CARDINALITY OF WELLORDERED DISJOINT UNIONS OF QUOTIENTS OF SMOOTH EQUIVALENCE RELATIONS

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Abstract. Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let $\approx$ denote the relation of being in bijection. Let $\kappa \in \mathrm{ON}$ and $\langle E_\alpha : \alpha < \kappa \rangle$ be such that for all $\alpha < \kappa$, E_α is an equivalence relation on $\mathbb{R}$ with all classes countable and $\mathbb{R}/E_\alpha \approx \mathbb{R}$. Then the disjoint union $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ is in bijection with $\mathbb{R} \times \kappa$ and $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ has the Jónsson property.

Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. A set $X \subseteq [\omega_1]^{<\omega_1}$ has a sequence $\langle E_\alpha : \alpha < \omega_1 \rangle$ of equivalence relations on $\mathbb{R}$ such that $\mathbb{R}/E_\alpha \approx \mathbb{R}$ and $X \approx \bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ if and only if $\mathbb{R} \sqcup \omega_1$ injects into X.

Assume AD. Suppose $R \subseteq [\omega_1]^\omega \times \mathbb{R}$ is a relation such that for all $f \in [\omega_1]^\omega$, $R_f = \{x \in \mathbb{R} : R(f,x)\}$ is nonempty and countable. Then there is an uncountable $X \subseteq \omega_1$ and function $\Phi : [X]^\omega \to \mathbb{R}$ which uniformizes R on $[X]^\omega$: that is, for all $f \in [X]^\omega$, $R(f, \Phi(f))$.

Under AD, if κ is an ordinal and $\langle E_\alpha : \alpha < \kappa \rangle$ is a sequence of equivalence relations on $\mathbb{R}$ with all classes countable, then $[\omega_1]^\omega$ does not inject into $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$.

1. Introduction

The original motivation for this work comes from the study of a simple combinatorial property of sets using only definable methods. Let X be a set. For each $n \in \omega$, let $[X]^n_= = \{f \in {}^nX : (\forall i,j \in n)(i \neq j \Rightarrow f(i) \neq f(j))\}$. Let $[X]^{<\omega}_= = \bigcup_{n\in\omega}[X]^n_=$. A set X has the Jónsson property if and only if for every function $F : [X]^{<\omega}_= \to X$, there is some Y such that $Y \approx X$ (where $\approx$ denotes the bijection relation) so that $F[[Y]^{<\omega}_=] \neq X$. That is, F can be made to miss an element of X when restricted to the collection of finite injective tuples through some subset Y of X of the same cardinality as X.

Under the axiom of choice, if there is a set with the Jónsson property, then large cardinal principles such as $0^\sharp$ hold. Using a measurable cardinal, one can construct models of ZFC in which $2^{\aleph_0}$ is Jónsson and is not Jónsson. Hence assuming the consistency of some large cardinals, the Jónsson property of $2^{\aleph_0}$ is independent of ZFC. Using AC, the sets $\mathbb{R}$, $\mathbb{R} \sqcup \omega_1$, $\mathbb{R} \times \omega_1$, and $\mathbb{R}/E_0$ are all in bijection. (E_0 is the equivalence relation defined on $\mathbb{R} = {}^\omega 2$ by $x \ E_0 \ y$ if and only if $(\exists m)(\forall n \geq m)(x(n) = y(n))$.)

From a definability perspective, the sets $\mathbb{R}$, $\mathbb{R} \times \omega_1$, $\mathbb{R} \sqcup \omega_1$, and $\mathbb{R}/E_0$ do not have definable bijections without invoking definable wellorderings of the reals which can exist in canonical inner models like L but in general can not exist if the universe satisfies more regularity properties for sets of reals. For example, there are no injections of $\mathbb{R}/E_0$ into $\mathbb{R}$ that is induced by a $\mathbf{\Delta}^1_1$ reduction $\Phi : \mathbb{R} \to \mathbb{R}$ of the $=$ relation into the E_0 equivalence relation. Such results for the low projective pointclasses can be extended to all sets assuming the axiom of determinacy, AD. Methods that hold in a determinacy setting are often interpreted to be definable methods. Moreover, the extension of AD called AD^+ captures this definability setting even better since AD^+ implies all sets of reals have a very absolute definition known as the ∞-Borel code.

Kleinberg [14] showed that $\aleph_n$ has the Jónsson property for all $n \in \omega$ under AD. [10] showed that under $\mathsf{ZF} + \mathsf{AD} + V = L(\mathbb{R})$, every cardinal below Θ has the Jónsson property. (Woodin showed that AD^+ alone can prove this result.)

Holshouser and Jackson began the study of the Jónsson property for nonwellorderable sets under AD such as $\mathbb{R}$. In [9], they showed that $\mathbb{R}$ and $\mathbb{R} \sqcup \omega_1$ have the Jónsson property. They also showed, using the fact that all $\kappa < \Theta$ have the Jónsson property, that $\mathbb{R} \times \kappa$ is Jónsson. [5] showed that $\mathbb{R}/E_0$ does not have the Jónsson property.

Holshouser and Jackson then asked if the Jónsson property of sets is preserved under the disjoint union operation: If $\kappa \in \mathrm{ON}$ and $\langle X_\alpha : \alpha < \kappa \rangle$ is a sequence of sets with the Jónsson property, then does the disjoint

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union $\bigsqcup_{\alpha<\kappa} X_\alpha$ have the Jónsson property? (Here $\bigsqcup$ will always refer to a formal disjoint union in contrast to the ordinary union $\bigcup$.) More specifically, does a disjoint union of sets, each in bijection with $\mathbb{R}$, have the Jónsson property? The determinacy axioms are particular helpful for studying sets which are surjective images of $\mathbb{R}$. Hence, a natural question would be if $\langle E_\alpha : \alpha < \kappa \rangle$ is a sequence of equivalence relations on $\mathbb{R}$ such that for each α, $\mathbb{R}/E_\alpha$ is in bijection with $\mathbb{R}$, then does $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ have the Jónsson property? An equivalence relation E on $\mathbb{R}$ is called smooth if and only if $\mathbb{R}/E$ is in bijection with $\mathbb{R}$. (Note that this term is used differently than the ordinary Borel theory which would define E to be smooth if $\mathbb{R}/E$ injects into $\mathbb{R}$. This will be referred as being weakly smooth.)

$\boldsymbol{\Delta}^1_1$ equivalence relations with all classes countable are very important objects of study in classical invariant descriptive set theory. One key property that make this investigation quite robust is the Lusin-Novikov countable section uniformization which can be used to show the Feldman-Moore theorem. [2, Theorem 4.13] showed that if $\langle E_\alpha : \alpha < \kappa \rangle$ is a sequence of equivalence relations with all classes countable (not necessarily smooth) and $F : [\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha]^{\leq\omega}_= \to \bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$, then there is a perfect tree p on 2 so that

$$F\left[[\bigsqcup_{\alpha<\kappa}[p]/E_\alpha]^{\leq\omega}_=\right] \neq \bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha.$$

(Here $\mathbb{R}$ refers to the Cantor space ${}^\omega 2$.) This "psuedo-Jónsson property" would imply the true Jónsson property if $\bigsqcup_{\alpha<\kappa}[p]/E_\alpha$ is in bijection with $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$. In general for nonsmooth equivalence relations, this can not be true since, for example, E_0 is an equivalence relation with all classes countable and $\mathbb{R}/E_0$ is not Jónsson ([5]). When each E_α is the identity relation $=$, then one can demonstrate these two sets are in bijection. Using this, [2, Theorem 4.15] shows that $\mathbb{R} \times \kappa$ has the Jónsson property, where κ is any ordinal, using only classical descriptive set theoretic methods and does not rely on any combinatorial properties of the ordinal κ.

[2, Question 4.14] asked if $\langle E_\alpha : \alpha < \kappa \rangle$ consists entirely of smooth equivalence relations on $\mathbb{R}$ with all classes countable and p is any perfect tree on 2, then is $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ and $\bigsqcup_{\alpha<\kappa}[p]/E_\alpha$ in bijection? Do such disjoint unions have the Jónsson property? The most natural attempt to show that wellordered disjoint unions of quotients of smooth equivalence relations with all classes countable is Jónsson would be to show it is, in fact, in bijection with $\mathbb{R} \times \kappa$, which has already been shown to possess the Jónsson property.

The computation of the cardinality of wellordered disjoint unions of quotients of smooth equivalence relations on $\mathbb{R}$ with all classes countable is the main result of the paper. Under AD^+, any equivalence relation on $\mathbb{R}$ has an ∞-Borel code. However, for the purpose of this paper, given a sequence of equivalence relations $\langle E_\alpha : \alpha < \kappa \rangle$ on $\mathbb{R}$, one will need to uniformly obtain ∞-Borel codes for each E_α. It is unclear if this is possible under AD^+ alone. To obtain this uniformity of ∞-Borel codes, one will need to work with natural models of AD^+, i.e. models satisfying $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Also the assumption that each equivalence relation has all classes countable is very important. Analogous to the role of the Lusin-Novikov countable section uniformization in the classical setting, Woodin's countable section uniformization under AD^+ will play a crucial role.

There are some things that can be said about $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ when $\langle E_\alpha : \alpha < \kappa \rangle$ is a sequence of smooth equivalence relations (with possibly uncountable classes). It is immediate that $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ will contain a copy of $\omega_1 \sqcup \mathbb{R}$. Hence $\omega_1 \sqcup \mathbb{R}$ is a lower bound on the cardinality of disjoint unions of quotients of smooth equivalence relations. An example of Holshouser and Jackson (Fact 4.2) produces a sequence $\langle F_\alpha : \alpha < \omega_1 \rangle$ of smooth equivalence relations such that $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/F_\alpha$ is in bijection with $\omega_1 \sqcup \mathbb{R}$. So this lower bound is obtainable. In fact, in natural models of AD^+, these two properties are equivalent.

Theorem 6.1 *Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Suppose $X \subseteq [\omega_1]^{<\omega_1}$ and $\mathbb{R} \sqcup \omega_1$ injects into X. Then there exists a sequence $\langle E_\alpha : \alpha < \omega_1 \rangle$ of smooth equivalence relations on $\mathbb{R}$ so that X is in bijection with $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$. Therefore, $X \subseteq [\omega_1]^{<\omega_1}$ has a sequence $\langle E_\alpha : \alpha < \omega_1 \rangle$ of smooth equivalence relations such that $X \approx \bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ if and only if $\mathbb{R} \sqcup \omega_1$ injects into X.*

$\mathbb{R} \times \kappa$ is a disjoint union coming from $\langle E_\alpha : \alpha < \kappa \rangle$ where each E_α is the $=$ relation, which is an equivalence relation with all classes countable. However, the proof of Theorem 6.1 uses equivalence relations with uncountable classes. Intuitively, it seems that $\mathbb{R} \sqcup \omega_1$, $[\omega_1]^\omega$, and $[\omega_1]^{<\omega_1}$ should not be obtainable using equivalence relations with countable classes. This motivates the conjecture that if $\langle E_\alpha : \alpha < \kappa\rangle$ is a sequence of smooth equivalence relations with all classes countable then the cardinality of $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ is $\mathbb{R} \times \kappa$.

Woodin [16] showed that there is elaborate structure of cardinals below $[\omega_1]^{<\omega_1}$. Theorem 6.1 shows that every cardinal above $\omega_1 \sqcup \mathbb{R}$ and below $[\omega_1]^{<\omega_1}$ is a disjoint union of quotients of smooth equivalence relations. A success in Holshouser and Jackson's original goal of establishing the closure of the Jónsson property under disjoint unions would yield the Jónsson property for many cardinals below $[\omega_1]^{<\omega_1}$. On the other hand, it difficult to see how one could establish the Jónsson property for every set that appears in this rich cardinal structure using solely the manifestation of these sets as a disjoint union of quotients of smooth equivalence relations.

The main tool for computing the cardinality of wellordered disjoint union of quotients of smooth equivalence relations with all countable classes is the Woodin's perfect set dichotomy which generalizes the Silver's dichotomy for $\boldsymbol{\Pi}^1_1$ equivalence relations. This perfect set dichotomy states that under AD^+, for any equivalence relation E on $\mathbb{R}$, either (i) $\mathbb{R}/E$ is wellorderable or (2) $\mathbb{R}$ injects into $\mathbb{R}/E$. Section 3 is dedicated to proving this result. A detailed analysis of the proof of this result will be needed for this paper. The proof for case (i) yields a uniform procedure which takes an ∞-Borel code for E and gives a wellordering of $\mathbb{R}/E$. Moreover, it shows that in this case, $\mathbb{R}/E \subseteq \mathrm{OD}_S$, where S is the ∞-Borel code for E. Under $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$, this will give a more general countable section uniformization (Fact 3.4). It will be seen that in the proof of case (ii), the injection of $\mathbb{R}$ will depend on certain parameters. If these parameters could be found uniformly for each equivalence relation from the sequence $\langle E_\alpha : \alpha < \kappa\rangle$, then the proof in case (ii) can uniformly produce injections of $\mathbb{R}$ into each $\mathbb{R}/E_\alpha$. Together, one would get an injection of $\mathbb{R}\times\kappa$ into $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$. In general this can not be done; for instance using the example from Fact 4.2. However, this can be done when all the equivalence relations are smooth and have all classes countable. This shows the following:

Theorem 4.5 *Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let $\kappa \in \mathrm{ON}$ and $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of smooth equivalence relations on $\mathbb{R}$ with all classes countable. Then $\mathbb{R} \times \kappa$ injects into $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$.*

This shows that $\mathbb{R} \times \kappa$ is a lower bound for the cardinality of $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$.

Section 5 will provide the proof of the relevant half of Hjorth's generalized E_0-dichotomy. Again, what is important from this result is the observation that if $\mathbb{R}/E_0$ does not inject into $\mathbb{R}/E$, then there is a wellordered separating family for E defined uniformly from the ∞-Borel code for E. If $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$ holds, then one has a uniform sequence of ∞-Borel codes for the sequence of equivalence relations $\langle E_\alpha : \alpha < \kappa\rangle$, where each E_α is smooth. Using the argument of Hjorth's dichotomy, one obtains uniformly a separating family for each E_α. This gives a sequence of injections of each $\mathbb{R}/E_\alpha$ into $\mathscr{P}(\delta)$ where δ is a possibly very large ordinal. If $\langle E_\alpha : \alpha < \kappa\rangle$ consists entirely of equivalence relations with all classes countable, then the generalized countable section uniformization can be used to uniformly obtain a selector for each $\mathbb{R}/E_\alpha$. This gives the desired injection into $\mathbb{R} \times \kappa$:

Theorem 5.4 *Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let κ be an ordinal and $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of smooth equivalence relations on $\mathbb{R}$ with all classes countable. Then there is an injection of $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ into $\mathbb{R} \times \kappa$.*

Theorem 5.8 *Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let $\kappa \in \mathrm{ON}$ and $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of smooth equivalence relations on $\mathbb{R}$ with all classes countable. Then $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha \approx \mathbb{R} \times \kappa$ and hence $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ has the Jónsson property.*

$[\omega_1]^\omega$ is the collection of increasing functions $f : \omega \to \omega_1$. [2] proved under AD^+ that $[\omega_1]^\omega$ does not inject into $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ when $\langle E_\alpha : \alpha < \kappa\rangle$ is a sequence of equivalence relations on $\mathbb{R}$ with all classes countable. (Of course, Theorem 5.8 asserts that such a disjoint union is in bijection with $\mathbb{R} \times \kappa$ under the assumption $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$.) The key ingredient is the ability to uniformize relations $R \subseteq [\omega_1]^\omega \times \mathbb{R}$ such that for all $f \in [\omega_1]^\omega$, $R_f = \{x \in \mathbb{R} : R(f, x)\}$ is nonempty and countable. Such a full uniformization is provable under AD^+. A careful inspection of the argument will show that one only needs to uniformize this relation on some $Z \subseteq [\omega_1]^\omega$ such that $Z \approx [\omega_1]^\omega$ to show that no such injection exists. [2, Question 4.21] asks whether such an almost full uniformization is provable in AD.

It should be noted that if one drops the demand that R_f be countable, then one cannot prove this in general. (See the discussion in Section 7.) The final section will show that such an almost full countable section uniformization for relations on $[\omega_1]^\omega \times \mathbb{R}$ is provable in AD:

Theorem 7.4 ($\mathsf{ZF} + \mathsf{AD}$) *Let $R \subseteq [\omega_1]^\omega \times \mathbb{R}$ be such that for all $f \in [\omega_1]^\omega$, R_f is nonempty and countable. Then there exists some uncountable $X \subseteq \omega_1$ and function Ψ which uniformizes R on $[X]^\omega$: For $f \in [X]^\omega$, $R(f, \Psi(f))$.*

Corollary 7.5 ($\mathsf{ZF} + \mathsf{AD}$) *Let $\langle E_\alpha : \alpha < \kappa \rangle$ be a sequence of equivalence relations on $\mathbb{R}$ with all classes countable, then $[\omega_1]^\omega$ does not inject into $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$.*

These methods also show that for an arbitrary function $\Phi : [\omega_1]^\omega \to \mathbb{R}$, one can find some uncountable $X \subseteq \omega_1$ and some reals σ and w so that $\Phi(f) \in L[\sigma, w, f]$, for all $f \in [X]^\omega$:

Theorem 7.6 ($\mathsf{ZF} + \mathsf{AD}$). *Let $\Phi : [\omega_1]^\omega \to \mathbb{R}$ be a function. Then there is an uncountable $X \subseteq \omega_1$, reals $\sigma, w \in \mathbb{R}$, and a formula ϕ so that for all $f \in [X]^\omega$, $\Phi(f) \in L[\sigma, w, f]$ and for all $z \in \mathbb{R}$, $z = \Phi(f)$ if and only if $L[\sigma, w, f, z] \models \phi(\sigma, w, f, z)$.*

2. Basics

See [3, Section 7 and 8] for more information on AD^+, ∞-Borel codes, and Vopěnka forcing.

Definition 2.1. Let S be a set of ordinals and φ be a formula in the language of set theory. (S, φ) is called an ∞-Borel code. For each $n \in \omega$, $\mathfrak{B}^n_{(S,\varphi)} = \{x \in \mathbb{R}^n : L[S, x] \models \varphi(S, x)\}$ is the subset of $\mathbb{R}^n$ coded by (S, φ).

A set $A \subseteq \mathbb{R}^n$ is ∞-Borel if and only if $A = \mathfrak{B}^n_{(S,\varphi)}$ for some ∞-Borel code (S, φ). (S, φ) is called an ∞-Borel code for A.

Definition 2.2. [17, Section 9.1] AD^+ consists of the following statements:
(1) $\mathsf{DC}_\mathbb{R}$.
(2) Every $A \subseteq \mathbb{R}$ has an ∞-Borel code.
(3) For every $\lambda < \Theta$, $A \subseteq \mathbb{R}$, and continuous function $\pi : {}^\omega\lambda \to \mathbb{R}$, $\pi^{-1}[A]$ is determined.

Models of the theory $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$ are known as natural models of AD^+. Natural models of AD^+ have several desirable properties. Woodin has shown that these models take one of two forms.

Fact 2.3. *(Woodin,* [1, Section 3.1]*) Suppose* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. *If* $\mathsf{AD}_\mathbb{R}$ *fails, then there is a set of ordinals J so that $V = L(J, \mathbb{R})$.*

If $V = L(J, \mathbb{R})$ for some set of ordinals J, then it is clear that every set is ordinal definable from J and a real. That is, every set is ordinal definable from some set of ordinals. This is also true in natural models of AD^+ in which $\mathsf{AD}_\mathbb{R}$ holds:

Fact 2.4. *(Woodin,* [1, Section 3.3]*) Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{AD}_\mathbb{R} + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. *Then every set is* OD *from some element of $\bigcup_{\lambda<\Theta} \mathscr{P}_{\omega_1}(\lambda)$.*

Fact 2.5. *(Woodin,* [1, Theorem 3.4]*) Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. *Let S be a set of ordinals. If $A \subseteq \mathbb{R}$ is OD_S, then A has an OD_S ∞-Borel code.*

Definition 2.6. If $x, y \in \mathbb{R}$, $x \leq_T y$ indicates x is Turing reducible to y. $x \equiv_T y$ denotes $x \leq_T y$ and $y \leq_T x$. If $x \in \mathbb{R}$, then $[x]_T$ denotes the equivalence class of x under $\equiv_T$. Let $\mathcal{D}$ denote the collection of $\equiv_T$ equivalence classes.

For $X, Y \in \mathcal{D}$, define $X \leq_T Y$ if and only if for all $x \in X$ and $y \in Y$, $x \leq_T y$. $U \subseteq \mathcal{D}$ contains a Turing cone with base $X \in \mathcal{D}$ if and only if for all $Y \in \mathcal{D}$, $X \leq_T Y$ implies that $Y \in U$.

Let $\mu \subseteq \mathscr{P}(\mathcal{D})$ be defined by $U \in \mu$ if and only if μ contains a Turing cone. Under AD, μ is a countably complete ultrafilter on $\mathcal{D}$. μ is called the Martin's measure.

Let $f, g : \mathcal{D} \to \omega_1$. Define $\sim_\mu$ by $f \sim_\mu g$ if and only if $\{X \in \mathcal{D} : f(X) = g(X)\} \in \mu$. $\sim_\mu$ is an equivalence relation on functions from $\mathcal{D}$ into ω_1. Let $[f]_{\sim_\mu}$ denote the equivalence class of f under $\sim_\mu$. Define $f <_\mu g$ if and only if $\{X \in \mathcal{D} : f(X) < g(X)\} \in \mu$. $\prod_{X \in \mathcal{D}} \omega_1/\mu$ is the collection of $\sim_\mu$-equivalence classes. The ultrapower ordering on $\prod_{X\in\mathcal{D}} \omega_1/\mu$ is defined by $F < G$ if and only if for any (or equivalently, there exist) representatives $f \in F$ and $g \in G$ so that $f <_\mu g$.

Fact 2.7. *(Woodin,* [1, Section 2.2]*)* $\mathsf{ZF} + \mathsf{AD}^+$ *proves* $\prod_{X\in\mathcal{D}} \mathrm{ON}/\mu$ *is wellfounded.*

Definition 2.8. For $n \in \omega$ and a set of ordinals S, let ${}_n\mathbb{O}_S$ denote the collection of nonempty OD_S subsets of $\mathbb{R}^n$. (${}_1\mathbb{O}_S$ will be denoted by $\mathbb{O}_S$.)

For $p, q \in {}_n\mathbb{O}_S$, let $p \leq_{{}_n\mathbb{O}_S} q$ if and only if $p \subseteq q$. Let $1_{{}_n\mathbb{O}_S} = \mathbb{R}^n$. $({}_n\mathbb{O}_S, \leq_{{}_n\mathbb{O}_S}, 1_{{}_n\mathbb{O}_S})$ is the n-dimensional S-Vopěnka's forcing.

By using an S-definable bijection of the collection of OD_S sets with ON, ${}_n\mathbb{O}_S$ can be considered as a set of ordinals. In this way, the forcing ${}_n\mathbb{O}_S$ is a forcing belonging to HOD_S.

For each $m \in \omega$, let $b_m \in \mathbb{O}_S$ be defined as $\{x \in \mathbb{R} : m \in x\}$. Let $\tau = \{(\check{m}, b_m) : m \in \omega\}$. τ is an $\mathbb{O}_S$-name for a real. (Similar definition exists for all ${}_n\mathbb{O}_S$.)

Fact 2.9. *(Vopěnka's theorem) Let M be a transitive inner model of* ZF*. Let $S \in M$ be a set of ordinals.*

For all $x \in \mathbb{R}^M$, there is an $\mathbb{O}_S^M$-generic filter over HOD_S^M, $G_x \in M$, so that $\tau[G_x] = x$.

Suppose K is an OD_S^M-set of ordinals and φ is a formula. Let N be some transitive inner model with $\mathrm{HOD}_S^M \subseteq N$. Suppose $p = \{x \in \mathbb{R}^M : L[K, x] \models \varphi(K, x)\}$ is a condition of $\mathbb{O}_S^M$ (i.e. is nonempty). Then $N \models p \Vdash_{\mathbb{O}_S^M} L[\check{K}, \tau] \models \varphi(\check{K}, \tau)$.

Proof. A proof of this can be found among [3, Fact 7.6 and Theorem 2.4], [11, Theorem 15.46], [8, Theorem 2.4], or [4, Fact 2.7]. ☐

Fact 2.10. *Let M be an inner model of* ZF*. Let $S \in M$ be a set of ordinals. Let N be an inner model of* ZF *such that $N \supseteq \mathrm{HOD}_S^M$. Let $n \geq 1$ be a natural number. Suppose $(g_0, ..., g_{n-1})$ is an ${}_n\mathbb{O}_S^M$-generic real over N. Then each g_0, ..., g_{n-1} is $\mathbb{O}_S^M$-generic over N.*

Proof. This is straightforward. See [3, Fact 8.2] and [4, Fact 2.8] for details. ☐

Definition 2.11. Suppose X and Y are sets. $X \approx Y$ indicates that there is a bijection between X to Y.

Definition 2.12. Let E be an equivalence relation on $\mathbb{R}$. E is smooth if and only if $\mathbb{R}/E \approx \mathbb{R}$. E is weakly smooth if and only if $\mathbb{R}/E$ injects into $\mathbb{R}$.

Under AD by the perfect set property, E is weakly smooth if and only if $\mathbb{R}/E$ is either countable or in bijection with $\mathbb{R}$.

(From the theory of Borel equivalence relations, "smooth" would usually refer to what is called weakly smooth above. In this article, one will reserve the term "smooth" for equivalence relations on $\mathbb{R}$ whose quotients are in bijection with $\mathbb{R}$.)

3. Perfect Set Dichotomy and Wellorderable Section Uniformization

The Silver's dichotomy ([15] and [7]) states the every $\mathbf{\Pi}_1^1$ equivalence relation E on $\mathbb{R}$ has countably many equivalence classes ($\mathbb{R}/E$ is hence wellorderable) or E has a perfect set of pairwise E-inequivalent elements ($\mathbb{R}$ injects into $\mathbb{R}/E$). Woodin's perfect set dichotomy states: under AD^+, for every equivalence relation E on $\mathbb{R}$, either $\mathbb{R}/E$ is wellorderable or $\mathbb{R}$ injects into $\mathbb{R}/E$. As a consequence, every set which is a surjective image of $\mathbb{R}$, either the set contains a copy of $\mathbb{R}$ or is wellorderable. (In natural models of AD^+, [1] more generally showed that every set either has a copy of $\mathbb{R}$ or is wellorderable.) Moreover, a consequence of the proof shows roughly that every wellorderable OD set contains only OD elements. This immediately yields wellorderable section uniformization for rather general relations (on sets) with each section wellorderable. This generalizes Woodin's countable section uniformization for relations on $\mathbb{R} \times \mathbb{R}$.

This section will provide a proof of Woodin's perfect set dichotomy and the wellorderable section uniformization. An observation about the uniformity of the proof will be necessary for studying disjoint unions of quotients of smooth equivalence relations with all classes countable. Later countable section uniformization (for relations on $\mathscr{P}(\delta) \times \mathbb{R}$ where $\delta \in \mathrm{ON}$) will also be needed. All results found in this section are due to Woodin or the authors of [1].

Definition 3.1. Let E be an equivalence relation on $\mathbb{R}$. An E-component is a nonempty set A so that for all $x, y \in A$, $x \; E \; y$. (An E-component is just a nonempty subset of an E-class.)

Theorem 3.2. *(Woodin) Assume $\mathsf{ZF} + \mathsf{AD}^+$. Let E be an equivalence relation on $\mathbb{R}$. Then either (i) or (ii) below holds.*
(i) $\mathbb{R}/E$ is wellorderable.
(ii) $\mathbb{R}$ injects into $\mathbb{R}/E$.

Proof. Silver [15] proved the $\mathbf{\Pi}^1_1$ version of this result. Harrington [7] produced a proof of this result using the Gandy-Harrington forcing of nonempty Σ^1_1 subsets of $\mathbb{R}$. This proof will replace Gandy-Harrington forcing with the Vopěnka forcing of nonempty OD subsets of $\mathbb{R}$. Arguments from [12, Chapter 10] and [8] will be adapted.

Using AD^+, let (S, φ) be an ∞-Borel code for E. In all models considered in this proof, E will always be understood to be the set defined by (S, φ).

Note that if $x \equiv_T y$, then $L[S, x] = L[S, y]$, $\text{HOD}_S^{L[S,x]} = \text{HOD}_S^{L[S,y]}$, and the canonical global wellordering of $\text{HOD}_S^{L[S,x]}$ and $\text{HOD}_S^{L[S,y]}$ are the same. Therefore, for each $X \in \mathcal{D}$, let $L[S, X] = L[S, x]$ and $\text{HOD}_S^{L[S,X]} = \text{HOD}_S^{L[S,x]}$ for any $x \in X$.

(Case I) For all $X \in \mathcal{D}$, for all $a \in \mathbb{R}^{L[S,X]}$, there is an $\text{OD}_S^{L[S,X]}$ E-component $A \in \mathbb{O}_S^{L[S,X]}$ containing a.

For each $F \in \prod_{X \in \mathcal{D}} \omega_1/\mu$, define A_F as follows: Let $f : \mathcal{D} \to \omega_1$ be such that $f \in F$, i.e. is a representative for F under the relation $\sim_\mu$. For $a \in \mathbb{R}$, $a \in A_F$ if and only if on a Turing cone of $X \in \mathcal{D}$, a belongs to the $f(X)^{\text{th}}$ E-component in $\mathbb{O}_S^{L[S,X]}$ according to the canonical global wellordering of $\text{HOD}_S^{L[S,X]}$. (This $f(X)^{\text{th}}$ set is said to be $\emptyset$ if there is no $f(X)^{\text{th}}$ E-component in $\mathbb{O}_S^{L[S,X]}$.) A_F is well defined in the sense that it is independent of the chosen representative.

A_F is an E-component: Suppose $a, b \in A_F$ and $\neg(a \; E \; b)$. Pick $f \in F$. Since $a, b \in A_F$, there is some $Z \geq_T [a \oplus b]_T$ so that for all $X \geq_T Z$, a and b belong to the $f(X)^{\text{th}}$ E-component in $\mathbb{O}_S^{L[S,X]}$. This would imply that $a \; E \; b$ holds in $L[S, X]$. Since E is defined by the ∞-Borel code (S, φ),

$$L[S, X] \models L[S, a, b] \models \varphi(S, a, b).$$

Then $V \models L[S, a, b] \models \varphi(S, a, b)$. Hence $V \models a \; E \; b$. Contradiction.

For all $a \in \mathbb{R}$, a belongs to some A_F: Let $f : \mathcal{D} \to \omega_1$ be defined as follows: For all $X \geq_T [a]_T$, let $f(X)$ be the least α so that a belongs to the α^{th} E-component of $\mathbb{O}_S^{L[S,X]}$. Such an α exists by the Case I assumption. If $F = [f]_{\sim_\mu}$, then $a \in A_F$.

By Fact 2.7, $\prod_{X \in \mathcal{D}} \omega_1/\mu$ is wellfounded. Hence $\langle A_F : F \in \prod_{X \in \mathcal{D}} \omega_1/\mu \rangle$ is a wellordered sequence of E-components so that every $a \in \mathbb{R}$ belongs to some A_F.

Let B_F be the E-closure of A_F. $\langle B_F : F \in \prod_{X \in \mathcal{D}} \omega_1/\mu \rangle$ is a surjection of a wellordered set onto the collection of E-classes. By removing duplicates by canonically choosing the least index for each E-class, one obtains a bijection $\langle C_\alpha : \alpha < \delta \rangle$ of some $\delta \in \text{ON}$ onto the collection of E-classes. Note for later purposes that $\langle C_\alpha : \alpha < \delta \rangle$ is obtained uniformly from the ∞-Borel code (S, φ). Moreover, each C_α is OD_S.

(Case II) There exists an $X \in \mathcal{D}$ and $a \in \mathbb{R}^{L[S,X]}$ which does not belong to any $\text{OD}_S^{L[S,X]}$ E-component of $\mathbb{O}_S^{L[S,X]}$.

Fix such an $X \in \mathcal{D}$. Let u be defined by

$$u = \{x \in \mathbb{R}^{L[S,X]} : x \text{ does not belong to any } \text{OD}_S^{L[S,X]} \; E\text{-component}\}.$$

The set u is nonempty by the case II assumption and is $\text{OD}_S^{L[S,X]}$. Hence $u \in \mathbb{O}_S^{L[S,X]}$.

Claim 1: Let τ_L and τ_R be the canonical $\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}$-names for the evaluation of the $\mathbb{O}_S^{L[S,X]}$-name τ according the left and right $\mathbb{O}_S^{L[S,X]}$-generic coming from an $\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}$-generic, respectively. Then

$$\text{HOD}_S^{L[S,X]} \models (u, u) \Vdash_{\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}} \neg(\tau_L \; E \; \tau_R).$$

Proof. Suppose not. Then there is some $(v, w) \leq_{\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}} (u, u)$ so that

$$\text{HOD}_S^{L[S,X]} \models (v, w) \Vdash_{\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}} \tau_L \; E \; \tau_R.$$

Subclaim 1.1: If G_0 and G_1 are $\mathbb{O}_S^{L[S,X]}$-generic over $\mathrm{HOD}_S^{L[S,X]}$ containing v (but not necessarily mutually generic), then $\tau[G_0]\ E\ \tau[G_1]$.

Proof. Since AD implies ω_1^V is inaccessible in every inner model of ZFC, $\mathbb{O}_S^{L[S,X]}$ and its power set in $\mathrm{HOD}_S^{L[S,X]}$ are countable in V. Hence, for any G_0 and G_1 which are $\mathbb{O}_S^{L[S,X]}$-generic filters over $\mathrm{HOD}_S^{L[S,X]}$ containing the condition v, there exists an H which is $\mathbb{O}_S^{L[S,X]}$-generic over $\mathrm{HOD}_S^{L[S,X]}[G_0]$ and $\mathrm{HOD}_S^{L[S,X]}[G_1]$ containing the condition w. Then by the forcing theorem, $\mathrm{HOD}_S^{L[S,X]}[G_0][H] \models \tau[G_0]\ E\ \tau[H]$ and $\mathrm{HOD}_S^{L[S,X]}[G_1][H] \models \tau[G_1]\ E\ \tau[H]$. Since E is defined by the ∞-Borel code (S,φ), this means

$$\mathrm{HOD}_S^{L[S,X]}[G_0][H] \models L[S,\tau[G_0],\tau[H]] \models \varphi(S,\tau[G_0],\tau[H])$$

and

$$\mathrm{HOD}_S^{L[S,X]}[G_1][H] \models L[S,\tau[G_1],\tau[H]] \models \varphi(S,\tau[G_1],\tau[H]).$$

But then

$$V \models L[S,\tau[G_0],\tau[H]] \models \varphi(S,\tau[G_0],\tau[H])$$

and

$$V \models L[S,\tau[G_1],\tau[H]] \models \varphi(S,\tau[G_1],\tau[H])$$

This means $V \models \tau[G_0]\ E\ \tau[H]$ and $V \models \tau[G_1]\ E\ \tau[H]$. Since E is an equivalence relation, $V \models \tau[G_0]\ E\ \tau[G_1]$ which completes the proof of subclaim 1.1. □

Note that there exist $a, b \in v$ so that $\neg(a\ E\ b)$ since otherwise v would be an E-component in $\mathbb{O}_S^{L[S,X]}$. Since $v \subseteq u$, this contradicts the definition of u. Let $p = (v \times v) \setminus E$. p is a nonempty $\mathrm{OD}_S^{L[S,X]}$ subset of $\mathbb{R}^2$. Hence $p \in {}_2\mathbb{O}_S^{L[S,X]}$.

Let τ^2 denote the canonical name for the element of $\mathbb{R}^2$ added by ${}_2\mathbb{O}_S^{L[S,X]}$. Let τ_0^2 and τ_1^2 be the canonical name for the first and second coordinate of τ^2, respectively. Let G be ${}_2\mathbb{O}_S^{L[S,X]}$-generic over $\mathrm{HOD}_S^{L[S,X]}$ containing p. Define the condition $q = \{(x,y) \in (\mathbb{R}^2)^{L[S,X]} : \neg(x\ E\ y)\}$. Observe that $p \leq_{{}_2\mathbb{O}_S^{L[S,X]}} q$ and so $q \in G$. Since E has (S,φ) as its ∞-Borel code, q can be expressed in the form for which the last statement of Fact 2.9 applies. Hence $\neg(\tau_0^2[G]\ E\ \tau_1^2[G])$. However Fact 2.10 states that $\tau_0^2[G]$ and $\tau_1^2[G]$ are the canonical reals added by some $\mathbb{O}_S^{L[S,X]}$-generic filter over $\mathrm{HOD}_S^{L[S,X]}$. Thus let G_0 and G_1 be $\mathbb{O}_S^{L[S,X]}$-generic filters over $\mathrm{HOD}_S^{L[S,X]}$ so that $\tau[G_0] = \tau_0^2[G]$ and $\tau[G_1] = \tau_1^2[G]$, respectively. Since $p \in G$, $\tau^2[G] \in p = (v \times v) \setminus E$. So $\tau[G_0] = \tau_0^2[G] \in v$ and $\tau[G_1] = \tau_1^2[G] \in v$. Hence $v \in G_0$ and $v \in G_1$. Now Subclaim 1.1 implies $\tau_0^2[G]\ E\ \tau_1^2[G]$. Contradiction. This proves Claim 1. □

Again by AD, $\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}$ and its power set are countable in V. Fix an enumeration $(D_n : n \in \omega)$ of all the dense open subsets of $\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}$ that belong to $\mathrm{HOD}_S^{L[S,X]}$. By intersecting if necessary, one may assume $D_{n+1} \subseteq D_n$ for each $n \in \omega$. One will define a sequence $\langle p_t : t \in {}^{<\omega}2\rangle$ in $\mathbb{O}_S^{L[S,X]}$ with the following properties.

- $p_\emptyset = u$.
- For all $s, t \in {}^{<\omega}2$ so that $s \subseteq t$, $p_t \leq_{\mathbb{O}_S^{L[S,X]}} p_s$.
- For all $n \in \omega$, for all $s, t \in {}^{n+1}2$ and $s \neq t$, $(p_s, p_t) \in D_n$.

Assume for a moment that such a sequence has been constructed. For each $f \in {}^\omega 2$, let K_f be the $\leq_{\mathbb{O}_S^{L[S,X]}}$-upward closure of $\{p_{f\restriction n} : n \in \omega\}$. Suppose $f, g \in {}^\omega 2$ and $f \neq g$. Let k be such that $f(k) \neq g(k)$. Let $m \in \omega$ be arbitrary and let $n = \max\{k, m\}$. Note that $p_{f\restriction n+1} \in K_f$ and $p_{g\restriction n+1} \in K_g$. Since $f \restriction n+1 \neq g \restriction n+1$, $(f \restriction n+1, g \restriction n+1) \in D_n \subseteq D_m$. This shows that $K_f \times K_g$ is a $\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}$-generic filter over $\mathrm{HOD}_S^{L[S,X]}$. Since $(u,u) = (p_\emptyset, p_\emptyset) \in K_f \times K_g$, Claim 1 implies that $\neg(\tau[K_f]\ E\ \tau[K_g])$. Thus the map $\Phi : {}^\omega 2 \to \mathbb{R}/E$ defined by $\Phi(f) = [\tau[K_f]]_E$ is an injection of ${}^\omega 2$ into $\mathbb{R}/E$.

It remains to construct such a sequence $\langle p_t : t \in {}^{<\omega}2\rangle$. Let $p_\emptyset = u$. Suppose for some $n \in \omega$, p_t has been defined for all $t \in {}^n 2$. Let $\langle (r_j, s_j) : j < K\rangle$ enumerate all pairs of $(r,s) \in {}^{n+1}2 \times {}^{n+1}2$ such that $r \neq s$, where $K = 2^{n+1}(2^{n+1}-1))$. For each $t \in {}^n 2$, let $q^0_{t\hat{}0} = q^0_{t\hat{}1} = p_t$. This defines $\langle q^0_t : t \in {}^{n+1}2\rangle$. Now one considers all possible pairs and extends these pairs into the dense open set D_n. The details are as follows. Suppose for $j < K$, $\langle q^j_t : t \in {}^{n+1}2\rangle$ has been defined. Since D_n is dense open, let $(q^{j+1}_{r_j}, q^{j+1}_{s_j})$ be the least pair $(q,q') \leq_{\mathbb{O}_S^{L[S,X]}} (q^j_{r_j}, q^j_{s_j})$ according to the canonical wellordering of $\mathrm{HOD}_S^{L[S,X]}$ so that $(q,q') \in D_n$. For t such that $t \neq r_j$ and $t \neq s_j$, let $q^{j+1}_t = q^j_t$. For each $t \in {}^{n+1}2$, let $p_t = q^K_t$. Since D_n is dense open, one can see that for any $s,t \in {}^{n+1}2$ so that $s \neq t$, $(p_s, p_t) \in D_n$. This completes the construction of the sequence $\langle p_t : t \in {}^{<\omega}2\rangle$.

For later purpose of this paper, note that once one chooses an $X \in \mathcal{D}$ witnessing Case II and the enumeration $(D_n : n \in \omega)$ of the dense open subsets of $\mathbb{O}_S^{L[S,X]} \times \mathbb{O}_S^{L[S,X]}$ in $\mathrm{HOD}_S^{L[S,X]}$, the embedding of $\mathbb{R}$ into $\mathbb{R}/E$ is given by the explicit procedure above. $\square$

Fact 3.3. *(1) Assume $\mathsf{ZF} + \mathsf{AD}^+$. If E is an equivalence relation on $\mathbb{R}$ with ∞-Borel code (S,φ) and $\mathbb{R}$ does not inject into the quotient $\mathbb{R}/E$, then $\mathbb{R}/E \subseteq \mathrm{OD}_S$. In particular, if $A \subseteq \mathbb{R}$ is a countable set of reals with ∞-Borel code (S,φ), then $A \subseteq \mathrm{HOD}_S$.*

(2) Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Suppose S is a set of ordinals. If E is an OD_S equivalence relation on $\mathbb{R}$ and $\mathbb{R}$ does not inject into $\mathbb{R}/E$, then $\mathbb{R}/E \subseteq \mathrm{OD}_S$. In particular, if $A \subseteq \mathbb{R}$ is a countable OD_S set of reals, then $A \subseteq \mathrm{HOD}_S$.

(3) Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. If A is any OD_S set such that $\mathbb{R}$ does not inject into A, then $A \subseteq \mathrm{OD}_S$ and hence wellorderable.

Proof. (1) First work in AD^+, the first statement comes from the observation at the end of the Case I argument in Theorem 3.2 that the sequence $\langle C_\alpha : \alpha < \delta\rangle$ is OD_S, produced uniformly from (S,φ), and each $C_\alpha \in \mathrm{OD}_S$. If $A \subseteq \mathbb{R}$ is countable with ∞-Borel code (S,φ), define the equivalence relation E on $\mathbb{R}$ by

$$x \; E \; y \Leftrightarrow (x = y) \vee (x, y \notin A).$$

E has an ∞-Borel code which is OD_S. Applying the first result to E, one has that $\mathbb{R}/E \subseteq \mathrm{OD}_S$. Thus for each $x \in A$, $[x]_E = \{x\} \in \mathrm{OD}_S$. This implies that $x \in \mathrm{OD}_S$ and in fact $x \in \mathrm{HOD}_S$ since $x \subseteq \omega$. Thus $A \subseteq \mathrm{HOD}_S$.

(2) Now work in $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. By Fact 2.5, every OD_S set of reals has an ∞-Borel code which is OD_S. The result then follows by applying the earlier statement.

(3) The idea is that in natural models of AD^+, one can break an arbitrary set A into a uniform sequence of subsets which are surjective images of $\mathbb{R}$. By the assumption that A does not contain a copy of $\mathbb{R}$, each of the pieces do not contain a copy of $\mathbb{R}$. Then Theorem 3.2 uniformly gives a wellordering of each piece. These wellorderings are then coherently patched together into a wellordering of the original set A. Recall that by Fact 2.3, natural models of AD^+ either take the form $L(J,\mathbb{R})$ for some set of ordinals J or satisfy $\mathsf{AD}_{\mathbb{R}}$. This patching for $L(J,\mathbb{R})$ is relatively straightforward. In the $\mathsf{AD}_{\mathbb{R}}$ case, it is more challenging and uses the unique supercompactness measure on $\mathscr{P}_{\omega_1}(\lambda)$ for each $\lambda < \Theta$. See [1] or [4] for the details. $\square$

Fact 3.4. *Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let δ be an ordinal and X be a set. Suppose $R \subseteq \mathscr{P}(\delta) \times X$ is a relation so that for each $N \in \mathscr{P}(\delta)$, the section $R_N = \{x \in X : (N,x) \in R\}$ is wellorderable, then R has a uniformization.*

In particular, if $R \subseteq \mathscr{P}(\delta) \times \mathbb{R}$ is a relation so that each R_N is countable, then R has a uniformization.

Proof. Using Fact 2.4 and the remarks preceding this fact, R is ordinal definable from some set of ordinals S. So each R_N is ordinal definable from the set of ordinals $\langle S, N\rangle$, where $\langle \cdot,\cdot\rangle$ refers to some fixed way of coding two sets of ordinals into a single set of ordinals. Because $\mathbb{R}$ can not inject into any wellorderable set, Fact 3.3 implies that $R_N \subseteq \mathrm{OD}_{\langle S,N\rangle}$. The canonical wellordering of $\mathrm{OD}_{\langle S,N\rangle}$ gives a canonical wellordering of R_N.

Although the hypothesis states that each R_N is wellorderable without, a priori, a uniform wellordering, one in fact does have a uniform wellordering of each section. The function that selects the least element of R_N using this canonical uniform wellordering of all sections of R is the desired uniformization function. $\square$

4. Lower Bound on Cardinality

The following section gives a lower bound on the cardinality of disjoint unions of smooth equivalence relations on $\mathbb{R}$. (Later it will be shown that this fact characterizes those subsets of $[\omega_1]^{<\omega_1}$ which are in bijection with a disjoint union of quotients of smooth equivalence relations.)

Fact 4.1. (ZF). *Let $\kappa > 0$ be an ordinal. Let $\langle G_\alpha : \alpha < \kappa\rangle$ be a sequence of smooth equivalence relations. Then $\mathbb{R} \sqcup \kappa$ injects into $\bigsqcup_{\alpha<\kappa} \mathbb{R}/G_\alpha$.*

Proof. Let $\bar{0} \in \mathbb{R}$ denote the constant 0 function. Let $\Phi : \mathbb{R} \to \mathbb{R}/G_0$ be an injection so that the image of Φ does not include $[\bar{0}]_{G_0}$. Let $\Psi : \mathbb{R} \sqcup \kappa \to \bigsqcup_{\alpha<\kappa} \mathbb{R}/G_\alpha$ be defined by $\Psi(r) = \Phi(r)$ if $r \in \mathbb{R}$ and $\Psi(\alpha) = [\bar{0}]_{G_\alpha}$ if $\alpha \in \kappa$. Ψ is an injection. ☐

This lower bound is optimal by the following example.

Fact 4.2. (ZF) *There is a sequence $\langle F_\alpha : \alpha < \omega_1\rangle$ of smooth equivalence relations such that $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/F_\alpha \approx \mathbb{R} \sqcup \omega_1$; and therefore under* AD*, $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/F_\alpha$ is not in bijection with $\mathbb{R} \times \omega_1$.*

Proof. Let WO denote the set of reals coding wellorderings. For each $\alpha < \omega_1$, let WO_α denote the set of reals coding wellorderings of ordertype α. Define F_α by

$$x \ F_\alpha \ y \Leftrightarrow (x \notin \mathrm{WO}_\alpha \wedge y \notin \mathrm{WO}_\alpha) \vee (x = y).$$

All elements of WO_α form singleton F_α-classes, and there is a single uncountable OD equivalence class consisting of $\mathbb{R} \setminus \mathrm{WO}_\alpha$.

Therefore, $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/F_\alpha \approx (\bigcup_{\alpha<\omega_1} \mathrm{WO}_\alpha) \sqcup \omega_1 = \mathrm{WO} \sqcup \omega_1 \approx \mathbb{R} \sqcup \omega_1$, where the copy of ω_1 comes from the large equivalence class for each $\alpha < \omega_1$.

The second statement follows from the fact that under AD, $\mathbb{R} \sqcup \omega_1$ is not in bijection with $\mathbb{R} \times \omega_1$. To see this fact, suppose $\Phi : \mathbb{R} \times \omega_1 \to \mathbb{R} \sqcup \omega_1$ is an injection. Let $\pi_2 : \mathbb{R} \times \omega_1 \to \omega_1$ be the projection onto the second coordinate. Since ω_1 is a wellordering, the preimage $\pi_2[\Phi^{-1}[\omega_1]]$ of the copy of ω_1 from the disjoint union $\mathbb{R} \sqcup \omega_1$ is a wellorderable set of reals. Since AD implies there are no uncountable wellorderable sets of reals, $\pi_2[\Phi^{-1}[\omega_1]]$ must be countable. Let $r \in \mathbb{R}$ with $r \notin \pi_2[\Phi^{-1}[\omega_1]]$. Since Φ is an injection, Φ must map $\{r\} \times \omega_1$ into the copy of $\mathbb{R}$ of the disjoint union $\mathbb{R} \sqcup \omega_1$. This is impossible since $\Phi[\{r\} \times \omega_1]$ would have to be an uncountable wellorderable set of reals. ☐

Using countable section uniformization for relations on $[\omega_1]^\omega \times \mathbb{R}$ given by Fact 3.4, [2, Fact 4.20] shows that $[\omega_1]^\omega$ can not inject into a disjoint union of quotients of smooth equivalence relations with all classes countable assuming $\mathsf{ZF} + \mathsf{AD}^+$. Section 7 will show under AD alone that $[\omega_1]^\omega$ can not inject into a disjoint union of quotients of equivalence relations with all section countable by proving an almost full countable section uniformization for relations on $[\omega_1]^\omega \times \mathbb{R}$.

Countable section uniformization seems to be a powerful tool that allows disjoint unions of quotients of smooth equivalence relations with all classes countable to be studied more easily. Each F_α from the sequence $\langle F_\alpha : \alpha < \omega_1\rangle$ from Fact 4.2 has only one uncountable class. Its disjoint union $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/F_\alpha$ is in bijection with $\mathbb{R} \sqcup \omega_1$. A natural question asked in [2] was whether it is necessary to use equivalence relations with uncountable classes to produce a disjoint union which is in bijection with $\mathbb{R} \sqcup \omega_1$. It is also natural to ask if it is possible to determine the cardinality of a disjoint union of quotients of smooth equivalence relations with all classes countable.

It will be shown later that many subsets of $[\omega_1]^{<\omega_1}$ are disjoint unions of quotients of smooth equivalence relations on $\mathbb{R}$. In particular, $[\omega_1]^{<\omega_1}$ itself is an ω_1-length disjoint union of quotients of smooth equivalence relations. Hence having all classes countable is necessary in the results of this paper.

Suppose $\langle E_\alpha : \alpha < \kappa\rangle$ is a wellordered sequence of equivalence relations where κ is any ordinal. The most natural presentation of such a sequence is as a relation $R \subseteq \kappa \times \mathbb{R} \times \mathbb{R}$ defined by $(\alpha, x, y) \Leftrightarrow x \ E_\alpha \ y$. Under AD^+, each $E_\alpha \subseteq \mathbb{R} \times \mathbb{R}$ has an ∞-Borel code. For the results of this paper, one will need to uniformly obtain an ∞-Borel code for each E_α. It is unclear this can be done for any wellordered sequence of equivalence relations under just AD^+. However, in models of $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$, this will be possible. This motivates the following definition.

Definition 4.3. Let $\kappa \in \mathrm{ON}$ and $R \subseteq \kappa \times \mathbb{R}$. An ∞-Borel code for R is a pair (S, φ) such that $(\alpha, r) \in R$ if and only if $L[S, r] \models \varphi(S, \alpha, r)$.

A sequence $\langle E_\alpha : \alpha < \kappa\rangle$ of equivalence relations on $\mathbb{R}$ has an ∞-Borel code if and only if the relation $R(\alpha, x, y) \Leftrightarrow x \ E_\alpha \ y$ has an ∞-Borel code.

Note that if (S, φ) is an ∞-Borel code for $\langle E_\alpha : \alpha < \kappa\rangle$, then uniformly from (S, φ), one can find formulas φ_α so that $(\langle S, \alpha\rangle, \varphi_\alpha)$ is an ∞-Borel code for E_α (in the ordinary sense). (Here $\langle S, \alpha\rangle$ is some fixed coding of sets of ordinals so that S and α can be recovered.)

Fact 4.4. *Under* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$*, for every* $\kappa \in \mathrm{ON}$ *and every relation* $R \subseteq \kappa \times \mathbb{R}$*,* R *has an* ∞*-Borel code.*

Proof. By Fact 2.3 and Fact 2.4, every set is ordinal definable from some set of ordinals. Let S be a set of ordinals so that R is OD_S. Let $R_\alpha = \{x : (\alpha, x) \in R\}$. Each R_α is OD_S. By Fact 2.5, each R_α has an ∞-Borel code in HOD_S. Let $(S_\alpha, \varphi_\alpha)$ be the least ∞-Borel code for R_α according to the canonical wellordering of HOD_S. Let $U = \{(\alpha, \beta) : \beta \in S_\alpha\}$. Then there is some φ so that (U, φ) is an ∞-Borel code for R in the sense of Definition 4.3. $\square$

Now suppose that $\langle E_\alpha : \alpha < \kappa\rangle$ is a sequence of smooth equivalence relations. Let (S, φ) be an ∞-Borel code for this sequence. Hence uniformly, E_α has some ∞-Borel code $(\langle S, \alpha\rangle, \varphi_\alpha)$.

By the remark at the end of the proof of Theorem 3.2, an embedding of $\mathbb{R}$ into $\mathbb{R}/E_\alpha$ can be produced uniformly from a choice of $X \in \mathcal{D}$ so that the Case II assumptions holds and a fixed enumeration $(D_n : n \in \omega)$ of the dense open subsets of $\mathbb{O}_{\langle S,\alpha\rangle}^{L[\langle S,\alpha\rangle,X]} \times \mathbb{O}_{\langle S,\alpha\rangle}^{L[\langle S,\alpha\rangle,X]}$. Since for any α, $L[\langle S, \alpha\rangle, X] = L[S, X]$ and $\mathbb{O}_{\langle S,\alpha\rangle}^{L[\langle S,\alpha\rangle,X]} = \mathbb{O}_S^{L[S,X]}$, one can drop the α. Thus one can uniformly find a sequence of injections of $\mathbb{R}$ into $\mathbb{R}/E_\alpha$ if one could find a single $X \in \mathcal{D}$ that witnesses the Case II assumption for all the equivalence relations E_α. With such a sequence, one could then inject $\mathbb{R} \times \kappa$ into $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$.

First, it is elucidating to see why a natural attempt to use the argument in Theorem 3.2 Case II is unable to produce a uniform sequence of embeddings of $\mathbb{R}$ into $\mathbb{R}/F_\alpha$, where $\langle F_\alpha : \alpha < \omega_1\rangle$ is the sequence of equivalence relations from Fact 4.2. Note that $\emptyset$ can serve as the ∞-Borel code for this sequence. Let $X \in \mathcal{D}$. $\mathbb{R}^{L[X]}$ is countable. Hence there is some $\alpha < \omega_1$ so that for all $\beta > \alpha$, $\mathbb{R}^{L[X]} \subseteq \mathbb{R} \setminus \mathrm{WO}_\beta$. Hence for all $\beta > \alpha$, every real of $L[X]$ belongs to the single ordinal definable uncountable class of E_β. So for all $\beta > \alpha$, X can not serve as the witness to the Case II assumption. This shows why the natural attempt to inject $\mathbb{R} \times \omega_1$ into $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/F_\alpha \approx \mathbb{R} \sqcup \omega_1$ must fail. However, when $\langle E_\alpha : \alpha < \omega_1\rangle$ is a sequence of smooth equivalence relations with all classes countable, this natural attempt does succeed.

For the next theorem, one will partition $\mathcal{D}$ into two disjoint sets $\mathcal{D} = H_0^\alpha \cup H_1^\alpha$ for various α's. One of the two sets belongs to the ultrafilter μ. The main task is to show that H_0^α is the one that belongs to μ. Supposing for the sake of contradiction that $H_1^\alpha \in \mu$. The main trick is to code information from all the local models $\mathrm{HOD}_S^{L[S,X]}$ into one single ordinal by using the wellfoundedness of the ultrapower $\prod_{X\in\mathcal{D}} \omega_1/\mu$. This magical ordinal contains so much information that it can then be used to give an OD_S definition for the E_α-class, $[a]_{E_\alpha}$, where $a \notin \mathrm{OD}_S$. Since $[a]_{E_\alpha}$ is a countable OD_S sets, this implies $a \in \mathrm{OD}_S$ by Fact 3.3. This yields the desired contradiction. The details follow below:

Theorem 4.5. *Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$*. Let* $\kappa \in \mathrm{ON}$ *and* $\langle E_\alpha : \alpha < \kappa\rangle$ *be a sequence of smooth equivalence relations on* $\mathbb{R}$ *with all classes countable. Then* $\mathbb{R} \times \kappa$ *injects into* $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$*.*

Proof. By Fact 4.4, let (S, φ) be an ∞-Borel code for the sequence $\langle E_\alpha : \alpha < \kappa\rangle$. By the description above, it suffices to find a single $X \in \mathcal{D}$ so that all $\alpha < \kappa$, there is some $a \in \mathbb{R}^{L[S,X]}$ so that a does not belong to any E_α-component in $\mathbb{O}_S^{L[S,X]}$.

Fix some $a \in \mathbb{R}$ which is not OD_S. Now for any $\alpha < \kappa$, let H_0^α be the set of $X \in \mathcal{D}$ so that a does not belong to any E_α-component of $\mathbb{O}_S^{L[S,X]}$. Let H_1^α be the set of $X \in \mathcal{D}$ so that a does belong to some E_α-component of $\mathbb{O}_S^{L[S,X]}$. Since μ is an ultrafilter, either $H_0^\alpha \in \mu$ or $H_1^\alpha \in \mu$. Suppose that $H_1^\alpha \in \mu$. Define $f : \mathcal{D} \to \omega_1$ by $f(X)$ is the least β so that a belongs to the β^{th} E_α-component in $\mathbb{O}_S^{L[S,X]}$ if $X \in H_1^\alpha$, and $f(X)$ is some default value otherwise. Let $F = [f]_{\sim_\mu}$. Then $[a]_{E_\alpha}$ is ordinal definable using S and F as parameters since $b \in [a]_{E_\alpha}$ if and only if for all representative f for F, for a cone of $X \in \mathcal{D}$, b is E_α related to some element in the $f(X)^{\text{th}}$ E_α-component in $\mathbb{O}_S^{L[S,X]}$. By Fact 2.7, $\prod_{X\in\mathcal{D}} \omega_1/\mu$ is wellfounded. Hence F is essentially an ordinal. This shows that $[a]_{E_\alpha}$ is OD_S. Then $[a]_{E_\alpha}$ is a countable OD_S set of reals. Fact 3.3 implies that $[a]_{E_\alpha}$ consists only of OD_S elements. But $a \notin \mathrm{OD}_S$ by the initial choice of a. Contradiction. This shows that $H_0^\alpha \in \mu$.

Since for each $\alpha < \kappa$, $[a]_{E_\alpha}$ is a countable $\mathrm{OD}^V_{\langle S,a\rangle}$ set of reals, Fact 3.3 implies $[a]_{E_\alpha} \subseteq \mathrm{HOD}^V_{\langle S,a\rangle}$. Using the canonical wellordering of $\mathrm{HOD}^V_{\langle S,a\rangle}$, let $\langle b^\alpha_\epsilon : \epsilon < \delta_\alpha\rangle$ be an injective enumeration of $[a]_{E_\alpha}$, where $\delta_\alpha \in \mathrm{ON}$. For each $\alpha < \kappa$, δ_α is less than $(2^{\aleph_0})^{\mathrm{HOD}^V_{\langle S,a\rangle}}$. Since $V \models \mathsf{AD}$, $(2^{\aleph_0})^{\mathrm{HOD}^V_{\langle S,a\rangle}}$ is countable in V. In V, choose a bijection of ω with $(2^{\aleph_0})^{\mathrm{HOD}^V_{\langle S,a\rangle}}$. This bijection then induces canonically bijections $\Sigma_\alpha : \omega \to \delta_\alpha$ for all

$\alpha < \kappa$. Define $r_\alpha \in \mathbb{R}$ by $r_\alpha(\langle n, k\rangle) = b^\alpha_{\Sigma_\alpha(n)}(k)$. Thus $[a]_{E_\alpha} = \{(r_\alpha)_n : n \in \omega\}$ where $(r_\alpha)_n(k) = r_\alpha(\langle n, k\rangle)$. But $\langle r_\alpha : \alpha < \kappa\rangle$ is a wellordered sequence of reals. Under AD, there are only countably many distinct r_α's. Thus $\{[a]_{E_\alpha} : \alpha < \kappa\}$ is a countable set.

Note that if $[a]_{E_\alpha} = [a]_{E_\beta}$, then $H_0^\alpha = H_0^\beta$. Therefore by countable additivity of μ, $\bigcap_{\alpha<\kappa} H_0^\alpha \in \mu$. Let $X \in \bigcap_{\alpha<\kappa} H_0^\alpha$. This is the desired degree X that witnesses the Case II assumption for all E_α. By the remarks above, this allows for the construction of an injection of $\mathbb{R} \times \kappa$ into $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ which completes the proof. $\square$

5. Upper Bound on Cardinality

Definition 5.1. Let E be an equivalence relation on $\mathbb{R}$. Let $\mathcal{S}$ be a collection of nonempty subsets of $\mathbb{R}$. $\mathcal{S}$ is a separating family for E if and only if for all $x, y \in \mathbb{R}$, $x\ E\ y$ if and only if for all $A \in \mathcal{S}$, $x \in A \Leftrightarrow y \in A$. In other words, for all $x, y \in \mathbb{R}$, $\neg(x\ E\ y)$ if and only if there is an $A \in \mathcal{S}$ which separate x and y in the sense that either $(x \in A \wedge y \notin A)$ or $(x \notin A \wedge y \in A)$.

Definition 5.2. E_0 is the equivalence relation ${}^\omega 2$ defined by $x\ E_0\ y$ if and only if $(\exists m)(\forall n \geq m)(x(n) = y(n))$.

The following is Hjorth's E_0-dichotomy in AD^+ which generalizes the classical E_0-dichotomy of Harrington-Kechris-Louveau [6].

Fact 5.3. [8, Theorem 2.5] *Assume $\mathsf{ZF} + \mathsf{AD}^+$. Let E be an equivalence relation on $\mathbb{R}$. Then either (i) or (ii) below holds.*

(i) There is a wellordered separating family for E.

(ii) There is a $\Phi : \mathbb{R} \to \mathbb{R}$ with the property that $x\ E_0\ y$ if and only if $\Phi(x)\ E\ \Phi(y)$.

Proof. Note that option (ii) implies that $\mathbb{R}/E_0$ injects in $\mathbb{R}/E$. Suppose option (i) holds. Let $\mathcal{S} = \langle B_\alpha : \alpha < \delta\rangle$, where δ is some ordinal, be the given separating family. For each $x \in \mathbb{R}$, let $\Psi(x) = \{\alpha : x \in B_\alpha\}$. Define $\Phi : \mathbb{R}/E \to \mathscr{P}(\delta)$ by $\Phi([x]_E) = \Psi(x)$. Φ is well defined since by the definition of $\mathcal{S}$ being a separating family for E, if $x\ E\ y$, then for all $\alpha < \delta$, $x \in B_\alpha$ if and only if $y \in B_\alpha$ which implies that $\Psi(x) = \Psi(y)$. Suppose $[x]_E \neq [y]_E$ which implies $\neg(x\ E\ y)$. Since $\mathcal{S}$ is a separating family, there exists an $\alpha < \delta$ so that $(x \in B_\alpha \wedge y \notin B_\alpha)$ or $(x \notin B_\alpha \wedge y \in B_\alpha)$. Thus $\Phi([x]_E) = \Psi(x) \neq \Psi(y) = \Phi([y]_E)$. This shows that Φ is an injection.

As usual in dichotomy results, there are two cases. One case yields the wellordered separating family and the other case yields an embedding. For the purpose of this paper, one is more concerned with producing the wellordered separating family. Moreover, one needs to observe that the wellordered separating family and its wellordering is produced uniformly from the ∞-Borel code for E. The following will give the argument to produce a wellordered separating family. The embedding case will be omitted as it is not relevant for this paper.

Let (S, φ) be an ∞-Borel code for E. Regardless of the universe in consideration, E will always be considered as the set defined by the ∞-Borel code (S, φ).

(Case I) For all $X \in \mathcal{D}$, for all $a, b \in \mathbb{R}^{L[S,X]}$, if $\neg(a\ E\ b)$, then there is an $\mathrm{OD}_S^{L[S,X]}$ $C \in \mathbb{O}_S^{L[S,X]}$ which is E-invariant in $L[S, X]$ and $a \in C$ and $b \notin C$.

For each $F \in \prod_{X\in\mathcal{D}} \omega_1/\mu$, define A_F as follows: Let $f : \mathcal{D} \to \omega_1$ be such that $f \in F$, that is f is a representative of F. Define A_F as follows. For $a \in \mathbb{R}$, $a \in A_F$ if and only if on a Turing cone of $X \in \mathcal{D}$, a belongs to the $f(X)^{\text{th}}$ E-invariant set in $\mathbb{O}_S^{L[S,X]}$. (If there is no $f(X)^{\text{th}}$ E-invariant $\mathrm{OD}_S^{L[S,X]}$-set, then let this set be $\emptyset$.) Note that A_F is well defined.

A_F is E-invariant: Suppose $a, b \in \mathbb{R}$, $a\ E\ b$ and $a \in A_F$. Pick $f \in F$. Since $a \in A_F$, there is some $Z \geq_T [a \oplus b]_T$ so that for all $X \geq_T Z$, a belongs to the $f(X)^{\text{th}}$ E-invariant set in $\mathbb{O}_S^{L[S,X]}$. Note $b \in L[S, X]$. $V \models a\ E\ b$ means that

$$V \models L[S, a, b] \models \varphi(S, a, b).$$

Hence

$$L[S, X] \models L[S, a, b] \models \varphi(S, a, b).$$

Thus b also belongs to the $f(X)^{\text{th}}$ E-invariant set in $\mathbb{O}_S^{L[S,X]}$. Hence $b \in A_F$.

By Fact 2.7, $\prod_{X\in\mathcal{D}}\omega_1/\mu$ is wellfounded. Hence $\mathcal{S} = \langle A_F : F \in \prod_{X\in\mathcal{D}}\omega_1/\mu\rangle$ is a wellordered set of E-invariant subsets of $\mathbb{R}$.

$\mathcal{S}$ is a separating family for E: Suppose $a, b \in \mathbb{R}$ and $\neg(a\ E\ b)$. Define $f : \mathcal{D} \to \omega_1$ as follows. Let $Z = [a \oplus b]_T$. If $X \geq_T Z$, then let $f(X)$ be the least ordinal $\alpha < \omega_1$ so that the α^{th} E-invariant set in $\mathbb{O}_S^{L[S,X]}$ contains a but not b. Such an α exists from the Case I assumption. If X is not Turing above Z, then let $f(X) = \emptyset$. Let $F = [f]_{\sim_\mu}$. Then $A_F \in \mathcal{S}$, $a \in A_F$, and $b \notin A_F$.

In conclusion, one has shown that $\mathcal{S}$ is a wellordered separating family for E.

(Case II) There is some $X \in \mathcal{D}$ and some $a, b \in \mathbb{R}^{L[S,X]}$ with $\neg(a\ E\ b)$ such that there are no E-invariant sets $C \in \mathbb{O}_S^{L[S,X]}$ so that $a \in C$ and $b \notin C$.

The idea is that this case assumption gives a natural condition in the forcing $\mathbb{O}_S^{L[S,X]}$ for which a perfect tree of mutual $\mathbb{O}_S^{L[S,X]}$-generics over $\mathrm{HOD}_S^{L[S,X]}$ below this condition serves as the desired embedding. The details can be found in [8] and are omitted since this case is not relevant for the rest of the paper. ☐

Theorem 5.4. *Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. *Let κ be an ordinal and $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of smooth equivalence relations on $\mathbb{R}$ with all classes countable. Then there is an injection of $\bigsqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha$ into $\mathbb{R}\times\kappa$.*

Proof. By Fact 4.4 (which holds for any ordinal κ), let (U, φ) be an ∞ code for $\langle E_\alpha : \alpha < \kappa\rangle$. Uniformly from (U, φ), one obtains ∞-Borel codes $(\langle U, \alpha\rangle, \varphi_\alpha)$ for each E_α. Since each E_α is smooth, $\mathbb{R}/E_\alpha \approx \mathbb{R}$. Since under AD, $\mathbb{R}/E_0$ does not inject into $\mathbb{R}$, Case I from the proof of Fact 5.3 must occur. The proof in Case I uniformly produces, from $(\langle U, \alpha\rangle, \varphi_\alpha)$, a separating family $\mathcal{S}_\alpha = \langle A_\gamma^\alpha : \gamma < \delta\rangle$ for E_α, where δ is the ordertype of $\prod_{X\in\mathcal{D}}\omega_1/\mu$.

Let $\Phi_\alpha : \mathbb{R} \to \mathscr{P}(\delta)$ be defined by $\Phi_\alpha(x) = \{\gamma : x \in A_\gamma^\alpha\}$. Let $\tilde{\Phi}_\alpha : \mathbb{R}/E_\alpha \to \mathscr{P}(\delta)$ be the induced injection defined by $\tilde{\Phi}_\alpha([x]_{E_\alpha}) = \Phi_\alpha(x)$ as in the beginning of the proof of Fact 5.3. Define a relation $R \subseteq \kappa \times \mathscr{P}(\delta) \times \mathbb{R}$ by

$$R(\alpha, B, x) \Leftrightarrow (B = \Phi_\alpha(x)) \vee ((\forall y)(B \neq \Phi_\alpha(y)) \wedge x = \bar{0})$$

where $\bar{0}$ is the constant 0 sequence. For each $(\alpha, B) \in \kappa \times \mathscr{P}(\delta)$, the section $R_{(\alpha,B)}$ is countable. Fact 3.4 implies that there is a uniformization function $F : (\kappa \times \mathscr{P}(\delta)) \to \mathbb{R}$.

Define $\Psi : \bigsqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha \to \mathbb{R}\times\kappa$ by $\Psi([x]_{E_\alpha}) = (F(\alpha, \tilde{\Phi}_\alpha([x]_{E_\alpha})), \alpha)$. Ψ is an injection. ☐

Definition 5.5. Let X be a set. For $n \in \omega$, let $[X]^n_= = \{f \in {}^nX : (\forall i, j \in n)(i \neq j \Rightarrow f(i) \neq f(j)\}$. Let $[X]^{\leq\omega}_= = \bigcup_{n\in\omega}[X]^n_=$.

A set X has the Jónsson property if and only if for all $f : [X]^{\leq\omega}_= \to X$, there is some $Y \subseteq X$ with $Y \approx X$ so that $f[[Y]^{\leq\omega}_=] \neq X$.

Fact 5.6. *Assume* $\mathsf{ZF} + \mathsf{AD}$.

([9]*, Holshouser and Jackson) $\mathbb{R}$, $\mathbb{R} \sqcup \omega_1$, and $\mathbb{R}\times\kappa$ where $\kappa < \Theta$ have the Jónsson property.*

([5]*) $\mathbb{R}/E_0$ does not have the Jónsson property.*

[2, Fact 4.23] *For any $\kappa < \Theta$, $(\mathbb{R}/E_0)\times\kappa$ does not have the Jónsson property.*

Fact 5.7. [2, Fact 4.13] *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let $\kappa \in \mathrm{ON}$. Let $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of equivalence relations on $\mathbb{R}$ with all classes countable. Let $f : [\bigsqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha]^{\leq\omega}_= \to \bigsqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha$. Then there is some perfect tree p so that $f[[\bigsqcup_{\alpha<\kappa}[p]/E_\alpha]^{\leq\omega}_=] \neq \bigsqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha$.*

[2, Theorem 4.15] *$\mathbb{R}\times\kappa$ has the Jónsson property for all $\kappa \in \mathrm{ON}$.*

If for every perfect tree p, $\bigsqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha \approx \bigsqcup_{\alpha<\kappa}[p]/E_\alpha$, then Fact 5.7 would imply that $\bigsqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha$ has the Jónsson property. However, in general these two sets can not be in bijection since $\mathbb{R}/E_0$ does not have the Jónsson property. In this particular case, the p satisfying fact 5.7 is not an E_0-trees (see [5, Definition 5.2]), i.e. a perfect tree with certain symmetry conditions.

Combining Theorem 4.5 and 5.4, one can determine the cardinality of disjoint unions of quotients of smooth equivalence relations with all classes countable and show that they have the Jónsson property.

Theorem 5.8. *Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. *Let $\kappa \in \mathrm{ON}$ and $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of smooth equivalence relations on $\mathbb{R}$ with all classes countable. Then $\bigsqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha \approx \mathbb{R}\times\kappa$ and hence $\bigsqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha$ has the Jónsson property.*

6. Disjoint Union of Quotients of Smooth Equivalence with Uncountable Classes

This section will show that in natural models of AD^+ many subsets below $[\omega_1]^{<\omega_1}$ are in bijection with disjoint unions of quotients of smooth equivalence relations on $\mathbb{R}$. It will be shown that any subset of $[\omega_1]^{<\omega_1}$ that contains $\mathbb{R} \sqcup \omega_1$ can be written in this way. The argument is similar to the example $\langle F_\alpha : \alpha < \omega_1 \rangle$ produced in the proof of Fact 4.2. Note that each F_α has one uncountable equivalence class that holds the reals that are not used for coding. In the following argument, the existence of a copy of ω_1 is again used to handle these classes.

Recall the distinction between smooth and weakly smooth from Definition 2.12. The first result of the next theorem is proved in just $\mathsf{ZF} + \mathsf{AD}$. The quotients of the weakly smooth but not smooth E_α's are (non-uniformly) in bijection with a countable ordinal. The uniformity of ∞-Borel code will be important in the argument to absorb these quotients of the weakly smooth but not smooth equivalence relations into ω_1. Thus it is unclear if the second statement of the next theorem is provable in just AD or AD^+.

Theorem 6.1. *(1) Assume $\mathsf{ZF} + \mathsf{AD}$. Suppose $X \subseteq [\omega_1]^{<\omega_1}$ and ω_1 injects into X. Then there exists a sequence $\langle E_\alpha : \alpha < \omega_1 \rangle$ of weakly smooth equivalence relations on $\mathbb{R}$ so that X is in bijection with $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$.*

(2) Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Suppose $X \subseteq [\omega_1]^{<\omega_1}$ and $\mathbb{R} \sqcup \omega_1$ injects into X. Then there exists a sequence $\langle E_\alpha : \alpha < \omega_1 \rangle$ of smooth equivalence relations on $\mathbb{R}$ so that X is in bijection with $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$.

(3) Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. $X \subseteq [\omega_1]^{<\omega_1}$ has a sequence $\langle E_\alpha : \alpha < \omega_1 \rangle$ of smooth equivalence relations such that $X \approx \bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ if and only if $\mathbb{R} \sqcup \omega_1$ injects into X.

Proof. Statement (2) will be shown (and statement (1) will follow from a part of the argument). Assume $\mathbb{R} \sqcup \omega_1$ injection into X. Since ω_1 injects into X, let $X = X_0 \sqcup X_1$ where X_1 is in bijection with ω_1. Henceforth, say $X = X_0 \sqcup \omega_1$.

Let WO denote the set of reals coding wellorderings on ω. Note that ${}^\omega\mathbb{R}$ is in bijection with $\mathbb{R}$. For $\alpha < \omega_1$, let WO^α be the set of $j \in {}^\omega\mathrm{WO}$ so that for all $m \neq n$, $\mathrm{ot}(j(m)) \neq \mathrm{ot}(j(n))$ and $\sup\{\mathrm{ot}(j(n)) : n \in \omega\} = \alpha$.

For $\alpha < \omega_1$, let $X_0^\alpha = \{f \in X : \sup(f) = \alpha\}$. For each $j \in \mathrm{WO}^\alpha$, let $\Psi(j) \in [\alpha+1]^{\leq\alpha}$ be the increasing enumeration of $\{\mathrm{ot}(j(n)) : n \in \omega\}$. Let $Y^\alpha = \Psi^{-1}[X_0^\alpha]$.

Define E_α on ${}^\omega\mathbb{R}$ by

$$x \; E_\alpha \; y \Leftrightarrow (x \notin Y^\alpha \wedge y \notin Y^\alpha) \vee (x \in Y^\alpha \wedge y \in Y^\alpha \wedge \Psi(x) = \Psi(y))$$

Note that each ${}^\omega\mathbb{R}/E_\alpha$ has one distinguished equivalence class corresponding to ${}^\omega\mathbb{R} \setminus Y^\alpha$. Denote this class by $\star_\alpha$. $({}^\omega\mathbb{R}/E_\alpha) \setminus \{\star_\alpha\}$ is in bijection with X_0^α in a canonical way. Thus canonically there is a bijection of $\bigsqcup_{\alpha<\omega_1} {}^\omega\mathbb{R}/E_\alpha$ with $X_0 \sqcup \omega_1$. Also since ${}^\omega\mathbb{R}/E_\alpha$ injects into $[\alpha+1]^{\leq\alpha}$, which is in bijection with $\mathbb{R}$, ${}^\omega\mathbb{R}/E_\alpha$ is either in bijection with $\mathbb{R}$ or is countable. (Note that this shows under ZF that any $X \subseteq [\omega_1]^{<\omega_1}$ which contains a copy of ω_1 is a disjoint union $\bigsqcup \mathbb{R}/E_\alpha$ where each $\mathbb{R}/E_\alpha$ is either countable or in bijection with $\mathbb{R}$ which establishes statement (1).)

By Fact 4.4, there is an ∞-Borel code (S, φ) for $\langle E_\alpha : \alpha < \omega_1 \rangle$ in the sense of Definition 4.3. As before, one can thus obtain uniformly the ∞-Borel code (in the ordinary sense) for each E_α.

Let $A = \{\alpha \in \omega_1 : |{}^\omega\mathbb{R}/E_\alpha| \leq \aleph_0\}$. The argument in Case I of Theorem 3.2 shows that uniformly in the ∞-Borel code for E_α for $\alpha \in A$, there is a wellordering of ${}^\omega\mathbb{R}/E_\alpha$.

Let $B = \omega_1 \setminus A$. $B \neq \emptyset$ since otherwise using the uniform wellordering of ${}^\omega\mathbb{R}/E_\alpha$ for all $\alpha \in A = \omega_1$, one could produce a bijection of $\bigsqcup_{\alpha<\omega_1} {}^\omega\mathbb{R}/E_\alpha$ with ω_1. It was shown above that $\bigsqcup_{\alpha<\omega_1} {}^\omega\mathbb{R}/E_\alpha$ is in bijection with $X_0 \sqcup \omega_1 = X$ and therefore X is in bijection with ω_1. However $\mathbb{R}$ injects into X by assumption but $\mathbb{R}$ is not wellorderable under AD. Contradiction.

Using the uniform wellordering of ${}^\omega\mathbb{R}/E_\alpha$ for all $\alpha \in A$, the set $K = \{\star_\alpha : \alpha \in \omega_1\} \cup \bigcup_{\alpha \in A} {}^\omega\mathbb{R}/E_\alpha$ (these two sets are not disjoint) is in bijection with ω_1. If B is countable, then $\bigsqcup_{\alpha<\omega_1} {}^\omega\mathbb{R}/E_\alpha = K \cup \bigsqcup_{\alpha\in B} {}^\omega\mathbb{R}/E_\alpha$ is in bijection with $\mathbb{R} \sqcup \omega_1$. By Fact 4.2, $\mathbb{R} \sqcup \omega_1$ is in bijection with a disjoint union of quotients of smooth equivalence relations on $\mathbb{R}$. Now suppose B is uncountable. Since $K \approx \omega_1$, pick a bijection of K with $\{\star_\alpha : \alpha \in B\}$. Since $\bigsqcup_{\alpha<\omega_1} {}^\omega\mathbb{R}/E_\alpha = K \sqcup \bigsqcup_{\alpha\in B} ({}^\omega\mathbb{R}/E_\alpha) \setminus \{\star_\alpha\}$, the map that sends K to $\{\star_\alpha : \alpha < \omega_1\}$ via the fixed bijection above and the identity on $\bigsqcup_{\alpha\in B} ({}^\omega\mathbb{R}/E_\alpha) \setminus \{\star_\alpha\}$ is a bijection of $\bigsqcup_{\alpha<\omega_1} {}^\omega\mathbb{R}/E_\alpha$ with $\bigsqcup_{\alpha\in B} {}^\omega\mathbb{R}/E_\alpha$.

It has been shown that X is in bijection with a disjoint union of quotients of smooth equivalence relations which gives statement (2). Statement (3) follows from statement (2) and Fact 4.1. $\square$

ω_1 and $\mathbb{R}$ cannot be written as an ω_1-length wellordered disjoint union of quotients of smooth equivalence relations. There are other examples.

Definition 6.2. Let $S_1 = \{f \in [\omega_1]^{<\omega_1} : \sup(f) = \omega_1^{L[f]}\}$.

The next result gives some cardinality comparisons between S_1, $\mathbb{R}$, and ω_1.

Fact 6.3. *([16]) Assume* $\mathsf{ZF} + \mathsf{AD}$. *$\mathbb{R}$ injects into S_1, and ω_1 does not inject into S_1.*

Proof. For each $r \in \mathbb{R}$, consider it as a subset of ω. Let $\Psi(r)$ be the subset of $\omega_1^{L[r]}$ consisting of r and all infinite ordinals less than $\omega_1^{L[r]}$. $\Psi(r) \in S_1$ and Ψ is injective as a function from $\mathbb{R}$ into S_1.

Suppose $\Phi : \omega_1 \to S_1$ is an injection. Φ can be coded as a subset of ω_1. Note that $\sup(\{\Phi(\alpha) : \alpha \in \omega_1\}) = \omega_1$ because otherwise Φ would be injecting into $[\alpha]^{<\alpha}$ for some $\alpha < \omega_1$. The latter is in bijection with $\mathbb{R}$. This would imply that there is an uncountable wellordered sequence of reals. Also note that since $L[\Phi] \models \mathsf{ZFC}$, $\omega_1^{L[\Phi]} < \omega_1$. Therefore choose some α so that $\sup(\Phi(\alpha)) > \omega_1^{L[\Phi]}$. However $\Phi(\alpha) \in L[\Phi]$ so $\omega_1^{L[\Phi(\alpha)]} \leq \omega_1^{L[\Phi]} < \sup(\Phi(\alpha))$. This implies that $\Phi(\alpha) \notin S_1$. Contradiction. $\square$

So in $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$, S_1 is not an ω_1-length disjoint union of quotients of smooth equivalence relations, but $S_1 \sqcup \omega_1$ is.

7. Almost Full Countable Section Uniformization for $[\omega_1]^\omega \times \mathbb{R}$

This section will show that $|[\omega_1]^\omega|$ is not below $|\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha|$ if $\langle E_\alpha : \alpha < \kappa\rangle$ is a sequence of equivalence relations on $\mathbb{R}$ with all classes countable under just AD. Note that by Theorem 5.8, $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$ is capable of proving that such a disjoint union is in bijection with $\mathbb{R} \times \kappa$. It is much more evident that $[\omega_1]^\omega$ does not inject into $\mathbb{R} \times \kappa$.

Fact 7.1. *([2])* ($\mathsf{ZF} + \mathsf{AD}$) *Let κ be an ordinal. Let $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of equivalence relations on $\mathbb{R}$. Let $\Phi : [\omega_1]^\omega \to \bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$. Let $R \subseteq [\omega_1]^\omega \times \mathbb{R}$ be defined by $R(f,x) \Leftrightarrow x \in \Phi(f)$. If there is a $Z \subseteq [\omega_1]^\omega$ with $Z \approx [\omega_1]^\omega$ and a $\Lambda : Z \to \mathbb{R}$ so that for all $f \in Z$, $R(f, \Lambda(f))$, then Φ is not an injection.*

Proof. See [2, Fact 4.19] and the subsequent discussions. $\square$

Using some of the ideas above, one can prove in AD^+, the (full) countable section uniformization for relations on $\mathbb{R} \times [\omega_1]^\omega$: For every $R \subseteq [\omega_1]^\omega \times \mathbb{R}$ such that $R_f = \{x \in \mathbb{R} : R(f,x)\}$ is nonempty and countable for all $f \in [\omega_1]^\omega$, there is a uniformization function for R. Then Fact 7.1 gives the following result:

Fact 7.2. [2, Fact 4.20] *Assume* $\mathsf{ZF}+\mathsf{AD}^+$. *Let κ be an ordinal and $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of equivalence relations on $\mathbb{R}$ such that each E_α has all classes countable. Then there is no injection $\Phi : [\omega_1]^\omega \to \bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$.*

For the Jónsson property, often uniformization on a sufficiently big set is enough for the desired result. AD can prove an almost full uniformization result for relation on $\mathbb{R} \times \mathbb{R}$. (For example, AD proves comeager uniformization.) Fact 7.1 only requires that one can uniformize relations on $[\omega_1]^\omega \times \mathbb{R}$ on a set $Z \subseteq [\omega_1]^\omega$ that has the same cardinality as $[\omega_1]^\omega$. In general, this is impossible. By Theorem 6.1, every cardinal below $[\omega_1]^{<\omega_1}$ that contains a copy of $\mathbb{R} \sqcup \omega_1$, for example $[\omega_1]^\omega$, can be written as a disjoint union of smooth equivalence relations. If this almost full uniformization exists for all relations on $[\omega_1]^\omega \times \mathbb{R}$, then Fact 7.1 would imply that there is no injection of $[\omega_1]^\omega$ into $[\omega_1]^\omega$, which is absurd.

The following will show in AD alone that one can prove almost full uniformization for relations on $[\omega_1]^\omega \times \mathbb{R}$ with all sections countable. This will suffice to prove the statement of Fact 7.2 in AD alone.

Definition 7.3. Fix a recursive bijection $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$. For $x \in \mathbb{R}$, let $(x)_m \in \mathbb{R}$ be defined by $(x)_m(n) = x(\langle m, n\rangle)$. Using the pairing function, one can also code relations on ω of various arity as a subset of ω.

Let $\mathrm{WO}^{[\omega]}$ be the set of reals x so that for all $n \in \omega$, $(x)_n \in \mathrm{WO}$ and for all $m < n$, $\mathrm{ot}((x)_m) < \mathrm{ot}((x)_n)$. If $x \in \mathrm{WO}^{[\omega]}$, then let $f_x \in [\omega_1]^\omega$ be defined by $f_x(n) = \mathrm{ot}((x)_n)$. (Every element of $[\omega_1]^\omega$ has a code in $\mathrm{WO}^{[\omega]}$).

Fix throughout, $W \in \mathrm{WO}$ to be a recursive wellordering of ordertype $\omega \cdot \omega$. In context, $\alpha < \omega \cdot \omega$ will refer to the element of ω which corresponds to the ordinal α according to the wellordering W. Similarly in context, $<$ will refer to the wellordering given by W.

Let $\mathrm{WO}^{[\omega\cdot\omega]}$ denote the set of $x \in \mathbb{R}$ so that for all $n \in \omega$, $(x)_n \in \mathrm{WO}$, and for all $\alpha, \beta \in \omega \cdot \omega$, $\alpha < \beta$ if and only if $\mathrm{ot}((x)_\alpha) < \mathrm{ot}((x)_\beta)$. (Here α and β refer to the natural numbers corresponding to α and β, respectively, according to W.) If $x \in \mathrm{WO}^{[\omega\cdot\omega]}$, let $g_x \in [\omega_1]^{\omega\cdot\omega}$ be defined by $g_x(\alpha) = \mathrm{ot}((x)_\alpha)$. Every element of $[\omega_1]^{\omega\cdot\omega}$ is of the form g_x for some $x \in \mathrm{WO}^{[\omega\cdot\omega]}$.

If $x \in \mathrm{WO}^{[\omega\cdot\omega]}$, let $h_x \in [\omega_1]^\omega$ be defined by $h_x(n) = \sup\{g_x(\omega \cdot n + i) : i \in \omega\}$.

If $g \in [\omega_1]^{\omega\cdot\omega}$, then let $\tilde{g} \in [\omega_1]^\omega$ be defined by $\tilde{g}(n) = \sup\{g(\omega \cdot n + i) : i \in \omega\}$.

If $X \subseteq \omega_1$, then WO_X be the collection of $x \in \mathrm{WO}$ so that $\mathrm{ot}(x) \in X$. $\mathrm{WO}_X^{[\omega]}$ and $\mathrm{WO}_X^{[\omega\cdot\omega]}$ are defined as above with WO replaced by WO_X.

Suppose $C \subseteq \omega_1$ is closed and unbounded. Let

$$_\omega C = \{\sup\{C(\omega \cdot \alpha + i) : i \in \omega\} : \alpha < \omega_1\}.$$

A C-witness for a $f \in [_\omega C]^\omega$, is a function $g \in [C]^{\omega\cdot\omega}$ so that $\tilde{g} = f$. A C-code for a function $f \in [_\omega C]^\omega$ is an $x \in \mathrm{WO}_C^{[\omega\cdot\omega]}$ so that $h_x = f$. So a code in $\mathrm{WO}_C^{[\omega\cdot\omega]}$ for any C-witness for f is a C-code. Every $f \in [_\omega C]^\omega$ has a C-code.

Theorem 7.4. (AD) *Let $R \subseteq [\omega_1]^\omega \times \mathbb{R}$ be such that for all $f \in [\omega_1]^\omega$, $R_f := \{x \in \mathbb{R} : (f, x) \in R\}$ is nonempty. There exists a $\sigma \in \mathbb{R}$ and a closed and unbounded $C \subseteq \omega_1$, so that for all $x \in \mathrm{WO}_C^{[\omega\cdot\omega]}$ so that $h_x \in [_\omega C]^\omega$, there is some $z \leq_T x \oplus \sigma$ so that $R(h_x, z)$. Moreover there is some formula φ so that for all $x \in \mathrm{WO}_C^{[\omega\cdot\omega]}$ with $h_x \in [_\omega C]^\omega$ and $z \in \mathbb{R}$, $L[\sigma, x, z] \models \varphi(\sigma, x, y)$ implies $R(h_x, z)$.*

Assume further that for all $f \in [\omega_1]^\omega$, $|R_f| \leq \aleph_0$. Then there exists some uncountable $X \subseteq \omega_1$ and function Ψ which uniformizes R on $[X]^\omega$: For $f \in [X]^\omega$, $R(f, \Psi(f))$.

Proof. The first half of this argument is similar to Martin's proof of the partition relation $\omega \to (\omega_1)_2^\omega$. The second half is similar to the proof of Woodin's countable section uniformization for relations on $\mathbb{R} \times \mathbb{R}$ as exposited in [13].

Consider the following game: Player 1 and 2 take turns playing integers. Player 1 produces a real x. Player 2 produces two reals y and z.

Player 1	x
Player 2	y, z

(As before, ordinals $\alpha < \omega \cdot \omega$ are considered natural numbers according to the fixed wellordering W when required in context.)

(Case A) If there is a least $\alpha < \omega \cdot \omega$ so that
(i) $(x)_\alpha \notin \mathrm{WO}$ or $\mathrm{ot}((x)_\alpha) \leq \sup\{\mathrm{ot}((x)_\beta), \mathrm{ot}((y)_\beta) : \beta < \alpha\}$ or
(ii) $(y)_\alpha \notin \mathrm{WO}$ or $\mathrm{ot}((y)_\alpha) \leq \sup\{\mathrm{ot}((x)_\beta), \mathrm{ot}((y)_\beta) : \beta < \alpha\}$.
Player 2 wins if and only if (i) holds.

(This means that one of the two players fails to ensure that every section of its own real codes a wellordering or fails to produce a wellordering larger than the wellordering of all the previous sections of both reals. In this case Player 2 wins if and only if Player 1 is the first to fail in this manner.)

(Case B) Suppose there is no such α as above. Define $h \in [\omega_1]^\omega$ by $h(\alpha) = \sup\{\mathrm{ot}((x)_\alpha), \mathrm{ot}((y)_\alpha)\}$. Player 1 wins if $\neg R(\tilde{h}, z)$.

This completes the definition of the payoff set of the game.

<u>Claim 1</u>: Player 1 does not have a winning strategy in this game.

Proof. Suppose σ is a winning strategy for Player 1. For each $\alpha < \omega \cdot \omega$ and $\beta < \omega_1$, let B_β^α be the set of $r \in \mathrm{WO}$ so that there exists some $y \in \mathbb{R}$ and $z \in \mathbb{R}$ so that for all $\gamma < \alpha$, $(y)_\gamma \in \mathrm{WO}$, $\sup\{\mathrm{ot}((y)_\gamma) : \gamma < \alpha\} < \beta$, and if x is the result of Player 1 in $\sigma * (y, z)$, then $r = (x)_\alpha$. B_β^α is $\boldsymbol{\Sigma}_1^1$ (using a code for β as a parameter). By the boundedness principle, there is some $\delta_\beta^\alpha < \omega_1$ so that for all $r \in B_\beta^\alpha$, $\mathrm{ot}(r) < \delta_\beta^\alpha$. Let C be the set of $\eta < \omega$ so that for all $\alpha < \omega \cdot \omega$ and $\beta < \eta$, $\delta_\beta^\alpha < \eta$.

Let $f \in [{}_\omega C]^\omega$ and z be such that $R(f, z)$. Pick some $g \in [C]^{\omega\cdot\omega}$ so that $\tilde{g} = f$. Let $y \in \mathrm{WO}_C^{[\omega\cdot\omega]}$ be such that $g_y = g$. Let $z \in R_f$. Play $\sigma * (y, z)$. Let x be the response produced by Player 1 according to σ. Define $h(\alpha) = \sup\{\mathrm{ot}((x)_\alpha), \mathrm{ot}((y)_\alpha)\}$. By definition of C, $\mathrm{ot}((x)_\alpha) < \mathrm{ot}((y)_\alpha)$. Thus $\tilde{h} = \tilde{g} = f$. Then $R(\tilde{h}, z)$. Player 2 won. This contradicts σ being a winning strategy for Player 1. This completes the proof of Claim 1. □

Now suppose that σ is a winning strategy for Player 2. For each $\alpha < \omega\cdot\omega$ and $\beta < \omega_1$, let B^α_β be the set of $r \in \mathrm{WO}$ so that there exists some $x \in \mathbb{R}$ so that for all $\gamma \leq \alpha$, $(x)_\gamma \in \mathrm{WO}$, $\sup\{\mathrm{ot}((x)_\gamma) : \gamma < \alpha\} < \beta$, and if (y, z) is the response of Player 2 from $x * \sigma$, then $r = (y)_\alpha$. Each $B^\alpha_\beta \subseteq \mathrm{WO}$ and is $\mathbf{\Sigma}^1_1$. By the boundedness lemma, there is a least ordinal δ^α_β so that for all $r \in B^\alpha_\beta$, $\mathrm{ot}(r) < \delta^\alpha_\beta$. Let C be the closed and unbounded set of η so that for all $\alpha < \omega\cdot\omega$ and $\beta < \eta$, $\delta^\alpha_\beta < \eta$.

Now suppose $x \in \mathrm{WO}_C^{[\omega\times\omega]}$ be such that $h_x \in [{}_\omega C]^\omega$. Use the player 2 strategy σ to play against x to produce the play $x*\sigma$. Let (y, z) be Player 2's response from the play $x*\sigma$. Let $h(\alpha) = \sup\{\mathrm{ot}((x)_\alpha), \mathrm{ot}((y)_\alpha)\}$. Using the definition of C as before, $\tilde{h} = h_x$. Since σ is winning for Player 2, one has that $R(h_x, z)$. Note that $z \leq_T x \oplus \sigma$. From this description, one can allow $\varphi(\sigma, x, z)$ to be the formula that asserts that there is some y so that (y, z) is Player 2's response in the play $x * \sigma$.

Now to prove the uniformization result: Assume that for all $f \in [\omega_1]^\omega$, $|R_f| \leq \aleph_0$. Let C be the club set from above. By a result of Solovay [14, Lemma 2.8], there is some $w \in \mathbb{R}$ so that C is definable in $L[w]$ from some fixed formula using w. Hence ${}_\omega C$ is also definable in $L[w]$. Now suppose that $f \in [{}_\omega C]^\omega$. Let $x \in \mathrm{WO}_C^{[\omega\cdot\omega]}$ be such that $h_x = f$. Suppose (y, z) is Player 2's response in the play $x * \sigma$.

Fix some $X \geq_T [x]_T$. In $L[\sigma, w, f, X]$, define the condition $p \in {}_2\mathbb{O}^{L[\sigma,w,X]}_{\sigma,w,f,X}$ by

$$p = \{(a, b) \in \mathbb{R}^2 : L[\sigma, w, f, a, b] \models \psi(\sigma, w, a, b)\}$$

where $\psi(\sigma, w, f, a, b)$ asserts that a is a C-code for f and $\varphi(\sigma, a, b)$. Note that one uses the real w to speak about C in $L[\sigma, w, f, a, b]$. Note that $p \neq \emptyset$ since $(x, z) \in p$. (Note that $z \leq_T \sigma \oplus x$ and hence $z \in L[\sigma, w, f, X]$.) This shows that p is indeed a condition in ${}_2\mathbb{O}^{L[\sigma,w,f,X]}_{\sigma,w,f}$.

Let τ denote the canonical name for the generic element of $\mathbb{R}^2$ added by ${}_2\mathbb{O}^{L[\sigma,w,f,X]}_{\sigma,w,f}$.

<u>Claim 2</u>: There is a dense set of conditions below p that determines the value of the second coordinate of τ.

Proof. Suppose not. Let $p' \leq p$ be some condition so that no $q \leq p'$ determines the second coordinate of τ. Since AD holds and $\mathrm{HOD}^{L[\sigma,w,f,X]}_{\sigma,w,f} \models \mathsf{AC}$, $(\mathscr{P}({}_2\mathbb{O}^{L[\sigma,w,f,X]}_{\sigma,w,f}))^{\mathrm{HOD}^{L[\sigma,w,f,X]}_{\sigma,w,f}}$ is countable in V. In V, let $\langle D_n : n \in \omega\rangle$ enumerate all the dense open subsets of ${}_2\mathbb{O}^{L[\sigma,f,w]}_{\sigma,f,w}$ that belong to $\mathrm{HOD}^{L[\sigma,f,w,X]}_{\sigma,f,w}$. Let $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the second coordinate.

Let $p_\emptyset \leq p'$ be the least element below p meeting D_0 according to the canonical wellordering of $\mathrm{HOD}^{L[\sigma,w,f,X]}_{\sigma,w,f}$. Let $m_\emptyset = 0$. Suppose for some $\sigma \in {}^{<\omega}2$, p_σ and m_σ have been defined. Since $p_\sigma \leq p'$, no condition extending p_σ can determine $\pi_2(\tau)$. Thus there is some $N > m_\sigma$ and some least pair $p_0, p_1 \leq p_\sigma$ so that, $p_0, p_1 \in D_{|\sigma|+1}$, p_0 and p_1 both decides $\pi_2(\tau) \restriction N$ and $p_i \Vdash \pi_2(\tau)(\check{N}) = \check{i}$ (that is, decides the value at N differently). Let $m_{\sigma\hat{}i} = N + 1$ and $p_{\sigma\hat{}i} = p_i$ for both $i \in 2$. This produces a sequence $\langle p_\sigma : \sigma \in {}^{<\omega}2\rangle$.

For each $r \in \mathbb{R} = {}^\omega 2$, let G_r be the upward closure of $\{p_{r\restriction n} : n \in \omega\}$. G_r is a ${}_2\mathbb{O}^{L[\sigma,w,f,X]}_{\sigma,w,f}$-generic filter over $\mathrm{HOD}^{L[\sigma,w,f,X]}_{\sigma,w,f}$. Also by construction, if $r \neq s$, then $\pi_2(\tau)[G_r] \neq \pi_2(\tau)[G_s]$. For all $r \in \mathbb{R}$, $p \in G_r$. Since p is a condition of the form to apply Fact 2.9, one has that $\mathrm{HOD}^{L[\sigma,w,f,X]}_{\sigma,w,f}[G_r] \models L[\sigma, w, f, \pi_1(\tau[G_r]), \pi_2(\tau[G_r])] \models \psi(\sigma, w, f, \pi_1(\tau[G_r]), \pi_2(\tau[G_r]))$. By the absoluteness of the coding, $\pi_1(\tau[G_f])$ is a C-code for f in V. By the property of the formula φ (namely its upward absoluteness), one has that $R(f, \pi_2(\tau[G_r]))$ holds in V. Thus it has been shown that for all $r \in \mathbb{R}$, $\pi_2(\tau[G_r]) \in R_f$. This contradicts $|R_f| \leq \aleph_0$. Claim 2 has been proved. □

<u>Claim 3</u>: $\mathrm{HOD}^{L[\sigma,w,f,X]}_{\sigma,w,f} \cap R_f \neq \emptyset$.

Proof. Since $X \geq_T [x]_T$, the real x and its associated $z \leq_T x \oplus \sigma$ (picked above) belong to $L[\sigma, f, w, X]$. By Fact 2.9, let G be the ${}_2\mathbb{O}^{L[\sigma,w,f,X]}_{\sigma,w,f}$-generic filter over $\mathrm{HOD}^{L[\sigma,w,f,X]}_{\sigma,w,f}$ so that $\tau[G] = (x, z)$. Note that $p \in G$. So Fact 2.9 implies that $R(f, z)$. Let D be the dense set below p from Claim 2. By genericity, $D \cap G \neq \emptyset$. Hence there is some $q \in D \cap G$. Since q completely determines $\pi_2(\tau)$, one has that for all $i \in 2$, $z(n) = i$ if and only if $q \Vdash \pi_2(\tau)(n) = i$. Since $q \in {}_2\mathbb{O}^{L[\sigma,w,f,X]}_{\sigma,w,f}$, q is essentially an ordinal. This shows that z is $\mathrm{OD}^{L[\sigma,w,f,X]}_{\sigma,w,f}$. Claim 3 has been proved. $\square$

It has been shown that for all $f \in [{}_\omega C]^\omega$, there is a cone of $X \in \mathcal{D}$ so that $R_f \cap \mathrm{HOD}^{L[\sigma,w,f,X]}_{\sigma,w,f} \neq \emptyset$.

<u>Claim 4</u>: There is a function $\Phi : [{}_\omega C]^\omega \to \mathbb{R}$ that uniformizes R on $[{}_\omega C]^\omega$.

Proof. Fix an $f \in [{}_\omega C]^\omega$. Let $Y \in \mathcal{D}$ be a base of a cone of $X \in \mathcal{D}$ so that $R_f \cap \mathrm{HOD}^{L[\sigma,w,f,X]}_{\sigma,w,f} \neq \emptyset$. For each $n \in \omega$ and $i \in 2$, let E^i_n be the set of $X \in \mathcal{D}$ such that $X \geq_T Y$ and if $z \in \mathbb{R}$ is the least element of $\mathrm{HOD}^{L[\sigma,w,f,X]}_{\sigma,w,f}$ belonging to R_f, then $z(n) = i$. Since $E^0_n \cap E^1_n = \emptyset$, $E^0_n \cup E^1_n$ is the set of all degrees above X, and Martin's measure μ is an ultrafilter, there is some a_n so that $E^{a_n}_n \in \mu$. Let $\Phi(f) \in \mathbb{R}$ be defined by $\Phi(f)(n) = a_n$. Using $\mathrm{AC}^{\mathbb{R}}_\omega$, one can find a sequence of reals $\langle x_i : i \in \omega\rangle$ so that the cone above $[x_i]_T$ is contained in $E^{a_i}_i$ for each $i \in \omega$. If $X \geq_T [\bigoplus_{i\in\omega} x_i]_T$, then $\Phi(f)$ is the $\mathrm{HOD}^{L[\sigma,w,f,X]}_{\sigma,w,f}$-least element of $R_f \cap \mathrm{HOD}^{L[\sigma,w,f,X]}_{\sigma,w,f}$. In particular, $R(f, \Phi(f))$. $\square$

The proof of the theorem is complete. $\square$

Corollary 7.5. ($\mathsf{ZF} + \mathsf{AD}$) *Let $\langle E_\alpha : \alpha < \kappa\rangle$ be a sequence of equivalence relations on $\mathbb{R}$ with all classes countable, then $[\omega_1]^\omega$ does not inject into $\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$.*

Proof. This follows from Fact 7.1 and Theorem 7.4. $\square$

The following result states that under AD, given any arbitrary function $\Phi : [\omega_1]^\omega \to \mathbb{R}$, one can find two reals σ and w and a set $X \subseteq \omega_1$ with $|X| = \omega_1$ so that for all $f \in [X]^\omega$, $\Phi(f)$ is constructible from σ, w, and f.

Theorem 7.6. ($\mathsf{ZF} + \mathsf{AD}$). *Let $\Phi : [\omega_1]^\omega \to \mathbb{R}$ be a function. Then there is an uncountable $X \subseteq \omega_1$, reals $\sigma, w \in \mathbb{R}$, and a formula ϕ so that for all $f \in [X]^\omega$, $\Phi(f) \in L[\sigma, w, f]$ and for all $z \in \mathbb{R}$, $z = \Phi(f)$ if and only if $L[\sigma, w, f, z] \models \phi(\sigma, w, f, z)$.*

Proof. Treating Φ as a relation, run the same argument as in Theorem 7.4. This produces the Player 2 winning strategy σ and a closed and unbounded set C. Let $X = {}_\omega C$. Using AD and a result of Solovay [14, Lemma 2.8], let w be a real so that C can be defined in $L[w]$.

Let $f \in [X]^\omega$. A C-code for f exists in any $\mathrm{Coll}(\omega, \sup(f))$-generic extension of $L[\sigma, w, f]$. Let τ be a homogeneous name for a C-code for f. Homogeneous here means that for any formula ς, either $L[\sigma, w, f] \models 1_{\mathrm{Coll}(\omega,\sup(f))} \Vdash \varsigma(\tau)$ or $L[\sigma, w, f] \models 1_{\mathrm{Coll}(\omega,\sup(f))} \Vdash \neg\varsigma(\tau)$. (The real w is needed to speak about C in $L[\sigma, w, f]$.) Define a real z by: $n \in z$ if and only $1_{\mathrm{Coll}}(\omega, \sup(f))$ forces that when Player 2 plays σ against τ, if (y', z') is the response from Player 2, then $n \in z'$. By the homogeneity of $\mathrm{Coll}(\omega, \sup(f))$ and the fact that Φ is a function, one can show that $1_{\mathrm{Coll}(\omega,\sup(f))}$ either forces the statement above or its negation. By the definability of the forcing relation $z \in L[\sigma, w, f]$. By definition of the game, $z = \Phi(f)$. The description above also provides the formula ϕ. $\square$

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