THE DESTRUCTION OF THE AXIOM OF DETERMINACY BY FORCINGS ON $\mathbb{R}$ WHEN Θ IS REGULAR

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ABSTRACT. $\mathsf{ZF} + \mathsf{AD}$ proves that for all nontrivial forcings $\mathbb{P}$ on a wellorderable set of cardinality less than Θ, $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \neg\mathsf{AD}$. $\mathsf{ZF} + \mathsf{AD} + \Theta$ is regular proves that for all nontrivial forcing $\mathbb{P}$ which is a surjective image of $\mathbb{R}$, $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \neg\mathsf{AD}$. In particular, $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$ proves that for every nontrivial forcing $\mathbb{P} \in L_{\Theta}(\mathbb{R})$, $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \neg\mathsf{AD}$.

1. INTRODUCTION

Paul Cohen [1] developed forcing which is a very flexible method of extending models of certain axioms of set theory (such as ZF or ZFC) so that the resulting structures continue to satisfy these axioms of set theory. This technique has become a powerful tool for showing statements are independent of ZFC. For example by [1], if ZFC is consistent, then $\mathsf{ZFC} + 2^{\aleph_0} = \aleph_1$ and $\mathsf{ZFC} + 2^{\aleph_0} > \aleph_1$ are both consistent.

Descriptive set theory is the study of the definable aspects of mathematics. Various interesting properties are commonly considered while employing definable techniques to study mathematical problems. Some of these include the perfect set property, Lebesgue measurability, the property of Baire, partition relations on ordinals, and certain properties of Turing degrees. These properties in their full generality are all incompatible with AC, the axiom of choice.

These properties are interesting and appeared naturally in classical descriptive set theory. Definable instances of these properties are provable in more basic axiom systems such as ZF, in the same way that definable instances of the axiom of choice, for example, coanalytic uniformization, is provable in ZF. This analogy justifies the study of the consequences of the full generalization of these properties just as one does with AC, the full generalization of definable selection principles.

The axiom of determinacy, AD, has developed into a comprehensive framework for studying the properties mentioned above in their full generality. As customary in descriptive set theory, $\mathbb{R}$ will denote the Baire space, ${}^{\omega}\omega$, of functions from ω into ω. For each $A \subseteq \mathbb{R}$, let G_A be the game where Player 1 and 2 take turns playing elements of ω. After infinitely many stages, a single $f \in \mathbb{R}$ has been produced. Player 1 wins this run of G_A if and only if $f \in A$. The axiom of determinacy states that for all $A \subseteq \mathbb{R}$, one of the two players has a winning strategy for G_A. AD implies the perfect set property, Lebesgue measurability, Baire property for all sets of reals, and there are many cardinals with various partition properties. As with all these properties, definable fragments of AD can be proved in ZF, for example, Martin showed all games G_A where A is Borel are determined under ZF.

One can wonder if the forcing construction which has been fruitful for studying consistency results over ZFC could be useful for AD. The most basic question would be to understand what forcings over AD could preserve AD. By the nature of AD, if one does not change $\mathbb{R}$ or $\mathscr{P}(\mathbb{R})$, then AD will be preserved. Therefore the question becomes what forcings which disturb $\mathbb{R}$ or $\mathscr{P}(\mathbb{R})$ can still preserve AD.

Ikegami and Trang initiated the study of the preservation of AD under forcing. They showed that many forcings, such as Cohen forcing, can never preserve AD. They also showed that if one is working with natural models of AD, i.e. models satisfying $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$, then any forcing which preserve AD must preserve Θ, where Θ is the supremum of the ordinals which are surjective images of $\mathbb{R}$. They also showed that the consistency of $\mathsf{ZF} + \mathsf{AD}^+ + \Theta > \Theta_0$ implies the consistency of $\mathsf{ZF} + \mathsf{AD}$ and there is a forcing which preserves AD and increases Θ. Thus necessarily this forcing must disturb $\mathscr{P}(\mathbb{R})$ by adding a new set of reals.

The following are some examples of concrete forcings applied within AD. They all destroy the axiom of determinacy for various reasons. These examples give some empirical evidence that most small forcings can not preserve AD and also motivate the general arguments presented throughout the paper.

Let $\mathbb{C}$ denote Cohen forcing. Cohen forcing adds a generic filter which is equiconstructible from a generic real, called the Cohen generic real. It is well known that if g is a Cohen generic real over V, then $V[g] \models$ "$\mathbb{R}^V$ does not have the Baire property". Hence $V[g] \models \neg$AD.

Note that Woodin has shown that if $V \models$ ZFC has a proper class of Woodin cardinals, then for any $\mathbb{P}$ and $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over V, $L(\mathbb{R})^V$ is elementarily equivalent to $L(\mathbb{R})^{V[G]}$. This setting implies that $L(\mathbb{R})^V \models$ AD. Let g be a Cohen real which is generic over V. Note $g \in L(\mathbb{R})^{V[g]}$ and by Woodin's result, $L(\mathbb{R})^{V[g]} \models$ AD. However, $L(\mathbb{R})[g] \models \neg$AD by the result mentioned in the previous paragraph. Observe that the elements of the ground model always belong to its forcing extension. Thus $\mathbb{R}^V \in L(\mathbb{R})[g]$; however, $\mathbb{R}^V \notin L(\mathbb{R})^{V[g]}$.

Assume ZF + $\mathsf{DC}_\mathbb{R}$. Let $\mathrm{Coll}(\omega_1, \omega_2)$ be the forcing consisting of countable partial functions from ω_1 into ω_2 ordered by reverse extension. Suppose there was a $G \subseteq \mathrm{Coll}(\omega_1, \omega_2)$ generic over V such that $V[G] \models$ AD. Since $\mathrm{Coll}(\omega_1, \omega_2)$ is countably closed and $\mathsf{DC}_\mathbb{R}$ holds, no new reals are added. $V[G]$ has a surjection of ω_1^V onto ω_2^V. Thus there is a new subset of ω_1^V which codes an ordering of ω_1^V of length ω_2^V. In V, let $\pi : \mathbb{R}^V \to \omega_1^V$ be a surjection. By the Moschovakis coding lemma in $V[G] \models$ AD, there is some real which codes this new subset of ω_1^V with respect to π. This is impossible if there are no new reals. Thus $V[G]$ can not satisfy AD.

Suppose κ is a cardinal. The partition relation $\kappa \to (\kappa)_2^\lambda$ is the statement that for all $\Phi : [\kappa]^\lambda \to 2$, there is a club $C \subseteq \kappa$ and an $i \in 2$ so that $\Phi(f) = i$ for all $f \in [\kappa]^\lambda$ of the correct type. The notion of correct type will be defined below and is needed to obtain a club set which is homogeneous. Martin showed that $\omega_1 \to (\omega_1)_2^{\omega_1}$ holds under AD.

Revisiting the Cohen forcing $\mathbb{C}$: Assume ZF. Suppose there was a Cohen generic real g over V such that $V[g] \models$ AD. Since $|\mathbb{C}| = \aleph_0$, every $D \subseteq \omega_1$ in $V[g]$ has a $C \in V$ which is a club subset of ω_1 so that $V[g] \models C \subseteq D$. (This phenomenon will be called the ground club property.) Note that $([\omega_1]^{\omega_1})^V \in V[g]$, so one may define a partition $\Phi : [\omega_1]^\omega \to 2$ in $V[g]$ as follows:

$$\Phi(f) = \begin{cases} 0 & f \in ([\omega_1]^\omega)^V \\ 1 & \text{otherwise} \end{cases}.$$

$V[g] \models$ AD, so by $\omega_1 \to (\omega_1)_2^\omega$, let $D \subseteq \omega_1$ be a club set homogeneous for Φ. Let $C \subseteq D$ be a club in V so that $V[g] \models C \subseteq D$. Taking any $f \in ([C]^\omega)^V$ of the correct type, one can show that in $V[g]$, C is homogeneous for Φ taking value 0. Let c_i denote the $(\omega \cdot i + \omega)^{\text{th}}$ element of C. As $C \in V$, $\langle c_i : i \in \omega \rangle \in V$. Pick $z \in \mathscr{P}(\omega)^{V[g]}$. Let $f_z \in ([C]^\omega)^{V[g]}$ be defined by letting f_z be the increasing enumeration of $\{c_i : i \in z\}$. The function f_z is of the correct type so $\Phi(f_z) = 0$. Thus $f_z \in V$. Since $z = \{i \in \omega : c_i \in f_z\}$, one has that $z \in V$. It has been shown that $\mathbb{R}^V = \mathbb{R}^{V[g]}$ which is impossible since $g \in \mathbb{R}^{V[g]} \setminus \mathbb{R}^V$.

These examples suggest that "small" nontrivial forcings should not be able to preserve AD. The examples also seem to indicate that the partition property and the ground model club phenomenon appears to be common aspects of these arguments.

The axiom of determinacy by its definition influences the sets which are surjective images of $\mathbb{R}$. It is reasonable to ask whether a nontrivial forcing which itself is within the realm of determinacy (i.e. is a surjective image of $\mathbb{R}$) must disturb $\mathbb{R}$ or $\mathscr{P}(\mathbb{R})$ and if so, can it preserve AD. More specifically, if $V \models$ AD, $L(\mathbb{R})$ is the smallest model of determinacy containing $\mathbb{R}^V$. One can ask if in $L(\mathbb{R})$, which is the most natural model of AD, can a nontrivial forcing within the realm of determinacy, i.e. in $L_\Theta(\mathbb{R})$, preserve AD. The following are the main questions:

Question 1.1. Assume ZF + AD. If $\mathbb{P}$ is a nontrivial forcing which is a surjective image of $\mathbb{R}$, is it possible that $1_\mathbb{P} \Vdash_\mathbb{P}$ AD?

Assume ZF + AD + $V = L(\mathbb{R})$. Is there any nontrivial $\mathbb{P}$ which is a surjective image of $\mathbb{R}$ so that $1_\mathbb{P} \Vdash_\mathbb{P}$ AD?

The first question will be answered negatively if the assumptions are augmented with the condition that Θ is regular. Since Θ is regular in $L(\mathbb{R})$, this immediately gives the negative answer to the second question.

The results of the paper are the following:

Theorem 3.2. *Assume* ZF. *If* $\mathbb{P}$ *is a nontrivial wellorderable forcing of cardinality less than* Θ, *then* $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \neg$AD.

The argument of the above theorem serves as a template for the main result. Its proof is a generalization of the example involving Cohen forcing. In discussion with Goldberg, a stronger result for wellorderable forcing can be shown using different techniques:

Theorem 3.5. *Assume* ZF + AD. *If* $\mathbb{P}$ *is a wellorderable forcing which adds a new real, the* $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \neg$AD.

The main results are:

Theorem 5.5. *Assume* ZF + AD+ Θ *is regular. Suppose* $\mathbb{P}$ *is a nontrivial forcing which is a surjective image of* $\mathbb{R}$. *Then* $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \neg$AD.

Corollary 5.6. *Assume* ZF + AD + V = L$(\mathbb{R})$. *No nontrivial forcing* $\mathbb{P} \in L_{\Theta}(\mathbb{R})$ *can preserve* AD.
   In fact, assume ZF + AD^{+} + $\neg$AD$_{\mathbb{R}}$ + V = L$(\mathscr{P}(\mathbb{R}))$. *No nontrivial forcing which is the surjective image of* $\mathbb{R}$ *can preserve* AD.

2. Ground Club Property

Recall that if $A \subseteq \mathbb{R} \times \mathbb{R}^n$ and $e \in \mathbb{R}$, $A_e = \{x \in \mathbb{R}^n : (e, x) \in A\}$.

Fact 2.1. *(Moschovakis) Assume* ZF + AD. *Let* Γ *be a nonselfdual pointclass closed under continuous substitution,* $\exists^{\mathbb{R}}$, $\wedge$, *and* $\mathbf{\Sigma}_1^1 \subseteq \Gamma$. *Let* $\prec \in \Gamma$ *be a strict prewellordering. For each* $a \in \mathrm{dom}(\prec)$, *let* $Q_a = \{b \in \mathrm{dom}(\prec) : a \preceq b \wedge b \preceq a\}$. *Let* $U \subseteq \mathbb{R}^3$ *be a* Γ-*universal set for subsets of* $\mathbb{R}^2$ *in* Γ. *Let* $Z \subseteq \mathrm{dom}(\prec) \times \mathbb{R}$. *Then there is an* $e \in \mathbb{R}$ *so that*
(1) $U_e \subseteq Z$.
(2) For all $a \in \mathrm{dom}(\prec)$, $(U_e)_a \neq \emptyset$ *if and only if* $Z_a \neq \emptyset$.

Proof. See [6] Section 7D. □

Fact 2.2. *Assume* ZF + AD. *Let* $X \subseteq \mathbb{R}$ *and* $\pi : X \to \kappa$ *be a surjection. Let* $\prec$ *be a strict prewellordering on* X *defined by* $x \prec y$ *if and only if* $\pi(x) < \pi(y)$. *Let* Γ *be a nonselfdual pointclass closed under continuous substitution,* $\exists^{\mathbb{R}}$, $\wedge$, $\prec \in \Gamma$, *and* $\mathbf{\Sigma}_1^1 \subseteq \Gamma$. *Let* U *be a fixed* Γ-*universal set for subsets of* $\mathbb{R}^2$ *in* Γ. *For each* $e \in \mathbb{R}$, *let* $S_e^{\pi} = \{\alpha < \kappa : (\exists a)(\pi(a) = \alpha \wedge U_e(a, 0))\}$.
   For all $C \subseteq \kappa$, *there is some* $e \in \mathbb{R}$ *so that* $S_e^{\pi} = C$.

Proof. Let $Z = \{(a, 0) : a \in X \wedge \pi(a) \in C\}$. Apply Fact 2.1. □

Definition 2.3. Assume ZF + AD. Let $A \subseteq \mathbb{R}$. Let δ_A be the least ordinal δ so that $L_{\delta}(A, \mathbb{R}) \prec_1 L(A, \mathbb{R})$, where $\prec_1$ denotes Σ_1 elementarity in a language that includes a predicate $\dot{A}$ and $\dot{\mathbb{R}}$, which are always interpreted as A and $\mathbb{R}$, respectively. It is also the least ordinal δ so that $L_{\delta}(A, \mathbb{R})$ is an elementary substructure of $L(A, \mathbb{R})$ with respect Σ_1 formulas in the above language using elements of $\mathbb{R}$, $\mathbb{R}$ itself, and A as parameters.
   Let $\Sigma_1(L(A, \mathbb{R}), \mathbb{R} \cup \{\mathbb{R}, A\})$ be the collection of sets in $L(A, \mathbb{R})$ which are Σ_1 definable in $L(A, \mathbb{R})$ using elements of $\mathbb{R}$, $\mathbb{R}$ itself, and A as parameters.

Definition 2.4. Following [4] Section 2.4 and 2.5, the following is an explicit prewellordering of a subset of $\mathbb{R}$ of length δ_A which is $\Sigma_1(L(A, \mathbb{R}), \mathbb{R} \cup \{\mathbb{R}, A\})$:
   Let T be the theory consisting of ZF without the power set axiom, "$\mathbb{R}$ exists", and countable choice for $\mathbb{R}$.
   Let $\varphi_A(x, A, \dot{\mathbb{R}})$ denote a Σ_1 formula that defines the $\Sigma_1(L(A, \dot{\mathbb{R}}), \{A, \dot{\mathbb{R}}\})$ set, denoted U_A, which is universal for $\Sigma_1(L(A, \dot{\mathbb{R}}), \dot{\mathbb{R}} \cup \{A, \dot{\mathbb{R}}\})$. For $x \in U_A$, let Θ_x be the least ordinal so that $L_{\Theta_x}(A, \dot{\mathbb{R}}) \models T$ and $L_{\Theta_x}(A, \dot{\mathbb{R}}) \models \varphi_A(x, A, \dot{\mathbb{R}})$. Define $\tilde{\rho}_A(x) = (\delta_A)^{L_{\Theta_x}(A, \dot{\mathbb{R}})}$. Let $\iota_A : \tilde{\rho}_A[U_A] \to \delta_A$ be the transitive collapse of $\tilde{\rho}[U_A]$. Let $\rho_A = \iota_A \circ \tilde{\rho}_A$. ρ_A is a $\Sigma_1(L(A, \dot{\mathbb{R}}), \{A, \dot{\mathbb{R}}\})$ surjection of U_A onto δ_A. In applications of the coding

lemma throughout the paper, the prewellordering and universal set used will always be the ones produced above.

Therefore there is a Σ_1 formula $\varsigma(\alpha, e, A, \dot{\mathbb{R}})$ so that for all $\alpha < \delta_A$, $L(A, \dot{\mathbb{R}}) \models \alpha \in S_e^{\rho_A} \Leftrightarrow \varsigma(\alpha, e, A, \dot{\mathbb{R}})$.

Definition 2.5. A function $f : \lambda \to \mathrm{ON}$ has uniform cofinality ω if and only if there is a $g : \lambda \times \omega \to \mathrm{ON}$ with the property that for all $\alpha < \lambda$ and $n \in \omega$, $g(\alpha, n) < g(\alpha, n+1)$ and $f(\alpha) = \sup\{g(\alpha, n) : n \in \omega\}$.

A function $f : \lambda \to \mathrm{ON}$ is of the correct type if and only if f is strictly increasing, for all $\alpha < \lambda$, $f(\alpha) > \sup\{f(\beta) : \beta < \alpha\}$, and f has uniform cofinality ω.

Let κ be an ordinal. For ordinals $\lambda \leq \kappa$, let $\kappa \to (\kappa)_2^\lambda$ denote that for all $\Phi : [\kappa]^\lambda \to 2$, there is a club $C \subseteq \kappa$ and $i \in 2$ so that for all $f : \lambda \to C$ of the correct type, $\Phi(f) = i$.

If $\kappa \to (\kappa)_2^\kappa$, then one says that κ has the strong partition property. If for all $\eta < \kappa$, $\kappa \to (\kappa)_2^\eta$, then κ is said to have the weak partition property. (Note that $\kappa \to (\kappa)_2^2$ implies that κ is regular.)

Fact 2.6. *Assume* ZF + AD. *Let* $A \subseteq \mathbb{R}$. *Then* δ_A *has the strong partition property in* $L(A, \mathbb{R})$ *and even in* V.

Proof. This is shown by following Martin's template for establishing partition properties. The reflection properties and the uniform coding lemma is used to produce a good coding system for functions $f : \delta_A \to \delta_A$. See [2] for more details. See [5] for the details of this specific result. $\square$

Definition 2.7. The ordinal Θ is the supremum of the ordinals which are surjective images of $\mathbb{R}$.

For $A, B \in \mathscr{P}(\mathbb{R})$, $A \leq_w B$ denotes that A is Wadge reducible to B. For each $r \in \mathbb{R}$, let Ξ_r denote the Wadge reduction coded by r. So $\Xi_r^{-1}[B]$ is the subset of $\mathbb{R}$ reducible to B via the Wadge reduction coded by r.

The Wadge lemma states that ZF + AD implies that for all $A, B \in \mathscr{P}(\mathbb{R})$ either $A \leq_w B$ or $B \leq_w (\mathbb{R} \setminus A)$.

Fact 2.8. *([3]) Assume* ZF + AD. *For all* $\lambda < \Theta$, *there exists some* κ *with* $\lambda < \kappa < \Theta$ *so that* κ *has the strong partition property.*

Proof. This result follows from Fact 2.6. [3] works with ZF + DC + AD as its base theory. [5] has a careful presentation of this result from just ZF + AD. $\square$

Definition 2.9. Let κ be a regular cardinal and $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}})$ be a forcing. $\mathbb{P}$ has the ground club property at κ if and only if for all $p \in \mathbb{P}$ and all $\mathbb{P}$-name $\dot{D}$ such that $p \Vdash_{\mathbb{P}}$ "$\dot{D}$ is a club subset of $\check{\kappa}$", there is some club $C \subseteq \kappa$ so that $p \Vdash_{\mathbb{P}} \check{C} \subseteq \dot{D}$.

If $A \subseteq \omega_1$ is an unbounded subset of ω_1, then for each $\alpha < \omega$, let $A(\alpha)$ denote the αth-element of A.

Lemma 2.10. *Assume* ZF. *Let* $\mathbb{P}$ *be a forcing and* $p \in \mathbb{P}$. *If* $\mathbb{P}$ *has the ground club property at* κ *and* $p \Vdash \kappa \to (\kappa)_2^\omega$, *then* $p \Vdash \dot{\mathbb{R}} = \check{\mathbb{R}}$.

Proof. Let $G \subseteq \mathbb{P}$ be any $\mathbb{P}$-generic filter over V containing p. Observe that every set of V belongs to $V[G]$, so in particular, $([\kappa]^\omega)^V \in V[G]$.

In $V[G]$, define $\Phi : [\kappa]^\omega \to 2$ by

$$\Phi(f) = \begin{cases} 0 & f \in ([\kappa]^\omega)^V \\ 1 & \text{otherwise} \end{cases}.$$

Let $D \subseteq \kappa$ be a club set homogeneous for Φ. By the ground club property at κ, there is some $C \subseteq D$ with $C \in V$ and is a club in V. Pick any $f \in ([C]^\omega)^V$ of correct type. Then $\Phi(f) = 0$. Thus D is homogeneous for Φ taking value 0. Therefore C is also homogeneous for Φ taking value 0. Any function $f \in ([C]^\omega)^{V[G]}$ of the correct type belongs to V.

Let $c_i = C(\omega \cdot i + \omega)$. Since $C \in V$, the sequence $(c_i : i \in \omega)$ belongs to V. Each $c_i \in C$ since C is club and each c_i has cofinality ω. Let $z \in \mathbb{R}^{V[G]}$. Let $f_z = \{c_i : i \in z\}$. Then $f_z \in [C]^\omega$ and is of correct type. So $\Phi(f_z) = 0$. $f_z \in V$. Then $z = \{i \in \omega : c_i \in f_z\}$. So $z \in \mathbb{R}^V$. $\square$

3. WELLORDERABLE FORCINGS OF CARDINALITY LESS THAN Θ

This section will show that a nontrivial forcing on a wellorderable set of cardinality less than Θ can not preserve AD. The results of this section are subsumed by the results of Section 5; however, the argument there is far less natural for wellorderable forcings.

Fact 3.1. *Assume* ZF. *Let* $\mathbb{P}$ *be a wellorderable forcing of size* λ. *Then* $\mathbb{P}$ *has the ground club property at* κ *for all regular* $\kappa > \lambda$.

Proof. Let $r \in \mathbb{P}$ and $\dot{D}$ be a $\mathbb{P}$ name such that $r \Vdash_\mathbb{P}$ "$\dot{D} \subseteq \check{\kappa}$ is a club". For each $\alpha < \kappa$, let $A_\alpha = \{p \in \mathbb{P} : (\exists \beta < \kappa)(p \Vdash_\mathbb{P} \dot{D}(\check{\alpha}) = \check{\beta})\}$. Let $B_\alpha = \{\beta : (\exists p \in A_\alpha)(p \Vdash_\mathbb{P} \dot{D}(\check{\alpha}) = \check{\beta})\}$. Since $|A_\alpha| \leq |\mathbb{P}| = \lambda < \kappa$ and κ is regular, $\sup B_\alpha < \kappa$. Let $F(\alpha) = \sup B_\alpha$. Note that $F(\alpha) \geq \alpha$ since $1_\mathbb{P} \Vdash_\mathbb{P} \dot{D}(\check{\alpha}) \geq \check{\alpha}$. Let $C = \{\alpha < \kappa : (\forall \eta < \alpha)(F(\eta) < \alpha)\}$. C is club.

Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter over V with $r \in G$. Let $D = \dot{D}[G]$. Suppose $\alpha \in C$. Since G is generic, for each $\eta < \alpha$, $G \cap A_\eta \neq \emptyset$. For any $q \in G \cap A_\eta$, $q \Vdash_\mathbb{P} \dot{D}(\check{\eta}) \leq F(\check{\eta}) < \check{\alpha}$. Hence $\eta \leq D(\eta) < \alpha$ for all $\eta < \alpha$. Since D is a club, $\alpha \in D$. This shows $C \subseteq D$ in $V[G]$. Since G was arbitrary with $r \in G$, $r \Vdash_\mathbb{P} \check{C} \subseteq \dot{D}$. $\square$

Theorem 3.2. *Assume* ZF. *If* $\mathbb{P}$ *is a nontrivial wellorderable forcing of cardinality less than* Θ, *then* $1_\mathbb{P} \Vdash_\mathbb{P} \neg$AD.

Proof. Suppose $|\mathbb{P}| = \delta$ where $\delta < \Theta$ is a cardinal. One may assume $\mathbb{P} \subseteq \delta$.

Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter over V. Assume that $V[G] \models$ AD. By Fact 2.8, let κ be a cardinal such that $\delta < \kappa < \Theta^{V[G]}$ and has the strong partition property in $V[G]$. Therefore, κ is regular in V. Let $p \in G$ be such that $p \Vdash \kappa \to (\kappa)_2^\omega$. Since $\delta < \Theta$, let $\pi : \mathbb{R}^V \to \delta$ be a surjection in V. Let Γ be a pointclass and U be a Γ-universal sets satisfying the conditions of Fact 2.2 for the surjection π.

Since $\mathbb{P} \subseteq \delta$, if $\mathbb{P}$ is nontrivial, then G is a new subset of δ. Since $V[G] \models$ AD, there is some $e \in \mathbb{R}^{V[G]}$ so that $S_e^\pi = G$ by Fact 2.2. If $\mathbb{R}^V = \mathbb{R}^{V[G]}$, then this would imply $G \in V$. Hence one must have that $\mathbb{R}^V \subsetneq \mathbb{R}^{V[G]}$.

Fact 3.1 implies that $\mathbb{P}$ has the ground club propery at κ. Lemma 2.10 implies that $p \Vdash_\mathbb{P} \dot{\mathbb{R}} = \check{\mathbb{R}}$. So $\mathbb{R}^V = \mathbb{R}^{V[G]}$. Contradiction. $\square$

The previous theorem illustrates the main ideas to be used in Section 5. The above proof uses the partition property $\kappa \to (\kappa)_2^\omega$. This requires the theorem to be restricted to wellorderable forcings of cardinality less than Θ. In discussion with Goldberg, the following more elementary argument was found which could apply to more wellorderable forcings:

Fact 3.3. *Assume* ZF, $\mathsf{AC}_\omega^\mathbb{R}$ *(the axiom of countable choice for* $\mathbb{R}$*), and all sets of reals have the Baire property and the perfect set property. Let* $\mathbb{P}$ *be a wellorderable forcing such that* $1_\mathbb{P} \Vdash_\mathbb{P} \check{\mathbb{R}} \subsetneq \dot{\mathbb{R}}$ *(adds new reals), then* $1_\mathbb{P} \Vdash_\mathbb{P}$ "$\check{\mathbb{R}}$ *has no perfect subset".*

Proof. Suppose there was a $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over V and $V[G] \models \mathbb{R}^V$ has a perfect subset. In $V[G]$, let T be a perfect tree so that $[T] \subseteq \mathbb{R}^V$. Let $\dot{T}$ be a name for T and $q \in G$ be such that $q \Vdash_\mathbb{P} \dot{T}$ is a perfect tree.

Work in V. For each $p \in \mathbb{P}$, let $A_p = \{x \in \mathbb{R}^V : p \Vdash_\mathbb{P} \check{x} \in [\dot{T}]\}$. Note that if $p \leq_\mathbb{P} q$, then each A_p is closed. To see this: Suppose z is a limit point of A_p. Let H be any $\mathbb{P}$-generic filter over V containing p. Since $[\dot{T}[H]]$ is a closed set and $A_p \subseteq [\dot{T}[H]]$, $z \in [\dot{T}[H]]$. Since H was arbitrary containing p, $p \Vdash_\mathbb{P} \check{z} \in [\dot{T}]$.

Let $q' \leq_\mathbb{P} q$, and note that $q' \Vdash_\mathbb{P} \dot{T}$ is a perfect tree. Observe that in $V[G]$, $[T] \subseteq \bigcup_{p \leq_\mathbb{P} q'} A_p$. Thus in V, $\bigcup_{p \leq_\mathbb{P} q'} A_p$ is an uncountable set. By the perfect set property in V, there is an injection $\Phi : \mathbb{R} \to \bigcup_{p \leq_\mathbb{P} q'} A_p$. Note that $\mathbb{R} = \bigcup_{p \leq_\mathbb{P} q'} \Phi^{-1}[A_p]$, which is a wellordered union of subsets of $\mathbb{R}$. By the Baire property in V for all sets of reals and the Kuratowski-Ulam, a wellordered union of meager sets is meager. Since $\mathbb{R}$ is not meager, there is must be a $p \leq_\mathbb{P} q'$ so that $\Phi^{-1}[A_p]$ is nonmeager and hence uncountable. Thus A_p is uncountable. Since A_p is a closed uncountable set (or by the perfect set property), there is some perfect tree U so that $[U] \subseteq A_p$. Note that for all $t \in U$, $p \Vdash_\mathbb{P} \check{t} \in \dot{T}$. Thus $p \Vdash_\mathbb{P} [\check{U}] \subseteq [\dot{T}]$.

Let D be the collection of $p \leq_\mathbb{P} q$ so that there is a perfect tree $U \in V$ so that $p \Vdash_\mathbb{P} [\check{U}] \subseteq [\dot{T}]$. Since $q' \leq_\mathbb{P} q$ was arbitrary in the above argument, one has shown that D is dense below $q \in G$. By the genericity of G, $G \cap D \neq \emptyset$. Let $p \in D$ and let $U \in V$ be a perfect tree so that $p \Vdash_\mathbb{P} [\check{U}] \subseteq [\dot{T}]$. Since $U \in V$ and $V[G]$ has a new real, $[U]$ must have a new real in $V[G]$. Because $V[G] \models [U] \subseteq [\dot{T}[G]]$ and $[\dot{T}[G]] = [T]$, one has that $[T]$ has a new real. But $[T] \subseteq \mathbb{R}^V$. Contradiction. $\square$

Fact 3.4. (ZF). *Let* $\mathbb{P}$ *be a forcing on a wellorderable set. If* $\mathbb{R}$ *is not wellorderable, then* $1_\mathbb{P} \Vdash_\mathbb{P} \check{\mathbb{R}}$ *is not wellorderable.*

Proof. Since $\mathbb{P}$ is wellorderable, let $|\mathbb{P}| = \delta$ where δ is some ordinal. One may assume $\mathbb{P} \subseteq \delta$. Suppose $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V and $V[G] \models \mathbb{R}^V$ is wellorderable. There is an injection $\Phi : \mathbb{R}^V \to \mathrm{ON}$. Let $\dot{\Phi}$ be a $\mathbb{P}$-name for Φ.

Work in V: For each $r \in \mathbb{R}$, let $A_r = \{\langle p, \beta \rangle : p \Vdash_{\mathbb{P}} \dot{\Phi}(\check{r}) = \check{\beta}\}$, where $\langle \cdot, \cdot \rangle$ denotes a definable bijection of $\mathrm{ON} \times \mathrm{ON}$ with ON. Each $A_r \neq \emptyset$ and if $r \neq s$, then $A_r \cap A_s = \emptyset$. In V, let $\Psi : \mathbb{R} \to \mathrm{ON}$ be defined by $\Psi(r) = \min A_r$. Ψ is an injection and hence $\mathbb{R}^V$ is wellorderable in V. Contradiction. $\square$

Theorem 3.5. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *If* $\mathbb{P}$ *is a wellorderable forcing which adds a new real, then* $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \neg \mathsf{AD}$.

Proof. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. By Fact 3.4, $V[G]$ must think that $\mathbb{R}^V$ is uncountable. By Fact 3.3, $\mathbb{R}^V$ is an uncountable set of reals without the perfect set property. Thus AD must fail. $\square$

Question 3.6. Assume $\mathsf{ZF} + \mathsf{AD}$. Can a nontrivial wellorderable forcing preserve AD?

If $\mathbb{P}$ is a nontrivial wellorderable forcing, then must $\mathbb{P}$ add a new real?

The proof of Theorem 3.2 used the Moschovakis coding lemma to show that nontrivial wellorderable forcing of cardinality less than Θ must add a new real.

4. Preservation of Θ

Trang and Ikegami showed that in natural models of AD^+, every forcing that preserves AD must preserve Θ:

Fact 4.1. *(Ikegami and Trang) Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. *If* $\mathbb{P}$ *is a nontrivial forcing and* $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \mathsf{AD}$, *then* $1_{\mathbb{P}} \Vdash \dot{\Theta} = \Theta^V$.

This section will show under $\mathsf{ZF} + \mathsf{AD}$ that any forcing which is a surjective image of $\mathbb{R}$ that preserves AD must preserve Θ. It will first be shown using Lemma 2.10 that any forcing that adds a new real and preserves AD must preserve Θ.

A nontrivial forcing adds the generic filter as a new object. If $\mathbb{P}$ is a surjective image of $\mathbb{R}$, then a new set of reals must be added. It will then be shown under $\mathsf{ZF} + \mathsf{AD}$ that any nontrivial forcing which is a surjective image of $\mathbb{R}$ which preserves AD must actually add a new real. Hence any nontrival forcing which is a surjective image of $\mathbb{R}$ must preserve Θ.

Lemma 4.3 and Fact 4.4 below have been known to Ikegami and Trang under $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$ for forcings more general than those which are surjective images of $\mathbb{R}$. An important aspect of their argument involves the sharps of sets of reals. It should be noted that the arguments below are for forcings which are surjective images of $\mathbb{R}$ proved under just $\mathsf{ZF} + \mathsf{AD}$ without $\mathsf{DC}_{\mathbb{R}}$. $\mathsf{DC}_{\mathbb{R}}$ is used in some classical arguments to produce sharps of sets of reals and to show the wellfoundedness of the Wadge hierarchy.

Fact 4.2. *Let* $\mathbb{P}$ *be a forcing which is a surjective image of* $\mathbb{R}$. *For each regular* $\kappa \geq \Theta$, $\mathbb{P}$ *has the ground club property at* κ.

Proof. Let $\pi : \mathbb{R} \to \mathbb{P}$ be a surjection. Let $\kappa \geq \Theta$ be regular.

Let $p \in \mathbb{P}$ and $\dot{D}$ be a $\mathbb{P}$-name so that $p \Vdash_{\mathbb{P}}$ "$\dot{D} \subseteq \check{\kappa}$ is a club". For each $\alpha < \kappa$, let $A_\alpha = \{p \in \mathbb{P} : (\exists \beta < \kappa)(p \Vdash_{\mathbb{P}} \dot{D}(\check{\alpha}) = \check{\beta})\}$. Let $B_\alpha = \{\beta : (\exists p \in A_\alpha)(p \Vdash \dot{D}(\check{\alpha}) = \check{\beta})\}$.

One will show that for all $\alpha < \kappa$, $\sup B_\alpha < \kappa$. Fix an $\alpha < \kappa$. Define in V, $\Phi : \mathbb{R} \to \kappa$ by

$$\Phi(r) = \begin{cases} 0 & \pi(r) \notin A_\alpha \\ \beta & \pi(r) \in A_\alpha \wedge \pi(r) \Vdash_{\mathbb{P}} \dot{D}(\check{\alpha}) = \check{\beta} \end{cases}$$

Φ induces a prewellordering on $\mathbb{R}$. Let $\delta < \Theta^V$ be the length of this prewellordering. Hence Φ induces a map $\Psi : \delta \to \kappa$. Since κ is regular in V, Ψ must be bounded below κ. Thus $\sup B_\alpha < \kappa$.

For each $\alpha < \kappa$, let $F(\alpha) = \sup B_\alpha$. Let $C = \{\alpha < \kappa : (\forall \eta < \alpha)(F(\eta) < \alpha)\}$. C is a club subset of κ in V. As in the proof of Fact 3.1, $p \Vdash_{\mathbb{P}} \check{C} \subseteq \dot{D}$. $\square$

Lemma 4.3. *Assume* ZF. *If* $\mathbb{P}$ *is a forcing which is a surjective image of* $\mathbb{R}$ *and adds a new real, then* $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \mathsf{AD}$ *implies that* $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \Theta = \Theta^V$.

Proof. Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter over $\mathbb{P}$. Suppose $V[G] \models \mathsf{AD}$ and $\Theta^{V[G]} > \Theta^V$.

By Fact 2.8 applied in $V[G]$, there is a κ such that $\Theta^V < \kappa < \Theta^{V[G]}$ and $\kappa \to (\kappa)^\omega_2$. Note $\kappa \to (\kappa)^2_2$ implies that κ is regular in $V[G]$. Hence κ is regular in V. By Fact 4.2, $\mathbb{P}$ has the ground club property at κ. Choose $p \in G$ so that $p \Vdash_\mathbb{P} \kappa \to (\kappa)^\omega_2$. Lemma 2.10 implies that $\mathbb{R}^V = \mathbb{R}^{V[G]}$. Contradiction. $\square$

Fact 4.4. *Assume* ZF. *Let* $\mathbb{P}$ *be a nontrival forcing which is a surjective image of* $\mathbb{R}$. *Suppose* $1_\mathbb{P} \Vdash_\mathbb{P} \mathsf{AD}$. *Then* $1_\mathbb{P} \Vdash_\mathbb{P} \check{\mathbb{R}} \subsetneq \dot{\mathbb{R}}$. *Hence* $1_\mathbb{P} \Vdash_\mathbb{P} \dot{\Theta} = \Theta^V$.

Proof. Let $\pi : \mathbb{R} \to \mathbb{P}$ be a surjection. Suppose there is some $p \in \mathbb{P}$ so that $p \Vdash_\mathbb{P} \check{\mathbb{R}} = \dot{\mathbb{R}}$. Since $\mathbb{P}$ is a nontrivial forcing, $\pi^{-1}[\dot{G}]$ is forced to be a new set of reals. Since $p \Vdash_\mathbb{P} \check{\mathbb{R}} = \dot{\mathbb{R}}$, for each $A \in \mathscr{P}(\mathbb{R})^V$, $p \Vdash_\mathbb{P} \check{A} \leq_w \pi^{-1}[\dot{G}]$.

In V, define $\Phi : \mathbb{R} \times \mathbb{R} \to \Theta$ by

$$\Phi(r,s) = \begin{cases} \alpha & \pi(r) \leq p, \, \pi(r) \Vdash_\mathbb{P} \text{``}\Xi_s^{-1}[\pi^{-1}[\dot{G}]] \in \check{V}, \text{ and is a prewellordering of length } \check{\alpha}\text{''} \\ 0 & \text{otherwise} \end{cases}$$

Thus in V, Φ is a surjection of $\mathbb{R} \times \mathbb{R}$ onto Θ. This is impossible. $\square$

Fact 4.5. *Assume that* $\mathbb{P}$ *is a forcing which is a surjective image of* $\mathbb{R}$. *Then there is a forcing* $\mathbb{Q}$ *on* $\mathbb{R}$ *so that for every* $G \subseteq \mathbb{P}$ *which is* $\mathbb{P}$-*generic over* V, *there is an* $H \subseteq \mathbb{Q}$ *which is* $\mathbb{Q}$-*generic over* V *so that* $V[G] = V[H]$.

Proof. Let $\pi : \mathbb{R} \to \mathbb{P}$ be a surjection. Define a forcing $\mathbb{Q}$ on $\mathbb{R}$ by $p \leq_\mathbb{Q} q$ if and only if $\pi(p) \leq_\mathbb{P} \pi(q)$. If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over V, then $\pi^{-1}[G] \subseteq \mathbb{Q}$ is a $\mathbb{Q}$-generic filter over V and $V[G] = V[\pi^{-1}[G]]$. $\square$

Lemma 4.6. *Assume* ZF *and there is an* $A \subseteq \mathbb{R}$ *such that* $V = L(A, \mathbb{R})$. *Let* $\mathbb{P}$ *be a forcing on* $\mathbb{R}$ *such that* $1_\mathbb{P} \Vdash_\mathbb{P} \mathsf{AD}$ *and* $\mathbb{P} \leq_w A$. *Let* $A \oplus \mathbb{R}^V$ *indicate some fixed recursive coding of the two sets of reals into a single set of reals. (Note that* $V = L(A \oplus \mathbb{R}^V, \mathbb{R}^V)$.) *Then* $1_\mathbb{P} \Vdash \dot{V} = L(\check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}})$.

Proof. Suppose not. Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter over $L(A, \mathbb{R})$ witnessing the failure of the conclusion of the lemma. Here $\mathbb{R}$ refers to $\mathbb{R}^{L(A,\mathbb{R})}$. Let $\mathbb{R}^* = \dot{\mathbb{R}}^{L(A,\mathbb{R})[G]}$. Note that $\mathbb{P}, \mathbb{R} \in L(A \oplus \mathbb{R}, \mathbb{R}^*)$. Therefore, $L(A, \mathbb{R})$ is a definable inner model of $L(A \oplus \mathbb{R}, \mathbb{R}^*)$. Thus $\Theta^{L(A,\mathbb{R})} \leq \Theta^{L(A \oplus \mathbb{R}, \mathbb{R}^*)}$. Since $\mathbb{P}, \mathbb{R} \in L(A \oplus \mathbb{R}, \mathbb{R}^*)$, $L(A \oplus \mathbb{R}, \mathbb{R}^*) \neq L(A, \mathbb{R})[G]$ implies that $G \notin L(A \oplus \mathbb{R}, \mathbb{R}^*)$. Since $L(A \oplus \mathbb{R}, \mathbb{R}^*)$ and $L(A, \mathbb{R})[G]$ have the same set of reals, G Wadge reduces every set of reals in $L(A \oplus \mathbb{R}, \mathbb{R}^*)$. In $L(A, \mathbb{R})[G]$, define $\Phi : \mathbb{R}^* \to \Theta^{L(A \oplus \mathbb{R}, \mathbb{R}^*)}$ by

$$\Phi(r) = \begin{cases} \text{length}(\Xi_r^{-1}[G]) & \Xi_r^{-1}[G] \in L(A \oplus \mathbb{R}, \mathbb{R}^*) \text{ and is a prewellordering on } \mathbb{R}^* \\ 0 & \text{otherwise} \end{cases}$$

Φ is a surjection in $L(A, \mathbb{R})[G]$ of $\mathbb{R}^*$ onto $\Theta^{L(A \oplus \mathbb{R}, \mathbb{R}^*)}$. This implies that $\Theta^{L(A,\mathbb{R})} \leq \Theta^{L(A \oplus \mathbb{R}, \mathbb{R}^*)} < \Theta^{L(A,\mathbb{R})[G]}$. This contradicts Fact 4.4 which asserts that $\Theta^{L(A,\mathbb{R})} = \Theta^{L(A,\mathbb{R})[G]}$. $\square$

Fact 4.7. *Assume* ZF. *If* $\mathbb{P}$ *is a forcing which is the surjective image of* $\mathbb{R}$ *and* Θ *is regular, then* $1_\mathbb{P} \Vdash_\mathbb{P} \mathsf{AD}$ *implies* $1_\mathbb{P} \Vdash \Theta$ *is regular.*

Proof. Let $\pi : \mathbb{R} \to \mathbb{P}$ be a surjection. Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter over V. By Fact 4.4, $V[G] \models \Theta^{V[G]} = \Theta^V$. Suppose Θ is not regular in $V[G]$. There is some $\eta < \Theta$ and a function $f : \eta \to \Theta$ which is cofinal. Let $\tau \in V$ be a $\mathbb{P}$-name so that $\tau[G] = f$.

Now work in V. Define $g : \eta \times \mathbb{R} \to \Theta$ by

$$g(\alpha, r) = \begin{cases} 0 & (\forall \beta < \Theta)(\pi(r) \not\Vdash_\mathbb{P} \tau(\check{\alpha}) = \check{\beta}) \\ \beta & \pi(r) \Vdash_\mathbb{P} \tau(\check{\alpha}) = \check{\beta} \end{cases}.$$

Let $\rho : \mathbb{R} \to \eta$ be a surjection. Define $h : \mathbb{R} \to \Theta$ by $h(x) = g(\rho(x_1), x_2)$, where $x = \langle x_1, x_2 \rangle$ under some standard pairing function. Let $x \preceq y$ if and only if $h(x) \leq h(y)$. As $\preceq$ is a prewellordering of $\mathbb{R}$, it has length some $\delta < \Theta$. Thus there is a map $\tilde{h} : \delta \to \Theta$ which is cofinal. This is impossible since Θ is regular in V. $\square$

5. Destroying AD When Θ Is Regular

By Fact 4.5, this section will assume that the forcing is on $\mathbb{R}$. For such a forcing $\mathbb{P}$, a name for a real consisting of elements of the form $(\check{n}, p)$ for $n \in \omega$ and $p \in \mathbb{P}$ can be considered subsets of $\mathbb{R}$. In this section, when one writes that a name $\sigma \in \mathscr{P}(\mathbb{R})$, it is understood that σ takes this form.

Definition 5.1. Let $\mathbb{P}$ be a forcing on $\mathbb{R}$. $\mathbb{P}$ has the name condition if and only if there is an $A \subseteq \mathbb{R}$ so that $\mathbb{P} \leq_w A$ and $1_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "for all $r \in \dot{\mathbb{R}}$, there is a $\mathbb{P}$-name $\sigma \in \mathscr{P}(\check{\mathbb{R}})^{L(\check{A},\check{\mathbb{R}})}$ so that $\sigma[\dot{G}] = r$ and $L(\check{A}, \check{\mathbb{R}}) \models \sigma \leq_w \check{A}$".

This means that there is a set $A \subseteq \mathbb{R}$ so that for all $G \subseteq \mathbb{P}$ which are $\mathbb{P}$-generic over V, for all $r \in \mathbb{R}^{V[G]}$, there is a set of reals σ in $L(A, \mathbb{R})$ which is also Wadge reducible to A in $L(A, \mathbb{R})$ so that when σ is construed as a $\mathbb{P}$-name, $\sigma[G] = r$.

Fact 5.2. *Assume* ZF. *Let $A \subseteq \mathbb{R}$. Let C_A be the set of ordinals α less than Θ so that A can Wadge reduce a prewellordering on $\mathbb{R}$ of length α. Then C_A is bounded below Θ.*

Proof. Suppose not. Define $\Psi : \mathbb{R} \times \mathbb{R} \to \Theta$ by

$$\Psi(r, s) = \begin{cases} \mathrm{rk}_{\Xi_r^{-1}[A]}(s) & \text{if } \Xi_r^{-1}[A] \text{ is a prewellordering on } \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

where if $\preceq$ is a prewellordering on $\mathbb{R}$, then $\mathrm{rk}_{\preceq}(s)$ denote the rank of s in the prewellordering $\preceq$. Ψ is a surjection of $\mathbb{R} \times \mathbb{R}$ onto Θ. Contradiction. $\square$

Fact 5.3. *Assume* ZF$+$AD$+\Theta$ *is regular. Every forcing $\mathbb{P}$ on $\mathbb{R}$ such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}}$ AD has the name condition.*

Proof. Let $p \in \mathbb{P}$ and $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter over V such that $p \in G$. By Fact 4.4 and Fact 4.7, $\Theta^{V[G]} = \Theta^V$ and Θ remains regular in $V[G]$.

Suppose $r \in \mathbb{R}^{V[G]}$. There is some $\mathbb{P}$-name $\tau \in V$ so that $r = \tau[G]$. Let $\sigma = \{(\check{n}, s) : s \Vdash_{\mathbb{P}} \check{n} \in \tau\}$. Note that $\sigma[G] = \tau[G]$ and σ can be considered as essentially a set of reals.

Since $\mathscr{P}(\mathbb{R})^V \in V[G]$, one can define a function $\Phi : \mathbb{R}^{V[G]} \to \Theta$ by

$$\Phi(r) = \min\{\sup(C_\sigma)^V + 1 : \sigma \in \mathscr{P}(\mathbb{R})^V \wedge \sigma[G] = r\}$$

where C_A, for $A \subseteq \mathbb{R}$, is defined in Fact 5.2.

In $V[G]$, define $x \sqsubseteq y$ if and only if $\Phi(x) \leq \Phi(y)$. $\sqsubseteq$ is a prewellordering on $\mathbb{R}$. There is some $\delta < \Theta^{V[G]} = \Theta^V$ so that $\sqsubseteq$ has length δ. Thus Φ induces a map $\tilde{\Phi} : \delta \to \Theta$. Since Θ is regular in $V[G]$, $\tilde{\Phi}$ and hence Φ is bounded below some $\gamma < \Theta$.

Fix a prewellordering $\preceq^*$ in V of length greater than or equal to γ. Let $r \in \mathbb{R}^{V[G]}$. Let $\sigma \in V$ be a set of reals so that when it is construed as a $\mathbb{P}$-name, $\sigma[G] = r$ and $\Phi(r) = (C_\sigma)^V + 1$. Since $\gamma > \sup(C_\sigma)^V$, σ can not Wadge reduce $\preceq^*$ in V. Hence by Wadge's lemma, $\sigma \leq_w \preceq^*$ in V.

It has been shown that in $V[G]$, there is some ordinal γ, so that for any prewellordering $\preceq^* \in V$ of length greater than or equal to γ, every $r \in \mathbb{R}^{V[G]}$ has a name $\sigma \in \mathscr{P}(\mathbb{R})^{L(\preceq^*,\mathbb{R})}$ so that $\sigma[G] = r$ and $L(\preceq^*, \mathbb{R}) \models \sigma \leq_w \preceq^*$. Find some $q \leq_{\mathbb{P}} p$, $q \in G$, and some $\gamma < \Theta$ so that q which forces this above statement about γ. Since $p \in \mathbb{P}$ was arbitrary, it has been shown that there is a dense set of q for which there is some γ so that q forces the above statement involving γ.

Define $\Psi : \mathbb{P} \to \Theta$ by $\Psi(q)$ is the least γ so that q forces the above statement involving γ if such a γ exists. Let $\Psi(q) = 0$ otherwise. Ψ induces a prewellordering on $\mathbb{R}$ of length $\delta < \Theta$. Since Θ is regular in V, Ψ is bounded below Θ by some γ. Let $\preceq^*$ be some prewellordering on $\mathbb{R}$ of length γ. Let $A = \preceq^*$. One has that A witnesses that $\mathbb{P}$ has the name condition. $\square$

Lemma 5.4. *Assume* ZF $+$ AD. *Let $\mathbb{P}$ be a forcing on $\mathbb{R}$ and $1_{\mathbb{P}} \Vdash$ AD. Assume that $\mathbb{P}$ has the name condition. Let $A \subseteq \mathbb{R}$ witness the name condition. Then in $L(A, \mathbb{R})$, $1_{\mathbb{P}} \Vdash_{\mathbb{P}}$ AD, δ_A has the ground club property, and $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\delta}_A$ has the strong partition property.*

Proof. Let A witness the name condition. Note that $L(A, \mathbb{R}) \models$ AD.

Throughout this proof, $\mathbb{R}$ denotes $\mathbb{R}^V$ and $\mathbb{R}^*$ denotes $\mathbb{R}^{V[G]}$ whenever G is $\mathbb{P}$-generic over V.

Let $p \in \mathbb{P}$. Let $G \subseteq \mathbb{P}$ be any $\mathbb{P}$-generic filter over V containing p. By definition of the name condition, $\dot{\mathbb{R}}^{L(A,\mathbb{R})[G]} = \dot{\mathbb{R}}^{V[G]}$. Thus since $V[G] \models$ AD, $L(A, \mathbb{R})[G] \models$ AD. Let $q \leq_{\mathbb{P}} p$ with $q \in G$ be such that

$L(A, \mathbb{R}) \models q \Vdash_{\mathbb{P}} \mathsf{AD}$. Since $p \in \mathbb{P}$ was arbitrary, there is a dense set of $q \in \mathbb{P}$ so that $L(A, \mathbb{R}) \models q \Vdash_{\mathbb{P}} \mathsf{AD}$. One has that $L(A, \mathbb{R}) \models 1_{\mathbb{P}} \Vdash_{\mathbb{P}} \mathsf{AD}$.

By Lemma 4.6, $L(A, \mathbb{R})[G] = L(A \oplus \mathbb{R}, \mathbb{R}^*)$ for any $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over V.

Let $p \in \mathbb{P}$. Let $G \subseteq \mathbb{P}$ be any $\mathbb{P}$-generic filter over V containing p.

Claim 1: $\delta_A = (\delta_{A \oplus \mathbb{R}})^{L(A \oplus \mathbb{R}, \mathbb{R}^*)}$.

Let $r \in \mathbb{R}^*$. By the name condition, there is some $\tau \subseteq \mathbb{R}$ which is Wadge reducible to A and $\tau[G] = r$ when τ is construed as a $\mathbb{P}$-name. Note that every set which is Wadge reducible to A appears at level $L_1(A, \mathbb{R})$. Let $\varphi(\dot{v}, A \oplus \mathbb{R}, \dot{\mathbb{R}})$ be a Σ_1 formula. Suppose that $L(A \oplus \mathbb{R}, \mathbb{R}^*) \models \varphi(r, A \oplus \mathbb{R}, \mathbb{R}^*)$. Since $L(A \oplus \mathbb{R}, \mathbb{R}^*) = L(A, \mathbb{R})[G]$, there is some $q_0 \leq_{\mathbb{P}} p$ so that $q_0 \in G$ and

$$L(A, \mathbb{R}) \models q_0 \Vdash_{\mathbb{P}} L(\check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}}) \models \varphi(\tau, \check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}}).$$

By replacement, the following is a true $\Sigma_1(L(A, \mathbb{R}), \mathbb{R} \cup \{\mathbb{R}, A\})$ formula: (Note that it is important that $\tau \leq_w A$.)

$$L(A, \mathbb{R}) \models (\exists \alpha)(L_\alpha(A, \mathbb{R}) \models q_0 \Vdash_{\mathbb{P}} L(\check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}}) \models \varphi(\tau, \check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}})).$$

By definition of δ_A, there exists some $\alpha < \delta_A$ so that

$$L(A, \mathbb{R}) \models L_\alpha(A, \mathbb{R}) \models q_0 \Vdash_{\mathbb{P}} L(\check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}}) \models \varphi(\tau, \check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}}).$$

Hence for some $\alpha < \delta_A$,

$$L_\alpha(A, \mathbb{R}) \models q_0 \Vdash_{\mathbb{P}} L(\check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}}) \models \varphi(\tau, \check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}}).$$

Since $q_0 \in G$, the forcing theorem gives

$$L_\alpha(A, \mathbb{R})[G] \models L(A \oplus \mathbb{R}, \dot{\mathbb{R}}) \models \varphi(r, A \oplus \mathbb{R}, \dot{\mathbb{R}}).$$

Also

$$(L(A \oplus \mathbb{R}, \dot{\mathbb{R}}))^{L_\alpha(A, \mathbb{R})[G]} = L_\alpha(A \oplus \mathbb{R}, \dot{\mathbb{R}}^{L_\alpha(A, \mathbb{R})[G]}).$$

Since A witnesses the name condition, every $t \in \mathbb{R}^*$ has a name in $L_1(A, \mathbb{R})$. Hence $\dot{\mathbb{R}}^{L_\alpha(A, \mathbb{R})[G]} = \mathbb{R}^*$. Thus one has

$$L_\alpha(A, \mathbb{R})[G] \models L_\alpha(A \oplus \mathbb{R}, \mathbb{R}^*) \models \varphi(r, A \oplus \mathbb{R}, \mathbb{R}^*).$$

Thus

$$L_\alpha(A \oplus \mathbb{R}, \mathbb{R}^*) \models \varphi(r, A \oplus \mathbb{R}, \mathbb{R}^*).$$

By upward absolute of Σ_1 formulas,

$$L_{\delta_A}(A \oplus \mathbb{R}, \mathbb{R}^*) \models \varphi(r, A \oplus \mathbb{R}, \mathbb{R}^*).$$

It has been established that $(\delta_{A \oplus \mathbb{R}})^{L(A \oplus \mathbb{R}, \mathbb{R}^*)} \leq \delta_A$.

Let $\varphi(\dot{v}, A, \dot{\mathbb{R}})$ be a Σ_1 formula and $r \in \mathbb{R}^V$. Note $\mathbb{R} \in L(A \oplus \mathbb{R}, \mathbb{R}^*)$. Suppose $L(A, \mathbb{R}) \models \varphi(r, A, \mathbb{R})$. Then

$$L(A \oplus \mathbb{R}, \mathbb{R}^*) \models L(A, \mathbb{R}) \models \varphi(r, A, \mathbb{R}).$$

The following is a true $\Sigma_1(L(A \oplus \mathbb{R}, \mathbb{R}^*), \mathbb{R}^* \cup \{A \oplus \mathbb{R}, \mathbb{R}^*\})$ sentence

$$L(A \oplus \mathbb{R}, \mathbb{R}^*) \models (\exists \alpha)(L_\alpha(A, \mathbb{R}) \models \varphi(r, A, \mathbb{R})).$$

By definition of $(\delta_{A \oplus \mathbb{R}})^{L(A \oplus \mathbb{R}, \mathbb{R}^*)}$, there is some $\alpha < (\delta_{A \oplus \mathbb{R}})^{L(A \oplus \mathbb{R}, \mathbb{R}^*)}$ so that

$$L(A \oplus \mathbb{R}, \mathbb{R}^*) \models L_\alpha(A, \mathbb{R}) \models \varphi(r, A, \mathbb{R}).$$

Thus

$$L_\alpha(A, \mathbb{R}) \models \varphi(r, A, \mathbb{R}).$$

By upward absoluteness

$$L_{(\delta_{A \oplus \mathbb{R}})^{L(A \oplus \mathbb{R}, \mathbb{R}^*)}}(A, \mathbb{R}) \models \varphi(r, A, \mathbb{R}).$$

This shows that $\delta_A \leq (\delta_{A \oplus \mathbb{R}})^{L(A \oplus \mathbb{R}, \mathbb{R}^*)}$. Claim 1 has been established.

By Claim 1, let $q \leq_{\mathbb{P}} p$ with $q \in G$ be such that $L(A, \mathbb{R}) \models q \Vdash \check{\delta}_A = \dot{\delta}_{A \oplus \check{\mathbb{R}}}$. Since $p \in \mathbb{P}$ was arbitrary, the set of $q \in \mathbb{P}$ such that $L(A, \mathbb{R}) \models q \Vdash \check{\delta}_A = \dot{\delta}_{A \oplus \check{\mathbb{R}}}$ is dense. Thus $L(A, \mathbb{R}) \models 1_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\delta}_A = \dot{\delta}_{A \oplus \check{\mathbb{R}}}$.

Fact 2.6, Lemma 4.6, and the fact that $L(A, \mathbb{R}) \models 1_{\mathbb{P}} \Vdash \check{\delta}_A = \dot{\delta}_{A \oplus \check{\mathbb{R}}}$ now imply that $L(A, \mathbb{R}) \models 1_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\delta}_A$ has the strong partition property. It remains to show that δ_A has the ground club property.

Claim 2: In $L(A, \mathbb{R})$, δ_A has the ground club property.

Let $p \in \mathbb{P}$ and G be $\mathbb{P}$-generic over $L(A, \mathbb{R})$ containing p. Recall it has been shown that $\delta_A^{L(A,\mathbb{R})} = \delta_{A\oplus\mathbb{R}}^{L(A\oplus\mathbb{R},\mathbb{R}^*)}$ and $L(A,\mathbb{R})[G] = L(A\oplus\mathbb{R}, \mathbb{R}^*)$ by Lemma 4.6. Let $D \in L(A,\mathbb{R})[G]$ be a club subset of $\delta_A^{L(A,\mathbb{R})} = \delta_{A\oplus\mathbb{R}}^{L(A\oplus\mathbb{R},\mathbb{R}^*)}$. Let $\rho_{A\oplus\mathbb{R}}$ and ς be those objects from Definition 2.4 for $A \oplus \mathbb{R}$ defined in $L(A,\mathbb{R})[G] = L(A \oplus \mathbb{R}, \mathbb{R}^*)$. Since $L(A,\mathbb{R})[G] = L(A \oplus \mathbb{R}, \mathbb{R}^*) \models \mathsf{AD}$ and Fact 2.2, there is some $e \in \mathbb{R}^*$ so that the graph of the increasing enumeration of D is $S_e^{\rho_{A\oplus\mathbb{R}}}$. By the name condition as witnessed by A, there is some $\mathbb{P}$-name $\dot{e} \subseteq \mathbb{R}$ so that $\dot{e} \leq_w A$ by a Wadge reduction coded in $L(A,\mathbb{R})$ and $\dot{e}[G] = e$. There is some $q_0 \leq_\mathbb{P} q$ with $q_0 \in G$ so that $q_0 \Vdash$ "$S_{\dot{e}}^{\rho_{A\oplus\mathbb{R}}}$ is the graph of an enumeration of a club subset of $\check{\delta}_A$".

By reflection, for each $\beta < \delta_A$, the following is a true Σ_1 statement in $L(A,\mathbb{R})$ using parameters among A, $\mathbb{R}$, and elements of $L_{\delta_A}(A,\mathbb{R})$:

$$L(A,\mathbb{R}) \models (\exists\alpha)(L_\alpha(A,\mathbb{R}) \models (\forall k \leq_\mathbb{P} q_0)(\exists j \leq_\mathbb{P} k)(\exists\gamma)(j \Vdash_\mathbb{P} \varsigma(\langle\check\beta, \tilde\gamma\rangle, \dot{e}, \check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}})),$$

where $\langle\cdot,\cdot\rangle$ refers to a fixed ordinal pairing function. This merely states that there is a dense set of conditions below q_0 which forces a value for the image of $\check\beta$ under the function whose graph is $S_{\dot{e}}^{\rho_{\check{A}\oplus\check{\mathbb{R}}}}$.

By the definition of δ_A in $L(A,\mathbb{R})$, there is some $\alpha < \delta_A$ so that

$$L_\alpha(A,\mathbb{R}) \models (\forall k \leq_\mathbb{P} q_0)(\exists j \leq_\mathbb{P} k)(\exists\gamma)(j \Vdash_\mathbb{P} \varsigma(\langle\check\beta, \tilde\gamma\rangle, \dot{e}, \check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}}).$$

Let ϵ_β be the least α with this property. By upward absoluteness of the Σ_1 formula ς,

$$L(A,\mathbb{R}) \models q_0 \Vdash_\mathbb{P} (\exists\gamma < \check\epsilon_\beta)(\varsigma(\langle\check\beta, \gamma\rangle, \dot{e}, \check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}})).$$

Thus for all $\mathbb{P}$-generic filter H containing q_0, the βth element of the club subset of $\delta_A^{L(A,\mathbb{R})}$ enumerated by the function whose graph is $S_{\dot{e}[H]}^{\rho_{A\oplus\mathbb{R}}}$ is less than ϵ_β. Define in $L(A,\mathbb{R})$, a function $g : \delta_A \to \delta_A$ by $g(\beta) = \epsilon_\beta$. Let $C = \{\mu < \delta_A : (\forall\gamma < \mu)(g(\gamma) < \mu)\}$. By the same argument as in the proof of Fact 3.1, $C \subseteq \delta_A$ is a club in $L(A,\mathbb{R})$ and $q_0 \Vdash$ "$\check{C}$ is a subset of the club enumerated by $S_{\dot{e}}^{\rho_{A\oplus\mathbb{R}}}$". Thus $L(A,\mathbb{R})[G] \models C \subseteq D$. This proves Claim 2 and completes the lemma. $\square$

Theorem 5.5. *Assume* $\mathsf{ZF}+\mathsf{AD}+ \Theta$ *is regular. Suppose* $\mathbb{P}$ *is a nontrivial forcing which is a surjective image of* $\mathbb{R}$*. Then* $1_\mathbb{P} \Vdash_\mathbb{P} \neg\mathsf{AD}$*.*

Proof. As throughout this section, one may assume $\mathbb{P}$ is a forcing on $\mathbb{R}$. Assume AD is preserved by the forcing. Fact 5.3 implies that $\mathbb{P}$ has the name condition. Let $A \subseteq \mathbb{R}$ witness the name condition.

Work in $L(A,\mathbb{R})$. Fact 4.4 states that a new real must be added. However Lemma 2.10 and Lemma 5.4 imply that the ground model and the forcing extension have the same reals. Contradiction. $\square$

Corollary 5.6. *Assume* $\mathsf{ZF} + \mathsf{AD} + V = L(\mathbb{R})$*. No nontrivial forcing* $\mathbb{P} \in L_\Theta(\mathbb{R})$ *can preserve* AD*.*

In fact, assume $\mathsf{ZF} + \mathsf{AD}^+ + \neg\mathsf{AD}_\mathbb{R} + V = L(\mathscr{P}(\mathbb{R}))$*. No nontrivial forcing which is the surjective image of* $\mathbb{R}$ *can preserve* AD*.*

Proof. If there is some set X so that every set is $\mathrm{OD}_{X,r}$ for some $r \in \mathbb{R}$, then Θ is regular. Hence if $L(\mathbb{R}) \models \mathsf{AD}$, then $L(\mathbb{R}) \models \Theta$ is regular. Woodin showed that if $\mathsf{ZF} + \mathsf{AD}^+ + \neg\mathsf{AD}_\mathbb{R} + V = L(\mathscr{P}(\mathbb{R}))$ holds, then there is some set of ordinals J so that $V = L(J,\mathbb{R})$. Hence in these natural models of $\mathsf{AD}^+ + \neg\mathsf{AD}_\mathbb{R}$, Θ is regular. $\square$

Question 5.7. Assume $\mathsf{ZF} + \mathsf{AD}$. If $\mathbb{P}$ is a nontrivial forcing which is a surjective image of $\mathbb{R}$, then does $1_\mathbb{P} \Vdash_\mathbb{P} \neg\mathsf{AD}$ hold?

By the above, it remains to consider the case when Θ is singular.

Let Θ_0 be the supremum of the ordinals which are the surjective image of $\mathbb{R}$ by OD surjections. Ikegami and Trang have informed the authors that the consistency of $\mathsf{ZF} + \mathsf{AD}^+$ and $\Theta > \Theta_0$ implies the consistency of the statement that there is a forcing $\mathbb{P}$ (which is not a surjective image of $\mathbb{R}$) such that $1_\mathbb{P} \Vdash_\mathbb{P} \mathsf{AD} \wedge \check\Theta < \dot\Theta$. This model also does not satisfy $\mathsf{ZF} + \mathsf{AD}^+ + V = L(\mathscr{P}(\mathbb{R}))$.

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