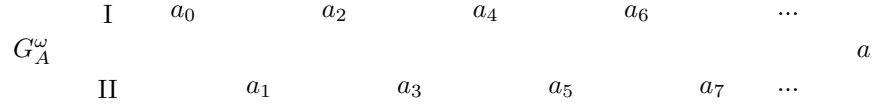# A LIPSCHITZ DETERMINACY PRINCIPLE EQUIVALENT TO WEAK KÖNIG LEMMA

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ABSTRACT. RCA_0 is a weak subsystem of second order arithmetic which will serve as the base theory. WKL_0 is the subsystem of second order arithmetic corresponding to the weak König lemma which states that every infinite tree on $\{0,1\}$ has an infinite path. $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)\text{-}\mathsf{LDET}^2$ is the principle asserting that one of the two players has a winning strategy in all Lipschitz game with moves from $\{0,1\}$ such that Player 1 owns a set which is the intersection of an open and a closed subset of $^{\mathbb{N}}2$ in its natural topology and Player 2 owns a clopen subset of $^{\mathbb{N}}2$. It will be shown that RCA_0 proves that WKL_0 is equivalent to $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)\text{-}\mathsf{LDET}^2$.

Let A be a subset of $^\omega\omega$, which is the set of functions of the form $f : \omega \to \omega$. Consider the game G_A^ω defined as in the following diagram.
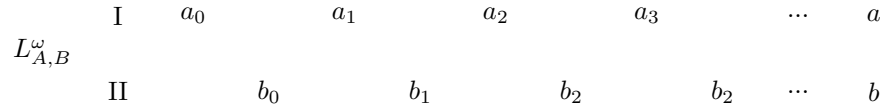
$$G_A^\omega \quad \begin{array}{c|cccccc} \text{I} & a_0 & & a_2 & & a_4 & & a_6 & & \cdots & \\ \hline \text{II} & & a_1 & & a_3 & & a_5 & & a_7 & \cdots & \end{array} \quad a$$

Player 1 and 2 take turns playing elements of ω with Player 1 making the even moves a_{2n} and Player 2 making the odd moves a_{2n+1}. Let $a = \langle a_n : n \in \omega \rangle$. Player 1 wins if and only if $a \in A$. A strategy is a function $\rho : {}^{<\omega}\omega \to \omega$ which uses the partial play to determines the next move. G_A^ω is determined if and only if one of the two players has a winning strategy.

The axiom of determinacy, AD, asserts that for all $A \subseteq {}^\omega\omega$, G_A^ω is determined. AD is incompatible with the axiom of choice, AC, but it is often a more suitable framework for studying definable mathematics in the sense of recursion theory or descriptive set theory. AD implies all sets of reals are Lebesgue measurable, have the Baire property, and have the perfect set property. Using notions of definability, AD implies many cardinals which are surjective images of $\mathbb{R}$ have interesting combinatorial properties such as the infinite partition relations (see [9] and [2]). For instance, combinatorial questions surrounding ω_1 under AD involve the pointclass of analytic ($\mathbf{\Sigma}_1^1$) set and its boundedness property. (See [2], [3], and [4] for more recent results concerning ω_1 under AD using these ideas.)

Given two subsets A and B of $^\omega\omega$, A is Wadge reducible to B, denoted $A \leq_W B$, if and only if there is a continuous function $\Phi : {}^\omega\omega \to {}^\omega\omega$ so that for all $x \in {}^\omega\omega$, $x \in A \Leftrightarrow \Phi(x) \in B$. (That is, $\Phi^{-1}[B] = A$.) Similarly, A is Lipschitz reducible to B, denoted $A \leq_L B$, if and only if there is a Lipschitz function $\Phi : {}^\omega\omega \to {}^\omega\omega$ so that for all $x \in {}^\omega\omega$, $x \in X \Leftrightarrow \Phi(x) \in B$.

The relevance of games comes from the following observations. If $\rho : {}^{<\omega}\omega \to \omega$ is considered as a Player 2 strategy, then define $\Xi_\rho^2 : {}^\omega\omega \to {}^\omega\omega$ by $\Xi_\rho^2(x) = y$ if and only if y is the sequence of Player 2 moves according to ρ when Player 1 simply plays the bits of x. Ξ_ρ^2 is a Lipschitz function and, in fact, all Lipschitz functions come from Player 2 strategies. (One can define an analogous Lipschitz function $\Xi_\rho^1 : {}^\omega\omega \to {}^\omega\omega$ using ρ as a strategy for Player 1.) Let A and B be subsets of $^\omega\omega$. Consider the Lipschitz game $L_{A,B}^\omega$ defined as follows.

$$L_{A,B}^\omega \quad \begin{array}{c|ccccccc} \text{I} & a_0 & & a_1 & & a_2 & & a_3 & & \cdots & a \\ \hline \text{II} & & b_0 & & b_1 & & b_2 & & b_2 & \cdots & b \end{array}$$

Player 1 plays the terms of the sequence of natural numbers $a = \langle a_n : n \in \omega \rangle$ and Player 2 plays the terms of the sequence of natural numbers $b = \langle b_n : n \in \omega \rangle$. Player 2 wins $L_{A,B}^\omega$ if and only if $a \in A \Leftrightarrow b \in B$. Thus if $\rho : {}^{<\omega}\omega \to \omega$ is a Player 2 winning strategy in $L_{A,B}^\omega$, then Ξ_ρ^2 is a Lipschitz function witnessing $A \leq_L B$.

If Player 1 has a winning strategy ρ, then Ξ^1_ρ is a Lipschitz function witnessing $B \leq_L \omega\omega \setminus A$. (There is an analogous Wadge game corresponding to Wadge reductions. However, this paper will be concern with only Lipschitz games.)

Thus under AD, $A \leq_L B$ or $B \leq_L \omega\omega \setminus A$ (also $A \leq_W B$ or $B \leq_W \omega\omega \setminus A$). This observation is known as Wadge's lemma which has various remarkable consequences. For instance, any set A and B can be compared via Lipschitz or Wadge reductions (after taking into account the nonselfdual sets): that is $(A \leq_L B \vee A \leq_L \omega\omega \setminus B) \vee (B \leq_L A \vee B \leq_L \omega\omega \setminus A)$. A boldface pointclass is a collection Γ of subsets of ${}^k\omega \times {}^j({}^\omega\omega)$ for various $k, j \in \omega$ with $j \geq 1$ which is closed under Wadge reductions, that is, if $A \leq_W B$ and $B \in \Gamma$, then $A \in \Gamma$. If Γ is a nonselfdual boldface pointclass and $\check{\Gamma}$ is its dual pointclass, then any set in $\Gamma \setminus \check{\Gamma}$ is complete for Γ with respect to Wadge reductions and even Lipschitz reductions. The Lipschitz degree of a set A is $[A]_L = \{B : A \leq_L B \wedge B \leq_L A\}$. Lipschitz degrees can be ordered by setting $\mathbf{a} \leq \mathbf{b}$ if and only if there exists an $A \in \mathbf{a}$ and a $B \in \mathbf{b}$ so that $A \leq_L B$. If $A \subseteq {}^\omega\omega$, let $[A]^*_L = [A]_L \cup [{}^\omega\omega \setminus A]_L$ be the modified Lipschitz degree of A. The Wadge lemma implies that the modified Lipschitz degree with its natural ordering is a linear ordering.

Let the axiom of Lipschitz determinacy assert that for all $A, B \subseteq {}^\omega\omega$, $L^\omega_{A,B}$ is determined. AD implies Lipschitz determinacy. Under ZF, $\mathsf{DC}_\mathbb{R}$ (dependent choice for $\mathbb{R}$), all sets of reals have the Baire property, and Lipschitz determinacy, Martin and Monk showed that the modified Lipschitz degrees and modified Wadge degrees are wellorderings under their natural orderings. Thus every $A \subseteq \mathbb{R}$ can be assigned two ordinals: the Lipschitz rank of $[A]^*_L$ and the Wadge rank of $[A]^*_W$. The use of Wadge rank and game arguments have applications to the study of the reduction, separation, prewellordering, and scale properties of pointclasses. (See [22], [23], and [25] for more information.)

Much work has been done to study the consistency strength of determinacy. Woodin ([**?**]howed that the consistency strength of AD is the existence of ω Woodin cardinals. Although the full axiom of determinacy is inconsistent with AC, local versions of determinacy restricted to specific pointclasses such as $\mathbf{\Delta}^1_1$, $\mathbf{\Sigma}^1_1$, and the projective sets can hold in ZFC assuming large cardinals. Martin and Steel established the consistency strength of determinacy for levels of the projective hierarchy from Woodin cardinals. (See [15], [10] and [18] for more information.)

Closer to the concerns of this article is the investigation of the strength of game principles in terms of equivalence with other principles. An open question is whether AD and Wadge (or Lipschitz) determinacy are equivalent over ZF. Results of Martin ([14]) and Harrington ([7]) showed the following are equivalent over ZFC

- $\mathbf{\Sigma}^1_1$ determinacy.
- For each $x \in \mathbb{R}$, $x^\sharp$ exists or equivalently there is a nontrivial elementary embedding $j : L[x] \to L[x]$.
- (Wadge's lemma for $\mathbf{\Sigma}^1_1$) Any $A \in \mathbf{\Sigma}^1_1 \setminus \mathbf{\Pi}^1_1$ is $\mathbf{\Sigma}^1_1$-complete.

Subsystems of second order arithmetic provide a rich setting for studying weaker determinacy principles. (See [20] and [8] for more on subsystems of second order arithmetic, which will be briefly reviewed in the next section.) Friedman [5] showed that the principle $\Sigma^0_5\text{-}\mathsf{DET}^\omega$ (corresponding to the determinacy of $\mathbf{\Sigma}^0_5$ subsets of ${}^\omega\omega$) is not provable in second order arithmetic. Martin showed that even $\Sigma^0_4\text{-}\mathsf{DET}^\omega$ is not provable in second order arithmetic. However, Martin showed $\Delta^0_4\text{-}\mathsf{DET}^\omega$ can be proved in second order arithmetic. Over RCA_0 which is a weak base system for second order arithmetic,

- (Steel [20] Theorem V.8.7) $\Sigma^0_1\text{-}\mathsf{DET}^\omega$, $\Delta^0_1\text{-}\mathsf{DET}^\omega$, and ATR_0 are equivalent.
- (Tanaka [24] or [20] Theorem VI.5.4) $(\Sigma^0_1 \wedge \Pi^0_1)\text{-}\mathsf{DET}^\omega$ (which corresponds to the determinacy of games with payoff sets which are intersections of an open and closed subset of ${}^\omega\omega$) and $\Pi^1_1\text{-}\mathsf{CA}$ are equivalent.

Montalbán and Shore studied in detail the limits of determinacy given by the results of Friedman and Martin. Let $n\text{-}\Pi^0_3$ denote boolean combinations of n-many Π^0_3 sets. They [16] showed that $\Pi_{n+2}\text{-}\mathsf{CA}$ proves $n\text{-}\Pi^0_3\text{-}\mathsf{DET}^\omega$ but $\Delta_{n+2}\text{-}\mathsf{CA}$ cannot prove $n\text{-}\Pi^0_3\text{-}\mathsf{DET}^\omega$. Montalbán and Shore [17] showed that these principles actually have different consistency strength.

To obtain determinacy principles equivalent to subsystems weaker than ATR_0, one needs to consider determinacy principles on ${}^\omega 2$. It is shown in [19] that over RCA_0,

- WKL_0, $\Sigma^0_1\text{-}\mathsf{DET}^2$ (open determinacy on ${}^\omega 2$), and $\Delta^0_1\text{-}\mathsf{DET}^2$ are equivalent.
- ACA_0 and $(\Sigma^0_1 \wedge \Pi^0_1)\text{-}\mathsf{DET}^2$ are equivalent.

See [26] for a more detail account of determinacy in second order arithmetic.

A natural question early in the study of determinacy was whether Borel determinacy is needed to establish Wadge's lemma within the Borel sets. Louveau and Saint-Raymond [13] showed that Borel Wadge determinacy on $^\omega\omega$ can be proved in second order arithmetic. Since Friedman showed Borel determinacy cannot be proved in second order arithmetic, this answers the above question negatively. The result of Louveau and Saint-Raymond suggests a more detail analysis of Borel Wadge determinacy within subsystems of second order arithmetic could be fruitful.

The main result of the paper is that over RCA_0, WKL_0 and $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)\text{-}\mathsf{LDET}^2$ are equivalent. WKL_0 is a subsystem that includes the statement of weak König lemma which asserts infinite binary trees (on $\{0, 1\}$) have infinite paths. $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)\text{-}\mathsf{LDET}^2$ is the principle corresponding to determinacy for Lipschitz games on $^\omega 2$ where Player 1 owns a set which is an intersection an open and closed subset of $^\omega 2$ and Player 2 owns a clopen subset of $^\omega 2$.

This article is a polished version of a draft produced in 2011-2012 which had been largely forgotten until the author encountered some recent slides of a talk by Loureiro [11]. These slides announced some work by Cordón-Franco, Lara-Martín, and Loureiro on the strength of Lipschitz and Wadge determinacy and the Wadge's lemma within second order arithmetic. They announced the following result which appears in Loureiro's thesis ([12] Theorem 4.41) that over RCA_0, the following are equivalent

- ACA_0
- $(\Sigma_1^0 \wedge \Pi_1^0, \Sigma_1^0 \wedge \Pi_1^0)\text{-}\mathsf{LDET}^2$.
- the Wadge lemma for $\Sigma_1^0 \wedge \Pi_1^0$ subsets of $^\omega 2$, that is, for any $\Sigma_1^0 \wedge \Pi_1^0$ subsets $A, B \subseteq {}^\omega 2$, either $A \leq_L B$ or $B \leq_L {}^\omega 2 \setminus A$.

In [12] Proposition 4.18, they also show WKL_0 proves $(\Sigma_1^0, \Delta_1^0)\text{-}\mathsf{LDET}^2$. In the slides [11] and in Loureiro's thesis ([12] Section 6.2.1), he asked for a "reversal for WKL_0 in terms of Lipschitz or Wadge determinacy". Given [12] Proposition 4.18, he conjectured in his thesis that $(\Delta_1^0, \Delta_1^0)\text{-}\mathsf{LDET}^2$ or $(\Sigma_1^0, \Delta_1^0)\text{-}\mathsf{LDET}^2$ could be possible candidates.

The author had first shown Theorem 2.1 that $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)\text{-}\mathsf{LDET}^2$ fails in REC and hence RCA_0 cannot prove $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)\text{-}\mathsf{LDET}^2$. The argument uses a pair of disjoint recursively enumerable sets which are recursively inseparable and a recursively enumerable simple set (or a Π_1^0 immune set). Motivate by the analysis of Lipschitz determinacy in the model REC, an answer to Loureiro's question will be provided in Theorem 2.2, Theorem 2.3, and Theorem 2.4 by showing that over RCA_0, WKL_0 is equivalent to the slightly stronger Lipschitz determinacy principle, $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)\text{-}\mathsf{LDET}^2$. The pair of disjoint recursively enumerable sets which are recursively inseparable motivates the use of Fact 1.4, which roughly states that WKL_0 is equivalent to the compactness of Δ_1^0 subsets of $^\omega 2$. The ideas from the Π_1^0 immune sets reappear in the form of Fact 1.2 (4). The reversal in this paper combined with the result of Loureiro and his coauthors mentioned above gives an interesting observation that $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)\text{-}\mathsf{LDET}^2$ is strictly weaker than $(\Sigma_1^0 \wedge \Pi_1^0, \Sigma_1^0 \wedge \Pi_1^0)\text{-}\mathsf{LDET}^2$.

There have been several new developments concerning Lipschitz and Wadge determinacy. In a recent preprint [1] of Cordón-Franco, Lara-Martín, and Loureiro, they have verified the conjecture of Loureiro which improves the reversal obtained in this paper from Theorem 2.4. [1] Theorem 3.11 showed that over RCA_0, WKL_0 and $(\Delta_1^0, \Delta_1^0)\text{-}\mathsf{LDET}^2$ are equivalent. They also improved their above equivalence with ACA_0 by showing in [1] Theorem 5.5 that over RCA_0, ACA_0 and $(\Sigma_1^0, \Sigma_1^0)\text{-}\mathsf{LDET}^2$ are equivalent. In a different direction, the result of Louveau and Saint-Raymond mentioned above implies that the Wadge lemma for Borel sets can be proved in second order arithmetic. Recently, Greenberg announced, in a slide [6], some joint work with Day, Harrison-Trainor, and Turetsky which showed that the Wadge lemma for Borel sets can be proved in the subsystem $\mathsf{ATR}_0 + \Pi_1^1\text{-}\mathsf{IND}$.

The work in this paper followed a topic course by Montalbán covering the results of Steel and Tanaka and a reading course with Montalbán covering portions of [19] at the University of Chicago in 2011. The author seems to recall that the suggestion to investigate Lipschitz and Wadge determinacy in second order arithmetic came from Takako Nemoto. The author would especially like to thank Antonio Montalbán for introducing the author to the determinacy axioms.

1. Basics

The reader should consult [20] and [8] for more information about the syntax and semantics of the first order theory of second order arithmetic.

Let $\mathscr{L}_2 = \{\mathbb{N}, \mathscr{P}(\mathbb{N}), +, \cdot, <, 0, 1, \in\}$ denote the first order language of second order arithmetic. $\mathbb{N}$ and $\mathscr{P}(\mathbb{N})$ are unary relation symbols. The arity and type of the other symbols should be clear from their frequent usage in mathematics. Let P denote the basic $\mathscr{L}_2$ axioms for second order arithmetics. These include all the usual axioms of arithmetic and order. One often writes $x \in \mathbb{N}$ and $x \in \mathscr{P}(\mathbb{N})$ rather than $\mathbb{N}(x)$ and $\mathscr{P}(\mathbb{N})(x)$. P should also include sentences which assert there are two types: $\mathbb{N}$ is the first order object called the numbers. $\mathscr{P}(\mathbb{N})$ are the second order objects representing subsets of the numbers. Some examples of sentences asserting these type-axioms may include the following.

$$(\forall x)(x \in \mathbb{N} \vee x \in \mathscr{P}(\mathbb{N}))$$

$$(\forall x)\neg(x \in \mathbb{N} \wedge x \in \mathscr{P}(\mathbb{N}))$$

$$(\forall x)(\forall y)(x \in y \Rightarrow (x \in \mathbb{N} \wedge y \in \mathscr{P}(\mathbb{N}))).$$

Note that $\mathbb{N}$ refers to the "number" part of an arbitrary $\mathscr{L}_2$-structure, and ω will be used to denote the standard natural numbers. P also contains full set induction:

$$(\forall X)[\{(0 \in X \wedge (\forall n)(n \in X \Rightarrow n + 1 \in X)\} \Rightarrow (\forall n)(n \in X))].$$

Suppose $\varphi(v)$ is a formula with one free variable v (and assume that n and X do not occur free in $\varphi(v)$), then φ-comprehension is the following sentence.

$$(\exists X)(\forall n)(n \in X \Leftrightarrow \varphi(n)).$$

Let Z_2 denote the axiom system consisting of P and φ-comprehension for all suitable formula φ. Z_2 is the $\mathscr{L}_2$-theory of second order arithmetic.

Suppose Γ is a collection of formulas $\varphi(v)$ as above, then Γ-comprehension (denoted Γ-CA) is the collection of sentences assert φ-comprehension for each $\varphi \in \Gamma$.

A formula is $\Delta_0^0 = \Sigma_0^0 = \Pi_0^0$ if and only if it has only bounded quantifiers. For $k \geq 1$, a formula φ is Σ_n^0 if and only if it takes the form $(\exists n_1)(\forall n_2)...(Qn_k)\phi$ where $Q = \exists$ if k is odd, $Q = \forall$ if k is even, and ϕ is a formula with only bounded quantifiers. A formula is Π_n^0 if and only if it takes the form $(\forall n_1)(\exists n_2)...(Qn_k)\phi$ where $Q = \forall$ if k is odd, $Q = \exists$ if k is even, and ϕ is a formula with only bounded quantifiers. Over P, negations of Σ_n^0 formulas are equivalent to Π_n^0 formulas and negations of Π_n^0 are equivalent to Σ_n^0 formulas. For instance, one will often say that if φ is Σ_1^0, then $\neg\varphi$ is Π_1^0 although formally it is not. A formula is arithmetic if it has no bounded set quantifier. Thus a formula is arithmetic if and only if there is an $n \in \omega$ so that the formula is Σ_n^0. A formula φ is $\Sigma_n^0 \wedge \Pi_n^0$ if and only if there a formula $\varphi_0 \in \Sigma_n^0$ and $\varphi_1 \in \Pi_n^0$ so that φ is $\varphi_0 \wedge \varphi_1$.

ACA_0 is the axiom system including P along with φ-comprehension for every arithmetic formula φ.

For each formula $\varphi(v)$ with one free variable v, φ-induction is the following sentence.

$$[\varphi(0) \wedge (\forall n)(\varphi(n) \Rightarrow \varphi(n + 1))] \Rightarrow (\forall n)\varphi(n).$$

If Γ is a collection of formulas, then Γ-IND is the collection of sentences stating φ-induction for each $\varphi \in \Gamma$.

Roughly a formula is Δ_n^0 if and only if it is "equivalent" to a Σ_n^0 formula and to a Π_n^0 formula, but this is not precise since two formulas being equivalent is not syntactical. The concept of a Δ_n^0 formula usually requires an equivalence of Σ_n^0 and Π_n^0 formula over some theory or inside a particular model. Concepts involving "Δ_n^0 formulas" needs to be defined carefully. For each $n \in \omega$, one will say that Δ_n^0-comprehension, denoted Δ_n^0-CA, is the following collection of sentences: For any formulas $\varphi_0 \in \Sigma_n^0$ and $\varphi_1 \in \Pi_n^0$, one has

$$(\forall n)(\varphi_0(n) \Leftrightarrow \varphi_1(n)) \Rightarrow (\exists X)(\forall n)(n \in X \Leftrightarrow \varphi_0(n)).$$

If $\varphi(v)$ is a formula with one free variable v, then bounded φ-comprension is the following sentence

$$(\forall j)(\exists X)(\forall n)(n \in X \Leftrightarrow [n < j \wedge \varphi(n)]).$$

If Γ is a collection of formulas, then bounded Γ-CA is the axiom scheme consisting of bounded φ-comprehension for each $\varphi \in \Gamma$.

Let RCA_0 consists of P, Δ_1^0-CA, and Σ_1^0-IND. RCA_0 will serve as the weak base system. [20] Theorem II.3.9 shows that RCA_0 proves bounded Σ_1^0-CA. Thus RCA_0 can prove bounded Π_1^0-CA and bounded $(\Sigma_1^0 \wedge \Pi_1^0)$-CA.

Let REC be the following $\mathscr{L}_2$-structure: Let $\mathcal{C}$ consists of the recursive subsets of ω. The domain or universe of REC is $\omega \cup \mathcal{C}$. $\mathbb{N}^{\mathsf{REC}} = \omega$. $\mathscr{P}(\mathbb{N})^{\mathsf{REC}} = \mathcal{C}$. The other symbols are interpreted as their usual arithmetic operations on ω. [20] Corollary II.18 shows that $\mathsf{REC} \models \mathsf{RCA}_0$ and in fact, for any $\mathcal{M} \models \mathsf{RCA}_0$ such that $\mathbb{N}^{\mathcal{M}} = \omega$, one has that $\mathscr{P}(\mathbb{N})^{\mathsf{REC}} \subseteq \mathscr{P}(\mathbb{N})^{\mathcal{M}}$.

RCA_0 can formalize basic notions concerning arithmetic and sequence of numbers. For instance, RCA_0 implies for every number n, there is a k so that $n = 2k$ or $n = 2k + 1$. $^{<\mathbb{N}}\mathbb{N}$ will refer to the collection of numbers used to code finite sequences of numbers although in practice, one will consider this the set of such finite sequences. If $K \subseteq \mathbb{N}$, then $^{<\mathbb{N}}K$ will refer to the set of sequences that only take values from K. For instance, $^{<\mathbb{N}}2$ is the collection of binary strings (taking value 0 or 1). There are functions that determines the length of strings and the values that appear along the string. Let $^{\mathbb{N}}\mathbb{N}$ refer to the class of infinite strings. If $f \in {}^{\mathbb{N}}\mathbb{N}$ and $n \in \omega$, then $f \restriction n \in {}^{<\mathbb{N}}\mathbb{N}$ is the initial segment of f of length n. See [20] Chapter II for details concerning sequences and their coding.

Next, one will show some basic facts in RCA_0 that will be needed.

Fact 1.1. RCA_0 *proves the following statements.*

(1) *For all $f : \mathbb{N} \to \mathbb{N}$, if f is an increasing function (meaning if $m < n$, then $f(m) < f(n)$), then for all n, $f(n) \geq n$. If $f : \mathbb{N} \to \mathbb{N}$ is injective, then for all n, there exists a k so that $f(k) > n$.*

(2) *For all $f : \mathbb{N} \to \mathbb{N}$, if f is an increasing function, then $\mathrm{rang}(f)$ exists.*

(3) *If $f : \mathbb{N} \to \mathbb{N}$ is injective, then there is an infinite set X so that $(\forall n)(n \in X \to (\exists m)(f(m) = n))$.*

(4) *If $\varphi(v)$ is a Σ_1^0 formula (in one distinguished free variable v and possible has paramters), then either there is a finite set F so that $(\forall n)(n \in F \leftrightarrow \varphi(n))$ or there is an injective function $f : \mathbb{N} \to \mathbb{N}$ so that $(\forall n)(\varphi(n) \Leftrightarrow (\exists m)(f(m) = n))$.*

(5) *If $\varphi(v)$ is a Σ_1^0 formula (possibly with parameters), then*

$$(\forall m)(\exists n > m)\varphi(n) \Rightarrow (\exists X)(\forall k)(k \in X \Rightarrow \varphi(k)).$$

In other words, if there are infinitely many n so that $\varphi(n)$, then there is an infinite set X consisting of solutions to φ.

(6) *There is no function $f : \mathbb{N} \to \mathbb{N}$ so that $(\forall n)(f(n + 1) < f(n))$.*

(7) *Suppose $f : \mathbb{N} \to \mathbb{N}$ is an injective function. Consider the Π_1^0 formula $\varphi(n)$ defined by $(\forall m > n)(f(m) > f(n))$. Then there are infinitely many n so that $\varphi(n)$.*

(8) *Suppose $f : \mathbb{N} \to \mathbb{N}$ is an injective function. For any $M \in \mathbb{N}$, there is an $N \in \mathbb{N}$ so that for all $n > N$, $f(n) > M$.*

Proof. (1) This can be shown by Σ_1^0-IND.

(2) Define a Σ_0^0 formula $\varphi(n)$ by $(\exists m \leq n)(f(m) = n)$. By Δ_1^0-CA, there is a set X such that for all n, $n \in X$ if and only if $\varphi(n)$. By (1), X is the range of f.

(3) By using (1) and an application of minimization ([20] Theorem II.3.5), there is a function $g_0 : \mathbb{N} \to \mathbb{N}$ so that $g_0(n)$ is the least k so that $f(k) > f(n)$. By primitive recursion ([20] Theorem II.3.4), there is a function g_1 such that $g_1(n + 1) = g_0(g_1(n))$. Let $g_2 : \mathbb{N} \to \mathbb{N}$ be defined by $g_2(n) = f(g_1(n))$. g_2 is an increasing function. By (2), $\mathrm{rang}(g_2)$ exists, is infinite, and consists only of elements which are in the "range of f" (although the range of f is not assumed to be a set).

(4) This is [20] Lemma II.3.7.

(5) This follows from (3) and (4).

(6) Suppose there is an $f : \mathbb{N} \to \mathbb{N}$ so that $f(n + 1) < f(n)$ for all $n \in \mathbb{N}$. Consider the Π_1^0 formula $\varphi(n)$ defined as $(\forall m)(f(m) \neq n)$. Note that $\varphi(0)$ holds since if there was an m so that $f(m) = 0$, then $f(m + 1) < f(m) = 0$ which is impossible. Suppose $\varphi(k)$ holds for all $k \leq n$. Suppose there is an m so that $f(m) = n + 1$. Then $f(m + 1) \leq n$. By the induction hypothesis, one has $\varphi(f(m + 1))$ but by definition, one has $\neg\varphi(f(n + 1))$. Since RCA_0 proves Π_1^0-IND by [20] Corollary II.3.10, one has $(\forall n)\varphi(n)$. This is impossible since $\neg\varphi(f(0))$ holds.

(7) Consider the Π_1^0 formula $\varphi(n)$ defined by $(\forall m > n)(f(m) > f(n))$. Suppose there exists an N so that for all n, $\varphi(n)$ implies $n < N$. Thus for all $n \geq N$, there exists an $m > n$ so that $f(m) \leq f(n)$ and since f is injective, one actually has $f(m) < f(n)$. Using minimization, there is a function $g_0 : \mathbb{N} \to \mathbb{N}$ with the property that $g_0(n)$ is the least $m > n$ so that $f(m + N) < f(n + N)$. By primitive recursion, let $g_1 : \mathbb{N} \to \mathbb{N}$

have the property that $g_1(0) = 0$ and $g_1(n + 1) = g_0(g_1(n))$ for all $n \in \mathbb{N}$. Let $h : \mathbb{N} \to \mathbb{N}$ be defined by $h(n) = f(g_1(n) + N)$. Note that h has the property that $h(n + 1) < h(n)$ for all $n \in \mathbb{N}$ which contradicts (6).

(8) Let φ be the Π_1^0 formula for f given in (7). By (7), there is some $K \in \mathbb{N}$ so that $F = \{n < K : \varphi(k)\}$ (which is a set under $\mathsf{RCA_0}$ by bounded Π_1^0-CA) has size $M + 2$. Let $r : M + 2 \to F$ be the increasing enumeration of F. Let $s : M + 2 \to \mathbb{N}$ be defined by $s(n) = f(s(n))$. By definition of φ, s is a finite increasing function. Using (1), it can be shown that $s(M + 1) > M$. By definition of $\varphi(r(M + 1))$, one has that for all $n > r(M + 1)$, $f(n) > f(r(M + 1)) = s(m + 1) > M$. Thus $N = r(M + 1)$ suffices. $\qquad\square$

Fact 1.2. $\mathsf{RCA_0}$ *proves the following are equivalent.*

(1) $\mathsf{ACA_0}$.
(2) Σ_1^0-CA.
(3) *For every injective function $f : \mathbb{N} \to \mathbb{N}$, there exists a set X so that $(\forall n)(n \in X \Leftrightarrow (\exists m)(f(m) = n))$. (That is, $\mathrm{rang}(f)$ exists as a set.)*
(4) *For every Π_1^0 formula $\varphi(v)$ with one free variable v (and possibly has parameters),*

$$(\forall m)(\exists n > m)\varphi(n) \Rightarrow (\exists X)[(\forall m)(\exists n > m)(n \in X) \wedge (\forall k)(k \in X \Rightarrow \varphi(k))].$$

In other words, if there are infinitely many n so that $\varphi(n)$, then there is an infinite set X consisting of solutions to φ.

Proof. The equivalence of (1), (2), and (3) is [20] Lemma III.1.3.

(1) $\Rightarrow$ (4) Fix such a Π_1^0 formulas. By $\mathsf{ACA_0}$, there is an X so that $(\forall n)(n \in X \Leftrightarrow \varphi(n))$. This set X satisfies the condition of (4).

(4) $\Rightarrow$ (3) Let $f : \mathbb{N} \to \mathbb{N}$ be an injective function. Consider the Π_1^0 formula $\varphi(n)$ defined by $(\forall m > n)(f(m) > f(n))$. By Fact 1.1 (7), one must have that for all N, there exists an $n > N$ so that $\varphi(n)$. Thus (4) applies to give an infinite set X with the property that $(\forall k)(k \in X \Rightarrow \varphi(k))$. Using minimization and primitive recursion, let $r : \mathbb{N} \to X$ be the increasing enumeration of X. Let $h = f \circ r$ which is an increasing function by the property of X and φ. Thus by Fact 1.1 (1), one has that $h(n) \geq n$. Let ψ be defined by $\psi(n)$ if and only if $(\exists m \leq r(n))(f(m) = n)$. By the property of X and φ and the fact that $h(n) \geq n$ for all $n \in \mathbb{N}$, one has that $(\exists m)(f(m) = n)$ if and only if $\psi(n)$. Since ψ is Σ_0^0 and using Δ_1^0-CA, let Y be a set so that $(\forall n)(n \in Y \Leftrightarrow \psi(n))$. Thus Y is $\mathrm{rang}(f)$ and hence $\mathrm{rang}(f)$ exists as a set. $\qquad\square$

A tree on $2 = \{0, 1\}$ is a nonempty set $T \subseteq {}^{<\mathbb{N}}2$ which is closed under the string extension relation $\subseteq$. That is, if $t \in T$ and $s \subseteq t$, then $s \in T$. If $f \in {}^{\mathbb{N}}2$, then one writes $f \in [T]$ if and only if for all $n \in \mathbb{N}$, $f \restriction n \in T$, and one will say that f is a path through T. Weak König lemma, abbreviated WKL, is the statement that every infinite tree T on 2 has a path. $\mathsf{WKL_0}$ is the axiom system consisting of $\mathsf{RCA_0}$ with WKL.

Let $T \subseteq {}^{<\mathbb{N}}2$. Using $\mathsf{ACA_0}$, one can define $\hat{T} = \{s \in T : (\forall n)(\exists t)(s \subseteq t \wedge t \in T \wedge |t| > n\}$ which is an infinite subtree tree of T with no dead branches. By primitive recursion, one can define the usual leftmost branch of $\hat{T}$. Thus $\mathsf{ACA_0}$ can prove $\mathsf{WKL_0}$.

The next result shows $\mathsf{WKL_0}$ can prove that ${}^{\mathbb{N}}2$ is compact.

Fact 1.3. $\mathsf{WKL_0}$ *can prove the following: Suppose U is a set so that for all $f : \mathbb{N} \to 2$, there exists an $n \in \mathbb{N}$ so that $f \restriction n \in U$. Then there is a finite set $F \subseteq U$ so that for all $f : \mathbb{N} \to 2$, there is an $n \in \mathbb{N}$ so that $f \restriction n \in F$.*

Now suppose $\varphi(X)$ is a Σ_1^0 formula with one free set variable. Identifying $\mathscr{P}(\mathbb{N})$ and ${}^{\mathbb{N}}2$, $\mathsf{RCA_0}$ can prove that the solutions to the Σ_1^0 formula $\varphi(X)$ is an open set. Precisely, [20] Theorem II.2.7 shows that in $\mathsf{RCA_0}$, if $\varphi(X)$ is a Σ_1^0 formula with free set variable X, then there is a Σ_0^0 formula $\phi(v)$ with number variable v so that for all $f : \mathbb{N} \to 2$, $\varphi(f)$ if and only if $(\exists n)\phi(f \restriction n)$, where ${}^{<\mathbb{N}}2$ is coded using $\mathbb{N}$. By Δ_1^0-CA, there is a set $U \subseteq {}^{<\mathbb{N}}2$ so that $(\forall s \in {}^{<\mathbb{N}}2)(s \in U \Leftrightarrow \phi(s))$. It has been shown that if $\varphi(f)$ is Σ_1^0 with one free set variable f, then there is set $U \subseteq {}^{<\mathbb{N}}2$ so that $(\forall f)(\varphi(f) \Leftrightarrow (\exists n)(f \restriction n \in U))$.

Thus in $\mathsf{RCA_0}$, if $\psi(f)$ is Π_1^0 in one free set variable f, then there is a set $U \subseteq {}^{<\mathbb{N}}2$ so that $(\forall f)(\psi(f) \Leftrightarrow (\forall n)(f \restriction n \notin U))$. Let $T = \{s \in {}^{<\mathbb{N}}2 : (\forall n \leq |s|)(s \restriction n \notin U)\}$ which is a set by Δ_1^0-CA and is a tree. Note that $\psi(f)$ if and only if $f \in [T]$. It has been shown in $\mathsf{RCA_0}$ that if $\psi(f)$ is a Π_1^0-formula in one free set variable f, then there is a tree T on 2 so that $(\forall f)(\psi(f) \Leftrightarrow f \in [T])$.

Fact 1.4. RCA_0 *proves the following are equivalent.*

    (1) WKL_0.

    (2) *For all injective functions $g_0, g_1 : \mathbb{N} \to \mathbb{N}$ with the property that $(\forall m)(\forall n)(g_0(m) \neq g_1(n))$, there exists a set X so that $(\forall n)(g_0(n) \notin X \wedge g_1(n) \in X)$.*

    (3) *Suppose $\phi(f)$ is a Σ_1^0 formula and $\psi(f)$ is a Π_1^0 formula in the free set variable f. If $(\forall f)(\phi(f) \Leftrightarrow \psi(f))$, then there exists a finite set $F \subseteq {}^{<\mathbb{N}}2$ so that for all $f : \mathbb{N} \to 2$, $\phi(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in F)$.*

Proof. The equivalence of (1) and (2) is shown in [20] Lemma IV.4.4.

(1) $\Rightarrow$ (3). Using the comments above, since $\phi(f)$ is Σ_1^0, there is a set $U_0 \subseteq {}^{<\mathbb{N}}2$ so that $(\forall f)(\phi(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in U_0))$. Since $\neg\psi(f)$ is Σ_1^0, there is also a $U_1 \subseteq {}^{<\mathbb{N}}2$ so that $(\forall f)(\neg\psi(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in U_1))$. Also for all $(\forall f)(\exists n)(f \upharpoonright n \in U_0 \cup U_1)$ and $U_0 \cap U_1 = \emptyset$. By Fact 1.3, there is a finite $E \subseteq U_0 \cup U_1$ so that $(\forall f)(\exists n)(f \upharpoonright n \in E)$. Let $F = E \cap U_0$ which is a finite set. Then one has $(\forall f)(\phi(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in F))$.

(3) $\Rightarrow$ (2). Suppose (2) fails. Let $g_0, g_1 : \mathbb{N} \to \mathbb{N}$ be two injective functions so that for all m and n, $g_0(m) \neq g_1(n)$ and there is no separating set X with the property that $(\forall n)(g_0(n) \notin X \wedge g_1(n) \in X)$.

If $f : \mathbb{N} \to 2$, then f naturally is the characteristic function of a set $X_f = \{n \in \mathbb{N} : f(n) = 1\}$. Say that f fails to be a separation first for g_0 if and only if $\phi_0(f)$ holds, where $\phi_0(f)$ is defined by

$$(\exists n)[(\forall m < n)(f(g_0(m)) = 0 \wedge f(g_1(m)) = 1) \wedge f(g_0(n)) = 1 \wedge f(g_1(n)) = 1].$$

Intuitively, the witness n to the existential quantifier in $\phi_0(f)$ states that X_f fails to be a separation for g_0 and g_1 with n being the first error in the sense that $g_0(n) \in X_f$ and n does not cause an error for g_1 in the sense that $g_1(n) \in X_f$. Note that ϕ_0 is Σ_1^0.

Say that f fails to be a separation first for g_1 or fails for both g_0 and g_1 at the same time if and only if $\phi_1(f)$ holds, where $\phi_1(f)$ is defined by

$$(\exists n)[(\forall m < n)(f(g_0(m)) = 0 \wedge f(g_1(m)) = 1) \wedge f(g_1(n)) = 0].$$

Intuitively, the witness n to the existential quantifier in $\phi_1(f)$ states that X_f fails to be a separation for g_0 and g_1 first at n in the sense that $g_1(n) \notin X_f$ (and it is possible that at this n, f also has an error for g_0 in the sense that $g_0(n) \in X_f$). Note that ϕ_1 is Σ_1^0.

Observe that if $\neg\phi_0(f) \wedge \neg\phi_1(f)$, then one would have that $X_f = \{n \in \mathbb{N} : f(n) = 1\}$ is a separation for g_0 and g_1. Since one has assumed that no such separation exists, one has $(\forall f)(\phi_0(f) \vee \phi_1(f))$. Moreover, one also has $(\forall f)(\neg\phi_0(f) \vee \neg\phi_1(f))$. This shows that $(\forall f)(\phi_0(f) \Leftrightarrow \neg\phi_1(f))$.

Now let $F \subseteq {}^{<\mathbb{N}}2$ be a finite set. Let $N_0 \in \mathbb{N}$ be the length of the longest string in F. Fact 1.1 (8) implies there is an $N_1 \in \mathbb{N}$ so that for all $n > N_1$, $g_0(n) > N_0$ and $g_1(n) > N_0$. Define $s : N_0 \to 2$ by

$$s(n) = \begin{cases} 1 & (\exists t \leq N_1)(g_1(t) = n) \\ 0 & \text{otherwise} \end{cases}.$$

Note that s exists by Δ_1^0-CA. The main property of s is that for any $f : \mathbb{N} \to 2$ extending s, the least n for which X_f fails to be a separation in the sense that either $f(g_0(n)) = 1$ or $f(g_1(n)) = 0$ must have the property that $g_0(n) \geq N_0$ or $g_1(n) \geq N_0$.

(Case I) Suppose there is an initial segment of s which belongs to F. Using Δ_1^0-CA, let $f_0 : \mathbb{N} \to 2$ be defined by $f_0 = s\hat{~}\bar{0}$ which is the function with s as its initial segment followed by the constant 0 sequence. As noted above, the least n such $f_0(g_0(n)) = 1$ or $f_0(g_1(n)) = 0$ must have the property that $g_0(n) \geq N_0$ or $g_1(n) \geq N_0$. Since $f_0(k) = 0$ for all $k \geq N_0$, the latter must occur. This means $\phi_1(f_0)$ holds and therefore $\neg\phi_0(f_0)$. It has been shown that there is a function f_0 so that an initial segment of f_0 belongs to F but $\neg\phi_0$.

(Case II) There is no initial segment of s in F. Let $f_1 : \mathbb{N} \to 2$ be defined by $f_1 = s\hat{~}\bar{1}$, i.e. the function with s as an initial segment followed by the constant 1 function. As noted above, the least n so that $f_1(g_0(n)) = 1$ or $f_1(g_1(n)) = 0$ must have the property that $g_0(n) \geq N_0$ or $g_1(n) \geq N_0$. Since $f_1(k)$ takes only value 1 at $k \geq N_0$, the former must occur. Thus one has $\phi_0(f)$. Note that no initial segment of f_1 belongs to F since $f_1 \upharpoonright N_0 = s$ and N_0 is the length of the longest string in F. It has been shown that there is an f_1 so that no initial segment of f_1 belongs to F but $\phi_0(f)$.

The conclusions from Case I and II show that F does not witness the condition in (3) for the formula ϕ_0. Since $F \subseteq {}^{<\mathbb{N}}2$ was an arbitrary finite set, (3) fails. $\qquad\square$

The following are the basic notions of determinacy. See [20] V.8 for more information.

Definition 1.5. ($\mathsf{RCA_0}$) Let $K = \mathbb{N}$ or $K \in \mathbb{N}$ (thinking of $K = \{n \in \mathbb{N} : n < K\}$). (For example, $K = 2 = \{0, 1\}$.) A stategy in K is a function $\rho : {}^{<\mathbb{N}}K \to K$.

Now suppose ρ_0 and ρ_1 are two strategies in K. By primitive recursion in $\mathsf{RCA_0}$ ([20] Theorem II.3.4), let $\rho_0 * \rho_1 \in {}^{\mathbb{N}}K$ be the unique function $f \in {}^{\mathbb{N}}K$ so that for all $k \in \mathbb{N}$, $f(2k) = \rho_0(f \restriction 2k)$ and $f(2k+1) = \rho_1(f \restriction 2k + 1)$.

If $\varphi(X)$ is an $\mathscr{L}_2$-formula with one free "set" variable X. One says that the game G_φ^K is determined if and only if one of the following holds.

(1) There exists a strategy (in K) ρ_0 so that for all strategies ρ_1 (in K), $\varphi(\rho_0 * \rho_1)$.
(2) There exists a strategy (in K) ρ_1 so that for all strategies ρ_0 (in K), $\neg\varphi(\rho_0 * \rho_1)$.

The following diagram illustrates the game G_φ^K. Here ρ_0 acts as Player 1's strategy, and ρ_1 acts as Player 2's strategy. $\rho_0 * \rho_1$ is the infinite run of the corresponding play.

$$G_\varphi^K \qquad \begin{array}{llllll} \rho_0 \ \mathrm{I} & a_0 & & a_2 & & a_4 & & \cdots \\ \\ \rho_1 \ \mathrm{II} & & a_1 & & a_3 & & a_5 & & \cdots \end{array} \qquad a = \rho_0 * \rho_1$$

In the above diagram, Player 1 produces the even moves $a_{2k} \in K$ and Player 2 produces the odd moves $a_{2k+1} \in K$. Together they produce an infinite sequence $a \in {}^{\mathbb{N}}K$. Player 1 is said to win G_φ^K if and only if $\varphi(a)$ holds. In the definition of the determinacy of G_φ^K, the first case indicates ρ_0 is a Player 1 winning strategy. The second case indicates that ρ_1 is a Player 2 winning strategy.

If Γ is a collection of formulas with one free set variable, then $\Gamma\text{-}\mathsf{DET}^K$ is the statement that for all $\varphi \in \Gamma$, the game G_φ^K is determined.

This article will be mostly concerned with Lipschitz games.

Definition 1.6. ($\mathsf{RCA_0}$) If $f \in {}^{\mathbb{N}}\mathbb{N}$, then let $f_{\mathrm{even}} \in {}^{\mathbb{N}}\mathbb{N}$ be defined by $f_{\mathrm{even}}(n) = f(2n)$ and $f_{\mathrm{odd}} \in {}^{\mathbb{N}}\mathbb{N}$ be defined by $f_{\mathrm{odd}}(n) = f(2n+1)$. Similarly, if $s \in {}^{<\mathbb{N}}\mathbb{N}$, let $s_{\mathrm{even}}(n) = s(2n)$ for all n such that $2n < |s|$ and $s_{\mathrm{odd}}(n) = s(2n+1)$ for all n such that $2n + 1 < |s|$.

Let $K = \mathbb{N}$ or $K \in \mathbb{N}$. Let $\varphi_0(X)$ and $\varphi_1(X)$ be two formulas with X as its only free set variable. One says that the Lipshitz game $L_{\varphi_0,\varphi_1}^K$ is determined if and only if one of the following holds.

(1) There exists a strategy ρ_0 (in K) so that for all strategies ρ_1 (in K),

$$\varphi_0((\rho_0 * \rho_1)_{\mathrm{even}}) \Leftrightarrow \neg\varphi_1((\rho_0 * \rho_1)_{\mathrm{odd}}).$$

(2) There exists a strategy ρ_1 (in K) so that for all strategies ρ_0 (in K),

$$\varphi_0((\rho_0 * \rho_1)_{\mathrm{even}}) \Leftrightarrow \varphi_1((\rho_0 * \rho_1)_{\mathrm{odd}}).$$

The following diagram illustrates the game $L_{\varphi_0,\varphi_1}^K$.

$$L_{\varphi_0,\varphi_1}^K \qquad \begin{array}{lllll} \rho_0 \ \mathrm{I} & a_0 & & a_1 & & a_2 & & \cdots \\ \\ \rho_1 \ \mathrm{II} & & b_0 & & b_1 & & b_2 & & \cdots \end{array} \qquad \begin{array}{l} a = (\rho_0 * \rho_1)_{\mathrm{even}} \\ \\ b = (\rho_0 * \rho_1)_{\mathrm{even}} \end{array}$$

Player 1 and Player 2 take turns making moves from K. Player 1 plays $a_n \in K$ and Player 2 plays $b_n \in K$ for all $n \in \omega$. Thus Player 1 independently produces $a \in {}^{\mathbb{N}}K$ and Player 2 independently produces $b \in {}^{\mathbb{N}}K$. Player 2 wins the game $L_{\varphi_0,\varphi_1}^K$ if and only if $\varphi_0(a) \Leftrightarrow \varphi_1(b)$. In the definition of the determinacy of the Lipschitz game $L_{\varphi_0,\varphi_1}^K$, the first case indicates Player 1 has a winning strategy ρ_0 and the second case indicates that Player 2 has a winning strategy ρ_1.

Suppose Γ_0 and Γ_1 are two collections of formulas with one free set variable. Then $(\Gamma_0, \Gamma_1)\text{-}\mathsf{LDET}^K$ is the statement that for all $\varphi_0(X) \in \Gamma_0$ and $\varphi_1(X) \in \Gamma_1$, the Lipschitz game $L_{\varphi_0,\varphi_1}^K$ is determined.

Again Lipschitz determinacy principles involving the classes Δ_n^0 require more care to define. The main Lipschitz determinacy principle of concern in this article is $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)\text{-}\mathsf{LDET}^2$. For explicitness, this is the following axiom schema: For each formula $\psi \in \Sigma_1^0 \wedge \Pi_1^0$ (i.e. there is a formula $\zeta_0 \in \Sigma_1^0$ and a formula $\zeta_1 \in \Pi_1^0$ so that $\psi = \zeta_0 \wedge \zeta_1$) and any pair of formulas $\varphi_0 \in \Sigma_1^0$ and $\varphi_1 \in \Pi_1^0$, one has the following statement: if $(\forall f)(\varphi_0(f) \Leftrightarrow \varphi_1(f))$, then the Lipschitz game (on $2 = \{0, 1\}$) L_{ψ,φ_0}^2 is determined.

Suppose φ_0 and φ_1 are two formulas. Note that ρ_0 is a Player 1 winning strategy in $L^2_{\varphi_0,\varphi_1}$ if and only if ρ_0 is a Player 1 winning strategy in $L^2_{\neg\varphi_0,\neg\varphi_1}$. Similarly, ρ_1 is a Player 2 winning strategy in $L^2_{\varphi_0,\varphi_1}$ if and only if ρ_1 is a Player 2 winning strategy in $L^2_{\neg\varphi_0,\neg\varphi_1}$. Thus $L^2_{\varphi_0,\varphi_1}$ is determined if and only if $L^2_{\neg\varphi_0,\neg\varphi_1}$ is determined. In particular, this can be used to show that $(\Sigma^0_1 \wedge \Pi^0_1, \Delta^0_1)$-LDET^2 is equivalent to $(\Sigma^0_1 \vee \Pi^0_1, \Delta^0_1)$-LDET^2.

2. Lipschitz Determinacy and WKL$_0$

First one will show that RCA$_0$ cannot prove $(\Sigma^0_1 \wedge \Pi^0_1, \Delta^0_1)$-LDET^2. This will be done by showing that $(\Sigma^0_1 \wedge \Pi^0_1, \Delta^0_1)$-LDET^2 fails in the model REC $\models$ RCA$_0$. This result is important as it motivates the framework of the main theorems.

Theorem 2.1. REC $\not\models (\Sigma^0_1 \wedge \Pi^0_1, \Delta^0_1)$-LDET^2. *Hence* RCA$_0$ *does not prove* $(\Sigma^0_1 \wedge \Pi^0_1, \Delta^0_1)$-LDET^2.

Proof. Recall that lightface Δ^0_1 subset of ω in the sense of descriptive set theory or recursion theory is equivalently a recursive subset of ω. A lightface Σ^0_1 subset of ω is a recursively enumerable subset of ω which is also the range of an injective recursive function. Let $\mathcal{C}$ denote the collection of recursive subsets of ω. Recall that REC is the $\mathscr{L}_2$ structure where $\mathbb{N}^{\mathsf{REC}} = \omega$ and $\mathscr{P}(\mathbb{N})^{\mathsf{REC}} = \mathcal{C}$.

Let $\langle \Phi_e : e \in \omega \rangle$ denote a recursive enumeration of all partial recursive functions. Let $B_0 = \{e \in \omega : \Phi_e(e) = 0\}$ and $B_1 = \{e \in \omega : \Phi_e(e) = 1\}$. One can show that B_0 and B_1 are recursively enumerable disjoints sets which which are recursively inseparable, meaning that there is no recursive set $X \subseteq \omega$ so that $B_1 \subseteq X$ and $B_0 \cap X = \emptyset$. (Note B_0 and B_1 are not computable so they do not belong to $\mathscr{P}(\mathbb{N})^{\mathsf{REC}} = \mathcal{C}$.) Thus there are recursive functions $g_0 : \omega \to \omega$ and $g_1 : \omega \to \omega$ so that $g_0[\omega] = B_0$ and $g_1[\omega] = B_1$ (in the real world). Note g_0 and g_1 are functions that belong to REC. Let $\varphi_0(f)$ be

$$(\exists n)[(\forall m < n)(f(g_0(m)) = 0 \wedge f(g_1(m)) = 1) \wedge f(g_0(n)) = 1 \wedge f(g_1(n)) = 1].$$

Let $\varphi_1(f)$ be

$$(\exists n)[(\forall m < n)(f(g_0(m)) = 0 \wedge f(g_1(m)) = 1) \wedge f(g_1(n)) = 0].$$

As argued in Fact 1.4 using the assumption that REC has no separation for g_0 and g_1, one has that REC $\models (\forall f)(\varphi_0(f) \vee \varphi_1(f))$ and REC $\models (\forall f)(\neg\varphi_0(f) \vee \neg\varphi_1(f))$. Thus REC $\models (\forall f)(\varphi_0(f) \Leftrightarrow \neg\varphi_1(f))$ (although in the real world this equivalence does not hold). As argued in Fact 1.4, there are no finite sets F so that $(\forall f)(\varphi_0(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in F))$. Since φ_0 and φ_1 are both lightface Σ^0_1 subsets of $^\omega 2$, there are computable sets $U_0, U_1 \subseteq \, ^{<\omega}2$ (and hence U_0 and U_1 are sets in REC) so that for all f, $\varphi_0(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in U_0)$ and $\varphi_1(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in U_1)$. Note that $U_0 \cap U_1 = \emptyset$. Let $\varpi(s)$ be the following Σ^0_1 formula

$$(\exists t_0 \in U_0)(\exists t_1 \in U_1)(s \subseteq t_0 \wedge s \subseteq t_1)\}.$$

The set $A = \{s \in \, ^{<\omega}2 : \varpi(s)\}$ is a recursively enumerable set (which may not belong to REC). Intuitively, $\varpi(s)$ means that s has not decided φ_0 in the sense there are extensions of s which belong to U_0 and there are extensions of s which belong U_1. A is infinite and in fact, REC proves that ϖ has infinitely many solutions. To see this, suppose that REC thinks there is an $N \in \omega$ so that for all $s \in \, ^{<\omega}2$, $\varpi(s)$ implies $|s| < N$. Then since $(\forall f)(\varphi_0(f) \vee \varphi_1(f))$, one has that for all $s \in \, ^N 2$, there is an $i \in \{0, 1\}$ so that for all $f \supseteq s$, f has an initial segment in U_i and therefore $\varphi_i(f)$. (That is, all extension of f have been decided in the same way.) Let F be the finite set of $s \in \, ^N 2$ so that there exists a $t \in U_0$ so that $t \subseteq s$ or $s \subseteq t$. Since F is finite, F is recursive and therefore belongs to REC. Note that for all $f : \omega \to 2$, $\varphi_0(f)$ if and $(\exists n)(f \upharpoonright U_0)$ if and only if $f \upharpoonright N \in F$ if and only if $(\exists n)(f \upharpoonright n \in F)$. However, REC does not think that such a finite set exists for φ_0. It has been shown that A is an infinite recursively enumerable set and since every infinite recursively enumerable set has an infinite computable subset, let $Y \subseteq A$ be an infinite computable set. Note that Y is a set belonging to REC. Also REC thinks Y is infinite and for all $n \in Y$, $\varpi(n)$ holds.

Let S be a recursively enumerable set whose complement does not have any infinite recursively enumerable subsets. (S is called a simple set and $\omega \setminus S$ is a Π^0_1 immune set. Post ([21] Theorem 5.2.3) showed simple sets exist.) Let ζ be a Σ^0_1 formula so that $n \in S$ if and only if $\zeta(n)$.

Define a formula $\psi(f)$ in the free set variable f as follows.

$$(\exists n)[(f(n) = 1) \wedge \zeta(n) \wedge (\forall m < n)(f(m) = 0)] \vee (\forall n)(f(n) = 0).$$

Intuitively, one can think of $f : \omega \to 2$ as the characteristic function of the set $X_f = \{n \in \omega : f(n) = 1\}$. $\psi(f)$ holds means either $X_f = \emptyset$ or the least element of X_f belongs to the simple set S. Note that ψ is a $\Sigma_1^0 \vee \Pi_1^0$ formula.

Now one will consider the Lipschitz game L^2_{ψ, φ_0}.

(Case I) Suppose Player 2 has a winning strategy ρ_1^* for L^2_{ψ, φ_0} in REC.

Let $\tilde{\rho}_0 : {}^{<\omega}2 \to 2$ be the constant 0 function. (That is, $\tilde{\rho}_0$ has Player 1 put down 0 regardless of Player 2's move.) $\tilde{\rho}_0$ is recursive and therefore $\tilde{\rho}_0$ belongs to REC. $(\tilde{\rho}_0 * \rho_1^*)_{\text{even}} = \bar{0}$, the constant 0 sequence. Thus $\psi((\tilde{\rho}_0 * \rho_1^*)_{\text{even}})$ by the second disjunct in the definition of ψ. Since ρ_1^* is a Player 2 winning strategy in L^2_{ψ, φ_0}, one must have $\varphi_0((\tilde{\rho}_0 * \rho_1^*)_{\text{odd}})$. Thus there is some $n \in \omega$ so that $(\tilde{\rho}_0 * \rho_1^*)_{\text{odd}} \upharpoonright n \in U_0$. Since S is a simple set, $\omega \setminus S$ is an immune set and hence infinite. Thus there exists some $m > n$ so that $\neg\zeta(m)$. Let $\hat{\rho}_0 : {}^{<\omega}2 \to 2$ be the Player 1's strategy that put down 1 on Player 1's $(m+1)^{\text{st}}$ move and 0 for its other moves. That is,

$$\hat{\rho}_0(s) = \begin{cases} 1 & |s| = 2m \\ 0 & \text{otherwise} \end{cases}.$$

$\hat{\rho}_0$ is recursive and thus belongs to REC. By definition of $\hat{\rho}_0$,

$$(\hat{\rho}_0 * \rho_1^*)_{\text{even}}(k) = \begin{cases} 1 & k = m \\ 0 & \text{otherwise} \end{cases}$$

So m is the least k so that $(\hat{\rho}_0 * \rho_1^*)_{\text{even}}(k) = 1$. Since $\neg\zeta(m)$ holds, one has that $\neg\psi((\hat{\rho}_0 * \rho_1^*)_{\text{even}})$ holds. Since $m > n$, one has that $\tilde{\rho}_0$ and $\hat{\rho}_0$ agree on all strings of length less than n. Thus $(\hat{\rho}_0 * \rho_1^*)_{\text{odd}} \upharpoonright n = (\tilde{\rho}_0 * \rho_1^*)_{\text{odd}} \upharpoonright n \in U_0$. Hence $\varphi_0((\hat{\rho}_0 * \rho_1^*)_{\text{odd}})$ holds. Player 1 using $\hat{\rho}_0$ defeats Player 2 using ρ_1^*. This contradicts ρ_1^* being a Player 2 winning strategy in L^2_{ψ, φ_0}.

(Case II) Suppose Player 1 has a winning strategy ρ_0^* for L^2_{ψ, φ_0} in REC.

Since ρ_0^* belongs to REC, ρ_0^* is a recursive function. Since Y is an infinite computable subset of the recursively enumerable set A (defined by ϖ), one has that for each $s \in Y$, there exists some $t_0 \in U_0$ so that $s \subseteq t_0$. Define $\Psi : Y \to U_0$ by $\Psi(s)$ is the least $t \in U_0$ so that $s \subseteq t$. (Sequences can be coded by natural numbers so "least" refers to the ordering on ω.) Because for each $s \in Y$, there does exists such a t, this implies that Ψ is well defined on Y and Ψ is a recursive function. Hence Ψ belongs to REC. $\Psi[Y] = \{t : (\exists s)(s \in Y \wedge \Psi(s) = t\}$ is an infinite recursively enumerable set since Y is infinite and recursive. Thus there is an infinite recursive set $Z \subseteq \Psi[Y]$ and thus Z is a set in REC.

Next, one will define a recursive family of Player 2 strategies $\langle \rho_1^u : u \in Z \rangle$ so that for each $u \in Z$, ρ_1^u is the player 2 strategy that simply puts down u and then 0 forever afterward. More precisely, define a function $\Theta : {}^{<\omega}2 \times {}^{<\omega}2 \to 2$ as follows.

$$\Theta(u, s) = \begin{cases} u(k) & u \in Z \wedge |s| = 2k+1 \wedge k < |u| \\ 0 & \text{otherwise} \end{cases}$$

Note that Θ is a computable function. Θ defines a Z-index family of Player 2 strategies as follows. For each $u \in Z$, let $\rho_1^u : {}^{<\omega}2 \to 2$ be defined by $\rho_1^u(s) = \Theta(u, s)$. Let $\varsigma(n)$ be the following formula.

$$(\exists u \in Z)[(\rho_0^* * \rho_1^u)_{\text{even}}(n) = 1 \wedge (\forall m < n)((\rho_0^* * \rho_1^u)_{\text{even}}(m) = 0)]$$

Note that $\varsigma(n)$ is a Σ_1^0 formula involving the computable function Θ. Intuitively $\varsigma(n)$ holds if and only if there is some $u \in Z$ so that n is the least k so that $(\rho_0^* * \rho_1^u)_{\text{even}}(k) = 1$. Let $L = \{n \in \omega : \varsigma(n)\}$. L is a recursively enumerable set.

First, one seeks to show that $L \subseteq \omega \setminus S$. Suppose $n \in L$. Let $u \in Z$ be such that n is the least natural number k so that $(\rho_0^* * \rho_1^u)_{\text{even}}(k) = 1$. If $n \in S$, then $\zeta(n)$ holds. Thus by definition of ψ, $\psi((\rho_0^* * \rho_1^u)_{\text{even}})$ holds. However, $u \in Z$ implies that $u \in \Psi[Y] \subseteq U_0$. By the definition of ρ_1^u, $(\rho_0^* * \rho_1^u)_{\text{odd}} = u\hat{\ }\bar{0}$. Since $u \subset (\rho_0^* * \rho_1^u)_{\text{odd}}$ and $u \in U_0$, one has that $\varphi_0((\rho_0^* * \rho_1^u)_{\text{odd}})$. Thus Player 2 has won. Contradiction. This shows that $n \in L$ implies that $n \notin S$.

Next, one seeks to show that L is infinite. Suppose L is finite. Then there is a $P \in \omega$ so that for all $n \in \omega$, $n \in L$ implies $n < P$. Let $G = \{s \in Y : |s| \le P\}$ which is a finite set. Thus $\Psi[G] = \{\Psi(s) : s \in G\}$ is a finite set. Since Z is infinite, $Z \setminus \Psi[G]$ is nonempty. Let $u \in Z \setminus \Psi[G]$. There is some $v \in Y$ so that $\Psi(v) = u$ and one must have that $|v| > P$. Since $u \in U_0$, $u \subseteq u\hat{\ }\bar{0}$, and $(\rho_0^* * \rho_1^u)_{\text{odd}} = u\hat{\ }\bar{0}$, one has $\varphi_0((\rho_0^* * \rho_1^u)_{\text{odd}})$. Since

ρ_0^* is a Player 1 winning strategy in L_{ψ,φ_0}^2, one must have that $\neg\psi((\rho_0^* * \rho_1^u)_{\text{even}})$. First, this means that there exists an n so that $(\rho_0^* * \rho_1^u)_{\text{even}}(n) = 1$. Let n^* be the least such n. By definition of L, $n^* \in L$. Since one was assuming that L is bounded by P, one must have that $n^* < P$. Also note that $\neg\psi((\rho_0^* * \rho_1^u)_{\text{even}})$ implies that $\neg\zeta(n^*)$. Since $v \in Y$ and hence $\varpi(v)$, one has that there exists $\hat{u} \in U_1$ so that $v \subset \hat{u}$. Since $|v| > P$, $|\hat{u}| > P$. Let $\hat{\rho}_1 : {}^{<\omega}2 \to 2$ be the Player 2 strategy which simply has Player 2 put down $\hat{u}$ and then 0 forever. Precisely

$$\hat{\rho}_1(s) = \begin{cases} \hat{u}(k) & |s| = 2k+1 \wedge k < |\hat{u}| \\ 0 & \text{otherwise} \end{cases}$$

Note that $\hat{\rho}_1$ is recursive and hence belongs to REC. Now since $v \subseteq \hat{u}$ and $|v| > P$, one has that ρ_1^u and $\hat{\rho}_1$ agree on all strings of length less than P. Thus $(\rho_0^* * \hat{\rho}_1)_{\text{even}} \upharpoonright P = (\rho_0^* * \rho_1^u)_{\text{even}} \upharpoonright P$. Since $n^* < P$, n^* is also the least n so that $(\rho_0^* * \hat{\rho}_1)_{\text{even}}(n) = 1$. Since it was noted above that $\neg\zeta(n^*)$, one must have that $\neg\psi((\rho_0^* * \hat{\rho}_1)_{\text{even}})$. Also by definition of $\hat{\rho}_1$, one has that $(\rho_0^* * \hat{\rho}_1)_{\text{odd}} = \hat{u}^\frown \bar{0}$. Since $\hat{u} \in U_1$ and $\hat{u} \subseteq (\rho_0^* * \hat{\rho}_1)_{\text{odd}}$, one has that $\varphi_1((\rho_0^* * \hat{\rho}_1)_{\text{odd}})$. Thus $\neg\varphi_0((\rho_0^* * \hat{\rho}_1)_{\text{odd}})$. This implies that Player 2 using $\hat{\rho}_1$ has defeated Player 1 using ρ_0^*. This contradicts that ρ_0^* was a Player 1 winning strategy in L_{ψ,φ_0}^2. Thus this shows that L is infinite.

It has been shown that L is an infinite recursively enumerable subset of $\omega \setminus S$. Since S is a simple set, $\omega \setminus S$ is immune meaning $\omega \setminus S$ cannot have an infinite recursively enumerable subset. Contradiction.

Since neither Case I nor Case II can occur, one has that L_{ψ,φ_0}^2 is not determined in REC. ψ is $\Sigma_1^0 \vee \Pi_1^0$ so $\neg\psi$ is $\Sigma_1^0 \wedge \Pi_1^0$. Since the determinacy of L_{ψ,φ_0}^2 is equivalent to the determinacy of $L_{\neg\psi,\neg\varphi_0}^2$, one has that $L_{\neg\psi,\neg\varphi_0}^2$ is not determined. Thus $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-LDET^2 fails in REC. $\qquad\square$

Theorem 2.1 motivates why Fact 1.4 (3) is the preferred form of WKL$_0$. The argument for Theorem 2.1 suggests an abstract argument to show $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-LDET^2 implies WKL$_0$. Note if one assumes the failure of WKL and replaces the Π_1^0 immune set with an arbitrary Π_1^0-formula with infinitely many solutions, the argument gives a procedure for producing an infinite set of solutions to the Π_1^0 formula. However Fact 1.2 implies this is equivalent to ACA$_0$. This is impossible since ACA$_0$ can prove WKL$_0$. The formal details is given in the next theorem whose proof is very similar to the argument in Theorem 2.1.

Theorem 2.2. RCA$_0$ *proves that* $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-LDET^2 *implies* WKL$_0$.

Proof. Suppose WKL$_0$ fails. By Fact 1.4 (3), there exist Σ_1^0-formulas $\varphi_0(f)$ and $\varphi_1(f)$ in one free set variable f so that $(\forall f)(\varphi_0(f) \Leftrightarrow \neg\varphi_1(f))$ and there is no finite set F with the property that $(\forall f)(\varphi_0(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in F))$. As mentioned above, RCA$_0$ proves there are quantifier free formulas θ_0 and θ_1 so that $(\forall f)(\varphi_0(f) \Leftrightarrow (\exists n)\theta_0(f \upharpoonright n))$ and $(\forall f)(\varphi_1(f) \Leftrightarrow (\exists n)\theta_1(f \upharpoonright n))$. By Δ_1^0-CA, let $U_0 = \{s \in {}^{<\mathbb{N}}2 : \theta_0(s)\}$ and $U_1 = \{s \in {}^{<\mathbb{N}}2 : \theta_1(s)\}$. Thus $(\forall f)(\varphi_0(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in U_0))$ and $(\forall f)(\varphi_1(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in U_1))$. Since $(\forall f)(\varphi_0(f) \Leftrightarrow \neg\varphi_1(f))$, one has that $U_0 \cap U_1 = \emptyset$. Let $\varpi(s)$ be the following Σ_1^0 formula

$$(\exists t_0 \in U_0)(\exists t_1 \in U_1)(s \subseteq t_0 \wedge s \subseteq t_1).$$

$\varpi(s)$ means that s has not decided φ_0 in the sense that s has an extension that belongs to U_0 and has an extension that belongs to U_1. Note that ϖ has infinitely many solution. To see this, suppose ϖ has only finitely many solutions. Then there is an $N \in \mathbb{N}$ so that for all $s \in {}^{<\mathbb{N}}2$, $\varpi(s)$ implies $|s| < N$. Thus if $|s| = N$, then there is an $i \in \{0,1\}$ so that for all $f_0, f_1 \supset s$, $\varphi_i(f_0)$ and $\varphi_i(f_1)$. To see this, suppose there are $f_0, f_1 \supseteq s$ so that $\varphi_0(f_0)$ and $\varphi_1(f_1)$. Then there exists $n_0, n_1 > |s|$ so that $f_0 \upharpoonright n_0 \in U_0$ and $f_1 \upharpoonright n_1 \in U_1$. Thus $\varpi(s)$ and $|s| = N$ which contradiction the assumption that all solutions to $\varpi(s)$ have length less than N. It has been shown that if $|s| = N$, then all extensions $f \supset s$ decides φ_0 in the same way. Using bounded Σ_1^0-CA which holds in RCA$_0$, let $F = \{s \in {}^N2 : (\exists t)(t \in U_0 \wedge t \subseteq s \vee s \subseteq t)\}$ which is a finite set. By the observation above that all s of length N have the property that all extensions of s decide membership in φ_0 in the same way, if $s \in F$, then for any $f \supset s$, $\varphi_0(f)$. Thus $\varphi_0(f)$ if and only if $f \upharpoonright N \in F$ if and only if $(\exists n)(f \upharpoonright n \in F)$. This contradicts the assumptions that no such finite set F exists for φ_0. This completes the argument that ϖ has infinitely many solutions. Since ϖ is a Σ_1^0 formula with infinitely many solutions, Fact 1.1 (5) implies that there is an infinite set Y so that $(\forall n)(n \in Y \Rightarrow \varpi(n))$.

Since one is assuming WKL$_0$ fails and ACA$_0$ proves WKL$_0$, one must also have that ACA$_0$ fails. By Fact 1.2 (4), there is a Σ_1^0-formula ζ so that $\neg\zeta$ has infinitely many solutions and there is no infinite set X so that $(\forall n)(n \in X \Rightarrow \neg\zeta(n))$, i.e. no infinite set of solutions for $\neg\zeta$.

Define the formula $\psi(f)$ in the free set variable f by

$$(\exists n)((f(n) = 1) \wedge \zeta(n) \wedge (\forall m < n)(f(m) = 0)) \vee (\forall n)(f(n) = 0).$$

Thus $\psi(f)$ means that either $f = \bar{0}$, the constant 0 function, or the first n so that $f(n) = 1$ satisfies $\zeta(n)$.

Now one will consider the Lipschitz game L^2_{ψ, φ_0}.

(Case I) Suppose Player 2 has a winning strategy ρ_1^* for L^2_{ψ, φ_0}.

Let $\tilde{\rho}_0 : {}^{<\mathbb{N}}2 \to 2$ be the constant 0 function. Since $(\tilde{\rho}_0 * \rho_1^*)_{\text{even}} = \bar{0}$, one has that $\psi((\tilde{\rho}_0 * \rho_1^*)_{\text{even}})$ holds. Since ρ_1^* is a Player 2 winning strategy in L^2_{ψ, φ_0}, one must have that $\varphi_0((\tilde{\rho}_0 * \rho_1^*)_{\text{odd}})$. So there is some $n \in \omega$ so that $(\tilde{\rho}_0 * \rho_1^*)_{\text{odd}} \upharpoonright n \in U_0$. Since $\neg\zeta$ has infinitely many solutions, there is an $m > n$ so that $\neg\zeta(m)$. Define $\hat{\rho}_0 : {}^{<\mathbb{N}}2 \to 2$ by

$$\hat{\rho}_0(s) = \begin{cases} 1 & |s| = 2m \\ 0 & \text{otherwise} \end{cases}.$$

Thus $\hat{\rho}_0$ is the strategy that plays 1 on the $(m+1)^{\text{st}}$-move and 0 on all other moves. Thus m is the least k so that $(\hat{\rho}_0 * \rho_1^*)_{\text{even}}(k) = 1$. Since $\neg\zeta(m)$ holds, one has that $\neg\psi((\hat{\rho}_0 * \rho_1^*)_{\text{even}})$. Since $m > n$, $\tilde{\rho}_0$ and $\hat{\rho}_0$ agree on all strings of length less than n. Thus $(\hat{\rho}_0 * \rho_1^*)_{\text{odd}} \upharpoonright n = (\tilde{\rho}_0 * \rho_1^*)_{\text{odd}} \upharpoonright n \in U_0$. Thus $\varphi_0((\hat{\rho}_0 * \rho_1^*)_{\text{odd}})$. Player 1 using $\hat{\rho}_0$ has defeated Player 2 using ρ_1^*. Contradiction.

(Case II) Suppose Player 1 has a winning strategy ρ_0^* for L^2_{ψ, φ_0}.

By definition of Y, which is an infinite set of solutions for ϖ, for each $s \in Y$, there exists a $t \in U_0$ so that $s \subseteq t$. Thinking of ${}^{<\mathbb{N}}2$ as coded by $\mathbb{N}$, minimization ([20] Theorem II.3.5) can be used to show there is a function $\Psi : Y \to U_0$ so that $\Psi(s)$ is the least $t \in U_0$ so that $s \subseteq t$. Define a Σ^0_1 formula $\xi(t)$ by $(\exists s)(s \in Y \wedge \Psi(s) = t)$. Since Y is infinite, for each n, there is an $s \in Y$ so that $|s| > n$. Hence for this s, $|\Psi(s)| > n$ and $\xi(\Psi(s))$. Thus ξ is a Σ^0_1 formula with infinitely many solutions. By Fact 1.1 (5), there is an infinite set $Z \subseteq {}^{<\mathbb{N}}2$ so that $(\forall s)(s \in Z \Rightarrow \xi(s))$.

Using Δ^0_1-CA, define $\Theta : {}^{<\mathbb{N}}2 \times {}^{<\mathbb{N}}2 \to 2$ by

$$\Theta(u, s) = \begin{cases} u(k) & u \in Z \wedge |s| = 2k+1 \wedge k < |u| \\ 0 & \text{otherwise} \end{cases}$$

Define $\rho_1^u : {}^{<\mathbb{N}}2 \to 2$ by $\rho_1^u(s) = \Theta(u, s)$. For each $u \in {}^{<\mathbb{N}}2$, ρ_1^u is considered a Player 2 strategy that simply puts down the bits of u and then plays 0 forever. Define the formula $\varsigma(n)$ by

$$(\exists u \in Z)[(\rho_0^* * \rho_1^u)_{\text{even}}(n) = 1 \wedge (\forall m < n)((\rho_0^* * \rho_1^u)_{\text{even}}(m) = 0)].$$

$\varsigma(n)$ is a Σ^0_1 formula expressed using Θ.

First, one seeks to show that $(\forall n)(\varsigma(n) \Rightarrow \neg\zeta(n))$. Suppose $\varsigma(n)$. There is a $u \in Z$ so that n is the least k so that $(\rho_0^* * \rho_1^u)_{\text{even}}(k) = 1$. Suppose that $\zeta(n)$ holds. Then $\psi((\rho_0^* * \rho_1^u)_{\text{even}})$ holds. However, since $u \in Z$, one has $\xi(u)$ holds. This means there is some $v \in Y$ so that $\Psi(v) = u$. Since Ψ maps into U_0, one has that $u \in U_0$. Since $(\rho_0^* * \rho_1^u)_{\text{odd}} = u^\frown\bar{0}$ and $u \subseteq (\rho_0^* * \rho_1^u)_{\text{odd}}$, one has that $\varphi_0((\rho_0^* * \rho_1^u)_{\text{odd}})$. This shows that Player 2 using ρ_1^u defeated the Player 1 winning strategy ρ_0^*. Contradiction. Thus one must have $\neg\zeta(n)$ holds.

Next, one will show that ς has infinitely many solutions. Suppose there is a P so that $(\forall n)(\varsigma(n) \Rightarrow n < P)$. Let $G = \{s \in Y : |s| \leq P\}$ and let $\Psi[G] = \{\Psi(s) : s \in G\}$ which are both finite sets. Since Z is infinite, $Z \setminus \Psi[G]$ is nonempty so fix $u \in Z \setminus \Psi[G]$. Since $\xi(u)$ holds, there is a $v \in Y$ so that $u = \Psi(v)$. One must have that $|v| > P$. Since Ψ maps into U_0, one has that $u \in U_0$. Since $(\rho_0^* * \rho_1^u)_{\text{odd}} = u^\frown\bar{0}$ and $u \in U_0$, one has that $\varphi_0((\rho_0^* * \rho_1^u)_{\text{odd}})$. Because ρ_0^* is a Player 1 winning strategy in L^2_{ψ, φ_0}, one has $\neg\psi((\rho_0^* * \rho_1^u)_{\text{even}})$. Hence letting n^* be the least n so that $(\rho_0^* * \rho_1^u)_{\text{even}}(n^*) = 1$, one has that $\varsigma(n^*)$ and $\neg\zeta(n^*)$. Since P bounds the solutions to ς, one has $n^* < P$. Since $v \in Y$ which means $\varpi(v)$, there exists a $\hat{u} \in U_1$ so that $v \subseteq \hat{u}$. Using Δ^0_1-CA, define $\hat{\rho}_1 : {}^{<\mathbb{N}}2 \to 2$ by

$$\hat{\rho}_1(s) = \begin{cases} \hat{u}(k) & |s| = 2k+1 \wedge k < |\hat{u}| \\ 0 & \text{otherwise} \end{cases}.$$

Note that $\hat{\rho}_1$ is the Player 2 strategy that puts down $\hat{u}$ and then plays 0 forever. Since $|v| > P$, one has $|\hat{u}| > P$. Thus ρ_1^u and $\hat{\rho}_1$ agree on all strings of length less than P. Thus $(\rho_0^* * \hat{\rho}_1)_{\text{even}} \upharpoonright P = (\rho_0^* * \rho_1^u)_{\text{even}} \upharpoonright P$. This implies n^* is also the least k so that $(\rho_0^* * \hat{\rho}_1)_{\text{even}}(k) = 1$. Since $\neg\zeta(n^*)$, one has that $\neg\psi((\rho_0^* * \hat{\rho}_1)_{\text{even}})$. Since $(\rho_0^* * \hat{\rho}_1)_{\text{odd}} = \hat{u}^\frown\bar{0}$ and $\hat{u} \in U_1$, one has $\varphi_1((\rho_0^* * \hat{\rho})_{\text{odd}})$ and therefore $\neg\varphi_0((\rho_0^* * \hat{\rho}_1)_{\text{odd}})$. This shows that

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Player 2 using $\hat{\rho}_1$ has defeated ρ_0^*. This contradicts ρ_0^* being a Player 1 winning strategy. This concludes the argument showing that ς has infinitely many solutions.

Since ς is an infinite Σ_1^0 set, Fact 1.1 (5) implies that there is an infinite set K so that $(\forall n)(n \in K \Rightarrow \varsigma(k))$. However, one has already shown that $(\forall n)(\varsigma(n) \Rightarrow \neg\zeta(n))$. Thus $(\forall n)(n \in K \Rightarrow \neg\zeta(n))$. However, $\neg\zeta$ was taken to be a formula with infinitely many solution but no infinite set of solutions. Contradiction.

Since neither Case I nor Case II can occur, one has shown that L_{ψ,φ_0}^2 is an undetermined game. Since the determinacy of L_{ψ,φ_0}^2 is equivalent to the determinacy of $L_{\neg\psi,\neg\varphi_0}^2$, one has that $L_{\neg\psi,\neg\varphi_0}^2$ is undetermined. Since ψ is $\Sigma_1^0 \vee \Pi_1^0$ formula, $\neg\psi$ is a $\Sigma_1^0 \wedge \Pi_1^0$ formula. Hence $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-$\mathsf{LDET}^2$ fails. $\qquad\square$

Theorem 2.3. WKL_0 *proves* $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-$\mathsf{LDET}^2$.

Proof. Suppose $\psi(f)$ is a $\Sigma_1^0 \wedge \Pi_1^0$ formula in one free set variable f. Let $\varphi_0(f)$ and $\varphi_1(f)$ be Σ_1^0 formulas such that $(\forall f)(\varphi_0(f) \Leftrightarrow \neg\varphi_1(f))$. One will show the determinacy of the Lipschitz game L_{ψ,φ_0}^2.

By Fact 1.4 (3) applied to φ_0 and φ_1, there are finite sets $F_0 \subseteq {}^{<\mathbb{N}}2$ and $F_1 \subseteq {}^{<\mathbb{N}}2$ with the property that $(\forall f)(\varphi_0(f) \Leftrightarrow (\exists n)(f \restriction n \in F_0))$ and $(\forall f)(\varphi_1(f) \Leftrightarrow (\exists n)(f \restriction n \in F_1))$. Note $F_0 \cap F_1 = \emptyset$.

First, suppose $F_0 = \emptyset$ or $F_1 = \emptyset$. Without loss of generality, suppose $F_1 = \emptyset$. Then $(\forall f)(\neg\varphi_1(f))$ and hence $(\forall f)\varphi_0(f)$. First, suppose $(\exists f)(\neg\psi(f))$. Let g be such that $\neg\psi(g)$. Using Δ_1^0-CA, define ρ_0^* to be a strategy that puts downs the bits of g regardless of what Player 2 plays. Let ρ_1 be any strategy for Player 2. Then $(\rho_0^* * \rho_1)_{\text{even}} = g$ and hence $\neg\psi((\rho_0^* * \rho_1)_{\text{even}})$. By assumption, $\varphi_0((\rho_0^* * \rho_1)_{\text{odd}})$. Thus Player 1 using ρ_0^* has won L_{ψ,φ_0}^2. ρ_0^* is a winning strategy for Player 1. Suppose now that $(\forall f)\psi(f)$. In this case, any strategy for Player 2 is a winning strategy. To see this, let ρ_1^* be any strategy for Player 2. Let ρ_0 be any strategy for Player 1. By the assumptions, $\psi((\rho_0 * \rho_1^*)_{\text{even}})$ and $\varphi_0((\rho_0 * \rho_1^*)_{\text{odd}})$. Player 2 using ρ_1^* has won L_{ψ,φ_0}^2. ρ_1^* is a winning strategy for Player 2.

So one may assume that $F_0 \neq \emptyset$ and $F_1 \neq \emptyset$. Let

$$A = \{s \in {}^{<\mathbb{N}}2 : (\exists t_0 \in F_0)(\exists t_1 \in F_1)(s \subseteq t_0 \wedge s \subseteq t_1)\}$$

which exists by Δ_1^0-CA and is a finite set since the length of all strings in A are less than or equal to the length of the longest string in the finite set $F_0 \cup F_1$. Moreover, since $F_0 \neq \emptyset$, $F_1 \neq \emptyset$, and $F_0 \cap F_1 = \emptyset$, one has that the empty string belongs to A and hence A is not empty. Intuitively, the elements of A consists of those strings s which are undecided for φ_0 meaning there are extensions of s belonging to F_0 and extensions of s belonging to F_1. Since A is finite, let M be the length of the longest string in A. Thus any string s of length greater than M has been decided meaning that there is an $i \in \{0, 1\}$ so that for all $f : \mathbb{N} \to 2$ with $s \subset f$, $\varphi_i(f)$. In particular, if $|s|$ has length greater than M, then if $i \in \{0, 1\}$ is such that s has an initial segment in F_i or s has an extension in F_i, then for all $f \supset s$, $\varphi_i(f)$. Using Δ_1^0-CA, define the following two finite sets.

$$K_0 = \{s \in {}^{M+1}2 : (\exists t \in F_0)(t \subseteq s \vee s \subseteq t)\} \quad K_1 = \{s \in {}^{M+1}2 : (\exists t \in F_1)(t \subseteq s \vee s \subseteq t)\}.$$

Note that $K_0 \cap K_1 = \emptyset$, $K_0 \cup K_1 = {}^{M+1}2$, and for all $i \in \{0, 1\}$, for all $s \in {}^{M+1}2$, and for all $f : \mathbb{N} \to 2$, if $s \in K_i$ and $f \supset s$, then $\varphi_i(f)$.

Consider the sentence θ asserting

$$(\exists s)[s \in {}^{<\mathbb{N}}2 \wedge |s| > M \wedge (\exists f_0)(\exists f_1)(\neg\psi(f_0) \wedge \psi(f_1) \wedge s \subset f_0 \wedge s \subset f_1)].$$

(Case I) θ holds.

Let u, g_0, and g_1 witness the existential quantifiers on the variables s, f_0, and f_1 respectively in the sentence θ. Intuitively, this means the string u (which has length longer than M) is undecided for ψ specifically because $g_0, g_1 : \mathbb{N} \to 2$ extend u but $\neg\psi(g_0)$ and $\psi(g_1)$. Player 1 wins L_{ψ,φ_0}^2 using the following idea for a winning strategy ρ_0^*. The Player 1 strategy ρ_0^* will play the bits of u for its first $|u|$ moves. After Player 2 moves, let t be the $|u|$ string produced thus far by Player 2. Since $|t| = |u| > M$, t has decided φ_0 or $\neg\varphi_0$ in the sense that $t \restriction M + 1$ belongs to K_0 or K_1. However, u has not yet decided to satisfy ψ or $\neg\psi$ since Player 1 still has the option to continue to play g_0 or g_1. If $t \restriction M + 1 \in K_0$, then ρ_0^* will continue by playing the bits of g_0 regardless of Player 2's moves. If $t \restriction M + 1 \in K_1$, then ρ_0^* will continue to play the bits of g_1 regardless of Player 2's moves. Let $h : \mathbb{N} \to 2$ be Player 2's infinite sequence of moves. In the first case, since $t \subset h$ and $t \restriction M + 1 \in K_0$, one has that $\varphi_0(h)$ holds. However Player 1 using ρ_0^* produced g_0 as its infinite sequence of moves. Thus $\neg\psi(g_0)$. Player 1 wins. Similarly, in the second case, since $t \subseteq h$ and

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$t \upharpoonright M + 1 \in K_1$, one has that $\varphi_1(h)$ and hence $\neg\varphi_0(h)$. However Player 1 using ρ_0^* produced g_1 as its infinite sequence of moves. Thus $\psi(g_1)$. Player 1 wins. Thus ρ_0^* is a Player 1 winning strategy.

The formal details are as follows. Using Δ_1^0-CA, define $\rho_0^* : {}^{<\mathbb{N}}2 \to 2$ by

$$\rho_0^*(s) = \begin{cases} u(k) & |s| = 2k \wedge k < |u| \\ g_0(k) & |s| = 2k \wedge k \geq |u| \wedge s_{\mathrm{odd}} \upharpoonright M + 1 \in K_0 \\ g_1(k) & |s| = 2k \wedge k \geq |u| \wedge s_{\mathrm{odd}} \upharpoonright M + 1 \in K_1 \\ 0 & |s| = 2k + 1 \text{ (irrelevant if used as a Player 1's strategy)} \end{cases}.$$

Now suppose ρ_1 is a Player 2 strategy. Let $\rho_0^* * \rho_1$ be the joint run. Suppose $(\rho_0^* * \rho_1)_{\mathrm{odd}} \upharpoonright M + 1 \in K_0$ and therefore $\varphi_0((\rho_0^* * \rho_1)_{\mathrm{odd}})$. Now since $|u| \geq M + 1$, $u \subset g_0$, and by the definition of ρ_0^*, one has that $(\rho_0^* * \rho_1)_{\mathrm{even}} = g_0$. Since $\neg\psi(g_0)$, one has $\neg\psi((\rho_0^* * \rho_1)_{\mathrm{odd}})$. Player 1 has won this run of L_{ψ,φ_0}^2. Suppose $(\rho_0^* * \rho_1)_{\mathrm{odd}} \upharpoonright M + 1 \in K_1$ and hence $\neg\varphi_0((\rho_0^* * \rho_1)_{\mathrm{odd}})$. Again by definition of ρ_0^*, one has that $(\rho_0^* * \rho_1)_{\mathrm{even}} = g_1$. Since $\psi(g_1)$, one has that $\psi((\rho_0^* * \rho_1)_{\mathrm{even}})$. Player 1 has won. This shows that ρ_0^* is a Player 1 winning strategy.

(Case II) $\neg\theta$ holds.

This implies that for all strings s with $|s| = M + 1$, s has decided ψ in the sense that every f_0 and f_1 extending s, either $\psi(f_0) \wedge \psi(f_1)$ or $\neg\psi(f_0) \wedge \neg\psi(f_1)$. For each $s \in {}^{<\omega}2$, let $s\hat{\,}\bar{0}$ be the sequence which has s as its initial segment followed by the constant 0 function. Using bounded $(\Sigma_1^0 \wedge \Pi_1^0)$-CA and bounded $(\Sigma_1^0 \vee \Pi_1^0)$-CA (which both follow from bounded Σ_1^0-CA of RCA$_0$), let

$$H_0 = \{s \in {}^{M+1}2 : \neg\psi(s\hat{\,}\bar{0})\} \quad H_1 = \{s \in {}^{M+1}2 : \psi(s\hat{\,}\bar{0})\}.$$

Note that $H_0 \cap H_1 = \emptyset$ and $H_0 \cup H_1 = {}^{M+1}2$. If $s \in H_0$, then for all $f \supset s$, $\neg\psi(f)$, and if $s \in H_1$, then for all $f \supset s$, $\psi(f)$. Since M is the length of the longest string in A, let $u \in U$ be a string of length M and note that u has the property there is an $i_0 \in \{0, 1\}$ so that $u\hat{\,}i_0 \in K_0$ and $u\hat{\,}i_1 \in K_1$ where $i_1 = 1 - i_0$.

The following is the intuitive idea of how to construct a Player 2 winning strategy ρ_1^*. For the first $|u| = M$ many moves, ρ_1^* will just put down the bits of u. After Player 1 makes the next move, let t be the length $M + 1$ string consisting of Player 1's moves thus far. Since $H_0 \cup H_1 = {}^{M+1}2$, either $t \in H_0$ or $t \in H_1$. If $t \in H_0$, then t decides ψ in the sense that all extensions $f \supset t$ satisfy $\neg\psi(f)$. Thus ρ_1^* will have Player 2 play i_1 forever afterward. Let h be Player 1's infinite sequence of moves. Player 2 following ρ_1 will have $u\hat{\,}\bar{i_1}$ as its infinite sequence of moves. Since $t \subset h$ and $t \in H_0$, one has that $\neg\psi_0(h)$ holds. Since $u\hat{\,}i_1 \subset u\hat{\,}\bar{i_1}$ and $u\hat{\,}i_1 \in K_1$, one has $\varphi_1(u\hat{\,}\bar{i_1})$ and thus $\neg\varphi_0(u\hat{\,}\bar{i_1})$. Hence Player 2 has won. Similarly if $t \in H_1$, then t decides ψ in the sense that all extensions $f \supset t$ satisfy $\psi(f)$. Thus ρ_1^* will have Player 2 play i_0 forever. Let h be Player 1's infinite sequence of moves. Player 2 following ρ_1^* will have $u\hat{\,}\bar{i_0}$ as its infinite sequence of moves. Since $t \subseteq h$ and $t \in H_1$, one has that $\psi(h)$ holds. Since $u\hat{\,}i_0 \in K_0$ and $u\hat{\,}i_0 \subseteq u\hat{\,}\bar{i_0}$, one has that $\varphi_0(u\hat{\,}\bar{i_0})$. Thus Player 2 has won. This shows ρ_1^* is a Player 2 winning strategy.

The formal details are as follows. Using Δ_1^0-CA, define $\rho_1^* : {}^{<\mathbb{N}}2 \to 2$ by

$$\rho_1^*(s) = \begin{cases} u(k) & |s| = 2k + 1 \wedge k < M \\ i_1 & |s| = 2k + 1 \wedge k \geq M \wedge s_{\mathrm{even}} \upharpoonright M + 1 \in H_0 \\ i_0 & |s| = 2k + 1 \wedge k \geq M \wedge s_{\mathrm{even}} \upharpoonright M + 1 \in H_1 \\ 0 & |s| = 2k \text{ (irrelevant if used as a Player 2's strategy)} \end{cases}.$$

Let ρ_0 be a strategy for Player 1. Let $\rho_0 * \rho_1^*$ be the joint run. Suppose $\psi((\rho_0 * \rho_1^*)_{\mathrm{even}})$. Thus $\rho_0 * \rho_1^* \upharpoonright M + 1 \in H_1$ and hence $(\rho_0 * \rho_1^*)_{\mathrm{odd}} = u\hat{\,}\bar{i_0}$. Since $u\hat{\,}i_0 \subseteq (\rho_0 * \rho_1^*)_{\mathrm{odd}}$ and $u\hat{\,}i_0 \in K_0$, one has that $\varphi_0((\rho_0 * \rho_1^*)_{\mathrm{odd}})$. Thus Player 2 has won. Suppose $\neg\psi((\rho_0 * \rho_1^*)_{\mathrm{even}})$. Thus $(\rho_0 * \rho_1^*)_{\mathrm{even}} \upharpoonright M + 1 \in H_0$ and hence $(\rho_0 * \rho_1^*)_{\mathrm{odd}} = u\hat{\,}\bar{i_1}$. Since $u\hat{\,}i_1 \subseteq (\rho_0 * \rho_1^*)_{\mathrm{odd}}$ and $u\hat{\,}i_1 \in K_1$, one has that $\varphi_1((\rho_0 * \rho_1^*)_{\mathrm{odd}})$ and thus $\neg\varphi_0((\rho_0 * \rho_1^*)_{\mathrm{odd}})$. Thus Player 2 has won. It has been shown that ρ_1^* is a Player 2 winning strategy for L_{ψ,φ_0}^2.

The game L_{ψ,φ_0}^2 is determined. $\qquad\square$

Theorem 2.4. RCA$_0$ *proves that* $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-LDET^2 *is equivalent to* WKL$_0$.

Proof. This follows from Theorem 2.2 and Theorem 2.3. $\qquad\square$

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