CODING AND ANTICODING OF A CARDINAL BY BOUNDED SUBSETS OF THE CARDINAL

WILLIAM CHAN

Abstract. This paper will consider combinatorial properties related to coding a cardinal by its bounded subsets. These properties have traditionally been studied in the context of very large cardinals and variations of these properties either reach the level of Kunen inconsistency or are very close to it. Within the descriptive set theoretic framework with determinacy or partition properties, these combinatorial properties are quite robust and have numerous natural examples.

Let κ be a cardinal, $\epsilon \leq \kappa$, and $X \subseteq \kappa$. $\mathsf{Bl}_\kappa(\epsilon, X)$ is the set of all subsets of κ of ordertype ϵ which are bounded below κ. $\mathsf{Bl}_\kappa(< \epsilon, X)$ is the set of all subsets of X of ordertype less than ϵ bounded below κ. The following will be shown which answer or address several questions of Ben-Neria and Garti.

- Let $\mu^1_{\omega_1}$ be the club filter on ω_1. Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_{<\omega_1}$ and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$. For any function $\Phi : \mathsf{Bl}(< \omega_1, \omega_\omega) \to \omega_\omega$, there is an $X \subseteq \omega_\omega$ with $|X| = |\omega_\omega|$ so that $\Phi[\mathsf{Bl}_{\omega_\omega}(< \omega_1, X)] \neq \omega_\omega$.
- Let μ^1_κ be the ω-club filter on κ. If $\kappa \to_* (\kappa)^{<\omega\cdot\omega}_2$, then for any $\epsilon < \omega \cdot \omega$ and any function $\Phi : \mathsf{Bl}_\kappa(< \epsilon, \kappa) \to \kappa$, there is an $X \in \mu^1_\kappa$ so that $\Phi[\mathsf{Bl}_\kappa(< \epsilon, X)] \neq \kappa$.
- Let Θ be the supremum of the ordinal onto which $\mathbb{R}$ surjects. For any cardinal κ with $\omega_1 \leq \kappa < \Theta$, there is a function $\Phi : \mathsf{Bl}_\kappa(\omega \cdot \omega, \kappa) \to \kappa$ so that for all $X \in \mu^1_\kappa$, $\Phi[\mathsf{Bl}_\kappa(\omega \cdot \omega, X)] = \kappa$.
- Assume AD and $\mathsf{DC}_\mathbb{R}$. For any uniform countably complete filter $\mathcal{F}$ on ω_1, there is a function $\Phi : \mathsf{Bl}_{\omega_1}(\omega \cdot \omega, \omega_1) \to \omega_1$ so that for all $X \in \mathcal{F}$, $\Phi[\mathsf{Bl}_{\omega_1}(\omega \cdot \omega, X)] = \omega_1$.
- Assume AD. Let $\boldsymbol{\delta}^1_\omega = \sup\{\boldsymbol{\delta}^1_n : n \in \omega\}$ be the supremum of the projective ordinals. For any $\epsilon < \boldsymbol{\delta}^1_\omega$ and $\Phi : \mathsf{Bl}_{\boldsymbol{\delta}^1_\omega}(< \epsilon, \boldsymbol{\delta}^1_\omega) \to \boldsymbol{\delta}^1_\omega$, there is an $X \subseteq \boldsymbol{\delta}^1_\omega$ with $|X| = |\boldsymbol{\delta}^1_\omega|$ so that $\Phi[\mathsf{Bl}_{\boldsymbol{\delta}^1_\omega}(< \epsilon, X)] \neq \boldsymbol{\delta}^1_\omega$. There is also a uniform filter $\mathcal{F}$ on $\boldsymbol{\delta}^1_\omega$ so that for all $\epsilon < \omega \cdot \omega$ and function $\Phi : \mathsf{Bl}_{\boldsymbol{\delta}^1_\omega}(< \epsilon, \boldsymbol{\delta}^1_\omega) \to \boldsymbol{\delta}^1_\omega$, there is an $X \in \mathcal{F}$ so that $\Phi[\mathsf{Bl}_{\boldsymbol{\delta}^1_\omega}(< \epsilon, X)] \neq \boldsymbol{\delta}^1_\omega$.

1. Introduction

This paper will be concerned with the existence or non-existence of strong coding functions for cardinals by certain subsets of the cardinal. These properties have been extensively studied under the axiom of choice where they are closely related to very strong large cardinal axioms reaching the level the Kunen's inconsistency or very close to it. Ben-Neria and Garti [1] began the investigation of these properties under the axiom of determinacy, AD. This paper will show that these concepts are quite robust under determinacy assumptions and many familiar cardinals of determinacy possess these properties. Several questions of Ben-Neria and Garti from [1] will be answered.

Let κ be a cardinal and $\epsilon \leq \kappa$. An ϵ-Jónsson function for κ is a function $\Phi : [\kappa]^\epsilon \to \kappa$ with the property that for all $A \subseteq \kappa$ with $|A| = \kappa$, $\Phi[[A]^\epsilon] = \kappa$. κ is said to be ϵ-Jónsson if and only if there are no ϵ-Jónsson functions for κ. A $(< \epsilon)$-Jónsson function for κ is a function $\Phi : [\kappa]^{<\epsilon} \to \kappa$ so that for all $A \subseteq \kappa$ with $|A| = \kappa$, $\Phi[[A]^{<\epsilon}] = \kappa$. κ is $(< \epsilon)$-Jónsson if and only if there are no $(< \epsilon)$-Jónsson functions for κ. A Jónsson function for κ is a $(< \omega)$-Jónsson function for κ. κ is a Jónsson cardinal if and only if κ is $(< \omega)$-Jónsson.

Under ZFC, the existence of a Jónsson cardinal implies $0^\sharp$ exists. Erdős and Hajnal ([13], [12]) showed that under ZFC and CH, 2^ω (the cardinal in bijection with $\mathbb{R}$) is not Jónsson. Solovay showed that assuming the consistency of a measurable cardinal, 2^ω can be real-valued measurable and hence Jónsson. Erdős and Hajnal ([13]) showed under ZFC that every infinite set has an ω-Jónsson function and thus there are no ω-Jónsson cardinals. ω-Jónsson functions appear in Kunen's original proof of the Kunen's inconsistency and is an important aspect of the proof which requires the axiom of choice.

The ordinary partition relation $\kappa \to (\kappa)^\epsilon_\gamma$ states that every function $P : [\kappa]^\epsilon \to \gamma$, there is an $A \subseteq \kappa$ with $|A| = \kappa$ and $\beta < \gamma$ so that for all $f \in [\kappa]^\epsilon$, $P(f) = \beta$. The partition relation $\omega \to (\omega)^\omega_2$ is commonly called

October 31, 2025. This research was funded in whole or in part by the Austrian Science Fund (FWF) 10.55776/Y1498.

the Ramsey property. A uncountable cardinal satisfying $\kappa \to (\kappa)^\kappa_2$ is called a strong partition cardinal. An uncountable cardinal satisfying $\kappa \to (\kappa)^\epsilon_2$ for all $\epsilon < \kappa$ is called a weak partition cardinal.

There has been substantial work on Jónsson cardinals under determinacy hypothesis. It is easily seen that ω is never Jónsson (Fact 2.2). Any cardinal κ which possesses a uniform normal ultrafilter is Jónsson (Fact 2.11). $\omega_1 \to (\omega_1)^{\omega_1}_2$ and $j_{\mu^1_\omega}(\omega_1) = \omega_2$ (where $\mu^1_{\omega_1}$ is the club filter on ω_1) imply that ω_2 is a weak partition cardinal. This hypothesis implies that the club filter on ω_1 is a uniform normal measure on ω_1 and the ω-club filter on ω_2 is a uniform normal ultrafilter on ω_2. Fact 2.11 implies ω_1 and ω_2 are Jónsson cardinals under these hypothesis (and in particular under AD). Kleinberg then showed that these same hypothesis implies for all $n < \omega$, ω_n are Jónsson cardinals. Let Θ be the supremum of the ordinals onto which $\mathbb{R}$ surjects. Jackson-Ketchersid-Schlutzenberg-Woodin ([17]) showed under AD^+ that every uncountable cardinal $\kappa < \Theta$ is Jónsson.

Jackson, Holshouser, Meehan, Trang, and the author have investigated Jónssonness property of non-wellorderable sets. Greater care needs to be made in the definition of Jónssonness when X is not wellorderable by using injective tuples. (See [11] and [4] for the relevant definitions.) Holshouser and Jackson showed that $\mathbb{R}$ is Jónsson (also see [11]) assuming AD. Let E_0 be the equivalence relation on ${}^\omega 2$ defined by $x \ E_0 \ y$ if and only if there exists an $m \in \omega$ so that for all $m \leq n < \omega$, $x(n) = y(n)$. Holshouser and Jackson showed that $\mathbb{R}/E_0$ is 2-Jónsson. Meehan and the author ([11]) showed that $\mathbb{R}/E_0$ is not 3-Jónsson and hence not Jónsson under AD. $\mathbb{R}/E_0$ and minor variations are essentially the only known example of sets which are not Jónsson in the determinacy context. Jackson, Trang, and the author ([9]) showed that AD implies ${}^\omega\omega_1$ is Jónsson. This argument essentially shows that for any cardinal κ satisfying $\kappa \to_* (\kappa)^{\omega\cdot\omega}_2$ (see below for the definition of the correct type partition relation), ${}^\omega\kappa$ is Jónsson. The methods of Jackson, Trang, and the author from [7] can show that for all cardinals $\kappa \leq \omega_\omega$, ${}^\omega\kappa$ is Jónsson. Using a higher dimensional analog of generalized Namba forcing (or diagonal Prikry forcing) over HOD-type models as developed by the author in [2], the author can show under AD^+ that ${}^\omega\kappa$ for $\kappa < \Theta$ with $\text{cof}(\kappa) = \omega$ is Jónsson. The Hjorth E_0-dichotomy ([16]) states that under AD^+, if X is a surjective image of $\mathbb{R}$, then exactly one of the following holds:

- X injects into the power set of an ordinal (and hence X is linearly orderable).
- $\mathbb{R}/E_0$ injects into X (and hence X is not linearly orderble).

In light of the known Jónssonness results and the Hjorth's dichotomy, an appealing conjecture is that under AD^+, a set X is Jónsson if and only if X is linearly orderable.

In contrast to the ZFC setting where there are no ω-Jónsson cardinals, suitable partition relations can yield even greater degree of Jónssonness:

(Proposition 2.4) Let κ be a cardinal and $\epsilon \leq \kappa$. If $\kappa \to (\kappa)^{1+\epsilon}_2$, then κ is ϵ-Jónsson.

The partition relation $\kappa \to (\kappa)^2_2$ already implies that κ must be a regular cardinal. Singular cardinals below Θ cannot be ω-Jónsson.

(Theorem 2.15) If $\kappa < \Theta$ and κ is a singular cardinal, then κ is not ϵ-Jónsson for all $\omega \leq \epsilon \leq \kappa$.

The primary subject of this paper are Magidor cardinals which were introduced and studied under ZFC in [14] by Garti, Hayut, and Shelah. Ben-Neria and Garti in [1] further investigated Magidor cardinals under AD in [1].

The Magidor properties are variations of the Jónssonness properties involving bounded subsets of a cardinal. If κ is a cardinal, $X \subseteq \kappa$, and $\epsilon < \kappa$, then define $\mathsf{BI}_\kappa(\epsilon, X)$ to be the set of increasing functions $f : \epsilon \to X$ so that $\sup(f) < \kappa$. ($\mathsf{BI}_\kappa(\epsilon, X)$ may be regarded as the set of bounded subsets of κ of order-type ϵ.) An ϵ-Magidor function for κ is a function $\Phi : \mathsf{BI}_\kappa(\epsilon, \kappa) \to \kappa$ so that for all $A \subseteq \kappa$ with $|A| = \kappa$, $\Phi[\mathsf{BI}_\kappa(\epsilon, A)] = \kappa$. κ is an ϵ-Magidor cardinal if and only if there are no ϵ-Magidor functions for κ. If $\epsilon \leq \kappa$, then let $\mathsf{BI}_\kappa(< \epsilon, X) = \bigcup_{\gamma<\epsilon} \mathsf{BI}_\kappa(\gamma, X)$. A function $\Phi : \mathsf{BI}_\kappa(< \epsilon, \kappa) \to \kappa$ is a $(< \epsilon)$-Magidor function if and only if for all $A \subseteq \kappa$ with $|A| = \kappa$, $\Phi[\mathsf{BI}_\kappa(< \epsilon, A)] = \kappa$. A cardinal κ is $(< \epsilon)$-Magidor if and only if there are no $(< \epsilon)$-Magidor functions for κ. A cardinal κ is called a Magidor cardinal if and only if κ is a $(< \omega_1)$-Magidor cardinal. A cardinal is called a super-Magidor cardinal if and only if κ is $(< \epsilon)$-Magidor for all $\epsilon < \kappa$.

It is easy to see that ω_1 is not Magidor (see Fact 2.25). Any singular cardinal $\kappa < \Theta$ with $\text{cof}(\kappa) > \omega$ is not ω-Magidor and hence not Magidor (see Fact 2.26). Thus under ZF, the only cardinals that can potentially be Magidor cardinals are regular cardinals and singular cardinals of cofinality ω. With the axiom of choice, every cardinal of uncountable cofinality is not ω-Magidor and hence not Magidor (see Fact 2.27). Under ZFC, the only cardinal that can potentially be a Magidor cardinal are singular cardinals of countable cofinality. Although ω-Jónsson cardinals cannot exists in ZFC, ω-Magidor and even Magidor cardinals can exist in ZFC assuming very large cardinals: Magidor observed that if λ witnesses the large cardinal axiom $I1$ in the sense that there is a nontrivial element embedding of $V_{\lambda+1}$ into $V_{\lambda+1}$, then λ is a Magidor cardinal.

In the presense of suitable partition properties, there are regular Magidor cardinals.

(Proposition 2.28) Let κ be a cardinal, $1 \leq \epsilon < \kappa$, and $\kappa \to (\kappa)^{1+\epsilon}_2$. Then κ is ϵ-Magidor.

Thus under AD, ω_2 is the least Magidor cardinal. Moreover, every weak partition cardinal is a super-Magidor cardinal.

As mentioned before, under ZF, only regular cardinal and singular cardinals of countable cofinality can possibly be Magidor cardinals. Ben-Neria and Garti ([1]) were interested in the possibility that ω_ω, the least singular cardinal of countable cofinality, could be a Magidor cardinal under AD. Ben-Neria and Garti ([1]) showed that ω_ω is ω-Magidor under AD. They asked in [1] Question 2.7 whether AD implies ω_ω is Magidor. The following answers their question positively.

(Theorem 3.40) Assume $\omega_1 \to (\omega_1)^{\omega_1}_\omega$ and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$. Then ω_ω is a Madigor cardinal.

The proof of Theorem 3.40 in Section 3 is purely combinatorial and self-contained. Martin developed an ultrapower representation for the ordinals below ω_ω under these combinatorial hypotheses. Martin showed these combinatorial hypotheses hold under AD. Jackson, Trang, and the author have recently applied these techniques involving the ultrapower analysis below ω_ω to investigate a number of combinatorial properties below ω_ω including the boldface GCH below ω_ω ([6]), the calibration of the cofinality of certain subsets of $\mathscr{P}(\omega_\omega)$ ([8]), and the primeness property for sets of ω-sequences below ω_ω ([7]). The paper [6] is a good introduction to the "sliding" arguments used here in an especially simple setting.

If κ is a cardinal and $\epsilon < \kappa$, then a uniform ultrafilter $\mathcal{F}$ on κ is an ϵ-Magidor filter for κ if and only if for all $\Phi : \mathsf{BI}_\kappa(\epsilon, \kappa) \to \kappa$, there is an $A \in \mathcal{F}$ so that $\Phi[\mathsf{BI}_\kappa(\epsilon, A)] \neq \kappa$. Similarly, a uniform ultrafilter $\mathcal{F}$ is a $(< \epsilon)$-Magidor filter for κ if and only if for all $\Phi : \mathsf{BI}_\kappa(< \epsilon, \kappa) \to \kappa$, there is an $A \in \mathcal{F}$ so that $\Phi[\mathsf{BI}_\kappa(< \epsilon, A)] \neq \kappa$. A Magidor filter is a $(< \omega_1)$-Magidor filter. Although ZFC with large cardinal axioms might have Magidor cardinals, Garti, Hayut, and Shelah ([14]) showed that there are no cardinals κ which possess a Magidor filter under AC. Ben-Neria and Garti [1] were interested in whether there are cardinals (especially singular cardinals of countable cofinality) possessing Magidor filters under AD. They showed that if AD holds and there is a strong partition cardinal above Θ, then in a Prikry generic extension, AD holds and there is a singular cardinal of countable cofinality (above Θ of the forcing extension) which possesses an ω-Magidor filter.

Section 4 will provide some remarks concerning Magidor filters. Suitable partition properties imply the ω-club filter on κ, μ^1_κ, is a $(< \omega \cdot \omega)$-Magidor filter.

(Proposition 4.3) Let κ be an uncountable cardinal and assume $\mathsf{AC}^{\mathscr{P}(\kappa)}_\omega$. Assume $\kappa \to (\kappa)^{<\omega\cdot\omega}_2$ holds. Then μ^1_κ is a $(< \omega \cdot \omega)$-Magidor ultrafilter on κ.

The ω-club filter can never be an $\omega \cdot \omega$-Magidor filter.

(Proposition 4.5) If κ is a cardinal with $\omega_1 \leq \kappa < \Theta$, then the ω-club filter μ^1_κ on κ is not an $\omega \cdot \omega$-Magidor filter.

Under AD, using Kunen's filter extension theorem and Kunen's classification of measures on ω_1, one can provide some evidence that perhaps there might be no $\omega \cdot \omega$-Magidor filter on ω_1. The following shows under AD, no countably complete filter on ω_1 can ever be an $\omega \cdot \omega$-Magidor filter for ω_1.

(Theorem 4.10) Assume AD and $\mathsf{DC}_\mathbb{R}$. If $\mathcal{F}$ is a countably complete nonprincipal ultrafilter on ω_1, then $\mathcal{F}$ is not an $\omega \cdot \omega$-Magidor filter for ω_1.

Although Theorem 3.40 shows that ω_ω is a Magidor cardinal, it is not known if ω_ω is a super-Magidor cardinal. Ben-Neria and Garti asks in [1] Question 2.8 whether there is a singular super-Magidor cardinal under AD. The final section will answer this question. As mentioned earlier, Ben-Neria and Garti showed that if AD holds and there is a strong partition cardinal above Θ, then there is forcing extension such that AD holds and there is cardinal κ possessing an ω-Magidor filter. The hypothesis that there is a strong partition cardinal above Θ is not known to be consistent. Moreover, the cardinal which possesses this ω-Magidor filter in the forcing extension is above Θ of the forcing extension satisfying AD and thus is not an image of $\mathbb{R}$. There is a consensus that sets which are not images of $\mathbb{R}$ may generally be beyond the influence of determinacy. A natural question would be whether AD alone can prove that a cardinal below Θ has an ω-Magidor or $(< \omega \cdot \omega)$-Magidor filter. The last section will also answer this question.

The last section will show that any cardinal κ which is a limit of an ω-sequence $\langle \kappa_n : n \in \omega \rangle$ of very nice strong partition cardinals are super-Magidor cardinals and possess $(< \omega \cdot \omega)$-Magidor filters. The only known set-theoretic setting possessing strong partition cardinals is the axiom of determinacy. The only known proof for the existence of strong partition cardinals is Martin's proof under AD that a cardinal κ which possesses a descriptive set theoretic good coding system satisfies the strong partition property. Roughly, these nice strong partition cardinals $\langle \kappa_n : n \in \omega \rangle$ are those that possesses good coding systems such that the associated pointclasses of the good coding system for κ_m and κ_n are sufficiently "far apart". There are unboundedly many such cardinals below Θ and the smallest such cardinal for which the results of the last section apply is the supremum of the projective ordinals $\boldsymbol{\delta}^1_\omega$.

> (Theorem 5.30 and Theorem 5.35) Assume AD. The supremum of the projective ordinals $\boldsymbol{\delta}^1_\omega$ is super-Magidor and possesses a $(< \omega \cdot \omega)$-Magidor filter.
>
> (Theorem 5.31 and Theorem 5.36) Assume AD. There are unboundedly many singular super-Magidor cardinals below Θ which possess a $(< \omega \cdot \omega)$-Magidor filter.

The last section is more descriptive set theoretic than the earlier sections of the paper. However, it should be accessible to a reader willing to accept the existence of the good coding systems produced by Jackson ([18], [19], [20]) and Kechris-Kleinberg-Moschovakis-Woodin ([22]). An important tool to prove the results of the last section is an independently interesting multi-cardinal partition property which appears in Theorem 5.24. The techniques here should be very useful in extending recent combinatorial results of Jackson, Trang, and the author at the first singular cardinal of countable cofinality, ω_ω, to the supremum of the projective ordinals, $\boldsymbol{\delta}^1_\omega$.

2. Jónsson and Magidor Properties

The paper will work over ZF and all other assumptions will be made explicit. If X and Y are two sets, then ${}^X Y$ is the set of all functions $f : X \to Y$. Let ON be the class of ordinals. If $X \subseteq \mathrm{ON}$ and $\epsilon \in \mathrm{ON}$, then $[X]^\epsilon$ is the set of all order preserving function $f : \epsilon \to X$. Let $[X]^{<\epsilon} = \bigcup_{\delta < \epsilon} [X]^\delta$.

Definition 2.1. Let κ be a cardinal and $\epsilon \leq \kappa$. A ϵ-Jónsson function for κ is a function $\Phi : [\kappa]^\epsilon \to \kappa$ with the property that for all $A \subseteq \kappa$ with $|A| = \kappa$, $\Phi[[A]^\epsilon] = \kappa$. κ is said to be ϵ-Jónsson if and only if there are no ϵ-Jónsson functions for κ.

A Jónsson function for κ is a function $\Phi : [\kappa]^{<\omega} \to \kappa$ so that for all $A \subseteq \kappa$ with $|A| = \kappa$, $\Phi[[A]^{<\omega}] = \kappa$. κ is Jónsson if and only if there are no Jónsson functions for κ.

A function $\Phi : [\kappa]^{<\epsilon} \to \kappa$ is a $(< \epsilon)$-Jónsson function for κ if and only if for all $A \subseteq \kappa$ with $|A| = \kappa$, $\Phi[[A]^{<\epsilon}] = \kappa$. (Note that a Jónsson function for κ is a $(< \omega)$-Jónsson function.) κ is $(< \epsilon)$-Jónsson if and only if there are no $(< \epsilon)$-Jónsson functions.

Fact 2.2. *For any infinite cardinal κ, κ is not $(< \kappa)$-Jónsson. In particular, ω is not Jónsson.*

Proof. Define $\Phi : [\kappa]^{<\kappa} \to \kappa$ be defined by $\Phi(f) = \mathrm{dom}(f)$. (That is, if $f \in [\kappa]^{<\kappa}$ and $\epsilon < \kappa$ with $f : \epsilon \to \kappa$, then $\Phi(f) = \epsilon$.) For any $A \subseteq \kappa$ with $|A| = \kappa$, $\Phi[[A]^{<\kappa}] = \kappa$. Thus Φ is a $(< \kappa)$-Jónsson function. ☐

Definition 2.3. (Ordinary partition relation) Let κ be a cardinal, $\epsilon \leq \kappa$, and $\gamma < \kappa$. $\kappa \to (\kappa)^\epsilon_\gamma$ is the assertion that for all $P : [\kappa]^\epsilon \to \gamma$, there is an $A \subseteq \kappa$ with $|A| = \kappa$ and a $\beta < \gamma$ so that for all $f \in [A]^\epsilon$, $P(f) = \beta$. (In this situation, one says that A is a homogeneous set for P taking value β.)

For a cardinal κ, $\epsilon \leq \kappa$, and $\gamma < \kappa$, $\kappa \to (\kappa)^{<\epsilon}_\gamma$ is the assertion that for all $\epsilon' < \epsilon$, $\kappa \to (\kappa)^{\epsilon'}_\gamma$.

For an uncountable cardinal κ, $\epsilon \leq \kappa$, and $\gamma \leq \kappa$, $\kappa \to (\kappa)^\epsilon_{<\gamma}$ is the assertion that for all $\gamma' < \gamma$, $\kappa \to (\kappa)^\epsilon_{\gamma'}$.

For an uncountable cardinal κ, $\epsilon \leq \kappa$, and $\gamma \leq \kappa$, $\kappa \to (\kappa)^{<\epsilon}_{<\gamma}$ is the assertion that for all $\epsilon' < \epsilon$ and $\gamma' < \gamma$, $\kappa \to (\kappa)^{\epsilon'}_{\gamma'}$.

κ is a weak partition cardinal if and only if $\kappa \to (\kappa)^{<\kappa}_2$. κ is a strong partition cardinal if and only if $\kappa \to (\kappa)^\kappa_2$. κ is a very strong partition cardinal if and only if $\kappa \to (\kappa)^\kappa_{<\kappa}$.

Note that $\kappa \to (\kappa)^2_2$ implies that κ must be regular.

The Ramsey theorem states that for all $1 \leq n < \omega$ and $1 \leq m < \omega$, $\omega \to (\omega)^n_m$. Under ZFC, if κ is an uncountable cardinal satisfying $\kappa \to (\kappa)^2_2$, then κ is called weakly compact cardinal. For any infinite cardinal κ, one can show that $\kappa \to (\kappa)^\omega_2$ implies $[\kappa]^\omega$ is not wellorderable and thus the axiom of choice must fail. $\omega \to (\omega)^\omega_2$ is often called the Ramsey property. Mathias ([25]) showed that assuming the consistency of an inaccessible cardinal, $\omega \to (\omega)^\omega_2$ holds in the Solovay model obtained from Lévy collapsing the inaccessible cardinal to ω_1. Mathias's argument used highly absolute codes for partitions $P : [\omega]^\omega \to 2$ which exists in the Solovay model to produce homogeneous sets using generics for Mathias forcing over an inner model of choice containing the code set. AD is the axiom of determinacy which states all infinite games of a suitable form has a winning strategy for one of the two players. AD^+ is Woodin's extension of the axiom of determinacy. Among the postulates of AD^+ is the assertion that all subsets of $\mathbb{R}$ have ∞-Borel code. Woodin observed that these ∞-Borel codes can be used in the same manner as Mathias's argument in the Solovay model to show that AD^+ proves $\omega \to (\omega)^\omega_2$. It is open if AD proves $\omega \to (\omega)^\omega_2$.

Mitchell ([26]) used Radin forcing to show the consistency of ZF, DC, and the club filter on ω_1 is countably complete ultrafilter from the consistency of a measure sequence with suitable repeat point properties. Woodin then used Radin forcing to show the consistency of ZF, DC, and the weak partition property $\omega_1 \to (\omega_1)^{<\omega_1}_2$ from the consistency of a measure sequence with suitable repeat point properties. The axiom of determinacy using Martin's good coding system for functions by reals which satisfies strong definability conditions relative to a pointclass is the only known setting with any strong partition cardinals. (Good coding system will be briefly reviewed in Section 5. See [21], [20], [5], and [3] for more about the good coding systems.) Martin's method of good coding always establishes the very strong partition property. It is open if the strong partition property at κ ($\kappa \to (\kappa)^\kappa_2$) always implies the very strong partition property ($\kappa \to (\kappa)^\kappa_{<\kappa}$). Martin showed AD proves that ω_1 is a very strong partition cardinal, $\omega_1 \to (\omega_1)^{\omega_1}_{<\omega_1}$. Martin also showed that AD implies that ω_2 is a weak partition cardinal satisfying $\omega_2 \to (\omega_2)^{<\omega_2}_2$. Martin and Paris showed ω_2 is not a strong partition cardinal. (See [21], [20], [5], and [3].) This result and many other properties of cardinals below ω_ω were established by Martin by analyzing the ultrapower of ω_1 by partition filters on ω_1 using the strong partition property (which will be discussed further below). Let $\mu^1_{\omega_1}$ be the club filter on ω_1. Suitable partition properties imply $\mu^1_{\omega_1}$ is a normal ultrafilter. Kleinberg ([24]) derived many of the results of Martin and many other combinatorial results (discussed below) under the combinatorial assumption that $\omega_1 \to (\omega_1)^{\omega_1}_2$ and the ultrapower $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ (which does hold under AD). (AD also seems to be the only known theory in which $\mu^1_{\omega_1}$ is a countably complete ultrafilter and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$.) Let Θ be the supremum of the ordinals onto which $\mathbb{R}$ surjects. Under ZFC and CH, $\Theta = \omega_2$. Under AD, Θ is very large. Sets which are images of $\mathbb{R}$ are under the influences of determinacy and hence Θ can be regarded as the ordinal height of the determinacy world. Using Martin's good coding methods, Kechris-Kleinberg-Moschovakis-Woodin ([22]) showed that there are unboundedly many strong partition cardinals below Θ. Kechris and Woodin ([23], [24]) showed that if $V = L(\mathbb{R})$ then AD holds if and only if there are unboundedly many strong partition cardinals below Θ. Jackson ([19], [20], [18]) showed that all the odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$ are very strong partition cardinals and the even projective ordinals $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$ are weak partition cardinal which are not strong partition cardinals.

Suitable ordinary partition properties imply appropriate degrees of Jónssonness.

If $f : \epsilon \to \mathrm{ON}$ and $\delta < \epsilon$, then let $\mathrm{drop}(f, \delta) : \epsilon - \delta \to \mathrm{ON}$ be defined by $\mathrm{drop}(f, \delta)(\alpha) = f(\delta + \alpha)$ (where $\epsilon - \delta$ is the unique ordinal γ so that $\epsilon = \delta + \gamma$).

Proposition 2.4. *Let κ be a cardinal and $\epsilon \leq \kappa$. If $\kappa \to (\kappa)^{1+\epsilon}_2$, then κ is ϵ-Jónsson.*

Proof. Let $\Phi : [\kappa]^\epsilon \to \kappa$. Define $P : [\kappa]^{1+\epsilon} \to 2$ by $P(g) = 0$ if and only if $\Phi(\mathrm{drop}(g, 1)) < g(0)$. By $\kappa \to (\kappa)^\epsilon_2$, there is an $A \subseteq \kappa$ with $|A| = \kappa$ which is homogeneous for P. Let $\bar{\alpha} < \bar{\beta}$ be the first two elements of A. Let $B = A \setminus (\bar{\beta} + 1)$. Suppose $f \in [B]^\epsilon$.

- A is homogeneous for P taking value 0: Let $g_f \in [A]^{1+\epsilon}$ be defined so that $g_f(0) = \bar{\alpha}$ and $\mathrm{drop}(g_f, 1) = f$. $P(g_f) = 0$ implies that $\Phi(f) = \Phi(\mathrm{drop}(g_f, 1)) < g_f(0) = \bar{\alpha}$. So $\bar{\alpha} \notin \Phi[[B]^\epsilon]$.
- A is homogeneous for P taking value 1: Let $g_f \in [A]^{1+\epsilon}$ be defined so that $g_f(0) = \bar{\beta}$ and $\mathrm{drop}(g_f, 1) = f$. $P(g_f) = 1$ implies that $\Phi(f) = \Phi(\mathrm{drop}(g_f, 1)) \geq g_f(0) = \bar{\beta} > \bar{\alpha}$. So $\bar{\alpha} \notin \Phi[[B]^\epsilon]$.

Thus $\Phi[[A]^\epsilon] \neq \kappa$. Since Φ was arbitrary, this shows that κ is ϵ-Jónsson. $\square$

Without the axiom of choice, there can exists ω-Jónsson functions.

Fact 2.5. *If $\omega \to (\omega)^\omega_2$, then ω is an ω-Jónsson cardinal.*

Proof. By Proposition 2.4. $\square$

Fact 2.6. *Assume $\omega_1 \to (\omega_1)^{\omega_1}_2$ and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$. ω_1 and ω_2 are ω-Jónsson.*

Proof. Martin and Kleinberg showed this hypothesis implies ω_2 is a weak partition cardinals. The result follow from Proposition 2.4. $\square$

If κ is a cardinal and μ is an ultrafilter on κ, then μ is uniform if and only if every $A \in \mu$, $|A| = \kappa$. If κ is an uncountable cardinal and μ is an ultrafilter on κ, then κ is normal if and only if for all $f : \kappa \to \kappa$ such that $\{\alpha \in \kappa : f(\alpha) < \alpha\} \in \mu$, then there is a $\delta < \kappa$ such that $\{\alpha \in \kappa : f(\alpha) = \delta\} \in \mu$. Let $\vec{A} = \langle A_\alpha : \alpha < \kappa\rangle \subseteq \mu$. Define $\triangle \vec{A} = \{\alpha \in \kappa : (\forall \beta < \alpha)(\alpha \in A_\beta)\}$. The normality of μ is equivalent to the fact that for all $\vec{A} = \langle A_\alpha : \alpha < \kappa\rangle \subseteq \mu$, $\triangle \vec{A} \in \mu$. Note that a uniform normal ultrafilter on κ is κ-complete.

Definition 2.7. Let κ be an uncountable cardinal, μ be a normal uniform ultrafilter on κ, $1 \leq n \in \omega$, and $\gamma < \kappa$. Let $\kappa \to_\mu (\kappa)^\epsilon_\gamma$ be the assertion that for all $P : [\kappa]^n \to \gamma$, there is a (unique) $\beta < \gamma$ and an $A \in \mu$ so that for all $\ell \in [A]^n$, $\Phi(\ell) = \beta$.

Fact 2.8. *(Rowbottom lemma) Assume κ is an uncountable cardinal, μ is a uniform normal ultrafilter on κ, $1 \leq n < \omega$, and $\gamma < \kappa$. Then $\kappa \to_\mu (\kappa)^n_\gamma$.*

Definition 2.9. Let κ be an uncountable cardinal, μ be a uniform normal ultrafilter on κ, and let $1 \leq n < \omega$. Define μ^n to be the filter on $[\kappa]^n$, defined by $X \in \mu^n$ if and only if there is an $A \in \mu$ so that $[A]^n \subseteq X$.

The Rowbottom lemma (Fact 2.8) implies that μ^n is an κ-complete ultrafilter on $[\kappa]^n$ for all $1 \leq n < \omega$. Let $\mu^{\otimes n}$ denote the n-fold product of μ which is an ultrafilter on ${}^n\kappa$. The Rowbottom lemma can be used to show that μ^n is equal to $\mu^{\otimes n}$ restricted to $[\kappa]^n$.

With $\mathsf{AC}^{\mathscr{P}(\kappa)}_\omega$, the Rowbottom lemma (Fact 2.8), and Proposition 2.4, it is easy to see that any κ which possess a uniform normal ultrafilter on κ is Jónsson. Under AD, $\mathsf{AC}^{\mathbb{R}}_\omega$ holds by a simple game argument. The Moschovakis coding lemma implies that if $\kappa < \Theta$, then there is a surjection of $\mathbb{R}$ onto $\mathscr{P}(\kappa)$. Thus AD proves $\mathsf{AC}^{\mathscr{P}(\kappa)}_\omega$ for all $\kappa < \Theta$. However, no form of countable choice is necessary to show that a cardinal κ which possesses a normal uniform ultrafilter is Jónsson if one carefully observe the uniformity in the proof of Rowbottom's lemma. This will be stated explicitly as follows:

Fact 2.10. *Let κ be an uncountable cardinal and let μ be a uniform normal ultrafilter on κ. There is a sequence $\langle \mathfrak{C}^n_\mu : 1 \leq n < \omega\rangle$ such that for all $1 \leq n < \omega$, $\mathfrak{C}^n_\mu : \mu^n \to \mu$ has the property that for all $B \in \mu^n$, $\mathfrak{C}^n_\mu(B) \in \mu$ and $[\mathfrak{C}^n_\mu(B)]^n \subseteq B$. (In other words, $\mathfrak{C}^n_\mu$ picks for each $B \in \mu^n$, a homogeneous set in μ for B.)*

Proof. The function $\langle \mathfrak{C}^n_\mu : 1 \leq n < \omega\rangle$ will be defined by recursion on n. Let $\mathfrak{C}^1_\mu : \mu \to \mu$ be the identity function. Suppose $1 \leq n < \omega$ and $\mathfrak{C}^n_\mu : \mu^n \to \mu$ has been defined with the property that for all $B \in \mu^n$, $\mathfrak{C}^n_\mu(B) \in \mu$ and $[\mathfrak{C}^n_\mu(B)]^n \subseteq B$. Let $B \in \mu^{n+1}$. This implies there is some $A \in \mu$ so that $[A]^{n+1} \subseteq B$. For each $\alpha < \kappa$, let $B_\alpha = \{\iota \in [\kappa]^n : \alpha < \iota(0) \wedge \langle\alpha\rangle\hat{\ }\iota \in B\}$. For each $\alpha \in A$, $B_\alpha \in \mu^n$ since $[A \setminus (\alpha+1)]^n \subseteq B_\alpha$. Thus $D_B = \{\alpha \in \kappa : B_\alpha \in \mu\} \in \mu$ since $A \subseteq D_B$. For all $\alpha \in D_B$, $[\mathfrak{C}^n_\mu(B_\alpha)]^n \subseteq B_\alpha$ by the inductive assumption. For each $\alpha < \kappa$, let $E^B_\alpha = \mathfrak{C}^n_\mu(B_\alpha)$ if $\alpha \in D_B$ and $E^B_\alpha = \kappa$ otherwise. Let $\vec{E}^B = \langle E^B_\alpha : \alpha < \kappa\rangle$. Define $\mathfrak{C}^{n+1}_\mu(B) = D_B \cap \triangle \vec{E}^B$. Note that $\mathfrak{C}^{n+1}_\mu(B) \in \mu$ since μ is normal. Suppose $\ell \in [\mathfrak{C}^{n+1}_\mu(B)]^{n+1}$. Then $\ell(0) \in D_B$. By definition of $\ell(k) \in \triangle \vec{E}^B$ for all $1 \leq k < n+1$, one has that $\ell(k) \in E_{\ell(0)} = \mathfrak{C}^n_\mu(B_{\ell(0)})$ since $\ell(0) \in D_B$. Thus $\text{drop}(\ell, 1) \in [\mathfrak{C}^n_\mu(B_{\ell(0)}]^n \subseteq B_{\ell(0)}$. Thus $\ell = \langle \ell(0)\rangle\hat{\ }\text{drop}(\ell, 1) \in B$. Since $\ell \in [\mathfrak{C}^{n+1}_\mu(B)]^{n+1}$ was arbitrary, one has shown that $[\mathfrak{C}^{n+1}_\mu(B)]^{n+1} \subseteq B$. This completes the construction. $\square$

Fact 2.11. *Let κ be an uncountable cardinal such that there is a uniform normal ultrafilter on κ. Then κ is Jónsson.*

Proof. Fix a uniform normal ultrafilter μ on κ. Let $\langle \mathfrak{C}^n_\mu : 1 \leq n < \omega\rangle$ be obtained by Fact 2.10 with the properties stated there. Let $\Phi : [\kappa]^{<\omega} \to \kappa$. For each $1 \leq n < \omega$, define $P_n : [\kappa]^{n+1} \to 2$ by $P_n(\ell) = 0$ if and only if $\Phi(\text{drop}(\ell, 1)) < \ell(0)$. By $\kappa \to_\mu (\kappa)^{n+1}_2$, there is a unique $i_n \in 2$ so that $P_n^{-1}[\{i_n\}] \in \mu^{n+1}$. Let $A = \bigcap_{1 \leq n < \omega} \mathfrak{C}^{n+1}_\mu(P_n^{-1}[\{i_n\}])$. Note that $A \in \mu$ since μ is κ-complete. Note that $[\kappa]^0 = \{\emptyset\}$. Let

$\bar{\alpha}$ be the least element of A greater than $\Phi(\emptyset)$. Let $\bar{\beta}$ be the least element of A greater than $\bar{\alpha}$. Let $B = A \setminus (\bar{\beta}+1)$. Let $n < \omega$. If $n = 0$, note that $\bar{\alpha} \neq \Phi(\emptyset)$. Suppose $1 \leq n < \omega$. Suppose $i_n = 0$. For any $\iota \in [A]^n$, let $\ell^n_\iota \in [B]^{n+1}$ be defined so that $\ell^n_\iota(0) = \bar{\alpha}$ and $\text{drop}(\ell^n_\iota, 1) = \iota$. Then $P_n(\ell^n_\iota) = i_n = 0$ implies that $\Phi(\iota) = \Phi(\text{drop}(\ell^n_\iota, 1)) < \ell^n_\iota(0) = \bar{\alpha}$. Now suppose $i_n = 1$. For any $\iota \in [B]^n$, let $\tau^n_\iota \in [A]^{n+1}$ be defined so that $\tau^n_\iota(0) = \bar{\beta}$ and $\text{drop}(\tau^n_\iota, 1) = \iota$. Then $P(\tau^n_\iota) = i_n = 1$ implies that $\bar{\alpha} < \bar{\beta} = \tau^n_\iota(0) \leq \Phi(\text{drop}(\tau^n_\iota, 1)) = \Phi(\iota)$. In any case, $\bar{\alpha} \notin \Phi[[B]^n]$. Since $n \in \omega$ was arbitrary, $\bar{\alpha} \notin \Phi[[B]^{<\omega}]$. This shows that Φ is not a Jónsson function. Since Φ was arbitrary, there are no Jónsson function for κ. κ is a Jónsson cardinal. $\square$

Next, one will show that singular cardinals below Θ cannot be ω-Jónsson.

Definition 2.12. Let κ be a cardinal $\epsilon \leq \kappa$, and X be a set. A (κ, ϵ, X)-coding function is a function $\Phi : [\kappa]^\epsilon \to X$ so that for all $A \subseteq \kappa$ with $|A| = \kappa$, $\Phi[[A]^\epsilon] = X$. Note that an ϵ-Jónsson function is a $(\kappa, \epsilon, \kappa)$-coding function.

Fact 2.13. *Let κ be a cardinal.*

(1) Let X be a set and $\epsilon_0 \leq \epsilon_1 \leq \kappa$. If there is a (κ, ϵ_0, X)-coding function, then there is a (κ, ϵ_1, X)-coding function.
(2) Let X be a set, Y be a set that X surjections onto, and $\epsilon \leq \kappa$. If there is a (κ, ϵ, X)-coding function, then there is a (κ, ϵ, Y)-coding function.
(3) Let X be a set which surjects onto κ. If there is a (κ, ϵ, X)-coding function, then κ is not ϵ-Jónsson.

Proof. (1) If Φ is a (κ, ϵ_0, X)-coding function, then $\Psi : [\kappa]^{\epsilon_1} \to X$ defined by $\Psi(f) = \Phi(f \upharpoonright \epsilon_0)$ is a (κ, ϵ_1, X)-coding function.

(2) Let $\pi : X \to Y$ be a surjection and $\Phi : [\kappa]^\epsilon \to X$ be a (κ, ϵ, X)-coding function. Then $\Psi : [\kappa]^\epsilon \to Y$ defined by $\Psi(f) = \pi(\Phi(f))$ is a (κ, ϵ, Y)-coding function.

(3) If X surjects into κ, then (2) implies there is a $(\kappa, \epsilon, \kappa)$-coding function or equivalently an ϵ-Jónsson function. $\square$

Theorem 2.14. *If κ is a singular cardinal with $\delta = \text{cof}(\kappa)$, then for all limit ordinals $\epsilon \leq \delta$, there is a $(\kappa, \epsilon, \mathscr{P}(\epsilon))$-coding function.*

Proof. Fix κ a singular cardinal, $\delta = \text{cof}(\kappa)$, and $\epsilon \leq \delta$ be a limit ordinal. Let $\rho : \delta \to \kappa$ be an increasing cofinal function. Let $\varphi : \kappa \to \delta$ be defined by $\varphi(\alpha)$ is the unique $\gamma < \delta$ so that $\sup(\rho \upharpoonright \gamma) \leq \alpha < \rho(\gamma)$. Let $f \in [\kappa]^\epsilon$. Note that $\varphi \circ f : \epsilon \to \delta$ is a non-decreasing sequence. Let $\xi_f = \text{ot}((\varphi \circ f)[\epsilon])$. Note that $\xi_f \leq \epsilon$. Let $\varpi(f) : \xi_f \to \delta$ be the increasing enumeration of $\varphi \circ f$. Let $\Phi(f) : [\kappa]^\epsilon \to \mathscr{P}(\epsilon)$ be defined by $\Phi(f) = \{\eta < \xi_f : |(\varphi \circ f)^{-1}[\{\varpi(f)(\eta)\}]| \geq 2\}$. The following intuitively describes $\Phi(f)$. For each $\eta < \xi_f$, $\varpi(f)(\eta)$ appears in the non-decreasing sequence $\varphi \circ f$. If $\varpi(f)(\eta)$ only appears once in $\varphi \circ f$, then $\eta \notin \Phi(f)$. If $\varpi(f)(\eta)$ appears more than once in $\varphi \circ f$, then $\eta \in \Phi(f)$.

Suppose $A \subseteq \kappa$ with $|A| = \kappa$. For each $\gamma < \delta$, let $A_\gamma = \{\alpha \in A : \sup(\rho \upharpoonright \gamma) \leq \alpha < \rho(\gamma)\}$. Note that $\text{ot}(A_\gamma) \leq \rho(\gamma)$ and $A = \bigcup_{\gamma < \delta} A_\gamma$. Let $B = \{\gamma < \delta : |A_\gamma| \geq 2\}$. B must be unbounded in δ. To see this, suppose B is bounded and let $\chi = \sup\{2, \rho(\gamma) : \gamma \in B\}$. Note that $\chi < \kappa$ and for all $\gamma < \delta$, $\text{ot}(A_\gamma) \leq \chi$. For all $\gamma < \delta$, let $\mathfrak{m}_\gamma : A_\gamma \to \text{ot}(A_\gamma)$ be the Mostowski collapse map. Since $\text{ot}(A_\gamma) < \chi$, one may regard $\mathfrak{m}_\gamma : A_\gamma \to \chi$. Define $\Psi : A \to \delta \times \chi$ by $\Psi(\alpha) = (\gamma, \mathfrak{m}_\gamma(\alpha))$ where γ is unique so that $\alpha \in A_\gamma$. Ψ is an injection and so $|\kappa| = |A| \leq |\delta \times \chi| \leq \max\{|\delta|, |\chi|\} < |\kappa|$ which is a contradiction. This shows that B is unbounded in δ. Since δ is regular, $\text{ot}(B) = \delta$. Since $\epsilon \leq \delta$, let $\langle \gamma_\eta : \eta < \epsilon \rangle$ be the increasing enumeration of the first ϵ-many elements of B. For each $\eta \in \epsilon$, let $\alpha^0_\eta < \alpha^1_\eta$ be the first two elements of A_{γ_η}. Note that for all $i, j \in \omega$ and $\eta_0 < \eta_1$, $\alpha^i_{\eta_0} < \rho(\gamma_{\eta_0}) \leq \sup(\rho \upharpoonright \gamma_{\eta_1}) \leq \alpha^j_{\eta_1}$. Fix $E \in \mathscr{P}(\epsilon)$. Let $F_E = \{\alpha^0_\eta : \eta \notin E\} \cup \{\alpha^0_\eta, \alpha^1_\eta : \eta \in E\}$. Note that $\text{ot}(F_E) = \epsilon$ using the assumption that ϵ is a limit ordinal. Let $f_E \in [\kappa]^\epsilon$ be the increasing enumeration of F_E. Note that $\varpi(f_E) = \langle \gamma_\eta : \eta \in \epsilon \rangle$. For all $\eta \notin E$, $|(\varphi \circ f_E)^{-1}[\{\varpi(f_E)(\eta)\}]| = |(\varphi \circ f_E)^{-1}[\{\gamma_\eta\}]| = |f_E^{-1}[\{\alpha^0_\eta\}]| = 1$ and thus $\eta \notin \Phi(E)$. For all $\eta \in E$, $|(\varphi \circ f_E)^{-1}[\{\varpi(f_E)(\eta)\}]| = |(\varphi \circ f_E)^{-1}[\{\gamma_\eta\}]| = |f_E^{-1}[\alpha^0_\eta, \alpha^1_\eta]| = 2$ and thus $\eta \in \Phi(E)$. This shows that $\Phi(f_E) = E$. Since $E \in \mathscr{P}(\epsilon)$ was arbitrary. $\Phi[[A]^\kappa] = \mathscr{P}(\epsilon)$. Since $A \subseteq \kappa$ with $|A| = \kappa$ was arbitrary, this shows that Φ is a $(\kappa, \omega, \mathscr{P}(\epsilon))$-coding function. $\square$

Theorem 2.15. *If $\kappa < \Theta$ and κ is a singular cardinal, then κ is not ϵ-Jónsson for all $\omega \leq \epsilon \leq \kappa$.*

Proof. By Theorem 2.14, κ has a $(\kappa, \omega, \mathscr{P}(\omega))$-coding function. Since $\kappa < \Theta$ means κ is a image of $\mathbb{R}$, Fact 2.13 (2) implies there is a (κ, ω, κ)-coding function. Then by Fact 2.13 (1), for all $\omega \leq \epsilon \leq \kappa$, there is a $(\kappa, \epsilon, \kappa)$-coding function. Since a $(\kappa, \epsilon, \kappa)$-coding function is an ϵ-Jónsson function, this shows that κ is not ϵ-Jónsson for all $\omega \leq \epsilon \leq \kappa$. $\square$

Fact 2.16. *Assume $\omega_1 \to (\omega_1)^{\omega_1}_2$ and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ (which holds under AD). For all $3 \leq n < \omega$, ω_n is not ϵ-Jónsson for any $\omega \leq \epsilon \leq \kappa$.*

Proof. Under these hypothesis, Martin showed that $\text{cof}(\omega_n) = \omega_2$ for all $2 \leq n < \omega$. Thus ω_n is singular for all $3 \leq n < \omega$. The result now follows from Theorem 2.15. $\square$

Under AD, if κ is a regular cardinal below the supremum of the projective ordinals, $\sup\{\boldsymbol{\delta}^1_n : n < \omega\}$, then Jackson has verified that $\kappa \to (\kappa)^\epsilon_2$ for all $\epsilon < \omega_1$. Thus every regular cardinal below the supremum of the projective ordinals is ω-Jónsson by Proposition 2.4. Steel ([28] Theorem 8.27) and Woodin ([29] Theorem 2.18) showed that AD^+ implies that the ω-club filter on any regular cardinal below Θ has a normal uniform ultrafilter on κ. Thus the Rowbottom lemma implies that under AD^+, for every regular cardinal $\kappa < \Theta$ and $n < \omega$, $\kappa \to (\kappa)^n_2$. It seem at least plausible that under AD^+ every regular cardinal $\kappa < \Theta$ satisfies $\kappa \to (\kappa)^\omega_2$. If this conjecture is true, then Proposition 2.4 and Theorem 2.15 together would imply under AD^+ that the set of ω-Jónsson cardinals below Θ is exactly the set of regular cardinals below Θ.

The correct type partition relation is often more practically useful when handling infinite exponent as it directly influence the behavior of the (correct type) partition filter. These partition filters will be essential for the analysis of ω_ω and the cardinals below ω_ω.

Definition 2.17. Let $\epsilon \in \text{ON}$ and $f : \epsilon \to \text{ON}$.

- f is discontinuous everywhere if and only if for all $\alpha < \epsilon$, $\sup(f \upharpoonright \alpha) < f(\alpha)$ (and thus f is an increasing function).
- f has uniform cofinality ω if and only if there is a function $F : \epsilon \times \omega \to \text{ON}$ so that for all $k \in \omega$ and $\alpha < \epsilon$, $F(\alpha, k) < F(\alpha, k+1)$ and $f(\alpha) = \sup\{F(\alpha, k) : k \in \omega\}$.
- f has the correct type if and only if f is both discontinuous everywhere and has uniform cofinality ω.

If $X \subseteq \text{ON}$, then let $[X]^\epsilon_*$ be the set of all functions $f : \epsilon \to X$ of the correct type. Note that $[\kappa]^1_*$ is just the set of $\alpha < \kappa$ with $\text{cof}(\alpha) = \omega$.

Note that if a function $f : \epsilon \to \text{ON}$ has uniform cofinality ω, then in particular, for all $\alpha < \epsilon$, $f(\alpha) \geq \omega$ and $\text{cof}(f(\alpha)) = \omega$. These notions are nontrivial only for uncountable cardinals. Thus the partition relation on ω cannot be formulated using the correct type notion and must be formulated using the ordinary partition relation.

Definition 2.18. (Correct type partition relation) Let κ be an uncountable cardinal, $\epsilon \leq \kappa$, and $\gamma < \kappa$. $\kappa \to_* (\kappa)^\epsilon_\gamma$ is the assertion that for all $P : [\kappa]^\epsilon \to \gamma$, there is a (unique) $\beta < \kappa$ and a club $C \subseteq \kappa$ so that for all $f \in [C]^\epsilon_*$, $P(f) = \beta$.

One can similarly define the following notion: $\kappa \to_* (\kappa)^{<\epsilon}_\gamma$ for all $\epsilon \leq \kappa$ and $\gamma < \kappa$. $\kappa \to_* (\kappa)^\epsilon_{<\gamma}$ for all $\epsilon \leq \kappa$ and $\gamma \leq \kappa$. and $\kappa \to_* (\kappa)^{\leq\epsilon}_{<\gamma}$ for all $\epsilon \leq \kappa$ and $\gamma \leq \kappa$.

The following indicates the relation between the ordinary and the correct type partition relation.

Fact 2.19. *Let κ be an uncountable cardinal, $\epsilon \leq \kappa$, $\gamma < \kappa$.*

- *$\kappa \to_* (\kappa)^\epsilon_\gamma$ implies $\kappa \to (\kappa)^\epsilon_\gamma$.*
- *$\kappa \to (\kappa)^{\omega\cdot\epsilon}_\gamma$ implies $\kappa \to_* (\kappa)^\epsilon_2$.*

In particular, $\kappa \to_* (\kappa)^{<\kappa}_2$ is equivalent to $\kappa \to (\kappa)^{<\kappa}_2$, $\kappa \to_* (\kappa)^\kappa_2$ is equivalent $\kappa \to (\kappa)^\kappa_2$, and $\kappa \to_* (\kappa)^\kappa_{<\kappa}$ is equivalent to $\kappa \to (\kappa)^\kappa_{<\kappa}$. That is, the weak partition property, the strong partition property, and the very strong partition property can be equivalently formulated using the ordinary partition relation or the correct type partition relation.

For the correct type partition relation, the homogeneous sets are now clubs rather than simply sets of large cardinalities. One nice benefit is that the homogeneous value for a partition is unique independent of the choice of homogeneous set. Correct type partition relation are more directly related to the (correct type) partition filter. The price to pay is that one cannot use simply increasing functions but must use functions of the correct type. Sometimes one will need to put in effort to make and show functions are discontinuous everywhere and have uniform cofinality ω. (The type of the functions becomes especially important in Section 4 when considering Magidor filters.)

Definition 2.20. If κ is an uncountable cardinal and $1 \leq \epsilon \leq \kappa$, then let μ^ϵ_κ be the (correct type) partition filter on $[\kappa]^\epsilon$ defined by $X \in \mu^\epsilon_\kappa$ if and only if there is a club $C \subseteq \kappa$, $[C]^\epsilon_* \subseteq X$. (Note that μ^1_κ is just the ω-club filter.)

Fact 2.21. *Let κ be an uncountable cardinal.*

1. *For all $\epsilon \leq \kappa$, $\kappa \rightarrow_* (\kappa)^\epsilon_2$ implies μ^ϵ_κ is an ultrafilter.*
2. *For all $\epsilon < \kappa$, $\kappa \rightarrow_* (\kappa)^{\epsilon+\epsilon}_2$ implies $\kappa \rightarrow_* (\kappa)^\epsilon_{<\kappa}$. (Thus $\kappa \rightarrow_* (\kappa)^{<\kappa}_2$ implies $\kappa \rightarrow_* (\kappa)^{<\kappa}_{<\kappa}$.)*
3. *For all $\epsilon \leq \kappa$ and $\gamma < \kappa$, $\kappa \rightarrow_* (\kappa)^\epsilon_\gamma$ implies that μ^ϵ_κ is γ^+-complete.*
4. *$\kappa \rightarrow_* (\kappa)^2_2$ implies the ω-club filter μ^1_κ is a normal ultrafilter.*

The ordinary partition relation $\kappa \rightarrow (\kappa)^2_2$ at an uncountable cardinal κ is consistent with ZF (assuming the consistency of a weakly compact cardinal). However, using the normality of $\mu^1_{\omega_1}$, one can show that $\kappa \rightarrow_* (\kappa)^2_2$ implies ${}^\omega\kappa$ is not wellorderable. The finite correct type partition relations already exhibit many of the properties of the infinite exponent ordinary partition relation. Also the normality of $\mu^1_{\omega_1}$ can be used to show that the identity function $\mathsf{id} : \kappa \rightarrow \kappa$ does not have uniform cofinality ω.

If $X \subseteq \text{ON}$, then let $\mathsf{enum}_X : \text{ot}(X) \rightarrow X$ be the increasing enumeration of X.

Fact 2.22. *Let κ be a cardinal and $C \subseteq \kappa$ be a club. Let $E = \{\mathsf{enum}_C(\omega \cdot \alpha + \omega) : \alpha < \kappa\}$. For any $\epsilon < \kappa$, $[E]^\epsilon = [E]^\epsilon_*$.*

Proof. It is clear that $[E]^\epsilon_* \subseteq [E]^\epsilon$. Let $f \in [E]^\epsilon$. Let $g \in [\kappa]^\epsilon$ be so that for all $\alpha < \epsilon$, $f(\alpha) = \mathsf{enum}_C(\omega \cdot g(\alpha) + \omega)$. For any $\alpha < \epsilon$, $\sup(f \restriction \alpha) = \sup\{f(\beta) : \beta < \alpha\} = \sup\{\mathsf{enum}_C(\omega \cdot g(\beta) + \omega) : \beta < \alpha\} \leq \mathsf{enum}_C(\omega \cdot g(\alpha)) < \mathsf{enum}_C(\omega \cdot g(\alpha) + \omega) = f(\alpha)$. This shows that f is discontinuous everywhere. Let $F : \epsilon \times \omega \rightarrow \kappa$ be defined by $F(\alpha, n) = \mathsf{enum}_C(\omega \cdot g(\alpha) + n)$. F witnesses that f has uniform cofinality ω. Thus $f \in [E]^\epsilon_*$. This shows $[E]^\epsilon \subseteq [E]^\epsilon_*$. □

Proposition 2.23. *Suppose κ is an uncountable cardinal, $\epsilon \leq \kappa$, and $\kappa \rightarrow_* (\kappa)^\epsilon_\epsilon$. Then κ is $(< \epsilon)$-Jónsson.*

Proof. Let $\Phi : [\kappa]^{<\epsilon} \rightarrow \kappa$. For each $\gamma < \epsilon$, let $P_\gamma : [\kappa]^\epsilon \rightarrow 2$ be defined by $P_\gamma(f) = 0$ if and only if $\Phi(\text{drop}(f, 1) \restriction \gamma) < f(0)$. By $\kappa \rightarrow_* (\kappa)^\epsilon_2$, there is a unique $i_\gamma \in 2$ so that there is a club which is homogeneous for P_γ taking value i_γ. Define $Q : [\kappa]^\epsilon \rightarrow 2$ by $Q(f) = 0$ if and only if for all $\gamma < \epsilon$, $P_\gamma(f) = i_\gamma$. By $\kappa \rightarrow_* (\kappa)^\epsilon_2$, there is club $C_0 \subseteq \kappa$ which is homogeneous for Q. Suppose C_0 is homogeneous for Q taking value 1. Define $\Psi : [C_0]^\epsilon \rightarrow \epsilon$ by $\Psi(f)$ is the least $\gamma < \epsilon$ so that $P_\gamma(f) \neq i_\gamma$. By $\kappa \rightarrow_* (\kappa)^\epsilon_\epsilon$, there is a club $C_1 \subseteq C_0$ and a $\bar{\gamma} < \epsilon$ so that for all $f \in [C_1]^\epsilon_*$, $\Psi(f) = \bar{\gamma}$. Thus C_1 is homogeneous for $P_{\bar{\gamma}}$ taking value $1 - i_{\bar{\gamma}}$. This is impossible since by definition, $i_{\bar{\gamma}}$ is the unique homogeneous value for $P_{\bar{\gamma}}$. Thus C_0 must be homogeneous for Q taking value 0. Let $\bar{\alpha} < \bar{\beta}$ be the first two elements of $[C_0]^1_*$ (i.e. $\bar{\alpha}$ and $\bar{\beta}$ are the first two elements of C_0 having cofinality ω). Let $D = C_0 \setminus (\bar{\beta} + 1)$. Let $\gamma < \epsilon$. First, suppose $i_\gamma = 0$. For each $g \in [C_0]^\gamma_*$, let $f_g \in [D]^\epsilon_*$ be defined by

$$f_g(\xi) = \begin{cases} \bar{\alpha} & \xi = 0 \\ g(\zeta) & 1 \leq \xi \leq 1 + \gamma \wedge \xi = 1 + \zeta \\ \mathsf{next}_C^{\omega \cdot \xi + \omega}(\sup(g)) & 1 + \gamma < \xi < \epsilon \end{cases}.$$

Note that $f_g(0) = \bar{\alpha}$ and $\text{drop}(f_g, 1) \restriction \gamma = g$. (Note that $\bar{\alpha}$ was chosen to have cofinality ω in order to ensure f_g has the correct type.) Since $Q(f_g) = 0$, one has that $P_\gamma(f_g) = i_\gamma = 0$ which implies $\Phi(g) = \Phi(\text{drop}(f_g, 1) \restriction \gamma) < f_g(0) = \bar{\alpha}$. Next, suppose $i_\gamma = 1$. If $g \in [D]^\gamma_*$, then let $h_g \in [C_0]^\epsilon_*$ be defined by

$$h_g(\xi) = \begin{cases} \bar{\beta} & \xi = 0 \\ g(\zeta) & 1 \leq \xi \leq 1 + \gamma \wedge \xi = 1 + \zeta \\ \mathsf{next}_C^{\omega \cdot \xi + \omega}(\sup(g)) & 1 + \gamma < \xi < \epsilon \end{cases}.$$

Note that $h_g(0) = \bar{\beta}$ and $\text{drop}(h_g, 1) \restriction \gamma = g$. Since $Q(h_g) = 0$, $P_\gamma(h_g) = i_\gamma = 1$ implies $\bar{\alpha} < \bar{\beta} = h_g(0) \leq \Phi(\text{drop}(h_g, 1) \restriction \gamma) = \Phi(g)$. So $\bar{\alpha} \notin \Phi[[D]^\gamma_*]$. Since $\gamma < \epsilon$ was arbitrary, $\bar{\alpha} \notin \Phi[[D]^{<\epsilon}_*]$. Let $E =$

$\{\mathsf{enum}_D(\omega\cdot\alpha+\omega) : \alpha < \kappa\}$. Note that $E \subseteq D$ and $[E]^{<\epsilon} = [E]^{<\epsilon}_*$ by Fact 2.22. Since $[E]^{<\epsilon} = [E]^{<\epsilon}_* \subseteq [D]^{<\epsilon}_*$, one has that $\bar{\alpha} \notin \Phi[[E]^{<\epsilon}]$. It has been shown that Φ is not a $(<\epsilon)$-Jónsson function. Since Φ was arbitrary, κ is $(<\epsilon)$-Jónsson. ☐

Definition 2.24. Let κ be a cardinal, $X \subseteq \kappa$, and $\epsilon < \kappa$. Define $\mathsf{BI}_\kappa(\epsilon, X)$ to be the set of bounded increasing functions $f : \epsilon \to X$ such that $\sup(f) < \kappa$. (Note that $\mathsf{BI}_\kappa(\epsilon, X)$ can be regarded as the bounded subsets of X of ordertype ϵ.) Let $\mathsf{BI}_\kappa(<\epsilon, X) = \bigcup_{\gamma<\epsilon} \mathsf{BI}_\kappa(\gamma, X)$.

Let κ be a cardinal and $\epsilon < \kappa$. A function $\Phi : \mathsf{BI}_\kappa(\epsilon, \kappa) \to \kappa$ is an ϵ-Magidor function for κ if and only if for all $A \subseteq \kappa$ with $|A| = \kappa$, $\Phi[\mathsf{BI}_\kappa(\epsilon, A)] = \kappa$. κ is ϵ-Magidor if and only if there are no ϵ-Magidor function for κ.

Let κ be a cardinal. κ is lower-Magidor if and only if for all $\epsilon < \kappa$, κ is ϵ-Magidor.

Let κ be a cardinal and $\epsilon \leq \kappa$. A function $\Phi : \mathsf{BI}_\kappa(<\epsilon, \kappa) \to \kappa$ is an $(<\epsilon)$-Magidor function for κ if and only if for all $A \subseteq \kappa$ with $|A| = \kappa$, $\Phi[\mathsf{BI}_\kappa(<\epsilon, A)] = \kappa$. A cardinal κ is $(<\epsilon)$-Magidor if and only if there are no $(<\epsilon)$-Magidor function for κ. A cardinal κ is Magidor if and only if κ is $(<\omega_1)$-Magidor.

A cardinal κ is super-Magidor if and only if for all $\epsilon < \kappa$, κ is $(<\epsilon)$-Magidor.

Fact 2.25. *For any cardinal κ, κ is not $(<\kappa)$-Magidor. In particular, ω_1 is not Magidor.*

Proof. Let $\Phi : \mathsf{BI}_\kappa(<\kappa, \omega) \to \kappa$ be defined by $\Phi(f) = \mathrm{dom}(f)$. Φ is a Magidor function for κ. ☐

Fact 2.26. *A singular cardinal $\kappa < \Theta$ of uncountable cofinality is not ω-Magidor and hence not Magidor.*

Proof. A singular cardinal $\kappa < \Theta$ is not ω-Jónsson by Propostion 2.15. Since $\mathrm{cof}(\kappa) > \omega$, $[\kappa]^\omega = \mathsf{BI}_\kappa(\omega, \kappa)$. Thus any ω-Jónsson function for κ is an ω-Magidor function for κ. . ☐

By Fact 2.26, under ZF, the only cardinals below Θ which could potentially be Magidor cardinals are regular cardinal above ω_1 and singular cardinals of countable cofinality. With the axiom of choice, only singular cardinals of countably cofinality can be Magidor.

Fact 2.27. *Assume the axiom of choice, AC. A cardinal of uncountable cofinality is not ω-Magidor and hence not Magidor.*

Proof. Erdős and Hajnal [13] showed that every infinite set has an ω-Jónsson function. If $\mathrm{cof}(\kappa) > \omega$, then $[\kappa]^\omega = \mathsf{BI}_\kappa(\omega, \kappa)$. Thus any ω-Jónsson function for κ is an ω-Magidor function for κ. ☐

Magidor observed that if λ witnessed the axiom I1 in the sense that there is a nontrivial elementary embedding from $V_{\lambda+1}$ into $V_{\lambda+1}$, then λ is a Magidor cardinal (and necessarily has countable cofinality). Thus assuming very strong large cardinals, there can be Magidor cardinals in ZFC.

Proposition 2.28. *Let κ be a cardinal, $1 \leq \epsilon < \kappa$, and $\kappa \to (\kappa)^{1+\epsilon}_2$. Then κ is ϵ-Magidor.*

Proof. By Proposition 2.4, κ is ϵ-Jónsson. Since the partition relation implies κ is regular, $B_\kappa(\epsilon, \kappa) = [\kappa]^\epsilon$. Thus being ϵ-Jónsson is equivalent to being ϵ-Magidor. ☐

Proposition 2.29. *If κ is a cardinal and $\kappa \to (\kappa)^{<\kappa}_2$. Then κ is lower-Magidor.*

Proof. This follow from Proposition 2.28 ☐

Proposition 2.30. *ω is lower-Magidor.*

Assume $\omega_1 \to (\omega_1)^{\omega_1}_2$ and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ (so in particular, under AD). ω_1 and ω_2 are lower Magidor.

Proof. The Ramsey theorem implies for each $n < \omega$, $\omega \to (\omega)^n_2$. ω_1 is lower-Magidor by Proposition 2.29.

Under AD, ω_1 and ω_2 are weak partition cardinals. Thus ω_1 and ω_2 are lower-Magidor by Proposition 2.29. ☐

Thus ω_1 is never Magidor, but ω_1 is lower-Magidor assuming the weak partition property on ω_1. Note that the notion of lower-Magidor and Magidor (and super-Magidor) have a key different. To establish that κ is lower-Magidor, one needs to show κ is ϵ-Magidor individually for each $\epsilon < \kappa$. To establish a cardinal κ is Magidor, one needs to simulteneously verify ϵ-Magidorness for all $\epsilon < \omega_1$ by showing no function $\Phi : \mathsf{BI}_\kappa(<\omega_1, \kappa) \to \kappa$ is a Magidor function. It seems potentially possible that a cardinal $\kappa > \omega_1$ could be lower-Magidor and yet not Magidor. However, no example is known to the author.

Without the axiom of choice, there are settings with regular Magidor cardinals. For example, AD has an abundance of regular Magidor cardinals and even very small regular cardinals such as ω_2 can be Magidor.

Proposition 2.31. *Let $\kappa > \omega_1$ be an uncountable cardinal satisfying $\kappa \to_* (\kappa)^{\omega_1}_{\omega_1}$. Then κ is Madigor.*

Proof. Note $\kappa \to_* (\kappa)^{\omega_1}_{\omega_1}$ implies κ is regular. Since $\kappa > \omega_1$, $[\kappa]^{<\omega_1} = \mathsf{Bl}_{\omega_1}(< \omega_1, \kappa)$. Thus κ is $(< \omega_1)$-Jónsson if and only κ is Magidor. By Proposition 2.23, κ is $< \omega_1$-Jónsson. Thus κ is Magidor. $\square$

Proposition 2.32. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$. Then ω_2 is the least Magidor cardinal.*

Thus AD implies ω_2 is the least Magidor cardinal.

Proof. By Fact 2.25, ω and ω_1 are not Magidor. Martin showed that the hypothesis implies ω_2 is a weak partition cardinal and in particular satisfies $\omega_2 \to_* (\omega_2)^{\omega_1}_{\omega_1}$. Proposition 2.31 implies ω_2 is Magidor. $\square$

Proposition 2.33. *Suppose κ is an uncountable cardinal, $1 \leq \epsilon < \kappa$ and $\kappa \to_* (\kappa)^\epsilon_\epsilon$. Then κ is $(< \epsilon)$-Magidor.*

Proof. Again since κ is regular by the partition relation, one will identify $\mathsf{Bl}_\kappa(< \epsilon, \kappa)$ with $[\kappa]^{<\epsilon}$. Let $\Phi : [\kappa]^{<\epsilon} \to \kappa$. Define $P_\delta : [\kappa]^{1+\delta} \to 2$ by $P_\delta(\ell) = 0$ if and only if $\Phi(\text{drop}(\ell, 1)) < \ell(0)$. By $\kappa \to_* (\kappa)^{1+\delta}_2$, there is a unique $i_\delta \in 2$ so that there is a club homogeneous for P_δ taking value i_δ. Let $A_\alpha = \{\ell \in [\kappa]^\epsilon : P_\delta(\ell \restriction 1 + \delta) = i_\delta\}$. Note that $A_\delta \in \mu^\epsilon_\kappa$. Since $\kappa \to_* (\kappa)^\epsilon_\epsilon$ implies that μ^ϵ_κ is ϵ-complete, $A = \bigcap_{\delta<\epsilon} A_\delta \in \mu^\epsilon_\kappa$. Let $C \subseteq \kappa$ be a club so that $[C]^\epsilon_* \subseteq A$. Let $\bar{\alpha} < \bar{\beta}$ be the first two elements of $[C]^1_*$. Let $D = \{\mathsf{enum}_{C \setminus (\bar{\beta}+1)}(\omega \cdot \alpha + \omega) : \alpha < \kappa\}$. Note that $|D| = \kappa$ and $\min(D) > \bar{\beta}$. Also observe that $[D]^{<\epsilon} = [D]^{<\epsilon}_*$ by Fact 2.22. Pick $\iota \in [D]^{<\epsilon}$. Let $\delta = |\iota|$.

- Suppose $i_\delta = 0$. Let $\ell = \langle \bar{\alpha} \rangle \hat{\ } \iota$ and note that $\ell \in [C]^{1+\delta}_*$. Then $P_\delta(\ell) = 0$ implies that $\Phi(\iota) = \Phi(\text{drop}(\ell, 1)) < \ell(0) = \bar{\alpha}$.
- Suppose $i_\delta = 1$. Let $\ell = \langle \bar{\beta} \rangle \hat{\ } \iota$ and note that $\ell \in [C]^{1+\delta}_*$. Then $P_\delta(\ell) = 1$ implies that $\bar{\alpha} < \bar{\beta} = \ell(0) \leq \Phi(\text{drop}(\ell, 1)) = \Phi(\iota)$.

Since $\iota \in [D]^{<\epsilon}$ was arbitrary, one has that $\bar{\alpha} \notin \Phi[[D]^{<\epsilon}]$. So $\Phi[[D]^{<\epsilon}] \neq \kappa$. Since Φ was arbitrary, this shows that κ is $(< \epsilon)$-Magidor. $\square$

Proposition 2.34. *Suppose κ is a weak partition cardinal (satisfies $\kappa \to_* (\kappa)^{<\kappa}_2$). Then κ is a super-Magidor cardinal.*

Proof. For any $\epsilon < \kappa$, $\kappa \to (\kappa)^{\epsilon+\epsilon}_2$ implies $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$ by Fact 2.21. The result now follows from Proposition 2.33. $\square$

Proposition 2.35. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$. ω_1 and ω_2 are super-Magidor.*

In particular, under AD, ω_1 and ω_2 are super-Magidor.

Note that ω_1 is not Magidor (that is, not $(< \omega_1)$-Magidor) but is lower-Magidor and even super-Magidor. This akwardness is due to some incompatibility with the older definition of a Magidor cardinal and the definition of a lower-Magidor and super-Magidor cardinal presented here.

Using the finite Ramsey theorem (for all $1 \leq n < \omega$, $\omega \to (\omega)^n_2$), one can show that ω is also super-Magidor using similar combinatorial arguments under just ZF.

Proposition 2.36. *ω is super-Magidor.*

3. ω_ω is Magidor

This section (and Section 5) will address the existence of Magidor cardinals of countable cofinality under AD. This section will specifically answer Question 2.7 from [1] of Ben-Neria and Garti about the consistency of ω_ω being Magidor. First, one will need a more complete survey of the Martin's ultrapower analysis below ω_ω and the combinatorial hypothesis $\omega_1 \to_* (\omega_1)^{\omega_1}_{<\omega_1}$ and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$.

There is a more practically useful equivalence of $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$.

Definition 3.1. Let $\prod_{\alpha\in\omega_1}\alpha = \{(\alpha,\beta) : \beta < \alpha\}$. A function $\mathcal{K} : \prod_{\alpha\in\omega_1}\alpha \to \omega_1$ is a Kunen function if and only if for all $\alpha < \omega_1$, $\{\mathcal{K}(\alpha,\beta) : \beta < \alpha\}$ is an ordinal which will be denoted $\Xi^{\mathcal{K}}(\alpha)$. If $f : \omega_1 \to \omega_1$, then the Kunen function $\mathcal{K}$ bounds f if and only if $\{\alpha \in \omega_1 : f(\alpha) \leq \Xi^{\mathcal{K}}(\alpha)\} \in \mu^1_{\omega_1}$. $\mathcal{K}$ strictly bounds f if and only if $\{\alpha < \omega_1 : f(\alpha) < \Xi^{\mathcal{K}}(\alpha)\} \in \mu^1_{\omega_1}$. If $\gamma < \omega_1$, then let $\mathcal{K}^\gamma : \omega \setminus (\gamma+1) \to \omega_1$ be defined by $\mathcal{K}^\gamma(\alpha) = \mathcal{K}(\alpha,\gamma)$. Let $\bigstar$ be the assumption that for all $f : \omega_1 \to \omega_1$, there is a Kunen function bounding f.

Under AD, Kunen defined the eponymous Kunen tree whose sections by different reals can be used to create Kunen functions bounding any $f : \omega_1 \to \omega_1$. This uniformity is needed for deeper analysis of the projective ordinals. Here, it suffices to know that every function has a Kunen function non-uniformly.

Fact 3.2. *(Kunen; [20] Lemma 4.1)* AD *implies* $\bigstar$. *(For every function $f : \omega_1 \to \omega_1$, there is a Kunen function bounding f.)*

Martin and Kleinberg showed that $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ implies many of the basic combinatorial properties at and below ω_ω. The assumption $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ is equivalent to $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. The fact that AD implies $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ is proved by through $\bigstar$.

One will show that $\omega_1 \to_* (\omega_1)^2_2$ and $j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$ is equivalent to $\omega_1 \to_* (\omega_1)^2_2$ and $\bigstar$. Also using Fact 2.21 (4) and the ideas from the Rowbottom lemma, one can also show $\omega_1 \to_* (\omega_1)^2_2$ is equivalent to $\mu^1_{\omega_1}$ being a normal ultrafilter.

Fact 3.3. *Assume $\omega_1 \to_* (\omega_1)^2_2$. $j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$ implies $\bigstar$.*

Proof. $\omega_1 \to_* (\omega_1)^2_2$ implies $\mu^1_{\omega_1}$ is a normal ultrafilter. Let $f : \omega_1 \to \omega_1$. Thus $[f]_{\mu^1_{\omega_1}} < j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$. There is a surjection $\Phi : \omega_1 \to [f]_{\mu^1_{\omega_1}}$. Define a wellordering on ω_1 by $\alpha \prec \beta$ if and only if $\Phi(\alpha) < \Phi(\beta)$. Let $\mathcal{W} = (\omega_1, \prec)$ and note that $\mathrm{ot}(\mathcal{W}) = [f]_{\mu^1_{\omega_1}}$. For each $\alpha < \omega_1$, let $\mathcal{W}_\alpha = (\alpha, \prec\restriction\alpha)$ be the restriction of $\prec$ to α. For any $\alpha \in \omega_1$, let $\mathrm{rk}(\mathcal{W},\alpha)$ be the rank of α in $\mathcal{W}$. For any $\alpha \in \omega_1$ and $\beta < \alpha$, let $\mathrm{rk}(\mathcal{W}_\alpha,\beta)$ be the rank of β in $\mathcal{W}_\alpha$. For $\beta < \alpha$, let $\mathcal{K}(\alpha,\beta) = \mathrm{rk}(\mathcal{W}_\alpha,\beta)$. It is clear that for all $\alpha < \omega_1$, $\{\mathcal{K}(\alpha,\beta) : \beta < \alpha\} = \mathrm{ot}(\mathcal{W}_\beta) \in \mathrm{ON}$. Thus $\mathcal{K}$ is a Kunen function and $\Xi^{\mathcal{K}}(\alpha) = \mathrm{ot}(\mathcal{W}_\alpha)$ for all $\alpha < \omega_1$. Let $\eta < [f]_{\mu^1_{\omega_1}} = \mathrm{ot}(\mathcal{W})$. Let $\gamma_\eta \in \omega_1$ be the unique γ such that $\mathrm{rk}(\mathcal{W},\gamma) = \eta$. Let $g_\eta : \omega_1 \setminus (\gamma_\eta + 1) \to \omega_1$ be defined by $g_\eta(\alpha) = \mathrm{rk}(\mathcal{W}_\alpha, \gamma_\eta)$. Note that for all $\alpha < \omega_1$, $g_\eta(\alpha) < \mathrm{ot}(\mathcal{W}_\alpha) = \Xi^{\mathcal{K}}(\alpha)$. Suppose $\eta_0 < \eta_1 < [f]_{\mu^1_{\omega_1}}$. Then $\{\alpha \in \omega_1 : g_{\eta_0}(\alpha) < g_{\eta_1}(\alpha)\} \supseteq \omega_1 \setminus (\max\{\gamma_{\eta_0},\gamma_{\eta_1}\}+1)$ and thus $\{\alpha \in \omega_1 : g_{\eta_0}(\alpha) < g_{\eta_1}(\alpha)\} \in \mu^1_{\omega_1}$. This shows that $\eta_0 < \eta_1$ implies $[g_{\eta_0}]_{\mu^1_{\omega_1}} < [g_{\eta_1}]_{\mu^1_{\omega_1}}$. The map $\Psi : [f]_{\mu^1_{\omega_1}} \to [\Xi^{\mathcal{K}}]_{\mu^1_{\omega_1}}$ defined by $\Psi(\eta) = [g_\eta]_{\mu^1_{\omega_1}}$ is an order embedding. Thus $[f]_{\mu^1_{\omega_1}} \leq [\Xi^{\mathcal{K}}]_{\mu^1_{\omega_1}}$. This implies $\{\alpha \in \omega_1 : f(\alpha) \leq \Xi^{\mathcal{K}}(\alpha)\} \in \mu$. Thus $\mathcal{K}$ is a Kunen function for f. $\square$

Now one will see the converse of Fact 3.3. Note that when one writes $j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$, this supposes that the ultrapower $j_{\mu^1_{\omega_1}}(\omega_1)$ is even wellfounded. Here one will never assume any form of dependent choice or even any form of countable choice. The most salient feature of Kunen function is that it allows the ability to select representatives. Since Magidor functions involves countable bounded subsets, to address the main question of this section, one will need to be able to choose representatives for all countable sets $A \subseteq j_{\mu^n_{\omega_1}}(\omega_1) = \omega_{n+1}$ for all $n < \omega$. If one works under AD, $\mathsf{AC}^{\mathbb{R}}_\omega$ and the Moschovakis coding lemma give $\mathsf{AC}^{\mathscr{P}(\omega_1)}_\omega$ which will be sufficient to choose representative for countable sets. However, the relevant subtheory of AD is already able to choose representative for ω_1-size subsets of $j_{\mu^n_{\omega_1}}(\omega_1) = \omega_{n+1}$.

Fact 3.4. *Assume $\omega_1 \to_* (\omega_1)^2_2$. Suppose $\mathcal{K}$ is a Kunen function. Suppose $\mathcal{K}$ strictly bounds f (or equivalently $[f]_{\mu^1_{\omega_1}} \prec_{\mu^1_{\omega_1}} [\Xi^{\mathcal{K}}]_{\mu^1_{\omega_1}}$ in the ultrapower ordering). Then there is a $\gamma < \omega_1$ so that $[f]_{\mu^1_{\omega_1}} = [\mathcal{K}^\gamma]_{\mu^1_{\omega_1}}$.*

Proof. $\omega_1 \to_* (\omega_1)^2_2$ implies $\mu^1_{\omega_1}$ is a normal ultrafilter. Let $A_0 = \{\alpha \in \omega_1 : f(\alpha) < \Xi^{\mathcal{K}}(\alpha)\} \in \mu^1_{\omega_1}$. For all $\alpha \in A_0$, $f(\alpha) \in \{\mathcal{K}(\alpha,\beta) : \beta < \alpha\}$. Let $h : A_0 \to \omega_1$ be defined by $h(\alpha)$ is the least $\beta < \alpha$ such that $f(\alpha) = \mathcal{K}(\alpha,\beta)$. Thus $A_0 = \{\alpha \in A_0 : h(\alpha) < \alpha\} \in \mu^1_{\omega_1}$. Since $\mu^1_{\omega_1}$ is normal, there is an $A_1 \subseteq A_0$ and a $\gamma < \omega_1$ so that $A_1 \in \mu^1_{\omega_1}$ and for all $\alpha \in A_1$, $h(\alpha) = \gamma$. Thus for all $\alpha \in A_1$, $f(\alpha) = \mathcal{K}(\alpha, h(\alpha)) = \mathcal{K}(\alpha,\gamma) = \mathcal{K}^\gamma(\alpha)$. $\square$

Fact 3.5. *Assume $\omega_1 \to_* (\omega_1)^2_2$. $\bigstar$ implies $j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$.*

Proof. First, one needs to show $j_{\mu^1_{\omega_1}}(\omega_1)$ under the ultrapower ordering $\prec_{\mu^1_{\omega_1}}$ is wellfounded. Suppose $j_{\mu^1_{\omega_1}}(\omega_1)$ is not wellfounded. Let $X \subseteq j_{\mu^1_{\omega_1}}(\omega_1)$ be a set with no minimal element under $\prec_{\mu^1_{\omega_1}}$. Pick any element $x \in X$ and $f : \omega_1 \to \omega_1$ such that $x = [f]_{\mu^1_{\omega_1}}$. By $\bigstar$, let $\mathcal{K}$ be a Kunen function strictly bounding f. By Fact 3.4, let $\delta_0 < \omega_1$ be least such δ such that $[\mathcal{K}^\delta]_{\mu^1_{\omega_1}} = [f]_{\mu^1_{\omega_1}} = x$. Suppose $n \in \omega$ and $\delta_n < \omega_1$ has been defined so that $[\mathcal{K}^{\delta_n}]_{\mu^1_{\omega_1}} \in X$. Since X is not wellfounded, there is a $y \in X$ and $y \prec_{\mu^1_{\omega_1}} [\mathcal{K}^{\delta_n}]_{\mu^1_{\omega_1}}$. Let $g : \omega_1 \to \omega_1$ be such that $y = [g]_{\mu^1_{\omega_1}}$. Thus $\mathcal{K}$ strictly bounds g. By Fact 3.4, there is a $\delta < \omega_1$ so that $[\mathcal{K}^\delta]_{\mu^1_{\omega_1}} = [y]_{\mu^1_{\omega_1}} \prec_{\mu^1_{\omega_1}} [\mathcal{K}^{\delta_n}]_{\mu^1_{\omega_1}}$. Let δ_{n+1} be the least $\delta < \omega_1$ such that $[\mathcal{K}^\delta]_{\mu^1_{\omega_1}} \prec_{\mu^1_{\omega_1}} [\mathcal{K}^{\delta_n}]_{\mu^1_{\omega_1}}$. This completes the construction of $\langle \delta_n : n \in \omega\rangle$ with the property that for all $n \in \omega$, $[\mathcal{K}^{\delta_{n+1}}]_{\mu^1_{\omega_1}} \prec_{\mu^1_{\omega_1}} [\mathcal{K}^{\delta_n}]_{\mu^1_{\omega_1}}$. For each $n \in \omega$, let $A_n = \{\alpha \in \omega_1 : \mathcal{K}^{\delta_{n+1}}(\alpha) < \mathcal{K}^{\delta_n}(\alpha)\} \in \mu^1_{\omega_1}$. Note $A = \bigcap_{n\in\omega} A_n \in \mu^1_{\omega_1}$ since $\mu^1_{\omega_1}$ is countably complete by $\omega_1 \to_* (\omega_1)^2_2$. In particular, $A \neq \emptyset$. Let $\bar{\alpha} \in A$. For all $n \in \omega$, $\bar{\alpha} \in A_n$ implies $\mathcal{K}^{\delta_{n+1}}(\bar{\alpha}) < \mathcal{K}^{\delta_n}(\alpha)$. Thus $\langle \mathcal{K}^{\delta_n}(\bar{\alpha}) : n \in \omega\rangle$ is an infinite descending sequence of ordinals under the usual ordinal ordering which is a contradiction. This shows that $j_{\mu^1_{\omega_1}}(\omega_1)$ is a wellordering.

Thus one can identify $j_{\mu^1_{\omega_1}}(\omega_1)$ with an ordinal. Let $x \in j_{\mu^1_{\omega_1}}(\omega_1)$ and let $f : \omega_1 \to \omega_1$ be such that $x = [f]_{\mu^1_{\omega_1}}$. By $\bigstar$, let $\mathcal{K}$ be a Kunen function bounding f. By Fact 3.4, every $y \prec_{\mu^1_{\omega_1}} x$, there is a $\delta < \omega_1$ so that $[\mathcal{K}^\delta]_{\mu^1_{\omega_1}} = y$. Let $\mathsf{init}_{\mu^1_{\omega_1}}(x) = \{y \in j_{\mu^1_{\omega_1}}(\omega_1) : y \prec_{\mu^1_{\omega_1}} x\}$. Let $\Gamma : \mathsf{init}_{\mu^1_{\omega_1}}(x) \to \omega_1$ be defined by $\Gamma(y)$ is the least $\delta < \omega_1$ be such that $[\mathcal{K}^\delta]_{\mu^1_{\omega_1}} = y$. Γ is an injection of the initial segment of x into ω_1. Since $j_{\mu^1_{\omega_1}}(\omega_1)$ is a wellordering and essentially an ordinal, this implies $j_{\mu^1_{\omega_1}}(\omega_1) \leq (\omega_1)^+ = \omega_2$. ☐

Thus Fact 3.3 and Fact 3.5 imply that over $\omega_1 \to_* (\omega_1)^2_2$, $j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$ is equivalent to $\bigstar$.

If one further assumes the strong partition property $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, one can prove that $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ and $j_{\mu^1_{\omega_1}}(\omega_1)$ is regular by a result of Martin concerning ultrapowers of strong partition cardinals. See [6] for a proof.

Fact 3.6. *(Martin) Assume $\kappa \to_* (\kappa)^\kappa_*$.*

- *If μ is a measure on κ such that $j_\mu(\kappa)$ is a wellordering, then $j_\mu(\kappa)$ is a cardinal.*
- *If μ is a normal measure on κ such that $j_\mu(\kappa)$ is a wellordering, then $j_\mu(\kappa)$ is a regular cardinal.*

Fact 3.7. *(Martin) Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. Then $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ and ω_2 is regular.*

Proof. Fact 3.5 already implies $j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$. $\omega_1 \to_* (\omega_1)^2_2$ implies $\mu^1_{\omega_1}$ is a normal ultrafilter. Thus $\omega_1 = [\mathsf{id}]_{\mu^1_{\omega_1}} < j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$. Fact 3.6 implies $j_{\mu^1_{\omega_1}}(\omega_1)$ must be a cardinal above ω_1 and less than or equal to ω_2. Hence $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ and ω_2 is regular. ☐

Thus over $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, $\bigstar$ is equivalent to $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$.

Fact 3.8. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. If $A \subseteq \omega_2$ with $|A| \leq \omega_1$, then there is a function Γ on A so that for all $x \in A$, $\Gamma(x) : \omega_1 \to \omega_1$ and $x = [\Gamma(x)]_{\mu^1_{\omega_1}}$.*

Proof. Since $\mathrm{cof}(\omega_2) = \omega_1$ and $|A| \leq \omega_1$, $\sup(A) < \omega_1$. Let $f : \omega_1 \to \omega_1$ be such that $[f]_{\mu^1_{\omega_1}} = \sup(A)$. By $\bigstar$, let $\mathcal{K}$ be a Kunen function strictly bounding f. Let $x \in A$ and pick any $g : \omega_1 \to \omega_1$ so that $x = [g]_{\mu^1_{\omega_1}}$. Then $\mathcal{K}$ is a Kunen function strictly bounding g. By Fact 3.4, there is a $\gamma < \omega_1$ so that $x = [g]_{\mu^1_{\omega_1}} = [\mathcal{K}^\gamma]_{\mu^1_{\omega_1}}$. It has been shown that for all $x \in A$, there is a $\gamma \in \omega_1$ so that $x = [\mathcal{K}^\gamma]_{\mu^1_{\omega_1}}$. For each $x \in A$, let γ_x be the least such γ. Define $\Gamma(x) = \mathcal{K}^{\gamma_x}$. Γ is the desired function. ☐

If κ is a regular cardinal, $X \subseteq \kappa$ with $\mathrm{ot}(X) = \kappa$, and $\alpha < \kappa$, let $\mathsf{next}^\alpha_X : \kappa \to X$ be defined by $\mathsf{next}^\alpha_X(\beta)$ is the $(1+\alpha)^{\text{th}}$-element of X greater than β. Given a club $C \subseteq \kappa$, then the following subclub is very useful for many constructions.

Fact 3.9. *If $C \subseteq \kappa$ is a club consisting of indecomposable ordinals, then let $D = \{\alpha \in C : \mathsf{enum}_C(\alpha) = \alpha\}$. Then D is a club subset of κ and for all $\epsilon \in D$ and all $\alpha, \beta, \gamma, \delta < \epsilon$, $\mathsf{next}^{\alpha\cdot\beta+\gamma}_C(\delta) < \epsilon$.*

Fact 3.10. *Let κ be a regular cardinal, $\epsilon < \kappa$, and $\kappa \to_* (\kappa)^{\epsilon+1}_2$ holds. Let $\Phi : [\kappa]^\epsilon \to \kappa$. Then there is a club $C \subseteq \kappa$ so that for all $\iota \in [C]^\epsilon_*$, $\Phi(\iota) < \mathsf{next}^\omega_C(\sup(\iota))$.*

Proof. Define $P : [\kappa]^{\epsilon+1} \to 2$ by $P(\ell) = 0$ if and only if $\Phi(\ell \upharpoonright \epsilon) < \ell(\epsilon)$. By $\kappa \to_* (\kappa)^{\epsilon+1}_2$, there is a club $C \subseteq \kappa$ which is homogeneous for P. Pick any $\iota \in [C]^\epsilon_*$. Let $\ell \in [C]^{\epsilon+1}_*$ be such that $\ell \upharpoonright \epsilon = \iota$ and $\ell(\epsilon) > \max\{\sup(\iota), \Phi(\iota)\}$. Then $\Phi(\ell \upharpoonright \epsilon) = \Phi(\iota) < \ell(\epsilon)$. Thus $P(\ell) = 0$. Since C is homogeneous for P, one has that C is homogeneous for P taking value 0. For any $\iota \in [C]^\epsilon$, let $\ell_\iota \in [C]^{\epsilon+1}_*$ be defined by $\ell_\iota = \iota\hat{\ }\langle \mathsf{next}^\omega_C(\sup(\iota))\rangle$. Then $P(\iota_\ell) = 0$ implies that $\Phi(\iota) = \Phi(\ell_\iota \upharpoonright \epsilon) < \ell_\iota(\epsilon) = \mathsf{next}^\omega_C(\sup(\iota))$. ☐

Definition 3.11. Let κ be an uncountable cardinal and $\Phi : [\kappa]^\epsilon \to \kappa$. Say that a club C is Φ-bounding if and only if C consists only of indecomposable ordinals and for all $\ell \in [C]^{\epsilon+1}_*$, $\Phi(\ell \upharpoonright \epsilon) < \ell(\epsilon)$.

Fact 3.12. *Let κ be a regular cardinal, $\epsilon < \kappa$, and $\kappa \to_* (\kappa)^{\epsilon+1}_2$ holds. Let $\Phi : [\kappa]^\epsilon \to \kappa$. Then there is a Φ-bounding club.*

Proof. By Fact 3.10, there is a club C_0 so that for all $\iota \in [C_0]^\epsilon_*$, $\Phi(\iota) < \mathsf{next}^\omega_{C_0}(\sup(\iota))$. One may assume C_0 consists only of indecomposable ordinals. Let $C_1 = \{\alpha \in C_0 : \mathsf{enum}_{C_0}(\alpha) = \alpha\}$. For any $\ell \in [C_1]^{\epsilon+1}_*$, $\Phi(\ell \upharpoonright \epsilon) < \mathsf{next}^\omega_{C_0}(\sup(\ell \upharpoonright \epsilon)) < \ell(\epsilon)$ since $\sup(\ell \upharpoonright \epsilon) \in C_0$, $\ell(\epsilon) \in C_1$, and using Fact 3.9. ☐

Fact 3.13. *Let κ be an uncountable cardinal, $\delta < \epsilon \leq \kappa$, $\kappa \to (\kappa)^{\delta+1+(\epsilon-\delta)}_2$, and $\kappa \to_* (\kappa)^{\epsilon-\delta}_{<\kappa}$. Let $\Phi : [\kappa]^\epsilon \to \kappa$ be such that $\{\ell \in [\kappa]^\epsilon : \Phi(\ell) < \ell(\delta)\} \in \mu^\epsilon_\kappa$. Then there is a club $C \subseteq \kappa$ and a function $\Psi : [C]^\delta_* \to \kappa$ so that for all $\ell \in [C]^\epsilon_*$, $\Phi(\ell) = \Psi(\ell \upharpoonright \delta)$.*

Proof. Let $C_0 \subseteq \kappa$ be a club so that $\Phi(\ell) < \ell(\delta)$ for all $\ell \in [C_0]^\epsilon_*$. If $g \in [\kappa]^{\delta+1+(\epsilon-\delta)}$, then let $\hat{g} \in [\kappa]^\epsilon$ be defined by

$$\hat{g}(\alpha) = \begin{cases} g(\alpha) & \alpha < \delta \\ g(\delta + 1 + (\alpha - \delta)) & \delta \leq \alpha \end{cases}.$$

Define $P : [\kappa]^{\delta+1+(\epsilon-\delta)} \to 2$ by $P(g) = 0$ if and only if $\Phi(\hat{g}) < g(\delta)$. By $\kappa \to_* (\kappa)^{\delta+1+(\epsilon-\delta)}_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for P. Let $C_2 = \{\alpha \in C_1 : \mathsf{enum}_{C_1}(\alpha) = \alpha\}$. Let $f \in [C_2]^\epsilon_*$. By the property of C_0, $\Phi(f) < f(\delta)$. By Fact 3.9, one has that $\mathsf{next}^\omega_{C_1}(\max\{\sup(f \upharpoonright \delta), \Phi(f)\}) < f(\delta)$. Let $g \in [C_1]^{\delta+1+(\epsilon-\delta)}_*$ be defined by

$$g(\alpha) = \begin{cases} f(\alpha) & \alpha < \delta \\ \mathsf{next}^\omega_{C_1}(\max\{\sup(f \upharpoonright \delta), \Phi(f)\}) & \alpha = \delta \\ f(\delta + (\alpha - (\delta + 1))) & \delta < \alpha \end{cases}.$$

Since $\Phi(\hat{g}) = \Phi(f) < \mathsf{next}^\omega_{C_1}(\max\{\sup(f \upharpoonright \delta), \Phi(f)\}) = g(\delta)$, one has that $P(g) = 0$. Thus C_1 must be homogeneous for P taking value 0. Let $f \in [C_2]^\epsilon_*$, let $g_f \in [C_1]^{\delta+1+(\epsilon-\delta)}_*$ be defined by

$$g_f(\alpha) = \begin{cases} f(\alpha) & \alpha < \delta \\ \mathsf{next}^\omega_{C_1}(\sup(f \upharpoonright \delta)) & \alpha = \delta \\ f(\delta + (\alpha - (\delta + 1)) & \delta < \alpha \end{cases}.$$

Then $P(g_f) = 0$ implies that $\Phi(f) = \Phi(\hat{g}_f) < g_f(\delta) = \mathsf{next}^\omega_{C_1}(\sup(f \upharpoonright \delta))$. It has been shown that for all $f \in [C_2]^\epsilon_*$, $\Phi(f) < \mathsf{next}^\omega_{C_1}(\sup(f \upharpoonright \delta))$. For each $\tau \in [C_2]^\delta_*$, let $\Phi_\tau : [C_2 \setminus (\sup(\tau) + 1)]^{\epsilon-\delta}_* \to \kappa$ be defined by $\Phi_\tau(\sigma) = \Phi(\tau\hat{\ }\sigma)$. By the discussion above, for all $\sigma \in [C_2 \setminus (\sup(\tau)+1)]^{\epsilon-\delta}_*$, $\Phi_\tau(\sigma) = \Phi(\tau\hat{\ }\sigma) < \mathsf{next}^\omega_{C_1}(\sup(\tau))$. By $\kappa \to_* (\kappa)^{\epsilon-\delta}_{<\kappa}$, there is a $\zeta_\tau \in \kappa$ so that for $\mu^{\epsilon-\delta}_\kappa$-almost all σ, $\Phi_\tau(\sigma) = \zeta_\tau$. Define $Q : [C_2]^\epsilon_* \to 2$ by $Q(f) = 0$ if and only if $\Phi(f) = \zeta_{f \upharpoonright \delta}$. By $\kappa \to_* (\kappa)^\epsilon_2$, let $C_3 \subseteq C_2$ be a club homogeneous for Q. Pick any $\tau \in [C_3]^\delta_*$. There is a club $D \subseteq C_3$ so that for all $\sigma \in [D]^{\epsilon-\delta}_*$, $\Phi_\tau(\sigma) = \zeta_\tau$. Pick any $\sigma \in [D]^{\epsilon-\delta}_*$ with $\sup(\tau) < \sigma(0)$. Let $f = \tau\hat{\ }\sigma$ and note that $f \in [C_3]^\epsilon_*$. Then $\Phi(f) = \Phi_{f \upharpoonright \delta}(\mathrm{drop}(f, \delta)) = \Phi_\tau(\sigma) = \zeta_\tau = \zeta_{f \upharpoonright \delta}$. So $Q(f) = 0$. This shows that C_3 is homogeneous for Q taking value 0. Define $\Psi : [C_3]^\delta_* \to \kappa$ by $\Psi(\tau) = \zeta_\tau$. It has been shown that for all $f \in [C_3]^\epsilon_*$, $\Phi(f) = \Psi(f \upharpoonright \delta)$. ☐

Fact 3.14. *Suppose κ is an uncountable cardinal, $\delta < \epsilon \leq \kappa$, $\kappa \to_* (\kappa)^{\delta+1+(\epsilon-\delta)}_2$, and $\kappa \to_* (\kappa)^{\epsilon-\delta}_{<\kappa}$. Let $\Sigma^\epsilon_\delta : [\kappa]^\epsilon \to \kappa$ be defined by $p^\epsilon_\delta(\ell) = \ell(\delta)$. For $\Phi : [\kappa]^\delta \to \kappa$, let $\hat{\Phi} : [\kappa]^\epsilon \to \kappa$ be defined by $\hat{\Phi}(\ell) = \Phi(\ell \upharpoonright \delta)$. Define $\Gamma : j_{\mu^\delta_\kappa}(\kappa) \to j_{\mu^\epsilon_\kappa}(\kappa)$ by $\Gamma(x) = [\hat{\Phi}]_{\mu^\epsilon_\kappa}$ for any $\Phi : [\kappa]^\delta \to \kappa$ such that $[\Phi]_{\mu^\delta_\kappa} = x$. Γ is a well defined order preserving bijection into $\mathsf{init}_{\mu^\epsilon_\kappa}([\Sigma^\epsilon_\delta]_{\mu^\epsilon_\kappa})$.*

Proof. It is clear that Γ is well defined and order preserving. Let $\Phi : [\kappa]^\delta \to \kappa$. By Fact 3.12, there is a club $C \subseteq \kappa$ which is Φ-bounding. For all $\ell \in [C]^\epsilon_*$, $\hat{\Phi}(\ell) = \Phi(\ell \restriction \delta) < \ell(\delta) = \Sigma^\epsilon_\delta(\ell)$. So $\Gamma([\Phi]_{\mu^\delta_\kappa}) \in \mathsf{init}_{\mu^\epsilon_\kappa}([\Sigma^\epsilon_\delta]_{\mu^\epsilon_\kappa})$. Now suppose $\Upsilon : [\kappa]^\epsilon \to \kappa$ such that $[\Upsilon]_{\mu^\epsilon_\kappa} \in \mathsf{init}_{\mu^\epsilon_\kappa}([\Sigma^\epsilon_\delta]_{\mu^\epsilon_\kappa})$. This means $\{\ell \in [\kappa]^\epsilon_* : \Upsilon(\ell) < \Sigma^\epsilon_\delta(\ell) = \ell(\delta)\} \in \mu^\epsilon_\kappa$. By Fact 3.13, there is a $\Psi : [\kappa]^\delta_* \to \kappa$ and a club $D \subseteq \kappa$ so that for all $\ell \in [D]^\epsilon_*$, $\Upsilon(\ell) = \Psi(\ell \restriction \delta)$. For all $\ell \in [D]^\epsilon_*$, $\hat{\Psi}(\ell) = \Psi(\ell \restriction \delta) = \Upsilon(\ell)$. Thus $\Gamma([\Psi]_{\mu^\delta_{\omega_1}}) = [\Upsilon]_{\mu^\epsilon_\kappa}$. This shows that Γ is a bijection onto $\mathsf{init}_{\mu^\epsilon_\kappa}([\Sigma^\epsilon_\delta]_{\mu^\epsilon_\kappa})$. $\square$

Fact 3.15. *Let κ be an uncountable cardinal, $\epsilon < \kappa$, and $\kappa \to_* (\kappa)^{\epsilon+1}_2$. If $f : \kappa \to \kappa$, let $\hat{f} : [\kappa]^\epsilon \to \kappa$ be defined by $\hat{f}(\ell) = f(\sup(\ell))$. Define $\rho : j_{\mu^1_\kappa}(\kappa) \to j_{\mu^\epsilon_\kappa}(\kappa)$ by $\rho([f]_{\mu^1_{\omega_1}}) = [\hat{f}]_{\mu^\epsilon_\kappa}$. Then $\hat{\rho}$ is a well defined increasing cofinal map of $j_{\mu^1_\kappa}(\kappa)$ into $j_{\mu^\epsilon_\kappa}(\kappa)$ (in the ultrapower orderings).*

Proof. Let $\Phi : [\kappa]^\epsilon \to \kappa$. By Fact 3.10, there is a club $C \subseteq \kappa$ so that for all $\ell \in [C]^\epsilon_*$, $\Phi(\ell) < \mathsf{next}^\omega_C(\sup(\ell))$. Let $f : \kappa \to \kappa$ be defined by $f = \mathsf{next}^\omega_C$. Thus $[\Phi]_{\mu^\epsilon_\kappa} < [\hat{f}]_{\mu^\epsilon_\kappa} = \rho([f]_{\mu^1_\kappa})$. $\square$

Definition 3.16. Let $1 \leq n < \omega$ and $h : [\omega_1]^n \to \omega_1$. Define the partial function $\mathcal{K}^{n,h} : [\omega_1]^{n+1} \to \omega_1$ by $\mathcal{K}^{n,h}(\ell) = \mathcal{K}(\ell(n), h(\ell \restriction n))$ for all $\ell \in [\omega_1]^{n+1}$ such that $h(\ell \restriction n) < \ell(n)$.

Assume $\omega_1 \to_* (\omega_1)^{n+1}_2$, by Fact 3.12, any function $h : [\omega_1]^n \to \omega_1$ has an h-bounding club C. Thus for any $\ell \in [C]^{n+1}_*$, $\mathcal{K}^{n,n}(\ell)$ is defined. Also for $n = 0$, $[\omega_1]^0 = \{\emptyset\}$ so $h : [\omega_1]^0 \to \omega_1$ may be regarded as a constant γ. Then $\mathcal{K}^{0,h}$ is $\mathcal{K}^\gamma$ of the earlier notation.

For the main question, one will need Fact 3.18 (4) for just countable $A \subseteq \omega_{n+1}$. (Again, under AD, this can be obtained by $\mathsf{AC}^{\mathscr{P}(\omega_1)}_\omega$ which follows from $\mathsf{AC}^{\mathbb{R}}_\omega$ and the Moschovakis coding lemma.) It seems that one needs to inductive prove all four statements in Fact 3.18 even if one is only interested in statement (4). The proof of Fact 3.18 only need statement (4) for countable $A \subseteq j_{\mu^n_{\omega_1}}(\omega_1) = \omega_{n+1}$, but many other combinatorial problems below ω_ω (such as the weak partition property on ω_2) requires this result for A with $|A| \leq \omega_1$.

Definition 3.17. For any $f : [\omega_1]^{n+1} \to \omega_1$, let $J_f : \omega_1 \to \omega_1$ be defined by $J_f(\alpha) = \sup\{f(\ell) : \ell(n) = \alpha\}$.

Fact 3.18. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. For all $1 \leq n < \omega$, one has the following:*

(1) $j_{\mu^n_{\omega_1}}(\omega_1)$ is a wellordering.
(2) $j_{\mu^n_{\omega_1}}(\omega_1) = \omega_{n+1}$.
(3) $\mathrm{cof}(\omega_{n+1}) = \omega_2$.
(4) If $A \subseteq j_{\mu^n_{\omega_1}}(\omega_1) = \omega_{n+1}$ and $|A| \leq |\omega_1|$, then there is a function Γ on A so that for all $x \in A$, $\Gamma(x) : [\omega_1]^n \to \omega_1$ and $x = [\Gamma(x)]_{\mu^n_{\omega_1}}$.

Proof. This result is proved by induction on n. For $n = 1$, this has already been shown by Fact 3.7 and Fact 3.8. Now suppose all four properties hold at n.

First, one will show that $j_{\mu^{n+1}_{\omega_1}}(\omega_1)$ is a wellordering. Suppose not. Let $X \subseteq j_{\mu^{n+1}_{\omega_1}}(\omega_1)$ be a nonempty set with no minimal element in the ultrapower ordering $\prec_{\mu^{n+1}_{\omega_1}}$. Pick any $x \in X$ and let $f : [\omega_1]^{n+1} \to \omega_1$ be such that $[f]_{\mu^{n+1}_{\omega_1}} = x$. By $\bigstar$, let $\mathcal{K}$ be any Kunen function bounding J_f. Let $y \prec_{\mu^{n+1}_{\omega_1}} x$. Let $\tilde{g} : [\omega_1]^{n+1} \to \omega_1$ be any representative for y. Since $y \prec_{\mu^{n+1}_{\omega_1}} x$, the set $E = \{\ell \in [\omega_1]^{n+1} : \tilde{g}(\ell) < f(\ell)\} \in \mu^{n+1}_{\omega_1}$. Define $g : [\omega_1]^{n+1} \to \omega_1$ by $g(\ell) = \tilde{g}(\ell)$ if $\ell \in E$ and $g(\ell) = 0$ if otherwise. Then $y = [\tilde{g}]_{\mu^{n+1}_{\omega_1}} = [g]_{\mu^{n+1}_{\omega_1}}$ and $g(\ell) \leq f(\ell)$ for all $\ell \in [\omega_1]^{n+1}_*$. Then $[J_g]_{\mu^1_{\omega_1}} \leq [J_f]_{\mu^1_{\omega_1}}$. Hence $\mathcal{K}$ is also a Kunen function strictly bounding J_g. Let $C = \{\alpha \in \omega_1 : J_g(\alpha) < \Xi^{\mathcal{K}}(\alpha)\}$. For any $\ell \in [C]^{n+1}$, $g(\ell) < J_g(\ell(n)) < \Xi^{\mathcal{K}}(\ell(n)) = \{\mathcal{K}(\ell(n), \beta) : \beta < \ell(n)\}$. Let $\hat{h} : [\omega_1]^{n+1} \to \omega_1$ be defined by $\hat{h}(\ell)$ is the least $\beta < \ell(n)$ so that $g(\ell) = \mathcal{K}(\ell(n), \beta)$. For all $\ell \in [C]^{n+1}$, $\hat{h}(\ell) < \ell(n)$. By Fact 3.13, there is an $h : [\omega_1]^n \to \omega_1$ and a club $D \subseteq C$ so that for all $\ell \in [D]^{n+1}$, $\hat{h}(\ell) = h(\ell \restriction n)$. Note that for all $\ell \in [D]^{n+1}$, $g(\ell) = \mathcal{K}(\ell(n), h(\ell \restriction n)) = \mathcal{K}^{n,h}(\ell)$. By the inductive hypothesis, $j_{\mu^n_{\omega_1}}(\omega_1) = \omega_{n+1}$ and thus $[h]_{\mu^n_{\omega_1}} \in \omega_{n+1}$. It has been shown that for all $y \prec_{\mu^{n+1}_{\omega_1}} x$, there is a $\gamma < \omega_{n+1}$ so that for all $h : [\omega_1]^n \to \omega_1$ with $[h]_{\mu^n_{\omega_1}} = \gamma$, $y = [\mathcal{K}^{n,h}]_{\mu^{n+1}_{\omega_1}}$. Let γ_y be the least such $\gamma < \omega_{n+1}$ with the previous property for y. Let $A = \{\gamma_y : y \in X\}$. Let δ_0 be the least member of A. Suppose δ_k has been defined so $y_k \in X$ where $y_k = [\mathcal{K}^{n,h}]_{\mu^{n+1}_{\omega_1}}$ for any $h : [\omega_1]^n \to \omega_1$ with $\delta_k = [h]_{\mu^n_{\omega_1}}$. Since X has no minimal element, there is some $y \in X$ with $y \prec_{\mu_{\omega_1}^{n+1}} y_n$. Thus there is some $\delta \in A$ so that $y = [\mathcal{K}^{n,h}]_{\mu_{\omega_1}^{n+1}}$ for any h such that $[h]_{\mu_{\omega_1}^n} = \delta$. Let δ_{k+1} be the least $\delta \in A$ so that $[\mathcal{K}^{n,h}]_{\mu_{\omega_1}^{n+1}} < y_k$ for any $h : [\omega_1]^n \to \omega_1$ with $[h]_{\mu_{\omega_1}^n} = \delta$. Note that $y_{k+1} = [\mathcal{K}^{n,h}]_{\mu_{\omega_1}^{n+1}} \in X$ for any $h : [\omega_1]^n \to \omega_1$ with $\delta_{k+1} = [h]_{\mu_{\omega_1}^n}$ since $\delta_{k+1} \in A$. Let $B = \{\delta_k : k \in \omega\}$. Since $B \subseteq \omega_{n+1}$ and $|B| \leq |\omega| < |\omega_1|$, by the induction hypothesis at n, there is a function Γ on B so that for all $\delta \in B$, $\Gamma(\delta) : [\omega_1]^n \to \omega_1$ and $\delta = [\Gamma(\delta)]_{\mu_{\omega_1}^n}$. Let $h_k = \Gamma(\delta_n)$. One has defined a sequence $\langle h_k : n \in \omega\rangle$ with the property that for all $n \in \omega$, $E_n = \{\ell \in [\omega_1]^{n+1} : \mathcal{K}^{n,h_{k+1}}(\ell) < \mathcal{K}^{n,h_k}(\ell)\} \in \mu_{\omega_1}^{n+1}$. Then $E = \bigcap_{k\in\omega} E_k \in \mu_{\omega_1}^{n+1}$ since $\mu_{\omega_1}^{n+1}$ is countably complete. Pick any $\bar{\ell} \in E$. Then $\langle \mathcal{K}^{n,h_k}(\bar{\ell}) : k \in \omega\rangle$ is an infinite descending sequence of ordinals in the usual ordinal ordering. This is a contradiction. This shows $j_{\mu_{\omega_1}^{n+1}}(\omega_1)$ is a wellordering.

By Fact 3.14, $j_{\mu_{\omega_1}^n}(\omega_1)$ order embeds into a proper initial segment of $j_{\mu_{\omega_1}^{n+1}}(\omega_1)$. Thus $\omega_{n+1} = j_{\mu_{\omega_1}^n}(\omega_1) < j_{\mu_{\omega_1}^{n+1}}(\omega_1)$. Let $x \in j_{\mu_{\omega_1}^{n+1}}(\omega_1)$. Let $f : [\omega_1]^{n+1} \to \omega_1$ be such that $[f]_{\mu_{\omega_1}^{n+1}} = x$. By $\bigstar$, let $\mathcal{K}$ be a Kunen function bounding J_f. By the argument above, for each $y < x$, there is a $\delta < \omega_{n+1}$ so that for any $h : [\omega_1]^n \to \omega_1$ with $\delta = [h]_{\mu_{\omega_1}^n}$, $y = [\mathcal{K}^{n,h}]_{\mu_{\omega_1}^n}$. Let δ_y be the least such δ. Let $\Phi : \mathsf{init}_{\mu_{\omega_1}^{n+1}}(x) \to \omega_{n+1}$ be defined by $\Phi(y) = \delta_y$. Φ is an injection and thus, $|\mathsf{init}_{\mu_{\omega_1}^{n+1}}(x)| \leq \omega_{n+1}$. Since $j_{\mu_{\omega_1}^{n+1}}(\omega_1)$ has been shown to be a wellordering and hence an ordinal, this implies that $j_{\mu_{\omega_1}^{n+1}}(\omega_1) \leq (\omega_{n+1})^+ = \omega_{n+2}$. By Fact 3.6, $j_{\mu_{\omega_1}^{n+1}}(\omega_1)$ must be a cardinal strictly greater than ω_{n+1} and less than or equal to ω_{n+2}. Thus $j_{\mu_{\omega_1}^{n+1}}(\omega_1) = \omega_{n+2}$.

Note that $\mathrm{cof}(\omega_{n+2}) = \omega_2$ follows from Fact 3.15.

Let $A \subseteq \omega_{n+2} = j_{\mu_{\omega_1}^{n+1}}(\omega_1)$ with $|A| \leq \omega_1$. Since it has just been shown that $\mathrm{cof}(\omega_{n+2}) = \omega_2$, $\sup(A) < \omega_{n+2}$. Let $f : [\omega_1]^{n+1} \to \omega_1$ be such that $\sup(A) = [f]_{\mu_{\omega_1}^{n+1}}$. By $\bigstar$, let $\mathcal{K}$ be a Kunen function bounding J_f. By the argument above, for each $x \in A$, there is a $\delta < \omega_{n+1}$ so that for any $h : [\omega_1]^n \to \omega_1$ with $\delta = [h]_{\mu_{\omega_1}^n}$, $x = [\mathcal{K}^{n,h}]_{\mu_{\omega_1}^{n+1}}$. Let δ_x be the least such δ. Let $B = \{\delta_x : x \in A\}$. Note that $B \subseteq \omega_{n+1}$ and $|B| \leq \omega_1$. By the induction hypothesis at n, there is a function Σ on B so that for all $\delta \in B$, $\Sigma(\delta) : [\omega_1]^n \to \omega_1$ and $[\Sigma(\delta)]_{\mu_{\omega_1}^n} = \delta$. For each $x \in A$, let $\Gamma(x) = \mathcal{K}^{n,\Sigma(\delta_x)}$. Then $x = [\Gamma(x)]_{\mu_{\omega_1}^{n+1}}$ for all $x \in A$.

The result has been shown at $n+1$. The full result follows by induction. $\square$

Fact 3.19. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. If $A \subseteq \omega_\omega$ with $\sup(A) < \omega_\omega$ and $|A| \leq \omega_1$, then there is a function Γ on A with the following properties:*

(1) If $\alpha \in A$ and $\alpha < \omega_1$, then $\Gamma(\alpha) = \alpha$.
(2) If there is an $1 \leq n < \omega$ so that $\alpha \in \omega_{n+1} \setminus \omega_n$, then $\Gamma(\alpha) : [\omega_1]^n \to \omega_1$ and $\alpha = [\Gamma(\alpha)]_{\mu_{\omega_1}^n}$.

Proof. Since $\sup(A) < \omega_\omega$, let $\bar{n}$ be least $n \in \omega$ such that $A \subseteq \omega_{n+1}$. Let $A_0 = \{\alpha \in A : \alpha < \omega_1\}$. For $1 \leq n \leq \bar{n}$, let $A_n = \{\alpha \in A : \omega_n \leq \alpha < \omega_{n+1}\}$. Let Γ_0 be the identity function on A_0. For $1 \leq n < \bar{n}$, let Γ_n be a function on A_n with the property that for all $\alpha \in A_n$, $\alpha = [\Gamma_n(\alpha)]_{\mu_{\omega_1}^n}$ obtained from Fact 3.18 (4) applied to A_n. Define Γ on A by $\Gamma(\alpha) = \Gamma_n(\alpha)$ where n is unique such that $\alpha \in A_n$. $\square$

Jackson showed that for any ordinal $\alpha < \omega_\omega$, α has a unique type. That is, for any $1 \leq n < \omega$ and ordinal $\alpha \in \omega_{n+1} \setminus \omega_n$, there is permutation of n inducing a wellordering on $[\omega_1]^n$ and a particular uniform cofinality so that $\alpha = [f]_{\mu_{\omega_1}^n}$ where $f : [\omega_1]^n \to \omega_1$ is a function respecting the given wellordering on $[\omega_1]^n$ and has the specified uniform cofinality. This analysis of type for ordinals is important for Jackson's description theory and the measures on ω_ω roughly corresponds to these possible types. For the purpose of this section, one will only need some nice types which will be described below.

Definition 3.20. Suppose $\mathcal{X} = (X, \prec)$ be a linear ordering. The lexicographic ordering $<^{\mathcal{X}}_{\mathrm{lex}}$ on ${}^{<\omega}X$ is defined by

- $\iota \subsetneq \ell$ (ι is a proper substring of ℓ).
- If $k < |\iota|$ is least so that $\iota(k) \neq \ell(k)$, then $\iota(k) \prec \ell(k)$.

Definition 3.21. For $1 \leq n < \omega$. When one writes $(\alpha_0, ..., \alpha_{n-1}) \in [\omega_1]^n$, the implicit assumption is that $\alpha_0 < \alpha_1 < ... < \alpha_{n-1}$. Define $\sqsubset_n$ on $[\omega_1]^n$ by $(\alpha_0, ..., \alpha_{n-1}) \sqsubset_n (\beta_0, ..., \beta_{n-1})$ if and only if the least $i < n$ such that $\alpha_{n-1-i} \neq \beta_{n-1-i}$, then $\alpha_{n-1-i} < \beta_{n-1-i}$. ($\sqsubset_n$ is the reverse lexicographic ordering on $[\omega_1]^n$ which can be more explicitly be written as $(\alpha_0, ..., \alpha_{n-1}) \sqsubset_n (\beta_0, ..., \beta_{n-1})$ if and only if $(\alpha_{n-1}, \alpha_{n-2}, ..., \alpha_0) <^{\omega_1}_{\mathrm{lex}} (\beta_{n-1}, \beta_{n-2}, ..., \beta_0)$.) Let $\mathcal{T}_n = ([\omega_1]^n, \sqsubset_n)$. Note that $\mathrm{ot}(\mathcal{T}_n) = \omega_1$.

A function $f : [\omega_1]^n \to \omega_1$ has type n if and only if the following holds:

- f is order preserving between $\mathcal{T}_n$ into $(\omega_1, <)$ with the usual ordering.
- f is discontinuous everywhere: for all $\ell \in [\omega_1]^n$, $\sup(f \restriction \ell) = \sup\{f(\iota) : \iota \sqsubset_n \ell\} < f(\ell)$.
- f has uniform cofinality ω: there is a function $F : [\omega_1]^n \times \omega \to \omega_1$ so that for all $\ell \in [\omega_1]^n$ and $k \in \omega$, $F(\ell, k) < F(\ell, k+1)$ and $f(\ell) = \sup\{F(\ell, k) : k \in \omega\}$.

For $1 \leq n < \omega$, let $\mathfrak{B}_{n+1}$ be the set of $[f]_{\mu^n_{\omega_1}}$ such that $f : [\omega_1]^n \to \omega_1$ has type n. Note that $\mathfrak{B}_{n+1} \subseteq \omega_{n+1} \setminus \omega_n$. If $C \subseteq \omega_1$ is a club, then let $\mathfrak{B}^C_{n+1}$ be the set of $[f]_{\mu^n_{\omega_1}}$ such that $f : [\omega_1]^n \to C$ has type n.

Definition 3.22. Let $1 \leq n < \omega$. Suppose $f : [\omega_1]^n \to \omega_1$ be a function which is order preserving on $\mathcal{T}_n = ([\omega_1]^n, \sqsubset_n)$. For each $1 \leq k \leq n$, define $I^k_f : [\omega_1]^k \to \omega_1$ by $I^k_f(\iota) = \sup\{f(\tau\hat{}\iota) : \tau \in [\kappa]^{n-k} \wedge \sup(\tau) < \iota(0)\}$. (Note that $I^n_f = f$.)

Definition 3.22 is made only for functions $f : [\omega_1]^n \to \omega_1$ which are order preserving with respect to $\sqsubset_n$. There is a more general invariant for any function $f : [\omega_1]^n \to \omega_1$ in [20] but it will not be needed here.

If $f, g : [\omega_1]^n \to \omega_1$ are two function such that $[f]_{\mu^n_{\omega_1}} = [g]_{\mu^n_{\omega_1}}$, then it is possible that $[J_f]_{\mu^1_{\omega_1}} \neq [J_g]_{\mu^1_{\omega_1}}$ (where recall J_f was defined in Definition 3.17). However, if one assumed f and g are everywhere monotonic, then one will have $[J_f]_{\mu^1_{\omega_1}} = [J_g]_{\mu^1_{\omega_1}}$.

Definition 3.23. Let κ be an uncountable cardinal and $\epsilon \leq \kappa$. A function $\Phi : [\kappa]^\epsilon \to \mathrm{ON}$ is everywhere monotonic if and only if for all $\iota, \ell \in [\kappa]^\epsilon \to \mathrm{ON}$, if for all $\alpha < \epsilon$, $\iota(\alpha) \leq \ell(\alpha)$, then $\Phi(\iota) \leq \Phi(\ell)$. Φ is μ^ϵ_κ-almost everywhere monotonic if and only if there is a club $C \subseteq \kappa$ so that for all $\iota, \ell \in [C]^\epsilon_*$, if for all $\alpha < \epsilon$, $\iota(\alpha) \leq \ell(\alpha)$, then $\Phi(\iota) \leq \Phi(\ell)$

[10] showed that an appropriate partition relation κ will imply that every function $\Phi : [\kappa]^\epsilon \to \mathrm{ON}$ is μ^ϵ_κ-almost everywhere monotonic. For the purpose here, one will only need monotonicity for functions $\Phi : [\kappa]^n \to \mathrm{ON}$ where n is finite which can be shown by a simple partition arguments which can be found in full detail in [6].

Fact 3.24. *Let κ be a cardinal, $1 \leq n < \omega$, and $\kappa \to_* (\kappa)^{n+1}_2$ holds. Every function $\Phi : [\kappa]^n \to \mathrm{ON}$ is μ^n_κ-almost everywhere monotonic. There is an everywhere monotonic function $\Psi : [\kappa]^n \to \mathrm{ON}$ so that $\{\ell \in [\kappa]^n_* : \Phi(\ell) = \Psi(\ell)\} \in \mu^n_\kappa$.*

Fact 3.25. *Let $1 \leq n < \omega$ and $\omega_1 \to_* (\omega_1)^n_2$.*

- *If $f, g : [\omega_1]^n \to \omega_1$ are everywhere monotonic and $[f]_{\mu^n_{\omega_1}} = [g]_{\mu^n_{\omega_1}}$, then $[J_f]_{\mu^1_{\omega_1}} = [J_g]_{\mu^1_{\omega_1}}$.*
- *If $f, g : [\omega_1]^n \to \omega_1$ have type n and $[f]_{\mu^n_{\omega_1}} = [g]_{\mu^n_{\omega_1}}$, then for all $1 \leq k \leq n$, $[I^k_f]_{\mu^k_{\omega_1}} = [I^k_g]_{\mu^k_{\omega_1}}$.*

Proof. The first statement will be shown. Let $C_0 \subseteq \kappa$ be a club such that for all $\ell \in [C_0]^n$, $f(\ell) = g(\ell)$. Let $C_1 = \{\alpha \in C_0 : \mathsf{enum}_{C_0}(\alpha) = \alpha\}$. Suppose $\gamma \in C_1$. Let $\ell \in [\omega_1]^n$ be such that $\ell(n-1) = \gamma$. Let $\hat{\ell} \in [C_0]^n$ be defined by $\hat{\ell} = \mathsf{enum}_{C_0} \circ \ell$ and note that $\hat{\ell}(n-1) = \mathsf{enum}_{C_0}(\ell(n-1)) = \mathsf{enum}_{C_0}(\gamma) = \gamma$. Since f is everywhere monotonic, for all $k < n$, $\ell(k) \leq \hat{\ell}(k)$, and $\hat{\ell} \in [C_0]^n$, one has that $f(\ell) \leq f(\hat{\ell}) = g(\hat{\ell})$. So for every $\ell \in [\omega_1]^n$ with $\ell(n-1) = \gamma$, there is a $\hat{\ell} \in [\omega_1]^n$ with $\hat{\ell}(n-1) = \gamma$ so that $f(\ell) \leq g(\hat{\ell})$. Thus $J_f(\gamma) \leq J_g(\gamma)$ for all $\gamma \in C_1$. By a symmetric argument, $J_g(\gamma) \leq J_f(\gamma)$ for all $\gamma \in [C_1]$. This shows that $J_f(\gamma) = J_g(\gamma)$ for all $\gamma \in C_1$. Hence $[J_f]_{\mu^1_{\omega_1}} = [J_g]_{\mu^1_{\omega_1}}$. ☐

Definition 3.26. If $\alpha \in \mathfrak{B}_{n+1}$ and $1 \leq k \leq n$, then let $\mathcal{I}^k_\alpha = [I^k_f]_{\mu^k_{\omega_1}}$ for any $f : [\omega_1]^n \to \omega_1$ of type n such that $[f]_{\mu^n_{\omega_1}} = \alpha$. (Note that this is well defined using Fact 3.25.)

Fact 3.27. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. Let $1 \leq n < \omega$. If $\delta \in \omega_{n+1} \setminus \omega_n$, then there is an everywhere monotonic $f : [\omega_1]^n \to \omega_1$ so that $\delta = [f]_{\mu^n_{\omega_1}}$, and for all $\iota_0, \iota_1 \in [\omega_1]^n$, if $\iota_0(n-1) < \iota_1(n-1)$, then $f(\iota_0) < f(\iota_1)$.*

Proof. Let $\delta \in \omega_{n+1} \setminus \omega_n$. Using Fact 3.25, let $g : [\omega_1]^n \to \omega_1$ be such that g is everywhere monotonic and $[g]_{\mu^n_{\omega_1}} = \delta$. Let $P : [\omega_1]^n_* \to \omega_1$ be defined by $P(\ell) = 0$ if and only if $g(\ell) \geq \ell(n-1)$. By $\omega_1 \to_* (\omega_1)^n_2$, there is a club $C_0 \subseteq \omega_1$ which is homogeneous for P. If C_0 is homogeneous for P taking value 1, then for all $\ell \in [C_0]^n_*$, one has that $g(\ell) < \ell(n-1)$. By Fact 3.13, there is an $h : [\omega_1]^{n-1} \to \omega_1$ and club $C_1 \subseteq C_0$

so that for all $\ell \in [C_1]^n_*$, $g(\ell) = h(\ell \restriction n)$. Then $\delta = [h]_{\mu^{n-1}_{\omega_1}} < \omega_n$. This contradicts $\delta \in \omega_{n+1} \setminus \omega_n$. Thus C_0 is homogeneous for P taking value 0. By Fact 3.10, there is club $C_2 \subseteq C_0$ so that for all $\ell \in [C_2]^n_*$, $f(\ell) < \mathsf{next}^\omega_{C_2}(\ell(n-1))$. Let $C_3 = \{\alpha \in C_2 : \mathsf{enum}_{C_2}(\alpha) = \alpha\}$. Pick $\ell_0, \ell_1 \in [C_3]^n_*$ with $\ell_0(n-1) < \ell_1(n-1)$. Then $g(\ell_0) < \mathsf{next}^\omega_{C_2}(\ell_0(n-1)) < \ell_1(n-1) \leq g(\ell_1)$ by the property of C_2 and since $P(\ell_1) = 0$. Let $f : [\omega_1]^n \to \omega_1$ be defined by $f(\ell) = g(\mathsf{enum}_{C_3} \circ \ell)$. Let $\ell_0, \ell_1 \in [\omega_1]^n$ be such that $\ell_0(n-1) < \ell_1(n-1)$. Then $\mathsf{enum}_{C_3}(\ell_0(n-1)) < \mathsf{enum}_{C_3}(\ell_1(n-1))$. Thus $f(\ell_0) = g(\mathsf{enum}_{C_3} \circ \ell_0) < g(\mathsf{enum}_{C_3} \circ \ell_1) = f(\ell_1)$. Let $C_4 = \{\alpha \in C_3 : \mathsf{enum}_{C_3}(\alpha) = \alpha\}$. For all $\ell \in [C_3]^n$, $\mathsf{enum}_{C_3} \circ \ell = \ell$ and thus $f(\ell) = g(\ell)$. So $[f]_{\mu^n_{\omega_1}} = [g]_{\mu^n_{\omega_1}} = \delta$. □

Definition 3.28. Let $1 \leq n < \omega$. Let U^n be the set of tuples $(\alpha_{n-1}, ..., \alpha_0, \gamma)$ where $\alpha_0 < ... < \alpha_{n-1}$ and $\gamma < \alpha_{n-1}$. Let $\mathcal{U}^n = (U^n, <^{\omega_1}_{\text{lex}})$ where $<^{\omega_1}_{\text{lex}}$ is the lexicographic ordering on ${}^{<\omega}(\omega_1)$ induced from the usual ordering on ω_1. Note that $\text{ot}(\mathcal{U}^n) = \omega_1$. A function $H : U^n \to \omega_1$ has the correct type if and only if the following hold:

- H is order preserving between $\mathcal{U}^n$ and $(\omega_1, <)$.
- H is discontinuous everywhere: For all $x \in U^n$, $\sup(H \restriction x) = \sup\{H(y) : y <^{\omega_1}_{\text{lex}} x\} < H(x)$.
- H has uniform cofinality ω: There is a function $\bar{H} : U^n \times \omega \to \omega_1$ so that for all $x \in U^n$ and $k \in \omega$, $\bar{H}(x, k) < \bar{H}(x, k+1)$ and $H(x) = \sup\{\bar{H}(x, k) : k \in \omega\}$.

Fact 3.29. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. For all $1 \leq n < \omega$ and club $C \subseteq \omega_1$, there is an order embedding of ω_{n+1} into $\mathfrak{B}^C_{n+1}$.*

Proof. Fix $H : U^n \to C$ which has the correct type from $\mathcal{U}^n$ into $(C, <)$. Suppose $\delta \in \omega_{n+1} \setminus \omega_n$. By Fact 3.27, there is an everywhere monotonic $f : [\omega_1]^n \to \omega_1$ such that $[f]_{\mu^n_{\omega_1}} = \delta$ and for all $\ell_0, \ell_1 \in [\omega_1]^n$, if $\ell_0(n-1) < \ell_1(n-1)$, then $f(\ell_0) < f(\ell_1)$. Note that this implies that for all $\gamma_0 < \gamma_1 < \omega_1$, $J_f(\gamma_0) < J_f(\gamma_1)$. Let $H : [\omega_1]^{n+1} \to C$ be any function of type $n+1$. Define $\hat{f} : [\omega_1]^n \to \omega_1$ by

$$\hat{f}(\alpha_0, ..., \alpha_{n-1}) = H(J_f(\omega \cdot \alpha_{n-1}), J_f(\omega \cdot \alpha_{n-2}), ..., J_f(\omega \cdot \alpha_0), f(\alpha_0, ..., \alpha_{n-1}))$$

Suppose $(\alpha_0, ..., \alpha_{n-1}) \sqsubset_n (\beta_0, ..., \beta_{n-1})$. Let $k < n$ be largest such that $\alpha_k \neq \beta_k$. For all $k < j < n$, $J_f(\omega \cdot \alpha_j) = J_f(\omega \cdot \alpha_j)$ and $J_f(\omega \cdot \alpha_k) < f(0, 1, ..., n-2, \omega \cdot \alpha_k + 1) < J_f(\omega \cdot \beta_k)$ using the property of f. Since H is order preserving on $\mathcal{U}^n$, it is clear that $\hat{f}(\alpha_0, ..., \alpha_{n-1}) < \hat{f}(\beta_0, ..., \beta_{n-1})$. $\hat{f}$ is discontinuous and has uniform cofinality since H is discontinuous and has uniform cofinality ω. Thus $\hat{f}$ has type n. Thus $[\hat{f}]_{\mu^n_{\omega_1}} \in \mathfrak{B}^C_{n+1}$. Define $\Phi : (\omega_{n+1} \setminus \omega_n) \to \mathfrak{B}^C_{n+1}$ by $\Phi(\delta) = [\hat{f}]_{\mu^n_{\omega_1}}$ and note that this is independent of the choice of everywhere monotonic f with the above properties representing δ using Fact 3.25. Suppose $\delta_0 < \delta_1$. Let $\tilde{f}_0, \tilde{f}_1 : [\omega_1]^n \to \omega_1$ be two everywhere monotonic functions representing δ_0 and δ_1, respectively, with the property that for all $i \in 2$ and $\ell_0, \ell_1 \in [\omega_1]^n$, $\ell_0(n-1) < \ell_1(n-1)$ implies $\tilde{f}_i(\ell_0) < \tilde{f}_i(\ell_1)$. Let $A = \{\ell \in [\omega_1]^n : f_0(\ell) < f_1(\ell)\} \in \mu^n_{\omega_1}$. Let $D_0 \subseteq \omega_1$ be a club such that $[D_0]^n_* \subseteq A$. Define $f_i(\ell) = \tilde{f}_i(\mathsf{enum}_{D_0} \circ \ell)$ for $i \in 2$. Note that $[f_i]_{\mu^n_{\omega_1}} = [\tilde{f}_i]_{\mu^n_{\omega_1}} = \delta_i$, $f_0(\ell) < f_1(\ell)$ for all $\ell \in [\omega_1]^n$, and for all $\ell_0, \ell_1 \in [\omega_1]^n$, if $\ell_0(n-1) < \ell_1(n-1)$, then $f_i(\ell_0) < f_i(\ell_1)$ for all $i \in 2$. For all $(\alpha_0, ..., \alpha_{n-1}) \in [\omega_1]^n$, for all $k < n$, $J_{f_0}(\alpha_k) \leq J_{f_1}(\alpha_k)$ and $f_0(\alpha_0, ..., \alpha_{n-1}) < f_1(\alpha_0, ..., \alpha_{n-1})$. Thus $\hat{f}_0(\ell) < \hat{f}_1(\ell)$ for all $\ell \in [\omega_1]^n$. This shows that Φ is an order preserving map. □

Definition 3.30. Let $\diamond$ be a new symbol. Let F be the linear ordering $(\omega_1 \cup \{\diamond\}, <^F)$ where $\diamond$ is $<^F$-less than all elements of ω_1 and $<^F$ restricted to ω_1 is the usual order on ω_1. Let V be the set of all $(\alpha_{n-1}, \alpha_{n-2}, ..., \alpha_0, \diamond, \gamma)$ such that $1 \leq n < \omega$, $\alpha_0 < ... < \alpha_{n-1}$, and $\gamma < \alpha_{n-1}$. Let $\mathcal{V} = (V, <^F_{\text{lex}})$ (where $<^F_{\text{lex}}$ is the lexicographic ordering induced from F). Note that $\text{ot}(\mathcal{V}) = \omega_1$. A function $H : V \to \omega_1$ has the correct type if and only if the following conditions holds:

- H is order preserving from $\mathcal{V}$ into $(\omega_1, <)$.
- H is discontinuous everywhere: For all $x \in V$, $\sup(H \restriction x) = \sup\{H(y) : y <^{\omega_1}_{\text{lex}} x\} < H(x)$.
- H has uniform cofinality ω: There is a function $\bar{H} : V \times \omega \to \omega_1$ so that for all $x \in V$ and $k \in \omega$, $\bar{H}(x, k) < \bar{H}(x, k+1)$ and $H(x) = \sup\{\bar{H}(x, k) : k \in \omega\}$.

If $X \subseteq \omega_1$, let $[X]^{\mathcal{V}}_*$ be the set of all increasing correct type function $H : V \to X$.

Fix $H \in [\omega_1]^{\mathcal{V}}_*$. Let $\bar{H} : V \times \omega \to \omega_1$ witness that H has uniform cofinality ω.

- Let $h_n^H : [\omega_1]^n \to \omega_1$ be defined by $h_n^H(\alpha_0,...,\alpha_{n-1}) = \sup\{H(\alpha_{n-1},...,\alpha_0,\gamma,\diamond,0) : \gamma < \alpha_0\}$. Let $\delta_n^H = [h_n^H]_{\mu_{\omega_1}^n}$.
- Let $\Phi_H : (\omega_\omega \setminus \omega_1) \to \omega_\omega$ be defined as follows: Let $1 \leq n < \omega$ and $\eta \in \omega_{n+1} \setminus \omega_n$. Let $f : [\omega_1]^n \to \omega_1$ be such that $\eta = [f]_{\mu_*^n}$. Let $\hat{f} : [\omega]^{n+1} \to \omega_1$ be defined by $\hat{f}(\alpha_0,...,\alpha_n) = H(\alpha_n,...,\alpha_0,\diamond,f(\alpha_0,...,\alpha_{n-1}))$ whenever $f(\alpha_0,...,\alpha_{n-1}) < \alpha_n$. Let $\Phi_H(\eta) = [\hat{f}]_{\mu_{\omega_1}^{n+1}}$. (It will be check below that Φ is well defined.)
- Define $\Psi_H : (\omega_\omega \setminus \omega_1) \to \omega_\omega$ as follow: Let $\eta < \omega_\omega \setminus \omega_1$. Let $1 \leq n < \omega$ be so that $\eta \in \omega_{n+1} \setminus \omega_n$. Let $f : [\omega_1]^n \to \omega_1$ be such that $[f]_{\mu_{\omega_1}^n} = \eta$. Define $\check{f} : [\omega_1]^{n+1} \to \omega_1$ by $\check{f}(\alpha_0,...,\alpha_n) = \sup\{H(\alpha_n,...,\alpha_0,\diamond,\gamma) : \gamma < f(\alpha_0,...,\alpha_{n-1})\}$. Let $\Psi_H(\eta) = [\check{f}]_{\mu_{\omega_1}^{n+1}}$.
- Define $\Upsilon_{H,\bar{H}} : (\omega_\omega \setminus \omega_1) \times \omega \to \omega_\omega$ be defined as follow: Let $\eta < \omega_\omega \setminus \omega_1$. Let $1 \leq n < \omega$ be so that $\eta \in \omega_{n+1} \setminus \omega_n$. Let $f : [\omega_1]^n \to \omega_1$ be such that $[f]_{\mu_{\omega_1}^n} = \eta$. For $k < \omega_1$, let $\tilde{f}^k : [\omega_1]^{n+1} \to \omega_1$ be defined by $\tilde{f}^k(\alpha_0,...,\alpha_n) = \bar{H}((\alpha_n,...,\alpha_0,\diamond,f(\alpha_0,...,\alpha_{n-1})),k)$. Let $\Upsilon_{H,\bar{H}}(\eta,k) = [\tilde{f}^k]_{\mu_{\omega_1}^{n+1}}$.

Lemma 3.31. *(With Jackson and Trang) Assume $\omega_1 \to_* (\omega_1)_2^{\omega_1}$ and $\bigstar$. Let $C \subseteq \omega_1$ be a club. Let $H \in [C]_*^{\mathcal{V}}$ be a function of the correct type which is order preserving from $\mathcal{V}$ into $(C,<)$ and let $\bar{H} : V \times \omega \to \omega_1$ witness that H has uniform cofinality ω. Then $\langle \delta_n^H : 1 \leq n < \omega \rangle$, Φ_H, Ψ_H, and $\Upsilon_{H,\bar{H}}$ have the following properties*

(1) For all $n \in \omega$, $\delta_n^H \in \omega_{n+1}$. For all $1 \leq m < n < \omega$, $\mathcal{I}_{\delta_n^H}^m = \delta_m^H$

(2) Let $1 \leq n < \omega$, $f : [\omega_1]^n \to \omega_1$, and $D \subseteq \omega_1$ be an f-bounding club. Then $\hat{f}$ is defined on $[D]_*^{n+1}$ and for all $1 \leq m \leq n$, and $\ell \in [D]_*^m$, $I_f^m(\ell) = h_m^H(\ell)$. For all $1 \leq n < \omega$, if $\eta \in \omega_{n+1} \setminus \omega_n$, $\Phi_H(\eta)$ is well defined independent of choice of representative of η and $\mathcal{I}_{\Phi_H(\eta)}^n = \delta_n^H$.

(3) For all $1 \leq n < \omega$, if $\eta \in \omega_{n+1} \setminus \omega_n$, then $\Phi_H(\eta) \in \mathfrak{B}_{n+2}^C$. $\Phi_H : (\omega_\omega \setminus \omega_1) \to \omega_\omega$ is an increasing function (and hence an injection). For all $\eta \in (\omega_\omega \setminus \omega_1)$, $\sup(\Phi_H \restriction \eta) = \Psi_H(\eta) < \Phi_H(\eta)$. Φ_H has uniform cofinality ω as witnessed by $\Upsilon_{H,\bar{H}}$. (Thus Φ_H is a function of the correct type.)

Proof. Fix the objects from above and use the notation from Definition 3.30.

(1) It is clear that $\delta_n^H \in \omega_{n+1}$ for each $1 \leq n < \omega$. Now suppose $1 \leq m < n < \omega$. Let $(\beta_0,...,\beta_{m-1}) \in [\omega_1]_*^m$.

$$I_{h_n^H}^m(\beta_0,...,\beta_{m-1}) = \sup\{h_n^H(\gamma_0,...,\gamma_{n-m-1},\beta_0,...,\beta_{m-1}) : \gamma_0 < ... < \gamma_{n-m-1} < \beta_0\}$$

$$= \sup\{\sup\{H(\beta_{m-1},...,\beta_0,\gamma_{n-m-1},...,\gamma_0,\zeta,\diamond,0) : \zeta < \gamma_0\} : \gamma_0 < ... < \gamma_{n-m-1} < \beta_0\}$$

$$= \sup\{H(\beta_{m-1},...,\beta_0,\zeta,\diamond,0) : \zeta < \beta_0\} = h_m^H(\beta_0,...,\beta_{m-1})$$

To see the two supremum are the same: For all $\zeta < \gamma_0 < ... < \gamma_{n-m-1} < \beta_0 < ... < \beta_{m-1}$ with $(\beta_0,...,\beta_{m-1}) \in [\omega_1]_*^m$, let $\xi = \gamma_{n-m-1} + 1$ and note that $\xi < \beta_0$ since β_0 is a limit ordinal. Then one has

$$(\beta_{m-1},...,\beta_0,\gamma_{n-m-1},...,\gamma_0,\zeta,\diamond,0) <_{\text{lex}}^{\digamma} (\beta_{m-1},...,\beta_0,\xi,\diamond,0).$$

For $\zeta < \beta_0$ with $(\beta_0,...,\beta_{m-1}) \in [\omega_1]_*^m$, one can find $\zeta < \xi < \gamma_0 < ... < \gamma_{m-n-1} < \beta_0$ since β_0 is a limit ordinal. Then

$$(\beta_{m-1},...,\beta_0,\zeta,\diamond,0) <_{\text{lex}}^{\digamma} (\beta_{m-1},...,\beta_0,\gamma_{m-n-1},...,\gamma_0,\xi,\diamond,0).$$

(2) Fix $1 \leq n < \omega$, $f : [\omega_1]^n \to \omega_1$, and $D \subseteq \omega_1$ be an f-bounding club (which exists by Fact 3.12). By the definition of $\hat{f}$, $\hat{f}$ is defined on $[D]_*^{n+1}$. Let $(\alpha_0,...,\alpha_{n-1}) \in [D]_*^n$.

$$I_{\hat{f}}^n(\alpha_0,...,\alpha_{n-1}) = \sup\{\hat{f}(\gamma,\alpha_0,...,\alpha_{n-1}) : \gamma < \alpha_0\} = \sup\{H(\alpha_{n-1},...,\alpha_0,\gamma,\diamond,f(\alpha_0,...,\alpha_{n-1})) : \gamma < \alpha_0\}$$

$$= \sup\{H(\alpha_{n-1},...,\alpha_0,\gamma,\diamond,0) : \gamma < \alpha_0\} = h_n(\alpha_0,...,\alpha_{n-1})$$

This shows that $I_{\hat{f}}^n = h_n^H$ on $[D]_*^n$. The same argument shows that for all $1 \leq m \leq n$, $I_{\hat{f}}^m = h_m^H$.

If $\eta \in \omega_{n+1} \setminus \omega_n$, it is clear that $\Phi_H(\eta)$ is independent of the choice of $f : [\omega_1]^n \to \omega_1$ so that $[f]_{\mu_{\omega_1}^n} = \eta$. The above implies that $\mathcal{I}_{\Phi_H(\eta)}^n = \delta_n^H$.

(3) Fix $\eta \in \omega_{n+1} \setminus \omega_n$. Let $f : [\omega_1]^n \to \omega_1$ with the property that $[f]_{\mu^n_{\omega_1}} = \eta$. By Fact 3.12, let $D_0 \subseteq \omega_1$ be an f-bounding club. Let $D_1 = \{\alpha \in D_0 : \mathsf{enum}_{D_0}(\alpha) = \alpha\}$. Let $g : [\omega_1]^{n+1} \to C$ be defined by $g(\ell) = \hat{f}(\mathsf{enum}_{D_0} \circ \ell) = H(\mathsf{enum}_{D_0}(\ell(n)), ..., \mathsf{enum}_{D_0}(\ell(0)), \diamond, f(\mathsf{enum}_{D_0} \circ \ell))$. For all $\ell \in [D_1]^{n+1}_*$, $\mathsf{enum}_{D_0} \circ \ell = \ell$ by Fact 3.9. Thus $[g]_{\mu^{n+1}_{\omega_1}} = [\hat{f}]_{\mu^{n+1}_{\omega_1}} = \Phi_H(\eta)$. It is clear that $g : [\omega_1]^{n+1}_* \to C$ has type $n+1$. Thus $\Phi_H(\eta) \in \mathfrak{B}^C_{n+2}$.

Let $\omega_n < \eta_0 < \eta_1 < \omega_{n+1}$. Let $f_0, f_1 : [\omega_1]^n \to \omega_1$ be such that $[f_0]_{\mu^n_{\omega_1}} = \eta_0$ and $[f_1]_{\mu^n_{\omega_1}} = \eta_1$. There is a club $D \subseteq \omega_1$ which is f_0-bounding, f_1-bounding, and for all $\iota \in [D]^n_*$, $f_0(\iota) < f_1(\iota)$. Then for all $(\alpha_0, ..., \alpha_n) \in [D]^{n+1}_*$,

$$\hat{f}_0(\alpha_0, ..., \alpha_n) = H(\alpha_n, ..., \alpha_0, \diamond, f_0(\alpha_0, ..., \alpha_{n-1}) < H(\alpha_n, ..., \alpha_0, \diamond, f_1(\alpha_0, ..., \alpha_{n-1}) = \hat{f}_1(\alpha_0, ..., \alpha_n).$$

This shows $\Phi_H(\eta_0) = [\hat{f}_0]_{\mu^{n+1}_{\omega_1}} < [\hat{f}_1]_{\mu^{n+1}_{\omega_1}} = \Phi_H(\eta_1)$. Φ_H is an increasing function.

Let $\omega_n < \eta < \omega_{n+1}$, $[f]_{\mu^n_{\omega_1}} = \eta$, and D is a f-bounding club. Note that for all $(\alpha_0, ..., \alpha_n) \in [D]^{n+1}_*$,

$$\begin{aligned} \check{f}(\alpha_0, ..., \alpha_n) &= \sup\{H(\alpha_n, ..., \alpha_0, \diamond, \gamma) : \gamma < f(\alpha_0, ..., \alpha_{n-1})\} \\ &< H(\alpha_n, ..., \alpha_0, \diamond, f(\alpha_0, ..., \alpha_{n-1})) = \hat{f}(\alpha_0, ..., \alpha_n) \end{aligned}$$

since H was assumed to be discontinuous. Thus $\Psi_H(\eta) < \Phi_H(\eta)$. It is clear that if $\eta_0 < \eta_1$, then $\Phi_H(\eta_0) \leq \Psi_H(\eta_1) < \Phi_H(\eta)$. This also shows that Φ_H is discontinuous everywhere.

Let $\omega_n < \eta < \omega_{n+1}$ and $\zeta < \Phi_H(\eta)$. Let $f : [\omega_1]^n \to \omega_1$ be such that $[f]_{\mu^n_{\omega_1}} = \eta$ and $g : [\omega_1]^{n+1} \to \omega_1$ be such that $[g]_{\mu^{n+1}_{\omega_1}} = \zeta$. For $\mu^{n+1}_{\omega_1}$-almost all ℓ, $g(\ell) < \hat{f}(\ell) = H(\ell(n), ..., \ell(0), \diamond, f(\ell \upharpoonright n))$. Since $\bar{H}$ witness that H has uniform cofinality ω, let $p(\ell)$ be the least $k \in \omega$ so that $g(\ell) < \bar{H}((\ell(n), ..., \ell(0), \diamond, f(\ell \upharpoonright n)), k)$. By the countably completeness of $\mu^{n+1}_{\omega_1}$, there is a $\bar{k}$ so that $p(\ell) = \bar{k}$ for $\mu^{n+1}_{\omega_1}$-almost all ℓ. Then $\zeta = [g]_{\mu^{n+1}_{\omega_1}} < \Upsilon_{H,\bar{H}}(\eta, \bar{k})$. $\Upsilon_{H,\bar{H}}$ witnesses that Φ_H has uniform cofinality ω.

This completes the proof ☐

Fact 3.32. *Suppose $1 \leq n < \omega$ and $\omega_1 \to_* (\omega_1)^{\max\{n,2\}}_2$. Suppose $g : \omega_1 \to \omega_1$ is a function of type 1. Suppose $f : [\omega_1]^n \to \omega_1$ is a function of type n. Assume $[g]_{\mu^1_{\omega_1}} < [I^1_f]_{\mu^1_{\omega_1}}$. Then there is a club $C \subseteq \omega_1$ with the following properties.*

- *For all $\alpha \in [C]^1_*$ and $\ell \in [C]^n_*$, if $\alpha \leq \ell(n-1)$, then $g(\alpha) < f(\ell)$.*
- *For all $\alpha \in [C]^1_*$ and $\ell \in [C]^n_*$, if $\ell(n-1) < \alpha$, then $f(\ell) < \alpha < g(\alpha)$.*

Proof. Let C_0 be a club so that for all $\alpha \in [C_0]^1_*$, $g(\alpha) < I^1_f(\alpha)$. Define $P : [C_0]^n \to 2$ by $P(\ell) = 0$ if and only if $g(\ell(n-1)) < f(\ell)$. By $\omega_1 \to_* (\omega_1)^n_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for P. Let $C_2 = \{\alpha \in C_1 : \mathsf{enum}_{C_1}(\alpha) = \alpha\}$. Pick any $\bar{\alpha} \in C_2$. Since $g(\bar{\alpha}) < I^1_f(\bar{\alpha})$, there is some $\iota \in [\omega_1]^n$ with $\iota(n-1) = \bar{\alpha}$ and $f(\iota) > g(\bar{\alpha})$. Let $\ell \in [C_1]^n$ be defined by $\ell = \mathsf{enum}_{C_1} \circ \iota$ and note that $\ell(n-1) = \mathsf{enum}_{C_1}(\iota(n-1)) = \mathsf{enum}_{C_1}(\bar{\alpha}) = \bar{\alpha}$. Since f has type n, $g(\bar{\alpha}) < f(\iota) < f(\ell)$. Then $P(\ell) = 0$. Thus C_1 is homogeneous for P taking value 0. C_1 is the desired club satisfying the first property. Using Fact 3.12 and $\omega_1 \to_* (\omega_1)^2_2$, let $C_3 \subseteq C_2$ be a club which is I^1_f-bounding. Suppose $\ell \in [C_3]^n_*$ and $\alpha \in [C_3]^n_*$ with $\ell(n-1) < \alpha$. Since C_3 is I^1_f-bounding, one has that $f(\ell) \leq I^1_f(\ell(n-1)) < \alpha < g(\alpha)$. C_3 is a club which also has the second property. ☐

For the main result of this section, one will need ω_1-many partitions of (essentially) $[\omega_1]^{\omega_1}$. Each partition will be defined from one of ω_1-many instructions for how to create partitions.

Definition 3.33. An instruction $\mathfrak{i}$ is a tuple $(\epsilon^{\mathfrak{i}}, \varphi^{\mathfrak{i}})$ satisfying the following properties.

- $\epsilon^{\mathfrak{i}} < \omega_1$.
- $\varphi^{\mathfrak{i}} : \epsilon^{\mathfrak{i}} \to \omega \setminus \{0\}$ is a nondecreasing function strictly bounded below ω.

Let $\mathfrak{I}$ be the set of all instructions. Note that $|\mathfrak{I}| = \omega_1$.

Definition 3.34. Let $\clubsuit_0$, $\clubsuit_1$, and $\clubsuit_2$ be three new formal symbols. Let $\omega_1^\clubsuit = \{\clubsuit_0, \clubsuit_1, \clubsuit_2\} \cup \omega_1$. Define $\ll$ on $\omega_1^\clubsuit$ by $\clubsuit_0 \ll \clubsuit_1 \ll \clubsuit_2 \ll \alpha \ll \beta$ for all $\alpha < \beta < \omega_1$. Let $\Omega = (\omega_1^\clubsuit, \ll)$. Note that Ω is simply three new elements put before a copy of the ordinary ordering on ω_1. Thus $\mathrm{ot}(\Omega) = 3 + \omega_1 = \omega_1$.

Each instruction has a corresponding linear ordering which is order isomorphic to the usual ordering on ω_1.

Definition 3.35. Suppose $\mathfrak{i} \in \mathfrak{I}$ is an instruction of the form $\mathfrak{i} = (\epsilon^{\mathfrak{i}}, \varphi^{\mathfrak{i}})$. Let $T^{\mathfrak{i}}$ consists of the following objects:

(1) $(\alpha, \clubsuit_0)$ for each $\alpha < \omega_1$.
(2) $(\alpha_{\varphi^{\mathfrak{i}}(\eta)}, \clubsuit_1, \alpha_{\varphi^{\mathfrak{i}}(\eta)-1}, ..., \alpha_0, \clubsuit_2, \eta)$ for all $\eta < \epsilon^{\mathfrak{i}}$ and all $\alpha_0 < \alpha_1 < ... < \alpha_{\varphi^{\mathfrak{i}}(\eta)-1} < \alpha_{\varphi^{\mathfrak{i}}(\eta)} < \omega_1$. (Note that $\varphi^{\mathfrak{i}}$ take nonzero value by the definition of $\mathfrak{i}$ being an instruction.)

Let $\mathcal{T}^{\mathfrak{i}} = (T^{\mathfrak{i}}, <^{\Omega}_{\text{lex}})$ which is the linear ordering on $T^{\mathfrak{i}}$ with the lexicographic ordering induced from Ω (restricted to $T^{\mathfrak{i}}$). Note that $\text{ot}(\mathcal{T}^{\mathfrak{i}}) = \omega_1$.

Now one can intuitively explain the purpose of the three new formal symbols, $\clubsuit_0$, $\clubsuit_1$, and $\clubsuit_2$. $\clubsuit_0$ and $\clubsuit_1$ ensure that tuple of type (1) starting with $\alpha < \omega_1$ will be $<^{\Omega}_{\text{lex}}$-smaller than any tuple of type (2) also starting with the same α. Suppose $\eta_0 < \eta_1$ with $m = \varphi^{\mathfrak{i}}(\eta_0) < \varphi^{\mathfrak{i}}(\eta_1) = n$. The purpose of $\clubsuit_2$ is to serve as a barrier point to distinguish tuple of type (2) of different length starting with the same ordinals. More precisely, suppose $\alpha_0 < \alpha_1 < ... < \alpha_m < \omega_1$ and $\beta_0 < \beta_1 < ... < \beta_n < \omega_1$ with the property that for all $k \leq m$, $\alpha_{m-k} = \beta_{n-k}$. The $\clubsuit_2$ of the first tuple ensures that $(\alpha_m, \clubsuit_1, \alpha_{m-1}, ..., \alpha_0, \clubsuit_2, \eta_0) = (\beta_n, \clubsuit_1, \beta_{n-1}, ..., \beta_{n-m}, \clubsuit_2, \eta_0) <^{\Omega}_{\text{lex}} (\beta_n, \clubsuit_1, \beta_{n-1}, ..., \beta_0, \clubsuit_2, \eta_1)$.

Definition 3.36. Let $\mathfrak{i}$ be an instruction. A function $F : T^{\mathfrak{i}} \to \omega_1$ has type $\mathfrak{i}$ if and only if the following holds:

- F is order preserving between $\mathcal{T}^{\mathfrak{i}}$ into $(\omega_1, <)$.
- F is discontinuous everywhere: For all $x \in T^{\mathfrak{i}}$, $\sup(F \upharpoonright x) = \sup\{F(y) : y <^{\Omega}_{\text{lex}} x\} < F(x)$.
- F has uniform cofinality ω: There is a function $G : T^{\mathfrak{i}} \times \omega \to \omega_1$ so that for all $k \in \omega$ and $x \in T^{\mathfrak{i}}$, $G(x, k) < G(x, k+1)$ and $F(x) = \sup\{G(x, k) : k \in \omega\}$.

If $X \subseteq \omega_1$, then let $[X]^{\mathcal{T}^{\mathfrak{i}}}_*$ be the collection of all functions of type $\mathfrak{i}$.

Definition 3.37. Let $\mathfrak{i}$ be an instruction and $F \in [\omega_1]^{\mathcal{T}^{\mathfrak{i}}}_*$. Let $F^{\mathfrak{i},\triangle} : \omega_1 \to \omega_1$ be defined by $F^{\mathfrak{i},\triangle}(\alpha) = F(\alpha, \clubsuit_0)$. For each $\eta < \epsilon^{\mathfrak{t}}$, let $F^{\mathfrak{i},\eta} : [\omega_1]^{\varphi^{\mathfrak{i}}(\eta)+1} \to \omega_1$ be defined as follows: for any $(\alpha_0, ..., \alpha_{\varphi^{\mathfrak{i}}(\eta)}) \in [\omega_1]^{\varphi^{\mathfrak{i}}(\eta)+1}$, $F^{\mathfrak{i},\eta}(\alpha_0, ..., \alpha_{\varphi^{\mathfrak{i}}(\eta)}) = F(\alpha_{\varphi^{\mathfrak{i}}(\eta)}, \clubsuit_1, \alpha_{\varphi^{\mathfrak{i}}(\eta)-1}, ..., \alpha_0, \clubsuit_2, \eta)$. Define $\triangle^{\mathfrak{i},F} \in \omega_2$ by $\triangle^{\mathfrak{i},F} = [F^{\mathfrak{i},\triangle}]_{\mu^1_{\omega_1}}$. Define $p^{\mathfrak{i},F}(\eta) \in \omega_{\varphi^{\mathfrak{i}}(\eta)+2}$ by $p^{\mathfrak{i},F}(\eta) = [F^{\mathfrak{i},\eta}]_{\mu^{\varphi^{\mathfrak{i}}(\eta)+1}_{\omega_1}}$. Note that $p^{\mathfrak{i},F} : \epsilon^{\mathfrak{i}} \to (\omega_{\sup(\varphi^{\mathfrak{i}})+2} \setminus \omega_2)$.

Lemma 3.38. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. Suppose $C \subseteq \omega_1$ is a club, $H \in [C]^{\mathcal{V}}_*$, and $\bar{H} : V \times \omega \to \omega_1$ witness that H has uniform cofinality ω. Let $Z = \Phi_H[(\omega_\omega \setminus \omega_1)]$. Let $p \in \mathsf{Bl}_{\omega_\omega}(< \omega_1, Z)$ and $\chi \in \mathfrak{B}^C_2$ so that $\chi < \delta^H_1$. Let $\epsilon = \text{dom}(p)$ and $\varphi : \epsilon \to \omega$ be defined by $\varphi(\eta)$ be the least $1 \leq n < \omega$ so that $\omega_{n+1} \leq p(\eta) < \omega_{n+2}$. Let $\mathfrak{i} = (\epsilon, \varphi)$. Then there is an $F \in [C]^{\mathcal{T}^{\mathfrak{i}}}_*$ so that $\Delta^{\mathfrak{i},F} = \chi$ and $p^{\mathfrak{i},F} = p$.*

Proof. Note that $\mathfrak{i}$ as defined above is an instruction since $p \in \mathsf{Bl}_{\omega_\omega}(< \omega_1, Z)$. Let $g : \omega_1 \to C$ be a function of type 1 so that $\chi = [g]_{\mu^1_{\omega_1}}$. Let $G : \omega_1 \times \omega \to \omega_1$ witness that g has uniform cofinality ω. By Lemma 3.31 (2), $\Phi_H[\omega_\omega \setminus \omega_1] \in \omega_\omega \setminus \omega_2$. For $\eta < \epsilon$, let $n_\eta = \varphi(\eta)$. Let $\zeta_\eta = \Phi_H^{-1}(p(\eta))$ and note that by Lemma 3.31, $\zeta_\eta \in \omega_{\varphi(\eta)+1}$. Apply Fact 3.19, for each $\eta < \epsilon$, there is an $f_\eta : [\omega_1]^{n_\eta} \to \omega_1$ so that $[f_\eta]_{\mu^{n_\eta}_{\omega_1}} = \zeta_\eta$. Then $p(\eta) = \Phi_H(\zeta_\eta) = [\hat{f}_\eta]_{\mu^{n_\eta+1}_{\omega_1}}$. For each $\eta < \epsilon$, let $A^0_\eta = \{\tau \in [\omega_1]^\omega : f(\tau \upharpoonright n_\eta) < \tau(n_\eta)\}$. By Fact 3.11, there is a club $D \subseteq \omega_1$ which is f_η-bounding. Then $A^0_\eta \in \mu^\omega_{\omega_1}$ since $[D]^\omega_* \subseteq A_\eta$. For $\eta_0 < \eta_1 < \epsilon$, let $A^1_{\eta_0,\eta_1} = \{\tau \in [\omega_1]^\omega : f_{\eta_0}(\tau \upharpoonright n_{\eta_0}) < f_{\eta_1}(\tau \upharpoonright n_{\eta_1})\}$. Since Φ_H is increasing by Fact 3.31 and p is an increasing function, $[f_{\eta_0}]_{\mu^{n_{\eta_0}}_{\omega_1}} < [f_{\eta_1}]_{\mu^{n_{\eta_1}}_{\omega_1}}$. This implies that $A^1_{\eta_0,\eta_1} \in \mu^\omega_{\omega_1}$. For each $\eta < \epsilon$, let

$$A^2_\eta = \{\tau \in [\omega_1]^\omega : g(\tau(n_\eta)) < \hat{f}_\eta(\tau \upharpoonright n_\eta + 1) \wedge \hat{f}_\eta(\tau \upharpoonright n_\eta + 1) < \tau(n_\eta + 1) < g(\tau(n_\eta + 1))\}.$$

By Fact 3.32, $A^2_\eta \in \mu^\omega_{\omega_1}$ since $[g]_{\mu^1_{\omega_1}} = \chi < \delta^H_1 = [I^1_{\hat{f}_\eta}]_{\mu^1_{\omega_1}}$. Note that $\mu^\omega_{\omega_1}$ is countably complete by $\omega_1 \to_* (\omega_1)^{\omega+\omega}_2$ and Fact 2.21. Let $A = \bigcap\{A^0_{\eta_0}, A^1_{\eta_0,\eta_1}, A^2_{\eta_0} : \eta_0 < \eta_1 < \epsilon\}$. Note that $A \in \mu^\omega_{\omega_1}$ since it is a countable intersection of sets from $\mu^\omega_{\omega_1}$. Let $D \subseteq \omega_1$ be a club so that $[D]^{\omega_1}_* \subseteq A$.[1] To summarize, D has the following properties:

[1]Countable choice of club subsets of ω_1 generally may not be possible from these hypotheses. The purpose of using $\mu^\omega_{\omega_1}$ is to find this club D using the countably completeness of $\mu^\omega_{\omega_1}$.

(1) For all $\eta < \epsilon$, D is f_η-bounding. Thus $\hat{f}_\eta$ is defined on $[D]^{n_\eta+1}_*$ and for all $1 \leq m \leq n_\eta$, $I^m_{f_\eta} = h^H_m$ on $[D]^m_*$ by Lemma 3.31.
(2) For all $\eta_0 < \eta_1 < \epsilon$ and $\ell \in [D]^{n_{\eta_1}}_*$, $f_{\eta_0}(\ell \upharpoonright n_{\eta_0}) < f_{\eta_1}(\ell)$.
(3) For all $\eta < \epsilon$, $\alpha \in [D]^1_*$, and $\ell \in [D]^{n_\eta+1}_*$,
  (i) If $\alpha \leq \ell(n_\eta)$, then $g(\alpha) < \hat{f}_\eta(\ell)$.
  (ii) If $\ell(n_\eta) < \alpha$, then $\hat{f}_\eta(\ell) < \alpha < g(\alpha)$.

Let $\mathfrak{e} = \mathsf{enum}_D$. Define $F : T^{\mathfrak{i}} \to \omega_1$ be defined as follows:
(a) For all $\alpha < \omega_1$, let $F(\alpha, \clubsuit_0) = g(\mathfrak{e}(\alpha))$.
(b) For all $\eta < \epsilon$, $\alpha_0 < ... < \alpha_{n_\eta} < \omega_1$, let

$$F(\alpha_{n_\eta}, \clubsuit_1, \alpha_{n_\eta-1}, ..., \alpha_0, \clubsuit_2, \eta) = \hat{f}_\eta(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n_\eta})).$$

Note that F is defined everywhere on $T^{\mathfrak{i}}$ since it is defined in all instance of (b) using property (1) of the club D. Since g maps into C and $\hat{f}_\eta$ maps into C (since H maps into C), one has that $F : T^{\mathfrak{i}} \to C$. Define $\tilde{F} : T^{\mathfrak{i}} \times \omega \to \omega_1$ as follows:

$$\tilde{F}(x,k) = \begin{cases} G(\mathfrak{e}(\alpha), k) & x = (\alpha, \clubsuit_0) \\ \bar{H}((\mathfrak{e}(\alpha_{n_\eta}), ..., \mathfrak{e}(\alpha_0), \diamond, f(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n_\eta-1}))), k) & x = (\alpha_{n_\eta}, \clubsuit_1, \alpha_{n_\eta-1}, ..., \alpha_0, \clubsuit_0, \eta) \end{cases}.$$

$\tilde{F}$ witnesses that F has uniform cofinality ω. Next, one will show that F is order preserving from $\mathcal{T}^{\mathfrak{i}}$ into $(C, <)$. Suppose $x, y \in T^{\mathfrak{i}}$ with $x <^\Omega_{\text{lex}} y$.

(A) $x = (\alpha, \clubsuit_0)$ and $y = (\beta, \clubsuit_0)$ with $\alpha < \beta$: Then $F(x) = g(\mathfrak{e}(\alpha)) < g(\mathfrak{e}(\beta)) = F(y)$ since g is an increasing function (since g has type 1).
(B) $x = (\alpha, \clubsuit_0)$ and $y = (\beta_{n_\eta}, \clubsuit_1, \beta_{n_\eta-1}, ..., \beta_0, \clubsuit_2, \eta)$ with $\alpha \leq \beta_{n_\eta}$ and $\eta < \epsilon$: Then

$$F(x) = g(\mathfrak{e}(\alpha)) < \hat{f}_\eta(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n_\eta})) = F(y)$$

by property (3i) of the club D.
(C) $x = (\alpha_{n_\eta}, \clubsuit_1, \alpha_{n_\eta-1}, ..., \alpha_0, \clubsuit_2, \eta)$ and $y = (\beta, \clubsuit_0)$ with $\alpha_{n_\eta} < \beta$ and $\eta < \epsilon$: Then

$$F(x) = \hat{f}_\eta(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n_\eta})) < g(\mathfrak{e}(\beta)) = F(y)$$

by property (3ii) of the club D.
(D) $x = (\alpha_{n_{\eta_0}}, \clubsuit_1, \alpha_{n_{\eta_0}-1}, ..., \alpha_0, \clubsuit_2, \eta_0)$ and $y = (\beta_{n_{\eta_1}}, \clubsuit_1, \beta_{n_{\eta_1}-1}, ..., \beta_0, \clubsuit_2, \eta_1)$ and there is some $j < \min\{n_{\eta_0}, n_{\eta_1}\}$ so that $\alpha_{n_{\eta_0}-j} < \beta_{n_{\eta_1}-j}$ and for all $i < j$, $\alpha_{n_{\eta_0}-i} = \beta_{n_{\eta_1}-i}$. Then

$$F(x) = \hat{f}_{\eta_0}(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n_{\eta_0}})) = H(\mathfrak{e}(\alpha_{n_{\eta_0}}), ..., \mathfrak{e}(\alpha_0), \diamond, f(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n_{\eta_0}-1})))$$
$$< H(\mathfrak{e}(\beta_{n_{\eta_1}}), ..., \mathfrak{e}(\beta_0), \diamond, f(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n_{\eta_1}-1}))) = \hat{f}_{\eta_1}(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n_{\eta_1}})) = F(y)$$

with the inequality coming from comparing to the j^{th} position consisting of $\mathfrak{e}(\alpha_{n_{\eta_0}-j})$ and $\mathfrak{e}(\beta_{n_{\eta_1}-j})$.
(E) $x = (\alpha_{n_{\eta_0}}, \clubsuit_1, \alpha_{n_{\eta_0}-1}, ..., \alpha_0, \clubsuit_2, \eta_0)$ and $y = (\beta_{n_{\eta_1}}, \clubsuit_1, \beta_{n_{\eta_1}-1}, ..., \beta_0, \clubsuit_2, \eta_1)$ with $n_{\eta_0} < n_{\eta_1}$, and for all $j \leq n_{\eta_0}$, $\alpha_{n_{\eta_0}-j} = \beta_{n_{\eta_1}-j}$. Then

$$F(x) = \hat{f}_{\eta_0}(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n_{\eta_0}})) = H(\mathfrak{e}(\alpha_{n_{\eta_0}}), ..., \mathfrak{e}(\alpha_0), \diamond, f(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n_{\eta_0}-1})))$$
$$= H(\mathfrak{e}(\beta_{n_{\eta_1}}), ..., \mathfrak{e}(\beta_{n_{\eta_1}-n_{\eta_0}}), \diamond, f(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n_{\eta_0}-1})))$$
$$< H(\mathfrak{e}(\beta_{n_{\eta_1}}), ..., \mathfrak{e}(\beta_0), f(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n_{\eta_1}-1}))) = \hat{f}_{\eta_1}(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n_{\eta_1}})) = F(y)$$

where the inequality comes from comparing $\diamond <^F \mathfrak{e}(\beta_{n_{\eta_1}-n_{\eta_0}-1})$ and using the fact that H is order preserving on $<^F_{\text{lex}}$.
(F) $x = (\alpha_{n_{\eta_0}}, \clubsuit_1, \alpha_{n_{\eta_0}-1}, ..., \alpha_0, \clubsuit_2, \eta_0)$ and $y = (\beta_{n_{\eta_1}}, \clubsuit_1, \beta_{n_{\eta_1}-1}, ..., \beta_0, \clubsuit_2, \eta_1)$ with $\eta_0 < \eta_1$, $n_{\eta_0} = n_{\eta_1}$, $\alpha_j = \beta_j$ for all $j \leq n_{\eta_0} = n_{\eta_1}$. Then

$$F(x) = \hat{f}_{\eta_0}(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n_{\eta_0}})) = H(\mathfrak{e}(\alpha_{n_{\eta_0}}), ..., \mathfrak{e}(\alpha_0), \diamond, f(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n_{\eta_0}-1})))$$
$$< H(\mathfrak{e}(\beta_{n_{\eta_1}}), ..., \mathfrak{e}(\beta_0), \diamond, f(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n_{\eta_1}-1}))) = \hat{f}_{\eta_1}(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n_{\eta_1}})) = F(y)$$

since $f_{\eta_0}(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n_{\eta_0}-1})) < f_{\eta_1}(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n_{\eta_1}-1}))$ by property (2) of the club D.

It has been shown that F is order preserving from $\mathcal{T}^{\mathfrak{i}}$ into $(C,<)$. Next, one will show that F is discontinuous everywhere. Let $x \in T^{\mathfrak{i}}$.

(I) Suppose $x = (\alpha, \clubsuit_0)$.

$$\sup(F \upharpoonright x) = \sup\{F(y) : y <^F_{\mathrm{lex}} x\} = \sup\{\hat{f}_\eta(\mathfrak{e} \circ \ell) : \eta < \epsilon \wedge \ell \in [\omega_1]^{n_\eta+1} \wedge \ell(n_\eta) < \alpha\}$$
$$\leq \mathfrak{e}(\alpha) < g(\mathfrak{e}(\alpha)) = F(x)$$

using property (3ii) of the club D.

(II) Suppose $x = (\alpha_{n_0}, \clubsuit_1, n_0 - 1, n_0 - 2, ..., 0, \diamond, 0)$. The immediate $<^F_{\mathrm{lex}}$ predecessor of x is $(\alpha_{n_0}, \clubsuit_0)$.

$$\sup(F \upharpoonright x) = F(\alpha_{n_0}, \clubsuit_0) = g(\mathfrak{e}(\alpha_{n_0})) < \hat{f}(\mathfrak{e}(0), ..., \mathfrak{e}(n_0 - 1), \mathfrak{e}(\alpha_{n_0})) = F(x)$$

by property (3i) of the club D.

(III) Suppose x is not as in Case (I) or Case (II). Say $x = (\alpha_{n_\eta}, \clubsuit_1, \alpha_{n_\eta - 1}, ..., \alpha_0, \clubsuit_0, \eta)$. Let E be the set of $y \in T^{\mathfrak{i}}$ so that $y <^\Omega_{\mathrm{lex}} x$ and y takes the form $(\beta_{n_{\bar{\eta}}}, \clubsuit_1, \beta_{n_{\bar{\eta}}-1}, ..., \beta_0, \clubsuit_2, \bar{\eta})$. Note that $\sup(F \upharpoonright x) = \sup\{F(y) : y \in E\}$. If $y \in E$ and $y = (\beta_{n_{\bar{\eta}}}, \clubsuit_1, \beta_{n_{\bar{\eta}}-1}, ..., \beta_0, \clubsuit_2, \bar{\eta})$, then

$$(\mathfrak{e}(\beta_{n_{\bar{\eta}}}), ..., \mathfrak{e}(\beta_0), \diamond, f_{\bar{\eta}}(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n_{\bar{\eta}}-1}))) <^F_{\mathrm{lex}} (\mathfrak{e}(\alpha_{n_\eta}), ..., \mathfrak{e}(\alpha_0), \diamond, f_\eta(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n_\eta - 1})))$$

using property (2) of the club D (when $n_{\bar{\eta}} = n_\eta$). Thus

$$\sup(F \upharpoonright x) = \sup\{H(\mathfrak{e}(\beta_{n_{\bar{\eta}}}), ..., \mathfrak{e}(\beta_0), \diamond, f_{\bar{\eta}}(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n_{\bar{\eta}}-1}))) : (\beta_{n_{\bar{\eta}}}, \clubsuit_1, \beta_{n_{\bar{\eta}}-1}, ..., \beta_0, \clubsuit_2, \bar{\eta}) \in E\}$$
$$< H(\mathfrak{e}(\alpha_{n_\eta}), ..., \mathfrak{e}(\alpha_0), \diamond, f_\eta(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n_\eta - 1}))) = F(x)$$

using the discontinuity of H.

It has been shown that F is discontinuous everywhere. This shows that F has the correct type and thus $F \in [C]^{\mathcal{T}^{\mathfrak{i}}}_*$. For all $\alpha \in \omega_1$,

$$F^{i,\triangle}(\alpha) = F(\alpha, \clubsuit_0) = g(\mathfrak{e}(\alpha)).$$

For all $\eta < \epsilon$ and $\ell \in [\omega_1]^{n_\eta+1}$,

$$F^{\mathfrak{i},\eta}(\ell) = F(\mathfrak{e}(\ell(n_\eta)), \clubsuit_1, \mathfrak{e}(\ell(n_\eta - 1)), ..., \mathfrak{e}(\ell(0)), \clubsuit_2, f(\mathfrak{e} \circ (\ell \upharpoonright n_\eta))) = \hat{f}(\mathfrak{e} \circ \ell).$$

Let $\tilde{D} = \{\alpha \in D : \mathsf{enum}_D(\alpha) = \alpha\}$. For all $\alpha \in [\tilde{D}]^1_*$, $\mathfrak{e}(\alpha) = \mathsf{enum}_D(\alpha) = \alpha$ so $F^{\mathfrak{i},\triangle}(\alpha) = g(\mathfrak{e}(\alpha)) = g(\alpha)$. For all $\eta < \epsilon$ and $\ell \in [\tilde{D}]^{n_\eta+1}_*$, $\mathfrak{e} \circ \ell = \mathsf{enum}_D \circ \ell = \ell$ and so $F^{\mathfrak{i},\eta}(\ell) = \hat{f}_\eta(\mathfrak{e} \circ \ell) = \hat{f}_\eta(\ell)$. Thus $\triangle^{\mathfrak{i},F} = [F^{\mathfrak{i},\triangle}]_{\mu^1_{\omega_1}} = [g]_{\mu^1_{\omega_1}} = \chi$ and for all $\eta < \epsilon$, $p^{\mathfrak{i},F}(\eta) = [F^{\mathfrak{i},\eta}]_{\mu^{n_\eta+1}_{\omega_1}} = [\hat{f}_\eta]_{\mu^{n_\eta+1}_{\omega_1}} = p(\eta)$. This completes the proof. $\square$

Observe that the set of instruction $\mathfrak{I}$ has cardinality ω_1. Each element $\mathfrak{i} \in \mathfrak{I}$ will induces a certain partition on $P_{\mathfrak{i}} : [\omega_1]^{\omega_1} \to 2$ in the main theorem below. One will need to be able to choose homogeneous club for ω_1-many partitions in order to construct the relevant objects. In many combinatorial constructions involving partition relations, one often needs to choose clubs for a large family of partitions possibly indexed by uncountable and even nonwellorderable sets. [3] has an extensive study of club uniformization principles. Here, one will need a form of wellordered club uniformization. AD implies $\mathsf{AC}^{\mathbb{R}}_\omega$ and thus by the Moschovakis coding lemma, one can choose clubs from a countable family of club subsets of ω_1. However, here one will formulate all result in a setting that does not assume any form of countable choice. The very strong partition property of ω_1 will allow the ability to choose ω_1-many clubs.

For an uncountable cardinal κ, club_κ will denote the set of all club subsets of κ. If X is a set and $R \subseteq X \times \mathsf{club}_\kappa$, then R is said to be $\subseteq$-downward closed in the club_κ-coordinate if and only if for all $x \in X$ and clubs $C \subseteq D$, if $R(x, D)$ holds, then $R(x, C)$ holds.

Fact 3.39. *Assume κ is an uncountable cardinal satisfying the very strong partition relation $\kappa \to_* (\kappa)^\kappa_{<\kappa}$.*

- *If $R \subseteq \kappa \times \mathsf{club}_\kappa$ is $\subseteq$-downward closed in the club_κ-coordinate, then there is a club $C \subseteq \kappa$ so that for all $\alpha \in \mathrm{dom}(R)$, $R(\alpha, C \setminus (\alpha + 1))$.*
- *If $R \subseteq [\kappa]^2 \times \mathsf{club}_\kappa$ is $\subseteq$-downward closed in the club_κ-coordinate, then there is a club $C \subseteq \kappa$ so that for all $(\alpha, \beta) \in \mathrm{dom}(R)$, $R((\alpha, \beta), C \setminus (\beta + 1))$.*

Proof. The second statement will be shown. If $f \in [\kappa]^\kappa$, then let $\mathcal{C}_f$ be the closure of $f[\kappa]$ which is a club subset of κ. Let $\prec$ be a wellordering in $[\kappa]^2$ defined by $(\alpha, \beta) \prec (\gamma, \delta)$ if and only if $(\beta < \delta) \vee (\beta = \delta \wedge \alpha < \gamma)$. Let $\mathfrak{g} : [\kappa]^2 \to \kappa$ be the Gödel pairing function defined by $\mathfrak{g}(\alpha, \beta)$ is the rank of (α, β) under $\prec$. Fix $R \subseteq [\kappa]^2 \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed in the club_κ-coordinate. Define $P : [\kappa]^\kappa \to 2$ by $P(f) = 0$ if and only if for all $(\alpha, \beta) \in [f(0)]^2$, $(\alpha, \beta) \in \mathrm{dom}(R) \Rightarrow R((\alpha, \beta), \mathcal{C}_f)$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_0 \subseteq \kappa$ which is homogeneous for P. One may assume C_0 is closed under the Gödel pairing function $\mathfrak{g}$ in the sense that for all $\gamma \in C_0$, for all $\alpha < \beta < \gamma$, $\mathfrak{g}(\alpha, \beta) < \gamma$. Suppose C_0 was homogeneous for P taking value 1. For any $f \in [C_0]^\kappa_*$, there is a $(\alpha, \beta) < [f(0)]^2$ with $(\alpha, \beta) \in \mathrm{dom}(R)$ and $R((\alpha, \beta), \mathcal{C}_f)$. Let $\Psi : [C_0]^\kappa_* \to \kappa$ be defined by $\Psi(f)$ is $\mathfrak{g}(\alpha, \beta)$ for the $\prec$-least such (α, β) with the previous property. By the property that $f(0)$ is closed under $\mathfrak{g}$, one has that for all $f \in [C_0]^\kappa_*$, $\Psi(f) < f(0)$. By $\kappa \to_* (\kappa)^\kappa_{<\kappa}$, Fact 3.13 implies there is a club $C_1 \subseteq C_0$ and a $\zeta < \kappa$ so that for all $f \in [C_1]^\kappa_*$, $\Psi(f) = \zeta$. Let $(\bar{\alpha}, \bar{\beta}) = \mathfrak{g}^{-1}(\zeta)$. By definition of Ψ, $(\bar{\alpha}, \bar{\beta}) \in \mathrm{dom}(R)$. Let $D \subseteq C_1$ be a club so that $R((\bar{\alpha}, \bar{\beta}), D)$. Let $f \in [D]^\kappa_*$ with $\bar{\beta} < f(0)$. Since $\mathcal{C}_f \subseteq D$ and R is $\subseteq$-downward closed, one has that $R((\bar{\alpha}, \bar{\beta}), \mathcal{C}_f)$. This contradicts $\Psi(f) = \zeta = \mathfrak{g}(\bar{\alpha}, \bar{\beta})$. So C_0 must have been homogenous for P taking value 0. Pick any $h \in [C_0]^\kappa_*$. Let $E = \mathcal{C}_h$ which is a club subset of κ. Suppose $(\alpha, \beta) \in \mathrm{dom}(R)$. Let η_β be the least $\eta < \kappa$ so that $\beta < h(\eta)$. Note that $\beta < h(\eta_\beta) = \mathrm{drop}(h, \eta_\beta)(0)$. Since $P(\mathrm{drop}(h, \eta_\beta)) = 0$, $(\alpha, \beta) \in \mathrm{dom}(R)$, and $\alpha < \beta < \mathrm{drop}(h, \eta_\beta)(0)$, one has that $R((\alpha, \beta), \mathcal{C}_{\mathrm{drop}(h,\eta_\beta)})$. Since $E \setminus (\beta + 1) = \mathcal{C}_{\mathrm{drop}(h,\eta_\beta)}$, one has that $R((\alpha, \beta), E \setminus \beta + 1)$. E is the desired club. ☐

So in the main argument, one will have a club C which is simultaneous homogeneous for $P_\mathfrak{i}$ for each $\mathfrak{i} \in \mathfrak{I}$ in the sense that for each $\mathfrak{i}$, there is an ordinal $\xi_\mathfrak{i}$ so that $C \setminus (\xi_\mathfrak{i} + 1)$ is homogeneous for $P_\mathfrak{i}$. The shift by $\xi_\mathfrak{i} + 1$ will cause no harm for the main argument because each ω_1-sequence through the homogeneous set is meant to represent ordinals in ultrapowers by various $\mu^n_{\omega_1}$. Thus shifting the representative up above $\xi_\mathfrak{i} + 1$ does not change the represented ordinal. The details follow next which answer [1] Question 2.7 of Ben-Neria and Garti.

Theorem 3.40. *Assume $\omega_1 \to (\omega_1)^{\omega_1}_{<\omega_1}$ and $\bigstar$. ω_ω is a Magidor cardinal.*

Proof. Let $\Psi : \mathsf{Bl}_{\omega_\omega}(< \omega_1, \omega_\omega) \to \omega_\omega$. Since $|\mathfrak{I}| = |\omega_1|$, let $\mathfrak{b} : \omega_1 \to \mathfrak{I}$ be a fixed bijection. For each $\mathfrak{i} \in \mathfrak{I}$, let $\xi_\mathfrak{i} = \mathfrak{b}^{-1}(\mathfrak{i})$. For each $\mathfrak{i} \in \mathfrak{I}$, let $P_\mathfrak{i} : [\omega_1]^{\mathcal{T}^\mathfrak{i}} \to 2$ be defined by $P_\mathfrak{i}(F) = 0$ if and only if $\mathcal{I}^1_{\Psi(p^{\mathfrak{i},F})} < \triangle^{\mathfrak{i},F}$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is a club homogeneous for $P_\mathfrak{i}$ taking value $j_\mathfrak{i} \in 2$. Define a relation $R \subseteq \omega_1 \times \mathsf{club}_{\omega_1}$ by $R(\alpha, C)$ if and only if C is homogeneous for $P_{\mathfrak{b}(\alpha)}$ (necessarily taking value $j_{\mathfrak{b}(\alpha)}$). Clearly R is $\subseteq$-downward closed in the club_{ω_1}-coordinate and $\mathrm{dom}(R) = \omega_1$ by the discussion above. Since $\omega_1 \to_* (\omega_1)^{\omega_1}_{<\omega_1}$ holds, Fact 3.39 implies there is a club $C \subseteq \omega_1$ so that for all $\alpha \in \omega_1$, $R(\alpha, C \setminus (\alpha + 1))$. In other words, for all instructions $\mathfrak{i} \in \mathfrak{I}$, $C \setminus (\xi_\mathfrak{i} + 1)$ is homogeneous for $P_\mathfrak{i}$ taking value $j_\mathfrak{i}$.

Pick $\eta_0 \in \mathfrak{B}^C_2$. Pick a function $J \in [\omega_1]^\mathcal{V}_*$ so that $\eta_0 < \delta^J_1$ (where δ^J_1 is defined for J as in definition 3.30). Since $|\mathfrak{B}^C_2| = |\omega_2|$, pick any $\eta_1 \in \mathfrak{B}^C_2$ with $\delta^J_1 < \eta_1$. Pick any $H \in [C]^\mathcal{V}_*$ so that $\eta_1 < \delta^H_1$ (where again δ^H_1 is defined in Definition 3.30 for H). Let $Z = \Phi_H[\omega_\omega \setminus \omega_1]$ and note that $|Z| = |\omega_\omega|$ since Φ_H is an injection. For any $\mathfrak{i} \in \mathfrak{I}$, let $H^\mathfrak{i} \in [C \setminus (\xi_\mathfrak{i} + 1)]^\mathcal{V}_*$ be defined by $H^\mathfrak{i}(x) = \mathsf{enum}_C(\xi_\mathfrak{i} + \mathsf{enum}^{-1}_C(H(x)))$. Note that H and $H^\mathfrak{i}$ only disagree on countably many $x \in V$. Thus $\delta^H_1 = \delta^{H^\mathfrak{i}}_1$ and $Z = \Phi_{H^\mathfrak{i}}[\omega_\omega \setminus \omega_1]$.

Suppose $p \in \mathsf{Bl}_{\omega_\omega}(< \omega_1, Z)$. Let $\epsilon = \mathrm{dom}(p)$ and $\varphi : \epsilon \to \omega$ be defined by $\varphi(\eta)$ is the least $1 \leq n < \omega$ so that $\omega_{n+1} \leq p(\eta) < \omega_{n+2}$. Let $\mathfrak{i} = (\epsilon, \varphi)$ and note that $\mathfrak{i} \in \mathfrak{I}$ is an instruction.

(1) ($j_\mathfrak{i} = 0$) Since $\eta_0 \in \mathfrak{B}^C_2 = \mathfrak{B}^{C \setminus (\xi_\mathfrak{i}+1)}_2$, $\eta_0 < \delta^H_1 = \delta^{H^\mathfrak{i}}_1$ and $p \in \mathsf{Bl}_{\omega_\omega}(< \omega_1, Z)$, Lemma 3.38 applied to $C \setminus (\xi_\mathfrak{i} + 1)$ and $H^\mathfrak{i}$ gives an $F \in [C \setminus (\xi_\mathfrak{i} + 1)]^{\mathcal{T}^\mathfrak{i}}_*$ so that $\Delta^{\mathfrak{i},F} = \eta_0$ and $p^{\mathfrak{i},F} = p$. Since $C \setminus (\xi_\mathfrak{i} + 1)$ is homogeneous for $P_\mathfrak{i}$ taking value $j_\mathfrak{i}$, one has $P_\mathfrak{i}(F) = j_\mathfrak{i} = 0$ implies that $\mathcal{I}^1_{\Psi(p)} = \mathcal{I}^1_{\Psi(p^{\mathfrak{i},F})} < \Delta^{\mathfrak{i},F} = \eta_0 < \delta^J_1$.
(2) ($j_\mathfrak{i} = 1$) Since $\eta_1 \in \mathfrak{B}^C_2 = \mathfrak{B}^{C \setminus (\xi_\mathfrak{i}+1)}_2$, $\eta_1 < \delta^H_1 = \delta^{H^\mathfrak{i}}_1$ and $p \in \mathsf{Bl}_{\omega_\omega}(< \omega_1, Z)$, Lemma 3.38 applied to $C \setminus (\xi_\mathfrak{i} + 1)$ and $H^\mathfrak{i}$ gives an $F \in [C \setminus (\xi_\mathfrak{i} + 1)]^{\mathcal{T}^\mathfrak{i}}_*$ so that $\Delta^{\mathfrak{i},F} = \eta_1$ and $p^{\mathfrak{i},F} = p$. Since $C \setminus (\xi_\mathfrak{i} + 1)$ is homogeneous for $P_\mathfrak{i}$ taking value $j_\mathfrak{i}$, one has $P_\mathfrak{i}(F) = j_\mathfrak{i} = 1$ implies that $\delta^J_1 < \eta_1 = \Delta^{\mathfrak{i},F} \leq \mathcal{I}^1_{\Psi(p^{\mathfrak{i},F})} = \mathcal{I}^1_{\Psi(p)}$.

In either case, $\mathcal{I}^1_{\Psi(p)} \neq \delta^J_1$. Let $K = \{\gamma \in \omega_\omega : \mathcal{I}^1_\gamma = \delta^J_1\}$. By Fact 3.31 applied to J, Φ_J is an injection of $(\omega_\omega \setminus \omega_1)$ into K. Thus $|K| = |\omega_\omega|$. Since $p \in \mathsf{Bl}_{\omega_\omega}(< \omega_1, Z)$ was arbitrary, $K \cap \Psi[\mathsf{Bl}_{\omega_\omega}(< \omega_1, Z)] = \emptyset$. Thus $\Psi[\mathsf{Bl}_{\omega_\omega}(< \omega_1, Z)] \neq \omega_\omega$ (and moreover, $\Psi[\mathsf{Bl}_{\omega_\omega}(< \omega_1, Z)]$ misses a subset of ω_ω of cardinality ω_ω). Ψ is not a Magidor function. Since Ψ was arbitary, ω_ω is Magidor. ☐

Similar arguments which more directly involves Kunen functions should be able to show that for all $\epsilon < \omega_2$, ω_ω is ϵ-Magidor and $(< \epsilon)$-Magidor. It is not known if ω_ω is $(< \omega_2)$-Magidor, ω_2-Magidor, lower-Magidor, or super-Magidor.

4. Remarks on Magidor Filters

Definition 4.1. Let κ be a cardinal and $\epsilon < \kappa$. A uniform filter $\mathcal{F}$ on κ (which means for all $A \in \mathcal{F}$, $|A| = |\kappa|$) is an ϵ-Magidor filter if and only if for all $\Phi : \mathsf{Bl}_\kappa(\epsilon, \kappa) \to \kappa$, there is an $A \in \mathcal{F}$ so that $\Phi[\mathsf{Bl}_\kappa(\epsilon, A)] \neq \kappa$.

Let κ be a cardinal and $\epsilon < \kappa$. A uniform filter $\mathcal{F}$ on κ is a $(< \epsilon)$-Magidor filter if and only if for all $\Phi : \mathsf{Bl}_\kappa(< \epsilon, \kappa) \to \kappa$, there is an $A \in \mathcal{F}$ so that $\Phi[\mathsf{Bl}_\kappa(< \epsilon, A)] \neq \kappa$.

A Magidor filter is a $(< \omega_1)$-Magidor filter.

Let κ be a cardinal. A uniform filter $\mathcal{F}$ on κ is a super-Magidor filter if and only if for all $\epsilon < \kappa$ and $\Phi : \mathsf{Bl}_\kappa(< \epsilon, \kappa) \to \kappa$, there is an $A \in \mathcal{F}$ so that $\Phi[\mathsf{Bl}_\kappa(< \epsilon, A)] \neq \kappa$.

[14] showed that under ZFC, there may exists Magidor cardinals assuming very powerful large cardinal axioms, but ZFC proves there cannot exist any Magidor filters. This section will have some remarks about the existence of Magidor filter (of various partial extent) in the choiceless framework.

Next, one will show that for partition cardinals κ, the ω-club filter μ^1_κ is a $(< \omega \cdot \omega)$-Magidor filter but is not a $\omega \cdot \omega$-Madidor filter. Recall that the correct type partition relation is formulated to have club homogeneous sets for functions of the correct type and being of the correct type means the function is discontinuous everywhere and has uniform cofinality ω. However, $[\kappa]^\epsilon$ (the set of all increasing ϵ-sequences) may contain functions which are not discontinuous everywhere. To handle non-discontinuous increasing functions while using the correct type partition relation, one will need to represent a non-discontinuous increasing function by the correct type function which induces it. If $\epsilon < \omega \cdot \omega$, then there are only finitely many limit ordinals below ϵ. Thus there are only finitely many "types" for functions on ϵ when $\epsilon < \omega \cdot \omega$. This is the key property that makes Proposition 4.2 possible. When $\epsilon \geq \omega \cdot \omega$, there will be infinitely many limit ordinals below ϵ and one will have an $\mathbb{R}$-index family of possible "types". Proposition 4.5 will show that this leads to a coding of $\mathbb{R}$ using these infinitely many limit ordinals and hence μ^1_κ cannot be a $\omega \cdot \omega$-Magidor filter.

Ben-Neria and Sharon [1] showed that that ω-club filter μ^1_κ is a ω-Magidor filter at suitable partition cardinals κ. The following generalization is the optimal extent that μ^1_κ can "serve as a Magidor filter".

Proposition 4.2. *Let κ be an uncountable cardinal and assume $\mathsf{AC}_\omega^{\mathscr{P}(\kappa)}$. Let $1 \leq \epsilon < \omega \cdot \omega$ and assume $\kappa \to (\kappa)^{1+\epsilon}_2$ holds. Then μ^1_κ is an ϵ-Magidor ultrafilter on κ.*

Proof. $\kappa \to_* (\kappa)^2_2$ implies that κ is regular and μ^1_κ is normal. Fix $\epsilon < \omega \cdot \omega$. Thus $\mathsf{Bl}_\kappa(\epsilon, \kappa)$ is equal to $[\kappa]^\epsilon$. Let $\Phi : [\kappa]^\epsilon \to \kappa$. Let L be the set of limit ordinals below ϵ. Since $\epsilon < \omega \cdot \omega$, L is a finite set. If $F \subseteq L$, let $\zeta_F = \mathrm{ot}(\epsilon \setminus F)$ and let $\mathfrak{e}_F : \zeta_F \to \epsilon \setminus F$ be the increasing enumeration of $\epsilon \setminus F$. If $f : \zeta_F \to \kappa$, then let $f^F : \epsilon \to \kappa$ be defined by

$$f^F(\alpha) = \begin{cases} f(\beta) & \alpha \notin L \wedge \beta = \mathfrak{e}_F^{-1}(\alpha) \\ \sup\{f(\beta) : \mathfrak{e}_F(\beta) < \alpha\} & \alpha \in L \end{cases}$$

Note that f^F is continuous precisely at the points $\alpha \in L$. For each $F \subseteq L$, let $P_F : [\kappa]^{1+\zeta_F} \to 2$ be defined by $P_F(g) = 0$ if and only if $\Phi(\mathrm{drop}(g,1)^F) < g(0)$. By $\kappa \to_* (\kappa)^{1+\epsilon}_2$, for each $F \subseteq L$, there is a club which is homogeneous for P^F taking value $i^F \in 2$. Since there are only finitely many $F \subseteq L$ because L is finite, there is a single club C which is homogeneous for all P_F for $F \subseteq L$. Let $D = [C]^1_*$ or in other words, $D = \{\alpha \in C : \mathrm{cof}(\alpha) = \omega\}$. Let $\bar{\alpha} < \bar{\beta}$ be the first two elements of D. Let $E = D \setminus (\bar{\beta}+1)$. Note that $E \in \mu^1_{\omega_1}$. The claim is that $\bar{\alpha} \notin \Phi[[E]^\epsilon] = \Phi[\mathsf{Bl}_\kappa(\epsilon, E)]$. To see this, let $h \in [E]^\epsilon$. Let $F = \{\alpha \in \epsilon : \sup(h \restriction \alpha) = h(\alpha)\}$ and note that $F \subseteq L$. Define $f : \zeta_F \to D$ by $f(\alpha) = h(\mathfrak{e}_F(\alpha))$. Note that f is an everywhere discontinuous function and $f^F = h$.

- Suppose $i_F = 0$. Let $g = \langle \bar{\alpha} \rangle^\frown f$. Note that $g : 1 + \zeta_f \to D$ is everywhere discontinuous. Since $1 + \zeta_F < \omega \cdot \omega < \omega_1$ and $D = [C]^1_*$, $\mathsf{AC}_\omega^{\mathscr{P}(\kappa)}$ implies g has uniform cofinality ω.[2] Thus $g \in [C]^{1+\zeta_F}_*$. $P_F(g) = i_F = 0$ implies that $\Phi(h) = \Phi(f^F) = \Phi(\mathrm{drop}(g,1)^F) < g(0) = \bar{\alpha}$.

[2]The use of $\mathsf{AC}_\omega^{\mathscr{P}(\kappa)}$ is important here to ensure any countable sequence through D has uniform cofinality ω.

- Suppose $i_F = 1$. Let $g = \langle\bar{\beta}\rangle\hat{\ } f$. As in the previous case, $g \in [C]^{1+\zeta_f}_*$. Thus $P_F(g) = i_F = 1$ implies that $\bar{\alpha} < \bar{\beta} = g(0) \leq \Phi(\mathrm{drop}(g,1)^F) = \Phi(f^F) = \Phi(h)$.

So in either case $\Phi(h) \neq \bar{\alpha}$. Since $h \in [E]^\epsilon$ was arbitary, $\bar{\alpha} \notin \Phi[[E]^\epsilon]$. Thus $\Phi[[E]^\epsilon] = \Phi[\mathsf{Bl}_\kappa(\epsilon, E)] \neq \kappa$ and $E \in \mu^1_\kappa$. Since Φ was arbitrary, this shows that μ^1_κ is an ϵ-Magidor filter. ☐

Proposition 4.3. *Let κ be an uncountable cardinal and assume $\mathsf{AC}_\omega^{\mathscr{P}(\kappa)}$. Assume $\kappa \to (\kappa)^{<\omega\cdot\omega}_2$ holds. Then μ^1_κ is a $(<\omega\cdot\omega)$-Magidor ultrafilter on κ.*

Proof. Fix a function $\Phi : \mathsf{Bl}_\kappa(<\omega\cdot\omega, \kappa) \to \kappa$. Since κ is regular, $\mathsf{Bl}_\kappa(<\omega\cdot\omega,\kappa) = [\kappa]^{<\omega\cdot\omega}$. Thus $\Phi : [\kappa]^{<\omega\cdot\omega} \to \kappa$. For each $\epsilon < \omega\cdot\omega$, let $\Phi^\epsilon : [\kappa]^\epsilon \to \kappa$ be defined by $\Phi^\epsilon(f) = \Phi(f)$. For $\epsilon < \omega\cdot\omega$, let L_ϵ be the set of limit ordinals below ϵ which is a finite set. For $F \subseteq \epsilon$, let $\zeta_{\epsilon,F} = \mathrm{ot}(\epsilon \setminus F)$ and $\mathfrak{e}_{\epsilon,F} : \zeta_F \to \epsilon \setminus F$ be the increasing enumeration of $\epsilon \setminus F$. If $f : \zeta_{\epsilon,F} \to \kappa$, then let $f^{\epsilon,F} : \epsilon \to \kappa$ and $P_{\epsilon,F}$ be as defined in the proof of Proposition 4.2 using ϵ. (In the proof of Proposition 4.2, ϵ was fixed but now one must consider all $\epsilon < \omega\cdot\omega$.) Let $i^{\epsilon,F}$ be the unique homogeneous value for $P_{\epsilon,F}$ for each $\epsilon < \omega\cdot\omega$ and $F \subseteq L_\epsilon$. Since $\omega\cdot\omega$ is countable and $\mathsf{AC}_\omega^{\mathscr{P}(\kappa)}$ holds, there is a sequence $\langle C^{\epsilon,F} : \epsilon < \omega\cdot\omega \wedge F \subseteq L_\epsilon\rangle$ with the property that $C^{\epsilon,F} \subseteq \kappa$ is a club and is homogeneous for $P_{\epsilon,F}$ taking value $i^{F,\epsilon}$. Let $C = \bigcap\{C^{\epsilon,F} : \epsilon < \omega\cdot\omega \wedge F \subseteq L_\epsilon\}$ which is still a club subset of κ as it is a countable intersection of club subsets of κ. Let $D = [C]^1_*$. Let $\bar{\alpha} < \bar{\beta}$ be the first two elements of D. Let $E = D \setminus (\bar{\beta}+1)$. Much as in the proof of Proposition 4.2 by considering all $P^{\epsilon,F}$ for $\epsilon < \omega\cdot\omega$ and $F \subseteq L_\epsilon$, one can show that $\bar{\alpha} \notin \Phi[\mathsf{Bl}_\kappa(<\omega\cdot\omega, E)]$. This shows μ^1_κ is a $(<\omega\cdot\omega)$-Magidor filter. ☐

Proposition 4.4. *Assume* AD. *If $\kappa < \Theta$ is an uncountable cardinal satisfying $\kappa \to_* (\kappa)^{<\omega\cdot\omega}_2$, then μ^1_κ is a $(<\omega\cdot\omega)$-Magidor filter.*

In particular, $\mu^1_{\omega_1}$ and $\mu^1_{\omega_2}$ are $(<\omega\cdot\omega)$-Magidor filters for ω_1 and ω_2, respectively.

Proof. AD implies $\mathsf{AC}^{\mathbb{R}}_\omega$. If $\kappa < \Theta$, then $\mathsf{AC}^{\mathbb{R}}_\omega$ implies $\mathsf{AC}_\omega^{\mathscr{P}(\kappa)}$ by the Moschovakis coding lemma. The result now follows from Proposition 4.3. ☐

Proposition 4.5. *If κ is a cardinal with $\omega_1 \leq \kappa < \Theta$, then the ω-club filter μ^1_κ on κ is not an $\omega\cdot\omega$-Magidor filter.*

Proof. Since $\kappa < \Theta$, let $\pi : {}^\omega 2 \to \kappa$ be a surjection. Define $\Phi : \mathsf{Bl}_\kappa(\omega\cdot\omega, \kappa) \to {}^\omega 2$ by

$$\Phi(f)(n) = \begin{cases} 0 & \sup(f \restriction \omega\cdot n + \omega) < f(\omega\cdot n + \omega) \\ 1 & \sup(f \restriction \omega\cdot n + \omega) = f(\omega\cdot n + \omega) \end{cases}$$

Define $\Psi : \mathsf{Bl}_\kappa(\omega\cdot\omega, \kappa) \to \kappa$ by $\pi \circ \Phi$.

Suppose $A \in \mu^1_\kappa$. Thus there is a club $C \subseteq \kappa$ so that $[C]^1_* \subseteq A$. Let $h : \omega\cdot\omega \to C$ be the enumeration of the first $\omega\cdot\omega$ element of $[C]^1_*$. Since C is club, note that for all $n \in \omega$, $h(\omega\cdot n + \omega) = \sup(h \restriction \omega\cdot n + \omega)$. Pick any $r \in {}^\omega 2$. Define $f_r \in [A]^{\omega\cdot\omega}$ as follows:

$$f_r(\omega\cdot m + n) = \begin{cases} h(n) & m = 0 \\ h(\omega\cdot m + n) & m > 0 \wedge r(m-1) = 1 \\ h(\omega\cdot m + 1 + n) & m > 0 \wedge r(m-1) = 0 \end{cases}$$

If $r(m) = 0$, then $\sup(f_r \restriction \omega\cdot m + \omega) = h(\omega\cdot m + \omega) < h(\omega\cdot m + \omega + 1) = f_r(\omega\cdot m + \omega)$ and thus $\Phi(f_r)(m) = 0 = r(m)$. If $r(m) = 1$, then $\sup(f_r \restriction \omega\cdot m + \omega) = h(\omega\cdot m + \omega) = f_r(\omega\cdot m + \omega)$ and thus $\Phi(f_r)(m) = 1 = r(m)$. This shows that $\Phi(f_r) = r$. It has been shown that $\Phi[\mathsf{Bl}_\kappa(\omega\cdot\omega, A)] = {}^\omega 2$. Thus $\Psi[\mathsf{Bl}_\kappa(\omega\cdot\omega, A)] = (\pi\circ\Phi)[\mathsf{Bl}_\kappa(\omega\cdot\omega, A)] = \kappa$. Since $A \in \mu^1_\kappa$ was arbitrary, μ^1_κ is not $\omega\cdot\omega$-Magidor. ☐

Let $\zeta < \kappa$ be a regular cardinal and let ν^ζ_κ be the ζ-club filter on κ. With the appropriate modification and a strengthened partition property, one may prove an analogs of the Proposition 4.3 that ν^ζ_κ is a $(<\omega\cdot\omega)$-Magidor filter. One can also prove an analog of Proposition 4.5 that ν^ζ_κ is not an $(\omega\cdot\omega)$-Magidor filter.

The author does not know if it is ever possible to have an $\omega\cdot\omega$-Magidor filter at a cardinal below Θ even at strong partition cardinals (like ω_1). Next, one will show under AD and $\mathsf{DC}_{\mathbb{R}}$ that no countably complete filter on ω_1 can ever be an $(\omega\cdot\omega)$-Magidor filter.

Definition 4.6. Let $1 \leq n < \omega$ and $\pi : n \to n$ be a permutation. Define $\prec^{n,\pi}$ on $[\omega_1]^n$ by $\iota \prec \ell$ if and only if $\iota \circ \pi <_{\text{lex}}^{\omega_1} \ell \circ \pi$. Let $\mathcal{L}^{n,\pi} = ([\omega_1]^n, \prec^{n,\pi})$. Note that $\text{ot}(\mathcal{L}^{n,\pi}) = \omega_1$.

Every injective function $\Phi : [\omega_1]^n \to \omega_1$ is almost everywhere order preserving on $\mathcal{L}^{n,\pi}$ for some permutation π such that $\pi(0) = n-1$. Note that if $\pi_n = (n-1, n-2, ..., 0)$, then a function of type n is order preserving on $\mathcal{L}^{n,\pi_n}$. The proof is a fairly straightforward partition argument.

Fact 4.7. *([20] Lemma 4.23) Let $1 \leq n < \omega$ and $\omega_1 \to_* (\omega_1)^{n+n}_2$. Let $C \subseteq \omega_1$ be a club and $\Phi : [C]^n_* \to \omega_1$ be an injective function, then there is a club $D \subseteq C$ and a permutation $\pi : n \to n$ with $\pi(0) = n-1$ so that $\Phi : [D]^n_* \to \omega_1$ is order preserving from $\mathcal{L}^{n,\pi} \restriction [D]^n_* = ([D]^n_*, \prec^{n,\pi})$ into $(\omega_1, <)$.*

Recall that under AD, there are no nonprincipal ultrafilters on ω and hence any ultrafilter on any set is countably complete. If $\mathcal{F}$ is a filter on a set X, Y is a set, and $\Phi : X \to Y$ is a function, then define the Rudin-Keisler pushforward of μ by Φ, $\Phi_*\mu$, which is a filter on Y by $A \in \Phi_*\mu$ if and only if $\Phi^{-1}[A] \in \mu$. If $\mathcal{F}$ is a filter on a set X and $A \in \mathcal{F}$, then $\mathcal{F} \restriction A = \{B \in \mathcal{F} : B \subseteq A\}$.

Fact 4.8. *(Kunen) Assume AD. If $\kappa < \Theta$ and $\mathcal{F}$ is a countably complete filter on κ, then there exists an ultrafilter μ on κ such that $\mathcal{F} \subseteq \mu$.*

Proof. If $x, y \in {}^\omega 2$, let $x \leq_{\mathsf{Turing}} y$ indicate that x is Turing reducible to y. Note that for any $x \in {}^\omega 2$, $\{y \in {}^\omega 2 : y \leq_{\mathsf{Turing}} x\}$ is countable. Let $\equiv_{\mathsf{Turing}}$ denote the Turing equivalence relation on ${}^\omega 2$ defined by $x \equiv_{\mathsf{Turing}} y$ if and only if $x \leq_{\mathsf{Turing}} y$ and $y \leq_{\mathsf{Turing}} x$. Let $\mathcal{D}_{\mathsf{Turing}} = {}^\omega 2 / \equiv_{\mathsf{Turing}}$ be the collection of Turing degrees. If $X, Y \in \mathcal{D}_{\mathsf{Turing}}$, then define $X \leq Y$ if and only if there exists $x \in X$ and $y \in Y$ so that $x \leq_{\mathsf{Turing}} y$. If $X \in \mathcal{D}_{\mathsf{Turing}}$, then the Turing cone $\mathcal{C}_X$ is $\{Y \in \mathcal{D}_{\mathsf{Turing}} : X \leq Y\}$. The Martin measure μ_{Turing} on $\mathcal{D}_{\mathsf{Turing}}$ is defined by $A \in \mu_{\mathsf{Turing}}$ if and only there is an $X \in \mathcal{D}_{\mathsf{Turing}}$ so that $\mathcal{C}_X \subseteq A$. Under AD, Martin showed that μ_{Turing} is an ultrafilter on $\mathcal{D}_{\mathsf{Turing}}$. Since $\kappa < \Theta$, there is a surjection of $\mathbb{R}$ onto $\mathscr{P}(\kappa)$ by the Moschovakis coding lemma. Since $\mathcal{F} \subseteq \mathscr{P}(\kappa)$, there is a surjection $\varpi : \mathbb{R} \to \mathcal{F}$. Define $\Pi : \mathcal{D}_{\mathsf{Turing}} \to \kappa$ by $\Pi(X) = \min \bigcap \{\varpi(z) : [z]_{\equiv_{\mathsf{Turing}}} \leq X\}$. Since the intersection of countably many elements of the countably complete filter $\mathcal{F}$ is in $\mathcal{F}$ and hence nonempty, Π is well defined. One can check that $\Pi_*\mu_{\mathsf{Turing}}$ (the Rudin-Keisler pushforward of μ_{Turing} by Π) is an ultrafilter which extends $\mathcal{F}$. ☐

Fact 4.9. *(Kunen; [20] Theorem 4.8) Assume AD and $\mathsf{DC}_{\mathbb{R}}$. Assume μ is a countably complete nonprincipal ultrafilter on ω_1. Then there is a $1 \leq n < \omega$ so that μ is Rudin-Keisler equivalent to $\mu^n_{\omega_1}$: There is a set $A \in \mu^n_{\omega_1}$, a set $B \in \mu$, and a bijection $\Phi : A \to B$ so that $\mu \restriction B = \Phi_*(\mu^n_{\omega_1} \restriction A)$ and $\mu^n_{\omega_1} \restriction A = (\Phi^{-1})_*(\mu \restriction B)$.*

Theorem 4.10. *Assume AD and $\mathsf{DC}_{\mathbb{R}}$. If $\mathcal{F}$ is a countably complete nonprincipal ultrafilter on ω_1, then $\mathcal{F}$ is not an $\omega \cdot \omega$-Magidor filter for ω_1.*

Proof. Since ω_1 is regular, $\mathsf{Bl}_{\omega_1}(\omega \cdot \omega, X) = [X]^{\omega \cdot \omega}$ for any $X \subseteq \omega_1$ so one will prefer to use the notation $[X]^{\omega \cdot \omega}$. By Fact 4.8, let μ be an ultrafilter on ω_1 which extends $\mathcal{F}$. By Fact 4.9, there is a $1 \leq n < \omega$, $A \in \mu^n_{\omega_1}$, $B \in \mu$, and bijection $\Pi : A \to B$ so that $\mu \restriction B = \Pi_*(\mu^n_{\omega_1} \restriction A)$ and $\mu^n_{\omega_1} \restriction A = (\Pi^{-1})_*(\mu \restriction B)$. Using Fact 4.7, let $\pi : n \to n$ be a bijection and $C \subseteq \omega_1$ be a club with $[C]^n_* \subseteq A$ so that $\Pi : [C]^n_* \to B$ is an order embedding of $\mathcal{L}^{n,\pi} \restriction [C]^n$ into $(B, <)$. Observe that $\text{ot}(\mathcal{L}^{n,\pi} \restriction [C]^n_*) = \omega_1$. If $E \subseteq [C]^n_*$ is countable, then let $\sup^*(E)$ denote the least element of $[C]^n_*$ which is $\prec^{n,\pi}$ greater than every element of E. Suppose $h \in [\omega_1]^{\omega \cdot \omega}$. Say that h is suitable if and only if for all $\alpha < \omega \cdot \omega$, $h(\alpha) \in \Pi[[C]^n_*]$. If h is suitable, let $\tilde{h} : \omega \cdot \omega \to [C]^n_*$ be defined by $\tilde{h}(\alpha) = \Pi^{-1}(h(\alpha))$. Let $m \in \omega$. Say that m is an h-limit if and only if $h(\omega \cdot m + \omega) = \Pi(\sup^*\{\tilde{h}(\omega \cdot m + k) : k < \omega\})$. Now define $\Psi : [\omega_1]^{\omega \cdot \omega} \to {}^\omega 2$ by

$$\Psi(h)(m) = \begin{cases} 0 & h \text{ is not suitable} \\ 0 & h \text{ is suitable and } m \text{ is not an } h\text{-limit} \\ 1 & h \text{ is suitable and } m \text{ is an } h\text{-limit} \end{cases}.$$

Let $X \in \mathcal{F}$. Since $\mathcal{F} \subseteq \mu$, $X \in \mu$. Note that $\Pi[[C]^n_*] \in \mu$ since $\mu \restriction B = \Pi_*(\mu^n_{\omega_1} \restriction A)$. Thus $X \cap \Pi[[C]^n_*] \in \mu \restriction B$. $\Pi^{-1}[X \cap \Pi[[C]^n_*]] \in \mu^n_{\omega_1}$. Let $D \subseteq C$ be a club so that $[D]^n_* \subseteq \Pi^{-1}[X \cap \Pi[[C]^n_*]]$. Let $u : \omega \cdot \omega \to [D]^n_*$ be any order preserving discontinuous map from $(\omega \cdot \omega, <)$ into $([D]^n, \prec^{n,\pi})$ where discontinuous means that for all $m < \omega$, $\sup^*\{u(\omega \cdot m + k) : k < \omega\} < u(\omega \cdot m + \omega)$. For each $m \in \omega$, let $v(m) = \sup^*\{u(\omega \cdot m + k) : k < \omega\}$.

Let $r \in {}^\omega 2$. Let $h_r \in [\omega_1]^{\omega \cdot \omega}$ be defined by

$$h_r(\alpha) = \begin{cases} \Pi(\alpha+1) & \alpha = 0 \vee \alpha \text{ is a successor ordinal} \\ \Pi(u(\omega \cdot m + \omega)) & \alpha = \omega \cdot m + \omega \wedge r(m) = 0 \\ \Pi(v(m)) & \alpha = \omega \cdot m + \omega \wedge r(m) = 1 \end{cases}.$$

Since $\Pi[[D]^n_*] \subseteq X$, one has that $h_r \in [X]^{\omega \cdot \omega}$. Since for all $\alpha < \omega \cdot \omega$, $u(\alpha) \in [D]^n_* \subseteq [C]^n_*$, $\Pi(u(\alpha)) \in \Pi[[C]^n_*]$ for all $\alpha < \omega \cdot \omega$. Also $v(m) \in [D]^n_* \subseteq [C]^n_*$. Hence $\Pi(v(m)) \in \Pi[[C]^n_*$ for all $m \in \omega$. Thus for all $r \in {}^\omega 2$, h_r is suitable. Note that for any $r \in \mathbb{R}$ and $m \in \omega$, $\sup^*\{\tilde{h}_r(\omega \cdot m + k) : k < \omega\} = \sup\{u(\omega \cdot m + 1 + k) : k < \omega\} = v(m)$. Suppose $r(m) = 0$. Since u is discontinuous, $u(\omega \cdot m + \omega) > v(m)$ and thus $h_r(\omega \cdot m + \omega) = \Pi(u(\omega \cdot m + \omega)) > \Pi(v(m)) = \Pi(\sup^*\{\tilde{h}_r(\omega \cdot m + 1 + k) : k < \omega\})$. m is not an h_r-limit. Thus $\Psi(h_r)(m) = 0 = r(m)$. Now suppose $r(m) = 1$. Then $h_r(\omega \cdot m + \omega) = \Pi(v(m)) = \Pi(\sup^*\{\tilde{h}_r(\omega \cdot m + 1 + k) : k < \omega\})$. Thus m is a h_r-limit. Thus $\Psi(h_r)(m) = 1 = r(m)$. It has been shown that $\Psi(h_r) = r$. This shows that $r \in \Psi[[X]^n]$. Since r was arbitrary, $\Psi[[X]^n] = \mathbb{R}$. Since $X \in \mathcal{F}$ was an arbitrary, it has been shown that for all $X \in \mathcal{F}$, $\Psi[[X]^{\omega \cdot \omega}] = \mathbb{R}$. Since $\kappa < \Theta$, let $\varpi : \mathbb{R} \to \kappa$ be surjection. Define $\Phi : [\omega_1]^{\omega \cdot \omega} \to \kappa$ by $\Phi = \varpi \circ \Psi$. It has been shown that for all $X \in \mathcal{F}$, $\Phi[[X]^{\omega \cdot \omega}] = \kappa$. $\mathcal{F}$ is not an $(\omega \cdot \omega)$-Magidor filter. Since $\mathcal{F}$ was an arbitrary countably complete filter on ω_1, it has been shown that no countably complete ultrafilter on ω_1 is an $(\omega \cdot \omega)$-Magidor filter. ☐

Jackson [20] has completely classified all the countably complete measures on any cardinal below the projective ordinal (and a bit beyond) and they are closely related to the partition properties on the odd projective ordinals. Similar argument to the above should show that for any cardinal below the supremum of the projective ordinals, no countably complete filter on that cardinal can be an $(\omega \cdot \omega)$-Magidor filter.

The natural question is whether ω_1 has an $(\omega \cdot \omega)$-Magidor filter under AD. If it exists, it must not be countably complete. Are there are any cardinals below Θ which possesses an $(\omega \cdot \omega)$-Magidor filter under AD?

5. Singular Super-Magidor Cardinals

Ben-Neria and Garti [1] asked whether there is a singular lower-Magidor cardinal below Θ. This section will answer this question by showing there are unboundedly many super-Magidor cardinals below Θ. Let $\boldsymbol{\delta}^1_\omega$ be the supremum of the projective ordinals. $\boldsymbol{\delta}^1_\omega$ is the smallest such cardinal for which the results of this section applies. Ben-Neria and Garti [1] also showed that assuming there is a strong partition cardinal above Θ, there is a Prikry-extension satisfying AD in which there is a singular cardinal possessing an ω-Magidor filter. It is not known if the existence of a strong partition cardinal above Θ is consistent. In fact, the existence of a cardinal $\kappa > \Theta$ with $\kappa \to_* (\kappa)^\omega_2$ would already suffice for their argument. To the author's knowledge, it is not known if even this is consistent with AD. However, the techniques here show that $\boldsymbol{\delta}^1_\omega$ will be a singular cardinal with an $(< \omega \cdot \omega)$-Magidor filter answering a question of Ben-Neria and Garti.

This section will use descriptive set theory under determinacy assumptions. [3] exposits some of the preliminary material of this section in more details.

One will need some notation associated to winning strategies.

Definition 5.1. A strategy on X is a function $\rho : {}^{<\omega}X \to X$. If σ and τ are strategies on X, then let $\sigma * \tau \in {}^\omega X$ be defined by recursion by $(\sigma * \tau) = \sigma(\sigma * \tau \upharpoonright n)$ if n is even and $(\sigma * \tau)(n) = \tau(\sigma * \tau \upharpoonright n)$ if n is odd.

If $f \in {}^\omega X$, then let $f_{\text{even}}, f_{\text{odd}} \in {}^\omega X$ be defined by $f_{\text{even}}(n) = f(2n)$ and $f_{\text{odd}}(n) = f(2n+1)$. If $f \in {}^\omega X$, then let $\rho_f : {}^{<\omega}X \to X$ be defined by $\rho(s) = f(|s|)$. If ρ is a strategy, then let $\Xi^1_\rho : {}^\omega X \to {}^\omega X$ be defined by $\Xi^1_\rho(f) = (\rho * \rho_f)_{\text{even}}$. If ρ is a strategy, then let $\Xi^2_\rho : {}^\omega X \to {}^\omega X$ be defined by $\Xi^2_\rho(f) = (\rho_f * \rho)_{\text{odd}}$. Note that Ξ^1_ρ and Ξ^2_ρ are Lipschitz continuous function and one can show that for every Lipschitz function $\Xi : {}^\omega X \to {}^\omega X$, there is a strategy ρ on X so that $\Xi = \Xi^2_\rho$.

The axiom of determinacy, AD, is the assertion that for all $A \subseteq {}^\omega\omega$, exactly one of the following holds:

- There is a strategy σ so that for all strategy τ, $\sigma * \tau \in A$ (and one will say that σ is a Player 1 winning strategy in the game G^ω_A).
- There is a strategy τ so that for all strategy σ, $\sigma * \tau \notin A$ (and one will say that τ is a Player 2 winning strategy in the game G^ω_A).

Definition 5.2. A pointclass Γ is a collection of subsets of spaces of the form $X_0 \times ... \times X_{n-1}$ where for each $i < n$, X_i is either ω or ${}^\omega\omega$ closed under continuous preimages (or Wadge reductions) (which means for continuous functions $\Phi : X \to Y$ and $B \subseteq Y$ with $B \in \Gamma$, $\Phi^{-1}[B] \in \Gamma$). (More generally, Γ could be a set of subsets of various Polish spaces.) If Γ is a pointclass, then $\check{\Gamma}$ refers to its dual pointclass (which consists of $X \setminus A$ for all $A \subseteq X$ and $A \in \Gamma$). Let $\Delta_\Gamma = \Gamma \cap \check{\Gamma}$. Γ is nonselfdual if and only if $\Gamma \neq \check{\Gamma}$. A set $P \in \Gamma$ is Γ-complete if and only if for for all $Q \in \Gamma$, Q is the preimage of P under some Lipschitz continuous function. By the Wadge lemma under AD, every nonselfdual pointclass Γ has a Γ-complete set. A set $U \in \Gamma$ with $U \subseteq \mathbb{R} \times X$ is Γ-universal for X if and only if for all $P \in \Gamma$ with $P \subseteq X$, there is an $e \in \mathbb{R}$ so that $P = U_e = \{x \in X : U(e, x)\}$. Every nonselfdual pointclass has a Γ-universal set for all Polish spaces X.

For simplicity, one will make the following definition for subsets of $\mathbb{R}$ (or ${}^\omega\omega$). The reader can adapt these definition to the more general Polish spaces.

Definition 5.3. A prewellordering on a set $P \subseteq \mathbb{R}$ is a wellfounded, reflexive, transitive, and total relation $\preceq$ on P. A norm on P is a function $\varphi : P \to \kappa$ for some ordinal κ. Prewellorderings can be uniquely identified with a surjective norm onto an ordinal.

If Γ is a pointclass, then $\boldsymbol{\delta}(\Gamma)$ is the supremum of the rank of the prewellordering $\preceq \in \Delta_\Gamma$. $\boldsymbol{\delta}(\Gamma)$ is called the prewellordering ordinal of Γ. The projective ordinals $\boldsymbol{\delta}^1_n$ are defined to be $\boldsymbol{\delta}(\boldsymbol{\Pi}^1_n)$. Familiar examples include $\boldsymbol{\delta}^1_1 = \omega_1$, $\boldsymbol{\delta}^1_2 = \omega_2$, $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, $\boldsymbol{\delta}^1_4 = \omega_{\omega+2}$.

If Γ is a pointclass, then a prewellordering $\varphi : P \to \kappa$ is a Γ-prewellordering if and only if there are relations $\leq^\varphi_\Gamma \in \Gamma$ and $\leq^\varphi_{\check{\Gamma}} \in \check{\Gamma}$ so that

$$(\forall y)\Big(P(y) \Rightarrow (\forall x)\Big[(P(x) \wedge \varphi(x) \leq \varphi(y)) \Leftrightarrow x \leq^\varphi_\Gamma y \Leftrightarrow x \leq^\varphi_{\check{\Gamma}} y\Big]\Big).$$

A pointclass Γ has the prewellordering property if and only if for all $P \in \Gamma$, there is a Γ-norm on P. For all $n \in \omega$, $\boldsymbol{\Pi}^1_{2n+1}$ and $\boldsymbol{\Sigma}^1_{2n+2}$ have the prewellordering property by the first periodicity theorem of Moschovakis ([27]).

Fact 5.4. *(Boundedness property) Let Γ be a pointclass closed under $\forall^\mathbb{R}$ and $\wedge$. Suppose there is a $P \in \Gamma$ which is Γ-complete and has a surjective Γ-norm $\varphi : P \to \kappa$ (onto some ordinal κ). If $A \subseteq P$ is $\check{\Gamma}$, then there is a $\delta < \kappa$ so that $\varphi[A] \subseteq \delta$.*

Fact 5.5. *(Moschovakis;* [20] *Lemma 2.13 and Lemma 2.16) Let Γ be a pointclass closed under $\wedge$, $\vee$, and $\forall^\mathbb{R}$. Suppose there is a Γ-complete set $P \in \Gamma$ and a surjective Γ-norm $\varphi : P \to \kappa$. Then the length of φ (namely κ) is $\boldsymbol{\delta}(\Gamma)$ and $\boldsymbol{\delta}(\Gamma)$ is a regular cardinal.*

The following is Solovay's method of coding a "dense" collection of clubs subsets of ω_1 by strategies.

Definition 5.6. Let Γ be a nonselfdual pointclass closed under $\wedge$, $\vee$, and $\forall^\mathbb{R}$. Suppose there is a Γ-complete set $P \in \Gamma$ and a surjective Γ-norm $\varphi : P \to \kappa$, where $\kappa = \boldsymbol{\delta}(\Gamma)$ by Fact 5.5. Let $\mathsf{clubcode}^\varphi_\kappa$ be the collection of strategies on ω with the property that

$$(\forall w)(w \in P \Rightarrow (\Xi^2_\rho(w) \in P \wedge \varphi(w) < \varphi(\Xi^2_\rho(w)))).$$

If $\rho \in \mathsf{clubcode}^\varphi_\kappa$, then define

$$\mathfrak{C}^{\varphi,\kappa}_\rho = \{\eta \in \kappa : (\forall w)((w \in P \wedge \varphi(w) < \eta) \Rightarrow \varphi(\Xi^2_\rho(w)) < \eta)\}.$$

The next several results follow from the boundedness property (Fact 5.4). See [3] for the details.

Fact 5.7. *Assume the setting of Definition 5.6. For each $\rho \in \mathsf{clubcode}^\varphi_\kappa$, $\mathfrak{C}^{\varphi,\kappa}_\rho$ is a club subset of κ.*

Fact 5.8. *(Solovay) Assume the setting of Definition 5.6 and AD. If $C \subseteq \kappa$ is a club subset of κ, then there is a $\rho \in \mathsf{clubcode}^\varphi_\kappa$ so that $\mathfrak{C}^{\varphi,\kappa}_\rho \subseteq C$.*

Proof. Only the game will be presented but see [3] or [5] Fact 4.6 for the full details. Fix a club $C \subseteq \kappa$. Consider the game S_C where Player 1 produces $v \in {}^\omega\omega$ and Player 2 produces $w \in {}^\omega\omega$, separately.

$$\begin{array}{c|cccccccc} & \text{I} & v(0) & & v(1) & & v(2) & & \cdots & v \\ S_C & & & & & & & & & \\ & \text{II} & & w(0) & & w(1) & & w(2) & \cdots & w \end{array}$$

Player 2 wins S_C if and only if $v \in P \Rightarrow (w \in P \wedge \varphi(v) < \varphi(w) \wedge \varphi(w) \in C)$. By the boundedness property (Fact 5.4) and AD, one can show that Player 2 has a winning strategy ρ and $\mathfrak{C}^{\varphi,\kappa}_\rho \subseteq C$. $\square$

The following is the most important tool for club selection in this section. See [3] or [5] Fact 4.7 for the proof.

Fact 5.9. *Assume the setting of Definition 5.6. Suppose $A \subseteq \mathsf{clubcode}^\varphi_\kappa$ and $A \in \check{\Gamma}$, then uniformly from A, there is a club $C \subseteq \kappa$ so that for all $\rho \in A$, $C \subseteq \mathfrak{C}^{\varphi,\kappa}_\rho$.*[3]

The only known method to establish strong partition cardinals in any set theoretic framework is through a descriptive set theoretic coding of functions by reals developed by Martin under determinacy called a good coding system. One will follow the notational convention developed in [3].

Definition 5.10. (Martin) Let κ be a cardinal and $\epsilon \leq \kappa$. A good coding system $\mathcal{G}$ for ${}^\epsilon\kappa$ is $\mathcal{G} = (\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \kappa)$ with the following properties:

- κ is a regular cardinal.
- Γ is a nonselfdual pointclass closed under $\forall^\mathbb{R}$.
- decode is a function of the form $\mathsf{decode} : \mathbb{R} \to \mathscr{P}(\epsilon \times \kappa)$ with the property that for all $f : \epsilon \to \kappa$, there is an $x \in \mathbb{R}$, $\mathsf{decode}(x) = f$. (One will often identify functions with their graph.)
- For all $\beta < \epsilon$ and $\gamma < \delta$, $\mathsf{GC}_{\beta,\gamma} \in \Delta_\Gamma$ and for all $x \in \mathbb{R}$, $x \in \mathsf{GC}_{\beta,\gamma}$ if and only if
$$\mathsf{decode}(x)(\beta,\gamma) \wedge (\forall \xi < \kappa)(\mathsf{decode}(x)(\beta,\xi) \Rightarrow \gamma = \xi).$$
- For each $\beta < \epsilon$, let $\mathsf{GC}_\beta = \bigcup_{\gamma<\kappa} \mathsf{GC}_{\beta,\gamma}$. For all $\beta < \epsilon$ and for all $A \in \exists^\mathbb{R}\Delta$, if $A \subseteq \mathsf{GC}_\beta$, then there is a $\delta < \kappa$ so that $A \subseteq \bigcup_{\gamma<\delta} \mathsf{GC}_{\beta,\gamma}$.

Let $\mathsf{GC} = \bigcap_{\beta<\epsilon} \mathsf{GC}_\beta$. Say that κ is ϵ-reasonable if and only if there is a good coding system for ${}^\epsilon\kappa$.

If one needs to emphasize the good coding system $\mathcal{G}$, one might write, $\Gamma^\mathcal{G}$, $\mathsf{decode}^\mathcal{G}$, $\mathsf{GC}^\mathcal{G}_{\beta,\gamma}$, $\mathsf{GC}^\mathcal{G}_\beta$, or $\mathsf{GC}^\mathcal{G}$.

The idea is that $x \in \mathsf{GC}_{\beta,\gamma}$ implies that $\mathsf{decode}(x)(\beta,\gamma)$ code the graph of a potential partial function which at least maps β to γ. $x \in \mathsf{GC}_\beta$ intuitively means that $\mathsf{decode}(x)$ codes the graph of a potential partial function which is defined at β taking some value below κ. $x \in \mathsf{GC}$ means $\mathsf{decode}(x)$ is the graph of a function from ϵ into κ.

The pointclass that appears in a good coding system can be shown to have many additional properties:

Fact 5.11. *([20] Remark 2.35) Assume AD. Let $\mathcal{G} = (\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \kappa)$ be a good coding system for ${}^\epsilon\kappa$. Then Γ is a nonselfdual pointclass closed under countable union, countable intersection, and $\forall^\mathbb{R}$, has the prewellordering property, and $\kappa = \boldsymbol{\delta}(\Gamma)$. Δ_Γ is closed under less than κ-length unions and intersections.*

The primary application of good coding systems is to prove partition properties:

Fact 5.12. *(Martin) If κ is $\omega \cdot \epsilon$-reasonable, then $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$.*

Good coding system supply an almost everywhere uniformization relative to the good codes. This will also be used later to select clubs.

Definition 5.13. Let $\epsilon \in \mathrm{ON}$ and $f : \omega \cdot \epsilon \to \mathrm{ON}$. Then let $\mathsf{block}(f) : \epsilon \to \mathrm{ON}$ be defined by $\mathsf{block}(f)(\alpha) = \sup\{f(\omega \cdot \alpha + n) : n \in \omega\}$.

Fact 5.14. *([3], Almost everywhere good code uniformization) Let $\epsilon \leq \kappa$ and $\mathcal{G} = (\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \omega \cdot \epsilon, \gamma < \kappa\rangle$ be a good coding system for ${}^{\omega\cdot\epsilon}\kappa$. Let $R \subseteq [\kappa]^\epsilon_* \times \mathbb{R}$. There exists a club $C \subseteq \kappa$ and a Lipschitz function $\Xi : \mathbb{R} \to \mathbb{R}$ so that for all $x \in \mathsf{GC}$ with $\mathsf{decode}(x) \in [C]^{\omega\cdot\epsilon}$, $R(\mathsf{block}(\mathsf{decode}(x)), \Xi(x))$.*

Definition 5.15. Let $\epsilon \leq \kappa$ and $\mathcal{G}$ be a good coding system for ${}^\epsilon\kappa$. If $X \subseteq \kappa$, then let $\mathrm{Inc}(X)$ be the set of all $x \in \mathsf{GC}$ so that $\mathsf{decode}(x) \in [X]^\epsilon$. If the good coding system $\mathcal{G}$ needs to be made explicit, one will write $\mathrm{Inc}^\mathcal{G}(X)$.

[3]Uniformly means there is a function $\Upsilon : \check{\Gamma} \to \mathscr{P}(\kappa)$ so that for all $A \in \check{\Gamma}$ with $A \subseteq \mathsf{clubcode}^\varphi_\kappa$, $\Upsilon(A)$ is a club subset of κ and for all $\rho \in A$, $\Upsilon(A) \subseteq \mathfrak{C}^{\varphi,\kappa}_\rho$.

One will need explicit applications of the Moschovakis coding lemma (rather than merely its consequence that if $\kappa < \Theta$, then $\mathbb{R}$ surjects into $\mathscr{P}(\kappa)$ which has been used previously).

Fact 5.16. *([20] Theorem 2.12) Assume* AD*. Let* Γ *be a pointclass closed under* $\exists^{\mathbb{R}}$ *and* $\wedge$*. Let* $P \in \Gamma$ *and* $\varphi : P \to \kappa$ *be a surjective norm so that the associated prewellordering* $\preceq$ *belongs to* Γ*. For any* $R \subseteq P \times \mathbb{R}$*, there is an* $S \in \Gamma$ *with the following properties:*

- $S \subseteq R$.
- *For all* $\alpha < \kappa$*, if there exists* $v \in \mathrm{dom}(R)$ *with* $\varphi(v) = \alpha$*, then there exists* $w \in \mathrm{dom}(S)$ *with* $\varphi(w) = \alpha$.

Fact 5.17. *Suppose* Γ *is a nonselfdual pointclass closed under* $\exists^{\mathbb{R}}$ *and* $\wedge$*.* Γ *is closed under less than* $\boldsymbol{\delta}(\Gamma)$*-length unions.*[4] *Thus* $\check{\Gamma}$ *is closed under less than* $\boldsymbol{\delta}(\Gamma)$*-length intersections.*

Proof. Let $\delta < \boldsymbol{\delta}(\Gamma)$. Let $\varphi : P \to \delta$ be a norm whose associated prewellordering belongs to Δ_Γ. Let $\langle A_\alpha : \alpha < \delta \rangle$ be a sequence of subsets of $\mathbb{R}$ in Γ. Let $U \subseteq \mathbb{R} \times \mathbb{R}$ be Γ-universal for subsets of $\mathbb{R}$. Let $R(w, e)$ if and only if $w \in P$ and $U_e = A_{\varphi(w)}$. By the Moschovakis coding lemma (Fact 5.16 applied to the pointclass Γ), there is an $S \in \Gamma$ with the property specified in the coding lemma. Then $x \in \bigcup_{\alpha<\delta} A_\alpha$ if and only if $(\exists w)(\exists e)(S(w, e) \wedge U(e, x))$. Thus $\bigcup_{\alpha<\delta} A_\alpha \in \Gamma$. ☐

Fact 5.18. *Assume* AD*. Let* $\epsilon \leq \kappa$ *and* $\mathcal{G} = (\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\alpha} : \beta < \epsilon \wedge \gamma < \kappa)$*. For all* $\beta < \epsilon$*,* $\mathsf{GC}_\beta \in \exists^{\mathbb{R}}\Gamma$*.* $\mathsf{GC} \in \forall^{\mathbb{R}}\exists^{\mathbb{R}}\Gamma$*. For all* $X \subseteq \kappa$*,* $\mathsf{Inc}(X) \in \forall^{\mathbb{R}}\exists^{\mathbb{R}}\Gamma$*.*

Proof. Since $\kappa = \boldsymbol{\delta}(\Gamma)$, there is a prewellordering of length κ in $\Gamma \subseteq \Delta_{\exists^{\mathbb{R}}\Gamma}$. Thus $\kappa < \boldsymbol{\delta}(\exists^{\mathbb{R}}\Gamma)$. For each $\beta < \epsilon$, $\mathsf{GC}_\beta = \bigcup_{\gamma<\kappa} \mathsf{GC}_{\beta,\gamma}$ which is a κ-length union of set from $\Delta_{\exists^{\mathbb{R}}\Gamma} \subseteq \exists^{\mathbb{R}}\Gamma$. Thus $\mathsf{GC}_\beta \in \exists^{\mathbb{R}}\Gamma$ by Fact 5.17. $\mathsf{GC} = \bigcap_{\beta<\epsilon} \mathsf{GC}_\beta$ and is thus an ϵ-length intersection of sets from $\exists^{\mathbb{R}}\Gamma \subseteq \forall^{\mathbb{R}}\exists^{\mathbb{R}}\Gamma$. Note that $\epsilon \leq \kappa < \boldsymbol{\delta}(\exists^{\mathbb{R}}\Gamma) \leq \boldsymbol{\delta}(\forall^{\mathbb{R}}\exists^{\mathbb{R}}\Gamma)$. Applying Fact 5.17, one has that $\forall^{\mathbb{R}}\exists^{\mathbb{R}}\Gamma$ is closed under ϵ-length intersections. So $\mathsf{GC} \in \forall^{\mathbb{R}}\exists^{\mathbb{R}}\Gamma$. Note that $\mathsf{Inc}(X) = \mathsf{GC} \cap \bigcup\{\mathsf{GC}_{\beta_0,\gamma_0} \cap \mathsf{GC}_{\beta_1,\gamma_1} : \beta_0 < \beta_1 < \epsilon \wedge \gamma_0 < \gamma_1 \wedge \gamma_0, \gamma_1 \in X\}$. GC was already shown to be $\forall^{\mathbb{R}}\exists^{\mathbb{R}}\Gamma$ and the latter part of the intersection is a κ-length union of sets in Δ_Γ which was already observed to belong to $\exists^{\mathbb{R}}\Gamma$. The total complexity is $\forall^{\mathbb{R}}\exists^{\mathbb{R}}\Gamma$. ☐

As an example: in one instance of the intended application of this section, one will have two good coding systems $\mathcal{G}_1$ for ${}^{\epsilon_0}\kappa_0$ and $\mathcal{G}_1$ for ${}^{\epsilon_1}\kappa_1$. One would like to have $\mathsf{GC}^{\mathcal{G}_0} \in \Delta_{\Gamma^{\mathcal{G}_1}}$. However Fact 5.18 is already too coarse for two successive projective ordinals. ω_1 has a good coding system $\mathcal{G}_0$ where $\Gamma^{\mathcal{G}_0} = \boldsymbol{\Pi}^1_1$ and $\omega_{\omega+1}$ has a good coding system $\mathcal{G}_1$ where $\Gamma^{\mathcal{G}_1} = \boldsymbol{\Pi}^1_3$. Fact 5.18 would imply $\mathsf{GC}^{\mathcal{G}_0} \in \boldsymbol{\Pi}^1_3 = \Gamma^{\mathcal{G}_1}$. This is already too high. In this case and many others, the complexity can be shown to be lower. By Fact 5.18, $\mathsf{GC}^{\mathcal{G}_0}_\beta$ is at most $\boldsymbol{\Sigma}^1_2$ for each $\beta < \epsilon$. $\boldsymbol{\Delta}^1_3$ can be shown to be closed under $< \omega_{\omega+1}$-length unions and intersections. Thus $\mathsf{GC}^{\mathcal{G}_0} = \bigcap_{\beta<\epsilon} \mathsf{GC}^{\mathcal{G}_0}_\beta$ is $\boldsymbol{\Delta}^1_3$ which is good enough for the purpose here. Harrington-Kechris ([15] Corollary 2.2) shows that $\boldsymbol{\Sigma}^1_{n+1}$, $\boldsymbol{\Pi}^1_{n+1}$, and $\boldsymbol{\Delta}^1_{n+1}$ are closed under ζ-length unions and intersection for all $\zeta < \boldsymbol{\delta}^1_n$ under AD. Thus $\boldsymbol{\Sigma}^1_2$ is closed under countable intersections. So when $\epsilon_0 < \omega_1$, $\mathsf{GC}^{\mathcal{G}_0}$ is $\boldsymbol{\Sigma}^1_2$. When $\epsilon_0 = \omega_1$, it can be shown that $\mathsf{GC}^{\mathcal{G}_0} \notin \boldsymbol{\Sigma}^1_2$ (see [3]]). However, by careful inspection of an explicit good coding systems on $\boldsymbol{\delta}^1_{2n+1}$, one can get even better complexity estimates. See [3] for the details for the good coding systems on ω_1 and [18] and [19] for the general odd projective ordinals.

Fact 5.19. *Assume* AD*. Let* $\epsilon \leq \omega_1$*. There is a good coding system* $\mathcal{G} = (\boldsymbol{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \omega_1)$ *for* ${}^{\epsilon}\omega_1$ *with the following properties:*

- *For all* $\beta < \epsilon$*,* $\mathsf{GC}_\beta \in \boldsymbol{\Pi}^1_1$*.*
- *If* $\epsilon < \omega_1$*, then* $\mathsf{GC} \in \boldsymbol{\Pi}^1_1$*. If* $\epsilon = \omega_1$*, then* $\mathsf{GC} \in \boldsymbol{\Pi}^1_2$*.*
- *For* $\epsilon < \omega_1$ *and club* $C \subseteq \omega_1$*,* $\mathrm{Inc}(C) \in \boldsymbol{\Pi}^1_1$*. For* $\epsilon = \omega_1$ *and club* $C \subseteq \omega_1$*,* $\mathrm{Inc}(C) \in \boldsymbol{\Pi}^1_2$*.*

Let $n \in \omega$ *and* $\epsilon \leq \boldsymbol{\delta}^1_{2n+1}$*. There is a good coding system* $\mathcal{G} = (\boldsymbol{\Pi}^1_{2n+1}, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \boldsymbol{\delta}^1_{2n+1})$ *with the following properties:*

- *For all* $\beta < \epsilon$*,* $\mathsf{GC}_\beta \in \boldsymbol{\Pi}^1_{2n+1}$*.*
- *If* $\epsilon < \boldsymbol{\delta}^1_{2n+1}$*, then* $\mathsf{GC} \in \boldsymbol{\Pi}^1_{2n+1}$*. If* $\epsilon = \omega_1$*, then* $\mathsf{GC} \in \boldsymbol{\Pi}^1_{2n+2}$*.*

[4] If Γ has the prewellordering property, then Γ is furthermore closed under wellordered unions. See [20] Lemma 2.21.

- *For $\epsilon < \boldsymbol{\delta}^1_{2n+1}$ and club $C \subseteq \boldsymbol{\delta}^1_{2n+1}$, $\mathrm{Inc}(C) \in \boldsymbol{\Pi}^1_{2n+1}$. For $\epsilon = \boldsymbol{\delta}^1_{2n+1}$ and club $C \subseteq \boldsymbol{\delta}^1_{2n+1}$, $\mathrm{Inc}(C) \in \boldsymbol{\Pi}^1_{2n+2}$.*

Definition 5.20. Let $1 \leq n < \omega$, $\langle \kappa_0, ..., \kappa_{n-1} \rangle$ be an increasing sequence of cardinals and $\langle \epsilon_0, ..., \epsilon_{n-1} \rangle$ be a sequence of ordinals such that for all $i < n$, $\epsilon_i \leq \kappa_i$. Define $\langle \kappa_0, ..., \kappa_{n-1} \rangle \to_* (\kappa_0, ..., \kappa_{n-1})_2^{\epsilon_0, ..., \epsilon_{n-1}}$ if and only if for all functions $P : \prod_{i<n} [\kappa_i]^{\epsilon_i} \to 2$, there is an $i \in 2$ and sequence $\langle C_0, ..., C_{n-1} \rangle$ so that for each $j < n$, $C_j \subseteq \kappa_j$ is a club subset of κ_i and for all $(f_0, ..., f_{n-1}) \in \prod_{j<n} [C_j]^{\epsilon_j}_*$, $P(f_0, ..., f_{n-1}) = i$.

Definition 5.21. Let $1 \leq n < \omega$, $\langle \kappa_0, ..., \kappa_{n-1} \rangle$ be an increasing sequence of cardinals, and let $\langle \epsilon_0, ..., \epsilon_{n-1} \rangle$ be a sequence of ordinals such that $\epsilon_i \leq \kappa_i$ for all $i < n$. Say that $\langle \kappa_0, ..., \kappa_{n-1} \rangle$ is an $\langle \epsilon_0, ... \epsilon_{n-1} \rangle$-reasonable sequence if and only if there is a sequence $\langle \mathcal{G}_0, ..., \mathcal{G}_{n-1} \rangle$ with the following properties:

- $\mathcal{G}_i$ is a good coding system for ${}^{\epsilon}\kappa_i$.
- For any $i < j < n$ and club $C_i \subseteq \kappa_i$, $\mathrm{Inc}^{\mathcal{G}_i}(C_i) \in \Delta_{\Gamma^{\mathcal{G}_j}}$.

Example 5.22. Assume AD. Let $1 \leq n < \omega$ and $\ell : n \to \omega$ be a strictly increasing sequence. Let $\langle \epsilon_i : i < n \rangle$ be a sequence of ordinals so that $\epsilon_i \leq \boldsymbol{\delta}^1_{2\ell(i)+1}$ for all $i < n$. Then $\langle \boldsymbol{\delta}^1_{2\ell(i)+1} : i < n \rangle$ is $\langle \epsilon_i : i < n \rangle$-reasonable using Fact 5.19.

Example 5.23. Assume AD. Let $A \in \mathscr{P}(\mathbb{R})$. Let $\boldsymbol{\Sigma}_1^{L(A,\mathbb{R})}$ be the subsets of $\mathbb{R}$ which are Σ_1-definable over $L(A, \mathbb{R})$ in the language with a symbol $\dot{\mathbb{R}}$ for $\mathbb{R}$ using parameters from $\mathbb{R}$. Let $\boldsymbol{\delta}_A$ the least A-stable ordinal which is the least ordinal δ so that $L_\delta(A, \mathbb{R})$ is a Σ_1-elementary substructure of $L(A, \mathbb{R})$. Kechris-Kleinberg-Moschovakis-Woodin ([22]) showed there is a good coding system $\mathcal{G}$ for ${}^{\boldsymbol{\delta}_A}\boldsymbol{\delta}_A$ so that $\Gamma^{\mathcal{G}} = \boldsymbol{\Sigma}_1^{L(A,\mathbb{R})}$. (Also see [3] for a construction of this good coding system.) Note that $\{\boldsymbol{\delta}_A : A \in \mathscr{P}(\mathbb{R})\}$ is a collection of strong partition cardinals which is unbounded in Θ.

Let $1 \leq n < \omega$ and $\ell : n \to \mathscr{P}(\mathbb{R})$ with the property that for all $i < j < n$, $\boldsymbol{\delta}_{\ell(i)} < \boldsymbol{\delta}_{\ell(j)}$. Let $\langle \epsilon_i : i < n \rangle$ be such that for all $i < n$, $\epsilon_i \leq \boldsymbol{\delta}_{\ell(i)}$. Then $\langle \boldsymbol{\delta}_{\ell(i)} : i < n \rangle$ is an $\langle \epsilon_i : i < n \rangle$-reasonable sequence using Fact 5.18 since the pointclasses $\boldsymbol{\Sigma}_1^{L(\ell(i),\mathbb{R})}$ and $\boldsymbol{\Sigma}_1^{L(\ell(j),\mathbb{R})}$ are sufficiently far apart from each other.

The following is an independently interesting multi-cardinal partition relation.

Theorem 5.24. *Assume* AD. *Let $1 \leq n < \omega$, $\langle \kappa_i : i < n \rangle$, and $\langle \epsilon_i : i < n \rangle$ be such that $\langle \kappa_i : i < n \rangle$ is $\langle \omega \cdot \epsilon_i : i < n \rangle$-reasonable. Then $\langle \kappa_0, ..., \kappa_{n-1} \rangle \to_* (\kappa_0, ..., \kappa_n)_2^{\epsilon_0, ..., \epsilon_{n-1}}$ holds.*

Proof. This result is proved by induction on the length $1 \leq n < \omega$.

For $n = 1$, the hypothesis simply states that κ_0 is $\omega \cdot \epsilon_0$-reasonable. Thus $\kappa_0 \to_* (\kappa_0)^{\epsilon_0}_2$ holds (by Fact 5.12) which is equivalent to $\langle \kappa_0 \rangle \to_* (\kappa_0)^{\epsilon_0}_2$.

Suppose the result has been shown for $1 \leq n < \omega$. Let $\langle \kappa_0, ..., \kappa_n \rangle$ and $\langle \epsilon_0, ..., \epsilon_n \rangle$ be such that $\langle \kappa_0, ..., \kappa_n \rangle$ is $\langle \omega \cdot \epsilon_0, ..., \omega \cdot \epsilon_n \rangle$-reasonable. Let $\langle \mathcal{G}_0, ..., \mathcal{G}_n \rangle$ be a sequence of good coding systems witnessing this. By Fact 5.11, for each $i < n$, let W_i be a $\Gamma^{\mathcal{G}_i}$-complete set and $\varphi_i : W_i \to \kappa_i$ be a surjective $\Gamma^{\mathcal{G}_i}$-norm. Fix a map $P : \prod_{i<n+1} [\kappa_i]^{\epsilon_i} \to 2$. For each $f_0 \in [\kappa_0]^{\epsilon_0}$, define $P_{f_0} : \prod_{i<n} [\kappa_{i+1}]^{\epsilon_{i+1}} \to 2$ by $P_{f_0}(g_1, ..., g_n) = P(f_0, g_1, ..., g_n)$. By the induction hypothesis at n, $\langle \kappa_1, ..., \kappa_n \rangle \to_* (\kappa_1, ..., \kappa_n)_2^{\epsilon_1, ..., \epsilon_n}$ holds. Thus for each $f_0 \in [\kappa]^{\epsilon_0}$, there is a unique $j_{f_0} \in 2$ for which there exists $\langle D_1, ..., D_n \rangle$ with the property that for all $1 \leq i \leq n$, $D_i \subseteq \kappa_i$ is a club subset of κ_i and for all $(g_1, ..., g_n)$ with $g_i \in [D_i]^{\kappa_i}_*$ for all $1 \leq i \leq n$, $P_{f_0}(g_1, ..., g_n) = j_{f_0}$. Define $Q : [\kappa_0]^{\epsilon_0} \to 2$ by $Q(f_0) = j_{f_0}$. Since the hypothesis implies κ_0 is $\omega \cdot \epsilon_0$-reasonable, Fact 5.12 implies $\kappa_0 \to_* (\kappa_0)^{\epsilon_0}_2$. Thus there is a club $C_0 \subseteq \kappa$ and a $\bar{j} \in 2$ so that for all $f_0 \in [C_0]^{\epsilon_0}_*$, $Q(f_0) = j_{f_0} = \bar{j}$. Define $R \subseteq [\kappa]^{\epsilon_0} \times {}^n\mathbb{R}$ by $R(f_0, (\rho_1, ..., \rho_n))$ if and only if for all $1 \leq i \leq n$, $\rho_i \in \mathsf{clubcode}^{\varphi_i}_{\kappa_i}$ and for all $(g_1, ..., g_n) \in [\mathfrak{C}^{\varphi_1, \kappa_1}_{\rho_1}]^{\epsilon_1}_* \times ... \times [\mathfrak{C}^{\varphi_n, \kappa_n}_{\rho_n}]^{\epsilon_n}_*$, $P_{f_0}(g_0, ..., g_n) = \bar{j}$. The first claim is that $\mathrm{dom}(R) = [C_0]^{\epsilon_0}_*$. To see this, by the observation above, for each $f_0 \in [C_0]^{\epsilon_0}_*$, there is a sequence $(D_1, ..., D_n)$ with each $D_i \subseteq \kappa_i$ club in κ_i for all $1 \leq i \leq n$ which is homogeneous for P_{f_0} taking value $j_{f_0} = \bar{j}$. By Fact 5.8, for each $1 \leq i \leq n$, there is a $\rho_i \in \mathsf{clubcode}^{\varphi_i}_{\kappa_i}$ so that $\mathfrak{C}^{\varphi_i, \kappa_i}_{\rho_i} \subseteq D_i$. Then $R(f_0, (\rho_1, ..., \rho_n))$ holds and hence $f_0 \in \mathrm{dom}(R)$. By the almost everywhere good code uniformization (Fact 5.14), there is a club $C_1 \subseteq C_0$ and a Lipschitz continuous function $\Xi : \mathbb{R} \to {}^n\mathbb{R}$ so that for all $e \in \mathrm{Inc}^{\mathcal{G}_0}(C_1)$, $R(\mathsf{block}(\mathsf{decode}^{\mathcal{G}_0}(e)), \Xi(e))$. Let $\pi^n_i : {}^n\mathbb{R} \to \mathbb{R}$ be the projection onto the i^{th}-coordinate for each $1 \leq i \leq n$. Let $\Xi_1, ..., \Xi_n : {}^n\mathbb{R} \to \mathbb{R}$ be defined by $\Xi^i = \pi^n_i \circ \Xi$ for each $1 \leq i \leq n$. Note Ξ_i are also Lipschitz functions (if the coding of tuples were chosen reasonably). (Ξ^i being continuous is enough.) By the hypothesis, $\mathrm{Inc}^{\mathcal{G}_0}(C_1) \in \Delta_{\Gamma^{\mathcal{G}_i}}$ for all $1 \leq i \leq n$. Thus $\Xi_i[\mathrm{Inc}^{\mathcal{G}_0}(C_1)] \in \exists^{\mathbb{R}} \Delta_{\Gamma^{\mathcal{G}_i}} \subseteq \check{\Gamma}^{\mathcal{G}_i}$. By the property of Ξ, one has that $\Xi_i[\mathrm{Inc}^{\mathcal{G}_0}(C_1)] \subseteq \mathsf{clubcode}^{\varphi_i}_{\kappa_i}$. By Fact

5.9, for each $1 \leq i \leq n$, there is a club $E_i \subseteq \kappa_i$ so that for all $\rho \in \Xi_i[\mathrm{Inc}^{\mathcal{G}_0}(C_1)]$, $E_i \subseteq \mathfrak{C}_\rho^{\varphi_i,\kappa_i}$. Let $E_0 \subseteq C_1$ be the club of limit points of C_1. The claim is that $(E_0, ..., E_n)$ is homogeneous for P taking value $\bar{j}$. Pick any $(f_0, f_1, ..., f_n) \in \prod_{i<n+1}[E_i]^{\epsilon_i}_*$. Since $f_0 \in [E_0]^{\epsilon_0}_* \subseteq [C_1]^{\epsilon_0}_* \subseteq \mathrm{dom}(R)$, one has that $f_0 \in \mathrm{dom}(R)$. Also since E_0 consists of limit points of C_1, pick any $h_0 \in [C_1]^{\omega\cdot\epsilon}$ so that $\mathsf{block}(h_0) = f_0$. By the property of the good coding system $\mathcal{G}_0$, there is some $e_0 \in \mathsf{GC}^{\mathcal{G}_0}$ so that $\mathsf{decode}^{\mathcal{G}_0}(e_0) = h_0$. Thus $e_0 \in \mathrm{Inc}^{\mathcal{G}_0}(C_1)$. Let $(\rho_1, ..., \rho_n) = \Xi(e_0)$. $R(f_0, (\rho_1, ..., \rho_n))$ holds since $R(\mathsf{block}(\mathsf{decode}^{\mathcal{G}_0}(e_0)), \Xi(e_0))$ holds. By definition of R, this means that for all $(g_1, ..., g_n) \in \prod_{1\leq i\leq n}[\mathfrak{C}_{\rho_i}^{\varphi_i,\kappa_i}]^{\epsilon_1}_*$, $P_{f_0}(g_1, ..., g_n) = \bar{j}$. Since $(f_1, ..., f_n) \in \prod_{1\leq i\leq n} E_i \subseteq \prod_{1\leq i\leq n}[\mathfrak{C}_{\rho_i}^{\varphi_i,\kappa_i}]^{\epsilon_1}_*$, one has that $P(f_0, f_1, ..., f_n) = P_{f_0}(f_1, ..., f_n) = \bar{j}$. Since $(f_0, ..., f_n) \in \prod_{i<n+1}[E_i]^{\epsilon_i}_*$ was arbitrary, this shows that $(E_0, ..., E_n)$ is homogeneous for P taking value $\bar{j}$. Since P was arbitrary, this establishes $\langle \kappa_0, ..., \kappa_n\rangle \to_* (\kappa_0, ..., \kappa_n)_2^{\epsilon_0,...,\epsilon_n}$. The result has been shown for $n+1$.

By induction, this completes the proof. □

Definition 5.25. A sequence of cardinals $\langle \kappa_n : n \in \omega\rangle$ is a reasonable sequence if and only if there are sequence $\langle \zeta_n : n \in \omega\rangle$ and $\langle \Gamma_n : n \in \omega\rangle$ with the following properties:

(1) For all $n \in \omega$, $\zeta_n \leq \kappa_n + 1$. $\langle \zeta_n : n \in \omega\rangle$ is an increasing sequence.
(2) $\sup\{\zeta_n : n \in \omega\} = \sup\{\kappa_n : n \in \omega\}$.
(3) For all $n \in \omega$, Γ_n is a pointclass.
(4) For all $n \in \omega$ and $\xi < \zeta_n$, there is a good coding system $\mathcal{G}$ for ${}^\xi\kappa_n$ with $\Gamma^{\mathcal{G}} = \Gamma_n$ and $\mathsf{GC}^{\mathcal{G}} \in \Delta_{\Gamma_m}$ for all $m > n$. [5]
(5) There is a set $Z \in \mathscr{P}(\mathbb{R})$ which Lipschitz reduces all sets in $\bigcup_{n\in\omega}\Gamma_n$.

Example 5.26. The sequence of odd projective ordinals $\langle \boldsymbol{\delta}^1_{2n+1} : n \in \omega\rangle$ is a resonable sequence. This is witnessed by $\langle \zeta_n : n \in \omega\rangle$ and $\langle \boldsymbol{\Pi}^1_{2n+1} : n \in \omega\rangle$ where $\zeta_n = \boldsymbol{\delta}^1_{2n+1} + 1$ for each $n \in \omega$. This follows from Fact 5.19.

Example 5.27. Let $\langle A_n : n \in \omega\rangle$ is a sequence in $\mathscr{P}(\mathbb{R})$ so that the corresponding sequence of stable ordinals $\langle \boldsymbol{\delta}_{A_n} : n \in \omega\rangle$ is a strictly increasing sequence. Then $\langle \boldsymbol{\delta}_{A_n} + 1 : n \in \omega\rangle$ and $\langle \boldsymbol{\Sigma}_1^{L(A_n,\mathbb{R})} : n \in \omega\rangle$ witness that $\langle \boldsymbol{\delta}_{A_n} : n \in \omega\rangle$ is a reasonable sequence. This follows from the discussion in Example 5.23.

The following definition is used in the proof of Theorem 5.29.

Definition 5.28. Let $\epsilon \in \mathrm{ON}$. $(<\epsilon)$-instruction $\mathfrak{i}$ is a triple $(n^{\mathfrak{i}}, \mathfrak{p}^{\mathfrak{i}}, \ell^{\mathfrak{i}})$ such that $1 \leq n^{\mathfrak{i}} < \omega$, $\mathfrak{p}^{\mathfrak{i}} : n^{\mathfrak{i}} \to \omega$ is a strictly increasing sequence, and $\ell^{\mathfrak{i}} : n^{\mathfrak{i}} \to \epsilon$ is sequence such that $\ell^{\mathfrak{i}}(0) + ... + \ell^{\mathfrak{i}}(n-1) < \epsilon$. If $m < \omega$, then a $(<\epsilon)$-instruction above m is a $(<\epsilon)$-instruction $\mathfrak{i}$ with $\mathfrak{p}^{\mathfrak{i}}(0) > m$.

Note that for any $\epsilon \in \mathrm{ON}$, the collection of $(<\epsilon)$-instructions has cardinality $\max\{|\omega|, |\epsilon|\}$.

Theorem 5.29. *Assume* AD. *If κ is the supremum of a reasonable sequence, then κ is a super-Magidor cardinal.*

Proof. Let $\langle \kappa_n : n \in \omega\rangle$ be a reasonable sequence with $\kappa = \sup\{\kappa_n : n \in \omega\}$. Let $\langle \Gamma_n : n \in \omega\rangle$ be a sequence of pointclass and let $\langle \zeta_n : n \in \omega\rangle$ be a sequence of ordinals witnessing that $\langle \kappa_n : n \in \omega\rangle$ is a reasonable sequence as in Definition 5.25. Pick $\epsilon < \kappa$. Let $\Phi : \mathsf{Bl}_{<\kappa}(<\epsilon, \kappa) \to \kappa$. Let $\bar{m}$ be the least m so that $\omega \cdot \epsilon < \zeta_m$. Let $\mathfrak{I}$ be the collection of all $(<\epsilon)$-instruction above $\bar{m}+1$. For each instruction $\mathfrak{i} \in \mathfrak{I}$, let $P_{\mathfrak{i}} : [\kappa_{\bar{m}+1}]^1 \times \prod_{i<n^{\mathfrak{i}}}[\kappa_{\mathfrak{p}^{\mathfrak{i}}(i)}]^{\ell^{\mathfrak{i}}(i)} \to 2$ be defined by $P(\alpha, f_0, ..., f_{n^{\mathfrak{i}}-1}) = 0$ if and only if $\Phi(f_0\hat{\ }...\hat{\ }f_{n^{\mathfrak{i}}-1}) < \alpha$. Then $\langle \kappa_{\bar{m}+1}, \kappa_{\mathfrak{p}^{\mathfrak{i}}(0)}, ..., \kappa_{\mathfrak{p}^{\mathfrak{i}}(n^{\mathfrak{i}}-1)}\rangle$ is $\langle \omega\cdot 1, \omega\cdot\ell^{\mathfrak{i}}(0), ..., \omega\cdot\ell^{\mathfrak{i}}(n^{\mathfrak{i}}-1)\rangle$-reasonable since $\omega\cdot\ell^{\mathfrak{i}}(i) < \omega\cdot\epsilon < \zeta_{\bar{m}} < \zeta_{\mathfrak{p}^{\mathfrak{i}}(i)}$ for all $i < n$ by the choice of $\bar{m}$ and since $\mathfrak{i} \in \mathfrak{I}$ is a $(<\epsilon)$-instruction above $\bar{m}+1$. Thus $\langle \kappa_{\bar{m}+1}, \kappa_{\mathfrak{p}^{\mathfrak{i}}(0)}, ..., \kappa_{\mathfrak{p}^{\mathfrak{i}}(n^{\mathfrak{i}}-1)}\rangle \to_* (\kappa_{\bar{m}+1}, \kappa_{\mathfrak{p}^{\mathfrak{i}}(0)}, ..., \kappa_{\mathfrak{p}^{\mathfrak{i}}(n^{\mathfrak{i}}-1)})_2^{1,\ell^{\mathfrak{i}}(0),...,\ell^{\mathfrak{i}}(n^{\mathfrak{i}}-1)}$ by Fact 5.24. So there is a unique $u_{\mathfrak{i}} \in 2$ which is the homogeneous value for $P_{\mathfrak{i}}$. Note that $|\mathfrak{I}| = |\epsilon|$ so let $\mathfrak{b} : \epsilon \to \mathfrak{I}$ be a bijection. For each $1 \leq m < n$, let $\Sigma^n : {}^n\mathbb{R} \to \mathbb{R}$ be a fixed bijection. Let $\Pi^n_m : \mathbb{R} \to \mathbb{R}$ be recursive bijection so that for all $(x_0, ..., x_{n-1}) \in {}^n\mathbb{R}$, $\Pi^n_m(\Sigma^n(x_0, ..., x_{n-1})) = x_m$. By the hypothesis of $\langle \kappa_n : n \in \omega\rangle$ being a reasonable sequence, there is a $Z \in \mathscr{P}(\mathbb{R})$ which Lipschitz reduces all sets in $\bigcup_{n\in\omega}\Gamma_n$. Define $R \subseteq \omega \times \mathbb{R}$ by $R(n, \rho)$ if and only $(\Xi^2_\rho)^{-1}[Z]$ is a Γ_n-norm on a Γ_n-complete set. By $\mathsf{AC}^{\mathbb{R}}_\omega$ (which holds under AD), let $\langle \rho_n : n \in \omega\rangle$ be

[5] Observe that this merely asserts the existence of good coding system but does not provide any ability to uniformly pick good coding system in $n \in \omega$ and $\xi < \zeta_n$.

such that for all $n \in \omega$, $R(n, \rho_n)$. Let $\varphi_n : W_n \to \kappa_n$ be the surjective Γ_n-norm on a complete Γ_n-set coded by $(\Xi^2_{\rho_n})^{-1}[Z]$. Let $\varphi : W \to \kappa_{\bar{m}+1}$ be a $\Gamma_{\bar{m}+1}$-norm on a complete $\Gamma_{\bar{m}+1}$-set. Since $\epsilon < \kappa_{\bar{m}}$, fix $\psi : Q \to \epsilon$ be a surjective norm in $\Delta_{\Gamma_{\bar{m}}}$. Define $S \subseteq Q \times \mathbb{R}$ by $S(q, x)$ if and only if the following holds:

- Let $\mathfrak{b}(\psi(q))$ be the instruction $\mathfrak{i} = (n, \mathfrak{p}, \ell)$.
- $\Pi^{n+1}_0(x) \in \mathsf{clubcode}^{\varphi}_{\kappa_{\bar{m}+1}}$.
- For all $i < n$, $\Pi^{n+1}_{i+1}(x) \in \mathsf{clubcode}^{\varphi_{\mathfrak{p}(i)}}_{\kappa_{\mathfrak{p}(i)}}$.
- For all $(\alpha, f_0, ..., f_{n-1}) \in [\mathfrak{C}^{\varphi,\kappa_{\bar{m}+1}}_{\Pi^{n+1}_0(x)}]^1_* \times \prod_{i<n} [\mathfrak{C}^{\varphi_{\mathfrak{p}(i)},\kappa_{\mathfrak{p}(i)}}_{\Pi^{n+1}_{i+1}(x)}]^{\ell(i)}_*$, $P_{\mathfrak{i}}(\alpha, f_0, ..., f_{n-1}) = u_{\mathfrak{i}}$.

By the discussion above, $\mathrm{dom}(S) = Q$. By the Moschovakis coding lemma (Fact 5.16) applied to $\check{\Gamma}_{\bar{m}}$ and ψ, there is a $T \subseteq S$ with $T \in \check{\Gamma}_{\bar{m}}$ and for all $\alpha < \epsilon$, there exists some $q \in \mathrm{dom}(T)$ with $\psi(q) = \alpha$. Fix $\alpha < \epsilon$ and suppose $\mathfrak{b}(\alpha) = \mathfrak{i} = (n, \mathfrak{p}, \ell)$. Let K^α be defined by

$$K^\alpha = \{z \in \mathbb{R} : (\exists w)(\exists x)(w \in Q \wedge \psi(w) = \alpha \wedge T(w, x) \wedge \Pi^{n+1}_0(x) = z)\}.$$

Note that $K^\alpha \subseteq \mathsf{clubcode}^{\varphi}_{\kappa_{\bar{m}+1}}$ and belongs to $\exists^{\mathbb{R}}\check{\Gamma}_{\bar{m}} = \check{\Gamma}_{\bar{m}}$. For each $i < n$, let K^α_i be defined by

$$K^\alpha_i = \{z \in \mathbb{R} : (\exists w)(\exists x)(w \in Q \wedge \psi(w) = \alpha \wedge T(w, x) \wedge z = \Pi^{n+1}_{i+1}(x))\}.$$

Note that for all $i < n$, $K^\alpha_i \subseteq \mathsf{clubcode}^{\varphi_{\mathfrak{p}(i)}}_{\kappa_{\mathfrak{p}(i)}}$ and belongs to $\check{\Gamma}_{\bar{m}}$. Note that $\check{\Gamma}_{\bar{m}} \subseteq \check{\Gamma}_{\bar{m}+1}$ and $\check{\Gamma}_{\bar{m}} \subseteq \check{\Gamma}_{\kappa_{\mathfrak{p}(i)}}$ for all $i < n$ since $\mathfrak{i}$ is an $(< \epsilon)$-instruction above $\bar{m} + 1$. Thus by Fact 5.9, one obtains clubs $D^\alpha \subseteq \kappa_{\bar{m}}$ and clubs $D^\alpha_i \subseteq \kappa_{\mathfrak{p}(i)}$ with the property that for all $z \in K^\alpha$, $D^\alpha \subseteq \mathfrak{C}^{\varphi,\kappa_{\bar{m}}}_z$ and for all $z \in K^\alpha_i$, $D^\alpha_i \subseteq \mathfrak{C}^{\varphi_{\mathfrak{p}(i)},\kappa_{\mathfrak{p}(i)}}_z$. Pick any $q \in \mathrm{dom}(T)$ with $\psi(q) = \alpha$. Pick any y with $T(q, y)$. Note that $\Pi^{n+1}_0(y) \in K^\alpha$ and for all $i < n$, $\Pi^{n+1}_{i+1}(y) \in K^\alpha_i$. Thus $D^\alpha \subseteq \mathfrak{C}^{\varphi,\kappa_{\bar{m}}}_{\Pi^{n+1}_0(y)}$ and for all $i < n$, $D^\alpha_i \subseteq \mathfrak{C}^{\varphi_{\mathfrak{p}(i)},\kappa_{\mathfrak{p}(i)}}_{\Pi^{n+1}_{i+1}(y)}$. By definition of $T(q, y)$, $(\mathfrak{C}^{\varphi,\kappa_{\bar{m}}}_{\Pi^{n+1}_0(y)}, \mathfrak{C}^{\varphi_{\mathfrak{p}(0)},\kappa_{\mathfrak{p}(0)}}_{\Pi^{n+1}_1(y)}, \mathfrak{C}^{\varphi_{\mathfrak{p}(n-1)},\kappa_{\mathfrak{p}(n-1)}}_{\Pi^{n+1}_n(y)})$ is homogeneous for $P_{\mathfrak{i}}$ taking value $u_{\mathfrak{i}}$. Thus $(D^\alpha, D^\alpha_0, ..., D^\alpha_n)$ is homogeneous for $P_{\mathfrak{i}}$ taking value $u_{\mathfrak{i}}$. Since everything was done uniformly from α and $\mathfrak{b} : \epsilon \to \mathfrak{I}$ is a bijection, one can restate what has been shown as follows: There exists a sequence $\langle (D^{\mathfrak{i}}, D^{\mathfrak{i}}_0, ..., D^{\mathfrak{i}}_{n^{\mathfrak{i}}-1}) : \mathfrak{i} \in \mathfrak{I} \rangle$ with the property that for all $\mathfrak{i} \in \mathfrak{I}$, $D^{\mathfrak{i}}$ is a club subset of $\kappa_{\bar{m}+1}$ and $D^{\mathfrak{i}}_i$ is a club subset of $\kappa_{\mathfrak{p}^{\mathfrak{i}}(i)}$ for all $i < n^{\mathfrak{i}}$, and $(D^{\mathfrak{i}}, D^{\mathfrak{i}}_0, ..., D^{\mathfrak{i}}_{n^{\mathfrak{i}}-1})$ is homogeneous for $P_{\mathfrak{i}}$ taking value $u_{\mathfrak{i}}$. Let $D = \bigcap\{D^{\mathfrak{i}} : \mathfrak{i} \in \mathfrak{I}\}$. Note that D is a club subset of $\kappa_{\bar{m}+1}$ since $\epsilon < \kappa_{\bar{m}+1}$ and D is an ϵ-size intersection of club subsets of $\kappa_{\bar{m}+1}$. For each $\bar{m} + 1 < n < \omega$, let $D_n = \bigcap\{D^{\mathfrak{i}}_i : \mathfrak{p}^{\mathfrak{i}}(i) = n\}$. Once again, $D_n \subseteq \kappa_n$ is a club subset of κ_n since $\epsilon < \kappa_{\bar{m}+1} < \kappa_n$ and D is an ϵ-length intersection of club subsets of κ_n. One has define a club $D \subseteq \kappa_{\bar{m}+1}$ and a sequence $\langle D_n : \bar{m} + 1 < n < \omega \rangle$ such that for all $\bar{m} + 1 < n < \omega$, $D_n \subseteq \kappa_n$ is a club subset of κ_n and for all $\mathfrak{i} \in \mathfrak{I}$, $(D, D_{\mathfrak{p}^{\mathfrak{i}}(0)}, ..., D_{\mathfrak{p}^{\mathfrak{i}}(n^{\mathfrak{i}}-1)})$ is homogeneous for $P_{\mathfrak{i}}$ taking value $u_{\mathfrak{i}}$. One may also assume that for all $\bar{m} + 1 < n < \omega$, $D_n \subseteq \kappa_n \setminus \kappa_{n-1}$. For all $\bar{m} + 1 < n < \omega$, let $E_n = \{\mathsf{enum}_{D_n}(\omega \cdot \alpha + \omega) : \alpha < \kappa_n\}$. Let $F = \bigcup_{\bar{m}+1<n<\omega} E_n$. Note that $|F| = \kappa$ and for all $\xi < \epsilon$, $[F]^\xi = [F]^\xi_*$ by Fact 2.22.[6] Let $\hat{\alpha} < \hat{\beta}$ be the first two elements of $[D]^1_*$. The claim is that $\hat{\alpha} \notin \mathsf{Bl}_\kappa(< \epsilon, F)$. Let $f \in \mathsf{Bl}_\kappa(< \epsilon, F)$. Let $\xi = \mathrm{dom}(f)$. Let $A = \{k \in \omega : (\exists \eta < \xi)(f(\eta) \in D_k)\}$. Since f is bounded below κ, A is finite. Let $n = |A|$. Let $\mathfrak{p} : n \to A$ be the increasing enumeration of A. For each $i < n$, let $A_i = \{\eta < \xi : f(\eta) \in D_{\mathfrak{p}(i)}\}$. Let $\ell(i) = \mathrm{ot}(A_i)$. Let $\mathfrak{i} = (n, \mathfrak{p}, \ell)$ which is an instruction. Note that for all $i < n$, $\bar{m}+1 < \mathfrak{p}(i) < \omega$ and $\ell(0)+...+\ell(n-1) = \xi < \epsilon$. Thus $\mathfrak{i}$ is a $(< \epsilon)$-instruction above $\bar{m} + 1$. Thus $\mathfrak{i} \in \mathfrak{I}$. For each $i < n$, let $f_i : \ell(i) \to F$ be defined by $f_i(\eta) = f(\sum_{j<i} \mathfrak{p}(j) + \eta)$. Note that $f = f_0\hat{\ }f_1\hat{\ }...\hat{\ }f_{n-1}$ and for all $i < n$, $f_i \in [D_{\mathfrak{p}(i)}]^{\ell(i)}_*$.

(1) Suppose $u_{\mathfrak{i}} = 0$. $(\hat{\alpha}, f_0, ..., f_{n-1}) \in [D]^1_* \times [D_{\mathfrak{p}(0)}]^{\ell(0)}_* \times ... \times [D_{\mathfrak{p}(n-1)}]^{\ell(n-1)}_*$. Thus $P_{\mathfrak{i}}(\hat{\alpha}, f_0, ..., f_{n-1}) = u_{\mathfrak{i}} = 0$ implies that $\hat{\alpha} > \Phi(f_0\hat{\ }...\hat{\ }f_{n-1}) = \Phi(f)$.
(2) Suppose $u_{\mathfrak{i}} = 1$. $(\hat{\beta}, f_0, ..., f_{n-1}) \in [D]^1_* \times [D_{\mathfrak{p}(0)}]^{\ell(0)}_* \times ... \times [D_{\mathfrak{p}(n-1)}]^{\ell(n-1)}_*$. Thus $P_{\mathfrak{i}}(\hat{\alpha}, f_0, ..., f_{n-1}) = u_{\mathfrak{i}} = 1$ implies that $\hat{\alpha} < \hat{\beta} \leq \Phi(f_0\hat{\ }...\hat{\ }f_{n-1}) = \Phi(f)$.

Since $f \in \mathsf{Bl}_\kappa(< \epsilon, F)$ was arbitrary, one has shown that $\hat{\alpha} \notin \Phi[\mathsf{Bl}_\kappa(< \epsilon, F)]$. Thus $\Phi[\mathsf{Bl}_\kappa(< \epsilon, F)] \neq \kappa$. Since $\epsilon < \kappa$ was arbitrary, this implies that κ is super-Magidor. $\square$

The next result answer Question 2.8 of Ben-Neria and Garti from [1].

Theorem 5.30. *Assume* AD. *The supremum of the projective ordinals* $\boldsymbol{\delta}^1_\omega$ *is super-Magidor.*

[6] Going from E to F obtains the property that $[F]^\xi = [F]^\xi_*$ which is important since all partitions above used functions of the correct type but $\mathsf{Bl}_\kappa(< \epsilon, F)$ refer to all increasing function.

Proof. Use Example 5.26 and Theorem 5.29. □

Theorem 5.31. *Assume* AD. *There are unboundedly many singular super-Magidor cardinals below* Θ.

Proof. Use Example 5.27 and Theorem 5.29. □

Next, one will show that the supremum κ of a reasonable sequence $\langle \kappa_n : n \in \omega \rangle$ has a $(< \omega \cdot \omega)$-Magidor filter. One will define the potential filters next.

Definition 5.32. Let $\vec{\kappa} = \langle \kappa_n : n \in \omega \rangle$ be a reasonable sequence and let $\kappa = \sup \vec{\kappa}$. Let $\vec{\zeta} = \langle \zeta_n : n \in \omega \rangle$ and $\vec{\Gamma} = \langle \Gamma_n : n \in \omega \rangle$ witness that $\vec{\kappa}$ is a very reasonable sequence. Assume that $\zeta_0 > \omega \cdot (\omega \cdot \omega)$. (One can always drop the first few terms from $\vec{\kappa}$ to obtain such a reasonable sequence.) Define $\mu^{\vec{\kappa}}$ to be a filter on κ by $X \in \mu^{\vec{\kappa}}$ if and only if there is a sequence $\langle D_n : n \in \omega \rangle$ so that for all $n < \omega$, D_n is a club subset of κ_n and for all $1 \leq n < \omega$ $D_n \subseteq \kappa_n \setminus \kappa_{n-1}$, and $\bigcup_{n \in \omega} D_n \subseteq X$.[7]

The following is the appropriate notion of instruction for partitions on ordinals below ϵ while accounting for limit behaviors.

Definition 5.33. Let $\epsilon < \omega \cdot \omega$. Let L_ϵ denote the finite set of limit ordinals below ϵ. If $F \subseteq L_\epsilon$ is a finite set. Let $\chi^\epsilon_F = \mathrm{ot}(\epsilon \setminus F)$. Let $\mathfrak{e}^\epsilon_F : \chi^\epsilon_F \to \epsilon \setminus F$ be the increasing enumeration of $\epsilon \setminus F$. An $(\epsilon, \star)$-instruction is $\mathfrak{i} = (\epsilon, F, n, \mathfrak{p}, \ell)$ such that $F \subseteq L_\epsilon$, $1 \leq n < \omega$, $\mathfrak{p} : n \to (\omega \setminus 1)$ is increasing, and $\ell : n \to \chi^\epsilon_F$ so that $\sum_{i<n} \ell(i) = \chi^\epsilon_F$. For $\epsilon < \omega \cdot \omega$, let $\mathfrak{I}^\epsilon$ denote the set of $(\epsilon, \star)$-instruction. Let $\mathfrak{I}^\star = \bigcup_{\epsilon < \omega \cdot \omega} \mathfrak{I}^\epsilon$. Note let $\mathfrak{I}^\star$ is countable.

Theorem 5.34. *Assume* AD. *If* κ *is the supremum of a reasonable sequence, then* κ *has a* $(< \omega \cdot \omega)$*-Magidor filter.*

Proof. Let $\vec{\kappa} = \langle \kappa_n : n \in \omega \rangle$ be a reasonable sequence such that $\kappa = \sup\{\kappa_n : n \in \omega\}$. Let $\langle \Gamma_n : n \in \omega \rangle$ and $\langle \zeta_n : n \in \omega \rangle$ witness that $\vec{\kappa}$ is very reasonable and one may assume that $\zeta_0 > \omega \cdot (\omega \cdot \omega)$. Let $\Phi : \mathsf{Bl}_\kappa(< \omega \cdot \omega, \kappa) \to \kappa$. Suppose $\mathfrak{i} \in \mathcal{I}^\star$. Say $\mathfrak{i}$ takes the form $\mathfrak{i} = (\epsilon, F, n, \mathfrak{p}, \ell)$. If $(f_0, ..., f_{n-1}) \in \prod_{i<n} [\kappa_{\mathfrak{p}(i)}]^{\ell(i)}$, then let $h^{\mathfrak{i}}_{f_0,...,f_{n-1}} : \epsilon \to \kappa$ be defined as follows: For any $\alpha \in \epsilon \setminus F$, let $i < n$ and $\eta < \ell(i)$ be such that $\alpha = \mathfrak{e}^\epsilon_F(\sum_{j<i} \ell(j) + \eta)$. Let $h^{\mathfrak{i}}_{f_0,...,f_{n-1}}(\alpha) = f_i(\eta)$. This defines $h^{\mathfrak{i}}_{f_0,...,f_{n-1}} \restriction (\epsilon \setminus F)$. For any $\alpha \in F$, let $h^{\mathfrak{i}}_{f_0,...,f_{n-1}}(\alpha) = \sup\{h^{\mathfrak{i}}_{f_0,...,f_{n-1}}(\beta) : \beta < \alpha \wedge \beta \in \epsilon \setminus F)\}$. Note that $h^{\mathfrak{i}}_{f_0,...,f_{n-1}}$ is continuous precisely at $\alpha \in F$. Define $P_{\mathfrak{i}} : [\kappa_0] \times \prod_{i<n} [\kappa_{\mathfrak{p}(i)}]^{\ell(i)} \to 2$ by $P_{\mathfrak{i}}(\alpha, f_0, ..., f_n) = 0$ if and only if $\Phi(h^{\mathfrak{i}}_{f_0,...,f_{n-1}}) < \alpha$. By Theorem 5.24, $\langle \kappa_0, \kappa_{\mathfrak{p}(0)}, ..., \kappa_{\mathfrak{p}(n-1)} \rangle \to_* (\kappa_0, \kappa_{\mathfrak{p}(0)}, ..., \kappa_{\mathfrak{p}(n-1)})^{1,\ell(0),...,\ell(n-1)}_2$. Thus there is a unique $u_{\mathfrak{i}} \in 2$ which is the homogeneous value for $P_{\mathfrak{i}}$. By the pointclass arguments in the proof of Theorem 5.29, there is a sequence $\langle D_n : n < \omega \rangle$ so that for all $n < \omega$, D_n is a club subset of κ_n and for all $\mathfrak{i} \in \mathfrak{I}^\star$ of the form $\mathfrak{i} = (\epsilon, n, F, \mathfrak{p}, \ell)$, $(D_0, D_{\mathfrak{p}(0)}, ..., D_{\mathfrak{p}(n-1)})$ is homogeneous for $P_{\mathfrak{i}}$.[8] Again, one can assume that for all $1 \leq n < \omega$, $D_n \subseteq \kappa_n \setminus \kappa_{n-1}$. Let $\hat{a} < \hat{b}$ be the first two elements of D_0. Let $E = \bigcup_{1 \leq n < \omega} D_n$. The claim is that $\hat{a} \notin \Phi[\mathsf{Bl}_\kappa(< \omega \times \omega, E)]$. Pick any $f \in \mathsf{Bl}(< \omega \cdot \omega, E)$. Let $\epsilon = |f|$. Let $F \subseteq L_\epsilon$ be those α such that $\sup(f \restriction \alpha) = f(\alpha)$. Let $A = \{k \in \omega : (\exists \eta < \chi^\epsilon_F)(f(\mathfrak{e}^\epsilon_F(\eta)) \in D_k)\}$. Let $n = |A|$. Let $\mathfrak{p} : n \to A$ be the increasing enumeration of A. For $i < n$, let $B_i = \{\eta < \chi^\epsilon_F : f(\mathfrak{e}_F(\eta)) \in D_{\mathfrak{p}(i)}\}$. Let $\ell(i) = \mathrm{ot}(B_i)$. Let $\mathfrak{i} = (\epsilon, n, \mathfrak{p}, \ell)$. Note that $\mathfrak{i} \in \mathfrak{I}^\star$. For each $i < n$, let $f_i : \ell(i) \to D_{\mathfrak{p}(i)}$ be defined by $f_i(\eta) = f(\mathfrak{e}(\mathsf{enum}_{B_i}(\alpha))$. Note that $f_i \in [D_{\mathfrak{p}(i)}]^{\ell(i)}_*$ (with the uniform cofinality ω given by the fact that $\ell(i)$ is countable and $\mathsf{AC}^{\mathbb{R}}_\omega$-holds). Thus $(f_0, .., f_n) \in \prod_{0 \leq i < n} [D_{\mathfrak{p}(i)}]^{\ell(i)}_*$ and $f = h^{\mathfrak{i}}_{f_0,...,f_{n-1}}$. By an argument similar to the proof of Theorem 5.29 considering the two possible value of $u_{\mathfrak{i}}$, one can show that $\Phi(f) \neq \hat{\alpha}$. Since $f \in \mathsf{Bl}_\kappa(< \omega \cdot \omega, E)$ was arbitrary, this shows that $\Phi[\mathsf{Bl}_\kappa(< \omega \cdot \omega, E)] \neq \kappa$. Let $\tilde{\kappa} = \langle \kappa_i : 1 \leq n < \omega \rangle$. Note that $E \in \mu^{\tilde{\kappa}}$. Since $\Phi : \mathsf{Bl}(< \omega \cdot \omega, \kappa) \to \kappa$ was arbitrary, it has been shown that $\mu^{\tilde{\kappa}}$ is a $(< \omega \cdot \omega)$-Magidor filter for κ. □

[7][1] demands that Magidor filter contain all tails. $\mu^{\vec{\kappa}}$ does not contain all tails but one can make a simple modification to the definition to make the filter contain all tails. One can then make an appropriate change in all the arguments below. However, this seems to be not particularly significant.

[8]Note that in the proof of Theorem 5.29 $\kappa_{\bar{m}}$ and $\kappa_{\bar{m}+1}$ were reserved and one considered instructions so that $\mathfrak{p}$ maps above $\bar{m}+1$. Here coordinate 0 plays the role of $\kappa_{\bar{m}+1}$. In Theorem 5.29, coordinate $\bar{m}$ was reserved to do the long ϵ-length selection of clubs. Here $\mathfrak{I}^\star$ is countable so one can use use $\mathsf{AC}^{\mathbb{R}}_\omega$ and the coarse Moschovakis coding lemma to make the corresponding selection.

The following answers [1] Question 3.4 (which is interpreted to mean ω-Magidor filter in light of the results of [1] Section 3 and the stronger Question 3.5).

Theorem 5.35. *Assume* AD. *The supremum of the projective ordinals* $\boldsymbol{\delta}^1_\omega$ *has a* $(<\omega\cdot\omega)$*-Magidor filter.*

Proof. Use Example 5.26 and Theorem 5.34. □

Theorem 5.36. *Assume* AD. *There are unboundedly many singular super-Magidor cardinals below* Θ *which possess an* $(<\omega\cdot\omega)$*-Magidor filter.*

Proof. Use Example 5.27 and Theorem 5.34. □

References

1. Omer Ben-Neria and Shimon Garti, *Jonsson and Magidor filters*, arXiv preprint arXiv:2503.13744 (2025).
2. William Chan, *Cardinality of the set of bounded subsets of a cardinal*, In preparation.
3. ———, *Definable combinatorics of strong partition cardinals*, In preparation.
4. William Chan, *Ordinal definability and combinatorics of equivalence relations*, J. Math. Log. **19** (2019), no. 2, 1950009, 24. MR 4014889
5. ———, *An introduction to combinatorics of determinacy*, Trends in Set Theory, Contemp. Math., vol. 752, Amer. Math. Soc., Providence, RI, 2020, pp. 21–75. MR 4132099
6. William Chan, Stephen Jackson, and Nam Trang, *Boldface* GCH *below the first uncountable limit cardinal*, In preparation.
7. William Chan, Stephen Jackson, and Nam Trang, *The primeness property for cardinalities*, In preparation.
8. ———, *A search for the cofinality of the power set of the first uncountable cardinal*, In preparation.
9. ———, *More definable combinatorics around the first and second uncountable cardinals*, J. Math. Log. **23** (2023), no. 3, Paper No. 2250029, 31. MR 4603918
10. ———, *Almost Everywhere Behavior of Functions According to Partition Measures*, Forum Math. Sigma **12** (2024), Paper No. e16. MR 4696011
11. William Chan and Connor Meehan, *Definable Combinatorics of Some Borel Equivalence Relations*, ArXiv e-prints (2017).
12. Keith J. Devlin, *Some weak versions of large cardinal axioms*, Ann. Math. Logic **5** (1972–1973), 291–325. MR 0363906
13. P. Erdős and A. Hajnal, *On a problem of B. Jónsson*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **14** (1966), 19–23. MR 0209161
14. Shimon Garti, Yair Hayut, and Saharon Shelah, *On the verge of inconsistency: Magidor cardinals and Magidor filters*, Israel J. Math. **220** (2017), no. 1, 89–102. MR 3666820
15. Leo A. Harrington and Alexander S. Kechris, *On the determinacy of games on ordinals*, Ann. Math. Logic **20** (1981), no. 2, 109–154. MR 622782
16. Greg Hjorth, *A dichotomy for the definable universe*, J. Symbolic Logic **60** (1995), no. 4, 1199–1207. MR 1367205
17. S. Jackson, R. Ketchersid, F. Schlutzenberg, and W. H. Woodin, *Determinacy and Jónsson cardinals in* $L(\mathbb{R})$, J. Symb. Log. **79** (2014), no. 4, 1184–1198. MR 3343535
18. Steve Jackson, AD *and the projective ordinals*, Cabal Seminar 81–85, Lecture Notes in Math., vol. 1333, Springer, Berlin, 1988, pp. 117–220. MR 960900
19. ———, *A computation of* δ^1_5, Mem. Amer. Math. Soc. **140** (1999), no. 670, viii+94. MR 1606035
20. ———, *Structural consequences of* AD, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 1753–1876. MR 2768700
21. Alexander S. Kechris, AD *and projective ordinals*, Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77), Lecture Notes in Math., vol. 689, Springer, Berlin, 1978, pp. 91–132. MR 526915
22. Alexander S. Kechris, Eugene M. Kleinberg, Yiannis N. Moschovakis, and W. Hugh Woodin, *The axiom of determinacy, strong partition properties and nonsingular measures*, Cabal Seminar 77–79 (Proc. Caltech-UCLA Logic Sem., 1977–79), Lecture Notes in Math., vol. 839, Springer, Berlin-New York, 1981, pp. 75–99. MR 611168
23. Alexander S. Kechris and W. Hugh Woodin, *Equivalence of partition properties and determinacy*, Proc. Nat. Acad. Sci. U.S.A. **80** (1983), no. 6, i, 1783–1786. MR 699440
24. Eugene M. Kleinberg, *Infinitary combinatorics and the axiom of determinateness*, Lecture Notes in Mathematics, Vol. 612, Springer-Verlag, Berlin-New York, 1977. MR 0479903
25. A. R. D. Mathias, *Happy families*, Ann. Math. Logic **12** (1977), no. 1, 59–111. MR 491197
26. William Mitchell, *How weak is a closed unbounded ultrafilter?*, Logic Colloquium '80 (Prague, 1980), Studies in Logic and the Foundations of Mathematics, vol. 108, North-Holland, Amsterdam-New York, 1982, pp. 209–230. MR 673794
27. Yiannis N. Moschovakis, *Descriptive set theory*, second ed., Mathematical Surveys and Monographs, vol. 155, American Mathematical Society, Providence, RI, 2009. MR 2526093
28. John R. Steel, *An outline of inner model theory*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 1595–1684. MR 2768698
29. ———, *Ordinal definability in models of determinacy. Introduction to Part V*, Ordinal definability and recursion theory: The Cabal Seminar. Vol. III, Lect. Notes Log., vol. 43, Assoc. Symbol. Logic, Ithaca, NY, 2016, pp. 3–48. MR 3469165

Institute for Discrete Mathematics and Geometry, Vienna University of Technology, Vienna, Austria
Email address: `William.Chan@tuwien.ac.at`