L[S, \sigma] \models \vartheta(S, \sigma, \beta).$$

If $\Phi : {}^{<\omega_1}\omega_1 \to {}^{\omega}\kappa$ is a function, then there is a set of ordinals S so that for all $\sigma \in {}^{<\omega_1}\omega_1$, $\Phi(\sigma) \in L[S, \sigma]$.

Proof. Since $\kappa < \Theta$, let $\preceq$ be a prewellordering on $\mathbb{R}$ of length κ. Let (J', ϕ') be an ∞-Borel code for $\preceq$. Let $\varphi : \mathbb{R} \to \kappa$ be the associated ranking function of $\preceq$.

Fix $R \subseteq {}^{<\omega_1}\omega_1 \times \kappa$. Let $\tilde{R} \subseteq \mathbb{R} \times \mathbb{R}$ be defined by

$$\tilde{R}(x, y) \Leftrightarrow x \in \mathsf{BS} \wedge R(\sigma_x, \varphi(y)).$$

Let (J'', ϕ'') be an ∞-Borel code for $\tilde{R}$.

Let J be a set of ordinals coding in some fixed constructible way the two sets of ordinals J' and J''. Let ${}_\omega\mathbb{O}_J$ be the finite support direct limit of the Vopěnka forcing $\langle {}_n\mathbb{O}_J, \pi_{n,m} : 0 < m \leq n < \omega\rangle$. Let S be a set of ordinals that codes $(J, {}_\omega\mathbb{O}_J)$.

Fix $\sigma \in {}^{<\omega}\omega_1$ and let $\mathbb{P}_\sigma$ denote the forcing $\mathrm{Coll}(\omega, \sup(\sigma))$. Observe that forcing with $\mathbb{P}_\sigma$ over $L[J, \sigma]$ canonically adds a surjection of ω onto $\sup(\sigma)$. From this, one can canonically obtain a bijection of ω with $\sup(f)$. Thus one can naturally produce an element of BS which codes σ in any $\mathbb{P}_\sigma$-generic extension of $L[S, \sigma]$. Let τ_σ be a $\mathbb{P}_\sigma$-name in $L[S, \sigma]$ for this naturally produce element of BS which codes σ.

Let ϑ be the following formula: $\vartheta(S, \sigma, \beta)$ if and only if

$$1_{\mathbb{P}_\sigma} \Vdash_{\mathbb{P}_\sigma} L[J, {}_\omega\mathbb{O}_J, \tau_\sigma] \models 1_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma}} \Vdash_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma}}$$
$$L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma, y] \models \phi''(J'', \tau_\sigma, y))$$

In the above, "$\varphi(y) = \beta$" is an abbreviation for a statement asserting that β is the rank of y in the prewellordering defined by the ∞-Borel code (J', ϕ').

It is very important that "$\varphi(y) = \beta$" is expressed in this way. The purpose of using $L(J, \mathbb{R})$ and Woodin's results on the symmetric collapse is to express "$\varphi(y) = \beta$," which can not be computed correctly by evaluating the prewellordering directly in an inner model of ZFC which can only contain countably many of the reals of the original universe satisfying determinacy.

Claim: For all $\sigma \in {}^{<\omega_1}\omega_1$, $R(\sigma, \beta)$ if and only if $L[S, \sigma] \models \vartheta(S, \sigma, \beta)$.

To see this: $(\Rightarrow)$ Let $p \in \mathbb{P}_\sigma$. Since $\sup(\sigma) < \omega_1$, the powerset of $\mathbb{P}_\sigma$ computed in $L[S, \sigma]$ is countable in the real universe satisfying determinacy. Thus there is a $G \subseteq \mathbb{P}_\sigma$ containing p which is $\mathbb{P}_\sigma$-generic over $L[S, \sigma]$. In $L[S, \sigma][G]$, $\tau_\sigma[G] \in \mathsf{BS}$ is a code for σ, that is $\sigma_{\tau_\sigma[G]} = \sigma$. In $L(J, \mathbb{R})$, there is a $y \in \mathbb{R}$ so that $\varphi(y) = \beta$. Hence $\tilde{R}(\tau_\sigma[G], y)$. Thus

$$L(J, \mathbb{R}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma[G], y] \models \phi''(J'', \tau_\sigma[G], y)).$$

By Fact 2.10,

$$L[J, {}_\omega\mathbb{O}_J, \tau_\sigma[G]] \models 1_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}} \Vdash_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}}$$
$$L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma[G], y] \models \phi''(J'', \tau_\sigma[G], y)).$$

In particular,

$$L[S, \sigma][G] \models L[J, {}_\omega\mathbb{O}_J, \tau_\sigma[G]] \models 1_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}} \Vdash_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}}$$
$$L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma[G], y] \models \phi''(J'', \tau_\sigma[G], y)).$$

By the forcing theorem and the fact that $p \in G$, there is a $q \leq_{\mathbb{P}_\sigma} p$ so that

$$L[S, \sigma] \models q \Vdash_{\mathbb{P}_\sigma} L[J, {}_\omega\mathbb{O}_J, \tau_\sigma] \models 1_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma}} \Vdash_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma}}$$
$$L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma, y] \models \phi''(J'', \tau_\sigma, y)).$$

Since $p \in \mathbb{P}_\sigma$ was arbitrary, one has that $L[S, \sigma]$ believes that $1_{\mathbb{P}_\sigma}$ forces the statement in the forcing language above. Thus $L[S, \sigma] \models \vartheta(S, \sigma, \beta)$.

$(\Leftarrow)$ Since the powerset of $\mathbb{P}_\sigma$ computed in $L[S, \sigma] \models \mathsf{ZFC}$ is countable in the real world satisfying AD, there exists a $G \in V$ which is $\mathbb{P}_\sigma$-generic over $L[S, \sigma]$. Note that by the explicit definition of the coding used in BS, one has $\tau_\sigma[G] \in \mathsf{BS}$ and $\sigma_{\tau_\sigma[G]} = \sigma$ by absoluteness. Since $L[S, \sigma] \models \vartheta(S, \sigma, \beta)$, one has

$$L[S, \sigma][G] \models L[J, {}_\omega\mathbb{O}_J, \tau_\sigma[G]] \models 1_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}} \Vdash_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}}$$
$$L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma[G], y] \models \phi''(J'', \tau_\sigma[G], y)).$$

Since G is a set in the real world V,

$$V \models L[J, {}_\omega\mathbb{O}_J, \tau_\sigma[G]] \models 1_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}} \Vdash_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}}$$
$$L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma[G], y] \models \phi''(J'', \tau_\sigma[G], y)).$$

Fact 2.10 implies

$$L(J,\mathbb{R}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma[G], y] \models \phi''(J'', \tau_\sigma[G], y))$$

Since (J'', ϕ'') is the ∞-Borel code for $\tilde{R}$, one has that $\tilde{R}(\tau_\sigma[G], y)$. By definition of $\tilde{R}$ and the fact that $\tau_\sigma[G] \in \mathsf{BS}$ is a code for σ, $R(\sigma,\beta)$ holds.

This concludes the proof of the claim and hence the first statement in the fact.

Now suppose $\Phi : {}^{<\omega_1}\omega_1 \to {}^{\omega}\kappa$ is a function. Let $R(\sigma, n, \beta)$ assert that $\Phi(\sigma)(n) = \beta$. By the first part, there is a set of ordinals $S \subseteq \mathrm{ON}$ and a formula ϑ so that

$$R(\sigma, n, \beta) \Leftrightarrow L[S,\sigma] \models \vartheta(S, \sigma, n, \beta).$$

Then by comprehension in $L[S,\sigma]$, one has that $\Phi(\sigma) \in L[S,\sigma]$. □

A consequence of Fact 5.5 is that (under $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ and all sets of reals have ∞-Borel codes) every subset A of $[\omega_1]^{<\omega_1}$ has an ∞-Borel code (S, φ) in the sense that $\sigma \in A$ if and only if $L[S,\sigma] \models \varphi(S,\sigma)$.

A key idea of the previous argument was to use ∞-Borel codes to go into a suitable $L(J,\mathbb{R}) \models \mathsf{ZF}+\mathsf{AD}+\mathsf{DC}$ and then by considering the forcing language $\mathrm{Coll}(\omega, \sup(\sigma))$, one can speak of a canonical real coding σ. For $f \in {}^{\omega}\kappa$, there are various ways to code f by a real; however, it is unclear where to find or how to uniformly speak of a real coding f within the ZFC model $\mathrm{HOD}_J^{L(J,\mathbb{R})} = L[J, {}_{\omega}\mathbb{O}_J]$.

One can only prove the following weaker result which is quite similar to Fact 4.6:

Fact 5.6. *Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ and all sets of reals have an ∞-Borel code. Let $\Phi : {}^{\omega}\kappa \to {}^{<\omega_1}\omega_1$ be a partial function, where $\kappa < \Theta$. Then there is a set of ordinals $S \subseteq \mathrm{ON}$ so that for all $z \in \mathbb{R}$, one has that for all $f \in \mathrm{dom}(\Phi) \cap L[S,z]$, $\Phi(f) \in L[S,z]$.*

Proof. Since $\kappa < \Theta$, let $\preceq$ be a prewellordering of $\mathbb{R}$ of length κ. Let φ be it associated ranking function. Let (J', ϕ') denote the ∞-Borel code for $\preceq$.

For each $x \in \mathbb{R}$, let x_n denote the n^{th} section of x. Define $f_x \in {}^{\omega}\kappa$ by $f_x(n) = \varphi(x_n)$. In this way, every $f \in {}^{\omega}\kappa$ has an $x \in \mathbb{R}$ so that $f_x = f$.

Define a relation $R \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by $R(x,v,w)$ if and only if

$$f_x \in \mathrm{dom}(\Phi) \wedge v, w \in \mathrm{WO} \wedge \mathrm{ot}(v) \in \mathrm{dom}(\Phi(f_x)) \wedge \Phi(f_x)(\mathrm{ot}(v)) = \mathrm{ot}(w).$$

Let (J'', ϕ'') be an ∞-Borel code for R.

Let J be a set of ordinals that codes J' and J'' in some fixed constructible manner.

Now work in $L(J,\mathbb{R}) \models \mathsf{ZF}+\mathsf{AD}+\mathsf{DC}$. In $L(J,\mathbb{R})$, R is OD_J. Let ς be a formula with ordinal parameters so that $L(J,\mathbb{R}) \models R(x,v,w) \Leftrightarrow L(J,\mathbb{R}) \models \varsigma(J,x,v,w)$. In $L(J,\mathbb{R})$, let ${}_{\omega}\mathbb{O}_J$ denote finite support direct limit of J-Vopěnka forcing.

Define $\vartheta(z, J, f, \alpha, \beta)$ by

$$1_{{}_{\omega}\mathbb{O}_J/G_z^1} \Vdash_{{}_{\omega}\mathbb{O}_J/G_z^1} L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists x, v, w)((\forall n)(\varphi(x_n) = f(n) \wedge \alpha = \mathrm{ot}(v) \wedge \beta = \mathrm{ot}(w) \wedge \varsigma(J,x,v,w))).$$

Then for any $z \in \mathbb{R}$, by Fact 2.10, one can conclude that for all $f \in L[J, {}_{\omega}\mathbb{O}_J, z]$ that $L(J,\mathbb{R}) \models \Phi(f)(\alpha) = \beta$ if and only if $L[J, {}_{\omega}\mathbb{O}_J, z] \models \vartheta(z, J, f, \alpha, \beta)$. By comprehension, one has that $\Phi(f) \in L[J, {}_{\omega}\mathbb{O}_J, z]$. □

Theorem 5.7. *Assume $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ and all sets of reals have ∞-Borel codes. Then there is no injection of S_1 into ${}^{\omega}\mathrm{ON}$, the class of ω-sequences of ordinals.*

Proof. Suppose $\Phi : S_1 \to {}^{\omega}\mathrm{ON}$ is an injection. Since $\mathbb{R}$ surjects onto ${}^{<\omega_1}\omega_1$ (for example, BS and the coding from Definition 5.4), one has that $\mathbb{R}$ surjects onto $S_1 \subseteq {}^{<\omega_1}\omega_1$. Thus one can show that $A = \bigcup\{\mathrm{rang}(\Phi(\sigma)) : \sigma \in S_1\}$ is a collection of ordinals which is a surjective image of $\mathbb{R}$. Thus the Mostowski collapse of A is some ordinal $\kappa < \Theta$. Hence from Φ, one can derive an injection $\Psi : S_1 \to {}^{\omega}\kappa$. Since Ψ is an injection, $\Psi^{-1} : {}^{\omega}\kappa \to S_1$ is a partial function.

Let $S \subseteq \mathrm{ON}$ be a set of ordinals satisfying Fact 5.5 for the function Ψ and Fact 5.6 for the partial function Ψ^{-1}.

Since ω_1 is measurable in $L[S] \models \mathsf{ZFC}$, let $\zeta < \omega_1$ be an inaccessible cardinal of $L[S]$. Let $\mathrm{Coll}(\omega, <\zeta)$ be the Lévy collapse of ζ. Since $\zeta < \omega_1$ and $L[S] \models \mathsf{ZFC}$, the powerset of $\mathrm{Coll}(\omega, <\zeta)$ is countable in the real world satisfying AD. Thus in the real world, there is a $G \subseteq \mathrm{Coll}(\omega, <\zeta)$ which is $\mathrm{Coll}(\omega, <\zeta)$-generic over $L[S]$.

From G and its generic surjection of ζ onto ζ, one can find a cofinal function $g : \zeta \to \zeta$ so that $L[g] = L[G]$. Since $L[g] = L[G]$, $\omega_1^{L[g]} = \omega_1^{L[G]} = \zeta = \sup(g)$. Thus $g \in S_1$.

By the property of S from Fact 5.5, $\Psi(g) \in L[S, g]$. Since $\Psi(g) \in {}^\omega\kappa$, and using the main property of the Lévy collapse $\text{Coll}(\omega, <\zeta)$, there exists some $\xi < \zeta$ so that $\Psi(g) \in L[S][G \upharpoonright \xi]$. By using the $\text{Coll}(\omega, \xi)$-generic obtained from G, one sees that there is a real $z \in L[S][G]$ so that $L[S][G \upharpoonright \xi] \subseteq L[S][z]$. Thus $\Psi(g) \in L[S, z]$. By the property of S from Fact 5.6 for the partial function Ψ^{-1}, one has that $g = \Psi^{-1}(\Psi(g)) \in L[S, z]$. Thus $L[S][G] = L[S][g] \subseteq L[S][z] \subseteq L[S][G \upharpoonright \xi + 1]$. It is impossible that $L[S][G] = L[S][G \upharpoonright \xi + 1]$ for any $\xi < \zeta$.

It has been shown that no such injection can exist. $\square$

Theorem 5.8. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$ *and all sets of reals have* ∞*-Borel codes. Then* $|\mathbb{R}| < |S_1|$ *and* $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$.

Proof. Since $|\mathbb{R}| = |{}^\omega\omega|$, Theorem 5.7 implies that there is no injection of S_1 into $\mathbb{R}$ or $[\omega_1]^\omega$. Thus $|\mathbb{R}| < |S_1|$. Since $S_1 \subseteq [\omega_1]^{<\omega_1}$ and S_1 does not inject into $[\omega_1]^\omega$, one has that $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$. $\square$

6. Countable Powerset Operation

Definition 6.1. Let X be a set. Let $\mathscr{P}_{\omega_1}(X) = \{A \subseteq X : |A| \leq \aleph_0\}$ be the collection of countable subsets of X.

This section will discuss the question of what cardinality properties of $\mathscr{P}_{\omega_1}(X)$ must have already been exhibited by X. For example, it will be shown that if κ is a cardinal and κ injects into $\mathscr{P}_{\omega_1}(X)$, then κ already injected into X. It will also be shown that if $\mathscr{P}(\omega_1)$ injects into $\mathscr{P}_{\omega_1}(X)$, then $\mathbb{R} \sqcup \omega_1$ already injects into X.

Fact 6.2. *(Woodin's perfect set dichotomy) Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$ *and all sets of reals have an* ∞*-Borel code. Let E be an equivalence relation on $\mathbb{R}$. Then exactly one of the following holds:*
(1) $\mathbb{R}/E$ is wellorderable.
(2) $\mathbb{R}$ injects into $\mathbb{R}/E$.

Morover, if $\mathbb{R}/E$ is wellorderable and if (S, φ) is an ∞-Borel code for E, then there is a uniform procedure that takes (S, φ) to an $\text{OD}_S^{L(S,\mathbb{R})}$ *wellordering of $\mathbb{R}/E$.*

Proof. This result is attributed to Woodin by Hjorth [10]. A proof of these results can be found [2] Section 8 and [4] which give particular attention to the uniformity aspects of (1) and (2). $\square$

Definition 6.3. Let X be a set. Let $\mathscr{P}_{\text{WO}}(X) = \{A \subseteq X : A \text{ is wellorderable}\}$. Note that $\mathscr{P}_{\omega_1}(X) \subseteq \mathscr{P}_{\text{WO}}(X)$.

Fact 6.4. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$ *and all sets of reals have* ∞*-Borel codes. Let $\kappa < \Theta$ and E be an equivalence relation on $\mathbb{R}$. Suppose $\Phi : \kappa \to \mathscr{P}_{\text{WO}}(\mathbb{R}/E)$ is a function. Then there is a sequence $\langle <_\alpha : \alpha < \kappa \rangle$ so that $<_\alpha$ is a wellordering of $\Phi(\alpha)$ for each $\alpha < \kappa$.*

Proof. Let (J_0, ϕ_0) be an ∞-Borel code for E. Let $\preceq$ be a prewellordering on $\mathbb{R}$ of length κ. Let $\varsigma : \mathbb{R} \to \kappa$ be the ranking function of $\preceq$. Let (J_1, ϕ_1) be an ∞-Borel code for $\preceq$. Define $R \subseteq \mathbb{R} \times \mathbb{R}$ by $R(x, y) \Leftrightarrow [y]_E \in \Phi(\varsigma(x))$. Let (J_2, ϕ_2) be an ∞-Borel code for R. Let J be a set of ordinals that codes J_0, J_1, and J_2.

Now work in $L(J, \mathbb{R}) \models \mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$. Note that from J, one can recover in $L(J, \mathbb{R})$, the sets E, $\preceq$, R, and Φ. In fact, all these sets are $\text{OD}_J^{L(J,\mathbb{R})}$. Thus for each $\alpha < \kappa$, $\Phi(\alpha)$ is $\text{OD}_J^{L(J,\mathbb{R})}$ with a witnessing definition obtained uniformly in α. Consider $\bigcup \Phi(\alpha) \subseteq \mathbb{R}$. Let $E_\alpha = E \upharpoonright \bigcup \Phi(\alpha)$. E_α is $\text{OD}_J^{L(J,\mathbb{R})}$ uniformly from the definitions witnessing E and $\Phi(\alpha)$ is $\text{OD}_J^{L(J,\mathbb{R})}$. The $\text{OD}_J^{L(J,\mathbb{R})}$ set E_α has an $\text{OD}_J^{L(J,\mathbb{R})}$ ∞-Borel code obtained uniformly from a definition witnessing that E_α is $\text{OD}_J^{L(J,\mathbb{R})}$. (This follows from an application of Fact 2.10.) If the ∞-Borel codes for each equivalence relation in $\langle E_\alpha : \alpha < \kappa \rangle$ can be obtained uniformly, then Fact 6.2 states that one can uniformly produce a sequence of wellorderings $\langle <_\alpha : \alpha < \kappa \rangle$ so that each $<_\alpha$ is a wellordering of $(\bigcup \Phi(\alpha))/E_\alpha$ which is $\Phi(\alpha)$. $\square$

The following is the "Boldface GCH". It was established first in $L(\mathbb{R})$ by Steel. Woodin extended this result to AD^+.

Fact 6.5. *(Woodin) Assume* $\mathsf{ZF} + \mathsf{AD}^+$. *Let* $\kappa < \Theta$ *be a cardinal. If* $X \subseteq \mathscr{P}(\kappa)$ *is wellorderable, then* $|X| \leq \kappa$.

Theorem 6.6. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and all sets of reals have* ∞*-Borel codes. Suppose* $\kappa < \Theta$ *is a cardinal with the property that for all* $\delta < \kappa$*, there is no* κ *length sequence of distinct subsets of* $\mathscr{P}(\delta)$*. Let* X *be a set so that there is a surjection* $\pi : \mathbb{R} \to X$*. Then* $\kappa \leq |\mathscr{P}_{\mathrm{WO}}(X)|$ *implies that* $\kappa \leq |X|$*. In particular,* $\kappa \leq |\mathscr{P}_{\omega_1}(X)|$ *implies* $\kappa \leq |X|$.

Assuming $\mathsf{ZF} + \mathsf{AD}^+$*, for all cardinals* $\kappa < \Theta$ *and all sets* X *which are surjective images of* $\mathbb{R}$*,* $\kappa \leq |\mathscr{P}_{\mathrm{WO}}(X)|$ *implies* $\kappa \leq |X|$*. In particular,* $\kappa \leq |\mathscr{P}_{\omega_1}(X)|$ *implies* $\kappa \leq |X|$.

Proof. Define an equivalence relation on $\mathbb{R}$ be $x\ E\ y$ if and only if $\pi(x) = \pi(y)$. Then X is in bijection with $\mathbb{R}/E$. Thus one will work with $\mathbb{R}/E$ rather than directly with X. If $\kappa \leq |\mathscr{P}_{\mathrm{WO}}(X)|$, then one has an injection $\Phi : \kappa \to \mathscr{P}_{\mathrm{WO}}(\mathbb{R}/E)$. By Fact 6.4, let $\langle <_\alpha : \alpha < \omega_1\rangle$ be a sequence so for each $\alpha < \kappa$, $<_\alpha$ is a wellordering of $\Phi(\alpha)$.

By using the usual wellordering on κ and the sequence of wellorderings $\langle <_\alpha : \alpha < \kappa\rangle$, one can define a wellordering of $\bigcup \Phi[\kappa] = \bigcup\{\Phi(\alpha) : \alpha < \kappa\}$. Thus $|\bigcup \Phi[\kappa]|$ is a wellordered cardinal.

The claim is that $|\bigcup \Phi[\kappa]| \geq \kappa$. To see this: Suppose $|\bigcup \Phi[\kappa]| = \delta$ for some $\delta < \kappa$. Let $\Psi : \bigcup \Phi[\kappa] \to \delta$ be a bijection. Then $\Gamma(\alpha) = \Psi[\Phi(\alpha)] = \{\Psi(x) : x \in \Phi(\alpha)\}$ is an injection of κ into $\mathscr{P}(\delta)$. However, by assumption, there are no κ-length sequences of distinct subsets of $\mathscr{P}(\delta)$. The claim has been shown.

The claim immediately implies that $\kappa \leq |\mathbb{R}/E| = |X|$.

In the setting of $\mathsf{ZF} + \mathsf{AD}^+$, Fact 6.5 implies that for every cardinal $\delta < \kappa$, every wellorderable set of subsets of δ has cardinality δ. Thus κ can not inject into $\mathscr{P}(\delta)$. The second result now follows from the first. ☐

Corollary 6.7. *Assume* $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \mathsf{AD}$ *and all sets of reals have* ∞*-Borel codes. Let* X *be a set which is a surjective image of* $\mathbb{R}$*. Then* $\omega_1 \leq |\mathscr{P}_{\mathrm{WO}}(X)|$ *implies* $\omega_1 \leq |X|$*. In particular,* $\omega_1 \leq |\mathscr{P}_{\omega_1}(X)|$ *implies* $\omega_1 \leq |X|$.

To analyze the structure of the cardinality of sets X so that $|[\omega_1]^{\omega_1}| \leq |\mathscr{P}_{\omega_1}(X)|$, one needs an almost everywhere (with respect to the strong partition measure) continuity result for functions $\Phi : [\omega_1]^{\omega_1} \to \omega_1$. The result holds in $\mathsf{ZF} + \mathsf{AD}$ and its proof is quite different from the method used in this article.

Fact 6.8. *([5]) Assume* $\mathsf{ZF} + \mathsf{AD}$*. For every function* $\Phi : [\omega_1]^{\omega_1} \to \omega_1$*, there is a club* $C \subseteq \omega_1$ *so that* $\Phi \restriction [C]^{\omega_1}_* \to \omega_1$ *is continuous.*

If $C \subseteq \omega_1$ is club, then $[C]^{\omega_1}_*$ is the colletion of $f \in [C]^{\omega_1}$ which are of the correct type, i.e. has uniform cofinality ω and discontinuous everywhere. One can check that $|[\omega_1]^{\omega_1}| = |[C]^{\omega_1}_*|$. $\Phi \restriction [C]^{\omega_1}_*$ being continuous means that for all $f \in [C]^{\omega_1}_*$, there is an $\alpha < \omega_1$ so that for all $g \in [C]^{\omega_1}_*$, if $f \restriction \alpha = g \restriction \alpha$, then $\Phi(f) = \Phi(g)$.

Zapletal had also asked the authors that if one partitions $[\omega_1]^{\omega_1}$ into ω_1 many sets, then must one of the pieces have cardinality $|[\omega_1]^{\omega_1}|$, under determinacy assumptions. The almost everywhere continuity property gives a positive answer.

Fact 6.9. *([5]) Assume* $\mathsf{ZF} + \mathsf{AD}$*. Let* $\langle X_\alpha : \alpha < \omega_1\rangle$ *be such that each* $X_\alpha \subseteq [\omega_1]^{\omega_1}$ *and* $\bigcup_{\alpha<\omega_1} X_\alpha = [\omega_1]^{\omega_1}$*, then there exists some* $\alpha < \omega_1$ *so that* $|X_\alpha| = |[\omega_1]^{\omega_1}|$.

Theorem 6.10. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and all sets of reals have an* ∞*-Borel code. Let* X *be a set which is a surjective image of* $\mathbb{R}$*. If* $|[\omega_1]^{\omega_1}| \leq |\mathscr{P}_{\omega_1}(X)|$*, then* $|\mathbb{R} \sqcup \omega_1| \leq |X|$.

Proof. Let $\pi : \mathbb{R} \to X$ be a surjection. Again define an equivalence relation on $\mathbb{R}$ by $x\ E\ y$ if and only if $\pi(x) = \pi(y)$. Since $|X| = |\mathbb{R}/E|$, one will work with the quotient of E. Now suppose $\Phi : [\omega_1]^{\omega_1} \to \mathscr{P}_{\omega_1}(\mathbb{R}/E)$ is an injection.

Note that $|[\omega_1]^{\omega_1}| \leq |\mathscr{P}_{\omega_1}(\mathbb{R}/E)|$ implies, in particular, that $\omega_1 \leq |\mathbb{R}/E|$ by Corollary 6.7. Suppose $\neg(|\mathbb{R}| \leq |\mathbb{R}/E|)$. Then the Woodin perfect set dichotomy (Fact 6.2) implies that $\mathbb{R}/E$ is wellorderable and hence there is some cardinal κ so that $|\mathbb{R}/E| = \kappa$. Let $\Lambda : \mathbb{R}/E \to \kappa$ be a bijection.

Let $\Gamma : [\omega_1]^{\omega_1} \to [\kappa]^{<\omega_1}$ be defined by $\Gamma(f) = \Lambda[\Phi(f)]$. $\Phi(f) \in \mathscr{P}_{\omega_1}(\mathbb{R}/E)$ so $\Phi(f)$ is a countable subset of $\mathbb{R}/E$. Thus $\Lambda[\Phi(f)] = \{\Lambda(x) : x \in \Phi(f)\}$ is a countable subset of κ.

Let $\mathrm{ot}(\Lambda[\Phi(f)])$ be the ordertype of this countable subset of κ in the usual ordering on κ, which of course is a countable ordinal. Note that $\mathrm{ot} \circ \Gamma : [\omega_1]^{\omega_1} \to \omega_1$.

By letting $X_\alpha = (\mathrm{ot} \circ \Gamma)^{-1}(\{\alpha\})$, one has that $[\omega_1]^{\omega_1} = \bigcup_{\alpha<\omega_1} X_\alpha$. By Fact 6.9, there is some $\alpha < \omega_1$ so that $|X_\alpha| = |[\omega_1]^{\omega_1}|$. Let $\Xi : [\omega_1]^{\omega_1} \to X_\alpha$ be a bijection.

Since $\alpha < \omega_1$, let $B : \omega \to \alpha$ be a bijection. For each $f \in [\kappa]^\alpha$, define $\Sigma(f) \in [\kappa]^\omega$ by recursion as follow: $\Sigma(f)(0) = f(B(0))$ and $\Sigma(f)(n+1) = \Sigma(f)(n) + f(B(n+1))$. The map $\Sigma : [\kappa]^\alpha \to [\kappa]^\omega$ is an injection. Then $\Sigma \circ \Gamma \circ \Xi : [\omega_1]^{\omega_1} \to [\kappa]^\omega$ is an injection. Since $|S_1| \leq |[\omega_1]^{\omega_1}|$, one can derive an injection of S_1 into $[\kappa]^\omega$. This violates Theorem 5.7.

It has been shown that $|\mathbb{R}| \leq |\mathbb{R}/E| = |X|$. Thus $|\mathbb{R} \sqcup \omega_1| \leq |\mathbb{R}/E| = |X|$. $\square$

7. The Cardinalities Below $\mathbb{R} \times \omega_1$

This section will investigate the cardinalities below $\mathbb{R} \times \omega_1$. Assuming $\mathsf{AD}_\mathbb{R}$, a uniformization argument will show there are only four uncountable cardinalities below $|\mathbb{R} \times \omega_1|$. In models of the form AD^+, $\neg\mathsf{AD}_\mathbb{R}$, and $V = L(\mathscr{P}(\mathbb{R}))$, this section will show that there are many intermediate cardinalities below $\mathbb{R} \times \omega_1$. This large family of cardinalities will correspond to the ultrapower of ω_1 by the J-constructibility degree measure for a certain set of ordinals J.

Definition 7.1. Let $\Phi : \mathbb{R} \to \omega_1$. Define $\bigsqcup \Phi = \{(r, \alpha) : \alpha < \Phi(r)\}$, which is a $\mathbb{R}$-index disjoint union of countable ordinals given by the function Φ.

Fact 7.2. *Assume* AD. *For every* $\Phi : \mathbb{R} \to \omega_1$, ω_1 *does not inject into* $\bigsqcup \Phi$. *If* $\{r : \Phi(r) > 0\}$ *is uncountable, then* $|\mathbb{R}| \leq |\bigsqcup \Phi|$.

Proof. Let $\pi_1 : \mathbb{R} \times \omega_1 \to \mathbb{R}$ denote the projection onto the first coordinate. Suppose $\Psi : \omega_1 \to \bigsqcup \Phi$ is an injection. Since for all $r \in \mathbb{R}$, $\Phi(r) < \omega_1$, the set of α so that $\pi_1(\Psi(\alpha)) = r$ is countable. Thus $X = \{r : (\exists \alpha < \omega_1)(\pi_1(\Psi(\alpha)) = r)\}$ is an uncountable set of reals. X is wellorderable by setting $x \sqsubset y$ if and only if the least α so that $\pi_1(\Psi(\alpha)) = x$ is less than the least α so that $\pi_1(\Psi(\alpha)) = y$. This is a contradiction since there are no uncountable wellorderable sequence of reals.

Suppose $Y = \{r : \Phi(r) > 0\}$ is uncountable. By the perfect set property, let $\Lambda' : \mathbb{R} \to Y$ be an injection. Then $\Lambda : \mathbb{R} \to \bigsqcup \Phi$ defined by $\Lambda(r) = (\Lambda'(r), 0)$ is an injection. $\square$

Fact 7.3. *For all* $X \subseteq \mathbb{R} \times \omega_1$ *such that* $\neg(\omega_1 \leq |X|)$, *there is a* $\Phi : \mathbb{R} \to \omega_1$ *so that* $X \approx \bigsqcup \Phi$.

Proof. For each $r \in \mathbb{R}$, let $X_r = \{\alpha : (r, \alpha) \in X\}$. Since ω_1 does not inject into X, X_r is countable. Let δ_r be the ordertype of X_r. Let $\varpi_r : X_r \to \delta_r$ be the associated collapse map. Let $\Phi : \mathbb{R} \to \omega_1$ be defined $\Phi(r) = \delta_r$. Define $\Lambda : X \to \bigsqcup \Phi$ by $\Lambda(x) = (\pi_1(x), \varpi_{\pi_1(x)}(\pi_2(x)))$, where $\pi_1 : \mathbb{R} \times \omega_1 \to \mathbb{R}$ and $\pi_2 : \mathbb{R} \times \omega_1 \to \omega_1$ are the projections onto the first and second coordinate, respectively. Λ is a bijection. $\square$

Fact 7.4. *Assume* AD. *For every* $X \subseteq \mathbb{R} \times \omega_1$, *one of the following holds:*

(1) $|X| = |\mathbb{R} \times \omega_1|$.
(2) $|X| = \aleph_1$.
(3) X *is an uncountable set such that* $\neg(\omega_1 \leq |X|)$.
(4) There is an uncountable Y *so that* $\neg(\omega_1 \leq |Y|)$ *and* $|X| = |Y \sqcup \omega_1|$.
(5) $|X| \leq \aleph_0$.

Proof. Let $X \subseteq \mathbb{R} \times \omega_1$. For each $r \in \mathbb{R}$, let $X_r = \{\alpha : (r, \alpha) \in X\}$. Let $\delta_r = \mathrm{ot}(X_r)$. For each $r \in \mathbb{R}$, let $\varpi_r : X_r \to \delta_r$ denote the collapse map.

Let $A = \{r : |X_r| = \aleph_1\}$.

Suppose A is uncountable. Let $\Psi : \mathbb{R} \to A$ be a bijection which exists by the perfect set property and the Cantor-Schröder-Bernstein theorem. Define $\Lambda : \mathbb{R} \times \omega_1 \to X$ by $\Lambda(r, \alpha) = (\Psi(r), \varpi^{-1}_{\Psi(r)}(\alpha))$. Λ is a bijection. Hence $|X| = |\mathbb{R} \times \omega_1|$. This gives possibility (1).

Hence assume A is countable. Then $\mathbb{R} \setminus A$ is uncountable. Let $\Phi : \mathbb{R} \to \omega_1$ be defined by

$$\Phi(r) = \begin{cases} \delta_r & r \notin A \\ 0 & \text{otherwise} \end{cases}$$

Let $\Lambda : \bigsqcup \Phi \to X$ be defined by $\Lambda(r, \alpha) = (r, \varpi_r^{-1}(\alpha))$. Λ is an injection. In fact, it is a bijection onto $X \cap (\mathbb{R} \setminus A \times \omega_1)$. Thus $X \cap (\mathbb{R} \setminus A \times \omega_1)$ does not contain a copy of ω_1 by Fact 7.2. If $B = \{r \in \mathbb{R} \setminus A : \Phi(r) > 0\}$

is uncountable, then $X \cap (\mathbb{R} \setminus A \times \omega_1)$ is an uncountable set without a copy of ω_1. If B is countable, then since a countable union of countable ordinals is countable, $X \cap (\mathbb{R} \setminus A \times \omega_1)$ is a countable set.

Suppose A is nonempty. One can show that a countable union of sets in bijection with ω_1 is in bijection with ω_1. Thus $X \cap (A \times \omega_1) \approx \omega_1$.

Note that $X = X \cap (A \times \omega_1) \sqcup X \cap (\mathbb{R} \setminus A \times \omega_1)$. If A is empty and B is countable, then $|X| \leq \aleph_0$ which gives case (5). If A is empty and B is uncountable, then X is an uncountable set without a copy of ω_1 which gives case (3). If A is nonempty and B is countable, then $|X| = \aleph_1$ which gives case (2). If A is nonempty and B is uncountable, then X is a union of two sets: one set which is in bijection with ω_1 and another set which is an uncountable set without a copy of ω_1, which gives case (4). ☐

Fact 7.5. *Assume* $\mathsf{AD}_{\mathbb{R}}$. *Every* $X \subseteq \mathbb{R} \times \omega_1$ *such that* $\neg(\omega_1 \leq |X|)$ *injects into* $\mathbb{R}$.

Proof. Let WO be the set of reals coding wellorderings with underlying domain ω.

Let $X_r = \{\alpha : (r, \alpha) \in X\}$. Let $\delta_r = \text{ot}(X_r)$. Let $\varpi_r : X_r \to \delta_r$ be the collapse map of X_r.

Define $R \subseteq \mathbb{R} \times \mathbb{R}$ by $R(x, w)$ if and only if $w \in \text{WO}$ and $\text{ot}(w) = \delta_x$. By $\mathsf{AD}_{\mathbb{R}}$, let $\Sigma : \mathbb{R} \to \mathbb{R}$ be a uniformization for R. For each $w \in \text{WO}$, for each $\alpha < \text{ot}(w)$, let α^w denote the element of ω with rank α according to w. (If w codes a finite ordinal, then let $n^w = n$.)

Define $\Lambda : X \to \mathbb{R} \times \omega$ by $\Lambda(x) = (\pi_1(x), (\varpi_{\pi_1(x)}(\pi_2(x))^{\Sigma(\pi_1(x))})$. Λ is an injection. Since $|\mathbb{R} \times \omega| = |\mathbb{R}|$, the proof is complete. ☐

Corollary 7.6. *Assume* $\mathsf{AD}_{\mathbb{R}}$. *The uncountable cardinals below* $|\mathbb{R} \times \omega_1|$ *are* $|\mathbb{R}|$, $\aleph_1$, $|\mathbb{R} \sqcup \omega_1|$, *and* $|\mathbb{R} \times \omega_1|$.

Proof. This follows from Fact 7.4 and Fact 7.5.

This is also a consequence of Woodin's dichotomy below $|[\omega_1]^\omega|$ ([20] Theorem 18) which is proved under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$. However, the proof above under $\mathsf{AD}_{\mathbb{R}}$ uses an elementary uniformization argument while Woodin's stronger result uses very sophisticated AD^+ techniques. ☐

One will need several facts about J-constructibility degrees and J-pointed perfect trees:

Definition 7.7. Let J be a set of ordinal. A perfect tree $p \subseteq {}^{<\omega}2$ is J-pointed if and only if for all $x \in [p]$, $p \leq_J x$.

Definition 7.8. Let p be a perfect tree on 2. $s \in p$ is a split node of p if and only if $s\hat{}0, s\hat{}1 \in p$.

By recursion, define $\Xi^p : {}^{<\omega}2 \to {}^{<\omega}2$ by: $\Xi^p(\emptyset)$ be the least split node of p. If $\Xi^p(s)$ has been defined, then let $\Xi^p(s\hat{}i)$ be the least split node of p extending $\Xi^p(s)\hat{}i$.

Define $\Upsilon^p : {}^\omega 2 \to [p]$ by letting $\Upsilon^p(r) = \bigcup_{n \in \omega} \Xi^p(r \upharpoonright n)$. Υ^p is called the canonical homeomorphism between ${}^\omega 2$ and $[p]$.

Fact 7.9. *(Martin) Assume* AD. *For all* $A \subseteq \mathbb{R}$, A *or* $\mathbb{R} \setminus A$ *contains the body of a Turing-pointed tree. Hence for any set of ordinals* J, A *or* $\mathbb{R} \setminus A$ *contains the body of a* J*-pointed tree.*

(Martin) The Martin Turing degree measure, μ, *and the* J*-degree measure,* μ_J, *is a countable complete ultrafilter.*

Proof. Let $A \subseteq \mathbb{R}$. Let G_A denote the game

$$G_A \quad \begin{array}{lccccccc} \text{I} & x_0 & & x_2 & & x_4 & & \cdots \\ \text{II} & & x_1 & & x_3 & & x_5 & \cdots \end{array} \quad x$$

where Player 1 wins if and only if $x \in A$.

Suppose Player 1 has a winning strategy σ. For any $r \in \mathbb{R}$, let $\sigma(r)$ be Player 1's response using σ when Player 2 plays r. Similarly, if $t \in {}^{<\omega}2$, then $\sigma(t)$ is Player 1's response using σ when Player 2 plays t in the finite partial run of G_A.

Thinking of σ as an element of ${}^\omega 2$, let σ_n denote the n^{th} bit of σ. Let $Z = \{x \in {}^\omega 2 : (\forall n)(x(2n) = \sigma_n)\}$. Note that Z is the body of a perfect tree.

Let p be the $\subseteq$-downward closure of $\{\sigma(x \upharpoonright n) \oplus (x \upharpoonright n) : n \in \omega \wedge x \in Z\}$. (Recall that if $s, t \in {}^{<\omega}\omega$ of the same length k, then $s \oplus t$ has length $2k$ where $(s \oplus t)(2j) = s(j)$ and $(s \oplus t)(2j+1) = t(j)$ whenever $j < k$. If $x, y \in {}^\omega\omega$, one can similarly define $x \oplus y$.) Observe that p is a perfect tree and p is Turing reducible to σ. Suppose $f \in [p]$. There is an $x \in Z$ so that $f = \sigma(x) \oplus x$. Since σ is a Player 1 winning strategy, $f = \sigma(x) \oplus x \in A$. This shows that $[p] \subseteq A$. Note that p is Turing reducible to f since $\sigma_n = f(4n+1)$ for all n. p is a Turing pointed tree. Every Turing pointed tree is a J-pointed tree.

If Player 2 has a winning strategy τ, then a similar argument shows that ${}^{\omega}2 \setminus A$ contains the body of a Turing pointed tree.

Suppose $C \subseteq \mathcal{D}_J$. Let $\tilde{C} = \{x \in {}^{\omega}2 : [x]_J \in C\}$. By the above, $\tilde{C}$ or $\mathbb{R} \setminus \tilde{C}$ contains the body of a J-pointed tree p. Without loss of generality, suppose $[p] \subseteq \tilde{C}$. Suppose $x \in \mathbb{R}$ is such that $p \leq_J x$. Note $\Upsilon^p(x) \leq_J p \oplus x \leq_J x$. Since $\Upsilon^p(x) \in [p]$ and p is J-pointed, $p \leq_J \Upsilon^p(x)$. With knowledge of p, $x = (\Upsilon^p)^{-1}(\Upsilon^p(x)) \leq_J \Upsilon^p(x)$. Thus $\Upsilon^p(x)$ has the same J-degree as x. It has been shown that for any $x \geq_J p$, there is a $y \in [p] \subseteq \tilde{C}$ with the same J-degree as x. Thus C contains the J-cone above the J-degree of p. If $\mathbb{R} \setminus \tilde{C}$ contains a J-pointed tree, then the same argument shows that $\mathcal{D}_J \setminus C$ contains a J-cone. This shows that μ_J is an ultrafilter.

Suppose $\langle A_n : n \in \omega\rangle$ is a countable sequence from μ_J. Using $\mathsf{AC}_\omega^\mathbb{R}$, let $\langle a_n : n \in \omega\rangle$ be a sequence of reals so that for all $n \in \omega$, $[a_n]_{\equiv_J}$ is the base of J-cone inside A_n. Let $a = \bigoplus a_n$, where $\bigoplus$ is some recursion coding of sequences of reals by a real. Then $[a]_{\equiv_J}$ is a base of a J-cone within $\bigcap_{n\in\omega} A_n$. This shows that μ_J is countably complete (in fact, AD alone implies every ultrafilter is countably complete). ☐

Lemma 7.10. *Let J be a set of ordinals. Suppose $\Sigma : {}^{\omega}2 \to {}^{\omega}2$ is a Lipschitz continuous function. Suppose p is a J-pointed tree such that $\Sigma \leq_J p$. Assume that Σ is not constant on any basic neighborhood of $[p]$. Then there is a J-pointed subtree $q \subseteq p$ so that for all $r \in [q]$, $\Sigma(r) \oplus q \equiv_J r$.*

Proof. Since Σ is a Lipschitz continuous function, Σ can be considered a Player 2 stategy in a game where both players make moves from $\{0,1\}$. In this way, one will consider Σ as a real. Since Σ is Lipschitz, for each $u \in {}^{<\omega}2$, let $\Sigma(u) \in {}^{|u|}2$ be the string t such that every $x \in {}^{\omega}2$ with $u \subseteq x$, $t \subseteq \Sigma(x)$. If one considers Σ as a Player 2 winning strategy, then $\Sigma(u)$ is just the response of Player 2 using Σ when Player 1 plays u.

Fix a J-pointed tree p. One will construct a sequence $\langle u_s : s \in {}^{<\omega}2\rangle$ in the tree p and a sequence of natural numbers $\langle n_s : s \in {}^{<\omega}2\rangle$ with the following properties:
(1) For all $s \in {}^{<\omega}2$, $u_s \subseteq u_{s\hat{}i}$ for both $i \in 2$.
(2) For all $s \in {}^{<\omega}2$, if $t \subsetneq s$, then $n_t < n_s$.
(3) For all $s \in {}^{<\omega}2$ and $i \in 2$, $\Sigma(u_{s\hat{}i})(n_s) = i$.
(4) Both $\langle u_s : s \in {}^{<\omega}2\rangle$ and $\langle n_s : s \in {}^{<\omega}2\rangle$ are Turing computable from $p \oplus \Sigma$. Since $\Sigma \leq_J p$, both sequences belong to $L[J, p]$.

First suppose that such sequences exist. Let q be the $\subseteq$-downward closure of $\{u_s : s \in {}^{<\omega}2\}$. q is a perfect subtree of p. q is Turing computable from $p \oplus \Sigma$ and therefore, $q \leq_J p$. Suppose $r \in [q]$. Then $r \in [p]$. Since p is J-pointed, $p \leq_J r$. Thus $q \leq_J r$. This shows that q is also a J-pointed tree.

Let f be the left-most branch of q, i.e. $\Upsilon^q(\bar{0})$ where $\bar{0} \in {}^{\omega}2$ is the constant 0 sequence. Note that $f \leq_J q$. Since $f \in [p]$, $p \leq_J f$. Thus $p \leq_J q$ and as a result $p \equiv_J q$. Hence Σ, $\langle u_s : s \in {}^{<\omega}2\rangle$, and $\langle n_s : s \in {}^{<\omega}2\rangle$ belong to $L[J, q]$.

Now suppose $r \in [q]$. As observed above, $p \leq_J r$. One seeks to define a sequence $\langle v_n : n \in \omega\rangle \leq_J q \oplus \Sigma(r)$ in ${}^{<\omega}2$ so that for all $n \in \omega$, $v_n \subseteq v_{n+1}$, $|v_n| = n$, and $u_{v_n} \subseteq r$.

Let $v_0 = \emptyset$. By construction of q, $u_{v_0} = u_\emptyset \subseteq r$. Suppose v_n has been defined. Let $v_{n+1} = v_n\hat{}(\Sigma(r)(n_{v_n}))$. By the induction hypothesis, $u_{v_n} \subseteq r$. If $r \in [q]$, then $u_{v_n\hat{}0}$ or $u_{v_n\hat{}1}$ is an initial segment of r. By construction, one can determine which of the two is an initial segment of r by determining the value of $\Sigma(r)(n_{v_s})$. This shows that $u_{v_{n+1}} \subseteq r$. This completes the construction of the sequence $\langle v_n : n \in \omega\rangle$ which is Turing computable from $\langle u_s : s \in {}^{<\omega}2\rangle$, $\langle n_s : s \in {}^{<\omega}2\rangle$, and $\Sigma(r)$. Thus $\langle v_n : n \in \omega\rangle \leq_J q \oplus \Sigma(r)$.

Note that $r = \bigcup_{n\in\omega} u_{v_n}$. Thus $r \in L[J, q, \Sigma(r)]$, i.e. $r \leq_J q \oplus \Sigma(r)$.

Also since $r \in [q]$ and q is J-pointed, $\Sigma \leq_J q \leq_J r$. Thus $q \oplus \Sigma(r) \leq_J r$. It has been shown that $r \equiv_J q \oplus \Sigma(r)$.

Therefore, it remains to show that one can construct the sequence $\langle u_s : s \in {}^{<\omega}2\rangle$ and $\langle n_s : s \in {}^{<\omega}2\rangle$.

Let $u_\emptyset = \emptyset$. Since Σ is not constant, find the least triple (u_0, u_1, m) so that $u_0 \in p$, $u_1 \in p$, $u_0(m) = 0$ and $u_1(m) = 1$. Let $n_\emptyset = m$, $u_{\langle 0\rangle} = u_0$, and $u_{\langle 1\rangle} = u_1$.

Let $s \in {}^{<\omega}2$ and $|s| > 0$. Suppose u_s and $n_{s\restriction|s|-1}$ have been defined. Since Σ is not constant on N_{u_s}, find the least triple (u_0, u_1, m) so that $u_0 \in p$, $u_1 \in p$, $u_s \subseteq u_0$, $u_s \subseteq u_1$, $m > n_{s\restriction|s|-1}$, $|u_0| > m$, $|u_1| > m$, $\Sigma(u_0)(m) = 0$, and $\Sigma(u_1)(m) = 1$. Let $u_{s\hat{}0} = u_0$, $u_{s\hat{}1} = u_1$, and $n_s = m$.

This produces the sequences $\langle u_s : s \in {}^{<\omega}2\rangle$ and $\langle n_s : s \in {}^{<\omega}2\rangle$ with the desired property. The proof is complete. $\square$

Definition 7.11. A function $F : \mathbb{R} \to \omega_1$ is J-invariant if and only if for all $x, y \in \mathbb{R}$, $x \equiv_J y$ implies $F(x) = F(y)$.

If $F : \mathbb{R} \to \omega_1$ is a J-invariant function, then let $\tilde{F} : \mathcal{D}_J \to \omega_1$ be the induced function on $\mathcal{D}_J$. That is $\tilde{F}(X) = F(x)$, where $x \in X$.

A J-invariant function F is everywhere increasing if and only if for all $x, y \in \mathbb{R}$, $x \leq_J y$ implies $F(x) \leq F(y)$.

A J-invariant function F is increasing μ_J-almost everywhere if and only if there there is an $a \in \mathbb{R}$ so that for all $x, y \in \mathbb{R}$ with $a \leq_J x$ and $a \leq_J y$, $x \leq_J y$ implies that $F(x) \leq F(y)$.

Definition 7.12. Let J be a set of ordinals. For each $\mathfrak{F}, \mathfrak{G} \in \prod_{X \in \mathcal{D}_J} \mathrm{ON}$, define $\mathfrak{F} =_{\mu_J} \mathfrak{G}$ if and only if $\{X \in \mathcal{D}_J : \mathfrak{F}(X) = \mathfrak{G}(X)\} \in \mu_J$. Let $\mathfrak{F} <_{\mu_J} \mathfrak{G}$ if and only if $\{X \in \mathcal{D}_J : \mathfrak{F}(X) < \mathfrak{G}(X)\} \in \mu_J$.

The ultraproduct $\prod_{X \in \mathcal{D}_J} \mathrm{ON}/\mu_J$ consists of the equivalence class of $\prod_{X \in \mathcal{D}_J} \mathrm{ON}$ under $=_{\mu_J}$. For two elements $\mathcal{F}, \mathcal{G} \in \prod_{X \in \mathcal{D}_J} \mathrm{ON}/\mu_J$, one lets $\mathcal{F} < \mathcal{G}$ if and only if for all $\mathfrak{F} \in \mathcal{F}$ and $\mathfrak{G} \in \mathcal{G}$, $\mathfrak{F} <_{\mu_J} \mathfrak{G}$.

Let $\prod_{\mathcal{D}_J} \omega_1/\mu_J$ consists of the equivalence classes having a representative which is a function $\mathfrak{F} : \mathcal{D}_J \to \omega_1$.

Fact 7.13. *(Woodin) Assume* $\mathsf{ZF} + \mathsf{AD}$*. Let J be a set of ordinals.* $\prod_{X \in \mathcal{D}_J} \omega_1^{L[J,X]}/\mu_J = \omega_1$.

Proof. For each $\alpha < \omega_1$, let $F_\alpha : \mathbb{R} \to \omega_1$ be the constant function taking value α. Note that $\tilde{F}_\alpha \in \prod_{X \in \mathcal{D}_J} \omega_1^{L[J,X]}$. By the countable additivity of μ_J, $[\tilde{F}_\alpha]_{\mu_J} = \alpha$. Thus $\omega_1 \subseteq \prod_{X \in \mathcal{D}_J} \omega_1^{L[J,X]}$.

Let $\mathcal{F} \in \prod_{X \in \mathcal{D}_J} \omega_1^{L[J,X]}/\mu_J$. Let $F : \mathbb{R} \to \omega_1$ be a J-invariant function such that $\tilde{F}$ is a representative of $\mathcal{F}$. Consider the following game from [14] Lemma 3.3:

$$
\begin{array}{llllllll}
 & \mathrm{I} & x_0 & & x_1 & & x_2 & \cdots & & x \\
G_F & & & & & & & & & \\
 & \mathrm{II} & & y_0, z_0 & & y_1, z_1 & & y_2, z_2 \;\cdots & & y, z
\end{array}
$$

Player 2 wins if and only if $x \leq_J y$, $z \in \mathrm{WO}^{L[J,y]}$, and $\mathrm{ot}(z) = F(y)$.

Claim 1: Player 2 has a winning strategy in this game.

To see this: Suppose otherwise that Player 1 has a winning strategy σ. Consider σ as both a real and as a strategy. Since $\tilde{F} \in \prod_{X \in \mathcal{D}_J} \omega_1^{L[J,X]}$, pick a $y \geq_J \sigma$ such that $F(y) < \omega_1^{L[J,y]}$. Pick a $z \in \mathrm{WO}^{L[J,y]}$ so that $\mathrm{ot}(z) = F(y)$. Note that $\sigma(y, z) \leq_J y$ since $\sigma, y, z \leq_J y$. Thus Player 2 has won which contradicts σ being a Player 1 winning strategy. This proves Claim 1.

Thus suppose τ is a Player 2 winning strategy. Let $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first and second coordinate, respectively. Since τ is a winning strategy for Player 2, $\pi_2[\tau[\mathbb{R}]]$ is a $\mathbf{\Sigma}_1^1$ subset of WO. By boundedness, there is a $\delta < \omega_1$ so that for all $v \in \pi_2[\tau[\mathbb{R}]]$, $\mathrm{ot}(v) < \delta$. Now take $x \geq_J \tau$. Then $\tau(x) \leq_J x$ and therefore $\pi_1(\tau(x)) \leq_J x$. Since τ is a winning strategy for Player 2, $x \leq_J \pi_1(\tau(x))$. So $x \equiv_J \pi_1(\tau(x))$. Since F is J-invariant, $F(x) = F(\pi_1(\tau(x))) = \mathrm{ot}(\pi_2(\tau(x))) < \delta$. Then by the countable additivity of μ_J, there is an $\alpha < \delta$ so that for μ_J-almost all x, $F(x) = \alpha$. Hence $[\tilde{F}]_{\mu_J} = \alpha$.

This shows that $\prod_{X \in \mathcal{D}_J} \omega_1^{L[J,X]}/\mu_J \subseteq \omega_1$ which completes the proof. $\square$

Fact 7.14. *Assume* $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \mathsf{AD}$*. Let J be a set of ordinals. Every J-invariant function is increasing μ_J-almost everywhere.*

Proof. Consider the set $A = \{x \in \mathbb{R} : (\forall y)(x \leq_J y \Rightarrow F(x) \leq F(y))\}$. Since F is a J-invariant function, A is a J-invariant set. Let $\tilde{A} = A/\equiv_J$ be the corresponding set of J-degrees. By Fact 7.9, $\tilde{A} \in \mu_J$ or $\mathcal{D}_J \setminus \tilde{A} \in \mu_J$.

(Case 1) Suppose $\mathcal{D}_J \setminus \tilde{A} \in \mu_J$. There is some $\iota \in \mathbb{R}$ so that for all $x \in \mathbb{R}$ with $\iota \leq x$, $x \notin A$. Let $C_\iota = \{x \in \mathbb{R} : \iota \leq x\}$. Thus for all $x \in C_\iota$, there is a $y \in \mathbb{R}$ with $x \leq_J y$ and $F(y) < F(x)$. Since $\iota \leq_J x \leq_J y$, one in fact has for all $x \in C_\iota$, there is some $y \in C_\iota$ so that $F(y) < F(x)$. Define a binary relation R on C_ι by $y \mathrel{R} x$ if and only if $F(y) < F(x)$. By $\mathsf{DC}_{\mathbb{R}}$, there is a sequence $\langle x_n : n \in \omega\rangle$ so that $F(x_{n+1}) < F(x_n)$. This contradicts the wellfoundedness of ON. Thus Case 1 can not occur.

(Case 2) Suppose $A \in \mu_J$. There is some $\iota \in \mathbb{R}$ so that for all $x \in \mathbb{R}$ with $\iota \leq_J x$, $x \in A$. Suppose $x, y \in \mathbb{R}$ is such that $\iota \leq_J x \leq_J y$. By definition of $x \in A$, $F(x) \leq F(y)$. F is increasing on the cone above ι.

Since only Case 2 can occur, F must be increasing μ_J-almost everywhere. $\square$

Fact 7.15. *Assume $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \mathsf{AD}$. Let J be a set of ordinals. Let $F : \mathbb{R} \to \omega_1$ be a J-invariant function. Then there is a $G : \mathbb{R} \to \omega_1$ which is a J-invariant everywhere increasing function such that $\tilde{F} \sim_{\mu_J} \tilde{G}$.*

Proof. By Fact 7.14, there is an $\iota \in \mathbb{R}$ so that F is increasing above the J-cone of ι. Define $G(x) = \sup\{F(z) : \iota \leq_J z \leq_J x\}$. (If this set is empty, then $G(x) = 0$.) G is J-invariant.

If $x \leq_J y$, then $\{z : \iota \leq_J z \leq_J x\} \subseteq \{z : \iota \leq_J z \leq_J y\}$. Thus $G(x) \leq G(y)$. G is everywhere increasing.

If $x \in \mathbb{R}$ is such that $\iota \leq_J x$, then $G(x) = \sup\{F(z) : \iota \leq_J z \leq x\} = F(x)$ since F is increasing on the cone above ι. $\square$

Fact 7.16. *(Woodin, [17] Theorem 5.9) Assume AD. Let J be a set of ordinals. For μ_J-almost all $x \in \mathbb{R}$, $L[J, x] \models \mathsf{CH}$.*

Fact 7.17. *Assume $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \mathsf{AD}$ and $V = L(J, \mathbb{R})$ for some set of ordinals J. There is also a set of ordinals $\mathbb{X}_J$ that absorbs every function on $\mathbb{R} \times \omega_1$ in the following sense: for every partial function $\Lambda : \mathbb{R} \times \omega_1 \to \mathbb{R} \times \omega_1$, there is a real z, a formula φ, and an ordinal ξ so that for all $(r, \alpha) \in \mathrm{dom}(f)$, $\Lambda(r, \alpha) \in L[\mathbb{X}_J, z, r]$ and $\Lambda(r, \alpha) = (s, \beta) \Leftrightarrow L[\mathbb{X}_J, z, r, s] \models \varphi(\mathbb{X}_J, z, \xi, r, \alpha, s, \beta)$. In this context, z is said to code Λ.*

Proof. The proof is quite similar to Fact 4.6 and Fact 5.6. As in those argments, one can take $\mathbb{X}_J$ to be $J \oplus_\omega \mathbb{O}_J$. $\square$

Remark 7.18. Next one will study the cardinals below $\mathbb{R} \times \omega_1$ under the failure of $\mathsf{AD}_{\mathbb{R}}$. By Fact 2.7, if one is working in the theory $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R})) + \neg\mathsf{AD}_{\mathbb{R}}$, then there is set of ordinals J so that $V = L(J, \mathbb{R})$. In the rest of this section, one will work with models of the form $L(J, \mathbb{R}) \models \mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. By Fact 7.17, there is an associated set of ordinals $\mathbb{X}_J \in L(J, \mathbb{R})$ which absorbs all functions $\Lambda : \mathbb{R} \times \omega_1 \to \mathbb{R} \times \omega_1$ in $L(J, \mathbb{R})$. Without loss of generality by replacing J with $\mathbb{X}_J$, one can assume that J is a set of ordinals that absorbs all function from $\mathbb{R} \times \omega_1$ into $\mathbb{R} \times \omega_1$.

Definition 7.19. Let J be a set of ordinals. Let $F : \mathbb{R} \to \omega_1$ be a J-invariant function. Define $\Phi_F : \mathbb{R} \to \omega_1$ by $\Phi_F(x) = \omega_{F(x)}^{L[J,x]}$. Let $W_F^J = \bigsqcup \Phi_F$.

Fact 7.20. *Assume $\mathsf{ZF} + \mathsf{AD}$. Let $F_1, F_2 : \mathbb{R} \to \omega_1$ be two everywhere increasing J-invariant functions so that $\tilde{F}_1 =_{\mu_J} \tilde{F}_2$. Then $W_{F_1}^J \approx W_{F_2}^J$.*

Proof. Let $\ell \in \mathbb{R}$ be such that for all $x \geq_J \ell$, $F_1(x) = F_2(x)$. By Fact 7.9, let p be a J-pointed tree such that $[p] \subseteq \{x \in \mathbb{R} : \ell \leq_J x\}$.

Define $\Lambda : W_{F_1}^J \to W_{F_2}^J$ by letting $\Lambda(x, \alpha) = (\Upsilon^p(x), \alpha)$. Since p is J-pointed, $p \leq_J \Upsilon^p(x)$. Hence $p \in L[J, \Upsilon^p(x)]$. Using p and $\Upsilon^p(x)$, one can Turing compute x. Thus $x \leq_J \Upsilon^p(x)$. Since $\Upsilon^p(x) \in [p]$, $F_1(\Upsilon^p(x)) = F_2(\Upsilon^p(x))$. Thus $\alpha < \omega_{F_1(x)}^{L[J,x]} \leq \omega_{F_1(x)}^{L[J,\Upsilon^p(x)]} \leq \omega_{F_1(\Upsilon^p(x))}^{L[J,\Upsilon^p(x)]} = \omega_{F_2(\Upsilon^p(x))}^{L[J,\Upsilon^p(x)]}$ since $x \leq_J \Upsilon^p(x)$, F_1 is everywhere increasing, and F_1 and F_2 are equal on $[p]$. This shows that Λ is well defined. Λ is an injection. Thus $|W_{F_1}^J| \leq |W_{F_2}^J|$.

By reversing the role of F_1 and F_2 in this argument, one has that $|W_{F_2}^J| \leq |W_{F_1}^J|$. Hence $W_{F_1}^J \approx W_{F_2}^J$. $\square$

Definition 7.21. Assume $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \mathsf{AD}$ and there is a set of ordinals J so that $V = L(J, \mathbb{R})$. For each $\mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J$, define the cardinality $Y_{\mathcal{F}}^J$ to be $|W_F^J|$ where $F : \mathbb{R} \to \omega_1$ is any J-invariant everywhere increasing function so that $\tilde{F} \in \mathcal{F}$. (Note that such an F exists by Fact 7.15 and this definition is well defined by Fact 7.20.)

Fact 7.22. *Let J be a set of ordinals. For every $\Phi : \mathbb{R} \to \omega_1$, there is an everywhere increasing J-invariant function F so that $|\bigsqcup \Phi| \leq |W_F^J|$.*

Thus every subset of $\mathbb{R} \times \omega_1$ without a copy of ω_1 injects into W_F^J for some everywhere increasing J-invariant function F. Of course, W_F^J also does not contain a copy of ω_1 since it is of the form $\bigsqcup \Phi$ for some function Φ.

Proof. Let $F' : \mathbb{R} \to \omega_1$ be defined by $F'(x)$ is the ordinal such that $L[J,x] \models |\Phi(x)| = \aleph_{F'(x)}$.

For each $x \in \mathbb{R}$, let $\Gamma^x : \Phi(x) \to \omega_{F'(x)}^{L[J,x]}$ be the $L[J,x]$-least bijection. Then $\Lambda' : \bigsqcup \Phi \to W_{F'}^J$ defined by $\Lambda'(x,\alpha) = (x, \Gamma^x(\alpha))$ is a bijection.

Let $F(x) = \sup\{F'(z) : z \leq_J x\}$. F' is everywhere increasing and $W_{F'}^J$ injects into W_F^J.

The last statement follows from Fact 7.3. $\square$

Example 7.23. Let J be a set of ordinals. Let $H_0, H_1 : \mathbb{R} \to \omega_1$ denote the constant 0 and constant 1 function, respectively. Then $|W_{H_0}^J| = |W_{H_1}^J| = |\mathbb{R}|$.

Proof. Note $W_{H_0}^J = \bigsqcup \omega_0^{L[J,x]} \approx \mathbb{R} \times \omega \approx \mathbb{R}$.

For each $x \in \mathbb{R}$, let $\Gamma^x : \omega_1^{L[J,x]} \to \mathbb{R}$ denote the $L[J,x]$-least injection of $\omega_1^{L[J,x]}$ into $\mathbb{R}^{L[J,x]}$. Define $\Lambda : W_{H_0}^J \to \mathbb{R} \times \mathbb{R}$ by $\Lambda(x,\alpha) = (x, \Gamma^x(\alpha))$. Λ is an injection witnessing $|W_{H_1}^J| \leq |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$. Thus $W_{H_1}^J \approx \mathbb{R}$. $\square$

Fact 7.24. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and* $V = L(J,\mathbb{R})$ *where J is a set of ordinals that absorbs all functions from $\mathbb{R} \times \omega_1$ into $\mathbb{R} \times \omega_1$ as in Fact 7.17 and Remark 7.18. Suppose $F_1, F_2 : \mathbb{R} \to \omega_1$ are two everywhere increasing J-invariant functions such that $\tilde{F}_1 <_{\mu_J} \tilde{F}_2$ and F_1 is not μ_J almost everywhere equal to* 0. *Then* $|W_{F_1}^J| < |W_{F_2}^J|$.

Proof. Since F_1 is not μ_J-almost everywhere 0 and $F_1 <_{\mu_J} F_2$, let $\ell \in \mathbb{R}$ be such that for all $x \in \mathbb{R}$ with $\ell \leq_J x$, $1 \leq F_1(x) < F_2(x)$. Let p be a J-pointed tree such that $[p] \subseteq \{x \in \mathbb{R} : \ell \leq_J x\}$. Define $\Lambda : W_{F_1}^J \to W_{F_2}^J$ by $\Lambda(x,\alpha) = (\Upsilon^p(x), \alpha)$. For all $(x,\alpha) \in W_{F_1}^J$, $\alpha < \omega_{F_1(x)}^{L[J,x]} \leq \omega_{F_1(\Upsilon^p(x))}^{L[J,\Upsilon^p(x)]} < \omega_{F_2(\Upsilon^p(x))}^{L[J,\Upsilon^p(x)]}$ since $x \leq_J \Upsilon^p(x)$, F_1 is everywhere increasing, and $\ell \leq_J \Upsilon^p(x)$. Thus Λ is a well defined injection witnessing $|W_{F_1}^J| \leq |W_{F_2}^J|$.

Suppose there was an injection $\Lambda : W_{F_2}^J \to W_{F_1}^J$. Since J absorbs all functions, let $z \in \mathbb{R}$ and φ be some formulas such that within $L[J,z]$, Λ is correctly defined in the sense of Fact 7.17. That is, for all $(r,\alpha) \in W_{F_2}^J$, $\Lambda(r,\alpha) \in L[J,z,r]$ and $\Lambda(r,\alpha) = (s,\beta) \Leftrightarrow L[J,z,r] \models \varphi(J,z,r,\alpha,s,\beta)$. By Fact 7.16, let $e \in \mathbb{R}$ be such that for all $x \in \mathbb{R}$, $e \leq_J x$ implies that $L[J,x] \models \mathsf{CH}$.

Let $w = z \oplus \ell \oplus e$. Within $L[J,w]$, Λ as defined by φ is a injection of $W_{F_2}^J \cap L[J,w]$ into $W_{F_1}^J \cap L[J,w]$. In particular, within $L[J,w]$, there is an injection of $\{w\} \times \omega_{F_2(w)}^{L[J,w]}$ into $W_{F_1}^J \cap L[J,w] \subseteq \mathbb{R}^{L[J,w]} \times \omega_{F_1(w)}^{L[J,w]}$ since F_1 is an everywhere increasing function. Since $L[J,w] \models \mathsf{CH}$, $|\mathbb{R}|^{L[J,w]} = \omega_1^{L[J,w]}$. By the definition of ℓ, for all x such that $\ell \leq_J x$, $F_1(x) \geq 1$. Thus in $L[J,w] \models |\mathbb{R} \times \omega_{F_1(w)}| = \omega_{F_1(w)}$. Thus within $L[J,w]$, one has an injection of $\omega_{F_2(w)}^{L[J,w]}$ into $\omega_{F_1(w)}^{L[J,w]}$. Since $\ell \leq_J w$, $F_2(w) > F_1(w)$. Such an injection can not exists in $L[J,w]$. Contradiction. This shows $|W_{F_1}^J| < |W_{F_2}^J|$. $\square$

Corollary 7.25. *(Woodin) Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}} + V = L(\mathscr{P}(\mathbb{R}))$. *There is a set $X \subseteq \mathbb{R} \times \omega_1$ so that $|\mathbb{R}| < |X|$ and $\neg(\omega_1 \leq |X|)$,*

Proof. By Fact 2.7, there is a set of ordinals J so that $V = L(J,\mathbb{R})$ and J absorbs functions. Let $F^1, F^2 : \mathbb{R} \to \omega_1$ be the constant funtion taking value 1 and 2, respectively. By Example 7.23, $W_{F^1}^J \approx \mathbb{R}$. Then by Fact 7.24, $|\mathbb{R}| = |W_{F^1}^J| < |W_{F^2}^J|$.

The set $W_{F^2}^J$ is essentially the example in [20] Theorem 25. $\square$

Theorem 7.26. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and $V = L(J,\mathbb{R})$ for some set of ordinals J which absorbs functions from $\mathbb{R} \times \omega_1$ into $\mathbb{R} \times \omega_1$. Let $\mathfrak{V}$ be the collection of $|X|$ such that $X \subseteq \mathbb{R} \times \omega_1$ and $\neg(\omega_1 \leq |X|)$; that is, $\mathfrak{V}$ is the collection of cardinalities of sets below $\mathbb{R} \times \omega_1$ that do not possess a copy of ω_1.*

The sequence $(Y_{\mathcal{F}}^J : \mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J \setminus \{0\})$ is an order-preserving injection of the wellordering $\prod_{\mathcal{D}_J} \omega_1/\mu_J \setminus \{0\}$ with the ultrapower ordering into $\mathfrak{V}$ with the natural cardinality ordering induced by injections. Moreover, this sequence is cofinal in $\mathfrak{V}$ in the sense that if $Y \in \mathfrak{V}$, then there is a $\mathcal{F} \in \prod_{\mathcal{D}} \omega_1/\mu \setminus \{0\}$ so that $Y \leq Y_{\mathcal{F}}^J$.

Proof. This is clear from Fact 7.22 and Fact 7.24. Also note that it is necessary to remove 0 for otherwise the sequence would not be injective since $Y_0^J = |\mathbb{R}| = Y_1^J$ by Example 7.23. $\square$

Fact 7.27. *(Woodin) Assume* $\mathsf{ZF}+\mathsf{DC}_{\mathbb{R}}+\mathsf{AD}$ *and* $V = L(J,\mathbb{R})$ *for some set of ordinals* J*. Let* $\mathbb{X}_J = J \oplus_\omega \mathbb{O}_J$*. Then* $\prod_{\mathcal{D}_{\mathbb{X}_J}} \omega_2^{L[\mathbb{X}_J,X]}/\mu_{\mathbb{X}_J} = \Theta^{L(J,\mathbb{R})}$*.*

Proof. This is shown in [14] Theorem 5.16. $\square$

As in Remark 7.18, if one has that $V = L(J,\mathbb{R})$, one could have always chosen the set of ordinals which absorbed functions to be $J\oplus_\omega \mathbb{O}_J$. Moreover $L(J,\mathbb{R}) = L(J\oplus_\omega\mathbb{O}_J,\mathbb{R})$. Thus the length of $(Y^J_{\mathcal{F}} : \mathcal{F} \in \prod_{\mathcal{D}_J}\omega_1/\mu_J)$ is quite long.

Let $\mathfrak{Y} = \{Y^J_{\mathcal{F}} : \mathcal{F} \in \prod_{\mathcal{D}_J}\omega_1/\mu_J \setminus \{0\}\}$. A natural question would be is $\mathfrak{V}$, the collection of uncountable cardinals below $\mathbb{R}\times\omega_1$ which does not contain a copy of ω_1, the same as $\mathfrak{Y} = \{Y^J_{\mathcal{F}} : \mathcal{F} \in \prod_{\mathcal{D}_J}\omega_1/\mu_J \setminus \{0\}\}$. Certainly, $\mathfrak{Y}\subseteq\mathfrak{V}$ and $\mathfrak{Y}$ is cofinal in $\mathfrak{V}$. Moreover, for all $\mathcal{Y}\in\mathfrak{Y}$ and $\mathcal{X}\in\mathfrak{V}$, either $\mathcal{X}\leq\mathcal{Y}$ or $\mathcal{Y}\leq\mathcal{X}$. This will follows from the next result. Moreover, the game in the proof is important for later results.

Theorem 7.28. *Assume* $\mathsf{ZF}+\mathsf{AD}$*. Let* J *be a set of ordinals. Let* $F:\mathbb{R}\to\omega_1$ *be an everywhere increasing* J*-invariant function so that for all* $x\in\mathbb{R}$*,* $F(x)\geq 1$*. Let* $\Phi:\mathbb{R}\to\omega_1$ *be any function. Consider the following game* S^Φ_F*:*

$$\begin{array}{c|ccccccccccc} & I & r_0 & & r_1 & & r_2 & & r_3 & & & r \\ S^\Phi_F & & & & & & & & & & & \\ & II & & x_0 & & x_1 & & x_2 & & x_3 & & x \end{array}$$

where Player 1 and Player 2 separately play natural numbers to produce reals r *and* x*. Player 2 wins* S^Φ_F *if and only if* $L[J,r,x]\models\Phi(r)<\omega_{F(r\oplus x)}$*. If Player 2 has a winning strategy in* S^Φ_F*, then* $|\bigsqcup\Phi|\leq|W^J_F|$*. If Player 1 has a winning strategy in* S^Φ_F*, then* $|W^J_F|\leq|\bigsqcup\Phi|$*.*

Thus either $|\bigsqcup\Phi|\leq|W^J_F|$ *or* $|W^J_F|\leq|\bigsqcup\Phi|$*.*

Proof. Statement 1: Suppose Player 2 has a winning strategy τ. For each $r\in\mathbb{R}$, let $\tau(r)$ denote the real that Player 2 produces using τ when Player 1 plays r.

Since τ is a Player 2 winning strategy, for all $r\in\mathbb{R}$, $L[J,r,\tau(r)]\models\Phi(r)<\omega_{F(r\oplus\tau(r))}$. Define $\Lambda:\bigsqcup\Phi\to W^J_F$ by $\Lambda(r,\alpha)=(r\oplus\tau(r),\alpha)$. Λ is an injection witnessing $|\bigsqcup\Phi|\leq|W^J_F|$.

Statement 2: Suppose Player 1 has a winning strategy σ. For each $x\in\mathbb{R}$, let $\sigma(x)$ be the response by Player 1 using σ when Player 2 plays x.

Since σ is a Player 1 winning strategy, for all $x\in\mathbb{R}$, $L[J,\sigma(x),x]\models\omega_{F(\sigma(x)\oplus x)}\leq\Phi(\sigma(x))$. Note that if $x_0,x_1\in\mathbb{R}$ are such that $\sigma(x_0)=\sigma(x_1)$ and $\sigma(x_0)\oplus x_0\equiv_J\sigma(x_1)\oplus x_1$, then $\omega^{L[J,\sigma(x_0),x_0]}_{F(\sigma(x_0)\oplus x_0)}=\omega^{L[J,\sigma(x_1),x_1]}_{F(\sigma(x_1)\oplus x_1)}$.

By Fact 7.16, let $e\in\mathbb{R}$ be such that for all $x\in\mathbb{R}$ with $e\leq_J x$, $L[J,x]\models\mathsf{CH}$. By Fact 7.9, let p be a J-pointed perfect tree such that $e\oplus\sigma\leq_J p$, i.e. $[p]$ is inside the cone above $e\oplus\sigma$.

Note that when one considers $\sigma:\mathbb{R}\to\mathbb{R}$ as a Lipschitz function, it cannot be constant on any neighborhood of $[p]$ since $\omega^{L[J,\sigma(x),x]}_{F(\sigma(x)\oplus x)}\leq\Phi(\sigma(x))$ and $F(x)\geq 1$ for all $x\in\mathbb{R}$. Thus by Lemma 7.10, there is a J-pointed perfect subtree $q\subseteq p$ with the property that for all $x\in[q]$, $\sigma(x)\oplus q\equiv_J x$.

Before proceeding, one should give intuition for the next function: σ as a Lipschitz function is not an injection; however, for any $r\in\sigma[[q]]$, one knows where the possible preimages of r come from. Precisely, for any $r\in\sigma[[q]]$, $\sigma^{-1}[\{r\}]\subseteq\mathbb{R}^{L[r\oplus q]}$. Thus there are at most $|\mathbb{R}|^{L[J,r\oplus q]}$ many $x\in\mathbb{R}$ so that $\sigma(x)=r$. Since $L[J,r\oplus q]\models\mathsf{CH}$, $L[J,r\oplus q]\models|\mathbb{R}|=\omega_1$. In anticipation of this many possible x sharing the same r as its image, one will split $\omega^{L[J,r\oplus q]}_{F(r\oplus q)}$ into $\mathbb{R}^{L[J,r\oplus q]}$ many disjoint pieces of size $\omega^{L[J,r\oplus q]}_{F(r\oplus q)}$. This makes room for each of the possible x such that $\sigma(x)=r$. The details are as follows:

For each $r\in\sigma[[q]]$, let $\Pi^r:\mathbb{R}^{L[J,r\oplus q]}\times\omega^{L[J,r\oplus q]}_{F(r\oplus q)}\to\omega^{L[J,r\oplus q]}_{F(r\oplus q)}$ be the $L[J,r\oplus q]$-least injection which exists since $L[J,r\oplus q]\models\mathsf{CH}$ and $F(x)\geq 1$ for all $x\in\mathbb{R}$. Define $\Lambda':\bigsqcup_{x\in[q]}\omega^{L[J,x]}_{F(x)}\to\bigsqcup\Phi$ by

$$\Lambda'(x,\alpha)=(\sigma(x),\Pi^{\sigma(x)}(x,\alpha)).$$

Note this is well defined since for all $x\in[q]$, $\sigma\leq_J q\leq_J x$ and thus $\sigma(x)\oplus x\equiv_J x\equiv_J\sigma(x)\oplus q$. If $x\in[q]$ and $\alpha<\omega^{L[J,x]}_{F(x)}$, then $x\in\mathbb{R}^{L[J,x]}=\mathbb{R}^{L[J,\sigma(x)\oplus q]}$ and $\alpha<\omega^{L[J,x]}_{F(x)}=\omega^{L[J,\sigma(x)\oplus q]}_{F(\sigma(x)\oplus q)}$. Thus (x,α) is in the domain of $\Pi^{\sigma(x)}$. Also $\Pi^{\sigma(x)}$ maps into $\omega^{L[J,\sigma(x)\oplus q]}_{F(\sigma(x)\oplus q)}=\omega^{L[J,\sigma(x)\oplus x]}_{F(\sigma(x)\oplus x)}\leq\Phi(\sigma(x))$.

Suppose $(x_0, \alpha_0) \neq (x_1, \alpha_1)$ belong to $\bigsqcup_{x \in [q]} \omega_{F(x)}^{L[J,x]}$. If $\sigma(x_0) \neq \sigma(x_1)$, then it is clear that $\Lambda'(x_0, \alpha_0) \neq \Lambda'(x_1, \alpha_1)$. Suppose $\sigma(x_0) = \sigma(x_1)$. Let r denote the common value $r = \sigma(x_0) = \sigma(x_1)$. As noted above, since $x_0, x_1 \in [q]$, one has $x_0 \equiv_J \sigma(x_0) \oplus q \equiv_J r \oplus q \equiv_J \sigma(x_1) \oplus q \equiv_J x_1$. Thus $x_0, x_1 \in \mathbb{R}^{L[J, r \oplus q]}$. Since $x_0 \neq x_1$, $\Pi^r(x_0, \alpha_0) \neq \Pi^r(x_1, \alpha_1)$ since Π^r is an injection. By definition of Λ', $\Lambda'(x_0, \alpha_0) \neq \Lambda'(x_1, \alpha_1)$. Thus Λ' is an injection.

Finally, define $\Lambda'' : W_F^J \to \bigsqcup_{x \in [q]} \omega_{F(x)}^{L[J,x]}$ by $\Lambda''(x, \alpha) = (\Upsilon^q(x), \alpha)$. Note $x \leq_J \Upsilon^q(x)$ since q is J-pointed. Thus $\omega_{F(x)}^{L[J,x]} \leq \omega_{F(x)}^{L[J,\Upsilon^q(x)]} \leq \omega_{F(\Upsilon^q(x))}^{L[J,\Upsilon^q(x)]}$ since F is everywhere increasing. Thus Λ'' is a well defined injection.

Thus $|W_F^J| \leq |\bigsqcup_{x \in [q]} \omega_{F(x)}^{L[J,x]}| \leq |\bigsqcup \Phi|$. $\square$

Corollary 7.29. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let J be a set of ordinals. Let $F : \mathbb{R} \to \omega_1$ be a J-invariant function so that $F(x) \geq 1$ for all $x \in \mathbb{R}$. Suppose $X \subseteq \mathbb{R} \times \omega_1$ and $\neg(\omega_1 \leq |X|)$. Then either $|X| \leq Y_F^J$ or $Y_F^J \leq |X|$.*

In other words, for all $\mathcal{X} \in \mathfrak{V}$ and $\mathcal{Y} \in \mathfrak{Y}$, $\mathcal{X} \leq \mathcal{Y}$ or $\mathcal{Y} \leq \mathcal{X}$.

Proof. By Fact 7.3, there is some $\Phi : \mathbb{R} \to \omega_1$ so that $|X| = |\bigsqcup \Phi|$. The result now follows from Theorem 7.28. $\square$

Theorem 7.30. *Assume* $\mathsf{ZF}+\mathsf{AD}$. *Let J be a set of ordinals. Let $F : \mathbb{R} \to \omega_1$ be an everywhere increasing J-invariant function. Let $X \subseteq W_{F+1}^J$, where $(F+1)(x) = F(x)+1$. Then either $|X| \leq |W_F^J|$ or $|W_{F+1}^J| = |X|$.*

Proof. By Fact 7.3, there is a $\Phi : \mathbb{R} \to \omega_1$ so that $|X| = |\bigsqcup \Phi|$.

Consider the game S_{F+1}^{Φ} from Theorem 7.28:

$$
\begin{array}{cccccccccccc}
 & \mathrm{I} & r_0 & & r_1 & & r_2 & & r_3 & & & r \\
S_F^X & & & & & & & & & & & \\
 & \mathrm{II} & & x_0 & & x_1 & & x_2 & & x_3 & & x
\end{array}
$$

where Player 1 and Player 2 separately play natural numbers to produce reals r and x. Player 2 wins S_F^X if and only if $L[J, r, x] \models \Phi(r) < \omega_{F(r \oplus x)+1}$. By AD, one of the two players has a winning strategy.

(Case 1) By Theorem 7.28, if player 1 has a winning strategy that $|W_{F+1}^J| \leq |\bigsqcup \Phi| = |X| \leq |W_{F+1}^J|$. Thus $|X| = |W_{F+1}^J|$.

(Case 2) Suppose Player 2 has a winning strategy τ. One will need a more careful look at the proof of statement 1 in Theorem 7.28.

For each $r \in \mathbb{R}$, let $\tau(r)$ denote the real that Player 2 produces using τ when Player 1 plays r.

Since τ is a Player 2 winning strategy, for all $r \in \mathbb{R}$, $L[J, r, \tau(r)] \models \Phi(r) < \omega_{F(r \oplus \tau(r))+1}$. That is, $L[J, r, \tau(r)] \models |\Phi(r)| \leq \omega_{F(r \oplus \tau(r))}$. Let $\Gamma^r : \Phi(r) \to \omega_{F(r \oplus \tau(r))}^{L[J,r,\tau(r)]}$ denote the $L[J, r, \tau(r)]$-least injection of $\Phi(r)$ into $\omega_{F(r \oplus \tau(r))}^{L[J,r,\tau(r)]}$.

Define $\Lambda : \bigsqcup \Phi \to W_F^J$ by $\Lambda(r, \alpha) = (r \oplus \tau(r), \Gamma^r(\alpha))$. Λ is an injection witnessing $|\bigsqcup \Phi| \leq |W_F^J|$. $\square$

Note that the assumption for Theorem 7.28 and Theorem 7.30 is just $\mathsf{ZF} + \mathsf{AD}$ and J is any set ordinals (with no assumption about function absorption although the two cardinalities may degenerate without these assumptions).

Corollary 7.31. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and $V = L(J, \mathbb{R})$ where J is a set of ordinals which absorbs functions from $\mathbb{R} \times \omega_1 \to \mathbb{R} \times \omega_1$. Then for all $n \in \omega \setminus \{0\}$, there are no cardinalities between Y_n^J and Y_{n+1}^J. In particular, there are no cardinalities between $|\mathbb{R}| = Y_1^J$ and Y_2^J.*

Theorem 7.32. *Assume* $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \mathsf{AD}$ *and $V = L(J, \mathbb{R})$, where J is a set of ordinals. Let $\mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1 / \mu_J \setminus \{0\}$ be such that $\mathrm{cof}(\mathcal{F}) = \omega$. Let $\langle \mathcal{F}_n : n \in \omega \rangle$ be any ω-cofinal sequence through $\mathcal{F}$. Then there exists everywhere increasing J-invariant functions from $\mathbb{R}$ into ω_1, F and $\langle F_n : n \in \omega \rangle$, so that $[\tilde{F}]_{\mu_J} = \mathcal{F}$ and for all $n \in \omega$, $[\tilde{F}_n]_{\mu_J} = \mathcal{F}_n$.*

Furthermore, assume J is a set of ordinals which absorbs functions from $\mathbb{R} \times \omega_1$ to $\mathbb{R} \times \omega_1$. Then for any $X \subseteq W_F^J$, either $|X| = |W_F^J|$ or there exists an $n \in \omega$ so that $|X| \leq |W_{F_n}^J|$.

Proof. By Fact 7.15, every $\mathcal{G} \in \prod_{\mathcal{D}_J} \omega_1 / \mu_J$ has an everywhere increasing J-invariant $G : \mathbb{R} \to \omega_1$ so that $\mathcal{G} = [\tilde{G}]_{\mu_J}$. Since AD implies $\mathsf{AC}_\omega^{\mathbb{R}}$ and every set in $L(J, \mathbb{R})$ is ordinal definable from J and a real, one has that $L(J,\mathbb{R})$ satisfies AC_ω, the full axiom of countable choice. Thus one can obtain F and $\langle F_n : n \in \omega\rangle$ as in the first statement of the theorem. One may assume that for all $n \in \omega$, for all $x \in \mathbb{R}$, $F_n(x) \geq 1$.

Now fix an $X \subseteq W^J_F$. Suppose there is no n so that $|X| \leq |W^J_{F_n}|$.

Let $m \in \omega$. Suppose $\langle X_k : k < m\rangle$ is a sequence of disjoint subset of X and $X_k \approx W^J_{F_k}$ for all $k < m$.

Let $Y = X \setminus (\bigcup_{k<m} X_k)$. For each $r \in \mathbb{R}$, let $\delta_r = \mathrm{ot}(Y_r)$. Let $\Phi : \mathbb{R} \to \omega_1$ be defined by $\Phi(r) = \delta_r$. Note that $Y \approx \bigsqcup \Phi$.

Consider the game $S^\Phi_{F_m}$ from Theorem 7.28.

(Case 1) Suppose Player 2 has a winning strategy in $S^\Phi_{F_m}$. By Theorem 7.28, there is injection $\Lambda : \bigsqcup \Phi \to W^J_{F_m}$. Since $Y \approx \bigsqcup \Phi$, there is an injection of Y into $W^J_{F_m}$.

Note that $W^J_{F_m}$ is in bijection with $\bigsqcup_{k\leq m} W^J_{F_m}$. Since $X_k \approx W^J_{F_k}$ and $|W_{F_k}| \leq |W_{F_m}|$ for all $k < m$, there are injections of X_k into $W^J_{F_m}$. Thus there is an injection of $X = Y \sqcup \bigsqcup_{k<m} X_k$ into $\bigsqcup_{k\leq m} W^J_{F_m} \approx W^J_{F_m}$. This contradicts the assumption that there is no $n \in \omega$ so that $|X| \leq |W^J_{F_n}|$. So Case 1 can not occur.

(Case 2) Player 1 has a winning strategy in $S^\Phi_{F_m}$. Theorem 7.28 states that there is an injection $\Lambda_m : W^J_m \to Y$. Let X_m be the image of this injection.

Consider the tree T of $(\Lambda_0, ..., \Lambda_{m-1})$ so that each $\Lambda_i : W^J_{F_i} \to X$ is an injection and for all $i < j < m$, $\Lambda_i[W^J_i] \cap \Lambda_j[W^J_j] = \emptyset$. Order this tree by extension. By the anaysis above, this tree has no dead branches. Since $L(J,\mathbb{R}) \models \mathsf{DC}_\mathbb{R}$ and all sets are ordinal definable from J and a real, $L(J,\mathbb{R}) \models \mathsf{DC}$. Thus let $\langle \Lambda_i : i \in \omega\rangle$ be a branch through the tree T.

Let $K : \mathbb{R} \times \omega_1 \to \mathbb{R} \times \omega_1$ by

$$K(r,\alpha) = \begin{cases} F_\alpha(r) & \alpha < \omega \\ 0 & \text{otherwise} \end{cases}$$

Since J absorbs function, as in Fact 7.17, there is an $\ell_0 \in \mathbb{R}$ so that for all $x \geq_J \ell_0$ and $\alpha < \omega_1$, $K(x,\alpha) \in L[J,x]$. In particular by absorbing K, one has that for all x with $\ell_0 \leq x$, $\langle F_n(x) : n \in \omega\rangle \in L[J,x]$.

Since $\langle \mathcal{F}_n : n \in \omega\rangle$ is cofinal through $\mathcal{F}$, one can use the countable additivity of μ_J to find an $\ell \geq_J \ell_0$ so that for all $x \in \mathbb{R}$ with $\ell \leq_J x$, $\langle F_n(x) : n \in \omega\rangle$ is a cofinal sequence through $F(x)$. Let p be a J-pointed tree such that $\ell \leq_J p$. For each $s \in [p]$, let $\Sigma^s : \omega^{L[J,s]}_{F(x)} \to \bigsqcup_{n\in\omega} \omega^{L[J,s]}_{F_n(s)}$ by the $L[J,s]$-least injection. (Note it is important that $\langle F_n(s) : s \in \omega\rangle \in L[J,s]$ to make sense of this.)

Let $\Lambda^* : \bigsqcup_{x\in[p]} \omega^{L[J,x]}_{F(x)} \to X$ be defined by

$$\Lambda^*(x,\alpha) = \Lambda_{\pi_1(\Sigma^x((x,\alpha)))}(x, \pi_2(\Sigma^x(x,\alpha))).$$

Here one thinks of $\bigsqcup_{n\in\omega} \omega^{L[J,s]}_{F_n(x)} = \{(n,\alpha) : n \in \omega \wedge \alpha < \omega^{L[J,x]}_{F_n(x)}\}$ as a subset of $\omega \times \omega_1$. The functions $\pi_1 : \omega \times \omega_1 \to \omega$ and $\pi_2 : \omega \times \omega_1 \to \omega_1$ are the projections onto the first and second coordinate, respectively. Here one considers $W^J_{F_i}$ as a subset of $\mathbb{R} \times \omega_1$. Observe that Λ^* is an injection.

As usual, $\Lambda^\star : W^J_{F(x)} \to \bigsqcup_{x\in[p]} \omega^{L[J,x]}_{F(x)}$ defined by $\Lambda^\star(x,\alpha) = (\Upsilon^p(x), \alpha)$ is an injection. It has been shown that $|W^J_F| \leq |X|$ and hence $|X| = |W^J_F|$. ☐

By Fact 7.13, the first ω_1 elements of $\prod_{\mathcal{D}_J} \omega_1/\mu_J$ are the elements $\prod_{X\in\mathcal{D}_J} \omega_1^{L[J,X]}/\mu_J$. For each $\alpha < \omega_1$, let $F^\alpha : \mathbb{R} \to \omega_1$ by the constant function α. Note $[\tilde{F^\alpha}]_{\mu_J}$ is α is the ultrapower. So $Y^J_\alpha = |W^J_{F^\alpha}|$. From the results shown so far, one can determine the first ω_1 initial segment of $\mathfrak{V}$, the collection of cardinalities below $|\mathbb{R} \times \omega_1|$ without a copy of ω_1:

Theorem 7.33. *Assume $\mathsf{ZF} + \mathsf{AD} + V = L(J,\mathbb{R})$ where J is a set of ordinals which absorbs functions from $\mathbb{R} \times \omega_1$ into $\mathbb{R} \times \omega_1$. The collection of cardinalities $\{Y^J_\alpha : 1 \leq \alpha < \omega_1\}$ is closed under the injection relation, $\leq$. That is if $\mathcal{X}$ is an uncountable cardinality and there is some $\alpha < \omega_1$ so that $\mathcal{X} \leq Y^J_\alpha$, then there is some $1 \leq \beta \leq \alpha$ so that $\mathcal{X} = Y^J_\beta$. Moreover, $\{Y^J_\alpha : 1 \leq \alpha < \omega_1\}$ is an initial segment of $\mathfrak{V}$ under the injection relation in the sense that for all $\mathcal{X} \in \mathfrak{V}$, either $\mathcal{X} \in \{Y^J_\alpha : 1 \leq \alpha < \omega_1\}$ or for all $\alpha < \omega_1$, $Y^J_\alpha \leq \mathcal{X}$.*

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