WHEN A RELATION WITH ALL BOREL SECTIONS WILL BE BOREL SOMEWHERE?

WILLIAM CHAN AND MENACHEM MAGIDOR

Abstract. In ZFC, if there is a measurable cardinal with infinitely many Woodin cardinals below it, then for every binary relation $R \in L(\mathbb{R})$ on $\mathbb{R}$ with all sections $\mathbf{\Delta}^1_1$ ($\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$) and every σ-ideal I on $\mathbb{R}$ so that the associated forcing $\mathbb{P}_I$ of I^+ $\mathbf{\Delta}^1_1$ subsets is proper, there exists some I^+ $\mathbf{\Delta}^1_1$ set C so that $R \cap (C \times \mathbb{R})$ is $\mathbf{\Delta}^1_1$ ($\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$, respectively). In $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} + V = L(\mathscr{P}(\mathbb{R}))$, for every binary relation R on $\mathbb{R}$ with all sections $\mathbf{\Delta}^1_1$ ($\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$) and every σ-ideal I on $\mathbb{R}$ so that the associated forcing $\mathbb{P}_I$ is absolutely proper, there is some I^+ $\mathbf{\Delta}^1_1$ set C so that $R \cap (C \times \mathbb{R})$ is $\mathbf{\Delta}^1_1$ ($\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$, respectively) relation.

1. Introduction

The basic question of interest is:

Question 1.1. If E is an equivalence relation on ${}^\omega\omega$, is E a simpler equivalence relation when restricted to some subset?

This question can also be asked for equivalence relations on arbitrary Polish spaces, but for simplicity, this paper will only consider ${}^\omega\omega$, the Baire space of functions from ω into ω. Usually, descriptive set theoretic results about ${}^\omega\omega$ have proofs that can be transfered to arbitrary Polish spaces. Besides equivalence relations, this question can also be asked for graphs and more generally for binary relations. (This paper will state all results for the more general case of binary relations. However, the original form of this question as well as many of the examples from [7] and [2] considered equivalence relations. This introduction will briefly focus on equivalence relations.)

What should be the measure of complexity and what should be the paragon of simplicity? The measure of complexity will vaguely be definability and there is no need to explicitly state what it is since the paper will only strive to reach the base of complexity. However, there are various useful notions of definability given by ideas from topology, recursion theory, logical complexity, and set theory. The base of definable complexity needs to be explicitly stated. The class of Borel sets (denoted $\mathbf{\Delta}^1_1$) is chosen to be this base since it is a simple class characterized by all the notions of definability mentioned above. Moreover, many natural mathematical concerns appear at this level, and $\mathbf{\Delta}^1_1$ objects seem to be well behaved and relatively well understood.

Now the question can be more precisely formulated:

Question 1.2. If E is an equivalence relation on ${}^\omega\omega$, is there a $\mathbf{\Delta}^1_1$ set $C \subseteq {}^\omega\omega$ so that $E \upharpoonright C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

Here, $E \upharpoonright C = E \cap (C \times C)$. However, there is one obvious triviality. If C is countable, then any equivalence relation restricted to C is $\mathbf{\Delta}^1_1$. Since countable subsets of ${}^\omega\omega$ belong to any σ-ideal on ${}^\omega\omega$ which contains all singletons, this egregious triviality disappears if one asks that, in the above question, C be $\mathbf{\Delta}^1_1$ and non-trivial according to a σ-ideal on ${}^\omega\omega$. Subsets of ${}^\omega\omega$ that are not in the ideal I are called I^+ sets. In this paper, σ-ideals will always contain all the singletons.

However, it is unclear how to approach this question for arbitrary σ-ideals. The collection of available techniques is greatly enriched by considering σ-ideals on ${}^\omega\omega$ so that the associated forcing $\mathbb{P}_I$ of $\mathbf{\Delta}^1_1$ I^+ sets is a proper forcing. Considering such σ-ideals makes available powerful tools from models of set theory and absoluteness. (In fact, the questions below all have negative answers when considering arbitrary σ-ideals. See Section 2.) Now a test question can be posed for a slightly more complicated class of equivalence relations

September 13, 2018. The first author was partially supported by NSF grants DMS-1464475, EMSW21-RTG DMS-1044448, and DMS-1703708.

than the $\mathbf{\Delta}^1_1$ equivalence relations: Analytic (denoted $\mathbf{\Sigma}^1_1$) sets are continuous images of $\mathbf{\Delta}^1_1$ or even closed sets.

Question 1.3. Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation on ${}^\omega\omega$. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is a proper forcing. Is there an I^+ $\mathbf{\Delta}^1_1$ set C so that $E \upharpoonright C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

Note that questions like the above are very familiar. For example, the ideal of Lebesgue null set and the ideal of meager sets have the property that their associated forcings are proper forcings. It is very common in mathematics to ask questions about properties that hold on positive measure sets (or Lebesgue almost everywhere) or on non-meager (or comeager) sets.

Unfortunately, Question 1.3 has a negative answer:

Proposition 1.4. *There is a $\mathbf{\Sigma}^1_1$ equivalence relation E and a σ-ideal I with $\mathbb{P}_I$ proper so that for all $\mathbf{\Delta}^1_1$ I^+ set C, $E \upharpoonright C$ is not $\mathbf{\Delta}^1_1$.*

Proof. See [7], Example 4.25. □

So a positive answer is not even possible for the simplest class of equivalence relations in the projective hierarchy just above $\mathbf{\Delta}^1_1$. A positive answer to any variation of the basic question will likely only be feasible if the equivalence relations bear at least some resemblance to $\mathbf{\Delta}^1_1$ equivalence relations. A positive answer does hold for many important examples:

Example 1.5. ([7]) Let I be a σ-ideal on a Polish space X with $\mathbb{P}_I$ proper.

If E is a $\mathbf{\Sigma}^1_1$ equivalence relation all classes countable or E is $\mathbf{\Delta}^1_1$ reducible to an orbit equivalence relation of a Polish group action, then there is some I^+ $\mathbf{\Delta}^1_1$ set C so that $E \upharpoonright C$ is $\mathbf{\Delta}^1_1$.

In both these examples, the equivalence relations have all $\mathbf{\Delta}^1_1$ classes. Of course, $\mathbf{\Delta}^1_1$ equivalence relations have all $\mathbf{\Delta}^1_1$ classes. Perhaps those two examples give evidence that a sufficient resemblance for a positive answer is the property of having all $\mathbf{\Delta}^1_1$ classes. [7] asked the following question:

Question 1.6. ([7] Question 4.28) Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation on ${}^\omega\omega$ with all $\mathbf{\Delta}^1_1$ classes. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is a proper forcing. Let B be an I^+ $\mathbf{\Delta}^1_1$ set. Is there some $C \subseteq B$ which is I^+ $\mathbf{\Delta}^1_1$ so that $E \upharpoonright C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

Further examples and partial results suggest that a positive answer is consistent: Let I_{meager} be the ideal of meager sets. Let I_{null} be the ideal of Lebesgue measure zero sets. The associated forcings are Cohen forcing and Random real forcing which both satisfy the countable chain condition and are hence proper.

Example 1.7. ([2]) Assume $\mathsf{ZFC} + \mathsf{MA} + \neg\mathsf{CH}$. Let I be I_{meager} (or I_{null}). Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation with all classes $\mathbf{\Delta}^1_1$. Then there exists comeager (or measure 1) $\mathbf{\Delta}^1_1$ set C such that $E \upharpoonright C$ is $\mathbf{\Delta}^1_1$.

Here coanalytic sets (denoted $\mathbf{\Pi}^1_1$) are complements of $\mathbf{\Sigma}^1_1$ sets. The following example gives a consistency result for a particular instance of the $\mathbf{\Pi}^1_1$ version of Question 1.6:

Example 1.8. ([2]) Let κ be a remarkable cardinal in L. Let G be $\text{Coll}(\omega, <\kappa)$-generic over L. In $L[G]$, if I is a σ-ideal with $\mathbb{P}_I$ proper and E is a $\mathbf{\Pi}^1_1$ equivalence relation with all classes countable, then there is a I^+ $\mathbf{\Delta}^1_1$ set C such that $E \upharpoonright C$ is $\mathbf{\Delta}^1_1$.

See [2] for more examples and partial answers when conditions are imposed on the equivalence relations and the ideals.

It was then shown that, under large cardinal assumptions, this question has a positive answer:

Theorem 1.9. *Suppose for all $X \in H_{(2^{\aleph_0})^+}$, $X^\sharp$ exists. Then for all $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ equivalence relations with all $\mathbf{\Delta}^1_1$ classes, any σ-ideal I on ${}^\omega\omega$ with $\mathbb{P}_I$ proper, and B an I^+ $\mathbf{\Delta}^1_1$ set, there exists some I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $E \upharpoonright C$ is $\mathbf{\Delta}^1_1$.*

Proof. See [2]. Also see [3] for a similar result proved using a measurable cardinal. □

It should be noted that the proofs of Theorem 1.9 in both [2] and [3] use an approximation of $\mathbf{\Sigma}^1_1$ equivalence relations by $\mathbf{\Delta}^1_1$ equivalence relations: Burgess showed that for every $\mathbf{\Sigma}^1_1$ equivalence relation E there is (in a uniform way) an ω_1-length decreasing sequence $(E_\alpha : \alpha < \omega_1)$ of $\mathbf{\Delta}^1_1$ equivalence relations so that $E = \bigcap_{\alpha<\omega_1} E_\alpha$. The strategy of the proof is to find some countable elementary $M \prec H_\Xi$, where Ξ is large enough to contain certain desired objects, and some countable ordinal α so that if C is the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic reals over M (which exists by properness of $\mathbb{P}_I$), then $E \restriction C = E_\alpha \restriction C$. The sharps are used to obtain the absoluteness necessary to determine the countable level α at which the E classes and E_α classes of all generic reals stabilize.

In conversation with the first author, Neeman asked the following generalization of Question 1.6: Projective sets are those obtainable by applying finitely many applications of complements and continuous images starting with the $\mathbf{\Delta}^1_1$ sets.

Question 1.10. Assume some large cardinal hypotheses. Let E be a projective equivalence relation with all $\mathbf{\Delta}^1_1$ classes. Let I be a σ-ideal on ${}^\omega\omega$ with $\mathbb{P}_I$ proper. Let $B \subseteq {}^\omega\omega$ be an I^+ $\mathbf{\Delta}^1_1$ subset. Does there exist some I^+ $\mathbf{\Delta}^1_1$ $C \subseteq B$ so that $E \restriction C$ is $\mathbf{\Delta}^1_1$?

It is unclear if the proofs of Theorem 1.9 can be generalized to give an answer to this question since there does not appear to be any form of $\mathbf{\Delta}^1_1$ approximation to arbitrary projective equivalence relations. Moreover, it is known to be consistent that there is a negative answer to Question 1.10 even when restricted to the next level of the projective hierarchy above $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$. A $\mathbf{\Sigma}^1_2$ set is a continuous image of a $\mathbf{\Pi}^1_1$ set. A $\mathbf{\Pi}^1_2$ set is the complement of a $\mathbf{\Sigma}^1_2$ set. A $\mathbf{\Delta}^1_2$ set is a set that is both $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_2$:

Example 1.11. In the constructible universe L, there is a $\mathbf{\Delta}^1_2$ equivalence relation with all classes countable so that for every σ-ideal I and every I^+ $\mathbf{\Delta}^1_1$ set B, $E \restriction B$ is not $\mathbf{\Delta}^1_1$.

Proof. The equivalence E_L on ${}^\omega\omega$ is roughly defined by $x\ E_L\ y$ if and only if the least admissible level of L for which x and y appear is the same. See [2] or [3] for more details and the complete proof. □

In fact, it is not known what is the status of Question 1.6 or its $\mathbf{\Pi}^1_1$ analog in L. An interesting open question in this area is whether it is consistent that Question 1.6 or its $\mathbf{\Pi}^1_1$ analog has a negative answer. See the conclusion section of [2] for some discussions on this question.

This paper will be concerned with extending a positive answer to these types of questions to larger classes of equivalence relations on ${}^\omega\omega$ with all $\mathbf{\Delta}^1_1$ classes. As mentioned above, some new methods will need to be developed to take the role of Burgess's approximation in Theorem 1.9. A certain game will be used to fulfill this role.

Question 1.10 will be answered by an even more general result. Like in Theorem 1.9, the results of this paper will be proved in an extension of ZFC, the standard axiom system of set theory. Here, ZFC will be augmented by large cardinal axioms. The large cardinal axioms used here are well accepted and have proven to be very useful in descriptive set theory.

The model $L(\mathbb{R})$ is the smallest inner model of ZF (possibly without the axiom of choice) containing all the reals of the original universe. It contains all the sets which are "constructible" (in the sense of Gödel) from the reals of the original universe. Nearly all objects of ordinary mathematics can be found in $L(\mathbb{R})$. In particular, all projective subsets of ${}^\omega\omega$ belong to $L(\mathbb{R})$. A main result of the paper is:

Theorem 4.3. Suppose there is a measurable cardinal with infinitely many Woodin cardinals below it. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Let $R \in L(\mathbb{R})$ be a binary relation on ${}^\omega\omega$.

If R has all $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$) sections, then for every I^+ $\mathbf{\Delta}^1_1$ set B, there is an I^+ $\mathbf{\Delta}^1_1$ $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$, respectively).

This gives a positive answer to Question 1.10. Moreover, it shows that for a large class of binary relations on ${}^\omega\omega$ so that all the sections belong to a particular pointclass of the first level of the projective hierarchy, the relation somewhere is as simple as its sections.

Example 1.12. Assume there is a measurable cardinal with infinitely many Woodin cardinals below it. Let I be a σ-ideal on ${}^\omega\omega$ with $\mathbb{P}_I$ proper. Define an equivalence relation E on ${}^\omega\omega$ by $x\ E\ y$ if and only if $L(\mathbb{R})^V \models x \in \mathrm{OD}_y \wedge y \in \mathrm{OD}_x$. (That is, $x\ E\ y$ if and only if in $L(\mathbb{R})$, x is ordinal definable from y and y is ordinal definable from x.)

Note that $E \in L(\mathbb{R})$ and, in fact, is $(\Sigma^2_1)^{L(\mathbb{R})}$. E has all classes countable (and hence $\mathbf{\Delta}^1_1$). Theorem 4.3 implies that there is an I^+ $\mathbf{\Delta}^1_1$ set C so that $E \restriction C$ is a $\mathbf{\Delta}^1_1$ equivalence relation.

Having answered Question 1.10 positively and even given a positive answer for the larger class of $L(\mathbb{R})$ equivalence relation with all $\mathbf{\Delta}^1_1$ classes, the ultimate natural question is the following:

Question 1.13. Is it consistent relative to some large cardinals, that (the axiom of choice fails and) for every equivalence relation E with all $\mathbf{\Delta}^1_1$ classes and every σ-ideal I on ${}^\omega\omega$ such that $\mathbb{P}_I$ is a proper forcing, there is an I^+ $\mathbf{\Delta}^1_1$ set C so that $E \upharpoonright C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

As it is often the case for various regularity properties like the perfect set property, Lebesgue measurability, or the property of Baire, the axiom of choice can be used with a diagonalization argument to produce a failure of this property. In fact, using the axiom of choice, there is an equivalence relation with classes of size at most two so that for any σ-ideal I and any I^+ $\mathbf{\Delta}^1_1$ set C, $E \upharpoonright C$ is not $\mathbf{\Delta}^1_1$.

For the regularity properties mentioned above, it is consistent that all sets have these properties in a choiceless model of ZF, like the model $L(\mathbb{R})$. For instance, if the axiom of determinacy, AD, holds then all sets are Lebesgue measurable and have the property of Baire.

Assuming determinacy for certain games on the reals, every equivalence relation with all $\mathbf{\Delta}^1_1$ classes can be canonicalized by certain σ-ideals whose associated forcings are proper:

Theorem 5.4. Assume $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$. Let R be a binary relation on ${}^\omega\omega$. If R has all $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$) sections, then for every nonmeager $\mathbf{\Delta}^1_1$ set B, there is a $\mathbf{\Delta}^1_1$ set $C \subseteq B$ which is comeager in B so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$, respectively).

Theorem 5.5. Assume $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} + V = L(\mathscr{P}(\mathbb{R}))$. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is absolutely proper. Let R be a binary relation on ${}^\omega\omega$. If R has all $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$) sections, then for every I^+ $\mathbf{\Delta}^1_1$ set B, there is an I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$, respectively).

The notion of an absolutely proper forcing is defined in [13]. Definable forcings which are proper under AC often are absolutely proper under AD. The proof of the above theorem requires some absoluteness given by an embedding theorem for absolutely proper forcing under determinacy assumptions which is analogous to the proper forcing embedding theorem shown in [14] which holds under AC with large cardinals. [13] has also used a stronger form of the embedding theorem for absolutely proper forcings to establish a positive answer under AD^+ to a more general form of Question 1.10 for σ-ideals with associated forcing absolutely proper. An exposition of Neeman and Norwood's method using absoluteness and the embedding theorem in a simple setting can be found in [1]. [1] also shows that the conclusion of Theorem 4.3 is consistent with the existence of a remarkable cardinal which is far weaker than the assumption that there is a measurable cardinal with infinitely many Woodin cardinals below it.

Section 2 will review the basics of idealized forcing, the theory of measure, and homogeneous trees. The relevant game concepts will be introduced here.

Section 3 will prove that certain types of binary relations can be $\mathbf{\Delta}^1_1$, $\mathbf{\Sigma}^1_1$, or $\mathbf{\Pi}^1_1$ relations on I^+ $\mathbf{\Delta}^1_1$ subsets of ${}^\omega\omega$ for any σ-ideal I so that $\mathbb{P}_I$ is proper, under three assumptions about absoluteness and tree representations. The main results of this section will be proved using a certain game. This section can be understood with just basic knowledge of set theory and forcing. The results of this section are stated for binary relations with $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$) sections; however, all the theorems in this paper have an analogous statement for equivalence relations with all classes $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$) or for graphs G so that for all $x \in {}^\omega\omega$, the set $G_x = \{y : x \; G \; y\}$ is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$).

Section 4 will mostly assume the axiom of choice and will give a general situation in which the three assumptions used in the previous section hold. This section will give a very brief survey of the theory of generic absoluteness and tree representations of subsets of ${}^\omega\omega$, especially the Martin-Solovay tree construction. Theorem 4.3 will be presented.

Section 5 will assume a bit more than the axiom of determinacy for the reals and will mention the necessary results about tree representations and generic absoluteness to show that the three assumptions from Section 3 hold for every binary relation with all $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, or $\mathbf{\Delta}^1_1$ sections and ideals with sufficient absoluteness. Finally, Theorem 5.4 and Theorem 5.5 will be presented.

The authors would like to thank Alexander Kechris, Itay Neeman, and Zach Norwood for many useful discussions about the contents of this paper.

2. Basics

In this paper, σ-ideals always contain all the singleton.

Definition 2.1. Let I be a σ-ideal on ${}^\omega\omega$. Let $\mathbb{P}_I = (\boldsymbol{\Delta}^1_1 \setminus I, \subseteq, {}^\omega\omega)$ be the forcing of I^+ $\boldsymbol{\Delta}^1_1$ subsets of ${}^\omega\omega$ ordered by $\leq_{\mathbb{P}_I} = \subseteq$ and has largest element $1_{\mathbb{P}_I} = {}^\omega\omega$. Often $\mathbb{P}_I$ is identified with $\boldsymbol{\Delta}^1_1 \setminus I$.

Fact 2.2. *Let I be a σ-ideal on ${}^\omega\omega$. There is a name $\dot{x}_{\text{gen}} \in V^{\mathbb{P}_I}$ so that for all $\mathbb{P}_I$-generic filters G over V and all $\boldsymbol{\Delta}^1_1$ sets B coded in V, $V[G] \models B \in G \Leftrightarrow \dot{x}_{\text{gen}}[G] \in B$.*

Proof. See [17], Proposition 2.1.2. ☐

Definition 2.3. Let I be a σ-ideal on ${}^\omega\omega$. Let $M \prec H_\Xi$ be a countable elementary substructure for some sufficiently large cardinal Ξ. $x \in {}^\omega\omega$ is $\mathbb{P}_I$-generic over M if and only if the collection $\{B \in \mathbb{P}_I \cap M : x \in B\}$ is a $\mathbb{P}_I$-generic filter over M.

The following results make available some very useful techniques for handling ideals whose associated forcings are proper forcings. For the purpose of this paper, the following may as well be taken as the definition of properness:

Proposition 2.4. *Let I be a σ-ideal on ${}^\omega\omega$. The following are equivalent:*
(i) $\mathbb{P}_I$ is a proper forcing.
(ii) For any sufficiently large cardinal Ξ, every $B \in \mathbb{P}_I$, and every countable $M \prec H_\Xi$ with $\mathbb{P}_I \in M$ and $B \in M$, the set $C = \{x \in B : x \text{ is } \mathbb{P}_I\text{-generic over } M\}$ is an I^+ $\boldsymbol{\Delta}^1_1$ set.

Proof. See [17], Proposition 2.2.2. ☐

This proposition shows that σ-ideals whose associated forcings are proper may be useful for answering Question 1.6 since it indicates how to produce I^+ $\boldsymbol{\Delta}^1_1$ sets. It should be noted that some restrictions on the type of σ-ideals considered in Question 1.6 are necessary:

Let F_{ω_1} denote the countable admissible ordinal equivalence relation defined by $x \ F_{\omega_1} \ y$ if and only if $\omega_1^x = \omega_1^y$. F_{ω_1} is a thin $\boldsymbol{\Sigma}^1_1$ equivalence relation with all $\boldsymbol{\Delta}^1_1$ classes. Thin means that F_{ω_1} does not have a perfect set of pairwise F_{ω_1}-inequivalent elements. Let I be the σ-ideal which is σ-generated by the F_{ω_1}-classes. Suppose there is an I^+ $\boldsymbol{\Delta}^1_1$ set C so that $F_{\omega_1} \upharpoonright C$ is $\boldsymbol{\Delta}^1_1$. By definition of I, each F_{ω_1}-class is in I. So since C is I^+, C must intersect nontrivially uncountably many classes of F_{ω_1}. So $F_{\omega_1} \upharpoonright C$ has uncountable many classes. Since F_{ω_1} is thin, there is also no perfect set of $F_{\omega_1} \upharpoonright C$ inequivalent elements. This contradicts Silver's dichotomy (see Fact 5.2).

Of course, I is not proper or even ω_1-preserving: Let $G \subseteq \mathbb{P}_I$ be a $\mathbb{P}_I$-generic filter over V. Fact 2.2 implies that $\dot{x}_{\text{gen}}[G]$ is not in any ground model coded $\boldsymbol{\Delta}^1_1$ set in I. $\omega_1^{\dot{x}_{\text{gen}}[G]}$ can not be a countable admissible ordinal of V since if it was countable then a theorem of Sacks shows that there is a $z \in ({}^\omega\omega)^V$ so that $\omega_1^z = \omega_1^{\dot{x}_{\text{gen}}[G]}$. Then $x \in [z]_{F_{\omega_1}}$. By definition of I, $[z]_{F_{\omega_1}}$ is a $\boldsymbol{\Delta}^1_1$ set coded in V that belongs to I. Hence $\omega_1^{\dot{x}_{\text{gen}}[G]}$ must be an uncountable admissible ordinal of V, but in $V[G]$, $\omega_1^{\dot{x}_{\text{gen}}[G]}$ is a countable admissible ordinal. Hence $\mathbb{P}_I$ collapses ω_1.

Definition 2.5. A measure μ on a set X is a nonprincipal ultrafilter on X. Nonprincipal means for all $x \in X$, $\{x\} \notin \mu$.

If κ is a cardinal, then μ is κ-complete if and only if for all $\beta < \kappa$ and sequences $(A_\alpha : \alpha < \beta)$ with each $A_\alpha \in \mu$, $\bigcap_{\alpha<\beta} A_\alpha \in \mu$. $\aleph_1$-completeness is often called countable completeness.

Let $\text{meas}_\kappa(X)$ be the set of all κ-complete ultrafilters on X.

Suppose $\mu \in \text{meas}_{\aleph_1}({}^{<\omega}X)$. By countable completeness, there is a unique m so that ${}^mX \in \mu$. In this case, m is called the dimension of μ and this is denoted by $\dim(\mu) = m$.

Definition 2.6. Let X be a set. For $m \leq n < \omega$, let $\pi_{n,m} : {}^nX \to {}^mX$ be defined by $\pi_{n,m}(f) = f \upharpoonright m$.

Let $m \leq n < \omega$. Let ν be a measure of dimension m and μ be a measure of dimension n. μ is an extension of ν (or ν is a projection of μ) if and only if for all $A \in \nu$ with $A \subseteq {}^mX$, $\pi_{n,m}^{-1}[A] \in \mu$.

A tower of measures over X is a sequence $(\mu_n : n \in \omega)$ so that
(i) For all n, $\mu_n \in \text{meas}_{\aleph_1}({}^{<\omega}X)$ and $\dim(\mu_n) = n$.
(ii) For all $m \leq n < \omega$, μ_n is an extension of μ_m.

A tower of measures over X, $(\mu_n : n \in \omega)$, is countably complete if and only if for all sequence $(A_n : n \in \omega)$ with the property that for $n \in \omega$, $A_n \in \mu_n$, there exists a $f : \omega \to X$ so that for all $n \in \omega$, $f \restriction n \in A_n$.

Definition 2.7. A tree T on X is a subset of ${}^{<\omega}X$ so that if $s \subseteq t$ and $t \in T$, then $s \in T$.

If $s \in {}^{n}(X \times Y)$ where $n \in \omega$, then in a natural way, s be may be considered as a pair (s_0, s_1) with $s_0 \in {}^{n}X$ and $s_1 \in {}^{n}Y$.

Let T be a tree on X. The body of T, denoted $[T]$, is the set of infinite paths through T, that is $[T] = \{f \in {}^{\omega}X : (\forall n \in \omega)(f \restriction n \in T)\}$.

Suppose T is a tree on $X \times Y$. For each $s \in {}^{<\omega}X$, define $T^s = \{t \in {}^{|s|}Y : (s,t) \in T\}$. If $f \in {}^{\omega}X$, then define $T^f = \bigcup_{n \in \omega} T^{f \restriction n}$.

Let T be a tree on $X \times Y$, then

$$p[T] = \{f \in {}^{\omega}X : T^f \text{ is ill-founded}\} = \{f \in {}^{\omega}X : [T^f] \neq \emptyset\}$$

Definition 2.8. For any $k \in \omega$, $A \subseteq {}^{k}({}^{\omega}\omega)$ is $\mathbf{\Sigma}^1_1$ if and only if there exists a tree on ${}^{k}\omega \times \omega$ so that $A = p[T]$. $A \subseteq {}^{k}({}^{\omega}\omega)$ is $\mathbf{\Pi}^1_1$ if and only if $A = {}^{k}({}^{\omega}\omega) \setminus B$ for some $\mathbf{\Sigma}^1_1$ set $B \subseteq {}^{k}({}^{\omega}\omega)$. $A \subseteq {}^{k}({}^{\omega}\omega)$ is $\mathbf{\Delta}^1_1$ if and only if A is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

Definition 2.9. Let γ be an ordinal and $k \in \omega$. A tree T on ${}^{k}\omega \times \gamma$ is homogeneous if and only if there is a collection $(\mu_s : s \in {}^{<\omega}({}^{k}\omega))$ so that

(i) For each $s \in {}^{<\omega}({}^{k}\omega)$, $\mu_s \in \text{meas}_{\aleph_1}({}^{<\omega}\gamma)$ and concentrates on T^s (that is, $T^s \in \mu_s$).

(ii) For all $s, t \in {}^{<\omega}({}^{k}\omega)$, if $s \subseteq t$, then μ_t is an extension of μ_s.

(iii) For all $f \in p[T]$, $(u_{f \restriction n} : n \in \omega)$ is a countably complete tower of measures on γ.

A collection $(u_s : s \in {}^{<\omega}({}^{k}\omega))$ which witnesses the homogeneity of T is called a homogeneity system for T.

Let κ be a cardinal. The homogeneous tree T is κ-homogeneous if and only if each μ_s is κ-complete.

Definition 2.10. For any $k \in \omega$, $A \subseteq {}^{k}({}^{\omega}\omega)$ is homogeneously Suslin if and only if there exists an ordinal γ and a homogeneous tree on ${}^{k}\omega \times \gamma$ so that $A = p[T]$.

If the tree T is κ-homogeneous, then A is said to be κ-homogeneously Suslin.

Homogeneously Suslin sets have an important role in the theory of determinacy. In particular, games on ${}^{\omega}\omega$ associated with homogeneously Suslin sets are determined. Later, the homogeneity system of homogeneous trees will be used to show a certain player has a winning strategy in a particular game using techniques that are very similar to Martin's proof of $\mathbf{\Sigma}^1_1$ determinacy from a measurable cardinal.

Below, the basic setting of the relevant games will be described:

Definition 2.11. Let X be some set. Let $A \subseteq {}^{\omega}X$. The game associated to A, denoted G_A, is the following: The game has two players, Player 1 and Player 2, who alternatingly take turns playing elements of X with Player 1 playing first. The picture below denotes a partial play where Player 1 plays the sequence $(a_i : i \in \omega)$ and Player 2 plays the sequence $(b_i : i \in \omega)$.

$$\begin{array}{c|cccccccc} & a_0 & & a_1 & & \ldots & a_{k-1} & \\ \hline & & b_0 & & b_1 & \ldots & & b_{k-1} \end{array}$$

Player 2 is said to win this play of G_A if and only if the infinite sequence $(a_0 b_0 a_1 b_1 \ldots) \in A$. Otherwise Player 1 wins.

A function $\tau : {}^{<\omega}X \to X$ is a winning strategy for Player 1 if and only if for all sequence $(b_i : i \in \omega)$ played by Player 2, Player 1 wins by playing $(a_i : i \in \omega)$ where this sequence is defined recursively by $a_0 = \tau(\emptyset)$ and $a_{k+1} = \tau(a_0 b_0 \ldots a_k b_k)$.

A winning stategy $\tau : {}^{<\omega}X \to X$ for Player 2 is defined similarly.

The game G_A is determined if Player 1 or Player 2 has a winning strategy.

Let X be a set. ${}^{\omega}X$ is given the topology with basis $\{U_s : s \in {}^{<\omega}X\}$, where $U_s = \{f \in {}^{\omega}X : s \subseteq f\}$.

Fact 2.12. *([5], Gale-Stewart) If X is wellorderable and $A \subseteq {}^{\omega}X$ is open, then G_A is determined. Hence if A is closed, then G_A is also determined.*

3. The Game

Definition 3.1. Let R be a relation on $({}^\omega\omega)^2$. Let $R_x = \{y : (x, y) \in R\}$ and $R^x = \{y : (y, x) \in R\}$

The following results will be stated using the vertical sections R_x; however, the results hold using horizontal sections with the appropriate changes.

Definition 3.2. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$, where γ is some ordinal.
Denote $p[S]$ by R_S. Denote $({}^\omega\omega \times {}^\omega\omega) \setminus p[S] = R^S$.

Definition 3.3. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$ for some ordinal γ. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper.
Assumption A_Σ asserts that $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} \check{S}$ is a homogeneous tree.
Assumption A_Π asserts $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} \check{S}$ is a homogeneous tree.

Assumption A_Σ and A_Π just assert that the tree S remains homogeneous in $\mathbb{P}_I$-generic extensions. (Since the completeness of countably complete measures is a measurable cardinal and $|\mathbb{P}_I|$ is always less than a measurable cardinal under AC, this is always true under AC.)

Definition 3.4. Let S be a homogeneously Suslin tree on $\omega \times \omega \times \gamma$ for some ordinal γ. Let I be a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is a proper forcing.
Let D_Σ be the formula on ${}^\omega\omega \times {}^\omega\omega$ asserting:

$$D_\Sigma(x, T) \Leftrightarrow (T \text{ is tree on } \omega \times \omega) \wedge (\forall y)(R_S(x, y) \Leftrightarrow T^y \text{ is ill-founded})$$

Let D_Π be the formula on ${}^\omega\omega \times {}^\omega\omega$ asserting:

$$D_\Pi(x, T) \Leftrightarrow (T \text{ is a tree on } \omega \times \omega) \wedge (\forall y)(\neg(R^S(x, y)) \Leftrightarrow T^y \text{ is ill-founded})$$

If $D_\Sigma(x, T)$ holds, then T is a tree which witnesses $(R_S)_x$ is $\mathbf{\Sigma}^1_1$. Similarly, if $D_\Pi(x, T)$ holds, then T is a tree which witnesses ${}^\omega\omega \setminus (R^S)_x$ is $\mathbf{\Sigma}^1_1$, i.e. $(R^S)_x$ is $\mathbf{\Pi}^1_1$.

Definition 3.5. Let S be a homogeneously Suslin tree on $\omega \times \omega \times \gamma$ for some ordinal γ. Let I be a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is a proper forcing.
Let assumption B_Σ say: $(\forall x)(\exists T)D_\Sigma(x, T)$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} (\forall x)(\exists T)D_\Sigma(x, T)$.
Let assumption B_Π say: $(\forall x)(\exists T)D_\Pi(x, T)$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} (\forall x)(\exists T)D_\Pi(x, T)$.

Assumption B_Σ states that all R_S sections are $\mathbf{\Sigma}^1_1$ and all R_S sections remain $\mathbf{\Sigma}^1_1$ in $\mathbb{P}_I$-generic extensions. Similarly, assumption B_Π states that all R^S sections are $\mathbf{\Pi}^1_1$ and all R^S sections remain $\mathbf{\Pi}^1_1$ in $\mathbb{P}_I$-generic extensions.

Definition 3.6. Let S be a homogeneously Suslin tree on $\omega \times \omega \times \gamma$ for some ordinal γ. Let I be a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is a proper forcing.
Let assumption C_Σ state: There is an ordinal ϵ and a tree U on $\omega \times \omega \times \epsilon$ so that $p[U] = \{(x, T) : D_\Sigma(x, T)\}$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} p[\check{U}] = \{(x, T) : D_\Sigma(x, T)\}$.
Let assumption C_Π state: There is an ordinal ϵ and a tree U on $\omega \times \omega \times \epsilon$ so that $p[U] = \{(x, T) : D_\Pi(x, T)\}$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} p[\check{U}] = \{(x, T) : D_\Pi(x, T)\}$.

Assumption C_Σ states that the set defined by D_Σ has a tree representation that continues to represent the formula D_Σ in $\mathbb{P}_I$-generic extensions. C_Π is similar.
The following game plays an important role the next theorem.

Definition 3.7. Let S be a tree on $\omega \times \omega \times \gamma$. Let T be a tree on $\omega \times \omega$. Let $g \in {}^\omega\omega$.
Consider the following game $G^{g,T}$:

$$\begin{array}{c|cccccc} & m_0, n_0 & & m_1, n_1 & & \ldots \; m_{k-1}, n_{k-1} & \\ \hline & & \alpha_0 & & \alpha_1 & \ldots & \alpha_{k-1} \end{array}$$

The rules are:
(1) Player 1 plays $m_i, n_i \in \omega$. Player 2 plays $\alpha_i < \gamma$.
(2) $(m_0 ... m_{k-1}, n_0 ... n_{k-1}) \in T$
(3) $(g \upharpoonright k, m_0 ... m_{k-1}, \alpha_0 ... \alpha_{k-1}) \in S$.
The first player to violate these rules loses. If the game continues forever, then Player 2 wins.
This game is open for Player 1 and hence closed for Player 2.

The following shows under certain assumptions a more general canonicalization property holds for relations. [3] defines this phenomenon as the rectangular canonization property.

Theorem 3.8. *Let γ be an ordinal. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Assume A_Σ, B_Σ, and C_Σ hold for S and I.*

Then for any I^+ $\mathbf{\Delta}^1_1$ set $B \subseteq {}^\omega\omega$, there exists an I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $R_S \cap (C \times {}^\omega\omega)$ is an $\mathbf{\Sigma}^1_1$ relation.

Proof. Let U be the tree on $\omega \times \omega \times \epsilon$ witnessing C_Σ for S and I.

Let $M \prec H_\Xi$ be a countable elementary substructure with Ξ sufficiently large and $B, I, \mathbb{P}_I, S, U \in M$.

Claim 1 : Let g be $\mathbb{P}_I$-generic over M. If $x, T \in M[g]$ and $M[g] \models D_\Sigma(x,T)$, then $V \models D_\Sigma(x,T)$.

Proof of Claim 1: By assumption C_Σ for S and I and the fact that $M \prec H_\Xi$, $M[g] \models D_\Sigma(x,T)$ implies $M[g] \models (x,T) \in p[U]$. There exists some $f \in M[g]$ with $f : \omega \to \epsilon$ so that $M[g] \models (x,T,f) \in [U]$. Hence for each $n \in \omega$, $M[g] \models (x \restriction n, T \restriction n, f \restriction n) \in U$. For each $n \in \omega$, $(x \restriction n, T \restriction n, f \restriction n) \in M$. So by absoluteness, $M \models (x \restriction n, T \restriction n, f \restriction n) \in U$. For all $n \in \omega$, $V \models (x \restriction n, T \restriction n, f \restriction n) \in U$. $V \models (x,T) \in p[U]$. $V \models D_\Sigma(x,T)$.

Now fix a $g \in {}^\omega\omega$ so that g is $\mathbb{P}_I$-generic over M.

As $M \prec H_\Xi$, $M \models (\forall x)(\exists T) D_\Sigma(x,T)$. $M[g] \models (\forall x)(\exists T) D_\Sigma(x,T)$ by assumption B_Σ and the fact that $M \prec H_\Xi$. So fix a tree T on $\omega \times \omega$ so that $M[g] \models D_\Sigma(g,T)$.

Claim 2 : In $M[g]$, Player 2 has a winning strategy in the game $G^{g,T}$.

Proof of Claim 2: Work in $M[g]$: By an appropriate coding, $G^{g,T}$ is equivalent to a game G_A, where $A \subseteq {}^\omega\gamma$ is an open subset.

Suppose Player 2 does not have a winning strategy. By Fact 2.12, Player 1 must have a winning strategy τ^*.

By assumption A_Σ, S is a homogeneous tree in $M[g]$. Let $(\mu_t : t \in {}^{<\omega}(\omega \times \omega))$ be a homogeneity system witnessing the homogeneity of S.

Now two sequences of natural numbers, $(a_i : i \in \omega)$ and $(b_i : i \in \omega)$, and a sequence $(A_n : n \in \omega)$ so that $A_n \subseteq {}^n\gamma$ will be constructed (in $M[g]$) by recursion:

Let $a_0, b_0 \in \omega$ so that $(a_0, b_0) = \tau^*(\emptyset)$. Let $A_0 = \{\emptyset\}$.

Suppose $a_0, ..., a_{k-1}$, $b_0, ..., b_{k-1}$, and $A_0, ..., A_{k-1}$ has been constructed. Define the function

$$h_k : S^{(g \restriction k, a_0 ... a_{k-1})} \to \omega \times \omega$$

defined by

$$h_k(\beta_0 ... \beta_{k-1}) = \tau^*(a_0, b_0, \beta_0, ..., a_{k-1}, b_{k-1}, \beta_{k-1})$$

$\mu_{(g \restriction k, a_0 ... a_{k-1})}$ concentrates on $S^{(g \restriction k, a_0 ... a_{k-1})}$ and is countably complete; therefore, there is a unique (a_k, b_k) so that

$$h_k^{-1}[\{(a_k, b_k)\}] \in \mu_{(g \restriction k, a_0 ... a_{k-1})}$$

Let $A_k = h_k^{-1}[\{(a_k, b_k)\}]$.

This completes the construction of $(a_i : i \in \omega)$, $(b_i : i \in \omega)$, and $(A_i : i \in \omega)$.

Let $L \in {}^\omega(\omega \times \omega)$ be such that for all $i \in \omega$, $L(i) = (a_i, b_i)$. Note that $L \in [T]$. To see this, suppose not. Then there is some least $k \in \omega$ so that $L \restriction (k+1) = (a_0 ... a_k, b_0 ... b_k) \notin T$. For $i \leq k$, define $\mu_i = \mu_{g \restriction i, a_0 ... a_{i-1}}$. For $0 \leq i \leq j \leq k$, let $\pi_{j,i} : {}^j\gamma \to {}^i\gamma$ be defined by $\pi_{j,i}(s) = s \restriction i$. By definition of the homogeneity system for S, for $0 \leq i \leq j \leq k$, μ_j is an extension of μ_i. Hence for all $0 \leq i \leq k$, $\pi_{k,i}^{-1}[A_i] \in \mu_k$. By countable completeness of μ_k, $\bigcap_{0 \leq i \leq k} \pi_{k,i}^{-1}[A_i] \in \mu_k$. Let $(\beta_0 ... \beta_{k-1}) \in \bigcap_{0 \leq i \leq k} \pi_{k,i}^{-1}[A_i]$. Consider the following play of $G^{g,T}$ where player 1 uses the strategy τ^* and Player 2 plays $(\beta_0 ... \beta_{k-1})$:

$$\begin{array}{c|cccccccc} & a_0, b_0 & & a_1, b_1 & & ... & a_{k-1}, b_{k-1} & & a_k, b_k \\ \hline & & \beta_0 & & \beta_1 & ... & & \beta_{k-1} & \end{array}$$

Note that for all $0 \leq i \leq k$, $(\beta_0 ... \beta_{i-1}) \in A_i = h_i^{-1}[\{(a_i, b_i)\}] \subseteq S^{(g \restriction i, a_0 ... a_{i-1})}$. So rule (3) of the game $G^{g,T}$ is not violated by Player 2. However, $(a_0 ... a_k, b_0 ... b_k) = L \restriction (k+1) \notin T$. Player 1 violates rule (2) and is the first player to violate any rules. Player 1 loses this game. This contradicts the assumption that τ^* is a winning strategy for Player 1. So this completes the proof that $L \in [T]$.

Let $\mathfrak{a} = (a_i : i \in \omega)$. Since $L \in [T]$ and $D_\Sigma(g, T)$, this implies that $R_S(g, \mathfrak{a})$.

Now let $J \in {}^\omega(\omega \times \omega)$ be such that for all $k \in \omega$, $J \upharpoonright k = (g \upharpoonright k, a_0 ... a_{k-1})$. Then by definition of S, $J \in p[S]$. Since S is a homogeneous tree via $(u_t : t \in {}^{<\omega}(\omega \times \omega))$, $(\mu_{J \upharpoonright k} : k \in \omega)$ is a countably complete tower of measures.

Each $A_k \in \mu_{g \upharpoonright k, a_0 ... a_{k-1}} = \mu_{J \upharpoonright k}$. So by the countable completeness of the tower, there exists some $\Phi : \omega \to \gamma$ so that for all $k \in \omega$, $\Phi \upharpoonright k \in A_k$. Now consider the play of $G^{g,T}$ where Player 1 uses its winning strategy τ^* and Player 2 plays Φ. By construction of the sequences $(a_i : i \in \omega)$, $(b_i, i \in \omega)$, and $(A_i : i \in \omega)$, the game looks as follows:

$$\begin{array}{cccccccc} a_0, b_0 & & a_0, b_1 & & ... & a_{k-1}, b_{k-1} & \\ \hline & \Phi(0) & & \Phi(1) & ... & & \Phi(k-1) \end{array}$$

Neither players violate any rules in this play. Hence the game continues forever, and so Player 2 wins this play of $G^{g,T}$. This contradicts the fact that τ^* was a winning strategy for Player 1.

So Player 1 could not have had a winning strategy. Player 2 must have a winning strategy in $G^{g,T}$. This completes the proof of Claim 2.

By Claim 2, Player 2 has a winning strategy $\tau \in M[g]$.

<u>Claim 3</u> : τ is a winning strategy for $G^{g,T}$ in V.

Proof of Claim 3: Suppose the following is a play of $G^{g,T}$ (in V) in which Player 2 uses τ and loses

$$\begin{array}{cccccccc} m_0, n_0 & & m_1, n_1 & & ... & m_{k-1}, n_{k-1} & \\ \hline & \alpha_0 & & \alpha_1 & ... & & \alpha_{k-1} \end{array}$$

Since $\tau \in M[g]$ and ${}^{<\omega}\omega \subseteq M[g]$, this entire finite play belongs to $M[g]$. So, Player 2 loses this game in $M[g]$, as well. This contradicts τ being a winning strategy in $M[g]$. This completes the proof of Claim 3.

<u>Claim 4</u> : For all $y \in {}^\omega\omega$, $R_S(g, y)$ if and only if $(S \cap M)^{(g,y)}$ is ill-founded.

Proof of Claim 4: By Claim 1, $M[g] \models D_\Sigma(g, T)$ implies $V \models D_\Sigma(g, T)$. Hence in V, T gives the $\mathbf{\Sigma}^1_1$ definition of $(R_S)_g$.

Suppose $R_S(g, y)$. Then T^y is ill-founded. Let $f \in [T^y]$. Consider the following play of the game $G^{g,T}$ where Player 1 plays y and f, and Player 2 responds using its winning strategy τ.

$$\begin{array}{cccccccc} y(0), f(0) & & y(1), f(1) & & ... & y(k-1), f(k-1) & \\ \hline & \alpha_0 & & \alpha_1 & ... & & \alpha_{k-1} \end{array}$$

Since $f \in [T^y]$, Player 1 can not lose. Since τ is a winning strategy for Player 2, Player 2 also does not lose at a finite stage. Hence Player 2 wins by having the game continue forever. Let $\Phi : \omega \to \gamma$ be the sequence coming from Player 2's response, i.e. for all k, $\Phi(k) = \alpha_k$.

Since $\tau \in M[g]$ and ${}^{<\omega}\omega \subseteq M[g]$, each finite partial play of $G^{g,T}$ above belongs to $M[g]$. Hence $\Phi \upharpoonright k \in M[g]$ for all $k \in \omega$. As $\mathrm{On}^M = \mathrm{On}^{M[g]}$, $(g \upharpoonright k, y \upharpoonright k, \Phi \upharpoonright k) \in (S \cap M)$ for all $k \in \omega$.

It has been shown that $R_S(g, y)$ implies $(S \cap M)^{(g,y)}$ is ill-founded.

Of course, if $(S \cap M)^{(g,y)}$ is ill-founded, then $S^{(g,y)}$ is ill-founded. By definition, $R_S(g, y)$.

This completes the proof of Claim 4.

Let $b : \omega \to \mathrm{On}^M$ be a bijection. Define a new tree S' on $\omega \times \omega \times \omega$ by $(s_1, s_2, s_3) \in S' \Leftrightarrow (s_1, s_2, b \circ s_3) \in S$.

By Fact 2.4, let $C \subseteq B$ be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic reals over M inside B. By Claim 4, for all $y \in {}^\omega\omega$, $R_S(g, y) \Leftrightarrow (S')^{(g,y)}$ is ill-founded. $R_S \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$. The proof of the theorem is complete. $\square$

Theorem 3.9. *Let γ be an ordinal. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Assume A_Π, B_Π, and C_Π hold for S and I.*

Then for any I^+ $\mathbf{\Delta}^1_1$ set $B \subseteq {}^\omega\omega$, there exists an I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $R^S \cap (C \times {}^\omega\omega)$ is a $\mathbf{\Pi}^1_1$ relation.

Proof. The proof of this is very similar to the proof of Theorem 3.8. $\square$

Theorem 3.10. *Let γ and ν be ordinals. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$. Let U be a homogeneous tree on $\omega \times \omega \times \nu$. Suppose $p[S] = ({}^\omega\omega \times {}^\omega\omega) \setminus p[U]$. Let $R = R_S = R^U$. Let I be a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is a proper forcing. Suppose A_Σ, B_Σ, and C_Σ holds for S and I. Suppose A_Π, B_Π, and C_Π holds for U and I.*

Then for any I^+ $\boldsymbol{\Delta}^1_1$ set $B \subseteq {}^\omega\omega$, there exists an I^+ $\boldsymbol{\Delta}^1_1$ set $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is a $\boldsymbol{\Delta}^1_1$ relation.

Proof. By Theorem 3.8, there is some I^+ $\boldsymbol{\Delta}^1_1$ set $C' \subseteq B$ so that $R \cap (C' \times {}^\omega\omega)$ is $\boldsymbol{\Sigma}^1_1$. By Theorem 3.9, there is some I^+ $\boldsymbol{\Delta}^1_1$ set $C \subseteq C'$ so that $R \cap (C \times {}^\omega\omega)$ is $\boldsymbol{\Pi}^1_1$. Therefore, $R \cap (C \times {}^\omega\omega)$ is $\boldsymbol{\Delta}^1_1$. $\square$

If the above assumptions holds and $R_S = E$ defines an equivalence relation with all $\boldsymbol{\Sigma}^1_1$ classes, then there is some I^+ $\boldsymbol{\Delta}^1_1$ set so that $E \restriction C$ is an $\boldsymbol{\Sigma}^1_1$ equivalence relation.

Similarly, suppose $R_S = G$ is a graph on ${}^\omega\omega$. Then $G_x = \{y : x \; G \; y\}$ is the set of neighbors of x. Suppose G_x is $\boldsymbol{\Sigma}^1_1$ for all x. Then there is an I^+ $\boldsymbol{\Delta}^1_1$ set C so that the induced subgraph $G \restriction C$ is an $\boldsymbol{\Sigma}^1_1$ graph.

4. Canonicalization for Relations in $L(\mathbb{R})$

The main results of this section are the following. The definition of some terms are stated further below.

Theorem 4.1. *Let λ be a limit of Woodin cardinals. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Let $R \in \mathrm{Hom}_{<\lambda}$ be a binary relation on ${}^\omega\omega$.*

If R has all $\boldsymbol{\Sigma}^1_1$ ($\boldsymbol{\Pi}^1_1$ or $\boldsymbol{\Delta}^1_1$) sections, then for every I^+ $\boldsymbol{\Delta}^1_1$ set B, there is an I^+ $\boldsymbol{\Delta}^1_1$ $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is $\boldsymbol{\Sigma}^1_1$ ($\boldsymbol{\Pi}^1_1$ or $\boldsymbol{\Delta}^1_1$, respectively).

Theorem 4.2. *Suppose there are infinitely many Woodin cardinals. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Let R be a projective binary relation on ${}^\omega\omega$.*

If R has all $\boldsymbol{\Sigma}^1_1$ ($\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Delta}^1_1$) sections, then for every I^+ $\boldsymbol{\Delta}^1_1$ set B, there is an I^+ $\boldsymbol{\Delta}^1_1$ $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is $\boldsymbol{\Sigma}^1_1$ ($\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Delta}^1_1$, respectively).

Theorem 4.3. *Suppose there is a measurable cardinal with infinitely many Woodin cardinals below it. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Let $R \in L(\mathbb{R})$ be a binary relation on ${}^\omega\omega$.*

If R has all $\boldsymbol{\Sigma}^1_1$ ($\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Delta}^1_1$) sections, then for every I^+ $\boldsymbol{\Delta}^1_1$ set B, there is an I^+ $\boldsymbol{\Delta}^1_1$ $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is $\boldsymbol{\Sigma}^1_1$ ($\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Delta}^1_1$, respectively).

See the remarks at the end of Section 3 indicating how to apply these results to answer Question 1.10 for equivalence relations and graphs.

This section will provide a brief description of the theory of tree representations of subsets of ${}^\omega\omega$ and absoluteness. This will be used to indicate some circumstances in which the assumptions A_Σ, B_Σ, C_Σ, A_Π, B_Π, and C_Π hold. The results of the previous section will be applied to some familiar classes of binary relations. The following discussion is in $\mathsf{ZF} + \mathsf{DC}$ until it is explicitly mentioned that AC will be assumed.

Definition 4.4. Let κ be a cardinal. A κ-weak homogeneity system with support some ordinal γ is a sequence of κ-complete measures on ${}^{<\omega}\gamma$, $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$, so that

(i) If $s \neq t$, then $\mu_s \neq \mu_t$.

(ii) $\dim(\mu_s) \leq |s|$.

(iii) If μ_s is an extension of some measure ν, then there exists some $k < |s|$ so that $\mu_{s \restriction k} = \nu$.

Define $W_{\bar{\mu}}$ by

$$W_{\bar{\mu}} = \{x \in {}^\omega\omega : (\exists f \in {}^\omega\omega)(f \text{ is an increasing sequence} \wedge (\mu_{x \restriction f(k)} : k \in \omega) \text{ is a countably complete tower})\}$$

A set $A \subseteq {}^\omega\omega$ is κ-weakly homogeneous if and only there is a κ-weak homogenity system $\bar{\mu}$ so that $A = W_{\bar{\mu}}$.

Definition 4.5. Let γ be an ordinal. A tree on $\omega \times \gamma$ is κ-weakly homogeneous if and only there is some κ-weak homogeneity system $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ so that $p[T] = W_{\bar{\mu}}$ and for all $s \in {}^{<\omega}\omega$, there is some $k \leq |s|$ so that μ_s concentrates on $T^{s \restriction k}$.

$A \subseteq {}^\omega\omega$ is κ-weakly homogeneously Suslin if and only if $A = p[T]$ for some tree T which is κ-weakly homogeneous.

Fact 4.6. *If $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ is a κ-weak homogeneity system with support γ, then there is a tree T on $\omega \times \gamma$ so that $\bar{\mu}$ witnesses T is κ-weakly homogeneously Suslin.*

Hence a set is κ-weakly homogeneous if and only if it is κ-weakly homogeneously Suslin.

Proof. See [16], Proposition 1.12. ☐

Definition 4.7. Let μ be a countably complete measure on ${}^{<\omega}X$. Let M_μ be the Mostowski collapse of the ultrapower $\mathrm{Ult}(V,\mu)$. Let $j_\mu : V \to M_\mu$ be the composition of the ultrapower map and the Mostowski collapse map.

Suppose ν and μ are countably complete measures on ${}^{<\omega}X$. Suppose for some $m \leq n$, $\dim(\mu) = m$ and $\dim(\nu) = n$, and ν is an extension of μ. Define $\Lambda_{m,n} : {}^{{}^mX}V \to {}^{{}^nX}V$ by $\Lambda_{m,n}(f)(s) = f(s \restriction m)$ for each $s \in {}^nX$. Define an elementary embedding $\mathrm{Ult}(V,\mu) \to \mathrm{Ult}(V,\nu)$ by $[f]_\mu \mapsto [\Lambda_{m,n}(f)]_\nu$. This induces an elementary embedding $j_{\mu,\nu} : M_\mu \to M_\nu$.

Definition 4.8. Let γ and θ be ordinals. Let $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ be a weak homogeneity system with support γ. The Martin-Solovay tree with respect to $\bar{\mu}$ below θ, denoted $\mathrm{MS}_\theta(\bar{\mu})$, is a tree on $\omega \times \theta$ defined by: for all $s \in {}^{<\omega}\omega$ and $h \in {}^{|s|}\theta$

$$(s,h) \in \mathrm{MS}_\theta(\bar{\mu}) \Leftrightarrow (\forall i < j < |s|)(\mu_{s\restriction j} \text{ is an extension of } \mu_{s\restriction i} \Rightarrow j_{\mu_{s\restriction i},\mu_{s\restriction j}}(h(i)) > h(j))$$

If $(u_n : n \in \omega)$ is a tower of measure, then the tower is countably complete if and only if the directed limit of the directed system $(M_{\mu_i} : j_{\mu_i,\mu_j} : i < j < \omega)$ is well-founded. Suppose $x \in p[\mathrm{MS}_\theta(\bar{\mu})]$. If $(x,\Phi) \in [\mathrm{MS}_\theta(\bar{\mu})]$, then Φ witnesses in a continuous way that the directed limit model is ill-founded. This shows that $x \in p[\mathrm{MS}_\theta(\bar{\mu})]$ implies that $x \notin W_{\bar{\mu}}$. In fact, the converse is also true giving the following result:

Fact 4.9. *($\mathsf{ZF} + \mathsf{DC}$) Let κ be a cardinal. Suppose $\bar{\mu}$ is a κ-weak homogeneity system with support γ. Then if $\theta > |\gamma|^+$, then $p[\mathrm{MS}_\theta(\bar{\mu})] = {}^\omega\omega \setminus W_{\bar{\mu}}$.*

Proof. See [16] Lemma 1.19, [8] Fact 1.3.12, or [6] Theorem 4.10. ☐

Let μ be a κ-complete ultrafilter on some set X. Let $\mathbb{P}$ be a forcing with $|\mathbb{P}| < \kappa$. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. It can be shown that if $f^* : X \to V$ is a function in $V[G]$, then there is a function $f \in V$ and $A \in \mu$ so that $V[G] \models (\forall x \in A)(f(x) = f^*(x))$.

In $V[G]$, define $\mu^* \subseteq \mathcal{P}(X)$ by $A \in \mu^*$ if and only there exists a $B \in \mu$ so that $B \subseteq A$. In $V[G]$, μ^* is a κ-complete ultrafilter on X. Let M_{μ^*} denote the Mostowski collapse of $\mathrm{Ult}(V[G],\mu^*)$. Let $j^*_{\mu^*} : V[G] \to M_{\mu^*}$ be the induced elementary embedding.

In $V[G]$, $\mathrm{Ult}(V,\mu)$ can be embedded into $\mathrm{Ult}(V[G],\mu^*)$ as follows: for all $f \in ({}^XV) \cap V$, $[f]_\mu \mapsto [f]_{\mu^*}$. If $f \in ({}^XV) \cap V$ and $g' \in {}^XV[G]$ are such that $\mathrm{Ult}(V[G],\mu^*) \models [g']_{\mu^*} \in [f]_{\mu^*}$, then $\{x \in X : g'(x) \in f(x)\} \in \mu^*$. Therefore, one can find a $g^* \in V[G]$ so that $g^* : X \to V$ and $[g']_{\mu^*} = [g^*]_{\mu^*}$. By the above observation, one can find a $g \in V$ so that $[g]_{\mu^*} = [g^*]_{\mu^*} = [g']_{\mu^*}$. This shows that $\mathrm{Ult}(V,\mu)$ is identified (via the embedding above) as an $\in^{\mathrm{Ult}(V[G],\mu^*)}$-initial segment of $\mathrm{Ult}(V[G],\mu^*)$. After Mostowski collapsing the ultrapowers, it can be seen that $j_{\mu^*} \restriction M_\mu = j_\mu$.

Suppose $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ is a κ-weak homogeneity system. Denote $\bar{\mu}^* = (\mu^*_s : s \in {}^{<\omega}\omega)$. $\bar{\mu}^*$ is a κ-weak homogeneity system. From the construction, the Martin-Solovay trees depends only on $j_{\mu^*_s} \restriction \mathrm{ON}$. So by the above discussion, $\mathrm{MS}_\theta(\bar{\mu})^V = \mathrm{MS}_\theta(\bar{\mu}^*)^{V[G]}$. Hence Fact 4.9 implies that $V[G] \models p[\mathrm{MS}_\theta(\bar{u})] = {}^\omega\omega \setminus W_{\bar{\mu}^*}$.

(The above argument can be applied to a κ-homogeneous tree S and its witnessing κ-homogeneity system $\bar{\mu}$ to show that if $|\mathbb{P}| < \kappa$, then $\bar{\mu}^*$ is a κ-weak homogeneity system for S in $V[G]$. Assuming the axiom of choice, this shows assumption A_Σ and A_Π.)

Now suppose that T is a κ-weakly homogeneous tree on $\omega \times \alpha$ witnessed by the κ-weak homogeneity system $\bar{\mu}$. This gives that $p[T] = W_{\bar{\mu}}$. One seeks to show that $p[\mathrm{MS}_\theta(\bar{\mu})^V]$ continues to represent ${}^\omega\omega \setminus p[T]$ in $V[G]$. By the previous paragraph, it suffices to show that $V[G] \models p[T] = W_{\bar{\mu}^*}$: If $x \in W_{\bar{\mu}^*}$, then there is an increasing function $f : \omega \to \omega$ so that $(\mu^*_{x\restriction f(n)} : n \in \omega)$ is a countably complete tower. For all n, $\mu^*_{x\restriction f(n)}$ concentrates on $T^{x\restriction n}$. So by countably completeness, there is a path $\Phi \in [T^x]$. So $x \in p[T]$. Conversely, suppose $x \in p[T]$. Fact 4.9 implies that in V, T and $\mathrm{MS}_\theta(\bar{\mu})$ are complementing trees. By the absoluteness of well-foundedness, $V[G] \models \emptyset = p[T] \cap p[\mathrm{MS}_\theta(\bar{\mu})] = p[T] \cap p[\mathrm{MS}_\theta(\bar{\mu}^*)]$. So $x \notin p[\mathrm{MS}_\theta(\bar{\mu}^*)]$. Then applying Fact 4.9 in $V[G]$ to the weak homogeneity system $\bar{\mu}^*$, one obtains that $x \in W_{\bar{\mu}^*}$.

So in summary:

Fact 4.10. *($\mathsf{ZF} + \mathsf{DC}$) Let κ be a cardinal. Let T be a κ-weakly homogeneous tree on $\omega \times \gamma$, for some ordinal γ, with κ-weak homogeneity system $\bar{\mu}$. Let $\theta > |\gamma|^+$. Let $\mathbb{P}$ be a forcing with $|\mathbb{P}| < \kappa$ and $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V.*

$V[G] \models \mathrm{MS}_\theta(\bar{\mu}^*) = \mathrm{MS}_\theta(\bar{\mu})^V$. $V[G] \models p[\mathrm{MS}_\theta(\bar{\mu})^V] = {}^\omega\omega \setminus p[T]$.

Proof. See [16], Section 1 and especially Lemma 1.19. Also see [8], Section 1.3. □

So if T is κ-weakly homogeneous, an appropriate Martin-Solovay tree will continue to represent the complement of $p[T]$ in generic extensions by forcings of cardinality less than κ. The Martin-Solovay trees give the generically-correct tree representations for complements of κ-weakly homogeneously Suslin sets. However, the formulas D_Σ and D_Π involve more negations and quantifications over ${}^\omega\omega$. Multiple iterations of the Martin-Solovay construction will be needed. The following results are useful for continuing the Martin-Solovay construction of generically-correct tree representation for more complex sets. In addition, these results will also imply that these representations are also homogeneously Suslin. Until the end of this section, the axiom of choice will be assumed.

Definition 4.11. If $B \subseteq {}^k({}^\omega\omega) \times {}^\omega\omega$, denote

$$\exists^\mathbb{R} B = \{x : (\exists y)((x,y) \in B)\}$$

$$\forall^\mathbb{R} B = \{x : (\forall y)((x,y) \in B)\}$$

If $A \subseteq {}^k({}^\omega\omega)$, then denote

$$\neg A = {}^k({}^\omega\omega) \setminus A$$

Fact 4.12. *Let $A \subseteq {}^\omega\omega$. A is κ-weakly homogeneously Suslin if and only if there is a κ-homogeneously Suslin set $B \subseteq {}^\omega\omega \times {}^\omega\omega$ so that $A = \exists^\mathbb{R} B$.*

Proof. See [16], Proposition 1.10. □

A Woodin cardinal is a technical large cardinal which has been very useful in descriptive set theory. (See [8], Section 1.5 for more information about Woodin cardinals.)

Fact 4.13. *Let δ be a Woodin cardinal. Let $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ be a δ^+-weak homogeneity system with support $\gamma \in \mathrm{ON}$. Then for sufficiently large θ, $\mathrm{MS}_\theta(\bar{\mu})$ is κ-homogeneous for all $\kappa < \delta$.*

Proof. See [9]. □

Definition 4.14. If κ is a cardinal, then let Hom_κ be the collection of κ-homogeneously Suslin subsets of ${}^\omega\omega$. Let $\mathrm{Hom}_{<\kappa} = \bigcap_{\gamma<\kappa} \mathrm{Hom}_\gamma$.

The following are some well-known results on what sets can be in $\mathrm{Hom}_{<\lambda}$ when λ is limit of Woodin cardinals.

Fact 4.15. *(Martin-Steel) Let λ be a limit of Woodin cardinals. Then $\mathrm{Hom}_{<\lambda}$ is closed under complements and $\forall^\mathbb{R}$.*

Proof. See Section 2 of [16]. □

Fact 4.16. *(Martin) If κ is a measurable cardinal, then every $\mathbf{\Pi}^1_1$ set is κ-homogeneously Suslin.*

Proof. See [12], Theorem 4.15. □

Fact 4.17. *(Martin-Steel) Let λ be a limit of Woodin cardinals, then all projective sets are in $\mathrm{Hom}_{<\lambda}$.*

Proof. Every Woodin cardinal has a stationary set of measurable cardinals below it. Hence every $\mathbf{\Pi}^1_1$ set is κ-homogeneously Suslin for all $\kappa < \lambda$. That is, all $\mathbf{\Pi}^1_1$ sets are in $\mathrm{Hom}_{<\lambda}$. Then by the closure properties given by Fact 4.15, all projective sets are in $\mathrm{Hom}_{<\lambda}$. □

In fact, an even larger class of sets of reals can be homogeneously Suslin: $L(\mathbb{R})$ is the smallest transitive class model of ZF containing all the reals of V, i.e. $({}^\omega\omega)^V \subseteq L(\mathbb{R})$.

Fact 4.18. *(Woodin) Suppose λ is a limit of Woodin cardinals and there is a measurable cardinal greater than λ. Then every subset of ${}^\omega\omega$ in $L(\mathbb{R})$ is in $\mathrm{Hom}_{<\lambda}$.*

In the previous section, sets given by projections of certain trees were essentially identified with their trees. Homogeneously Suslin sets were defined to be those sets that can be presented as projections of some trees satisfying certain properties. In the ground model, there could be many homogeneous trees representing the same homogeneously Suslin set A. When considering generic extensions of the ground model, there is a question of which tree should be used to represent A in the generic extension. For instance, suppose $\kappa_1 < \kappa_2$. In the ground model, suppose $A = p[T_1]$ where T_1 is a κ_1-homogeneous tree and $A = p[T_2]$ where T_2 is a κ_2-homogeneous tree. Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are two different forcings. Which tree should represent A in each forcing extension? Are there circumstances in which one tree may be preferable over another? What are the relations between $p[T_1]$ and $p[T_2]$ in various forcing extensions?

Absolutely complemented trees and universal Baireness provide a way to interpret homogeneously Suslin sets in a way which is independent of the homogeneous tree representation in some sense:

Definition 4.19. (See [4]) Let κ be an ordinal. Let T be a tree on $\omega \times X$ and let U be a tree on $\omega \times Y$, for some sets X and Y. T and U are κ-absolute complements if and only if for all forcings $\mathbb{P} \in V_\kappa$ and all $G \subseteq \mathbb{P}$ which are $\mathbb{P}$-generic over V, $V[G] \models p[T] = {}^\omega\omega \setminus p[U]$.

A tree T on $\omega \times X$ is κ-absolutely complemented if and only if there exists some tree U on $\omega \times Y$ (for some set Y) so that T and U are κ-absolute complements.

A set $A \subseteq {}^\omega\omega$ is κ-universally Baire if and only if $A = p[T]$ for some tree T which is κ-absolutely complemented.

Fact 4.20. *Let T_1 and T_2 be trees on $\omega\times\gamma_1$ and $\omega\times\gamma_2$ which are κ-absolutely complemented and $p[T_1] = p[T_2]$. If $\mathbb{P} \in V_\kappa$ and $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V, then $V[G] \models p[T_1] = p[T_2]$.*

So if A is a κ-universally Baire set and if T_1 and T_2 are two κ-absolutely complemented trees so that $V \models A = p[T_1] = p[T_2]$, then either tree can be used to represent A in forcing extensions by forcings in V_κ. As a matter of convention, if A is κ-universally Baire and $\mathbb{P} \in V_\kappa$, the set A will always refer to $p[T]$ for some and any κ-absolutely complemented tree $T \in V$ so that $V \models p[T] = A$.

Fact 4.21. *(*[16] *Corollary 1.21) Let κ be a cardinal. κ-weakly homogenously Suslin sets are κ-universally Baire.*

In particular, κ-homogeneously Suslin sets can be interpreted unambiguously in $\mathbb{P}$-extensions whenever $\mathbb{P} \in V_\kappa$.

Let λ be a limit of Woodin cardinals. Let $\dot{A}$ be a new unary relation symbol. Let $A \subseteq ({}^\omega\omega)^n$ be such that $A \in \mathrm{Hom}_{<\lambda}$. Let $(H_{\aleph_1}, \in, A)$ be the $\{\dot{\in}, \dot{A}\}$-structure with domain $H_{\aleph_1}$ (the hereditarily countable sets) and with $\dot{A}$ interpreted as A. Now let $\mathbb{P} \in V_\lambda$ be some forcing and $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter over V. $\mathbb{P} \in V_\kappa$ for some $\kappa < \lambda$. The structure $(H_{\aleph_1}, \in, A^{V[G]})$ is understood in the following way: It is a structure with domain $H_{\aleph_1}^{V[G]}$ (the hereditarily countable subsets of $V[G]$) and $A^{V[G]}$ is $p[T]^{V[G]}$ for any γ-homogeneous tree T so that $V \models A = p[T]$ and $\gamma \geq \kappa$. By the above discussion, this is independent of which tree T is chosen. Actually, in the proof of the fact below, depending on the quantifier complexity of a particular formula φ involving $\dot{A}$, A will be considered as $p[T]$ for a sufficiently homogeneous tree T so that after the appropriate number of applications of the Martin-Solovay tree construction, the resulting tree representation of φ will be at least κ-homogeneous.

Using ideas very similar to the proof of Fact 4.15 (also see the proof of Fact 5.12 for Cohen forcing), one has the following absoluteness result:

Fact 4.22. *(Woodin) Let λ be a limit of Woodin cardinals. Let $A \in Hom_{<\lambda}$. Let $\mathbb{P} \in V_\lambda$ and $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Then $(H_{\aleph_1}^V, \in, A)$ and $(H_{\aleph_1}^{V[G]}, \in, A^{V[G]})$ are elementarily equivalent.*

Proof. See [16], Theorem 2.6. $\square$

In this setting, V and $V[G]$ satisfy the same formulas involving $\dot{A}$ and quantifications over the reals with the above intended interpretation. In particular, V and $V[G]$ satisfy the same projective formulas.

Now, the above discussion will be applied to indicate when assumptions A_Σ, B_Σ, C_Σ, A_Π, B_Π, and C_Π hold.

Let λ be a limit of Woodin cardinals. By the above discussion about universal Baireness, one may speak about a binary relation $R \in \mathrm{Hom}_{<\lambda}$ without explicit reference to a fix tree defining R. By Fact 4.15, if

$R \in \mathrm{Hom}_{<\lambda}$, then ${}^\omega\omega \setminus R \in \mathrm{Hom}_{<\lambda}$. Given an κ-weakly homogeneous tree representation of R for sufficiently large γ, the associated Martin-Solovay tree will be a sufficiently homogeneous tree representation of ${}^\omega\omega \setminus R$ by Fact 4.13. Hence in this setting, $R_S = R^T$, where T is the appropriate Martin-Solovay tree using the homogeneity system on S. (So if R_S has all $\mathbf{\Pi}^1_1$ classes, then the results of Section 3 should be applied to R^T using assumption A_Π, B_Π, C_Π for T and I.) Fix a σ-ideal I on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper.

The formula D_Σ and D_Π both involve complements and real quantification over the homogeneously Suslin set R. By Fact 4.15, $D_\Sigma, D_\Pi \in \mathrm{Hom}_{<\lambda}$. Starting with an appropriate weakly homogeneous tree representation of R, the process described in the proof of Fact 4.15 produces a tree U representing D_Σ or D_Π that is generically correct for $\mathbb{P}_I$, in the sense that $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} p[\check{U}] = \{(x,T) : D_\Sigma(x,T)\}$. So assumption C_Σ holds for R and I. (A similar argument holds for C_Π.)

R having all $\mathbf{\Sigma}^1_1$ sections can be expressed as a formula using some real quantifiers over the relation $R \in \mathrm{Hom}_{<\lambda}$. Fact 4.22 implies that these statements are absolute to the $\mathbb{P}_I$-extension. The tree S remains homogeneous in the $\mathbb{P}_I$-extension by the remark mentioned before Fact 4.10. This shows that A_Σ and B_Σ holds for R and I.

Finally, using the above discussion and results of the previous section, the three main theorems of this section are obtained.

With the appropriate assumptions, even more sets of reals are homogeneously Suslin and these canonicalization results would hold for relations in those classes. For example, Chang's model $L({}^\omega\mathrm{ON}) = \bigcup_{\alpha\in\mathrm{ON}} L({}^\omega\alpha)$ is the smallest inner model of ZF containing all the countable sequences of ordinals of V. Woodin has shown that with a proper class of Woodin cardinals every set of reals in $L({}^\omega\mathrm{ON})$ is ∞-homogeneously Suslin. Hence under this assumption, the above result would hold for binary relations in $L({}^\omega\mathrm{ON})$ with all $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, or $\mathbf{\Delta}^1_1$ sections.

5. Canonicalization for All Relations

This section will consider Question 1.13: Is it consistent that for every equivalence relation E with all $\mathbf{\Delta}^1_1$ classes and every σ-ideal I such that $\mathbb{P}_I$ is proper, there is an I^+ $\mathbf{\Delta}^1_1$ subset C such that $E \restriction C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

As with other regularity properties, this question has a negative answer if the axiom of choice holds. First, a definition and a property of all $\mathbf{\Pi}^1_1$ equivalence relations:

Definition 5.1. An equivalence relation E on ${}^\omega\omega$ is thin if and only if there does not exist a perfect set $P \subseteq {}^\omega\omega$ such that $\neg(x\ E\ y)$ for all $x, y \in P$ with $x \neq y$.

There are $\mathbf{\Sigma}^1_1$ thin equivalence relations with uncountably many classes. In fact, there are $\mathbf{\Sigma}^1_1$ thin equivalence relations with all $\mathbf{\Delta}^1_1$ classes and uncountably many classes: for example, the countable admissible ordinal equivalence relation, F_{ω_1}, and any counterexamples to Vaught's conjecture (if they exist). The Silver's dichotomy implies that there are no $\mathbf{\Pi}^1_1$ thin equivalence relations:

Fact 5.2. *(Silver) If E is a $\mathbf{\Pi}^1_1$ equivalence relation on ${}^\omega\omega$, then either E has countably many classes or there exists a perfect set of pairwise E-inequivalent elements.*

Proof. See [15]. □

Proposition 5.3. (ZF) *If there is a well-ordering of ${}^\omega\omega$, then there is a thin equivalence relation E^* on ${}^\omega\omega$ with equivalence classes of size at most two.*

For any σ-ideal I on ${}^\omega\omega$ and any I^+ $\mathbf{\Delta}^1_1$ set C, $E^* \restriction C$ is not $\mathbf{\Delta}^1_1$.

Proof. First a remark: Proposition 1.11 is proved in a similar way by showing that in L, there is a thin Δ^1_2 equivalence relation with all countable classes.

Now the proof of the proposition: Using the well-ordering of ${}^\omega\omega$, let $\Phi : 2^{\aleph_0} \to {}^\omega\omega$ be bijection and let $\Psi : 2^{\aleph_0} \to {}^\omega\omega$ be an enumeration of all the perfect trees on ω.

The equivalence E^* is defined by stages through transfinite recursion as follows:

Let $A_0 = \emptyset$. $E^*_0 = \emptyset$.

Stage $\xi + 1$: Suppose A_ξ and E^*_ξ have been defined with $|A_\xi| < 2^{\aleph_0}$. Find some reals r_ξ and s_ξ so that $r_\xi, s_\xi \notin A_\xi$, $r_\xi \neq s_\xi$, and $r_\xi, s_\xi \in [\Psi(\xi)]$.

If $\Phi(\xi) \in A_\xi \cup \{r_\xi, s_\xi\}$, then define $A_{\xi+1} = A_\xi \cup \{r_\xi, s_\xi\}$ and

$$E^*_{\xi+1} = E^*_\xi \cup \{(r_\xi, r_\xi), (s_\xi, s_\xi), (r_\xi, s_\xi), (s_\xi, r_\xi)\}$$

If $\Phi(\xi) \notin A_\xi \cup \{r_\xi, s_\xi\}$, then define $A_{\xi+1} = A_\xi \cup \{r_\xi, s_\xi, \Phi(\xi)\}$ and

$$E^*_{\xi+1} = E^*_\xi \cup \{(\Phi(\xi), \Phi(\xi)), (r_\xi, r_\xi), (s_\xi, s_\xi), (r_\xi, s_\xi), (s_\xi, r_\xi)\}$$

At limit stage ξ: Let $A_\xi = \bigcup_{\eta<\xi} A_\eta$ and $E^*_\xi = \bigcup_{\eta<\xi} E^*_\eta$.

Note that $A_{2^{\aleph_0}} = {}^\omega\omega$. Let $E^* = E^*_{2^{\aleph_0}}$. E^* is an equivalence relation on ${}^\omega\omega$. E^* has classes of size at most two. E^* is thin: Suppose T is a perfect tree on ω. Then $T = \Psi(\xi)$ for some $\xi < 2^{\aleph_0}$. Then $r_\xi \ E^* \ s_\xi$ and $r_\xi, s_\xi \in [\Psi(\xi)] = [T]$.

Now let I be a σ-ideal on ${}^\omega\omega$. Suppose there was some I^+ $\mathbf{\Delta}^1_1$ set C so that $E^* \restriction C$ is $\mathbf{\Delta}^1_1$. Since C is I^+ and I is a σ-ideal, C must be uncountable. Since E^* has classes of size at most two, $E \restriction C$ can not have only countably many classes. Since $\mathbf{\Delta}^1_1$ equivalence relations are $\mathbf{\Pi}^1_1$, the Silver's dichotomy (Fact 5.2) implies that there is a perfect set $P \subseteq C$ of E^*-inequivalent elements. There is a perfect tree T so that $[T] = P$. Let $\xi < 2^{\aleph_0}$ be so that $\Psi(\xi) = T$. Then $r_\xi, s_\xi \in [T] = P \subseteq C$ and $r_\xi \ E^* \ s_\xi$. Contradiction. $\square$

Hence to get a positive answer to Question 1.13, there cannot exist a well-ordering of the reals, so the full axiom of choice must fail.

The main results of this sections are the following. Some of the terms will be defined in the dicussion that follows:

Theorem 5.4. *Assume* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_\mathbb{R}$. *Let R be a binary relation on ${}^\omega\omega$. If R has all $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$) sections, then for every nonmeager $\mathbf{\Delta}^1_1$ set B, there is a $\mathbf{\Delta}^1_1$ set $C \subseteq B$ which is comeager in B so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$, respectively).*

Theorem 5.5. *Assume* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_\mathbb{R} + V = L(\mathscr{P}(\mathbb{R}))$. *Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is absolutely proper. Let R be a binary relation on ${}^\omega\omega$. If R has all $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$) sections, then for every I^+ $\mathbf{\Delta}^1_1$ set B, there is an I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$, respectively).*

First, the immediate concern in the choiceless setting is the definition of properness: Since set may not have a cardinality, it is preferable to use V_Ξ rather than H_Ξ. Recall in ZFC, for any σ-ideal I on ${}^\omega\omega$, $\mathbb{P}_I$ was proper if and only if for all sufficiently large cardinals Ξ, any $B \in \mathbb{P}_I$, and all countable elementary $M \prec V_\Xi$ with $\mathbb{P}_I, B \in M$, the set $\{x \in B : x \text{ is } \mathbb{P}_I\text{-generic over } M\}$ is I^+ $\mathbf{\Delta}^1_1$. Without the axiom of choice, the downward Lowenheim-Skolem theorem may fail for structures in countable languages and so there may be no countable elementary substructure. Moreover, in the previous section, it was also important to be able to choose countable elementary substructures containing certain homogeneously Suslin trees.

However, only dependence choice (DC) is needed to prove the following form of the downward Lowenheim-Skolem theorem: Let $\mathscr{L}$ be a countable language. Let M be an $\mathscr{L}$-structure. Let $A \subseteq M$ be countable. Then there exists an $\mathscr{L}$-elementary substructure N of M so that $A \subseteq N$.

Hence with DC, the definition of properness and the ability to construct elementary substructure of V_Ξ with certain desired objects inside are still available.

Without the axiom of choice, determinacy for various games are useful for settling many questions in descriptive set theory: The axiom of determinacy (AD) asserts that all games of the form in Definition 2.11 where the moves are elements of ω are determined. The axiom of determinacy for the reals ($\mathsf{AD}_\mathbb{R}$) asserts that all games of the form in Definition 2.11 where the moves are elements of ${}^\omega\omega$ are determined.

$\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_\mathbb{R}$ is preferable over AD since it can prove that every subset of ${}^\omega\omega$ is homogeneously Suslin and can prove a strong form of absoluteness for proper forcings:

Fact 5.6. *(Martin, [11]) Under $\mathsf{ZF}+\mathsf{AD}_\mathbb{R}$, every tree on $\omega\times\lambda$, where λ is an ordinal, is weakly homogeneously Suslin.*

Fact 5.7. *(Martin; [10]) Under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$, for every $A \subseteq {}^\omega\omega$, A is homogeneously Suslin if and only if if A and ${}^\omega\omega \setminus A$ are Suslin. Moreover, one can find a homogeneously Suslin tree T on $\omega \times \kappa$, for $\kappa < \Theta$, so that $A = p[T]$.*

(Martin, Woodin) Under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$, $\mathsf{AD}_\mathbb{R}$ is equivalent to the statement that every subset of ${}^\omega\omega$ is Suslin.

Combining these results, one has under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_\mathbb{R}$, every subset of the ${}^\omega\omega$ is homogeneously Suslin.

In the previous section, an important aspect of analyzing tree representations in generic extensions was the fact that any κ-complete measure μ could be naturally extended to a κ-complete measure μ^* in a forcing extension by $\mathbb{P}$, whenever $|\mathbb{P}| < \kappa$.

Let I be a σ-ideal on ${}^\omega\omega$. $\mathbb{P}_I$ is in bijection with ${}^\omega\omega$ and hence is not well-ordered under AD. Also note that the measures produced using AD to witness homogeneity and weak homogeneity are $\aleph_1$-complete. For the general σ-ideal, it is not clear how to modify the arguments of the previous section in the context of $\mathsf{AD}_\mathbb{R}$.

However, there is one important σ-ideal for which the previous arguments will work with minor modifications: For the meager ideal, $\mathbb{P}_{I_{\text{meager}}}$ is forcing equivalent to Cohen forcing, denoted $\mathbb{C}$, which is a countable forcing.

Let T be an $\aleph_1$-weakly homogeneous tree on $\omega \times \gamma$ witnessed by the weak homogeneity system $\bar{\mu}$. Fact 4.9, which is provable in $\mathsf{ZF} + \mathsf{DC}$, implies that $V \models p[T] = {}^\omega\omega \setminus \mathrm{MS}_{\gamma^+}(\bar{u})$.

Since $|\mathbb{C}| = \aleph_0 < \aleph_1$, any $\aleph_1$-complete measure can be extended to an $\aleph_1$-complete measure in the $\mathbb{C}$-forcing extension. Likewise, every $\aleph_1$-weak homogeneity system $\bar{\mu}$ can be extended to an $\aleph_1$-weak homogeneity system. Fact 4.10 and the discussion before it holds when $\kappa = \aleph_1$ and $\mathbb{P} = \mathbb{C}$:

Fact 5.8. *Assume $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$. Let T be an $\aleph_1$-weakly homogeneous tree on $\omega \times \gamma$ witnessed by the weak homogeneity system $\bar{\mu}$. If $G \subseteq \mathbb{C}$ is $\mathbb{C}$-generic over V, then $V[G] \models \mathrm{MS}_{\gamma^+}(\bar{\mu}^V) = \mathrm{MS}_{\gamma^+}(\bar{\mu}^*)$ and $V[G] \models p[\mathrm{MS}_{\gamma^+}(\bar{\mu})^V] = {}^\omega\omega \setminus p[T]$.*

The notion of an absolutely proper forcing is defined in [13]. A strong absoluteness result for absolutely proper forcing due to Neeman and Norwood can be used to show that T and $\mathrm{MS}_{\gamma^+}(\bar{\mu})$ continue to complement each other in the $\mathbb{P}_I$ generic extension for an arbitrary σ-ideal I so that $\mathbb{P}_I$ is absolutely proper. This result is similar to [14]. A more general version of the following result proved under AD^+ appears in [13].

Fact 5.9. *(Neeman and Norwood) Under $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_\mathbb{R}+V = L(\mathscr{P}(\mathbb{R}))$, for every absolutely proper forcing $\mathbb{P}$, and $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over V, there is an elementary embedding $j : L(\mathscr{P}(\mathbb{R})) \to L(\mathscr{P}(\mathbb{R})^{V[G]})$ so that j does not move ordinals or reals.*

Fact 5.10. *Assume $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_\mathbb{R} + V = L(\mathscr{P}(\mathbb{R}))$. Let $\mathbb{P}$ be an absolutely proper forcing and $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Suppose T and S are trees on $\omega \times \gamma$ and $\omega \times \delta$ so that $p[T] = {}^\omega\omega \setminus p[S]$. Then $V[G] \models p[T] = {}^\omega\omega \setminus p[S]$.*

In particular, if T is a weakly homogeneous tree on $\omega \times \gamma$ witnessed by the weak homogeneity system $\bar{\mu}$, then $V[G] \models p[T] = {}^\omega\omega \setminus p[\mathrm{MS}_{\gamma^+}(\bar{\mu})]$.

Proof. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Let $j : L(\mathscr{P}(\mathbb{R})) \to L(\mathscr{P}(\mathbb{R})^{V[G]})$ be an elementary embedding which does not move ordinals or reals. Note that if T is a tree on $\omega \times \gamma$, then $j(T)$ is a tree on $j(\omega) \times j(\gamma) = \omega \times \gamma$ and for all $s \in {}^{<\omega}(\omega \times \gamma)$, $s \in T$ if and only if $j(s) \in j(T)$ if and only if $s \in j(T)$. Hence $T = j(T)$ and similarly $S = j(S)$. So by elementarity, $L(\mathscr{P}(\mathbb{R})^{V[G]}) \models p[T] = {}^\omega\omega \setminus p[S]$. As $V[G]$ and $L(\mathscr{P}(\mathbb{R})^{V[G]})$ have the same reals, $V[G] \models p[T] = {}^\omega\omega \setminus p[S]$.

For the second statement, note that under $\mathsf{ZF} + \mathsf{DC}$, Fact 4.9 implies that $L(\mathscr{P}(\mathbb{R})) \models p[T] = {}^\omega\omega \setminus p[\mathrm{MS}_{\gamma^+}(\bar{\mu})]$. The rest follows by applying the first part. □

Fact 5.11. *Assume $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_\mathbb{R}+V = L(\mathscr{P}(\mathbb{R}))$ (or just $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_\mathbb{R}$ for Cohen forcing, $\mathbb{C}$). Let $\mathbb{P}$ be an absolutely proper forcing. Suppose T is a tree on ${}^k\omega \times \gamma$ for some cardinal γ and $k \in \omega$. If $A \subseteq {}^j({}^\omega\omega)$, for some $j \leq k$, is defined by applying complementation and $\exists^\mathbb{R}$ over $p[T]$, then there is some tree U on ${}^j\omega \times \delta$ for some cardinal δ so that $A = p[U]$ and $1_\mathbb{P} \Vdash_\mathbb{P} A = p[\check{U}]$.*

Proof. This is proved by induction. Suppose B is some set defined by real quantifiers over $p[T]$ such that there is some tree L on ${}^l\omega \times \epsilon$ so that $B = p[L]$ and $1_\mathbb{P} \Vdash_\mathbb{P} B = p[L]$.

For the $\exists^\mathbb{R}$ case: Suppose $l = i+1$. Define the tree U on ${}^i\omega \times \epsilon$ as the induced tree defined by considering the tree L on ${}^{i+1}\omega \times \omega$ as a tree on ${}^i\omega \times (\omega \times \epsilon)$ with ϵ and $\omega \times \epsilon$ identified by some bijection. Then $\exists^\mathbb{R} B = p[U]$.

For complementation: By Fact 5.6, L is weakly homogeneously Suslin. Let $\bar{\mu}$ be some weak homogeneity system witnessing this for L. By Fact 4.9, $p[T] = {}^\omega\omega \setminus p[\mathrm{MS}_{\epsilon^+}(\bar{\mu})]$. By Fact 5.10 (or Fact 5.8), $1_\mathbb{P} \Vdash_\mathbb{P} p[T] = {}^\omega\omega \setminus p[\mathrm{MS}_{\epsilon^+}(\bar{\mu})]$. □

Fact 5.12. *Assume* $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_{\mathbb{R}}+V=L(\mathscr{P}(\mathbb{R}))$ *(or just* $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_{\mathbb{R}}$ *in the case of* $\mathbb{C}$*). Let* $\mathbb{P}$ *be a absolutely proper forcing. Let* T *be a tree on* ${}^k\omega\times\gamma$ *for some cardinal* γ*. Let* A *denote a predicate symbol for* $p[T]$ *which will always be interpreted as* $p[T]$ *in forcing extensions. Let* φ *be a formula on* $\mathbb{R}$ *using predicate* A*, complementation, and* $\exists^{\mathbb{R}}$*. Then for all* $r\in\mathbb{R}^V$*,* $V\models\varphi(r)\Leftrightarrow V[G]\models\varphi(r)$*, whenever* $G\subseteq\mathbb{P}$ *is* $\mathbb{P}$*-generic over* V*.*

Proof. In the $V=L(\mathscr{P}(\mathbb{R}))$ case, this is essentially immediate from the absoluteness result of Fact 5.9 and the fact that $L(\mathscr{P}(\mathbb{R})^{V[G]})\models\varphi(r)\Leftrightarrow V[G]\models\varphi(r)$.

So consider the case for $\mathbb{C}$: Let $G\subseteq\mathbb{C}$ be a $\mathbb{C}$-generic over V. By Fact 5.11, for some tree U, $V\models(\forall x)(\varphi(x)\Leftrightarrow x\in p[U])$ and $V[G]\models(\forall x)(\varphi(x)\Leftrightarrow x\in p[U])$. Then for any $x\in\mathbb{R}^V$,

$$V\models\varphi(x)\Leftrightarrow V\models x\in p[U]\Leftrightarrow V[G]\models x\in p[U]\Leftrightarrow V[G]\models\varphi(x)$$

where the second equivalence follows from the absoluteness of well-foundedness. □

Now assume $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_{\mathbb{R}}+V=L(\mathscr{P}(\mathbb{R}))$ (or just $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_{\mathbb{R}}$ when working with the meager ideal). Let R be a binary relation on ${}^\omega\omega$ with all $\mathbf{\Sigma}^1_1$ (or $\mathbf{\Pi}^1_1$) sections. Let I be a σ-ideal on ${}^\omega\omega$ so that the associated forcing $\mathbb{P}_I$ is an absolutely proper forcing.

By Fact 5.7, R is homogeneously Suslin. Let S be a homogeneously Suslin tree so that $R=p[S]$. In the case of the meager ideal and under $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_{\mathbb{R}}$: By the countability of $\mathbb{C}$, the argument above about weak homogeneity systems would show that the homogeneity system for S would lift to a homogeneity system for S in the $\mathbb{C}$-extension. Thus S would still be a homogeneous tree in the $\mathbb{C}$-extension. Under $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_{\mathbb{R}}+V=L(\mathscr{P}(\mathbb{R}))$, for the general σ-ideal I with $\mathbb{P}_I$ absolutely proper, Fact 5.9 gives an elementary embedding $j:L(\mathscr{P}(\mathbb{R}))\to L(\mathscr{P}(\mathbb{R})^{V[G]})$. So $L(\mathscr{P}(\mathbb{R})^{V[G]})\models S$ is homogeneously Suslin. This is not exactly the requirement of A_Σ or A_Π so the proof of Theorem 3.8 needs to be slightly modified: To prove Claim 2, first use the same argument with the fact that S is a homogeneous tree in $L(\mathscr{P}(\mathbb{R})^{M[g]})$ to show that Player 2 has a winning strategy in $L(\mathscr{P}(\mathbb{R})^{M[g]})$. This strategy is still a winning strategy for Player 2 in $M[g]$. This proves Claim 2 and the rest of the argument remains unchanged.

A_Σ (and similarly A_Π) holds for S and I. The formula $D_\Sigma(x,T)$ from Definition 3.5 can be expressed as a statement involving a predicate for $p[S]$, complementation, and real quantifiers. Fact 5.11 shows that there is some tree U representing D_Σ in V and in $\mathbb{P}_I$-extensions. Statement C_Σ (and similarly C_Π) holds for S and I. The statement $(\forall x)(\exists T)D_\Sigma(x,T)$ is true in V since R is a relation with all $\mathbf{\Sigma}^1_1$ classes. This formula is also expressed as a statement involving a predicate for $p[S]$, complementation, and real quantifiers, so Fact 5.12 implies that this statement remains true in the $\mathbb{P}_I$-extension. B_Σ (and similarly B_Π) holds for S and I.

As Section 3 works in $\mathsf{ZF}+\mathsf{DC}$, the arguments of that section can be carried out in the present context, with the changes mentioned above. (Recall the discussion earlier in this section about properness and elementary substructures under DC.)

Since Cohen forcing satisfies the $\aleph_1$-chain condition, one can obtain more than just canonicalization on a nonmeager set but in fact on a comeager set.

Finally the two main results of this section are obtained. Again, the analogous results for equivalence relation and graphs are obtained as before.

References

1. William Chan, *Canonicalization by absoluteness*, Notes.
2. William Chan, *Equivalence relations which are Borel somewhere*, J. Symb. Log. **82** (2017), no. 3, 893–930. MR 3694334
3. Ohad Drucker, *Borel Canonization of Analytic Sets with Borel Sections*, ArXiv e-prints (2015).
4. Qi Feng, Menachem Magidor, and Hugh Woodin, *Universally Baire sets of reals*, Set theory of the continuum (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., vol. 26, Springer, New York, 1992, pp. 203–242. MR 1233821 (94g:03095)
5. David Gale and F. M. Stewart, *Infinite games with perfect information*, Contributions to the theory of games, vol. 2, Annals of Mathematics Studies, no. 28, Princeton University Press, Princeton, N. J., 1953, pp. 245–266. MR 0054922
6. Steve Jackson, *Suslin cardinals, partition properties, homogeneity. Introduction to Part II*, Games, scales, and Suslin cardinals. The Cabal Seminar. Vol. I, Lect. Notes Log., vol. 31, Assoc. Symbol. Logic, Chicago, IL, 2008, pp. 273–313. MR 2463617
7. Vladimir Kanovei, Marcin Sabok, and Jindřich Zapletal, *Canonical Ramsey theory on Polish spaces*, Cambridge Tracts in Mathematics, vol. 202, Cambridge University Press, Cambridge, 2013. MR 3135065
8. Paul B. Larson, *The stationary tower*, University Lecture Series, vol. 32, American Mathematical Society, Providence, RI, 2004, Notes on a course by W. Hugh Woodin. MR 2069032

9. Donald A. Martin and John R. Steel, *A proof of projective determinacy*, J. Amer. Math. Soc. **2** (1989), no. 1, 71–125. MR 955605
10. ______, *The tree of a Moschovakis scale is homogeneous*, Games, scales, and Suslin cardinals. The Cabal Seminar. Vol. I, Lect. Notes Log., vol. 31, Assoc. Symbol. Logic, Chicago, IL, 2008, pp. 404–420. MR 2463620
11. Donald A. Martin and W. Hugh Woodin, *Weakly homogeneous trees*, Games, scales, and Suslin cardinals. The Cabal Seminar. Vol. I, Lect. Notes Log., vol. 31, Assoc. Symbol. Logic, Chicago, IL, 2008, pp. 421–438. MR 2463621
12. Itay Neeman, *Determinacy in $L(\mathbb{R})$*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 1877–1950. MR 2768701
13. Itay Neeman and Zach Norwood, *Happy and mad families in $L(\mathbb{R})$*, `http://www.math.ucla.edu/~ineeman`.
14. Itay Neeman and Jindřich Zapletal, *Proper forcing and $L(\mathbb{R})$*, J. Symbolic Logic **66** (2001), no. 2, 801–810. MR 1833479
15. Jack H. Silver, *Counting the number of equivalence classes of Borel and coanalytic equivalence relations*, Ann. Math. Logic **18** (1980), no. 1, 1–28. MR 568914
16. J. R. Steel, *The derived model theorem*, Logic Colloquium 2006, Lect. Notes Log., Assoc. Symbol. Logic, Chicago, IL, 2009, pp. 280–327. MR 2562557
17. Jindřich Zapletal, *Forcing idealized*, Cambridge Tracts in Mathematics, vol. 174, Cambridge University Press, Cambridge, 2008. MR 2391923 (2009b:03002)

Department of Mathematics, University of North Texas, Denton, TX 76203
Email address: `William.Chan@unt.edu`

Einstein Institute of Mathematics, Hebrew University of Jerusalem
Email address: `mensara@savion.huji.ac.il`