DEFINABLE COMBINATORICS OF STRONG PARTITION CARDINALS

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Abstract. A cardinal κ has the strong partition property if and only if every partition $P : [\kappa]^\kappa \to 2$ has a homogeneous set $X \subseteq \kappa$ with $|X| = \kappa$. The axiom of determinacy, AD, implies there are many uncountable strong partition cardinals. The strong partition property at κ yield partition measures μ^κ_ϵ on the partition spaces $[\kappa]^\epsilon_*$ for each $\epsilon \leq \kappa$.

This paper will isolate several club uniformization principles which can be used to prove combinatorial results concerning these partition measures. The almost everywhere short length club uniformization at κ is the assertion that for all $R \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed in the club_κ-coordinate (for all $\ell \in [\kappa]^{<\kappa}_*$ and clubs $C_0 \subseteq C_1$, if $R(\ell, C_1)$, then $R(\ell, C_0)$), there is a club $C \subseteq \kappa$ and a function $\Lambda : \mathrm{dom}(R) \cap [C]^{<\kappa}_* \to \mathsf{club}_\kappa$ so that for all $\ell \in \mathrm{dom}(R) \cap [C]^{<\kappa}_*$, $R(\ell, \Lambda(\ell))$. If the almost everywhere short length club uniformization holds at κ, then every function $\Phi : [\kappa]^\kappa_* \to \kappa$ is almost everywhere continuous even in the following finer sense: There is a club $C \subseteq \kappa$ so that for all $f \in [C]^\kappa_*$, if $\beta < \kappa$ is such that $\sup(f \upharpoonright \beta) \leq \Phi(f) < f(\beta)$, then $f \upharpoonright \beta$ is a minimal continuity point for Φ relative to C, meaning for all $g \in [C]^{\omega_1}_*$ with $f \upharpoonright \beta = g \upharpoonright \beta$, $\Phi(f) = \Phi(g)$.

A strengthening of Martin's good coding system called a good coding family will be developed. It will be shown under AD that good coding families can be used to prove the almost everywhere short length club uniformization principle as well as code the ultrapower by the strong partition measure. Many familiar strong partition cardinals of AD such as ω_1, $\omega_{\omega+1}$, the odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$, and $\boldsymbol{\delta}^2_1$ will be shown to have good coding families.

These results concerning purely combinatorial aspects of strong partition cardinals and definable aspects of strong partition cardinals under determinacy will be used to show under AD that the ultrapower of the first uncountable strong partition cardinal ω_1 by the strong partition measure on ω_1 is strictly less than the second strong partition cardinal $\omega_{\omega+1}$. This answers a question of Goldberg which had been considered in some form earlier by Henle.

1. Introduction

Partition relations are combinatorial properties that have been widely studied and applied in finite and infinitary combinatorics. Ramsey showed that the finite partition relations $\omega \to (\omega)^n_k$ holds for all natural numbers k and n, which means every partition $P : [\omega]^k \to k$ (of increasing n-tuple of natural numbers) into k pieces, there is an infinite set $A \subseteq \omega$ which is homogeneous for P, that is, there is an $i < k$ so that for all $\ell \in [A]^k$, $P(\ell) = i$. Generalizations of finite exponent partition relations to uncountable cardinals are very important in infinitary combinatorics and set theory. Under the axiom of choice AC, the partition relation $\kappa \to (\kappa)^2_2$ characterizes a weakly compact cardinal and is a feature of other large cardinals concepts such as the measurable cardinal. Finite exponent partition relations on uncountable cardinals cannot be established in ZFC and have strong consistency strength.

Infinite exponent partition relations yield structure to the universe that is substantially different than the structure in settings satisfying the axiom of choice. From certain definability perspectives, this structure is more natural and often excludes sets that have historically been considered pathological. The infinite exponent partition relation $\omega \to (\omega)^\omega_2$ is called the strong partition property for ω or the Ramsey property for all partitions on $[\omega]^\omega$. Galvin and Prikry [10] and Silver [28] showed that if $P : [\omega]^\omega \to 2$ is a Borel or analytically definable partition, then P has an infinite homogeneous set. Mathias [24] established the consistency of $\omega \to (\omega)^\omega_2$ by exhibiting this property in the Solovay model after Lévy collapsing an inaccessible cardinal. Mathias also showed using the Ramsey property that every function $\Phi : [\omega]^\omega \to \mathbb{R}$ is continuous Ramsey almost everywhere, that is, there is an infinite $A \subseteq \omega$ so that $\Phi \upharpoonright [A]^\omega$ is continuous. Such continuity properties have recently been used by Schrittesser and Törnquist [27] to answer an old question of Mathias of whether $\omega \to (\omega)^\omega_2$ implies there are no maximal almost disjoint families on ω.

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This paper will be concerned with strong partition properties for uncountable cardinals. At uncountable cardinals, the strong partition property is equivalent to a more practically useful form involving homogeneous sets which are closed and unbounded (club): If κ is an uncountable cardinal and $X \subseteq \kappa$, then let $[X]^\kappa_*$ denote the set of increasing functions $f : \kappa \to X$ which are of the correct type (discontinuous everywhere and have uniform cofinality ω; see Definition 2.1). For all $\epsilon \leq \kappa$ and $\gamma < \kappa$, $\kappa \to_* (\kappa)^\epsilon_\gamma$ is the assertion that for all $P : [\kappa]^\kappa_* \to \gamma$, there is a $\delta < \gamma$ and a club $C \subseteq \kappa$ so that for all $f \in [C]^\kappa_*$, $P(f) = \delta$. If $\kappa \to_* (\kappa)^\kappa_2$ holds, then κ is called a strong partition cardinal. If $\kappa \to_* (\kappa)^\kappa_{<\kappa}$ (meaning the subscript γ partition relation holds for all $\gamma < \kappa$), then κ will be called a very strong partition cardinal. κ is a weak partition cardinal if and only if $\kappa \to_* (\kappa)^{<\kappa}_2$ (meaning for every $\epsilon < \kappa$, then exponent ϵ partition relation holds). The correct type partition relation directly defines a measure μ^κ_ϵ on $[\kappa]^\epsilon_*$ by $X \in \mu^\kappa_\epsilon$ if and only if there is a club $C \subseteq \kappa$ so that $[C]^\epsilon_* \subseteq X$. This paper will study almost everywhere combinatorics with respect to these partition measures μ^κ_ϵ with particular interest on the strong partition measure μ^κ_κ.

Many purely combinatorial questions concerning the strong partition property are open since the consistency strength of the strong partition property is poorly understood. The axiom of determinacy, AD, is the assertion that for every two player game on ω, one of the two players has a winning strategy. The only known method for establishing strong partition properties uses determinacy and involves Martin's good coding system for functions on ordinals. (See Section 5.) These methods shows universes satisfying AD possess cofinally many strong partition below Θ, which is the supremum of the ordinals onto which $\mathbb{R}$ can surject. Martin showed under AD that ω_1 is a strong partition cardinal. Jackson [12] showed that $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ is the second uncountable strong partition cardinal. Kechris, Kleinberg, Moschovakis, and Woodin [16] showed that the prewellordering ordinals associated to pointclasses with sufficient closure conditions are strong partition cardinals. As a corollary, $\boldsymbol{\delta}^2_1$ and for all $A \subseteq \mathbb{R}$, the least Σ_1-stable ordinal $\boldsymbol{\delta}_A$ of $L(A, \mathbb{R})$ (which is the least ordinal δ so that $L_\delta(A, \mathbb{R}) \prec_1 L(A, \mathbb{R})$ in a language with a symbol for A, $\mathbb{R}$, and each element of $\mathbb{R}$) are strong partition cardinals.

The consistency strength of the strong partition property is not well calibrated. Woodin observed that if DC and $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ hold, then $\boldsymbol{\Sigma}^1_2$ determinacy holds and thus implies the consistency of one Woodin cardinal. An open question of Woodin asks whether $V = L(\mathbb{R})$ and $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ together imply AD. Kechris and Woodin ([18] and [19]) showed that if $V = L(\mathbb{R})$, then AD holds if and only if there are cofinally many strong partition cardinals below Θ. It is open if it is consistent to have a strong partition cardinal greater than or equal to Θ.

Martin's determinacy methods can extract many additional combinatorial properties of strong partition cardinals which are not known from purely combinatorial method. For instance, Martin's methods always establishes the very strong partition property ($\kappa \to_* (\kappa)^\kappa_{<\kappa}$ or equivalently the κ-completeness of the strong partition measure on κ) simultaneously with the strong partition property ($\kappa \to_* (\kappa)^\kappa_2$). An old open question is whether $\kappa \to_* (\kappa)^\kappa_2$ implies $\kappa \to_* (\kappa)^\kappa_{<\kappa}$. In contrast, from pure combinatorics, when $\epsilon < \kappa$, $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$ implies $\kappa \to (\kappa)^\epsilon_{<\kappa}$. Thus if κ is a weak partition cardinal, then the short partition measures μ^κ_ϵ are κ-complete for all $\epsilon < \kappa$. This paper will be especially interested in strong extensions of the κ-completeness of the strong partition measure on κ called club uniformization properties which are choice principles for families of $\subseteq$-downward closed collections of clubs indexed by the strong partition cardinal or its bounded sequences.

Ultrapowers by measures on strong partition cardinals are important tools for investigating combinatorics especially above the strong partition cardinals itself. Martin showed that if κ is a strong partition cardinal and μ is any countably complete measure on κ such that the ultrapower $\prod_\kappa \kappa/\mu$ is wellfounded, then $\prod_\kappa \kappa/\mu$ is a cardinal. Moreover, if μ is a normal measure and $\prod_\kappa \kappa/\mu$ is wellfounded, then $\prod_\kappa \kappa/\mu$ is a regular cardinal. (See [2] Fact 5.2 and Fact 5.4.) Martin showed under AD that for all $1 \leq n < \omega$, $\prod_{[\omega_1]^n} \omega_1/\mu^{\omega_1}_n = \omega_{n+1}$ (and $\mathsf{DC}_\mathbb{R}$ is not needed here since the wellfoundedness of the ultrapowers can be established by Kunen trees as in [2] Fact 5.8). With these ultrapower representations, Martin and Paris proved the weak partition property for ω_2 and showed $\mathrm{cof}(\omega_n) = \omega_2$ for all $2 \leq n < \omega$. Kleinberg [21] used these ultrapowers to establish Jónssonness of all ω_n with $n < \omega$. Under AD, Kunen ([29]) and Jackson ([12] and [13]) showed that partition properties and these partition measures can be used to completely characterize all countably complete measures for some initial segment of cardinals (for instance below the supremum of the projective ordinal and even ω_{ω_1} but presently quite far below the supremum of the Suslin cardinals and Θ). Jackson showed that the strong partition cardinals below the supremum of the projective ordinals are exactly the odd projective ordinals by using this measure analysis to construct good coding systems at these odd projective ordinals. From pure combinatorics without AD, Kleinberg [21] showed that if κ is a strong partition cardinal and $\lambda = \prod_\kappa \kappa/\mu^\kappa_\kappa$ is wellfounded, then for all $\epsilon < \omega_1$, $\lambda \rightarrow_* (\lambda)^\epsilon_2$. These results show that a strong partition cardinal generates some weaker partition cardinals above itself while the addition of determinacy can provide much more information about these larger cardinals.

Thus a natural question is whether a strong partition cardinal can generate another strong partition cardinal, possibly through an ultrapower by a measure. The most distinguished ultrapower is the ultrapower of the strong partition cardinal by its own strong partition measure. As determinacy is the best understood and essentially only known setting with strong partition cardinals and ω_1 is the first uncountable strong partition cardinal of AD, a concrete question is whether $\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1} \geq \omega_{\omega+1}$, that is, whether the ultrapower of ω_1 by its strong partition measure is greater than or equal to the second strong partition cardinal $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ of AD. Since Martin computed $\omega_{n+1} = \prod_{[\omega_1]^n} \omega_1/\mu^{\omega_1}_n$ for all $1 \leq n < \omega$, Henle considered an early instance of this question: [11] "One would hope that in the case of AD," the function $\Phi : [\omega_1]^{\omega_1}_* \rightarrow \omega_1$ defined by $\Phi(f) = f(\omega)$ "might represent $\aleph_{\omega+1}$" in the ultrapower by the strong partition measure $\mu^{\omega_1}_{\omega_1}$ "but its cofinality turns out to be $\aleph_2$ while $\aleph_{\omega+1}$ is regular". It is unclear from this passage whether Henle believed $\omega_{\omega+1}$ could be represented at all in this ultrapower. Since Kunen showed that all measures on ω_1 are equivalent to $\mu^{\omega_1}_n$ for some $n \in \omega$ and Martin computed these ultrapowers, Goldberg asked Jackson about computing the ultrapower of ω_1 by all its other partition measures $\mu^{\omega_1}_\epsilon$ when $\omega \leq \epsilon \leq \omega_1$. When $\omega \leq \epsilon \leq \omega_1$, $[\omega_1]^\epsilon_*$ is not wellorderable and thus μ^κ_ϵ cannot be equivalent to a measure on the ordinal ω_1. Thus Martin's result above cannot be applied to show that $\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu^{\omega_1}_\epsilon$ is a cardinal (assuming it is wellfounded which it is under AD alone; see Fact 9.2).

Initially, it seems that these ultrapowers could be very large. Soon afterward Jackson and the author showed that for each $\epsilon \leq \omega_1$, there is a direct coding of the representing functions $\Phi : [\omega_1]^\epsilon_* \rightarrow \omega_1$ by reals through the Martin's games so that the corresponding coding of the ultrapower is a fairly simple relation on $\mathbb{R}$. Martin's good coding system for ${}^\epsilon\omega_1$ (with $\epsilon \leq \omega_1$) consists of the pointclass $\boldsymbol{\Pi}^1_1$, certain $\boldsymbol{\Delta}^1_1$ sets of reals $\mathsf{GC}_{\beta,\gamma}$ with $\beta < \epsilon$ and $\gamma < \omega_1$, and a function $\mathsf{decode} : \mathbb{R} \rightarrow \mathscr{P}(\epsilon \times \omega_1)$ satisfying strict definability conditions. If $e \in \mathsf{GC}_{\beta,\gamma}$, then $\mathsf{decode}(e)$ resembles the graph of a partial function defined at β taking value γ. There is also a set $\mathsf{GC} \subseteq \mathbb{R}$ consisting of those reals so that $\mathsf{decode}(e)$ is the graph of a total function from ϵ into ω_1 and in addition every $f : \epsilon \rightarrow \omega_1$, there is an $e \in \mathsf{GC}$ so that $\mathsf{decode}(e)$ is the graph of f. The fundamental features of the Martin games has been isolated in the almost everywhere good code uniformization Theorem 5.9. As an application, every function $\Phi : [\omega_1]^\epsilon_* \rightarrow \omega_1$ has a Lipschitz function Ξ with the property that there is a club $C^* \subseteq \omega_1$ so that for all $e \in \mathsf{GC}$, if $\mathsf{decode}(e) \in [C^*]^{\omega\cdot\epsilon}$, then $\Xi(e) \in \mathrm{WO}$ (the $\boldsymbol{\Pi}^1_1$ collection of reals coding countable ordinals) and $\Phi(\mathsf{block}(\mathsf{decode}(e))) = \mathrm{ot}(\Xi(e))$ (where if $g : \omega\cdot\epsilon \rightarrow \omega_1$, then $\mathsf{block}(g) : \epsilon \rightarrow \omega_1$ is a function of the correct type defined by $\mathsf{block}(g)(\alpha) = \sup\{g(\omega\cdot\alpha + n) : n \in \omega\}$). Every Lipschitz function can be coded by a real ρ and Ξ_ρ will denote the Lipschitz function corresponding to ρ. By Solovay's game, there is a set of reals $\mathsf{clubcode}$ so that each $z \in \mathsf{clubcode}$ codes a club $\mathfrak{C}_z \subseteq \omega_1$ and for each club $C \subseteq \omega_1$, there is an $z \in \mathsf{clubcode}$ so that $\mathfrak{C}_z \subseteq C$. A pair (ρ, z) with $\Xi = \Xi_\rho$ and $\mathfrak{C}_z \subseteq C^*$ together forms an ϵ-function code for the given function Φ above. This coding methods yields a collection of reals called Fcode_ϵ so that each $(\rho, z) \in \mathsf{Fcode}_\epsilon$ induces a total function $\Phi^{(\rho,\epsilon)} : [\mathfrak{C}_z]^\epsilon_* \rightarrow \omega_1$. The almost everywhere good code uniformization mentioned above implies every $\Phi : [\omega_1]^\epsilon_* \rightarrow \omega_1$ has a $(\rho, z) \in \mathsf{Fcode}_\epsilon$ so that $[\Phi]_{\mu^{\omega_1}_\epsilon} = [\Phi^{(\rho,z)}]_{\mu^{\omega_1}_\epsilon}$. By quantifying over all suitable good codes GC and comparing the corresponding ordertype given by the Lipschitz functions, one obtains a natural ϵ-function code comparison relation $\mathfrak{F}_\epsilon$ which codes the ultrapower $\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu^{\omega_1}_\epsilon$. When $\epsilon < \omega_1$, the corresponding good codes GC for ${}^{\omega\cdot\epsilon}\omega_1$ forms a $\boldsymbol{\Pi}^1_1$ set. This implies the comparison relation $\mathfrak{F}_\epsilon$ is a wellfounded $\boldsymbol{\Sigma}^1_3$ and hence ω_ω-Suslin relation. By the Kunen-Martin theorem, the length of $\mathfrak{F}_\epsilon$ is strictly less than $(\omega_\omega)^+ = \omega_{\omega+1}$ and thus $\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu^{\omega_1}_\epsilon < \omega_{\omega+1}$ when $\epsilon < \omega_1$. The collection of good coding for ${}^{\omega_1}\omega_1$ is $\boldsymbol{\Pi}^1_2$ and the natural complexity calculation implies that $\mathfrak{F}_{\omega_1}$ is a wellfounded $\boldsymbol{\Sigma}^1_4$ and thus $\omega_{\omega+1}$-Suslin relation. The Kunen-Martin theorem implies $\mathfrak{F}_{\omega_1}$ has length strictly less than $(\omega_{\omega+1})^+ = \omega_{\omega+2}$ and hence $\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1} < \omega_{\omega+2} = \boldsymbol{\delta}^1_4$.

With this result, the perspective on Goldberg's question shifted to suspecting the ultrapower by the partition measures should be rather small and possible below the next strong partition cardinal $\omega_{\omega+1}$. For

$\epsilon < \omega_1$, Theorem 9.12 solves Goldberg's question by showing

$$\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu^{\omega_1}_\epsilon < \omega_{\omega+1} = \boldsymbol{\delta}^1_3$$

when $\epsilon < \omega_1$. In particular when $\omega \leq \epsilon < \omega_1$, $\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu^{\omega_1}_\epsilon$ is a non-cardinal ordinal between ω_ω and $\omega_{\omega+1}$ of cofinality ω_2. In fact, Theorem 9.14 strengthens this by showing $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ is a fixed point under the ultrapower by the short partition measures $\mu^{\omega_1}_\epsilon$ when $\epsilon < \omega_1$,

$$\prod_{[\omega_1]^\epsilon_*} \omega_{\omega+1}/\mu^{\omega_1}_\epsilon = \omega_{\omega+1}.$$

Thus its remains to compute better lower bounds on $\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_*$, the ultrapower of ω_1 by the strong partition measure on ω_1. The reason $\mathfrak{F}_{\omega_1}$, the ω_1-function code comparison relation on $\mathsf{Fcode}_{\omega_1}$, is $\boldsymbol{\Sigma}^1_4$ comes from a universal quantification over GC, the collection of good codes for ${}^{\omega_1}\omega_1$, which is a $\boldsymbol{\Pi}^1_2$ sets and in fact $\boldsymbol{\Pi}^1_2$-complete by Fact 8.15. The method of $\mathsf{Fcode}_{\omega_1}$ and $\mathfrak{F}_{\omega_1}$ which uses quantification over GC cannot be used to provide a wellfounded $\boldsymbol{\Sigma}^1_3$ coding of the ultrapower by the strong partition measure on ω_1.

To simplify the complexity, a suitable coding of functions of the form $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ and a corresponding comparison relation must be developed that does not quantify over good code for full length functions. An almost everywhere continuity property for function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ (in a sense analogous to the Mathias continuity property for functions $\Phi : [\omega]^\omega \to {}^\omega\omega$) would assert that there is a club $C \subseteq \omega_1$ with the property that for all $f \in [C]^{\omega_1}_*$, there is an $\alpha < \omega_1$ so that for all $g \in [C]^{\omega_1}_*$ with $g \restriction \alpha = f \restriction \alpha$, $\Phi(f) = \Phi(g)$. Potentially a reduction in complexity could be obtained by replacing a quantification of good code for full length function by a quantification over continuity points which are bounded functions. Jackson and the author [5] proved this almost everywhere continuity property for functions of the form $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ under AD. There are two immediate issues. First, recognizing that a bounded sequence $\ell \in [\omega_1]^{<\omega_1}_*$ is a continuity point for Φ relative to some club C a priori seems to require a quantification over all good codes since one needs to express that for all $g_0, g_1 \in [C]^{\omega_1}_*$, $\Phi(\ell\hat{\ }g_0) = \Phi(\ell\hat{\ }g_1)$. Thus quantifications over GC has not truly been avoided. A second substantial issue is through what mechanism does one use bounded sequences or continuity points to code functions $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$.

The most important step in establishing the almost everywhere continuity property is showing that a certain partition is homogeneous for the desired side. This is shown by a construction involving an ω_1-length dependent choice of clubs and arranging the tail of a function $h : \omega_1 \to \omega_1$ to land into each of these chosen clubs. Such constructions are possible using an almost everywhere short length club uniformization principle: For all $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed in the club_{ω_1}-coordinate (i.e. for all $\ell \in [\omega_1]^{<\omega_1}_*$ and clubs $C_0 \subseteq C_1$, if $R(\ell, C_1)$, then $R(\ell, C_0)$), there is a club $C \subseteq \omega_1$ and a function $\Lambda : \mathrm{dom}(R) \cap [C]^{<\omega_1}_* \to \mathsf{club}_{\omega_1}$ so that for all $\ell \in \mathrm{dom}(R) \cap [C]^{<\omega_1}_*$, $R(\ell, \Lambda(\ell))$. Section 3 will show that if almost everywhere short length club uniformization at κ holds, then the almost everywhere continuity property for functions of the form $\Phi : [\kappa]^\kappa_* \to \kappa$ follows by purely combinatorial methods. By a more careful analysis using this club uniformization principle, Theorem 3.2 establishes a finer continuity property that every function $\Phi : [\kappa]^\kappa_* \to \kappa$, there is a club $C \subseteq \kappa$ so that for all $\ell \in [C]^{<\kappa}_*$, ℓ is a continuity point for Φ if and only if there is a $g \in [C]^\kappa_*$ so that $\Phi(\ell\hat{\ }g) < g(0)$. This resolves the first issue by providing a simple method of recognizing continuity points without quantification over full ω_1-length functions: by fixing a single function $h \in [C]^{\omega_1}_*$, ℓ can be determined to be a continuity point relative to C by asking whether $\Phi(\ell\hat{\ }\mathsf{drop}(h, \alpha)) < h(\alpha)$ for some appropriate $\alpha < \omega_1$ where $\mathsf{drop}(h, \alpha)(\beta) = h(\alpha + \beta)$.

Establishing the club uniformization principle under AD is closely connected to resolving the second issue and the coding of the ultrapower of ω_1 by its strong partition measure. Unfortunately, the original method of establishing the almost everywhere short length club uniformization for ω_1 from [5] was not particularly insightful for this purpose. [5] Theorem 3.7 used category arguments and a simple generic coding function on ω_1 to show that if $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ is $\subseteq$-downward closed in the club_{ω_1}-coordinate and the coded version $\tilde{R} \subseteq \mathbb{R} \times \mathbb{R}$ for R has a uniformization (in the usual sense), then R can be uniformized everywhere (on its entire domain). As a consequence, $\mathsf{AD}_\mathbb{R}$ (even $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$) implies everywhere short length club uniformization for ω_1. Everywhere short length club uniformization is unprovable under AD as it fails in $L(\mathbb{R})$; however, for the purpose of establishing almost everywhere continuity, an almost everywhere version is sufficient. Using the almost everywhere good code uniformization, [5] showed that there is a club $C \subseteq \omega_1$ so that the code version $\tilde{R}$ for $R \upharpoonright [C]^{<\omega_1}_*$ is a projective relation and thus uniformizable under AD using the scale property. This methods establishes the almost everywhere club uniformization for ω_1 under AD alone; albeit indirectly from instances of uniformization. Naturally, one can ask if other strong partition cardinals (for instance, $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$) could possess the almost everywhere short length club uniformization property. Although Kechris and Woodin ([20]) showed there are generic coding functions for larger uncountable cardinals which are reliable (a concept closely connected to scales), these coding functions only handle ω-sequences through these reliable ordinals. The best club uniformization that one could hope for by the method of generic coding is an everywhere countable length club uniformization (i.e. uniformizing relations on ${}^{<\omega_1}\kappa \times \mathsf{club}_\kappa$) which has been verified at cardinals associated to pointclasses with scale property under stronger determinacy assumption in [7]. The generic coding methods are inadequate for handling families indexed by sequences of longer length. (This inadequacy of the generic coding method has been expressed by Becker at the end of [1] as an obstable to his techniques for producing λ-supercompactness measures for large λ's.) Moreover, there are strong partition cardinals like $\boldsymbol{\delta}_A$ for $A \subseteq \mathbb{R}$ which are generally not associated with pointclasses with scales.

Definition 5.10 defines a good coding family for κ which augments Martin's concept of a good coding system for ${}^\kappa\kappa$ with a coding $\mathsf{BS} \subseteq \mathbb{R}$ and surjection $\mathsf{seq} : \mathsf{BS} \to {}^{<\kappa}\kappa$ which interact with the good coding system under strict definability conditions. There is a continuous function $\mathsf{merge} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ so that whenever $u \in \mathsf{BS}$ and $e \in \mathsf{GC}$, then $\mathsf{merge}(u, e) \in \mathsf{GC}$ and $\mathsf{decode}(\mathsf{merge}(u, e))$ is the full length function which results from $\mathsf{seq}(u)$ overriding the corresponding initial segment of $\mathsf{decode}(e)$. Theorem 5.12 shows that the prewellordering ordinal $\delta(\Gamma)$ associated to a pointclass Γ which are closed under $\forall^{\mathbb{R}}$, has the prewellordering property, and $\Delta = \Gamma \cap \check{\Gamma}$ is closed under both $\forall^{\mathbb{R}}$ and $\exists^{\mathbb{R}}$ will possess good coding families. This implies that $\boldsymbol{\delta}^2_1$ and $\boldsymbol{\delta}_A$ for all $A \subseteq \mathbb{R}$ possess good coding families. Section 7 will provide a self-contained and detailed analysis of the good coding family for ω_1 necessary for making the complexity calculations used to compute the lower bounds on the ultrapower of ω_1 by its strong partition measure. Section 11 will produce good coding families for all the odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$ assuming consequences of Jackson's description analysis.

Suppose κ has a very good coding family. Using the uniform coding lemma of Moschovakis, for each $f \in [\kappa]^\kappa_*$, there is a relation T_f which is simple relative to the pointclass Γ (from the good coding family) so that T_f collects coherently $z \in \mathsf{clubcode}_\kappa$ so that $R(f \upharpoonright \alpha, \mathfrak{C}_z)$ for each $\alpha < \kappa$. Fix a single good code e for some $h : \kappa \to C$ (where $C \subseteq \kappa$ is a suitable club given by an application of the almost everywhere good code uniformization). For each $\ell \in [C]^{<\kappa}_*$ and all codes $u \in \mathsf{BS}$ so that $\mathsf{block}(\mathsf{seq}(u)) = \ell$, consider T_f where $f = \mathsf{decode}(\mathsf{merge}(u, e))$ which has ℓ as an initial segment (or more precisely, ℓ is an initial segment of $\mathsf{block}(f)$). Since the collection of all codes $u \in \mathsf{BS}$ with $\mathsf{block}(\mathsf{seq}(u)) = \ell$ belong to Δ, this permits a boundedness argument to uniformly select a club C so that $R(\ell, C)$, depending solely on ℓ. In this manner, Theorem 6.1 shows that if κ has a good coding family, then almost everywhere short length club uniformization holds at κ. Thus cardinals κ with a good coding family will also satisfy the almost everywhere continuity property.

It was soon realized that this technique of fixing a single good code for a full length function and merging codes from BS into this fixed good code when combined with the finer continuity property could provide a coding of functions $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ and an ultrapower comparison relation which avoids quantifying over GC. Section 10 develops the notion of a pseudo-function coding. A pseudo-function code consists of a triple (ρ, z, e) so that ρ codes a Lipschitz function, $z \in \mathsf{clubcode}$, $e \in \mathsf{GC}$, and satisfies additional conditions which allows (ρ, z, e) to define a meaningful partial function $\Phi^{(\rho,z,e)} : [\mathfrak{C}_z]^{\omega_1}_* \to \omega_1$. Let PFcode denote the collection of pseudo-function codes. Motivated by the characterization of continuity points given by the finer continuity property (Theorem 3.2), $u \in \mathsf{BS}$ is a pseudo-continuity point for (ρ, z, e) if roughly $\Xi_\rho(\mathsf{merge}(u, e)) \in \mathrm{WO}$ and codes an ordinal less than the $|\mathsf{length}(\mathsf{seq}(u))|^{\text{th}}$ term of $\mathsf{decode}(e)$. $\Phi^{(\rho,z,e)}(f)$ is defined if there is an initial segment of f which is coded by a pseudo-continuity point u for (ρ, z, e) and one says $\Phi^{(\rho,z,e)}(f) = \xi$ if $\xi = \mathrm{ot}(\Xi_\rho(\mathsf{merge}(u, e)))$. It is important to note that $\Phi^{(\rho,z,e)}$ may not be defined $\mu^{\omega_1}_{\omega_1}$-almost everywhere if the collection of pseudo-continuity points for (ρ, z, e) is not very dense. A pseudo-function code (ρ, z, e) belong to $\mathsf{tPFcode}$ (or said to be a true pseudo-function code) if and only if $\Phi^{(\rho,z,e)}$ is defined $\mu^{\omega_1}_{\omega_1}$-almost everywhere. The fine continuity property implies that every $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ has a $(\rho, z, e) \in \mathsf{tPFcode}$ so that $[\Phi]_{\mu^{\omega_1}_{\omega_1}} = [\Phi^{(\rho,z,e)}]_{\mu^{\omega_1}_{\omega_1}}$. Definition 10.6 and Definition 10.7 define the natural pseudo-function code comparison relation which roughly says (ρ_0, z_0, e_0) is less than (ρ_1, z_1, e_1) if and only if there is a club D so that for all common pseudo-function codes u with $\mathsf{seq}(u) \in [D]^{<\omega_1}$, $\mathrm{ot}(\Xi_{\rho_0}(\mathsf{merge}(u, e_0)) < \mathrm{ot}(\Xi_{\rho_1}(\mathsf{merge}(u, e_1)))$. The pseudo-function code comparison relation on all of PFcode is $\mathbf{\Sigma}^1_3$.

However since for some $(\rho, z, e) \in \mathsf{PFcode}$, $\Phi^{(\rho,z,e)}$ is not defined $\mu^{\omega_1}_{\omega_1}$-almost everywhere, one cannot show that this relation on PFcode is wellfounded and hence the Kunen-Martin is not applicable. This comparison relation can be shown to be wellfounded whenever it is restricted to subsets of $\mathsf{tPFcode}$, the collection of true pseudo-function codes. (A priori, to assert a pseudo-function code is a true pseudo-function code appears to require a universal quantification of GC which ostensibly makes $\mathsf{tPFcode}$ too complicated.) There are useful $\mathbf{\Sigma}^1_3$ subsets of $\mathsf{tPFcode}$ which can be used to produce local bounds on the ultrapower by the strong partition measure. Fix $\Psi : [\omega_1]^{\omega_1}_* \to \omega_1$. Ψ has a $(\rho^*, z^*) \in \mathsf{Fcode}_{\omega_1}$ so that $[\Phi^{(\rho^*,z^*)}]_{\mu^{\omega_1}_{\omega_1}} = [\Psi]_{\mu^{\omega_1}_{\omega_1}}$. Fix an $e^* \in \mathsf{GC}$ so that $\mathsf{decode}(e^*) \in [\mathfrak{C}_{z^*}]^{\omega_1}$. The triple (ρ^*, z^*, e^*) belongs to $\mathsf{tPFcode}$ but is much nicer since $(\rho^*, z^*) \in \mathsf{Fcode}_{\omega_1}$. Let A consists of those elements of $(\rho, z, e) \in \mathsf{PFcode}$ so that each pseudo-function code for (ρ^*, z^*, e^*) is a pseudo-continuity point for (ρ, z, e) and the (ρ^*, z^*, e^*) value is larger than the (ρ, z, e) value on each such pseudo-continuity point. This set A is $\mathbf{\Pi}^1_2$. Since $(\rho^*, z^*) \in \mathsf{Fcode}_{\omega_1}$ is a strong code for Φ, (ρ^*, z^*, e^*) has pseudo-continuity points abounding and thus $A \subseteq \mathsf{tPFcode}$. Therefore, the comparison relation restricted to A is a wellfounded $\mathbf{\Sigma}^1_3$ relation and thus has length less than $\omega_{\omega+1}$ by the Kunen-Martin theorem. Since the comparison relation restricted to A corresponds to the ultrapower below Ψ, this shows $\prod_{f \in [\omega_1]^{\omega_1}_*} \Psi(f)/\mu^{\omega_1}_{\omega_1} < \omega_{\omega+1}$. After ranging over all $\Psi : [\omega_1]^{\omega_1}_* \to \omega_1$, one obtains, as a corollary, Theorem 10.13 which asserts

$$\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1} \leq \omega_{\omega+1} = \boldsymbol{\delta}^1_3.$$

These methods described thus far use only properties of the good coding families for ω_1 which can be extended to all the odd projective ordinals. Section 11 shows that for $\epsilon < \boldsymbol{\delta}^1_{2n+1}$ and $n \in \omega$,

$$\prod_{[\boldsymbol{\delta}^1_{2n+1}]^{\epsilon}_*} \boldsymbol{\delta}^1_{2n+1}/\mu^{\boldsymbol{\delta}^1_{2n+1}}_{\epsilon} < \boldsymbol{\delta}^1_{2n+3}, \quad \prod_{[\boldsymbol{\delta}^1_{2n+1}]^{\epsilon}_*} \boldsymbol{\delta}^1_{2n+3}/\mu^{\boldsymbol{\delta}^1_{2n+1}}_{\epsilon} = \boldsymbol{\delta}^1_{2n+3}, \quad \text{and} \quad \prod_{[\boldsymbol{\delta}^1_{2n}]^{\boldsymbol{\delta}^1_{2n+1}}_*} \boldsymbol{\delta}^1_{2n+1}/\mu^{\boldsymbol{\delta}^1_{2n+1}}_{\boldsymbol{\delta}^1_{2n+1}} \leq \boldsymbol{\delta}^1_{2n+3}.$$

Showing $\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1} \leq \omega_{\omega+1}$ appears to be the limit of arguments purely using good coding families. It remains to consider the possibility that $\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1} = \omega_{\omega+1} = \boldsymbol{\delta}^1_3$. An equality would provide a remarkable ultrapower representation for $\boldsymbol{\delta}^1_3$ which could yield new alternative methods to Jackson's descriptions to study the third projective ordinals. Unfortunately, this is not the case. To show the ultrapower of ω_1 by the strong partition measure is strictly less than $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, it suffices to show that $\mathsf{tPFcode}$, the collection of true pseudo-function codes, is $\mathbf{\Sigma}^1_3$. This is obtained by considering iterable models (which will be called mice here) possessing a sequence of normal measures whose critical point forms a discontinuous sequence whose supremum has an external normal measure. Under AD, Woodin [22] showed there are inner models for strong large cardinal and thus mice do exist. More generally, Theorem 12.14 shows that the existence of a strong partition cardinal κ possessing the almost everywhere short length club uniformization at κ is sufficient to prove the existence of countable mouse containing any particular real. Fix a countable mouse $\mathcal{M}$ containing some triple (ρ, z, e). Using a $\mathbf{\Sigma}^1_1$ boundedness argument, there is a club C so that for any $f \in [C]^{\omega_1}_*$, $\mathcal{M}$ can be iterated in a linear manner so that the sequence of internal critical points of the resulting iterate $\mathcal{N}$ align with f. A unique feature of the explicit good coding family for ω_1 produced in Section 7 and the existence of generics filter for countable forcings over inner models of AC is that BS code for $f \restriction \alpha$ always exists in $\mathrm{Coll}(\omega, \sup(f \restriction \alpha))$-generic extensions of the iterate $\mathcal{N}$. Consider the assertion that there exists a countable mouse $\mathcal{M}$ so that $\mathcal{M}$ believes there is an ordinal ξ less than the critical point of the top external measure so that all conditions in $\mathrm{Coll}(\omega, \xi)$ force there is a $u \in \mathsf{BS}$ which codes the ξ-length initial segment of its own sequence of internal critical points and this u is a pseudo-continuity point for (ρ, z, e). This assertion is Σ^1_3 using the fact that the collection of countable mouse is Π^1_2 and the satisfaction relation for countable models is Δ^1_1. If this assertion holds and $C \subseteq \omega_1$ is the club for which $\mathcal{M}$ can be iterated to align its internal sequence of critical points with any $f \in [C]^{\omega_1}_*$, then elementarity of the iteration map will imply that each $f \in [C]^{\omega_1}_*$ will have initial segments which are pseudo-continuity points for (ρ, z, e) and thus $(\rho, z, e) \in \mathsf{tPFcode}$. In fact, $(\rho, z, e) \in \mathsf{tPFcode}$ is equivalent to this $\mathbf{\Sigma}^1_3$ assertions. Now since $\mathsf{tPFcode}$ is $\mathbf{\Sigma}^1_3$, the pseudo-function comparison relation when restricted to $\mathsf{tPFcode}$ is a wellfounded $\mathbf{\Sigma}^1_3$ relation which has length less than $\omega_{\omega+1}$ by the Kunen-Martin theorem. Since the entire ultrapower of ω_1 by the strong partition measure corresponded to this comparison relation, Theorem 13.2 finally resolves the ultrapower question by showing

$$\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1} < \omega_{\omega+1} = \boldsymbol{\delta}^1_3.$$

The ability to find codes in BS for elements of $[\omega_1]^{<\omega_1}$ in generic extensions of models of AC by filters that exist in the real world is unique to ω_1. The coding BS from good coding families for higher odd projective ordinals come from Jackson's description analysis and are more mysterious. It is open if the ultrapower of $\boldsymbol{\delta}^1_{2n+1}$ by the strong partition measure on $\boldsymbol{\delta}^1_{2n+1}$ is strictly less than the next strong partition cardinal $\boldsymbol{\delta}^1_{2n+3}$. The use of iterability or indiscernibility for studying descriptive set theoretic and combinatorial questions at ω_1 has been frequently observed to be quite special to ω_1. Sargsyan at the end of the introduction to [26] asserts "from a current point of view, it seems that the function $x \mapsto x^\sharp$ provides a singularly magical coding of the subsets of ω_1" which "doesn't yet have a proper inner model theoretic generalization to higher levels of the projective hierarchy and beyond".

This paper studies various combinatorial features of strong partition cardinals from both purely combinatorial methods and determinacy methods. Several properties of independent interests for the general strong partition cardinal are considered while this paper pursues a solution to Goldberg's question of whether the ultrapower of the first uncountable strong partition cardinal by its strong partition measure is less than the second uncountable strong partition cardinal under AD. The paper is fairly self-contained. The reader most interested in computing lower bounds for the ultrapower of ω_1 by the partition measures on ω_1 could understand all the material before Section 11 with basic knowledge of descriptive set theory including $\boldsymbol{\Sigma}^1_1$ bounding, the Moschovakis coding lemma, and the Kunen-Martin theorem (while skipping some examples concerning higher strong partition cardinals). A detailed presentation of the good coding family for ω_1 is provided based on an unpublished argument of Kechris which produced a particularly elegant good coding system for ${}^{\omega_1}\omega_1$ using category and generic coding functions. Although a proof of the strong partition property from good coding systems is not provided here, many of the mains ideas appear in the proof of the almost everywhere good code uniformization (Theorem 5.9) and the actual proof can be found in [2] using the same notation developed here. The full solution to the ultrapower question in section 12 and 13 requires some basic knowledge of forcing and the theory of linear iterations of measures in the style of Kunen.

2. Partition Properties

This section will introduce the partition property, the partition measures, and some purely combinatorial consequences of partition properties. This section will work under ZF.

Definition 2.1. Let ON be the collection of ordinals. Let $\epsilon \in \text{ON}$ be an ordinal. A function $f : \epsilon \to \text{ON}$ has uniform cofinality ω if and only there is a function $F : \epsilon \times \omega \to \text{ON}$ so that for all $n \in \omega$ and $\alpha < \epsilon$, $F(\alpha, n) < F(\alpha, n+1)$ and $f(\alpha) = \sup\{F(\alpha, n) : n \in \omega\}$. If $f : \epsilon \to \text{ON}$ and $\alpha \leq \epsilon$, then let $\sup(f \restriction \alpha) = 0$ if $\alpha = 0$ and $\sup(f \restriction \alpha) = \sup\{f(\beta) : \beta < \alpha\}$ if $\alpha > 0$. A function $f : \epsilon \to \text{ON}$ is discontinuous everywhere if and only if for all $\alpha < \epsilon$, $f(\alpha) > \sup(f \restriction \alpha)$. A function $f : \epsilon \to \text{ON}$ is of the correct type if and only if f has uniform cofinality ω and discontinuous everywhere.

If X is a set of ordinals, then let $[X]^\epsilon$ be the collection of increasing functions from ϵ into X. Let $[X]^\epsilon_*$ denote the subset of $[X]^\epsilon$ consisting of those functions of the correct type.

Let κ be a cardinal. A subset $A \subseteq \kappa$ is unbounded in κ if and only if for all $\alpha < \kappa$, there is a $\beta \in A$ with $\alpha < \beta$. A set $A \subseteq \kappa$ is bounded in κ if and only if it is not unbounded in κ. A subset $A \subseteq \kappa$ is closed in κ if and only if for any $B \subseteq A$ so that B is bounded in κ, then $\sup(B) \in A$. A set $C \subseteq \kappa$ is a club subset of κ if and only if C is both closed and unbounded in κ.

Definition 2.2. Let κ be a cardinal, $\epsilon \leq \kappa$, and $\gamma < \kappa$. The ordinary partition relation, $\kappa \to (\kappa)^\epsilon_\gamma$, indicates that for every partition $P : [\kappa]^\epsilon \to \gamma$, there exists an $\alpha < \gamma$ and an $X \subseteq \kappa$ with $|X| = \kappa$ so that for all $f \in [X]^\epsilon$, $P(f) = \alpha$. Let $\kappa \to (\kappa)^\epsilon_{<\kappa}$ be the assertion that for all $\gamma < \kappa$, $\kappa \to (\kappa)^\epsilon_\gamma$ holds.

The correct type partition relation, $\kappa \to_* (\kappa)^\epsilon_\gamma$, indicates that for every $P : [\kappa]^\epsilon_* \to \gamma$, there exists a club $C \subseteq \kappa$ and an $\alpha < \gamma$ so that for all $f \in [C]^\epsilon_*$, $P(f) = \alpha$. In this situation, C is said to be a club homogeneous for P taking value α. Let $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$ be the assertion that for all $\gamma < \kappa$, $\kappa \to_* (\kappa)^\epsilon_\gamma$ holds.

A cardinal κ is called a strong partition cardinal if and only if $\kappa \to_* (\kappa)^\kappa_2$ holds.

Definition 2.3. If κ is a cardinal and $X \subseteq \kappa$ with $|X| = \kappa$. Let $\mathsf{enum}_X : \kappa \to X$ be the increasing enumeration of X. Let $\mathsf{next}_X : \kappa \to X$ be defined by $\mathsf{next}_X(\alpha)$ is the least element of X strictly greater than α. Let $\mathsf{next}^0_X : \kappa \to \kappa$ be defined by $\mathsf{next}^0_X(\alpha) = \alpha$. If $0 < \gamma < \kappa$, then let $\mathsf{next}^\gamma_X : \kappa \to X$ be defined by $\mathsf{next}^\gamma_X(\alpha)$ is the γ^{th} element of X strictly greater than α.

Definition 2.4. Let $\kappa \in \text{ON}$ and $\epsilon \leq \kappa$. Let $\mathsf{block} : {}^{\omega\cdot\epsilon}\text{ON} \to {}^{\epsilon}\text{ON}$ be defined by $\mathsf{block}(f)(\alpha) = \sup\{f(\omega \cdot \alpha + n) : n \in \omega\}$.

The correct type partition relation will be used in this article. The next result shows that the ordinary and correct type partition relations are closely related.

Fact 2.5. *([2] Fact 2.6) Let κ be a cardinal and $\epsilon \leq \kappa$.*

(1) $\kappa \to_* (\kappa)^\epsilon_2$ implies $\kappa \to (\kappa)^\epsilon_2$.
(2) $\kappa \to (\kappa)^{\omega\cdot\epsilon}_2$ implies $\kappa \to_* (\kappa)^\epsilon_2$.

The correct type partition relation is preferable over the ordinary partition relation because it is directly related to partition measures.

Definition 2.6. Let κ be a regular cardinal and $\epsilon \leq \kappa$. Let μ^κ_ϵ be the filter on $[\kappa]^\epsilon_*$ defined by $X \in \mu^\kappa_\epsilon$ if and only if there exists a club $C \subseteq \kappa$ so that $[C]^\epsilon_* \subseteq X$. If μ^κ_ϵ is an ultrafilter, then it will be called the ϵ-partition measure on κ.

Fact 2.7. *If $\kappa \to_* (\kappa)^\epsilon_2$, then μ^κ_ϵ is an ultrafilter.*

Fact 2.8. *Let κ be cardinal and $\epsilon < \kappa$. If $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$, then $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$.*

Proof. Let $\lambda < \kappa$ and $\Phi : [\kappa]^\epsilon_* \to \lambda$. If $h \in [\kappa]^{\epsilon+\epsilon}_*$, then let $h^0 \in [\kappa]^\epsilon_*$ and $h^1 \in [\kappa]^\epsilon_*$ be defined by $h^0(\alpha) = h(\alpha)$ and $h^1(\alpha) = h(\epsilon + \alpha)$. Define $P : [\kappa]^{\epsilon+\epsilon} \to 2$ by $P(h) = 0$ if and only if $\Phi(h^0) = \Phi(h^1)$. By $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$, there is a club $C \subseteq \kappa$ which is homogeneous for P. Suppose that C is homogeneous for P taking value 1. Let $Q : [C]^{\epsilon+\epsilon}_* \to 2$ be defined by $Q(h) = 0$ if and only if $Q(h^0) < Q(h^1)$. By $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$, there is a club $D \subseteq C$ which is homogeneous for Q. Let $E = \{\mathsf{enum}_D(\omega \cdot \alpha + \omega) : \alpha \in \kappa\}$ and note that $[E]^\epsilon = [E]^\epsilon_*$. For each $\delta < \kappa$, let $g_\delta \in [E]^\epsilon_*$ be defined by $g_\delta(\alpha) = \mathsf{enum}_E(\epsilon \cdot \delta + \alpha)$. For each $\delta_0 < \delta_1 \in \kappa$, let $h_{\delta_0,\delta_1} \in [E]^\epsilon_*$ be defined so that $h^0_{\delta_0,\delta_1} = g_{\delta_0}$ and $h^1_{\delta_0,\delta_1} = g_{\delta_1}$.

Suppose D is homogeneous for Q taking value 1. Then $Q(h_{n,n+1}) = 1$ implies that $\Phi(g_{n+1}) < \Phi(g_n)$. This contradicts the wellfoundedness of λ. Suppose D is homogeneous for Q taking value 0. Then for each $\delta_0 < \delta_1$, $Q(h_{\delta_0,\delta_1}) = 0$ and $P(h_{\delta_0,\delta_1}) = 1$ imply that $\Phi(g_{\delta_0}) < \Phi(g_{\delta_1})$. So the map $\Gamma : \kappa \to \lambda$ defined by $\Gamma(\delta) = \Phi(g_\delta)$ is an injection, which is impossible since $\lambda < \kappa$. Thus Q is a partition with no homogeneous club which violates $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$.

Thus C must be homogeneous for P taking value 0. Suppose $f_0, f_1 \in [C]^\epsilon_*$. Let $g \in [C]^\epsilon_*$ so that $\sup(f_0) < g(0)$ and $\sup(f_1) < g(0)$. Let $h_0, h_1 \in [C]^\epsilon_*$ be such that $h^0_0 = f_0$, $h^0_1 = f_1$, and $h^1_0 = h^1_1 = g$. $P(h_0) = P(h_1) = 0$ implies that $\Phi(f_0) = \Phi(g) = \Phi(f_1)$. Thus there is an $\eta < \lambda$ so that for all $f \in [C]^\epsilon_*$, $\Phi(f) = \eta$. ☐

Fact 2.9. *If $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$, then μ^κ_ϵ is a κ-complete ultrafilter.*

Proof. Suppose μ^κ_ϵ is not κ-complete. Then there is a $\lambda < \kappa$ and a sequence $\langle A_\alpha : \alpha < \lambda\rangle$ in μ^κ_ϵ so that $\bigcap_{\alpha<\lambda} A_\alpha = \emptyset$. Define $\Phi : [\kappa]^\epsilon_* \to \lambda$ by $\Phi(f)$ is the least $\alpha < \lambda$ so that $f \notin A_\alpha$. By Fact 2.9, there is a $\eta < \lambda$ and a club $C \subseteq \kappa$ so that $\Phi[[C]^\epsilon_*] = \{\eta\}$. Then $[C]^\epsilon_* \cap A_\eta = \emptyset$. This contradicts $A_\eta \in \mu^\kappa_\epsilon$. ☐

An ordinal γ is indecomposable if and only if for all $\alpha < \gamma$ and $\beta < \gamma$, $\alpha + \beta < \gamma$ and $\alpha \cdot \beta < \gamma$. (Here, an indecomposable ordinal is both additively and multiplicatively indecomposable.) Indecomposable ordinals are limit ordinals. If γ is indecomposable, then for each $\alpha < \gamma$, $\text{ot}(\{\beta : \alpha < \beta < \gamma\}) = \gamma$. Also if $\alpha < \gamma$, then $\alpha + \gamma = \gamma$ and $\alpha \cdot \gamma = \gamma$.

The collection of indecomposable ordinals of a cardinal κ is a club subset of κ. The following lemma is useful for thinning out a club C_0 to a club subset $C_1 \subseteq C_0$ such that C_0 is sufficiently dense within C_1 for many constructions.

Lemma 2.10. *Let κ be a cardinal and $C_0 \subseteq \kappa$ is a club consisting entirely of indecomposable ordinals. Then $C_1 = \{\alpha \in C_0 : \mathsf{enum}_{C_0}(\alpha) = \alpha\}$ is a club subset of C_0 with the property that for all $\delta \in C_1$, $\beta < \delta$, and $\gamma < \delta$, $\mathsf{next}^{\beta}_{C_0}(\gamma) < \delta$.*

Proof. Since $\delta \in C_1 \subseteq C_0$ and C_0 consists entirely of indecomposable ordinals, δ is an indecomposable ordinal. Since $\delta = \mathsf{enum}_{C_0}(\delta)$, δ is a limit point of C_0. Since $\gamma < \delta$, there is an $\eta < \gamma$ so that $\gamma < \mathsf{enum}_{C_0}(\eta) < \mathsf{enum}_{C_0}(\gamma) = \delta$. Then $\mathsf{next}^{\beta}_{C_0}(\gamma) \leq \mathsf{enum}_{C_0}(\eta + \beta) < \mathsf{enum}_{C_0}(\delta) = \delta$ since $\eta + \beta < \delta$ as δ is indecomposable. $\square$

Definition 2.11. Let $\epsilon \in \mathrm{ON}$ and $\delta_0, \delta_1 \leq \epsilon$ be such that $\delta_0 + \delta_1 = \epsilon$. If $f : \epsilon \to \mathrm{ON}$, then define $\mathsf{drop}(f, \delta_0) : \delta_1 \to \mathrm{ON}$ by $\mathsf{drop}(f, \delta_0)(\alpha) = f(\delta_0 + \alpha)$.

Fact 2.12. *Suppose $\epsilon \leq \kappa$, $\kappa \to_* (\kappa)^{1+\epsilon}_2$, and $\kappa \to_* (\kappa)^{\epsilon}_{<\kappa}$. If $\Phi : [\kappa]^{\epsilon}_* \to \kappa$ is a function so that for μ^{κ}_{ϵ}-almost all f, $\Phi(f) < f(0)$, then there is a $\zeta < \kappa$ so that for μ^{κ}_{ϵ}-almost all f, $\Phi(f) = \zeta$.*

Proof. Let $C_0 \subseteq \kappa$ be a club consisting entirely of indecomposable ordinals so that $\Phi(f) < f(0)$ for all $f \in [C_0]^{\epsilon}_*$. Define $P : [C_0]^{1+\epsilon}_* \to 2$ by $P(g) = 0$ if and only if $\Phi(\mathsf{drop}(g, 1)) < g(0)$. By $\kappa \to_* (\kappa)^{1+\epsilon}_2$, let $C_1 \subseteq C_0$ be a club homogeneous for P. Let $C_2 = \{\alpha \in C_1 : \mathsf{enum}_{C_1}(\alpha) = \alpha\}$. Pick an $f \in [C_2]^{\epsilon}_* \subseteq [C_0]^{\epsilon}_*$ and thus $\Phi(f) < f(0)$. Since $f(0) \in C_2$, Lemma 2.10 implies that $\mathsf{next}^{\omega}_{C_1}(\Phi(f)) < f(0)$. Let $g \in [C_0]^{1+\epsilon}_*$ be defined by $g(0) = \mathsf{next}^{\omega}_{C_1}(\Phi(f))$ and for all $\alpha < \epsilon$, $g(1+\alpha) = f(\alpha)$. Then $\Phi(\mathsf{drop}(g, 1)) = \Phi(f) < f(0) = g(0)$ and thus $P(g) = 0$. This shows that C_1 is homogeneous for P taking value 0. For all $f \in [C_2]^{\epsilon}_*$, Lemma 2.10 implies that $\mathsf{next}^{\omega}_{C_1}(0) < f(0)$. Let $g_f \in [C_1]^{1+\epsilon}_*$ be defined by $g_f(0) = \mathsf{next}^{\omega}_{C_1}(0)$ and for all $\alpha < \epsilon$, $g_f(1+\alpha) = f(\alpha)$. $P(g_f) = 0$ implies that $\Phi(g) < \mathsf{next}^{\omega}_{C_1}(0)$. Thus it has been shown that for all $f \in [C_2]^{\epsilon}_*$, $\Phi(f) < \mathsf{next}^{\omega}_{C_1}(0)$. By $\kappa \to_* (\kappa)^{\kappa}_{<\kappa}$, there is a club $C_3 \subseteq C_2$ and a $\zeta < \mathsf{next}^{\omega}_{C_1}(0)$ so that for all $f \in [C_3]^{\epsilon}_*$, $\Phi(f) = \zeta$. $\square$

Fact 2.13. *(Solovay) Suppose κ is a cardinal and $\kappa \to_* (\kappa)^2_2$ holds. Then the ω-club filter on κ, μ^{κ}_1, is a κ-complete normal ultrafilter on κ.*

Proof. Fact 2.9 implies μ^{κ}_1 is κ-complete. Fact 2.8 implies $\kappa \to_* (\kappa)^1_{<\kappa}$. Let $\Phi : \kappa \to \kappa$ be a function which is μ^{κ}_1-almost everywhere regressive. Fact 2.12 implies there is a club $C_0 \subseteq \kappa$ and a $\zeta < \kappa$ so that for all $\beta \in [C_0]^1_*$, $\Phi(\beta) = \zeta$. Thus Φ is constant μ^{κ}_1-almost everywhere. $\square$

Partition properties are useful for analyzing functions on partition spaces to establish properties of cardinalities for these sets. A set X is said to have regular cardinality if and only if there are no sets $Z \subseteq X$ with $|Z| < |X|$ and no family $\langle A_z : z \in Z\rangle$ of subsets of X with $|A_z| < |X|$ for each $z \in Z$ so that $X = \bigcup A_z$. Under AD, $|\mathbb{R}| = |\mathscr{P}(\omega)|$ is a nonwellorderable regular cardinality by the perfect set property. It is open if $|\mathscr{P}(\omega_1)|$ is a nonwellorderable regular cardinality under AD.

Zapletal asked whether a weaker wellordered regularity holds for $\mathscr{P}(\omega_1)$: If κ is an ordinal and $\langle X_\alpha : \alpha < \kappa\rangle$ is a sequence so that $\bigcup_{\alpha<\kappa} X_\alpha = \mathscr{P}(\omega_1)$, then is there an $\alpha < \kappa$ so that $|X_\alpha| = |\mathscr{P}(\omega_1)|$? When $\kappa = \omega_1$, this was solved in [5] as a consequence of the almost everywhere continuity property for ω_1. Since $|\mathscr{P}(\omega_1)| = |[\omega_1]^{\omega_1}| = |[\omega_1]^{\omega_1}_*|$, the presentation as $[\omega_1]^{\omega_1}_*$ will be favored for the sake of the partition property.

Many concrete subsets of $\mathscr{P}(\omega_1)$ are not regular and in fact fail the wellordered regularity of Zapletal's question.

Example 2.14. For $\epsilon < \omega_1$, there is a sequence $\langle X_\alpha : \alpha < \omega_1\rangle$ of subsets of $[\omega_1]^{\epsilon}$ so that $\bigcup_{\alpha<\omega_1} X_\alpha = [\omega_1]^{\epsilon}$ and for all $\alpha < \omega_1$, $|X_\alpha| \leq |\mathbb{R}|$. There is a sequence $\langle X_\alpha : \alpha < \omega_1\rangle$ of subsets of $[\omega_1]^{<\omega_1}$ so that $\bigcup_{\alpha<\omega_1} X_\alpha = [\omega_1]^{<\omega_1}$ and for all $\alpha < \omega_1$, $|X_\alpha| \leq |\mathbb{R}|$.

Proof. As an example, $[\omega_1]^{<\omega_1} = \bigcup_{\alpha\leq\beta<\omega_1} [\beta]^{\alpha}$ and one can check that $|[\beta]^{\alpha}| \leq |\mathbb{R}|$ when $\alpha \leq \beta < \omega_1$. $\square$

If a set is a surjective image of $\mathbb{R}$, then it is a quotient of an equivalence relation on $\mathbb{R}$. The next result shows that any subset of $[\omega_1]^{<\omega_1}$ that contains a copy of $\mathbb{R} \sqcup \omega_1$ is an ω_1-length disjoint union of quotients of $\mathbb{R}$ by an equivalence relation on $\mathbb{R}$ where each quotient is in bijection with $\mathbb{R}$.

Fact 2.15. *([4]) Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let $X \subseteq [\omega_1]^{<\omega_1}$. $|\mathbb{R} \sqcup \omega_1| \leq |X|$ if and only if there is a sequence $\langle E_\alpha : \alpha < \omega_1\rangle$ of equivalence relations on $\mathbb{R}$ so that for all $\alpha < \omega_1$, $|\mathbb{R}/E_\alpha| = |\mathbb{R}|$ and $|X| = |\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha|$.*

The proof of Fact 2.15 uses equivalence relations E_α that have at least one class which is uncountable. $\mathbb{R} \times \omega_1$ has a more natural presentation as $|\mathbb{R} \times \omega_1| = |\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha|$ where each E_α is the identity relation on $\mathbb{R}$ which has all equivalence classes countable and even size one. $|\mathbb{R} \times \omega_1|$ is the only cardinality obtainable this way by the following result.

Fact 2.16. *([4]) Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. *Let* $\kappa < \mathrm{ON}$ *and* $\langle E_\alpha : \alpha < \kappa \rangle$ *be a sequence of equivalence relations on* $\mathbb{R}$ *with all classes countable and* $|\mathbb{R}/E_\alpha| = |\mathbb{R}|$. *Then* $|\bigsqcup_{\alpha<\kappa} \mathbb{R}/E_\alpha| = |\mathbb{R} \times \kappa|$.

Thus $\mathscr{P}(\omega_1)$ is the first natural cardinality after $\mathscr{P}(\omega)$ which could have this wellordered regularity. The next result establishes this for any strong partition cardinal.

Theorem 2.17. *Suppose* δ *satisfies* $\delta \to_* (\delta)^\delta_2$. *Let* $\kappa \in \mathrm{ON}$. *Then for every function* $\Phi : [\delta]^\delta \to \kappa$, *there is an* $\alpha < \kappa$ *so that* $|\Phi^{-1}[\{\alpha\}]| = |[\delta]^\delta|$.

Proof. Assume this result is not true. Let κ be the least ordinal so that there is a function $\Phi : [\delta]^\delta \to \kappa$ with the property that for each $\alpha < \kappa$, $|\Phi^{-1}[\{\alpha\}]| < |[\delta]^\delta|$.

Let $\mathcal{L}$ be $\delta \times 2$ with the lexicographic ordering. ($\mathcal{L}$ is isomorphic to δ.) If $F : \mathcal{L} \to \delta$, then let $F_0, F_1 : \delta \to \delta$ be defined by $F_i(\alpha) = F(\alpha, i)$. Define a partition $P_0 : [\delta]^{\mathcal{L}}_* \to 2$ by $P_0(F) = 0$ if and only if $\Phi(F_0) \leq \Phi(F_1)$. By $\delta \to_* (\delta)^\delta_2$, there is a club $C_0 \subseteq \delta$ which is homogeneous for P_0. C_0 must be homogeneous for P_0 taking value 0. To see this, suppose C_0 was homogeneous for P_0 taking value 1. Let $A = \{\mathsf{enum}_C(\omega\cdot\alpha+\omega) : \alpha \in \delta\}$. (Notice that every function $f \in [A]^\delta$ is of the correct type.) Let $g_n(\alpha) = \mathsf{enum}_A(\omega \cdot \alpha + n)$. Let $G^n : \mathcal{L} \to A$ be defined so that $G^n(\alpha, i) = g_{n+i}(\alpha)$ for both $i \in \{0, 1\}$. Note that for each $n \in \omega$, $G^n \in [A]^{\mathcal{L}}_* \subseteq [C_0]^{\mathcal{L}}_*$ and $G^n_i = g_{n+i}$ for $i \in \{0, 1\}$. Since $P_0(G^n) = 1$ for all $n \in \omega$, $\Phi(g_{n+1}) < \Phi(g_n)$. This violates the wellfoundedness of δ.

Now define $P_1 : [C_0]^{\mathcal{L}}_* \to 2$ by $P_1(F) = 0$ if and only if $\Phi(F_0) < \Phi(F_1)$. Again by $\delta \to_* (\delta)^\delta_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for P_1. Again let $A \subseteq C_1$ be defined by $A = \{\mathsf{enum}_{C_1}(\omega\cdot\alpha+\omega) : \alpha < \delta\}$. Let $B = \{\mathsf{enum}_A(\omega \cdot \alpha + i) : i \in \{0, 1\} \wedge \alpha < \delta\}$. Since $|[\delta]^\delta| = |\mathscr{P}(\delta)| = |{}^\delta 2|$, let $\Sigma : [\delta]^\delta \to {}^\delta 2$ be a bijection. Let $h : \delta \to A$ be defined by $h(\alpha) = \mathsf{enum}_A(\omega\cdot\alpha+2)$. Define $\Psi : [\delta]^\delta \to [B]^\delta$ by $\Psi(f)(\alpha) = \mathsf{enum}_A(\omega\cdot\alpha+\Sigma(f)(\alpha))$. Note that Ψ is an injection. For each $f \in [\delta]^\delta$, let $G^f \in [C_1]^{\mathcal{L}}_*$ be defined by

$$G^f(\alpha, i) = \begin{cases} \Psi(f)(\alpha) & i = 0 \\ h(\alpha) & i = 1 \end{cases}$$

Note that $G^f_0 = \Psi(f)$ and $G^f_1 = h$.

Now suppose C_1 was homogeneous for P_1 taking value 1. One would have $P_1(G^f) = 1$ which implies that $\Phi(\Psi(f)) = \Phi(h)$ (since recall that $C_1 \subseteq C_0$ and C_0 is homogeneous for P_0 taking value 0). Let $\alpha = \Phi(h)$. Since $f \in [\delta]^\delta$ was arbitrary, one has that $\Psi[[\delta]^\delta] \subseteq \Phi^{-1}[\{\alpha\}]$. Since Ψ is an injection, one has that $|\Phi^{-1}[\{\alpha\}]| = |[\delta]^\delta|$. This contradicts the hypothesis on Φ. C_1 must be homogeneous for P_1 taking value 0.

Thus for any $f \in [\delta]^\delta$, $P_1(G^f) = 0$ and this implies that $\Phi(\Psi(f)) < \Phi(h)$. Let $\lambda = \Phi(h)$. Define $\Lambda : [\delta]^\delta \to \lambda$ by $\Lambda(f) = \Phi(\Psi(f))$. Since $\lambda < \kappa$ and κ is minimal with the above property, one has that there is an $\alpha < \lambda$ so that $|\Lambda^{-1}[\{\alpha\}]| = |[\delta]^\delta|$. However since Ψ is an injection and $\Psi[\Lambda^{-1}[\{\alpha\}]] \subseteq \Phi^{-1}[\{\alpha\}]$, one has that $|\Phi^{-1}[\{\alpha\}]| = |[\delta]^\delta|$. This contradicts the assumption on Φ.

Thus P_1 has no homogeneous club, which violates $\delta \to_* (\delta)^\delta_2$. ☐

Fact 2.18. *If* κ *has a* κ*-complete nonprincipal ultrafilter, then for all* $\alpha \leq \beta < \kappa$, κ *does not inject into* ${}^\alpha\beta$, *which is the collection of functions from* α *into* β.

Proof. Let μ be a κ-complete nonprincipal ultrafilter on κ. Let $\alpha \leq \beta < \kappa$. Suppose $\Phi : \kappa \to {}^\alpha\beta$ is an injection. For each $\gamma < \alpha$, by the κ-completeness of μ, there is a $u_\gamma < \beta$ and a set $A_\gamma \in \mu$ so that for all $\xi \in A_\gamma$, $\Phi(\xi)(\gamma) = u_\gamma$. Let $A = \bigcap_{\gamma<\alpha} A_\gamma$ and let $f \in {}^\alpha\beta$ be defined by $f(\gamma) = u_\gamma$. By the κ-completeness of μ, $A \in \mu$ and therefore contains at least two elements since μ is nonprincipal. Let $\xi_0, \xi_1 \in A$ be two distinct elements. Then $\Phi(\xi_0) = f = \Phi(\xi_1)$. This contradicts the injectiveness of Φ. ☐

The wellordered regularity property (Theorem 2.17) of $[\delta]^\delta$ when δ is a strong partition cardinal yields the following cardinality computation.

Fact 2.19. *Let* δ *be a cardinal satisfying* $\delta \to_* (\delta)^\delta_2$. *Then* $|[\delta]^{<\delta}| < |[\delta]^\delta| = |\mathscr{P}(\delta)|$.

Proof. The partition relation $\delta \to_* (\delta)^\delta_2$ implies that μ^δ_1 is a δ-complete nonprincipal measure by Fact 2.9. If $|[\delta]^{<\delta}| = |[\delta]^\delta|$, then let $\Psi : [\delta]^\delta \to [\delta]^{<\delta}$ be an injection. Fix a bijection $\pi : \delta \to \delta \times \delta$. Let $\pi_1, \pi_2 : \delta \times \delta \to \delta$ be the projections onto the first and second coordinate, respectively. Observe that $[\delta]^{<\delta} = \bigcup_{\alpha \leq \beta < \delta} [\beta]^\alpha$ by the regularity of δ. Define $\Phi : [\delta]^\delta \to \delta$ by $\Phi(f)$ is the least γ so that $\Psi(f) \in [\pi_2(\pi(\gamma))]^{\pi_1(\pi(\gamma))}$. By Theorem 2.17, there is an $\gamma < \delta$ so that $|[\pi_2(\pi(\gamma))]^{\pi_1(\pi(\gamma))}| = |\Phi^{-1}[\{\gamma\}]| = |[\delta]^\delta|$. Fact 2.18 implies this is not possible. $\square$

Next, a few more club uniformization principles will be defined. Establishing some of these principles under AD at suitable cardinals will be the subject of later sections.

Fact 2.20. *For all $\epsilon \leq \kappa$, μ^κ_ϵ is κ-complete ultrafilter if and only if $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$.*

Proof. ($\Leftarrow$) This is the argument from Fact 2.9.

($\Rightarrow$) Suppose $\lambda < \kappa$ and $\Phi : [\kappa]^\epsilon_* \to \lambda$. For $\alpha < \lambda$, let $A_\alpha = \Phi^{-1}[\{\alpha\}]$. Since $\bigcup_{\alpha<\lambda} A_\alpha = [\kappa]^\epsilon_*$, the κ-completeness of the ultrafilter μ^κ_ϵ implies that there some $\delta < \lambda$ so that $A_\delta \in \mu^\kappa_\epsilon$. Thus there is a club $C \subseteq \kappa$ with $[C]^\epsilon_* \subseteq A_\delta$. For all $f \in [C]^\epsilon_*$, $\Phi(f) = \delta$. $\square$

It is not known if $\kappa \to_* (\kappa)^\kappa_2$ alone is sufficient to prove μ^κ_κ is κ-complete (or equivalently $\kappa \to_* (\kappa)^\kappa_{<\kappa}$). Some authors define κ to be a strong partition cardinal if $\kappa \to_* (\kappa)^\kappa_{<\kappa}$ holds. In this article, a strong partition cardinal will merely satisfy $\kappa \to_* (\kappa)^\kappa_2$. Under AD, Fact 4.16 uses pointclass arguments to establish an everywhere wellordered club uniformization which will imply in many cases μ^κ_κ is κ-complete.

If κ is a cardinal, then let club_κ denote the set of club subsets of κ. If $f \in [\kappa]^\kappa_*$, then let $\mathcal{C}_f$ be the closure of $f[\kappa]$, which is a club subset of κ. The following club uniformization principle is provable purely from the strong partition relation.

Fact 2.21. *(Almost everywhere fixed short length club uniformization) Suppose $\kappa \to_* (\kappa)^\kappa_2$ and $\epsilon < \kappa$. Let $R \subseteq [\kappa]^\epsilon_* \times \mathsf{club}_\kappa$ be $\subseteq$-downward closed in the club_κ-coordinate, which means that for all $\ell \in [\kappa]^\epsilon_*$, for all clubs $C \subseteq D$, if $R(\ell, D)$ holds, then $R(\ell, C)$ holds. There is a club $C \subseteq \kappa$ so that for all $\ell \in \mathrm{dom}(R) \cap [C]^\epsilon_*$, $R(\ell, C \setminus (\sup(\ell) + 1)$.*

Proof. Define a partition $P : [\kappa]^\kappa_* \to 2$ by $P(f) = 0$ if and only if $f \restriction \epsilon \in \mathrm{dom}(R)$ and $R(f \restriction \epsilon, \mathcal{C}_{\mathsf{drop}(f,\epsilon)})$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $D \subseteq \kappa$ which is homogeneous for P. Pick any $\ell \in \mathrm{dom}(R) \cap [D]^\epsilon_*$. There is a club $E \subseteq D$ so that $R(\ell, E)$. Pick any $h \in [E]^\kappa_*$ with $\sup(\ell) < h(0)$. Since R is $\subseteq$-downward closed, $\mathcal{C}_h \subseteq E$, and $R(\ell, E)$, one has $R(\ell, \mathcal{C}_h)$. Let $f \in [D]^\kappa_*$ be such that $f \restriction \epsilon = \ell$ and $\mathsf{drop}(f, \epsilon) = h$. Then $P(f) = 0$. Thus D is homogeneous for P taking value 0.

Let $h(\alpha) = \mathsf{enum}_D(\omega \cdot \alpha + \omega)$ and note that $h \in [D]^\kappa_*$. Let $C = \mathcal{C}_h$. Pick any $\ell \in [C]^\epsilon_*$. Let $\xi < \kappa$ be least so that $\sup(\ell) < h(\xi)$. Let $f = \ell\hat{\ }\mathsf{drop}(h, \xi)$. Now since $f \in [D]^\kappa_*$, $P(f) = 0$ and $\mathsf{drop}(f, \epsilon) = \mathsf{drop}(h, \xi)$ imply that $R(\ell, \mathcal{C}_{\mathsf{drop}(h,\xi)})$. Since $C \setminus (\sup(\ell) + 1) = \mathcal{C}_{\mathsf{drop}(h,\xi)}$, $R(\alpha, C \setminus (\sup(\ell) + 1))$ holds. $\square$

Definition 2.22. Let κ be a cardinal. The everywhere wellordered club uniformization at κ is the assert that for every $R \subseteq \kappa \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed in the club_κ-coordinate, there is a function $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_\kappa$ so that for all $\alpha \in \mathrm{dom}(R)$, $R(\alpha, \Lambda(\alpha))$ holds.

The strong everywhere wellordered club uniformization at κ is the assertion that for every $R \subseteq \kappa \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed in the club_κ-coordinate, there is a club $C \subseteq \kappa$ so that for $\alpha \in \mathrm{dom}(R)$, $R(\alpha, C \setminus (\alpha + 1))$.

Fact 2.23. *Let κ be a cardinal. The everywhere wellordered club uniformization at κ is equivalent to the strong everywhere wellordered club uniformization at κ.*

Proof. Assume the everywhere wellordered club uniformization holds for κ. Suppose $R \subseteq \kappa \times \mathsf{club}_\kappa$ is a relation which is $\subseteq$-downward closed in the club_κ-coordinate. Let $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_\kappa$ be a uniformization function with the property that for all $\alpha \in \mathrm{dom}(R)$, $R(\alpha, \Lambda(\alpha))$. For each $\alpha < \kappa$, let $C_\alpha = \Lambda(\alpha)$ if $\alpha \in \mathrm{dom}(R)$ and $C_\alpha = \kappa$ if $\alpha \notin \mathrm{dom}(R)$. Let $C = \triangle_{\alpha<\kappa} C_\alpha = \{\xi < \kappa : (\forall \alpha < \xi)(\xi \in C_\alpha)\}$ be the diagonal intersection of $\langle C_\alpha : \alpha < \kappa \rangle$ which is a club subset of κ. Note that for each $\alpha \in \mathrm{dom}(R)$, $C \setminus (\alpha + 1) \subseteq C_\alpha$ and $R(\alpha, C_\alpha)$. Since R is $\subseteq$-downward closed in the club_κ-coordinate, $R(\alpha, C \setminus (\alpha + 1))$ holds. $\square$

Fact 2.24. *Suppose κ is a cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$. Then $\kappa \to_* (\kappa)^\kappa_{<\kappa}$ is equivalent to the everywhere wellordered club uniformization at κ.*

Proof. ($\Leftarrow$) Suppose $\lambda < \kappa$ and $\Phi : [\kappa]^\kappa_* \to \lambda$. Assume that there is no $\alpha < \lambda$ with a club $C \subseteq \kappa$ so that for all $f \in [C]^\kappa_*$, $\Phi(f) = \alpha$. This implies for all $\alpha < \lambda$, $\Phi^{-1}[\{\alpha\}] \notin \mu^\kappa_\kappa$. Since $\kappa \to_* (\kappa)^\kappa_2$ implies μ^κ_κ is an ultrafilter, for all $\alpha < \lambda$, $[\kappa]^\kappa_* \setminus \Phi^{-1}[\{\alpha\}] \in \mu^\kappa_\kappa$. Define $R \subseteq \kappa \times \mathsf{club}_\kappa$ by $R(\alpha, C)$ if and only if $\alpha < \lambda$ and $[C]^\kappa_* \subseteq [\kappa]^\kappa_* \setminus \Phi^{-1}[\{\alpha\}]$. Observe $\mathrm{dom}(R) = \lambda$. By the hypothesis, there is a $\Lambda : \lambda \to \mathsf{club}_\kappa$ such that for all $\alpha < \lambda$, $R(\alpha, \Lambda(\alpha))$. Since the intersection of less than κ many club subsets of κ is a club, $C = \bigcap_{\alpha<\lambda} \Lambda(\alpha)$ is a club subset. $C \subseteq \bigcap_{\alpha<\lambda}[\kappa]^\kappa_* \setminus \Phi^{-1}[\{\alpha\}] = [\kappa]^\kappa_* \setminus \bigcup_{\alpha<\lambda} \Phi^{-1}[\{\alpha\}] = [\kappa]^\kappa_* \setminus [\kappa]^\kappa_* = \emptyset$ which is a contradiction.

($\Rightarrow$) Suppose $R \subseteq \kappa \times \mathsf{club}_\kappa$ is a relation which is $\subseteq$-downward closed in the club_κ-coordinate. Define $P_0 : [\kappa]^\kappa_* \to 2$ by $P_0(f) = 0$ if and only if for all $\alpha \in \mathrm{dom}(R)$, $R(\alpha, \mathcal{C}_{\mathsf{drop}(f,\alpha)})$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_0 \subseteq \kappa$ homogeneous for P_0. Suppose C_0 is homogeneous for P_0 taking value 1. Define $P_1 : [C_0]^\kappa_* \to 2$ by $P_1(f) = 0$ if and only if there exists an $\alpha < f(0)$ so that $\alpha \in \mathrm{dom}(R)$ and $\neg R(\alpha, \mathcal{C}_f)$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_1 \subseteq C_0$ homogeneous for P_1. Take any $f \in [C_1]^\kappa$. Since $P_0(f) = 1$, there is an $\alpha \in \mathrm{dom}(R)$ with $\neg R(\alpha, \mathcal{C}_{\mathsf{drop}(f,\alpha)})$. Then $P_1(\mathsf{drop}(f,\alpha)) = 0$ and since $\mathsf{drop}(f,\alpha) \in [C_1]^\kappa_*$, C_1 must be homogeneous for P_1 taking value 0. Define $\Psi : [C_1]^\kappa_* \to \kappa$ by $\Psi(f)$ is the least $\alpha < f(0)$ so that $\alpha \in \mathrm{dom}(R)$ and $\neg R(\alpha, \mathcal{C}_f)$. Ψ has the property that for all $f \in [C_1]^\kappa_*$, $\Psi(f) < f(0)$. By Fact 2.12, there is a club $C_2 \subseteq C_1$ and a $\zeta < \kappa$ so that for all $f \in [C_2]^\kappa_*$, $\Psi(f) = \zeta$. This implies that $\zeta \in \mathrm{dom}(R)$. There is some club $D \subseteq \kappa$ with $R(\zeta, D)$. Pick an $h \in [D \cap C_2]^\kappa_*$. Then $R(\zeta, \mathcal{C}_h)$ holds since $\mathcal{C}_h \subseteq D \cap C_2 \subseteq D$ and R is $\subseteq$-downward closed in the club_κ-coordinate. This contradicts $\Psi(h) = \zeta$. Thus C_0 must have been homogeneous for P_0 taking value 0. Take any $f \in [C_0]^\kappa_*$. Define $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_\kappa$ by $\Lambda(\alpha) = \mathcal{C}_{\mathsf{drop}(f,\alpha)}$. $P_0(f) = 0$ implies that for all $\alpha \in \mathrm{dom}(R)$, $R(\alpha, \Lambda(\alpha))$. ☐

The following summarizes some equivalences of everywhere wellordered club uniformization.

Fact 2.25. *Suppose κ is a cardinal and assume $\kappa \to_* (\kappa)^\kappa_2$. Then the following are equivalent.*

- $\kappa \to_* (\kappa)^\kappa_{<\kappa}$.
- *μ^κ_κ is a κ-complete ultrafilter.*
- *Everywhere wellordered club uniformization at κ*
- *Strong everywhere wellordered club uniformization at κ.*

Definition 2.26. Let κ be a cardinal.

- (Almost everywhere short length club uniformization at κ) For every relation $R \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed in the club_κ-coordinate, there is a club $C \subseteq \kappa$ and a function $\Lambda : \mathrm{dom}(R) \cap [C]^{<\kappa}_* \to \mathsf{club}_\kappa$ so that for all $\ell \in \mathrm{dom}(R) \cap [C]^{<\kappa}_*$, $R(\ell, \Lambda(\ell))$.
- (Strong almost everywhere short length club uniformization at κ) For every relation $R \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed in the club_κ-coordinate, there is a club $C \subseteq \kappa$ so that for all $\ell \in \mathrm{dom}(R) \cap [C]^{<\kappa}_*$, $R(\ell, C \setminus (\sup(\ell) + 1))$.

Fact 2.27. *Suppose κ is a cardinal and $\kappa \to_* (\kappa)^\kappa_2$. Then almost everywhere short length club uniformization at κ is equivalent to the strong almost everywhere short length club uniformization at κ.*

Proof. Assume the almost everywhere short length club uniformization at κ. Let $R \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ be a relation which is $\subseteq$-downward closed in the club_κ-coordinate. By the hypothesis, let $C_0 \subseteq \kappa$ be a club and $\Lambda : \mathrm{dom}(R) \cap [C_0]^{<\kappa}_*$ have the property that for all $\ell \in \mathrm{dom}(R) \cap [C_0]^{<\kappa}_*$, $R(\ell, \Lambda(\ell))$.

Define a partition $P : [C_0]^\kappa_* \to 2$ by $P(f) = 0$ if and only if for all $\alpha < \kappa$, if $f \restriction \alpha \in \mathrm{dom}(R)$, then $R(f \restriction \alpha, \mathcal{C}_{\mathsf{drop}(f,\alpha)})$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for P. Now suppose that C_1 is homogeneous for P taking value 1. This means for all $f \in [C_1]^\kappa_*$, there exists an $\alpha < \kappa$ so that $f \restriction \alpha \in \mathrm{dom}(R)$ and $\neg R(f \restriction \alpha, \mathcal{C}_{\mathsf{drop}(f,\alpha)})$. Define $\Phi : [C_1]^\kappa_* \to \kappa$ by $\Phi(f)$ is the least α with the above property.

A function $h \in [C_1]^\kappa_*$ will be defined by recursion. If $h \restriction 0 = \emptyset \notin \mathrm{dom}(R)$, then let $F_0 = C_1$. If $h \restriction 0 = \emptyset \in \mathrm{dom}(R)$, then let $F_0 = C_1 \cap \Lambda(\emptyset)$. In either case, $h \restriction 0$ has the property that for all $g \in [F_0]^\kappa_*$, $\Phi(h \restriction 0\hat{\ }g) > 0$ since if $h \restriction 0 \in \mathrm{dom}(R)$, then $R(h \restriction 0, \mathcal{C}_g)$ holds because $R(h \restriction 0, \Lambda(\emptyset))$, $\mathcal{C}_g \subseteq \Lambda(0)$, and R is $\subseteq$-downward closed. Let $h(0) = \mathsf{next}^\omega_{F_0}(0)$. Suppose for $\alpha < \kappa$, $h \restriction \alpha$ and $\langle F_\beta : \beta < \alpha \rangle$ have been defined with the property that for all $\beta < \alpha$, if $g \in [F_\beta]^\kappa_*$, then $\Phi(h \restriction \beta\hat{\ }g) > \beta$. If $h \restriction \alpha \notin \mathrm{dom}(R)$, then let $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$. If $h \restriction \alpha \in \mathrm{dom}(R)$, then let $F_\alpha = \bigcap_{\beta<\alpha} F_\beta \cap \Lambda(h \restriction \alpha)$. In either case, for all $g \in [F_\alpha]^\kappa_*$, $\Phi(h \restriction \alpha\hat{\ }g) > \alpha$. To see this: If $\beta < \alpha$, note that $\mathsf{drop}(h \restriction \alpha, \beta)\hat{\ }g \subseteq F_\beta$ which implies that $\Phi(h \restriction \alpha\hat{\ }g) = \Phi((h \restriction \beta)\hat{\ }\mathsf{drop}(h \restriction \alpha, \beta)\hat{\ }g) > \beta$ by the induction hypothesis. Thus $\Phi(h \restriction \alpha\hat{\ }g) \geq \alpha$.

If $h \restriction \alpha \in \mathrm{dom}(R)$, then $R(h \restriction \alpha, \mathcal{C}_g)$ holds because $R(h \restriction \alpha, \Lambda(h \restriction \alpha))$ holds, $\mathcal{C}_g \subseteq \Lambda(h \restriction \alpha)$, and R is $\subseteq$-downward closed. This implies $\Phi(h \restriction \alpha\hat{\ }g) > \alpha$. If $f \restriction \alpha \notin \mathrm{dom}(R)$, then by definition of Φ, $\Phi(h \restriction \alpha\hat{\ }g) > \alpha$. Let $h(\alpha) = \mathsf{next}^\omega_{F_\alpha}(\sup(h \restriction \alpha))$. This completes the definition of h and $\langle F_\alpha : \alpha < \kappa\rangle$. For any α, $\Phi(h) = \Phi(h \restriction \alpha\hat{\ }\mathsf{drop}(h, \alpha)) > \alpha$ since $\mathsf{drop}(h, \alpha) \in [F_\alpha]^\kappa_*$. As $\alpha < \kappa$ is arbitrary, $\Phi(h) \geq \kappa$ which contradicts the fact that $\Phi : [C_1]^\kappa_* \to \kappa$.

This implies that C_1 must be homogeneous for P taking value 0. Let $h \in [C_1]^\kappa_*$ be defined by $h(\alpha) = \mathsf{enum}_{C_1}(\omega \cdot \alpha + \omega)$. Let $C_2 = \mathcal{C}_h$. Suppose $\ell \in \mathrm{dom}(R) \cap [C_2]^{<\kappa}_*$. Let α be least so that $h(\alpha) > \sup(\ell)$. Let $f = \ell\hat{\ }\mathsf{drop}(h, \alpha)$. Since $f \in [C_1]^\kappa_*$, $P(f) = 0$. Since $\ell \in \mathrm{dom}(R)$ and $f \restriction |\ell| = \ell$, $R(f \restriction |\ell|, \mathcal{C}_{\mathsf{drop}(f,|\ell|)})$ holds and thus $R(\ell, \mathcal{C}_{\mathsf{drop}(h,\alpha)})$ holds. However $\mathcal{C}_{\mathsf{drop}(h,\alpha)} = C_2 \setminus (\sup(\ell)+1)$. Thus $R(\ell, C_2 \setminus (\sup(\ell)+1))$ holds. C_2 is the desired club. ☐

Fact 2.28. *Let κ be a cardinal. Almost everywhere short length club uniformization at κ implies the everywhere wellordered club uniformization at κ*

Proof. Suppose $R \subseteq \kappa \times \mathsf{club}_\kappa$ is $\subseteq$-downward closed in the club_κ-coordinate. Define $S \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ by $S(\ell, D)$ if and only if $|\ell| \in \mathrm{dom}(R)$ and $R(|\ell|, D)$. By the almost everywhere short length club uniformization and Fact 2.27, there is a club C so that for all $\ell \in \mathrm{dom}(S) \cap [C]^{<\kappa}_*$, $S(\ell, C \setminus (\sup(\ell)+1))$. For each $\alpha < \omega_1$, let $\ell_\alpha \in [C]^\alpha_*$ be defined by recursion as follows. Let $\ell_\alpha(0) = \mathsf{next}^\omega_C(0)$. If $\beta < \alpha$ and $\ell_\alpha \restriction \beta$ has been defined, then let $\ell_\alpha(\beta) = \mathsf{next}^\omega_C(\sup(\ell_\alpha \restriction \beta))$. Let $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_\kappa$ be defined by $\Lambda(\alpha) = C \setminus (\sup(\ell_\alpha)+1)$. Suppose $\alpha \in \mathrm{dom}(R)$. Since $|\ell_\alpha| = \alpha$, $\ell_\alpha \in \mathrm{dom}(S)$ and thus $S(\ell_\alpha, C \setminus (\sup(\ell_\alpha)+1))$. By definition of S, $R(\alpha, \Lambda(\alpha))$. ☐

These uniformization results can be used to prove a mixed everywhere wellordered and almost everywhere short length club uniformization.

Fact 2.29. *Suppose the almost everywhere short length club uniformization holds at κ. Let $R \subseteq \kappa \times [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ be $\subseteq$-downward closed in the club_κ-coordinate. Then there is a club $C \subseteq \kappa$ so that for all $\alpha < \kappa$ and $\ell \in [C]^{<\kappa}_*$, if $(\alpha, \ell) \in \mathrm{dom}(R)$, then $R(\alpha, \ell, C \setminus (\max\{\sup(\ell), \alpha\}+1))$.*

Proof. For each $\ell \in [\kappa]^{<\kappa}_*$, define $S_\ell \subseteq \kappa \times \mathsf{club}_\kappa$ by $S_\ell(\alpha, D)$ if and only if $R(\alpha, \ell, D)$. S_ℓ is $\subseteq$-downward closed in the club_κ-coordinate. By Fact 2.28 and Fact 2.25, there is a club $E \subseteq \kappa$ so that for all $\alpha \in \mathrm{dom}(S_\ell)$, $S_\ell(\alpha, E \setminus (\alpha+1))$. Define $T \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ by $T(\ell, E)$ if and only if for all $\alpha < \kappa$, if $\alpha \in \mathrm{dom}(S_\ell)$, then $S_\ell(E \setminus (\alpha+1))$. By the previous discussion, $\mathrm{dom}(T) = [\kappa]^{<\kappa}_*$. By Fact 2.27, there is a club $C \subseteq \kappa$ so that for all $\ell \in [C]^{<\kappa}_*$, $T(\ell, C \setminus (\sup(\ell)+1))$. Now suppose $\alpha < \kappa$ and $\ell \in [C]^{<\kappa}_*$ with $(\alpha, \ell) \in \mathrm{dom}(R)$. One has $T(\ell, C \setminus (\sup(\ell)+1))$. Since $\alpha \in \mathrm{dom}(S_\ell)$, $S_\ell(\alpha, (C \setminus (\sup(\ell)+1)) \setminus (\alpha+1))$. By definition of S_ℓ and since $(C \setminus (\sup(\ell)+1)) \setminus (\alpha+1) = C \setminus (\max\{\sup(\ell), \alpha\})+1)$, one has $R(\alpha, \ell, C \setminus (\max\{\sup(\ell), \alpha\}+1))$. ☐

3. Almost Everywhere Continuity Properties

Definition 3.1. Let $\Phi : [\kappa]^\kappa_* \to \kappa$ and $C \subseteq \kappa$ be a club. Say that $\sigma \in [C]^{<\kappa}_*$ is a continuity point for Φ relative to C if and only if for all $g_0, g_1 \in [C]^\kappa_*$ such that $g_0 \restriction |\sigma| = \sigma = g_1 \restriction |\sigma|$, $\Phi(g_0) = \Phi(g_1)$. Say that σ is a minimal continuity point for Φ relative to C if and only if no proper initial segment of σ is a continuity point for Φ relative to C.

Theorem 3.2. *Let κ be a cardinal so that $\kappa \to_* (\kappa)^\kappa_2$ and the almost everywhere short length club uniformization at κ holds. Let $\Phi : [\kappa]^\kappa_* \to \kappa$. Then there is a club $C \subseteq \kappa$ with the following properties.*

- *(a) $\Phi \restriction [C]^\kappa_*$ is continuous: For every $f \in [C]^\kappa_*$, there exists an $\alpha < \kappa$ so that for all $g \in [C]^\kappa_*$, if $f \restriction \alpha = g \restriction \alpha$, then $\Phi(g) = \Phi(f)$.*
- *(b) For any $f \in [C]^\kappa_*$, let β_f be the unique β so that $\sup(f \restriction \beta) \leq \Phi(f) < f(\beta)$. Then $f \restriction \beta_f$ is a minimal continuity point for Φ relative to C.*
- *(c) For any $\sigma \in [C]^{<\kappa}_*$, if there is a $g \in [C]^\kappa_*$ so that $\sup(\sigma) < g(0)$ and $\Phi(\sigma\hat{\ }g) < g(0)$, then σ is a continuity point of Φ relative to C.*

Proof. Under the hypothesis, Fact 2.27 implies strong almost everywhere short length club uniformization at κ. For each $\sigma \in [\kappa]^{<\kappa}_*$, let $\Phi_\sigma : [\kappa \setminus (\sup(\sigma)+1)]^\kappa_* \to \kappa$ be defined by $\Phi_\sigma(g) = \Phi(\sigma\hat{\ }g)$. Let K be the set of $\sigma \in [\kappa]^{<\kappa}_*$ so that for all club $D \subseteq \kappa$, there exists a $g \in [D]^\kappa_*$ with $\sup(\sigma) < g(0)$ and $\Phi_\sigma(g) < g(0)$.

Claim 1: For each $\sigma \in K$, there is unique $c_\sigma \in \kappa$ so that there exists a club D with the property that for all $g \in [D]^\kappa_*$, $\Phi_\sigma(g) = c_\sigma < g(0)$.

Proof. Let $Q_\sigma : [\kappa \setminus (\sup(\sigma)+1)]^\kappa_* \to 2$ by $Q_\sigma(g) = 0$ if and only if $\Phi_\sigma(g) < g(0)$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club D_0 which is homogeneous for Q_σ. Since $\sigma \in K$, there is a $g \in [D_0]^\kappa_*$ so that $\sup(\sigma) < g(0)$ and $\Phi_\sigma(g) < g(0)$. Thus D_0 is homogeneous for Q_σ taking value 0. For all $g \in [D_0]^\kappa_*$, $Q_\sigma(g) = 0$ implies that $\Phi_\sigma(g) < g(0)$. By Fact 2.12, there is a club $D_1 \subseteq D_0$ and $c_\sigma \in \kappa$ so that for all $g \in [D_1]^\kappa_*$, $\Phi_\sigma(g) = c_\sigma$. (Note that c_σ depends only on σ and does not depend on D_1.) $\square$

Let $R_0 \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ be defined by $R(\sigma, D)$ if and only if the conjunction of the following holds.

(1) If $\sigma \in K$, then for all $g \in [D]^\kappa_*$, $\sup(\sigma) < g(0)$ and $\Phi_\sigma(g) = c_\sigma < g(0)$.
(2) If $\sigma \notin K$, then for all $g \in [D]^\kappa_*$, $\sup(\sigma) < g(0)$ and $\Phi_\sigma(g) \geq g(0)$.

Note that R is $\subseteq$-downward closed in the club_κ-coordinate. If $\sigma \in K$, then Claim 1 implies $\sigma \in \mathrm{dom}(R_0)$. If $\sigma \notin K$, then by definition $\sigma \in \mathrm{dom}(R_0)$. Thus $\mathrm{dom}(R_0) = [\kappa]^{<\kappa}_*$. By Theorem 6.2, there is a club $C_0 \subseteq \kappa$ so that for all $\sigma \in [\kappa]^{<\kappa}_*$, $R_0(\sigma, C_0 \setminus (\sup(\sigma)+1))$.

Claim 2: If $\sigma \in K \cap [C_0]^{<\kappa}_*$, then σ is a continuity point for Φ relative to C_0. Moreover, for all $f \in [C_0]^\kappa_*$ with $\sigma = f \restriction |\sigma|$, $\Phi(f) = c_\sigma < f(|\sigma|)$.

Proof. Suppose $\sigma \in K \cap [C_0]^{<\kappa}_*$. Let $f, g \in [C_0]^\kappa_*$ be such that $f \restriction |\sigma| = \sigma = g \restriction |\sigma|$. Since $R_0(\sigma, C_0 \setminus (\sup(\sigma)+1))$, $\mathsf{drop}(f, |\sigma|) \in C_0 \setminus (\sup(\sigma)+1)$, and $\mathsf{drop}(g, |\sigma|) \in C_0 \setminus (\sup(\sigma)+1)$,

$$\Phi(f) = \Phi_{f\restriction|\sigma|}(\mathsf{drop}(f \restriction |\sigma|)) = \Phi_\sigma(\mathsf{drop}(f, |\sigma|)) = c_\sigma = \Phi_\sigma(\mathsf{drop}(g, |\sigma|)) = \Phi_{g\restriction|\sigma|}(\mathsf{drop}(g \restriction |\sigma|)) = \Phi(g).$$

The properties of R_0 also imply $\Phi(f) = c_\sigma < \mathsf{drop}(f, |\sigma|)(0) = f(|\sigma|)$. $\square$

Let K^* be the set of $\sigma \in K$ such that for all proper initial segments $\tau \subset \sigma$, $\tau \notin K$.

Claim 3: For any club $C \subseteq C_0$, if $\sigma \in K^* \cap [C]^{<\kappa}_*$, then σ is a minimal continuity point relative to C.

Proof. Fix $C \subseteq C_0$. Suppose $\sigma \in K^* \cap [C]^{<\kappa}_*$. Since $\sigma \in K$, Claim 2 implies that σ is a continuity point for Φ relative to C_0 and hence also relative to C which is subset of C_0. Let $\tau \subset \sigma$ be a proper initial segment. Then $\tau \notin K$. Let $g_0 \in [C \setminus (\sup(\tau)+1)]^\kappa_*$. Pick $g_1 \in [C \setminus (\sup(\tau)+1)]^\kappa_*$ so that $g_1(0) > \Phi_\tau(g_0)$. Since $C \subseteq C_0$, $R_0(\tau, C_0 \setminus (\sup(\tau)+1))$ holds, and R_0 is $\subseteq$-downward closed in the club_κ-coordinate, one has $R_0(\tau, C \setminus (\sup(\tau+1)))$. Since $\tau \notin K$, the definition of R_0 implies

$$\Phi(\tau\hat{\ }g_0) = \Phi_\tau(g_0) < g_1(0) \leq \Phi_\tau(g_1) = \Phi(\tau\hat{\ }g_1).$$

Thus $\Phi(\tau\hat{\ }g_0) \neq \Phi(\tau\hat{\ }g_1)$ and hence τ is not a continuity point for Φ relative to C. $\square$

Claim 4: There is a club $C_1 \subseteq C_0$ so that for all $f \in [C_1]^\kappa_*$, there exists some $\alpha < \kappa$ with $f \restriction \alpha \in K$.

Proof. Let $P_0 : [C_0]^\kappa_* \to 2$ be defined by $P_0(f) = 0$ if and only if there exists an $\alpha < \kappa$ so that $f \restriction \alpha \in K$. By the partition relation $\kappa \to_* (\kappa)^\kappa_2$, let $C_1 \subseteq C_0$ be homogeneous for P_0. Suppose C_1 is homogeneous for P_0 taking value 1. Fix an $f \in [C_1]^\kappa_*$. Let $\alpha < \kappa$. Since $P_0(f) = 1$, $f \restriction \alpha \notin K$. Because $R_0(f \restriction \alpha, C_0 \setminus (\sup(f \restriction \alpha)+1))$, $C_1 \subseteq C_0$, and R_0 is $\subseteq$-downward closed in the club_κ-coordinate, $R_0(f \restriction \alpha, C_1 \setminus (\sup(f \restriction \alpha)+1))$. Since $f \restriction \alpha \notin K$ and $\mathsf{drop}(f, \alpha) \in C_1 \setminus (\sup(f \restriction \alpha)+1)$, the definition of R_0 implies that $\Phi(f) = \Phi_{f\restriction\alpha}(\mathsf{drop}(f, \alpha)) \geq \mathsf{drop}(f, \alpha)(0) = f(\alpha)$. It has been shown that for all $\alpha < \kappa$, $\Phi(f) \geq \alpha$. Thus $\Phi(f) \geq \kappa$ which is impossible since Φ takes values in κ. C_0 is homogeneous for P_0 taking value 0 which establishes the claim. $\square$

Using Claim 4, for each $f \in [C_1]^\kappa_*$, let α_f be the least $\alpha < \kappa$ so that $f \restriction \alpha \in K$. Thus $f \restriction \alpha_f \in K^*$. Recall that for all $f \in [\kappa]^\kappa_*$, β_f is the unique β so that $\sup(f \restriction \beta) \leq \Phi(f) < f(\beta)$.

Claim 5: For all $f \in [C_1]^\kappa_*$, $\beta_f \leq \alpha_f$.

Proof. Let $f \in [C_1]^\kappa_*$. $f \restriction \alpha_f \in K$. By Claim 2, one has $\Phi(f) < f(\alpha_f)$. Thus the unique β so that $\sup(f \restriction \beta) \leq \Phi(f) < f(\beta)$ is less than or equal to α_f. Hence $\beta_f \leq \alpha_f$. $\square$

Claim 6: There is a club $C_2 \subseteq C_1$ so that for all $f \in [C_2]^\kappa_*$, $\alpha_f = \beta_f$.

Proof. Let $P_1 : [C_1]^\kappa_* \to 2$ by $P_1(f) = 0$ if and only if $\alpha_f = \beta_f$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_2 \subseteq C_1$ which is homogeneous for P_1. Suppose C_2 is homogeneous for P_1 taking value 1.

Subclaim 6.1: For any $\sigma \in K^* \cap [C_2]^{<\kappa}_*$, there is an ordinal $\gamma < |\sigma|$ so that $c_\sigma < \sigma(\gamma)$.

Proof. Let $f \in [C_2]^\kappa_*$ so that $\sigma \subseteq f$. Since $\sigma \in K^*$, $\alpha_f = |\sigma|$. Since $P_1(f) = 1$, one has that $\beta_f < \alpha_f = |\sigma|$. Thus using Claim 2,

$$\sup(f \restriction \beta_f) \leq \Phi(f) = \Phi_\sigma(\mathsf{drop}(f, \alpha_f)) = c_\sigma < f(\beta_f) \leq \sup(f \restriction \alpha_f).$$

So β_f is an ordinal $\gamma < |\sigma|$ so that $c_\sigma < \sigma(\gamma)$. □

Let $f \in [C_2]^\kappa_*$. Suppose $f \restriction 0 = \emptyset \in K$. Then $f \restriction 0 \in K^*$ and thus $\alpha_f = 0$. $P_1(f) = 1$ implies that $\beta_f < \alpha_f = 0$ which is impossible. It has been shown that $f \restriction 0 \notin K$.

Suppose $\epsilon < \kappa$ and for all $\delta < \epsilon$, it has been shown that $f \restriction \delta \notin K$. Suppose $f \restriction \epsilon \in K$. Thus $f \restriction \epsilon \in K^*$ and hence by Claim 2, $\Phi(f) = c_{f \restriction \epsilon}$. By Subclaim 6.1, there is a $\gamma < \epsilon$ so that $\Phi(f) = c_{f\restriction\epsilon} < f(\gamma)$. Since $\gamma < \epsilon$, $f \restriction \gamma \notin K$. Because $C_2 \subseteq C_1$, one has $R_0(f \restriction \gamma, C_2 \setminus (\sup(f \restriction \gamma)+1))$. Since $\mathsf{drop}(f, \gamma) \in [C_2 \setminus (\sup(f \restriction \gamma) + 1)]^\kappa_*$, the definition of R_0 implies that $\Phi(f) = \Phi_{f\restriction\gamma}(\mathsf{drop}(f, \gamma)) \geq \mathsf{drop}(f, \gamma)(0) = f(\gamma)$. Thus we have shown $\Phi(f) < f(\gamma)$ and $\Phi(f) \geq f(\gamma)$. Contradiction. Thus C_2 must be homogeneous for P_1 taking value 0. □

C_2 is the desired club satisfying property (a), (b), and (c):

Property (b) following from Claim 3 and Claim 6. Property (a) follows from property (b).

Suppose $\sigma \in [C_2]^{<\kappa}_*$ and there is a $g \in [C_2]^\kappa_*$ with $\sup(\sigma) < g(0)$ and $\Phi(\sigma\hat{\ }g) < g(0)$. Let $f = \sigma\hat{\ }g$. Then $\beta_f \leq |\sigma|$. By Claim 6, $\alpha_f = \beta_f \leq |\sigma|$. Thus $\sigma \restriction \alpha_f$ is a minimal continuity point for Φ relative to C_2 by Claim 3 and thus σ is also a continuity point for Φ relative to C_2. This establishes property (c). □

Fact 3.3. *Assume* ZF. *Suppose κ is a cardinal such that for all $\Phi : [\kappa]^\kappa_* \to \kappa$, there is a club $C \subseteq \kappa$ so that for all $f \in [C]^\kappa_*$, $f \restriction \beta_f$ is a minimal continuity point for Φ relative to C where β_f is the unique β so that $\sup(f \restriction \beta) \leq \Phi(f) < f(\beta)$. Then $\kappa \to_* (\kappa)^\kappa_{<\kappa}$ holds.*

Proof. Let $\lambda < \kappa$ and $\Phi : [\kappa]^\kappa_* \to \lambda$. Let C be a club satisfying the expressed continuity property with $\lambda < \min(C)$. For all $f \in [C]^\kappa_*$, $\Phi(f) < \lambda < f(0)$ and thus $\beta_f = 0$. For all $f \in [C]^\kappa_*$, $\emptyset$ is a minimal continuity point for Φ relative to C. For all $f, g \in [C]^\kappa_*$, $\Phi(f) = \Phi(g)$. Pick any $h \in [C]^\kappa_*$ and let $\delta = \Phi(h)$. Then for all $f \in [C]^\kappa_*$, $\Phi(f) = \delta$. □

[5] Theorem 5.3 shows that every function $\Phi : [\omega_1]^{\omega_1}_* \to {}^{\omega_1}\omega_1$ is continuous μ^κ_κ-almost everywhere in a natural sense. Using Fact 2.29 and the ideas of [5] Theorem 5.3, the following analogous almost everywhere continuity result can be shown. The proof is omitted since this result will not be used in the paper.

Theorem 3.4. *Assume $\kappa \to_* (\kappa)^\kappa_2$ and the almost everywhere short length club uniformization holds at κ. Let $\Phi : [\kappa]^\kappa_* \to {}^\kappa\kappa$. Then there is a club $C \subseteq \kappa$ so that for all $f \in [C]^\kappa_*$ and $\beta < \kappa$, there exists an $\alpha < \kappa$ so that for all $g \in [C]^\kappa_*$, if $f \restriction \alpha = g \restriction \alpha$, then $\Phi(f) \restriction \beta = \Phi(g) \restriction \beta$.*

4. Everywhere Wellordered Club Uniformization

The set ${}^\omega\omega$ is the collection of functions from ω into ω. As customary in descriptive set theory, this set will also be denoted by $\mathbb{R}$ and its elements will be called reals. For the rest of the paper, the axiom of determinacy, AD, will be implicitly assumed unless otherwise stated.

Definition 4.1. Strategies for games on ω are functions $\sigma : {}^{<\omega}2 \to \omega$. If σ and τ are strategies, then $\sigma * \tau$ is an element of ${}^\omega\omega$ defined by recursion as follows: Let $(\sigma * \tau)(0) = \sigma(\emptyset)$. If $(\sigma * \tau) \restriction 2n$ has been defined, then $(\sigma * \tau)(2n) = \sigma((\sigma * \tau) \restriction 2n)$. If $(\sigma * \tau) \restriction 2n + 1$ has been defined, then $(\sigma * \tau)(2n + 1) = \tau((\sigma * \tau) \restriction 2n + 1)$. Note that strategies can be coded as reals. In such a setting, σ is used as a Player 1 strategy and τ is used as a Player 2 strategy. (The terms Player 1 or Player 2 strategy will be used although formally there is no distinction.)

If $z \in \mathbb{R}$, then let ρ^1_z be a strategy defined as follows: if $s \in {}^{<\omega}2$ has even length $2n$, then $\rho^1_z(s) = z(n)$. If $s \in {}^{<\omega}2$ has odd length, then $\rho^1_z(s) = 0$. Similarly, let ρ^2_z be a strategy defined as follows: if $s \in {}^{<\omega}2$ has odd length $2n + 1$, then $\rho^2_z(s) = z(n)$. If $s \in {}^{<\omega}2$ has even length, then let $\rho^2_z(s) = 0$. Intuitively, ρ^1_z (considered as a Player 1 strategy) will simply put down the bits of z turn by turn. Similarly, ρ^2_z (considered as a Player 2 strategy) will put down the bits of z turn by turn.

If $x \in {}^\omega\omega$, let $x_{\text{even}} \in {}^\omega\omega$ be defined by $x_{\text{even}}(n) = x(2n)$ and let $x_{\text{odd}} \in {}^\omega\omega$ be defined by $x_{\text{odd}}(n) = x(2n+1)$.

If σ is considered a Player 1 strategy, then define $\Xi^1_\sigma : {}^\omega\omega \to {}^\omega\omega$ by $\Xi^1_\sigma(z) = (\sigma * \rho^2_z)_{\text{even}}$. Similarly, if τ is considered a Player 2 strategy, then define $\Xi^2_\tau : {}^\omega\omega \to {}^\omega\omega$ by $\Xi^2_\tau(z) = (\rho^1_z * \tau)_{\text{odd}}$. Note that for any σ and τ, Ξ^1_σ and Ξ^2_τ are Lipschitz continuous functions. It can be shown that if $\Xi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, then there is a strategy ρ so that $\Xi = \Xi^2_\rho$.

Definition 4.2. A prewellordering on a set $P \subseteq \mathbb{R}$ (or more generally a Polish space of the form $\omega^m \times \mathbb{R}^n$, where $m, n \in \omega$ and $n > 0$) is a binary relation $\preceq$ on P which is transitive, reflexive, total on P, and whose strict part $\prec$ is wellfounded. A norm on P is a map $\varphi : P \to \text{ON}$. Every prewellordering on P induces a norm on P, and every norm on P induces a prewellordering on P.

Definition 4.3. Let Γ be a boldface pointclass (meaning closed under Wadge reductions). Let $\check{\Gamma}$ refer to the dual pointclass of Γ. Let $\Delta = \Gamma \cap \check{\Gamma}$. Define $\delta(\Gamma)$ to be the supremum of the rank of the prewellorderings $\preceq$ which belong to Δ.

Definition 4.4. Let Γ be a pointclass and $P \subseteq \mathbb{R}$. $\preceq$ is a Γ-prewellordering on P if and only if there are two relations $\leq^{\preceq}_{\Gamma} \in \Gamma$ and $\leq^{\preceq}_{\check{\Gamma}} \in \check{\Gamma}$ so that

$$(\forall y)\Big(P(y) \Rightarrow (\forall x)\left[(x \in P \wedge x \preceq y) \Leftrightarrow x \leq^{\preceq}_{\Gamma} y \Leftrightarrow x \leq^{\preceq}_{\check{\Gamma}}\right]\Big).$$

If φ is a norm on P, then one says that φ is a Γ-norm on P if and only if its associated prewellordering $\preceq$ is a Γ-prewellordering on P. In this case, one may write $\leq^{\varphi}_{\Gamma}$ and $\leq^{\varphi}_{\check{\Gamma}}$ for $\leq^{\preceq}_{\Gamma}$ and $\leq^{\preceq}_{\check{\Gamma}}$, respectively.

It can also be shown that if $\preceq$ is a Γ-prewellordering, then there are relation $<^{\preceq}_{\Gamma} \in \Gamma$ and $<^{\preceq}_{\check{\Gamma}} \in \check{\Gamma}$ so that

$$(\forall y)\Big(P(y) \Rightarrow (\forall x)\left[(x \in P \wedge x \prec y) \Leftrightarrow x <^{\preceq}_{\Gamma} y \Leftrightarrow x <^{\preceq}_{\check{\Gamma}}\right]\Big).$$

A pointclass Γ is adequate if and only if Γ contains all recursive subsets (of all Polish spaces of the form $\omega^m \times \mathbb{R}^n$), closed under $\wedge$, $\vee$, $\exists^\omega$, $\forall^\omega$, and recursive substitution.

Let Γ be an adequate pointclass. Γ has the prewellordering property if and only every $P \in \Gamma$ has a Γ-prewellordering.

Fact 4.5. *(Boundedness Property) Let Γ be an adequate pointclass closed under $\forall^{\mathbb{R}}$. Suppose there is a $P \in \Gamma$ which is Γ-complete and has a surjective Γ-norm $\varphi : P \to \kappa$. If $A \subseteq P$ is in $\check{\Gamma}$, then there is a $\delta < \kappa$ so that $\varphi[A] \subseteq \delta$.*

Fact 4.6. *(Moschovakis) ([13] Lemma 2.13 and Lemma 2.16) Let Γ be an adequate pointclass closed under $\forall^{\mathbb{R}}$. Suppose there is a Γ-complete set P and a Γ-norm φ on P. Then the length of φ is $\delta(\Gamma)$ and $\delta(\Gamma)$ is a regular cardinal.*

The following is the everywhere wellordered club uniformization for ω_1:

Fact 4.7. *([2] Fact 4.8) Suppose $R \subseteq \omega_1 \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed in the sense that for all $\alpha \in \omega_1$ and $C, D \in \mathsf{club}_{\omega_1}$, if $C \subseteq D$ and $R(\alpha, D)$, then $R(\alpha, C)$. Then there is a $\Lambda : \text{dom}(R) \to \mathsf{club}_{\omega_1}$ so that for all $\alpha \in \text{dom}(R)$, $R(\alpha, \Lambda(\alpha))$.*

This section will generalize this result for other cardinals κ.

Definition 4.8. Let Γ be a nonselfdual (boldface) adequate pointclass closed under $\forall^{\mathbb{R}}$. Let $\kappa = \delta(\Gamma)$. Let $P \in \Gamma$ be Γ-complete. Let $\preceq$ be a Γ-norm and $\varphi : P \to \kappa$ be the associated surjective norm coming from Fact 4.6. Let $\mathsf{clubcode}^\varphi_\kappa$ be the collection of Player 2 strategies τ with the property that

$$(\forall w)[w \in P \Rightarrow (\Xi_\tau(w) \in P \wedge \varphi(\Xi^2_\tau(w)) > \varphi(w))].$$

For $\zeta < \kappa$, let $Q^\varphi_\zeta = \{a \in P : \varphi(a) = \zeta\}$ and $Q^\varphi_{<\zeta} = \{a \in P : \varphi(a) < \zeta\}$. For each $\tau \in \mathsf{clubcode}^\varphi_\kappa$, let

$$\mathfrak{C}^{\varphi,\kappa}_\tau = \{\eta \in \kappa : (\forall \xi < \eta)(\Xi^2_\tau[Q^\varphi_\xi] \subseteq Q^\varphi_{<\eta})\}.$$

Fact 4.9. *In the setting of Definition 4.8, $\mathfrak{C}^{\varphi,\kappa}_\tau$ is a club subset of κ.*

Proof. For $\xi < \kappa$, define $\Phi(\xi) = \sup\{\varphi(\Xi^2_\tau(a)) + 1 : a \in Q^\varphi_{<\xi+1}\}$. Note that $Q^\varphi_{<\xi+1} \in \Delta$ since φ is a Γ norm. Observe that since Ξ^2_τ is Lipschitz continuous, $\Xi^2_\tau[Q^\varphi_{<\xi+1}]$ is $\exists^\mathbb{R}\Delta \subseteq \check{\Gamma}$ since $\check{\Gamma}$ is closed under $\exists^\mathbb{R}$. Since $\tau \in \mathsf{clubcode}^\varphi_\kappa$, $\Xi^2_\tau[Q^\varphi_{<\xi+1}] \subseteq P$. By the boundedness principle (Fact 4.5), $\Phi(\xi) < \kappa$.

Hence it has been shown that $\Phi : \kappa \to \kappa$ is well defined. Also since $\tau \in \mathsf{clubcode}^\varphi_\kappa$, for all $\xi < \kappa$, $\Phi(\xi) > \xi$. It is clear from the definition of Φ that if $\xi_0 < \xi_1$, then $\Phi(\xi_0) \leq \Phi(\xi_1)$. Note that $\mathfrak{C}^{\varphi,\kappa}_\tau = \{\eta < \kappa : (\forall \xi < \eta)(\Phi(\xi) \leq \eta)\}$. To see this, observe that $\mathfrak{C}^{\varphi,\kappa}_\tau$ consists entirely of limit ordinals. This is because $\tau \in \mathsf{clubcode}^\varphi_\kappa$ and if $\eta = \xi + 1$ and $w \in P$ is such that $\varphi(w) = \xi$, then $\varphi(\Xi^2_\tau(w)) \geq \xi + 1 = \eta$ and hence $\eta \notin \mathfrak{C}^{\varphi,\kappa}_\tau$. Suppose $\eta \in \mathfrak{C}^{\varphi,\kappa}_\tau$, $\xi < \eta$, and $a \in Q^\varphi_{<\xi+1}$. Then $\varphi(\Xi^2_\tau(a)) < \eta$. Since η is a limit ordinal, $\varphi(\Xi^2_\tau(a)) + 1 < \eta$. Since $\xi < \eta$ and $a \in Q^\varphi_{<\xi+1}$ were arbitrary, $\Phi(\xi) \leq \eta$. Conversely, suppose η is such that for all $\xi < \eta$, $\Phi(\xi) \leq \eta$. Fix a $\xi < \eta$ and $a \in Q^\varphi_\xi$. Since $a \in Q^\varphi_{<\xi+1}$, $\varphi(\Xi^2_\tau(a)) < \Phi(\xi) \leq \eta$. Thus $\eta \in \mathfrak{C}^{\varphi,\kappa}_\tau$.

Suppose $\eta \in \kappa$ is a limit point of $\mathfrak{C}^{\varphi,\kappa}_\tau$. Suppose $\xi < \eta$. Then there is a $\zeta \in \mathfrak{C}^{\varphi,\kappa}_\tau$ with $\xi < \zeta < \eta$. Then $\Phi(\xi) \leq \zeta < \eta$. Thus $\eta \in \mathfrak{C}^{\varphi,\kappa}_\tau$. It has been shown that $\mathfrak{C}^{\varphi,\kappa}_\tau$ is closed.

For each $\zeta < \kappa$, let $\eta_0 = \zeta$. If η_n has been defined, let $\eta_{n+1} = \Phi(\eta_n)$. Let $\eta_\omega = \sup\{\eta_n : n \in \omega\}$. Suppose $\xi < \eta_\omega$. There some $n \in \omega$, so that $\xi < \eta_n$. Thus $\Phi(\xi) \leq \Phi(\eta_n) = \eta_{n+1} < \eta_\omega$. Thus $\zeta < \eta_\omega$ and $\eta_\omega \in \mathfrak{C}^{\varphi,\kappa}_\tau$. This shows that $\mathfrak{C}^{\varphi,\kappa}_\tau$ is unbounded. ☐

Fact 4.10. *Assume the setting of Definition 4.8. Let $C \subseteq \kappa$ be a club. Then there is a $\tau \in \mathsf{clubcode}^\varphi_\kappa$ so that $\mathfrak{C}^{\varphi,\kappa}_\tau \subseteq C$.*

Proof. Let $C \subseteq \kappa$ be a club. Define the game G_C so that Player 1 and Player 2 separately produce reals v and w (respectively), one integer per turn. Player 2 wins G_C if and only if $v \in P$ implies $w \in P$, $\varphi(w) > \varphi(v)$, and $\varphi(w) \in C$. By AD, one of the two players has a winning strategy.

The claim is that Player 2 has a winning strategy. Suppose otherwise that Player 1 has a winning strategy σ. For all $w \in \mathbb{R}$, $\Xi^1_\sigma(w) \in P$ for otherwise Player 2 would win. $\Xi^1_\sigma[\mathbb{R}]$ is a $\mathbf{\Sigma}^1_1$ subset of P. By the boundedness principle of Fact 4.5, there is a $\delta < \kappa$ so that for all $v \in \Xi^1_\sigma[\mathbb{R}]$, $\varphi(v) < \delta$. Since φ is a Γ-norm on a Γ-complete set, there is a $w \in P$ so that $\varphi(w) > \delta$ and $\varphi(w) \in C$. If Player 2 plays w, then Player 2 wins which contradicts that σ is a Player 1 winning strategy.

Thus Player 2 has a winning strategy τ. By the winning condition for Player 2, it is clear that $\tau \in \mathsf{clubcode}^\varphi_\kappa$. Now suppose that $\eta \in \mathfrak{C}^{\varphi,\kappa}_\tau$. For any $\xi < \eta$, pick $v \in Q^\varphi_\xi$. Then $\varphi(\Xi^2_\tau(v)) \in C$ and $\xi < \varphi(\Xi^2_\tau(v)) < \eta$. Since ξ was arbitrary, one has shown that η is a limit point of C. Since C is a club, one must have $\eta \in C$. Since $\eta \in \mathfrak{C}^{\varphi,\kappa}_\tau$ was arbitrary, $\mathfrak{C}^{\varphi,\kappa}_\tau \subseteq C$. ☐

Fact 4.11. *Assume the setting of Definition 4.8. $\mathsf{clubcode}^\varphi_\kappa$ is $\forall^\mathbb{R}\check{\Gamma}$. There is a Γ relation $\mathsf{inClub}^\varphi_\kappa(\tau, x)$ with the property that whenever $\tau \in \mathsf{clubcode}^\varphi_\kappa$, $\mathsf{inClub}^\varphi_\kappa(\tau, x)$ if and only if $x \in P$ and $\varphi(x) \in \mathfrak{C}^{\varphi,\kappa}_\tau$. There is a $\forall^\mathbb{R}\check{\Gamma}$ relation $\mathsf{clubSubset}^\varphi_\kappa(\tau_0, \tau_1)$ with the property that whenever $\tau_0, \tau_1 \in \mathsf{clubcode}^\varphi_\kappa$, $\mathsf{clubSubset}^\varphi_\kappa(\tau_0, \tau_1)$ if and only if $\mathfrak{C}^{\varphi,\kappa}_{\tau_0} \subseteq \mathfrak{C}^{\varphi,\kappa}_{\tau_1}$.*

Proof. Let $<^\varphi_\Gamma$ and $<^\varphi_{\check{\Gamma}}$ be the relations from Definition 4.4 which witness that $\varphi : P \to \kappa$ is a Γ-norm. Note that $\tau \in \mathsf{clubcode}^\varphi_\kappa$ if and only if

$$(\forall w)[w \in P \Rightarrow (\Xi^2_\tau(w) \in P \wedge w <^\varphi_{\check{\Gamma}} \Xi^2_\tau(w))]$$

The above expression is $\forall^\mathbb{R}\check{\Gamma}$.

Define $\mathsf{inClub}^\varphi_\kappa(t, x)$ by

$$x \in P \wedge (\forall w)(w <_{\check{\Gamma}} x \Rightarrow \Xi^2_\tau(w) <_\Gamma x)$$

This relation is Γ. If $\tau \in \mathsf{clubcode}^\varphi_\kappa$, then from the definition of $\mathfrak{C}^{\varphi,\kappa}_\tau$, one has $\mathsf{inClub}^\varphi_\kappa(\tau, x)$ if and only if $\varphi(x) \in \mathfrak{C}^{\varphi,\kappa}_\tau$.

Define $\mathsf{clubSubset}^\varphi_\kappa$ by $\mathsf{clubSubset}^\varphi_\kappa(t_0, t_1)$ if and only if

$$(\forall w)[(w \in P \wedge \mathsf{inClub}^\varphi_\kappa(t_0, w)) \Rightarrow \mathsf{inClub}^\varphi_\kappa(t_1, w)].$$

Therefore $\mathsf{clubSubset}^\varphi_\kappa$ is $\forall^\mathbb{R}\check{\Gamma}$. ☐

Fact 4.12. *Assume the setting of Definition 4.8. Suppose $A \subseteq \mathsf{clubcode}^\varphi_\kappa$ is $\check{\Gamma}$. Then uniformly from A, one can produce a club $C \subseteq \kappa$ so that for all $\tau \in A$, $C \subseteq \mathfrak{C}^{\varphi,\kappa}_\tau$. (This means there is a function $\Psi : \mathscr{P}(\mathbb{R}) \to \mathsf{club}_\kappa$ with the property that whenenever $A \in \check{\Gamma}$ and $A \subseteq \mathsf{clubcode}^\varphi_\kappa$, $\Psi(A) \subseteq \mathfrak{C}^{\varphi,\kappa}_\tau$ for all $\tau \in A$.)*

Proof. For each $\xi < \kappa$, let $B_\xi = \{b : (\exists a)(\exists \tau)(a \in Q^{\varphi}_{<\xi+1} \wedge \tau \in A \wedge b = \Xi^2_\tau(a))\}$. Since φ is a Γ-norm and Γ is closed under $\forall^{\mathbb{R}}$, $B_\xi \in \check{\Gamma}$. Let $\Phi : \kappa \to \kappa$ be defined by $\Phi(\xi) = \sup\{\varphi(b) + 1 : b \in B_\xi\}$. Again by Fact 4.5, $\Phi(\xi) < \kappa$ for all $\xi < \kappa$. For all ξ, $\xi < \Phi(\xi)$ since $\tau \in \mathsf{clubcode}^{\varphi}_{\kappa}$, and for all $\xi_0 \leq \xi_1 < \kappa$, $\Phi(\xi_0) \leq \Phi(\xi_1)$. Let $C = \{\eta < \kappa : (\forall \xi < \eta)(\Phi(\xi) \leq \eta)\}$. Note C is a club subset of κ.

Now fix a $\tau \in A$. Let $\Phi^\tau(\xi) = \sup\{\varphi(b) + 1 : b \in \Xi^2_\tau[Q^{\varphi}_{<\xi+1}]\}$. Note that $\Phi^\tau(\xi) \leq \Phi(\xi)$ for all $\xi < \kappa$. As shown in Fact 4.9, $\mathfrak{C}^{\varphi,\kappa}_{\tau} = \{\eta < \kappa : (\forall \xi < \eta)(\Phi^\tau(\xi) \leq \eta)\}$. Let $\eta \in C$. Let $\xi < \eta$. Then $\Phi^\tau(\xi) \leq \Phi(\xi) \leq \eta$ since $\eta \in C$. Thus $\eta \in \mathfrak{C}^{\varphi,\kappa}_{\tau}$. It has been shown that $C \subseteq \mathfrak{C}^{\varphi,\kappa}_{\tau}$ for all $\tau \in A$. It is clear from the construction of C that it depends only on A. ☐

Definition 4.13. Suppose $\dot{A}$ and $\dot{B}$ are two distinct unary predicate symbols (intended to represent two subsets of $\mathbb{R}$). $\Sigma^1_1(\dot{A}, \dot{B})$ is the smallest (lightface) pointclass containing Σ^1_1, $\dot{A}$, $\mathbb{R} \setminus \dot{A}$, $\dot{B}$, and $\mathbb{R} \setminus \dot{B}$, and closed under $\wedge$, $\vee$, $\exists^\omega$, $\forall^\omega$, and $\exists^{\mathbb{R}}$. $\Sigma^1_1(\dot{A}, \dot{B})$ has ω-universal sets which are defined uniformly from $\dot{A}$ and $\dot{B}$. (See [25] Section 7D for more information about this pointclass.)

Let $\langle U^{(n)}(\dot{A}, \dot{B}) : n \in \omega \setminus \{0\}\rangle$ be a sequence of good universal sets for $\boldsymbol{\Sigma}^1_1(\dot{A}, \dot{B})$ which is defined uniformly from $\dot{A}$ and $\dot{B}$. That is, for each $n \in \omega \setminus \{0\}$, $U^{(n)}(\dot{A}, \dot{B}) \subseteq \mathbb{R}^{n+1}$ is $\Sigma^1_1(\dot{A}, \dot{B})$ and is universal for the $\boldsymbol{\Sigma}^1_1(\dot{A}, \dot{B})$ subsets of $\mathbb{R}^n$. There are continuous functions $s_{m,n} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ so that

$$U^{(n)}(\dot{A}, \dot{B})(e, x_0, ..., x_{m-1}, x_m, ..., x_{n-1}) = U^{(n-m)}(\dot{A}, \dot{B})(s_{m,n}(e, x_0, ..., x_{m-1}), x_m, ..., x_{n-1}).$$

This sequence also satisfies the recursion theorem: For all $D \subseteq \mathbb{R} \times \mathbb{R}^n$ which belong to $\boldsymbol{\Sigma}^1_1(\dot{A}, \dot{B})$, there exists an $e \in \mathbb{R}$ so that

$$D(e, x_0, ..., x_{n-1}) = U^{(n)}(\dot{A}, \dot{B})(e, x_0, ..., x_{n-1}).$$

For $e \in \mathbb{R}$, let $U^{(2)}_e(\dot{A}, \dot{B}) = \{(x_0, x_1) : U^{(2)}(\dot{A}, \dot{B})(e, x_0, x_1)\}$.

The good universal sequences are the necessary ingredients for proving variations of the Moschovakis coding lemma. One will use the presentation of the uniform coding lemma found in [22].

Fact 4.14. *(Moschovakis; [22] Theorem 3.4) (Uniform Coding Lemma) Let $\preceq$ be a prewellordering on P with associated norm $\varphi : P \to \mathrm{ON}$. For each $a \in P$, let $Q^{\varphi}_{<a} = Q^{\preceq}_{<a} = \{x \in P : \varphi(x) < \varphi(a)\} = \{x \in P : x \preceq a \wedge \neg(a \preceq x)\}$ and $Q^{\varphi}_{a} = Q^{\preceq}_{a} = \{x \in P : \varphi(x) = \varphi(a)\} = \{x \in P : x \preceq a \wedge a \preceq x\}$. Let $Z \subseteq P \times \mathbb{R}$. Then there is an e so that*

(1) For all $a \in P$, $U^{(2)}_e(Q^{\varphi}_{<a}, Q^{\varphi}_{a}) \subseteq Z \cap (Q^{\varphi}_{a} \times \mathbb{R})$.
(2) For all $a \in P$, $U^{(2)}_e(Q^{\varphi}_{<a}, Q^{\varphi}_{a}) \neq \emptyset$ if and only if $Z \cap (Q^{\varphi}_{a} \times \mathbb{R})$.

Note that for any $a_0, a_1 \in P$ with $\varphi(a_0) = \varphi(a_1)$, one has that $Q^{\varphi}_{<a_0} = Q^{\varphi}_{<a_1}$ and $Q^{\varphi}_{a_0} = Q^{\varphi}_{a_1}$. Thus if κ is such that $\varphi : P \to \kappa$ is onto, then for each $\alpha < \kappa$, one will write $Q^{\varphi}_{<\alpha}$ and Q^{φ}_{α} for $Q^{\varphi}_{<a}$ and Q^{φ}_{a}, respectively, for any $a \in P$ such that $\varphi(a) = \alpha$.

Theorem 4.15. *Let Γ be an adequate nonselfdual boldface pointclass closed under $\forall^{\mathbb{R}}$ with the prewellordering property. Let $\kappa = \delta(\Gamma)$. For all relations $R \subseteq \kappa \times \mathsf{club}_\kappa$ which are $\subseteq$-downward closed, there exists a function $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_\kappa$ so that for all $\alpha \in \mathrm{dom}(R)$, $R(\alpha, \Lambda(\alpha))$.*

Proof. Let P be a Γ-complete set and $\varphi : P \to \kappa$ be a surjective Γ-norm on P. Let $Z \subseteq P \times \mathbb{R}$ be defined by $Z(a, z)$ if and only if $z \in \mathsf{clubcode}^{\varphi}_{\kappa} \wedge R(\varphi(a), \mathfrak{C}^{\varphi,\kappa}_{z})$. By the uniform coding lemma (Fact 4.14), there is some $e \in \mathbb{R}$ so that for all $\xi < \kappa$, $U^{(2)}_e(Q^{\varphi}_{<\xi}, Q^{\varphi}_{\xi}) \subseteq Z \cap (Q^{\varphi}_{\xi} \times \mathbb{R})$ and $U^{(2)}_e(Q^{\varphi}_{<\xi}, Q^{\varphi}_{\xi}) \neq \emptyset$ if and only if $Z \cap (Q^{\varphi}_{\xi} \times \mathbb{R}) \neq \emptyset$. Since φ is a Γ-norm, for all $\xi < \kappa$, Q^{φ}_{ξ}, $\mathbb{R} \setminus Q^{\varphi}_{\xi}$, $Q^{\varphi}_{<\xi}$ and $\mathbb{R} \setminus Q^{\varphi}_{<\xi}$ all belong to Δ. Since $\check{\Gamma}$ is closed under $\exists^{\mathbb{R}}$, for all $\xi < \kappa$, $\Sigma^1_1(Q^{\varphi}_{<\xi}, Q^{\varphi}_{\xi}) \subseteq \check{\Gamma}$ so in particular, $U^{(2)}_e(Q^{\varphi}_{<\xi}, Q^{\varphi}_{\xi}) \in \check{\Gamma}$. Let $\pi_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection onto the second coordinate. Since $\pi_2(Z) \subseteq \mathsf{clubcode}^{\varphi}_{\kappa}$, one has that for all $\xi < \kappa$, $\pi_2[U^{(2)}_e(Q^{\varphi}_{<\xi}, Q^{\varphi}_{\xi})]$ is a $\check{\Gamma}$ subset of $\mathsf{clubcode}^{\varphi}_{\kappa}$. By Fact 4.12, for each $\xi < \kappa$, there is a club $C_\xi \subseteq \kappa$ which is defined uniformly from $\pi_2[U^{(2)}_e(Q^{\varphi}_{<\xi}, Q^{\varphi}_{\xi})]$ with the property that for all $z \in \pi_2[U^{(2)}_e(Q^{\varphi}_{<\xi}, Q^{\varphi}_{\xi})]$, $C_\xi \subseteq \mathfrak{C}^{\varphi,\kappa}_{z}$. If $\xi \in \mathrm{dom}(R)$, then there is a $z \in \pi_2[U^{(2)}_e(Q^{\varphi}_{<\xi}, Q^{\varphi}_{\xi})]$ which means that $R(\xi, \mathfrak{C}^{\varphi,\kappa}_{z})$. Since R is $\subseteq$-downward closed and $C_\xi \subseteq \mathfrak{C}^{\varphi,\kappa}_{z}$, one has $R(\xi, C_\xi)$ holds. This shows that the function $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_\kappa$ defined by $\Lambda(\xi) = C_\xi$ is a uniformization. ☐

Fact 4.16. *Let Γ be an adequate nonselfdual boldface pointclass closed under $\forall^{\mathbb{R}}$ with the prewellordering property. Let $\kappa = \delta(\Gamma)$. Then μ^κ_κ is a κ-complete filter.*

Proof. Suppose $\lambda < \kappa$ and $\langle A_\alpha : \alpha < \lambda \rangle$ is a sequence in μ^κ_κ. Define $R \subseteq \kappa \times \mathsf{club}_\kappa$ by $R(\alpha, C)$ if and only if $\alpha < \Lambda$ and $[C]^\kappa_* \subseteq A_\alpha$. Note that $\mathrm{dom}(R) = \lambda$. By Fact 4.15, let $\Lambda : \lambda \to \mathsf{club}_\kappa$ be such that for all $\alpha < \lambda$, $R(\alpha, \Lambda(\alpha))$. Let $C = \bigcap_{\alpha<\lambda} \Lambda(\alpha)$ which is a club subset of κ. Then $[C]^\kappa_* \subseteq \bigcap_{\alpha<\lambda} A_\alpha$. $\bigcap_{\alpha<\lambda} A_\alpha \in \mu^\kappa_\kappa$. ☐

5. Good Coding Systems and Families

Definition 5.1. (Martin) Let $\kappa \in \mathrm{ON}$ and $\epsilon \leq \kappa$. $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \kappa)$ is a good coding system for ${}^\epsilon\kappa$ if and only if

- κ is a regular cardinal.
- Γ is a boldface pointclass closed under $\forall^{\mathbb{R}}$. Let $\Delta = \Gamma \cap \check{\Gamma}$.
- $\mathsf{decode} : \mathbb{R} \to \mathscr{P}(\epsilon \times \kappa)$ and has the property that for all $f \in {}^\epsilon\kappa$, there is an $x \in \mathbb{R}$ so that $\mathsf{decode}(x) = f$.
- For all $\beta < \epsilon$ and $\gamma < \kappa$, $\mathsf{GC}_{\beta,\gamma} \in \Delta$ and for all $x \in \mathbb{R}$, $x \in \mathsf{GC}_{\beta,\gamma}$ if and only if
$$\mathsf{decode}(x)(\beta, \gamma) \wedge (\forall \xi < \kappa)(\mathsf{decode}(x)(\beta, \xi) \Rightarrow \xi = \gamma).$$
- For each $\beta < \epsilon$, let $\mathsf{GC}_\beta = \bigcup_{\gamma<\kappa} \mathsf{GC}_{\beta,\gamma}$. Suppose $A \in \exists^{\mathbb{R}}\Delta$ and $A \subseteq \mathsf{GC}_\beta$ for some $\beta < \epsilon$. Then there exists a $\delta < \kappa$ so that $A \subseteq \bigcup_{\alpha<\delta} \mathsf{GC}_{\beta,\gamma}$.

Let $\mathsf{GC} = \bigcap_{\beta<\epsilon} \mathsf{GC}_\beta$. GC is called the collection of good codes for ${}^\epsilon\kappa$. Note $x \in \mathsf{GC}$ if and only if $\mathsf{decode}(x) \in {}^\epsilon\kappa$, i.e. is a total function from ϵ into κ.

See [2] Section 3 or [13] Section 2.6 for more information. In [2], Γ is closed under $\exists^{\mathbb{R}}$. Here for notational consistency, Definition 5.1 will assume Γ is closed under $\forall^{\mathbb{R}}$ which amounts to replacing the pointclass with its dual.

Definition 5.2. Let κ be a regular cardinal and $\epsilon \leq \kappa$. κ is said to be ϵ-reasonable if and only if there exists a good coding system for ${}^\epsilon\kappa$.

Fact 5.3. *Suppose κ is regular cardinal and $\epsilon \leq \kappa$. Suppose $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \kappa)$ is a good coding system for ${}^\epsilon\kappa$. Then Γ is a nonselfdual pointclass closed under $\forall^{\mathbb{R}}$, countable intersection, countable unions, and has the prewellordering property. Δ is closed under less than κ-unions. Also $\kappa = \delta(\Gamma)$.*

Proof. Γ is nonselfdual since otherwise the last boundedness condition of the good coding system will fail. The proof of [13] Theorem 2.34 shows that Δ is closed under less than κ length unions. [13] Remark 2.35 states that Γ is closed under countable union, countable intersection, and has the prewellordering property. Martin ([13] Lemma 2.15) showed that if Γ is a nonselfdual pointclass closed under $\forall^{\mathbb{R}}$, $\wedge$, $\vee$, and has the prewellordering property, then Δ is closed under less than $\delta(\Gamma)$ length unions. If $\kappa < \delta(\Gamma)$, then $\mathsf{GC}_\beta = \bigcup_{\gamma<\kappa} \mathsf{GC}_{\beta,\gamma}$ belongs to Δ. Then the last boundedness property of the good coding system fails (for $A = \mathsf{GC}_\beta$). Thus $\delta(\Gamma) \leq \kappa$. Suppose $\delta(\Gamma) < \kappa$. Let $P \in \Gamma$ be a Γ-complete set and φ be a Γ-norm on P of length necessarily $\delta(\Gamma)$ by Fact 4.6. Since Δ is closed under less than κ-unions, if $\delta(\Gamma) < \kappa$, then $P = \bigcup_{\alpha<\delta(\Gamma)} P_\alpha$, where each $P_\alpha = \{x \in P : \varphi(x) = \alpha\}$ is a set in Δ. This would imply $P \in \Delta$. Contradiction. This shows that $\kappa = \delta(\Gamma)$. ☐

The next result of Martin shows that existence of a good cooding system for ${}^{\omega\cdot\epsilon}\kappa$ implies the partition relation $\kappa \to_* (\kappa)^\epsilon_2$. The proof is similar to the argument for Theorem 5.9 provided below.

Fact 5.4. *(Martin, [2] Theorem 3.7 or [13] Theorem 2.34) Let κ be a regular cardinal and $\epsilon \leq \kappa$. If κ is $\omega \cdot \epsilon$-reasonable, then $\kappa \to_* (\kappa)^\epsilon_2$ (and in fact, $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$).*

Fact 5.5. *Let κ be a regular cardinal and $\epsilon \leq \kappa$. If κ is $\omega \cdot \epsilon$-reasonable, then μ^κ_ϵ is a κ-complete measure on $[\kappa]^\epsilon_*$.*

Proof. This follows from Fact 5.4 and Fact 4.16. (Also this follows directly from $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$.) ☐

Theorem 5.6. *Suppose κ is κ-reasonable and $R \subseteq \kappa \times \mathsf{club}_\kappa$ is $\subseteq$-downward closed in the club_κ-coordinate. Then there is a club $C \subseteq \kappa$ so that for all $\alpha \in \mathrm{dom}(R)$, $R(\alpha, C \setminus (\alpha + 1))$.*

Proof. This follows from Fact 5.3, Fact 5.4, Theorem 4.15, and Theorem 2.23. (This also follows from $\kappa \to_* (\kappa)^\kappa_{<\kappa}$ and Fact 2.24.) $\square$

Definition 5.7. For $x \in \mathbb{R}$, let $\mathsf{fail}(x)$ be the least $\beta < \omega \cdot \epsilon$ so that $x \notin \mathsf{GC}_\beta$ if such a β exists; otherwise, one writes $\mathsf{fail}(x) = \infty$ which implies that $x \in \mathsf{GC}$.

Let $\mathcal{S}^1$ be the set of Lipschitz continuous functions $\Xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ so that

$$(\forall y)(\forall z)[\mathsf{fail}(y) \leq \mathsf{fail}(\Xi(y, z)) \wedge (\mathsf{fail}(y) < \infty \Rightarrow \mathsf{fail}(y) < \mathsf{fail}(\Xi(y, z)))].$$

Let $\mathcal{S}^2$ be the set of Lipschitz continuous function $\Xi : \mathbb{R} \to \mathbb{R}$ so that $(\forall x)(\mathsf{fail}(x) \leq \mathsf{fail}(\Xi(x)))$.

Lemma 5.8. *If a Lipschitz continuous function $\Xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{S}^1$, then there is a club $C \subseteq \kappa$ (with $\min(C) > \omega \cdot \epsilon$ if $\epsilon < \kappa$) so that for all $\delta \in C$, $\beta < \delta$, and $\gamma < \delta$*

$$\Xi\left[\left(\bigcap_{\beta' < \beta} \bigcup_{\gamma' < \gamma} \mathsf{GC}_{\beta', \gamma'}\right) \times \mathbb{R}\right] \subseteq \bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < \delta} \mathsf{GC}_{\beta', \gamma'}.$$

If a Lipschitz continuous function $\Xi : \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{S}^2$, then there is a club $C \subseteq \kappa$ (with $\min(C) > \omega \cdot \epsilon$ if $\epsilon < \kappa$) so that for all $\delta \in C$, $\beta < \delta$, and $\gamma < \delta$

$$\Xi\left[\bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < \gamma} \mathsf{GC}_{\beta', \gamma'}\right] \subseteq \bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < \delta} \mathsf{GC}_{\beta', \gamma'}.$$

Proof. Since Δ is closed under wellordered unions and intersections of length less than κ by Fact 5.3, the following set

$$R_{\beta,\gamma} = \bigcap_{\beta' < \beta} \bigcup_{\gamma' < \gamma} \mathsf{GC}_{\beta', \gamma'}$$

belongs to Δ whenever $\beta < \omega \cdot \epsilon$ and $\gamma < \kappa$. Since Ξ is continuous, $\Xi[R_{\beta,\gamma} \times \mathbb{R}]$ belongs to $\exists^{\mathbb{R}}\Delta$. If $(y, z) \in R_{\beta,\gamma} \times \mathbb{R}$, $\mathsf{fail}(y) \geq \beta$. Since $\Xi \in \mathcal{S}^1$, $\mathsf{fail}(\Xi(y, z)) > \beta$. This shows that $\Xi[R_{\beta,\gamma} \times \mathbb{R}] \subseteq \mathsf{GC}_{\beta'}$ for each $\beta' \leq \beta$. By the boundedness property of the good coding system, for each $\beta' \leq \beta < \omega \cdot \epsilon$ and $\gamma < \kappa$, there is a least ordinal $\ell^{\beta,\gamma}_{\beta'} > \gamma$ so that $\Xi[R_{\beta,\gamma} \times \mathbb{R}] \subseteq \bigcup_{\gamma' < \ell^{\beta,\gamma}_{\beta'}} \mathsf{GC}_{\beta',\gamma'}$. Since κ is $\omega \cdot \epsilon$-reasonable, κ is regular and thus $\ell^{\beta,\gamma} = \sup\{\ell^{\beta,\gamma'}_{\beta'} : \beta' \leq \beta \wedge \gamma' \leq \gamma\}$ is less than κ and strictly greater than γ. Also if $\beta_0 < \beta_1$ and $\gamma_0 < \gamma_1$, then $\ell^{\beta_0,\gamma_0} \leq \ell^{\beta_1,\gamma_1}$.

If $\epsilon = \kappa$, then let $C = \{\delta < \kappa : (\forall \beta, \gamma < \delta)(\ell^{\beta,\gamma} < \delta)\}$. C is clearly closed. Suppose $\alpha < \kappa$. Let $\alpha_0 = \alpha$, $\alpha_{n+1} = \ell^{\alpha_n,\alpha_n}$, and $\alpha_\infty = \sup\{\alpha_n : n \in \omega\}$. Note that for all $n \in \omega$, $\alpha_n < \alpha_{n+1}$ and thus $\alpha_n < \alpha_\infty$. Now suppose $\beta, \gamma < \alpha_\infty$. There is some n so that $\beta, \gamma < \alpha_n$. Then $\ell^{\beta,\gamma} \leq \ell^{\alpha_n,\alpha_n} = \alpha_{n+1} < \alpha_\infty$. Thus $\alpha_\infty \in C$ and $\alpha_\infty > \alpha$. C is a club subset of κ in the case $\epsilon = \kappa$.

If $\epsilon < \kappa$, let $C = \{\delta < \kappa : \delta > \omega \cdot \epsilon \wedge (\forall \beta < \omega \cdot \epsilon)(\forall \gamma < \delta)(\ell^{\beta,\gamma} < \delta)\}$. Suppose $\alpha < \kappa$. Let $\alpha_0 = \alpha$, $\alpha_{n+1} = \sup_{\beta < \omega \cdot \epsilon} \ell^{\beta,\alpha_n}$, and $\alpha_\infty = \sup\{\alpha_n : n \in \omega\}$. A similar argument shows that $\alpha < \alpha_\infty$ and $\alpha_\infty \in C$. Again C is a club in the case $\epsilon < \kappa$.

Fix a $\delta \in C$, $\beta < \delta$, and $\gamma < \delta$. Suppose $(y, z) \in R_{\beta,\gamma} \times \mathbb{R}$. Let $\beta' \leq \beta$. Then

$$\Xi(y, z) \in \bigcup_{\gamma' < \ell^{\beta,\gamma}_{\beta'}} \mathsf{GC}_{\beta',\gamma'} \subseteq \bigcup_{\gamma' < \ell^{\beta,\gamma}} \mathsf{GC}_{\beta',\gamma'} \subseteq \bigcup_{\gamma' < \delta} \mathsf{GC}_{\beta',\gamma'}.$$

Since $\beta' \leq \beta$ was arbitrary,

$$\Xi(y, z) \in \bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < \delta} \mathsf{GC}_{\beta',\gamma'}.$$

This show the argument for $\Xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in $\mathcal{S}^1$. The argument for $\Xi : \mathbb{R} \to \mathbb{R}$ in $\mathcal{S}^2$ is very similar. This completes the proof of the lemma. $\square$

Theorem 5.9. *(Almost Everywhere Good Code Uniformization) Suppose κ is $\omega \cdot \epsilon$-reasonable for some $\epsilon \leq \kappa$. Let $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta \leq \omega \cdot \epsilon, \gamma < \kappa)$ be a good coding system for ${}^{\omega \cdot \epsilon}\kappa$. For any $R \subseteq [\kappa]^\epsilon_* \times \mathbb{R}$, there exists a club $C \subseteq \kappa$ and a Lipschitz function $\Xi : \mathbb{R} \to \mathbb{R}$ so that for all $x \in \mathsf{GC}$ with $\mathsf{decode}(x) \in [C]^{\omega \cdot \epsilon}$ and $\mathsf{block}(\mathsf{decode}(x)) \in \mathrm{dom}(R)$, $R(\mathsf{block}(\mathsf{decode}(x)), \Xi(x))$.*

Proof. If $f, g : \omega \cdot \epsilon \to \kappa$, let $\mathsf{joint}(f,g) : \epsilon \to \kappa$ be defined by $\mathsf{joint}(f,g)(\alpha) = \sup\{f(\omega \cdot \alpha + n), g(\omega \cdot \alpha + n) : n \in \omega\}$. Consider the game

$$G \quad \begin{array}{lcccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & & x \\ \mathrm{II} & & y(0), z(0) & & y(1), z(1) & & y(2), z(2) & & y, z \end{array}$$

Player 2 wins G if and only if the disjunction of the following holds

(1) $\mathsf{fail}(x) < \mathsf{fail}(y) \vee \mathsf{fail}(x) = \mathsf{fail}(y) < \infty$

(2) $(\mathsf{fail}(x) = \mathsf{fail}(y) = \infty) \wedge [\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(y)) \in \mathrm{dom}(R) \Rightarrow R(\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(y)), z)]$

Intuitively, in the game G, whichever player fails first loses and if both players fail at the same time (at some ordinal below κ), this tie is said to be a win for Player 2. If both players do not fail, then Player 2 wins if and only if whenever the joint object $\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(y))$ belongs to $\mathrm{dom}(R)$, $R(\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(y)), z)$ holds.

Therefore, Player 1 winning condition is the conjunction of the following:

(1) $\mathsf{fail}(y) < \mathsf{fail}(x) \vee \mathsf{fail}(x) = \mathsf{fail}(y) = \infty$

(2) $\mathsf{fail}(x) = \mathsf{fail}(y) = \infty \Rightarrow [\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(y)) \in \mathrm{dom}(R) \wedge \neg R(\mathsf{joint}(\mathsf{decode}(x), \mathsf{decode}(y)), z)]$

The claim is that Player 2 wins. Suppose otherwise that Player 1 wins with winning strategy σ. Let $\Xi^1_\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the Lipschitz continuous function associated to σ as in Definition 4.1. From the Player 1 winning condition, one has that $\Xi^1_\sigma \in \mathcal{S}^1$. Let $C \subseteq \kappa$ be the club from Lemma 5.8 which one may assume consists entirely of limit ordinals. Let $D \subseteq C$ be the club of limit points of C. Pick any $f \in [D]^\epsilon_*$. If $f \in \mathrm{dom}(R)$, pick a z so that $R(f, z)$. If $f \notin \mathrm{dom}(R)$, then let $z \in \mathbb{R}$ be arbitrary. Since f has the correct type, it is discontinuous and there is a function $F : \epsilon \times \omega \to \kappa$ witnessing that f has uniform cofinality ω. Let $g : \omega \cdot \epsilon \to C$ be defined as follows: For each $\alpha < \epsilon$, let $g(\omega \cdot \alpha)$ be the least element of C greater than $\max\{\sup(f \restriction \alpha), F(\alpha, 0)\}$. If $g(\omega \cdot \alpha + n)$ has been defined, then let $g(\omega \cdot \alpha + n + 1)$ be the least element of C greater than $\max\{g(\omega \cdot \alpha + n), F(\alpha, n+1)\}$. Since f is discontinuous, one can check that g is increasing, discontinuous, and $\mathsf{block}(g) = f$. Pick $y \in \mathsf{GC}$ so that $\mathsf{decode}(y) = g$. Since $\mathsf{fail}(y) = \infty$, $\mathsf{fail}(\Xi^1_\sigma(y, z)) = \infty$.

For each $\beta < \omega \cdot \epsilon$, $\sup(g \restriction \beta) < g(\beta)$ by the discontinuity of g and therefore $\sup(g \restriction \beta) + 1 < g(\beta)$ since C consists entirely of limit ordinals. Since $g(\beta) \in C$, one has that

$$\Xi^1_\sigma(y, z) \in \Xi\left[\left(\bigcap_{\beta' < \beta} \bigcup_{\gamma' < \sup(g \restriction \beta) + 1} \mathsf{GC}_{\beta', \gamma'}\right) \times \mathbb{R}\right] \subseteq \bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < g(\beta)} \mathsf{GC}_{\beta', \gamma'}.$$

Thus $\mathsf{decode}(\Xi^1_\sigma(y, z))(\beta) < g(\beta)$. Hence for each $\alpha < \epsilon$,

$$\sup\{\mathsf{decode}(\Xi^1_\sigma(y, z))(\omega \cdot \alpha + n), \mathsf{decode}(y)(\omega \cdot \alpha + n) : n \in \omega\} = \sup\{g(\omega \cdot \alpha + n) : n \in \omega\} = \mathsf{block}(g)(\alpha) = f(\alpha).$$

Thus $\mathsf{joint}(\mathsf{decode}(\Xi^1_\sigma(y, z)), \mathsf{decode}(y)) = \mathsf{block}(g) = f$. If $f \notin \mathrm{dom}(R)$, then Player 2 wins. If $f \in \mathrm{dom}(R)$, then z was chosen so that $R(f, z)$. Thus Player 2 wins as well. This contradicts σ being a Player 1 winning strategy.

Thus it has been shown that Player 2 has a winning strategy τ in the game G. Let $\Xi^2_\tau : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the associated Lipschitz continuous function. Let $\pi_1, \pi_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection onto the first and second coordinate, respectively. Let $\Xi' = \pi_1 \circ \Xi^2_\tau$ and let $\Xi = \pi_2 \circ \Xi^2_\tau$. From Player 2's winning condition, one has that $\Xi' \in \mathcal{S}^2$. Let $C \subseteq \kappa$ be the club obtained from Lemma 5.8 applied to Ξ'. Let $D \subseteq C$ be the club of limit points of C. Suppose $x \in \mathsf{GC}$ is such that $\mathsf{decode}(x) \in [C]^{\omega \cdot \epsilon}$ and $\mathsf{block}(g) \in [D]^\epsilon_* \cap \mathrm{dom}(R)$. Denote $g = \mathsf{decode}(x)$ and $f = \mathsf{block}(g) = \mathsf{block}(\mathsf{decode}(x))$. Since $\mathsf{fail}(x) = \infty$, one has that $\mathsf{fail}(\Xi'(x)) = \mathsf{fail}(\pi_1(\Xi^2_\tau(x))) = \infty$.

Let $\beta < \omega \cdot \epsilon$. Note that $g(\beta) + 1 < g(\beta + 1) \in C$. Thus

$$\Xi'(x) \in \Xi'\left[\bigcap_{\beta' \leq \beta} \bigcup_{\gamma' < g(\beta+1)} \mathsf{GC}_{\beta', \gamma'}\right] \subseteq \bigcup_{\beta' \leq \beta} \bigcup_{\gamma' < g(\beta+1)} \mathsf{GC}_{\beta', \gamma'}.$$

Thus $\mathsf{decode}(\Xi'(x))(\beta) < g(\beta + 1)$. So for all $\alpha < \epsilon$,

$$\sup\{\mathsf{decode}(x)(\omega \cdot \alpha + n), \mathsf{decode}(\Xi'(x))(\omega \cdot \alpha + n) : n \in \omega\} = \sup\{g(\omega \cdot \alpha + n) : n \in \omega\} = \mathsf{block}(g)(\alpha) = f(\alpha).$$

Since τ is a Player 2 winning strategy, $f \in \mathrm{dom}(R)$, and $\Xi(x) = \pi_2(\Xi^2_\tau(x))$, one has that $R(f, \Xi(x))$. In particular, $R(\mathsf{block}(\mathsf{decode}(x)), \Xi(x))$.

Thus the club D and Lipschitz continuous function Ξ are the objects required by the main result. The proof is complete. $\square$

Definition 5.10. Let κ be a regular cardinal. A good coding family for κ consists of

$$\langle (\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \kappa), \mathsf{BS}, \mathsf{seq}, \mathsf{nGC}, \mathsf{merge} \rangle$$

with the following properties:

(1) $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \kappa)$ is a good coding system for ${}^\kappa\kappa$.
(2) $\mathsf{BS} \subseteq \mathbb{R}$. $\mathsf{seq} : \mathsf{BS} \to {}^{<\kappa}\kappa$ is a surjection.
(3) For any $\epsilon < \kappa$, $\ell \in [\kappa]^\epsilon_*$, and $\sigma \subseteq \sup(\ell)$ so that for each $\alpha < \epsilon$, $\ell(\alpha)$ is a limit point of σ, the set $\mathsf{BBS}^{\ell,\sigma} = \{u \in \mathsf{BS} : \mathsf{seq}(u) \in [\sigma]^{\omega\cdot\epsilon} \wedge \mathsf{block}(\mathsf{seq}(u)) = \ell\}$ belongs to Δ.
(4) $\mathsf{nGC} \subseteq \mathsf{GC}$ and for all $f \in {}^\kappa\kappa$, there is an $e \in \mathsf{nGC}$ so that $\mathsf{decode}(e) = f$.
(5) $\mathsf{merge} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function so that if $u \in \mathsf{BS}$, $\mathsf{seq}(u)$ has length ϵ, and $e \in \mathsf{nGC}$, then $\mathsf{merge}(u, e) \in \mathsf{GC}$ and

$$\mathsf{decode}(\mathsf{merge}(u,e))(\beta) = \begin{cases} \mathsf{seq}(u)(\beta) & \beta < \epsilon \\ \mathsf{decode}(e)(\beta) & \epsilon \leq \beta \end{cases}$$

Say that κ is very reasonable if and only if κ has a good coding family.

Remark 5.11. Suppose ϵ, ℓ, and σ are as above, then $\mathsf{BBS}^{\ell,\sigma} \neq \emptyset$. To see this, let $F : \epsilon \times \omega \to \kappa$ be a function witnessing that ℓ has uniform cofinality ω. Define $\iota : \omega \cdot \epsilon \to \sigma$ as followings. Let $\alpha < \epsilon$ and note that $\sup(\ell \restriction \alpha) < \ell(\alpha)$ since ℓ is discontinuous everywhere. Let $\iota(\omega \cdot \alpha)$ be the least element of σ greater than $\max\{\sup(\ell \restriction \alpha), F(\alpha, 0)\}$ which can be found since $\ell(\alpha)$ is a limit point of σ. If $\iota(\omega \cdot \alpha + n)$ has been defined for $n \in \omega$, then let $\iota(\omega \cdot \alpha + n + 1)$ be the least element of σ greater than $\max\{\iota(\omega \cdot \alpha + n), F(\alpha, n+1)\}$. Note that $\mathsf{block}(\iota) = \ell$. Since $\mathsf{seq} : \mathsf{BS} \to {}^{<\kappa}\kappa$ is surjective, let $u \in \mathsf{BS}$ be such that $\mathsf{seq}(u) = \iota$. Then $u \in \mathsf{BBS}^{\ell,\sigma}$.

nGC is a collection of nice good codes e which are suitable for merging with elements $u \in \mathsf{BS}$ in the sense that $\mathsf{decode}(\mathsf{merge}(u,e))$ has the intended meaning given in (5). Even if $e \in \mathsf{nGC}$, $\mathsf{merge}(u,e)$ may not be a member of nGC and hence, there may exists $v \in \mathsf{BS}$ so that $\mathsf{decode}(\mathsf{merge}(v, \mathsf{merge}(u,e)))$ does not have the intended meaning expressed in (5). In all applications, one will have a single $e \in \mathsf{nGC}$ and merge $u \in \mathsf{BS}$ into this e. Iterated merges will never appear so the issue above will not occur.

Often nGC is not necessary. For example, in Theorem 5.12, the good coding family and merge will be very natural in the sense that $\mathsf{nGC} = \mathsf{GC}$. The good coding family for ω_1 created in Theorem 7.23 will have the property that $\mathsf{nGC} \subsetneq \mathsf{GC}$ and in fact, there exists $u, v \in \mathsf{BS}$ and $e \in \mathsf{nGC}$ so that $\mathsf{decode}(\mathsf{merge}(u, \mathsf{merge}(v, e)))$ will not have the intended meaning. One can define a good coding family for ω_1 which eliminates nGC (i.e. has $\mathsf{GC} = \mathsf{nGC}$); however, this example is unnecessarily difficult to exposit.

Section 7 will give concrete examples of reasonable and very reasonable cardinals along with their good coding systems and families. In particular, the very reasonableness of ω_1 (which will needed to study the ultrapower of ω_1 by the strong partition measure on ω_1) will be carefully investigated. The next result shows that the prewellordering ordinal of a pointclass with sufficient closure properties are very reasonable.

Theorem 5.12. *Let Γ be a (boldface) nonselfdual adequate pointclass closed under $\forall^\mathbb{R}$ with the prewellordering property such that $\Delta = \Gamma \cap \check{\Gamma}$ is adequate, $\mathbf{\Sigma}^1_1 \subseteq \Delta$, and Δ is closed under both $\exists^\mathbb{R}$ and $\forall^\mathbb{R}$. Then $\delta(\Gamma)$ is very reasonable.*

Proof. [16] Lemma 1.6 shows $\delta(\Gamma)$ is $\delta(\Gamma)$-reasonable. Here, it will be shown that $\delta(\Gamma)$ is very reasonable by creating a good coding family for $\delta(\Gamma)$.

Let $\langle U^{(n)}(\dot{A}, \dot{B}) : n \in \omega \setminus \{0\}\rangle$ and functions $s_{m,n}$ be the objects from the good sequence of universal sets for the pointclass $\Sigma^1_1(\dot{A}, \dot{B})$ which is defined uniformly from $\dot{A}$ and $\dot{B}$ as in Definition 4.13. Let $P \in \Gamma$ be a Γ-complete set. Let $\preceq$ be a Γ-norm on P with associated norm φ which must have length $\delta(\Gamma)$ by Fact 4.6. For each $\xi < \delta(\Gamma)$, let $Q^\varphi_\xi = \{x \in P : \varphi(x) = \xi\}$ and $Q^\varphi_{<\xi} = \{x \in P : \varphi(x) < \xi\}$. Note that for each $\xi < \delta(\Gamma)$, both Q^φ_ξ and $Q^\varphi_{<\xi}$ belong to Δ.

First one will define a good coding system $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \delta(\Gamma))$ for ${}^{\delta(\Gamma)}\delta(\Gamma)$ as follow:

Note that by the hypothesis Γ is closed under $\forall^{\mathbb{R}}$ and $\delta(\Gamma)$ is regular by Fact 4.6.

Fix $\beta, \gamma < \delta(\Gamma)$. Define $\mathsf{GC}_{\beta,\gamma}$ by $e \in \mathsf{GC}_{\beta,\gamma}$ if and only if

$$(\exists x)(\exists y)(x \in Q^{\varphi}_{\beta} \wedge y \in Q^{\varphi}_{\gamma} \wedge U^{(2)}_{e}(Q^{\varphi}_{<\beta}, Q^{\varphi}_{\beta})(x,y)) \wedge (\forall x)(\forall y)[U^{(2)}_{e}(Q^{\varphi}_{<\beta}, Q^{\varphi}_{\beta})(x,y) \Rightarrow (x \in Q^{\varphi}_{\beta} \wedge y \in Q^{\varphi}_{\gamma})].$$

Since $Q^{\varphi}_{<\beta}$ and Q^{φ}_{β} belong to Δ and Δ is closed under $\exists^{\mathbb{R}}$, one has that $\Sigma^1_1(Q^{\varphi}_{<\beta}, Q^{\varphi}_{\beta}) \subseteq \Delta$ and hence $U^{(2)}(Q^{\varphi}_{<\beta}, Q^{\varphi}_{\beta}) \in \Delta$. Using this observation, the fact that $Q^{\varphi}_{\beta}, Q^{\varphi}_{\gamma} \in \Delta$, and the fact that Δ is closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$, one can see that $\mathsf{GC}_{\beta,\gamma} \in \Delta$.

Let $\mathsf{decode}(e)(\beta,\gamma)$ if and only if $e \in \mathsf{GC}_{\beta,\gamma}$. From the definition of $\mathsf{GC}_{\beta,\gamma}$, $e \in \mathsf{GC}_{\beta,\gamma}$ if and only if $\mathsf{decode}(e)(\beta,\gamma)$ and $(\forall \gamma')(\mathsf{decode}(e)(\beta,\gamma') \Rightarrow \gamma' = \gamma)$.

For each $f \in {}^{\delta(\Gamma)}\delta(\Gamma)$. Let $G_f(x,y)$ if and only if $x, y \in P \wedge f(\varphi(x)) = \varphi(y)$. By the uniform coding lemma (Fact 4.14), there is an $e \in \mathbb{R}$ so that for all $\beta < \delta(\Gamma)$,

(1) $U^{(2)}_{e}(Q^{\varphi}_{<\beta}, Q^{\varphi}_{\beta}) \subseteq G_f \cap (Q^{\varphi}_{\beta} \times \mathbb{R})$
(2) $U^{(2)}_{e}(Q^{\varphi}_{<\beta}, Q^{\varphi}_{\beta}) \neq \emptyset$ if and only if $G_f \cap (Q^{\varphi}_{\beta} \times \mathbb{R}) \neq \emptyset$.

For this $e \in \mathbb{R}$, $\mathsf{decode}(e) = f$.

Now suppose $A \in \exists^{\mathbb{R}}\Delta = \Delta$ and $A \subseteq \mathsf{GC}_{\beta} = \bigcup_{\gamma < \delta(\Gamma)} \mathsf{GC}_{\beta,\gamma}$ for some $\beta < \delta(\Gamma)$. By definition of $\mathsf{GC}_{\beta,\gamma}$ above, if $e \in \mathsf{GC}_{\beta,\gamma}$, then $\pi_2[U^{(2)}_{e}(Q^{\varphi}_{<\beta}, Q^{\varphi}_{\beta})] \subseteq Q^{\varphi}_{\gamma} \subseteq P$, where $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the second coordinate. Thus $K = \{y \in \mathbb{R} : (\exists e)(\exists x)(e \in A \wedge U^{(2)}_{e}(Q^{\varphi}_{<\beta}, Q^{\varphi}_{\beta})(x,y))\}$ is $\exists^{\mathbb{R}}\Delta = \Delta \subseteq \check{\Gamma}$ and $K \subseteq P$. By Fact 4.5, there is a $\delta < \delta(\Gamma)$ so that $K \subseteq Q^{\varphi}_{<\delta}$. Thus $A \subseteq \bigcup_{\gamma<\delta} \mathsf{GC}_{\beta,\gamma}$.

So far, it has been shown that $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \delta(\Gamma))$ is a good system for ${}^{\delta(\Gamma)}\delta(\Gamma)$.

Let BS be the collection of all (z,k) so that $z \in P \wedge (\forall \beta < \varphi(z))(k \in \mathsf{GC}_{\beta})$. Let $\mathsf{seq}(z,k)(\beta,\gamma)$ if and only if $\beta < \varphi(z)$ and $k \in \mathsf{GC}_{\beta,\gamma}$. Suppose $\ell \in {}^{<\kappa}\kappa$. Let $f \in {}^{\kappa}\kappa$ be defined by $f(\alpha) = \ell(\alpha)$ if $\alpha < |\ell|$ and $f(\alpha) = 0$ if $\alpha \geq |\ell|$. Let $k \in \mathsf{GC}$ be such that $\mathsf{decode}(k) = f$ and $z \in \mathsf{GC}$ be such that $\varphi(z) = |\ell|$. Then $(z,k) \in \mathsf{BS}$ and $\mathsf{seq}(z,k) = \ell$.

Now suppose $\epsilon < \delta(\Gamma)$, $\ell \in [\delta(\Gamma)]^{\epsilon}_{*}$, and $\sigma \subseteq \sup(\ell)$ with the property that for each $\alpha < \epsilon$, $\ell(\alpha)$ is a limit point of σ. Recall again that by Fact 5.3, Δ is closed under less than $\delta(\Gamma)$-length wellordered union. Let $A = \{(z,k) : z \in Q^{\varphi}_{\omega\cdot\epsilon}\}$ which belongs to Δ. Let $B = \{(z,k) \in A : (\forall \alpha < \omega \cdot \epsilon)(\exists \xi \in \sigma)(k \in \mathsf{GC}_{\alpha,\xi})\}$ which belongs to Δ since Δ is closed under recursive substitution and less than $\delta(\Gamma)$-length unions and intersections. For $\beta < \omega \cdot \epsilon$ and $\gamma \in \sigma$, let $B_{\beta,\gamma} = \{(z,k) \in B : k \in \mathsf{GC}_{\beta,\gamma}\}$. Note that each $B_{\beta,\gamma} \in \Delta$. For $\beta_0 < \beta_1 < \omega \cdot \epsilon$, let

$$I_{\beta_0,\beta_1} = \bigcup\{B_{\beta_0,\xi_0} \cap B_{\beta_1,\xi_1} : \xi_0 < \xi_1 \wedge \xi_0, \xi_1 \in \sigma\}.$$

I_{β_0,β_1} is Δ. Let

$$I = \bigcap\{I_{\beta_0,\beta_1} : \beta_0 < \beta_1 < \omega \cdot \epsilon\}.$$

Note that $I \in \Delta$ since Δ is closed under less than $\delta(\Gamma)$-length intersections. For all $(z,k) \in I$, $\mathsf{seq}(z,k) \in [\sigma]^{\omega\cdot\epsilon}$, i.e. is an increasing function. For each $\alpha < \epsilon$ and $\nu < \ell(\alpha)$, let

$$D_{\alpha,\nu} = \{(z,k) \in I : (\exists^{\omega} n)(\exists \xi)(\nu < \xi < \ell(\alpha) \wedge k \in \mathsf{GC}_{\omega\cdot\alpha+n,\xi})\}.$$

By similar arguments, $D_{\alpha,\nu} \in \Delta$ for each $\nu < \ell(\alpha)$. Let $D_{\alpha} = \bigcap_{\nu<\ell(\alpha)} D_{\alpha,\nu}$ is also Δ. Note that for any $(z,k) \in D_{\alpha}$, $\mathsf{block}(\mathsf{seq}(z,k))(\alpha) = \ell(\alpha)$. Thus $\mathsf{BSS}^{\ell,\sigma} = \bigcap_{\alpha<\epsilon} D_{\alpha}$ which is also Δ.

Let $D(\dot{A},\dot{B}) \in \Sigma^1_1(\dot{A},\dot{B})$ be defined by $D(\dot{A},\dot{B})(e,z,k,x,y)$ if and only if

$$x \in B \wedge ((z \notin \dot{A} \cup \dot{B}) \Rightarrow U^{(2)}_{k}(\dot{A},\dot{B})(x,y)) \wedge ((z \in \dot{A} \cup \dot{B}) \Rightarrow U^{(2)}_{e}(\dot{A},\dot{B})(x,y)).$$

Since $U^{(5)}$ is universal for the $\boldsymbol{\Sigma}^1_1(\dot{A},\dot{B})$ subsets of $\mathbb{R}^5$, there is a $e_0 \in \mathbb{R}$ so that $D(\dot{A},\dot{B})(e,z,k,x,y) \Leftrightarrow U^{(5)}(\dot{A},\dot{B})(e_0,e,z,k,x,y)$. Then one has

$$D(\dot{A},\dot{B})(e,z,k,z,y) \Leftrightarrow U^{(5)}(e_0,e,z,k,x,y) \Leftrightarrow U^{(2)}(s_{3,5}(e_0,e,z,k),x,y).$$

Define $\mathsf{merge}((z,k),e) = s_{3,5}(e_0,e,z,k)$ which is a continuous function since $s_{3,5}$ is continuous. Let $\mathsf{nGC} = \mathsf{GC}$.

To understand the meaning, substitute $\dot{A}$ and $\dot{B}$ with its intended interpretation of $Q^{\varphi}_{<\beta}$ and Q^{φ}_{β}, respectively. Then for each $\beta < \delta(\Gamma)$, one has

$$U^{(2)}_{\mathsf{merge}((z,k),e)}(Q^{\varphi}_{<\beta}, Q^{\varphi}_{\beta})(x,y) = D(Q^{\varphi}_{<\beta}, Q^{\varphi}_{\beta})(e,z,k,x,y) = \begin{cases} U^{(2)}_{k}(Q^{\varphi}_{<\beta}, Q^{\varphi}_{\beta})(x,y) & \text{if } \varphi(x) < \varphi(z) \\ U^{(2)}_{e}(Q^{\varphi}_{<\beta}, Q^{\varphi}_{\beta})(x,y) & \text{if } \varphi(z) \leq \varphi(x) \end{cases}$$

Therefore, for $e \in \mathsf{GC} = \mathsf{nGC}$ and $(z,k) \in \mathsf{BS}$, one has

$$\mathsf{decode}(\mathsf{merge}((z,k),e))(\beta,\gamma) \Leftrightarrow \begin{cases} \mathsf{seq}(z,k)(\beta) = \gamma & \text{if } \beta < \varphi(z) \\ \mathsf{decode}(e)(\beta,\gamma) & \text{if } \varphi(z) \leq \beta \end{cases}$$

Observe that $\mathsf{merge}((z,k),e) \in \mathsf{GC}$, $\mathsf{seq}(z,k)$ is an initial segment of $\mathsf{merge}((z,k),e)$, and the remaining tail of $\mathsf{merge}((z,k),e)$ is the corresponding tail of $\mathsf{decode}(e)$.

This shows that $\langle(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta,\gamma < \delta(\Gamma)), \mathsf{BS}, \mathsf{seq}, \mathsf{merge}\rangle$ is a good coding family for $\delta(\Gamma)$. $\square$

The following important example shows that under AD alone there are cofinally many very reasonable cardinals below Θ. (This of course implies that there are cofinally many strong partition cardinals below Θ.)

Definition 5.13. Define the language $\mathscr{L} = \{\dot{\in}, \dot{A}, \dot{\mathbb{R}}\}$ where $\dot{\in}$ is a binary relation symbol, $\dot{A}$ is a unary predicate symbol, and $\dot{\mathbb{R}}$ is a unary relation symbol.

Let $A \subseteq \mathbb{R}$. Let $L(A,\mathbb{R})$ be the constructible hierarchy built over $\mathsf{tc}(\{A,\mathbb{R}\})$, the transitive closure of $\{A,\mathbb{R}\}$. The canonical $\mathscr{L}$-structure on $L(A,\mathbb{R})$ is given by defining $\dot{\in}^{L(A,\mathbb{R})} = \in \cap (L(A,\mathbb{R}) \times L(A,\mathbb{R}))$, $\dot{A}^{L(A,\mathbb{R})} = A$, and $\dot{\mathbb{R}}^{L(A,\mathbb{R})} = \mathbb{R}$. $L(A,\mathbb{R})$ comes with a Σ_1 stratification into $\langle L_\alpha(A,\mathbb{R}) : \alpha \in \mathrm{ON}\rangle$ and in fact, this sequence is Δ_1 in $L(A,\mathbb{R})$.

Let $\boldsymbol{\Sigma}_1^{L(A,\mathbb{R})}$ be the Σ_1-definable within $L(A,\mathbb{R})$ subsets of $\mathbb{R}$ allowing the elements of $\mathbb{R}$ as parameters. $\boldsymbol{\Pi}_1^{L(A,\mathbb{R})}$ cosists of those subsets of $\mathbb{R}$ whose complements are $\boldsymbol{\Sigma}_1^{L(A,\mathbb{R})}$. $\boldsymbol{\Delta}_1^{L(A,\mathbb{R})}$ consists of those subsets of $\mathbb{R}$ which are both $\boldsymbol{\Sigma}_1^{L(A,\mathbb{R})}$ and $\boldsymbol{\Pi}_1^{L(A,\mathbb{R})}$. $\Sigma_1^{L(A,\mathbb{R})}$, $\Pi_1^{L(A,\mathbb{R})}$, and $\Delta_1^{L(A,\mathbb{R})}$ correspond to the same notions as before in which no parameters from $\mathbb{R}$ are permitted.

Let δ_A be the least ordinal δ so that $L_\delta(A,\mathbb{R}) \prec_1 L(A,\mathbb{R})$, where $\prec_1$ denotes Σ_1-elementarity in the language $\mathscr{L}$.

Example 5.14. $\boldsymbol{\Sigma}_1^{L(A,\mathbb{R})}$ is a boldface adequate pointclass closed under $\forall^{\mathbb{R}}$ (and $\exists^{\mathbb{R}}$) with the prewellordering property and $\boldsymbol{\Delta}_1^{L(A,\mathbb{R})}$ is adequate, $\boldsymbol{\Sigma}_1^1 \subseteq \boldsymbol{\Delta}_1^{L(A,\mathbb{R})}$, and $\boldsymbol{\Delta}_1^{L(A,\mathbb{R})}$ is closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$. The prewellordering ordinal of $\boldsymbol{\Sigma}_1^{L(A,\mathbb{R})}$, i.e. $\delta(\boldsymbol{\Sigma}_1^{L(A,\mathbb{R})})$, is the ordinal δ_A. Therefore, δ_A is very reasonable (and hence a strong partition cardinal). ([16]) Thus there are cofinally many $\kappa < \Theta$ so that κ is a strong partition cardinal.

Proof. It is clear that $\boldsymbol{\Sigma}_1^{L(A,\mathbb{R})}$ has the closure properties of Theorem 5.12. It remains to show it has the prewellordering property. The following describes the prewellordering from the beginning of [22] Section 2.5.

Let $U \subseteq \mathbb{R} \times \mathbb{R}$ be a $\Sigma_1^{L(A,\mathbb{R})}$ set which is universal for $\boldsymbol{\Sigma}_1^{L(A,\mathbb{R})}$ subsets of $\mathbb{R}$. Let ϕ be a Σ_1 $\mathscr{L}$-formula so that $U(x)$ if and only if $L(A,\mathbb{R}) \models \phi(x)$. Let T be ZF without the power set axioms, and the power set of ω exists.

The formula $\varsigma(x)$ which states $(\exists\alpha \in \mathrm{ON})(L_\alpha(A,\mathbb{R}) \models \mathsf{T} \wedge L_\alpha(A,\mathbb{R}) \models \phi(x))$ is $\Sigma_1^{L(A,\mathbb{R})}$. If $U(x)$ holds, then $L(A,\mathbb{R}) \models \phi(x)$. By reflection, there exists an α so that $L_\alpha(A,\mathbb{R}) \models \mathsf{T}$ and $L_\alpha(A,\mathbb{R}) \models \phi(x)$. This means $L(A,\mathbb{R}) \models \varsigma(x)$. By definition of δ_A, $L_{\delta_A}(A,\mathbb{R}) \models \varsigma(x)$. By definition of $\varsigma(x)$, there is an $\alpha < \delta_A$ so that $L_\alpha(A,\mathbb{R}) \models \mathsf{T}$ and $L_\alpha(A,\mathbb{R}) \models \phi(x)$. Define $\rho : U \to \delta_A$ by letting $\rho(x)$ be the least α so that $L_\alpha(A,\mathbb{R}) \models \mathsf{T}$ and $L_\alpha(A,\mathbb{R}) \models \phi(x)$.

Let $\Gamma = \boldsymbol{\Sigma}_1^{L(A,\mathbb{R})}$ and $\check{\Gamma} = \boldsymbol{\Pi}_1^{L(A,\mathbb{R})}$. Define $\leq^{\rho}_{\Gamma}$ by $x \leq^{\rho}_{\Gamma} y$ if and only if

$$(\exists\alpha \in \mathrm{ON})(L_\alpha(A,\mathbb{R}) \models [\mathsf{T} \wedge \phi(x) \wedge \phi(y)] \wedge (\forall\beta < \alpha)(L_\beta(A,\mathbb{R}) \models \mathsf{T} \Rightarrow L_\beta(A,\mathbb{R}) \models \neg\phi(y))).$$

Using the fact that $\langle L_\alpha(A,\mathbb{R}) : \alpha < \mathrm{ON}\rangle$ is $\Delta_1^{L(A,\mathbb{R})}$, one can see that $\leq^{\rho}_{\Gamma}$ is $\Sigma_1^{L(A,\mathbb{R})}$.

Define $\leq^{\rho}_{\check{\Gamma}}$ by $x \leq^{\rho}_{\Gamma} y$ if and only

$$(\forall\alpha \in \mathrm{ON})(L_\alpha(A,\mathbb{R}) \models [\mathsf{T} \wedge \phi(y)] \Rightarrow L_\alpha(A,\mathbb{R}) \models \phi(x))$$

Note that $\leq^{\Gamma}_{\check{\Gamma}}$ belongs to $\Pi_1^{L(A,\mathbb{R})}$.

One can check that these two relations have the necessary properties from Definition 4.4 to witness that ρ is a $\Sigma_1^{L(A,\mathbb{R})}$-norm on U. Since U is $\Sigma_1^{L(A,\mathbb{R})}$ and universal for $\mathbf{\Sigma}_1^{L(A,\mathbb{R})}$, $\mathbf{\Sigma}_1^{L(A,\mathbb{R})}$ has the prewellordering property. Hence by Theorem 5.12, $\delta(\mathbf{\Sigma}_1^{L(A,\mathbb{R})})$ is very reasonable. In particular, $\delta(\mathbf{\Sigma}_1^{L(A,\mathbb{R})})$ is a strong partition cardinal.

Finally, one will give a brief sketch to show that $\delta(\mathbf{\Sigma}_1^{L(A,\mathbb{R})}) = \delta_A$. It suffices to show the rank of ρ is δ_A since U is universal. The ideas of [22] Section 2.4 show that for each $\alpha < \delta_A$, there is a $\mathbf{\Delta}_1^{L(A,\mathbb{R})}$ prewellordering $\preceq$ on $\mathbb{R}$ on length α. Using the Moschovakis coding lemma, one can show that δ_A is a regular cardinal.

Let $\alpha < \delta_A$ be arbitrary. Let α^+ be the least ordinal $\beta > \alpha$ so that $L_\beta(A,\mathbb{R}) \models \mathsf{T}$. Let $\preceq$ be a $\mathbf{\Delta}_1^{L(A,\mathbb{R})}$ prewellordering of $\mathbb{R}$ of length α^+. Thus there is a some $e \in \mathbb{R}$ so that $U_e = \{x \in \mathbb{R} : U(e,x)\} = \preceq$ (after identifying elements of $\mathbb{R}^2$ with elements of $\mathbb{R}$ by some recursive pairing function). The claim is that there is an $x \in \mathbb{R}$ with $U(e,x)$ so that $\rho(e,x) > \alpha$. Suppose for all x such that $U(e,x)$, $\rho(e,x) \leq \alpha$. By upward absoluteness of the Σ_1 formula ϕ, one has that $\preceq = \{(e,x) : L_\alpha(A,\mathbb{R}) \models \phi(x)\}$. However, within $L_{\alpha^+}(A,\mathbb{R})$, the set $\{(e,x) : L_\alpha(A,\mathbb{R}) \models \phi(x)\} = \preceq$ exists as a set. Thus $\preceq \in L_{\alpha^+}(A,\mathbb{R})$. Since $L_{\alpha^+}(A,\mathbb{R}) \models \mathsf{T}$, it can recover the ordertype of the prewellordering $\preceq$ which is α^+. However it is impossible that $\alpha^+ \in L_{\alpha^+}(A,\mathbb{R})$. This establishes the claim. Since δ_A is regular and the image of ρ is cofinal through δ_A, it has been shown that the rank of ρ (which is $\delta(\mathbf{\Sigma}_1^{L(A,\mathbb{R})})$) is δ_A. It has been shown that $\delta(\mathbf{\Sigma}^{L(A,\mathbb{R})}) = \delta_A$.

Note that $\{\delta_A : A \in \mathscr{P}(\mathbb{R})\}$ is cofinal below Θ. To see this, let $\theta < \Theta$. Let A be a prewellordering of length θ. The statement $(\exists\alpha)(L_\alpha(A,\mathbb{R}) \models \mathsf{T})$ is Σ_1. Thus by the definition δ_A, there exists an $\alpha < \delta_A$ so that $L_\alpha(A,\mathbb{R}) \models \mathsf{T}$. If $L_\alpha(A,\mathbb{R}) \models \mathsf{T}$, then $\theta \in L_\alpha(A,\mathbb{R})$. This implies that $\theta < \alpha < \delta_A$. □

6. Almost Everywhere Short Length Club Uniformizations

The good coding family is designed to prove the almost everywhere short length club uniformization.

Theorem 6.1. *If κ is very reasonable, then the almost everywhere short length club uniformization holds at κ: If $R \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ is $\subseteq$-downward closed in the club_κ-coordinate, then there is a club $C \subseteq \kappa$ and a function $\Lambda : \mathrm{dom}(R) \cap [C]^{<\kappa}_* \to \mathsf{club}_\kappa$ so that for all $\ell \in \mathrm{dom}(R) \cap [C]^{<\kappa}_*$, $R(\ell, \Lambda(\ell))$.*

Proof. Let $\langle(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta,\gamma < \kappa), \mathsf{BS}, \mathsf{seq}, \mathsf{nGC}, \mathsf{merge}\rangle$ be a good coding family for κ. Let P be a Γ-complete set and φ be a Γ-norm on P of length $\kappa = \delta(\Gamma)$ using Fact 5.3. Let $\langle U^{(n)}(\dot{A},\dot{B}) : n \in \omega \setminus \{0\}\rangle$ be a sequence of good universal sets for $\Sigma^1_1(\dot{A},\dot{B})$ (which is defined uniformly in $\dot{A}$ and $\dot{B}$ as in Definition 4.13). Let $R \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ be $\subseteq$-downward closed.

For each $f \in [\kappa]^\kappa_*$, let $T_f \subseteq P \times \mathsf{clubcode}^\varphi_\kappa$ be defined by $(z,v) \in T_f$ if and only if

$$z \in P \wedge f \restriction \varphi(z) \in \mathrm{dom}(R) \wedge v \in \mathsf{clubcode}^\varphi_\kappa \wedge R(f \restriction \varphi(z), \mathfrak{C}^{\varphi,\kappa}_v).$$

By the uniform coding lemma (Fact 4.14), there is a $w \in \mathbb{R}$ so that for all $\beta < \kappa$,

(1) $U^{(2)}_w(Q^\varphi_\beta, Q^\varphi_{<\beta}) \subseteq T_f \cap (Q^\varphi_\beta \times \mathbb{R})$.
(2) $U^{(2)}_w(Q^\varphi_\beta, Q^\varphi_{<\beta}) \neq \emptyset$ if and only if $T_f \cap (Q^\varphi_\beta \times \mathbb{R}) \neq \emptyset$.

In this setting, one will say that w is an f-selector. Define $S \subseteq [\kappa]^\kappa_* \times \mathbb{R}$ by $S(f,w)$ if and only if w is an f-selector. Note that $\mathrm{dom}(S) = [\kappa]^\kappa_*$ by the above argument. By Theorem 5.9, there is a club $D \subseteq \kappa$ and a Lipschitz function $\Xi : \mathbb{R} \to \mathbb{R}$ so that for all $e \in \mathsf{GC}$ with $\mathsf{decode}(e) \in [D]^\kappa$, $S(\mathsf{block}(\mathsf{decode}(e)), \Xi(e))$. One may assume D consists entirely of limit ordinals. Let $C = \{\alpha \in D : \mathsf{enum}_D(\alpha) = \alpha\}$.

Fix any $g \in [C]^\kappa$. Let $e \in \mathsf{nGC}$ be such that $\mathsf{decode}(e) = g$. Suppose $\ell \in [C]^{<\kappa}_*$ with $\ell \in \mathrm{dom}(R)$. Let ϵ be the length of ℓ. Let ν_ℓ be least so that $g(\nu_\ell) > \sup(\ell)$. Let γ_ℓ be the least indecomposable ordinal greater than both $\omega \cdot \epsilon$ and ν_ℓ. Let $\rho_\ell \in [D]^{\gamma_\ell}$ be defined by

$$\rho_\ell(\alpha) = \begin{cases} \ell(\alpha) & \alpha < \epsilon \\ g(\nu_\ell + \alpha) & \epsilon \leq \alpha < \gamma_\ell \end{cases}$$

Note that ρ_ℓ is produced uniformly from ℓ. By definition of the good coding family for κ, $\mathsf{BBS}^{\rho_\ell, D\cap\sup(\rho_\ell)}$ belongs to Δ and $\mathsf{BBS}^{\rho_\ell, D\cap\sup(\rho_\ell)} \neq \emptyset$ by the remarks after Definition 5.10. Let $v \in K_\ell$ if and only if

$$(\exists z')(\exists u)[u \in \mathsf{BBS}^{\rho_\ell, D\cap\sup(\rho_\ell)} \wedge U^{(2)}_{\Xi(\mathsf{merge}(u,e))}(Q^\varphi_{<\epsilon}, Q^\varphi_\epsilon)(z',v).$$

Since Ξ is Lipschitz, merge is continuous, $\mathsf{BBS}^{\rho_\ell, D\cap\sup(\rho_\ell)}, Q^\varphi_{<\epsilon}, Q^\varphi_\epsilon \in \Delta$, one can check that K_ℓ is $\exists^\mathbb{R}\Delta$. Note that K_ℓ is also produced uniformly from ℓ. (D and e are fixed and ϵ is just the length of ℓ.)

If $u \in \mathsf{BBS}^{\rho_\ell, D\cap\sup(\rho_\ell)}$, $\mathsf{seq}(u)$ has length $\gamma_\ell = \omega\cdot\gamma_\ell$, $\sup(\mathsf{seq}(u)) = \sup(\rho_\ell) \leq g(\gamma_\ell)$ (using the indecomposability of γ_ℓ), and $\mathsf{decode}(e)(\gamma_\ell) = g(\gamma_\ell)$. Thus $\mathsf{decode}(\mathsf{merge}(u,e)) \in [D]^\kappa$ (i.e., it is an increasing function through D). $\Xi(\mathsf{merge}(u,e))$ is a $\mathsf{block}(\mathsf{decode}(\mathsf{merge}(u,e)))$-selector. However, since $u \in \mathsf{BBS}^{\rho_\ell, D\cap\sup(\rho_\ell)}$, one has that ℓ is the initial segment of $\mathsf{block}(\mathsf{decode}(\mathsf{merge}(u,e)))$ of length ϵ. Since $u \in \mathsf{BBS}^{\rho_\ell, D\cap\sup(\rho_\ell)}$ was arbitrary, this implies that $K_\ell \subseteq \mathsf{clubcode}^\varphi_\kappa$ and for all $v \in K_\ell$, $R(\ell, \mathfrak{C}^{\varphi,\kappa}_v)$. Since $K_\ell \subseteq \mathsf{clubcode}^\varphi_\kappa$ and $K_\ell \in \exists^\mathbb{R}\Delta \subseteq \check{\Gamma}$, Fact 4.12 implies that there is a club C_ℓ (produced uniformly from K_ℓ which itself is produced uniformly from ℓ) so that for all $v \in K_\ell$, $C_\ell \subseteq \mathfrak{C}^{\varphi,\kappa}_v$. Since $\ell \in \mathrm{dom}(R)$, $K_\ell \neq \emptyset$ and therefore there is a $v \in K_\ell$ so that $C_\ell \subseteq \mathfrak{C}^{\varphi,\kappa}_v$ and $R(\ell, \mathfrak{C}^{\varphi,\kappa}_v)$. Since R is $\subseteq$-downward closed, one has that $R(\ell, C_\ell)$.

The function $\Lambda : \mathrm{dom}(R) \cap [C]^{<\kappa}_* \to \mathsf{club}_\kappa$ defined by $\Lambda(\ell) = C_\ell$ is the desired uniformization. ☐

Theorem 6.2. *(Strong almost everywhere short length club uniformization) Let κ be a very reasonable cardinal. Let $R \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed. Then there is a club $C \subseteq \kappa$ so that for all $\ell \in \mathrm{dom}(R) \cap [C]^{<\kappa}_*$, $R(\ell, C \setminus (\sup(\ell)+1))$.*

Proof. This follows from Theorem 6.1 and Fact 2.27. ☐

7. Reasonableness at the First Uncountable Cardinal

In the analysis of ultrapowers of ω_1 by its partition measures, one will need good coding systems and families satisfying further definability conditions. First, one will consider good coding systems for ${}^\epsilon\omega_1$ when $\epsilon < \omega_1$.

Definition 7.1. Let $\mathsf{pair} : \omega\times\omega \to \omega$ denote a fixed recursive bijection. If $x \in \mathbb{R} = {}^\omega\omega$ and $k \in \omega$, let $x^{[k]} \in {}^\omega\omega$ be defined by $x^{[k]}(n) = x(\mathsf{pair}(k,n))$. ($x^{[k]}$ is the k^{th} section of x.)

If $x \in \mathbb{R}$, let $\mathcal{R}_x \subseteq \omega\times\omega$ be defined by $\mathcal{R}_x(a,b) \Leftrightarrow x(\mathsf{pair}(a,b)) = 0$.

Let $\mathrm{WO} \subseteq \mathbb{R}$ consists of those $x \in \mathbb{R}$ so that $\mathcal{R}_x$ is a wellordering. Let $\mathsf{field}(x) = \mathsf{field}(\mathcal{R}_x) = \{a \in \omega : (\exists b)(\mathcal{R}_x(a,b) \vee \mathcal{R}_x(b,a))\}$. WO is a Π^1_1 set which is $\boldsymbol{\Pi}^1_1$-complete. Let $\mathrm{ot} : \mathrm{WO} \to \omega_1$ be the ordertype function. Note that if $\alpha < \omega_1$, then there is some $w \in \mathrm{WO}$ so that $\mathrm{ot}(w) = \alpha$. $\mathrm{ot} : \mathrm{WO} \to \omega_1$ is a Π^1_1-norm. Let $\leq^{\mathrm{ot}}_{\Sigma^1_1}, <^{\mathrm{ot}}_{\Sigma^1_1} \in \Sigma^1_1$ and $\leq^{\mathrm{ot}}_{\Pi^1_1}, <^{\mathrm{ot}}_{\Pi^1_1} \in \Pi^1_1$ be the relations witnessing that ot is a Π^1_1-norm.

For each $\alpha \in \omega_1$, let WO_α be those $w \in \mathrm{WO}$ so that $\mathrm{ot}(w) = \alpha$. Let $\mathrm{WO}_{<\alpha}$ be those $w \in \mathrm{WO}$ so that $\mathrm{ot}(w) < \alpha$. Similarly one can define $\mathrm{WO}_{\leq\alpha}$, $\mathrm{WO}_{\geq\alpha}$, and $\mathrm{WO}_{>\alpha}$. Note that WO_α, $\mathrm{WO}_{<\alpha}$, and $\mathrm{WO}_{\leq\alpha}$ are $\Delta^1_1(w)$ for any $w \in \mathrm{WO}_\alpha$. $\mathrm{WO}_{\geq\alpha}$ and $\mathrm{WO}_{>\alpha}$ are $\Pi^1_1(w)$ for any $w \in \mathrm{WO}_\alpha$.

Let $\mathsf{rest} : \mathbb{R}\times\omega \to \mathbb{R}$ be such that

$$\mathsf{rest}(x,n)(a,b) \Leftrightarrow \mathcal{R}_x(a,b) \wedge \mathcal{R}_x(a,n) \wedge \mathcal{R}_x(b,n).$$

Note that rest is a recursive function and whenever $w \in \mathrm{WO}$ and $n \in \omega$, $\mathsf{rest}(w,n) \in \mathrm{WO}$ and $\mathrm{ot}(\mathsf{rest}(w,n))$ is the ordertype of $\mathcal{R}_x \upharpoonright \{a : \mathcal{R}_x(a,n)\}$.

Suppose $w \in \mathrm{WO}$ and $\alpha < \mathrm{ot}(w)$. Let $\mathsf{num}(w,\alpha)$ be the unique $k \in \mathsf{field}(w)$ so that $\mathrm{ot}(\mathsf{rest}(w,k)) = \alpha$. If $k \in \mathsf{field}(w)$, then let $\mathrm{ot}(w,k)$ be the ordertype of $\mathsf{rest}(w,k)$, namely the ordertype of k in $\mathcal{R}_w$.

Remark 7.2. The rest of the paper will mostly focus on ω_1. Coding of club subsets of ω_1 will always be with respect to the Π^1_1-norm $\mathrm{ot} : \mathrm{WO} \to \omega_1$. In this context, one will often drop the reference to ω_1 and ot. That is, $\mathsf{clubcode} = \mathsf{clubcode}^{\mathrm{ot}}_{\omega_1}$, $\mathsf{inClub} = \mathsf{inClub}^{\mathrm{ot}}_{\omega_1}$, $\mathsf{clubSubset} = \mathsf{clubSubset}^{\mathrm{ot}}_{\omega_1}$, and $\mathfrak{C}_z = \mathfrak{C}^{\mathrm{ot},\omega_1}_z$.

Definition 7.3. Let $\epsilon < \omega_1$ and $(\boldsymbol{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \omega_1)$ be a good coding system for ${}^\epsilon\omega_1$. This good coding system is a short-uniformly good coding system for ${}^\epsilon\omega_1$ if and only if there is a $W \in \mathrm{WO}_\epsilon$ with $\mathsf{field}(W) = \omega$ and there are two relations $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1}$ and $\mathsf{uGC}_{\boldsymbol{\Pi}^1_1}$ with the following properties.

(1) $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1} \subseteq \mathbb{R}\times\omega\times\mathbb{R}$ is $\boldsymbol{\Sigma}^1_1$ and $\mathsf{uGC}_{\boldsymbol{\Pi}^1_1} \subseteq \mathbb{R}\times\omega\times\mathbb{R}$ is $\boldsymbol{\Pi}^1_1$.
(2) For all $x \in \mathbb{R}$, for all $n \in \omega$, there exists a $y \in \mathbb{R}$ so that $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1}(x,n,y)$.
(3) For all $x \in \mathbb{R}$, for all $n \in \omega$, for all $v \in \mathrm{WO}$, $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1}(x,n,v) \Leftrightarrow \mathsf{uGC}_{\boldsymbol{\Pi}^1_1}(x,n,v)$.
(4) For all $x \in \mathbb{R}$, for all $n \in \omega$, for all $v \in \mathrm{WO}$, for all $w \in \mathbb{R}$, if $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1}(x,n,v)$ and $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1}(x,n,w)$, then $w \in \mathrm{WO}_{\mathrm{ot}(v)}$.
(5) For all $x \in \mathbb{R}$, $n \in \omega$, and $w \in \mathrm{WO}$, $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1}(x,n,w)$ if and only if $x \in \mathsf{GC}_{\mathrm{ot}(W,n),\mathrm{ot}(w)}$.

Note that (4) and (5) could also be formulated using $\mathsf{uGC}_{\mathbf{\Pi}^1_1}$ by (3).

Definition 7.4. Suppose κ is a regular cardinal and $\epsilon \leq \kappa$. Let $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \kappa)$ be a good coding system for ${}^{\epsilon}\kappa$. If $A \subseteq \kappa$, then let $\mathsf{Inc}(A) = \{x \in \mathsf{GC} : \mathsf{decode}(x) \in [A]^{\epsilon}\}$, i.e. $x \in \mathsf{Inc}(A)$ if and only if $\mathsf{decode}(x)$ is an increasing function from ϵ into A.

Fact 7.5. *Let $\epsilon < \omega_1$ and $(\mathbf{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \omega_1)$ be a short-uniformly good coding system for ${}^{\epsilon}\omega_1$. Then the following holds.*

(1) GC and $\mathsf{Inc}(\omega_1)$ are $\mathbf{\Pi}^1_1$.
(2) There is a $\mathbf{\Pi}^1_1$ relation INC so that whenever $z \in \mathsf{clubcode}$, $\mathsf{INC}(z, x)$ if and only if $x \in \mathsf{GC}$ and $\mathsf{decode}(x) \in [\mathfrak{C}_z]^{\epsilon}$.
(3) For any $z \in \mathsf{clubcode}$, $\mathsf{Inc}(\mathfrak{C}_z)$ is $\mathbf{\Pi}^1_1$.
(4) There is a $\mathbf{\Pi}^1_1$ relation $\mathsf{sameBlock}$ so that $\mathsf{sameBlock}(x_0, x_1)$ if and only if $x_0, x_1 \in \mathsf{Inc}(\omega_1)$ and $\mathsf{block}(\mathsf{decode}(x_0))(\alpha) = \mathsf{block}(\mathsf{decode}(x_1))(\alpha)$ for all $\alpha < \epsilon$ with $\omega \cdot \alpha + \omega < \epsilon$.

Proof. Observe that $x \in \mathsf{GC}$ if and only if

$$(\forall^{\omega} n)(\forall^{\mathbb{R}} z)(\mathsf{uGC}_{\mathbf{\Sigma}^1_1}(x, n, z) \Rightarrow z \in \mathrm{WO}).$$

To see this, note that by Definition 7.3 (2) implies that for all $x \in \mathbb{R}$ and $n \in \omega$, there exists a $y \in \mathbb{R}$ so that $\mathsf{uGC}_{\mathbf{\Sigma}^1_1}(x, n, y)$. (Thus for each $n \in \omega$, the statement above cannot be vacuously true.) Then Definition 7.3 (5) implies $x \in \mathsf{GC}$. The other direction is clear from Definition 7.3 (5). The latter expression is $\mathbf{\Pi}^1_1$.

Observe that $x \in \mathsf{Inc}(\omega_1)$ if and only if the conjunction of the following holds.

(1) $x \in \mathsf{GC}$.
(2) $(\forall^{\omega} m)(\forall^{\omega} n)(\forall^{\mathbb{R}} v)(\forall^{\mathbb{R}} w)[\{\mathcal{R}_W(m, n) \wedge \mathsf{uGC}_{\mathbf{\Sigma}^1_1}(x, m, v) \wedge \mathsf{uGC}_{\mathbf{\Sigma}^1_1}(x, n, w)\} \Rightarrow (v <^{\mathrm{ot}}_{\Pi^1_1} w)]$

The intention of (2) is to say that $\mathsf{decode}(x)$ is increasing. This expression is $\mathbf{\Pi}^1_1$.

Let $\mathsf{INC}(z, x)$ be the conjunction of the following statement.

(1) $x \in \mathsf{Inc}(\omega_1)$.
(2) $(\forall^{\omega} m)(\forall^{\mathbb{R}} v)[\mathsf{uGC}_{\mathbf{\Sigma}^1_1}(x, m, v) \Rightarrow \mathsf{inClub}(z, v)]$.

Note that inClub is Π^1_1 by Fact 4.11. Thus INC is $\mathbf{\Pi}^1_1$. In the case that $z \in \mathsf{clubcode}$, $\mathsf{INC}(z, x)$ will give the correct interpretation.

Observe that $x \in \mathsf{Inc}(\mathfrak{C}_z)$ if and only if $\mathsf{INC}(z, x)$, which is $\mathbf{\Pi}^1_1$ since INC is $\mathbf{\Pi}^1_1$.

Let $N \subseteq \omega$ be defined by $N = \{n \in \omega : (\exists \alpha < \epsilon)(\mathrm{ot}(W, n) = \omega \cdot \alpha + \omega)\}$. Define $\mathsf{sameBlock}(x_0, x_1)$ if and only if

(1) $x_0, x_1 \in \mathsf{Inc}(\omega_1)$.
(2) For all $n \in \omega$, if $n \in N$, then the conjunction of the following holds
  (a) $(\forall^{\omega} a)(\forall v)[\{\mathcal{R}_W(a, n) \wedge \mathsf{uGC}_{\mathbf{\Sigma}^1_1}(x_0, a, v)\} \Rightarrow (\exists^{\omega} b)(\exists^{\mathbb{R}} w)(\mathcal{R}_W(b, n) \wedge \mathsf{uGC}_{\mathbf{\Pi}^1_1}(x_1, b, w) \wedge v \leq^{\mathrm{ot}}_{\Pi^1_1} w)]$
  (b) $(\forall^{\omega} a)(\forall v)[\{\mathcal{R}_W(a, n) \wedge \mathsf{uGC}_{\mathbf{\Sigma}^1_1}(x_1, a, v)\} \Rightarrow (\exists^{\omega} b)(\exists^{\mathbb{R}} w)(\mathcal{R}_W(b, n) \wedge \mathsf{uGC}_{\mathbf{\Pi}^1_1}(x_0, b, w) \wedge v \leq^{\mathrm{ot}}_{\Pi^1_1} w)]$

In (2a), the intention is that $\mathrm{ot}(v) = \mathsf{decode}(x_0)(\mathrm{ot}(W, a))$ and $\mathrm{ot}(w) = \mathsf{decode}(x_1)(\mathrm{ot}(W, b))$. The above expression of $\mathbf{\Pi}^1_1$. ∎

Remark 7.6. The next example will provide a short-uniformly good coding system for ${}^{\epsilon}\omega_1$ when $\epsilon < \omega_1$. In this case, the good coding system for ${}^{\epsilon}\omega_1$ is so simple that the abstract concept of a short-uniformly good coding system and the abstract proof of Fact 7.5 is a luxury. The next example will directly verify Fact 7.5.

Later one will also define an analogous concept of a long-uniformly good coding system for ${}^{\omega_1}\omega_1$. All known good coding system for ${}^{\omega_1}\omega_1$ are quite complicated. Perspicuity will be gained by extracting the essential aspects of the good coding system and ignoring the specific features. Furthermore, examples of short-uniformly good coding systems for the higher odd projective ordinals are very non-trivial. (See Section 11.) These abstractions will become a practical necessity later.

Example 7.7. (Martin) For each $\epsilon < \omega_1$, there is a short-uniform good coding system $(\mathbf{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \omega_1)$ for ${}^{\epsilon}\omega_1$. Thus for all $\epsilon < \omega_1$, ω_1 is ϵ-reasonable and $\omega_1 \rightarrow_* (\omega_1)^{\epsilon}_2$ holds.

Proof. Fix $W \in \mathrm{WO}$ so that $\mathrm{ot}(W) = \epsilon$ and $\mathsf{field}(W) = \omega$.

For $\beta < \epsilon$ and $\gamma < \omega_1$, let $\mathsf{GC}_{\beta,\gamma} = \{x \in \mathbb{R} : x^{[\mathsf{num}(W,\beta)]} \in \mathrm{WO}_\gamma\}$. Note that $\mathsf{GC}_{\beta,\gamma} \in \mathbf{\Delta}^1_1$ since $\mathsf{num}(W,\gamma)$ is a fixed number and $\mathrm{WO}_\gamma \in \mathbf{\Delta}^1_1$. Define $\mathsf{decode} : \mathbb{R} \to \mathscr{P}(\epsilon \times \omega_1)$ by $\mathsf{decode}(x)(\beta,\gamma)$ if and only if $x \in \mathsf{GC}_{\beta,\gamma}$. If $f \in {}^\epsilon\omega_1$, then using $\mathsf{AC}^{\mathbb{R}}_\omega$, find an $x \in \mathbb{R}$ so that for all $\beta < \epsilon$, $x^{[\mathsf{num}(W,\beta)]} \in \mathrm{WO}$ and $\mathrm{ot}(x^{[\mathsf{num}(W,\beta)]}) = f(\beta)$. Then $\mathsf{decode}(x)$ is the graph of f. It is clear that $x \in \mathsf{GC}_{\beta,\gamma}$ if and only if $\mathsf{decode}(x)(\beta,\gamma)$ and for all $\xi < \omega_1$, $\mathsf{decode}(x)(\beta,\xi)$ implies that $\xi = \gamma$.

Now suppose $A \subseteq \mathsf{GC}_\beta$ and $A \in \exists^{\mathbb{R}}\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1$. Let $B = \{r \in \mathbb{R} : (\exists x)(x \in A \wedge r = x^{[\mathsf{num}(W,\beta)]})\}$. Note that for each $x \in \mathsf{GC}_\beta = \bigcup_{\gamma<\omega_1} \mathsf{GC}_{\beta,\gamma}$, one has $x^{[\mathsf{num}(W,\beta)]} \in \mathrm{WO}$. Thus B is $\mathbf{\Sigma}^1_1$ and $B \subseteq \mathrm{WO}$. By the boundedness principle, there is a $\delta < \omega_1$ so that $A \subseteq \bigcup_{\gamma<\delta} \mathsf{GC}_{\beta,\gamma}$.

It has been verified that this is a good coding system for ${}^\epsilon\omega_1$.

Define $U \subseteq \mathbb{R} \times \omega \times \mathbb{R}$ by $U(x,n,y)$ if and only if $x^{[n]} = y$. U is Δ^1_1. Let $\mathsf{uGC}_{\mathbf{\Sigma}^1_1} = \mathsf{uGC}_{\Pi^1_1} = U$. Using the wellordering W above and these relations, this coding system for ${}^\epsilon\omega_1$ is a short-uniformly good coding system for ${}^\epsilon\omega_1$. Next, one will directly verify Fact 7.5.

Note that $x \in \mathsf{GC}$ if and only if $(\forall n)(x^{[n]} \in \mathrm{WO})$. Thus GC is Π^1_1.

Observe that $x \in \mathsf{Inc}(\omega_1)$ if and only if $x \in \mathsf{GC}$ and $(\forall^\omega m)(\forall^\omega n)(\mathcal{R}_W(m,n) \Rightarrow x^{[m]} <^{\mathrm{ot}}_{\Pi^1_1} x^{[n]})$. This shows $\mathsf{Inc}(\omega_1)$ is Π^1_1.

Let $\mathsf{INC}(z,x)$ be the conjunction of the following statements.

(1) $x \in \mathsf{GC}$
(2) $(\forall^\omega m)[\mathsf{inClub}(z, x^{[n]})]$
(3) $(\forall^\omega m)(\forall^\omega n)(\mathcal{R}_W(m,n) \Rightarrow x^{[m]} <^{\mathrm{ot}}_{\Pi^1_1} x^{[n]})$

Let $N \subseteq \omega$ be defined by $N = \{n \in \omega : (\exists \alpha < \epsilon)(\mathrm{ot}(W,n) = \omega\cdot\alpha + \omega)\}$. Define $\mathsf{sameBlock}(x_0,x_1)$ if and only if $x_0, x_1 \in \mathsf{Inc}(\omega_1)$ and for all $n \in \omega$, if $n \in N$, then the conjunction of the following holds

- $(\forall^\omega a)(\mathcal{R}_W(a,n) \Rightarrow (\exists^\omega b)(\mathcal{R}_W(b,n) \wedge (x_0)^{[a]} \leq^{\mathrm{ot}}_{\Pi^1_1} (x_1)^{[b]})$
- $(\forall^\omega a)(\mathcal{R}_W(a,n) \Rightarrow (\exists^\omega b)(\mathcal{R}_W(b,n) \wedge (x_1)^{[a]} \leq^{\mathrm{ot}}_{\Pi^1_1} (x_0)^{[b]})$

□

Martin showed the existence of a good coding system for ${}^{\omega_1}\omega_1$ using indiscernibles and sharps of reals. Next, an unpublished argument of Kechris which constructs a good coding system for ${}^{\omega_1}\omega_1$ using category ideas will be provided. The proof of Kechris has a pedagogical benefit of using only very classical descriptive set theory. The argument needs to be provided in some detail and explicitness for the analysis of the ultrapower of the strong partition measure in later sections. The next definition makes explicit the most important definability features of a good coding system for ${}^{\omega_1}\omega_1$ which will be used in this analysis.

Definition 7.8. Let $(\mathbf{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta,\gamma < \omega_1)$ be a good coding system for ${}^{\omega_1}\omega_1$. This good coding system for ${}^{\omega_1}\omega_1$ is said to be a long-uniformly good coding system for ${}^{\omega_1}\omega_1$ if and only if there are two relations $\mathsf{uGC}_{\mathbf{\Sigma}^1_1}$ and $\mathsf{uGC}_{\mathbf{\Pi}^1_1}$ with the following properties.

(1) $\mathsf{uGC}_{\mathbf{\Sigma}^1_1} \subseteq \mathbb{R}^3$ and $\mathsf{uGC}_{\mathbf{\Pi}^1_1} \subseteq \mathbb{R}^3$.
(2) For all $x \in \mathbb{R}$ and all $v, w \in \mathrm{WO}$, $\mathsf{uGC}_{\mathbf{\Sigma}^1_1}(x,v,w) \Leftrightarrow \mathsf{uGC}_{\mathbf{\Pi}^1_1}(x,v,w)$.
(3) For all $x \in \mathbb{R}$, for all $v, w \in \mathrm{WO}$, and all $y \in \mathbb{R}$, if $\mathsf{uGC}_{\mathbf{\Sigma}^1_1}(x,v,w)$ and $\mathsf{uGC}_{\mathbf{\Sigma}^1_1}(x,v,y)$, then $y \in \mathrm{WO}_{\mathrm{ot}(w)}$.
(4) For all $x \in \mathbb{R}$, $v, w \in \mathrm{WO}$, $\mathsf{uGC}_{\mathbf{\Sigma}^1_1}(x,v,w) \Leftrightarrow x \in \mathsf{GC}_{\mathrm{ot}(v),\mathrm{ot}(w)}$.

Note that by (2), (3) and (4) could also be formulated using $\mathsf{uGC}_{\mathbf{\Pi}^1_1}$.

Fact 7.9. *Suppose $(\mathbf{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta,\gamma < \omega_1)$ is a long-uniformly good coding system for ${}^{\omega_1}\omega_1$. Then the following holds.*

(1) GC *and* $\mathsf{Inc}(\omega_1)$ *are* $\mathbf{\Pi}^1_2$.
(2) There is a $\mathbf{\Pi}^1_2$ relation INC *so that whenever $z \in$* $\mathsf{clubcode}$, $\mathsf{INC}(z,x)$ *if and only if $x \in$* GC *and* $\mathsf{decode}(x) \in [\mathfrak{C}_z]^{\omega_1}$.
(3) For any $z \in$ $\mathsf{clubcode}$, $\mathsf{Inc}(\mathfrak{C}_z)$ *is* $\mathbf{\Pi}^1_2$.
(4) There is a $\mathbf{\Pi}^1_2$ relation $\mathsf{sameBlock}$ *so that* $\mathsf{sameBlock}(x_0,x_1)$ *if and only if* $x_0, x_1 \in \mathsf{Inc}(\omega_1)$ *and* $\mathsf{block}(\mathsf{decode}(x_0))(\alpha) = \mathsf{block}(\mathsf{decode}(x_1))(\alpha)$ *for all* $\alpha < \omega_1$.

Proof. Observe that $x \in \mathsf{GC}$ if and only if the conjunction of the following holds.

(1) $(\forall^{\mathbb{R}} v)[v \in \mathrm{WO} \Rightarrow (\exists^{\mathbb{R}} y)(\mathsf{uGC}_{\boldsymbol{\Sigma}_1^1}(x, v, y))]$
(2) $(\forall^{\mathbb{R}} v)(\forall^{\mathbb{R}} w)[(v \in \mathrm{WO} \wedge \mathsf{uGC}_{\boldsymbol{\Pi}_1^1}(x, v, w)) \Rightarrow w \in \mathrm{WO}]$

This expression is $\boldsymbol{\Pi}_2^1$.

Note that $x \in \mathsf{Inc}(\omega_1)$ if and only if

(1) $x \in \mathsf{GC}$
(2) $(\forall^{\mathbb{R}} v_0)(\forall^{\mathbb{R}} v_1)(\forall^{\mathbb{R}} w_0)(\forall^{\mathbb{R}} w_1)[\{v_0 \in \mathrm{WO} \wedge v_1 \in \mathrm{WO} \wedge v_0 <^{\mathrm{ot}}_{\Pi_1^1} v_1 \wedge \mathsf{uGC}_{\Pi_1^1}(x, v_0, w_0) \wedge \mathsf{uGC}_{\boldsymbol{\Pi}_1^1}(x, v_1, w_1)\} \Rightarrow w_0 <_{\boldsymbol{\Pi}_1^1} w_1]$.

Define INC by $\mathsf{INC}(z, x)$ if and only if the conjunction of the following holds.

(1) $x \in \mathsf{Inc}(\omega_1)$
(2) $(\forall^{\mathbb{R}} v)(\forall^{\mathbb{R}} w)[(v \in \mathrm{WO} \wedge \mathsf{uGC}_{\boldsymbol{\Pi}_1^1}(x, v, w)) \Rightarrow \mathsf{inClub}(z, w)]$.

Since inClub is Π_1^1 by Fact 4.11, the above expression is $\boldsymbol{\Pi}_2^1$.

If $z \in \mathsf{clubcode}$, then $x \in \mathsf{Inc}(\mathfrak{C}_z)$ if and only if $\mathsf{INC}(z, x)$. Therefore, $\mathsf{Inc}(\mathfrak{C}_z)$ is $\boldsymbol{\Pi}_2^1$.

There is a Δ_1^1 function $K : \mathbb{R} \to \mathbb{R}$ with the property that for all $w \in \mathrm{WO}$, $K(w) \in \mathrm{WO}$ and $\mathrm{ot}(K(w)) = \omega \cdot \mathrm{ot}(w) + \omega$. Define $\mathsf{sameBlock}$ by $\mathsf{sameBlock}(x_0, x_1)$ if and only if the conjunction of the following holds.

(1) $x_0 \in \mathsf{Inc}(\omega_1) \wedge x_1 \in \mathsf{Inc}(\omega_1)$.
(2) For all $u \in \mathbb{R}$, $u \in \mathrm{WO}$ implies that the conjunction of the following holds.
  (a) $(\forall^{\mathbb{R}} a)(\forall^{\mathbb{R}} v)[\{a <^{\mathrm{ot}}_{\Sigma_1^1} K(u) \wedge \mathsf{uGC}_{\boldsymbol{\Sigma}_1^1}(x_0, a, v)\} \Rightarrow (\exists^{\mathbb{R}} b)(\exists^{\mathbb{R}} w)(b <_{\Sigma_1^1} K(u) \wedge \mathsf{uGC}_{\boldsymbol{\Sigma}_1^1}(x_1, b, w) \wedge v \leq^{\mathrm{ot}}_{\Sigma_1^1} w)]$
  (b) $(\forall^{\mathbb{R}} a)(\forall^{\mathbb{R}} v)[\{a <^{\mathrm{ot}}_{\Sigma_1^1} K(u) \wedge \mathsf{uGC}_{\boldsymbol{\Sigma}_1^1}(x_1, a, v)\} \Rightarrow (\exists^{\mathbb{R}} b)(\exists^{\mathbb{R}} w)(b <_{\Sigma_1^1} K(u) \wedge \mathsf{uGC}_{\boldsymbol{\Sigma}_1^1}(x_0, b, w) \wedge v \leq^{\mathrm{ot}}_{\Sigma_1^1} w)]$

(2a) and (2b) are both $\boldsymbol{\Pi}_2^1$ which implies (2) is $\boldsymbol{\Pi}_2^1$. The entire expresssion defining $\mathsf{sameBlock}$ is $\boldsymbol{\Pi}_2^1$. ☐

Definition 7.10. A tree on a set X is a set $T \subseteq {}^{<\omega}X$ so that for all $s, t \in {}^{<\omega}X$, if $s \subseteq t$ and $t \in T$, then $s \in T$. The body of the tree T is $[T] = \{f \in {}^{\omega}X : (\forall n)(f \restriction n \in T)\}$.

Let κ be a cardinal. A set $A \subseteq {}^{\omega}X$ is κ-Suslin if and only if there is a tree T on $X \times \kappa$ so that $A = \{x \in {}^{\omega}X : (\exists f)(f \in {}^{\omega}\kappa \wedge (x, f) \in [T])\}$.

The Kunen-Martin theorem will be frequently used throughout the paper to produce upper bounds.

Fact 7.11. *([17] Theorem 7.1, Kunen-Martin) Assume $\mathsf{ZF} + \mathsf{AC}_{\omega}^{\mathbb{R}}$. Let κ be a cardinal. If $R \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$ is a κ-Suslin wellfounded relation, then the length of R is less than κ^+.*

Definition 7.12. Let $A \subseteq {}^{\omega}\omega$ and $B \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$ so that $A \subseteq \pi_1[B]$ where $\pi_1 : {}^{\omega}\omega \times {}^{\omega}\omega \to {}^{\omega}\omega$ is the projection onto the first coordinate. Consider the game $G^*_{A,B}$ which is defined as follows:

$$
\begin{array}{cccccccccccc}
 & \mathrm{I} & s_0 & & s_2 & & s_4 & & s_6 & & \cdots & \\
G^*_{A,B} & & & & & & & & & & & x \\
 & \multirow{2}{*}{II} & & s_1 & & s_3 & & s_5 & & s_7 & \cdots & \\
 & \mathrm{II} & & y_0 & & y_1 & & y_2 & & y_3 & \cdots & y
\end{array}
$$

For all $n \in \omega$, Player 1 plays $s_{2n} \in {}^{<\omega}\omega$, Player 2 plays $s_{2n+1} \in {}^{<\omega}\omega$, and $y_n \in \omega$. Let $x \in {}^{\omega}\omega$ be the concatenation $s_0\hat{\ }s_1\hat{\ }s_2\hat{\ }...$ and $y \in {}^{\omega}\omega$ be defined by $y(n) = y_n$. Player 2 wins $G^*_{A,B}$ if and only if $x \in A$ and $(x, y) \in B$. $G^*_{A,B}$ is called an unfolded Banach-Mazur game.

Under AD, the unfolded Banach-Mazur game can be used to prove many interesting properties about category such as all subsets of ${}^{\omega}\omega$ have the Baire property, wellordered union of meager sets are meager, and comeager uniformization for relations on $\mathbb{R} \times \mathbb{R}$. The following result is another example which is provable in just ZF (using the determinacy for closed games given by the Gale-Stewart theorem).

Fact 7.13. *Assume ZF. Let $A \subseteq {}^{\omega}\omega$ be $\boldsymbol{\Sigma}_1^1$ and $B \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$ be $\boldsymbol{\Pi}_1^0$ such that $\pi_1[B] = A$. Then A is comeager if and only if Player 2 has a winning strategy in $G^*_{A,B}$.*

Let $\forall^{\omega}_{*}$ denote the universal category quantifier defined as follows. Let $A \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$ and $x \in {}^{\omega}\omega$. Say that $(\forall^*_{\omega} y)A(x, y)$ holds if and only if $\{y \in {}^{\omega}\omega : A(x, y)\}$ is comeager in ${}^{\omega}\omega$.

Fact 7.14. *Assume* ZF*. Suppose* A*,* B*, and* C *are* $\mathbf{\Sigma}_1^1$*,* $\mathbf{\Pi}_1^1$*, and* $\mathbf{\Delta}_1^1$ *subsets of* ${}^\omega\omega \times {}^\omega\omega$*, respectively. Let* $A^*(x) \Leftrightarrow (\forall_\omega^* y)A(x,y)$*,* $B^*(x) \Leftrightarrow (\forall_\omega^* y)B(x,y)$*, and* $C^*(x) \Leftrightarrow (\forall_\omega^* y)C(x,y)$*. Then* A^**,* B^**, and* C^* *are* $\mathbf{\Sigma}_1^1$*,* $\mathbf{\Pi}_1^1$*, and* $\mathbf{\Delta}_1^1$*, respectively.*

Proof. Let $A' \subseteq {}^\omega\omega \times {}^\omega\omega \times {}^\omega\omega$ be a $\mathbf{\Pi}_1^0$ set so that $\pi_1[A'] = A$. Using Fact 7.13, one has the following.

$$A^*(x) \Leftrightarrow A_x \text{ is comeager} \Leftrightarrow \text{Player 2 has a winning strategy in } G^*_{A_x, A'_x}$$

if and only if there exists a Player 2 strategy τ so that for all Player 1 strategy σ, $A'(r, x_{\sigma*\tau}, y_{\sigma*\tau})$, where $x_{\sigma*\tau}$ is the joint real produced by both players and $y_{\sigma*\tau}$ is the real produced by Player 2 when σ and τ are played against each other. This is an existential real quantifier followed by a universal real quantifier over a $\mathbf{\Pi}_1^0$ set. Since the projection of an open set is open, a universal real quantifier over a $\mathbf{\Pi}_1^0$ set is $\mathbf{\Pi}_1^0$. Hence, the latter expression is $\mathbf{\Sigma}_1^1$.

Note that $\neg B^*(x)$ if and only if B_x is not comeager if and only if $\neg B_x$ is not meager. Since $\mathbf{\Sigma}_1^1$ sets have the Baire property, the latter is equivalent to the statement that there exists an $s \in {}^{<\omega}\omega$ so that $\neg B_x$ is comeager in N_s. Since $\neg B$ is $\mathbf{\Sigma}_1^1$, the earlier argument can be used to show that $\neg B^*$ is $\mathbf{\Sigma}_1^1$ and thus B^* is $\mathbf{\Pi}_1^1$. C^* is $\mathbf{\Delta}_1^1$ by applying the results in the $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$ cases. ◻

Definition 7.15. Suppose $1 < \alpha \leq \omega_1$. If $s \in {}^{<\omega}\alpha$, then let $N_s^\alpha = \{f \in {}^\omega\alpha : s \subseteq f\}$. Give ${}^\omega\alpha$ the topology generated by $\{N_s^\alpha : s \in {}^{<\omega}\alpha\}$ as a basis.

Under AD, if $1 \leq \alpha < \omega_1$, then every subset of $A \subseteq {}^\omega\alpha$ has the property of Baire and wellordered unions of meager subsets of ${}^\omega\alpha$ are meager. (Note that ${}^\omega\alpha$ is homeomorphic to ${}^\omega\omega$ and these topological properties transfer to ${}^\omega\alpha$.) Observe that $\mathsf{surj}_\alpha = \{f \in {}^\omega\alpha : f[\omega] = \alpha\}$ is a comeager subset of ${}^\omega\alpha$

If $A \subseteq {}^\omega\omega \times {}^\omega\alpha$, then define the universal category quantifier on ${}^\omega\alpha$ by $(\forall_\alpha^* f)A(x,f)$ if and only if $\{f \in {}^\omega\alpha : A(x,f)\}$ is comeager in ${}^\omega\alpha$.

The following is a simple form of the Kechris-Woodin generic coding function ([20]) for ω_1.

Fact 7.16. *There is a continuous function* $\mathfrak{G} : {}^\omega\omega_1 \to \mathrm{WO}$ *so that for all* $\alpha < \omega_1$*, if* $f \in \mathsf{surj}_\alpha$*, then* $\mathfrak{G}(f) \in \mathrm{WO}_\alpha$*.*

For each $w \in \mathrm{WO}$*, let* $B_w : \omega \to \mathrm{ot}(w)$ *be defined by* $B_w(n) = \mathrm{ot}(w,n)$*. There is a continuous function* $\mathcal{G} : {}^\omega\omega \times {}^\omega\omega \to {}^\omega\omega$ *so that for all* $w \in \mathrm{WO}$ *with* $\mathsf{field}(w) = \omega$ *and for all* $r \in {}^\omega\omega$*,* $\mathcal{G}(w,r) = \mathfrak{G}(B_w \circ r)$*.*

Proof. Recall $\mathsf{pair} : \omega \times \omega \to \omega$ is a fixed recursive pairing function. For each $f \in {}^\omega\omega_1$, let $A_f = \{n \in \omega : (\forall m < n)(f(m) \neq f(n))\}$. Define

$$\mathfrak{G}(f)(k) = \begin{cases} 0 & k = \mathsf{pair}(m,n) \wedge m,n \in A_f \wedge f(m) < f(n) \\ 1 & \text{otherwise} \end{cases}.$$

Note that $\mathrm{dom}(\mathcal{R}_{\mathfrak{G}(f)}) = A_f$ and $(A_f, \mathcal{R}_{\mathfrak{G}(f)})$ is isomorphic to $f[\omega]$ with the ordering induced from ω_1. Thus $\mathfrak{G}(f) \in \mathrm{WO}$ and if $f \in \mathsf{surj}_\alpha$, then $\mathfrak{G}(f) \in \mathrm{WO}_\alpha$. Suppose $m,n \in \omega$ and without loss of generality, $m < n$. Then for any $g \in {}^\omega\omega_1$ so that $g \restriction n+1 = f \restriction n+1$, one has that $\mathfrak{G}(g)(m,n) = \mathfrak{G}(f)(m,n)$. $\mathfrak{G}$ is continuous.

Let $K(w,r,m,n)$ if and only if the conjunction of the following holds:

(1) $(\forall j < m)(r(j) \neq r(m))$.
(2) $(\forall j < n)(r(j) \neq r(n))$.
(3) $\mathcal{R}_w(r(m), r(n))$.

Define

$$\mathcal{G}(w,r)(k) = \begin{cases} 0 & k = \mathsf{pair}(m,n) \wedge K(w,r,m,n) \\ 1 & \text{otherwise} \end{cases}$$

◻

Fact 7.17. *(Solovay) Assume* $\mathsf{ZF} + \mathsf{AD}$*. For all* $h : \omega_1 \to \omega_1$*, there exists a strategy* τ *with the property that for all* $v \in \mathrm{WO}$*, there is a* $\delta > \mathrm{ot}(v)$ *so that*

$$h \restriction \delta = \Big\{(\beta,\gamma) : (\exists^\omega n)\Big((\Xi_\tau^2(v)^{[n]})^{[0]} \in \mathrm{WO}_\beta \wedge (\Xi_\tau^2(v)^{[n]})^{[1]} \in \mathrm{WO}_\gamma\Big)\Big\}$$

In other words, Ξ_τ^2 *is a Lipschitz continuous function so that for any* $v \in \mathrm{WO}$*,* $\Xi_\tau^2(v)$ *is a real which codes (using its sections) an initial segment of* h *of length strictly greater than* $\mathrm{ot}(v)$*.*

Proof. Consider the game G_h so that Player 1 and Player 2 separately produce reals v and z (respectively), one natural number per turn. Player 2 wins G_h if and only if $v \in \mathrm{WO}$ implies there is a $\delta > \mathrm{ot}(v)$ so that

$$h \restriction \delta = \Big\{(\beta,\gamma) : (\exists^\omega n)\Big((z^{[n]})^{[0]} \in \mathrm{WO}_\beta \wedge (z^{[n]})^{[1]} \in \mathrm{WO}_\gamma\Big)\Big\}.$$

By AD, one of the players has a winning strategy. Suppose Player 1 has a winning strategy σ. This implies $\Xi^1_\sigma[\mathbb{R}] \subseteq \mathrm{WO}$ and by the boundedness principle, there exists a $\delta < \omega_1$ so that $\mathrm{ot}[\Xi^1_\sigma[\mathbb{R}]] \subseteq \delta$. Let $B : \omega \to \delta$ be a bijection. By $\mathsf{AC}^{\mathbb{R}}_\omega$, let $z \in \mathbb{R}$ be such that for all $n \in \omega$, $(z^{[n]})^{[0]} \in \mathrm{WO}_{B(n)}$ and $(z^{[n]})^{[1]} \in \mathrm{WO}_{h(B(n))}$. By choice of δ, $\Xi^1_\sigma(z) \in \mathrm{WO}_{<\delta}$ and Player 2 wins by playing z which contradicts σ being a winning strategy for Player 1. Thus Player 2 has a winning strategy τ which is the desired object. $\square$

Fact 7.18. *(Martin) There is a long-uniformly good coding system* $(\mathbf{\Pi}^1_1, \mathsf{decode}^*, \mathsf{GC}^*_{\beta,\gamma}, : \beta,\gamma < \omega_1)$ *for* ${}^{\omega_1}\omega_1$*. Thus* $\omega_1 \to_* (\omega_1)^{\omega_1}_2$*.*

Proof. The following is the good coding system for ${}^{\omega_1}\omega_1$ of Kechris.

Fix a recursive coding of strategies by reals. If $x \in \mathbb{R}$, then let τ_x refer to the strategy coded by x. For each $\beta, \gamma < \omega_1$, define $K^\beta_\gamma \subseteq \mathbb{R} \times \mathbb{R}$ by $(x,v) \in K^\beta_\gamma$ if and only if

$$(\exists^\omega n)\Big((\Xi^2_{\tau_x}(v)^{[n]})^{[0]} \in \mathrm{WO}_\beta \wedge (\forall^\omega m < n)((\Xi^2_{\tau_x}(v)^{[m]})^{[0]} \notin \mathrm{WO}_\beta) \wedge (\Xi^2_{\tau_x}(v)^{[n]})^{[1]} \in \mathrm{WO}_\gamma\Big).$$

Observe that $K^\beta_\gamma \in \mathbf{\Delta}^1_1$. Define $\mathsf{GC}^*_{\beta,\gamma}$ by

$$x \in \mathsf{GC}^*_{\beta,\gamma} \Leftrightarrow (\forall^*_{\omega+\beta} f) K^\beta_\gamma(x, \mathfrak{G}(f)).$$

where $\mathfrak{G} : {}^\omega\omega_1 \to \mathrm{WO}$ is the generic coding function from Fact 7.16. Let $B : \omega \to \omega + \beta$ be a bijection. Define $\mathfrak{G}^\beta : {}^\omega\omega \to \mathrm{WO}$ by $\mathfrak{G}^\beta(r) = \mathfrak{G}(B \circ r)$. Note that $\mathfrak{G}^\beta$ is continuous since $\mathfrak{G}$ is continuous. Note

$$x \in \mathsf{GC}^*_{\beta,\gamma} \Leftrightarrow (\forall^*_\omega r) K^\beta_\gamma(x, \mathfrak{G}^\beta(r)).$$

By closure of the pointclass $\mathbf{\Delta}^1_1$ under continuous substitution and the universal category quantifier (Fact 7.14), $\mathsf{GC}^*_{\beta,\gamma}$ is $\mathbf{\Delta}^1_1$. For $\beta, \gamma < \omega_1$, define $\mathsf{decode}^* : \mathbb{R} \to \mathscr{P}(\omega_1 \times \omega_1)$ by $(\beta,\gamma) \in \mathsf{decode}^*(x)$ if and only if $x \in \mathsf{GC}^*_{\beta,\gamma}$.

<u>Claim 1</u>: $x \in \mathsf{GC}^*_{\beta,\gamma}$ if and only if $\mathsf{decode}^*(x)(\beta,\gamma)$ and $(\forall \xi < \omega_1)(\mathsf{decode}^*(x)(\beta,\xi) \Rightarrow \gamma = \xi$.

Proof. Suppose that $\mathsf{decode}^*(x)(\beta,\gamma)$ and $(\forall \xi < \omega_1)(\mathsf{decode}^*(x)(\beta,\gamma) \Rightarrow \xi = \gamma)$. Then by definition, $x \in \mathsf{GC}^*_{\beta,\gamma}$.

Conversely, suppose that $x \in \mathsf{GC}^*_{\beta,\gamma}$. Then by definition $\mathsf{decode}^*(x)(\beta,\gamma)$. Suppose also that $\mathsf{decode}^*(x)(\beta,\xi)$ for some $\xi < \omega_1$. Then $(\forall^*_{\omega+\beta} f) K^\beta_\gamma(x, \mathfrak{G}(f))$ and $(\forall^*_{\omega+\beta} f) K^\beta_\xi(x, \mathfrak{G}(f))$. The sets $E = \{f \in {}^\omega(\omega+\beta) : K^\beta_\gamma(x, \mathfrak{G}(f))\}$ and $F = \{f \in {}^\omega(\omega+\beta) : K^\beta_\xi(x, \mathfrak{G}(f))\}$ are both comeager in ${}^\omega(\omega+\beta)$. Let $f \in E \cap F$. Let $n \in \omega$ be such that

$$(\Xi^2_{\tau_x}(\mathfrak{G}(f))^{[n]})^{[0]} \in \mathrm{WO}_\beta \wedge (\forall^\omega m < n)((\Xi^2_{\tau_x}(\mathfrak{G}(f))^{[m]})^{[0]} \notin \mathrm{WO}_\beta.$$

Then $K^\beta_\gamma(x, \mathfrak{G}(f))$ and $K^\beta_\xi(x, \mathfrak{G}(f))$ imply that $(\Xi^2_{\tau_x}(\mathfrak{G}(f))^{[n]})^{[1]} \in \mathrm{WO}_\gamma \cap \mathrm{WO}_\xi$. This implies $\gamma = \xi$. $\square$

<u>Claim 2</u>: For all $h : \omega_1 \to \omega_1$, there is an $x \in \mathbb{R}$ such that $\mathsf{decode}^*(x) = h$.

Proof. Let $x \in \mathbb{R}$ be such that τ_x satisfies Fact 7.17 for the function h. Suppose $\beta < \omega_1$. For all $f \in \mathsf{surj}_{\omega+\beta}$, $\mathfrak{G}(f) \in \mathrm{WO}_{\omega+\beta}$ and thus the sections of $\Xi^2_{\tau_x}(\mathfrak{G}(f))$ codes $h \restriction \delta$ for some $\delta > \omega + \beta$. So for each $f \in \mathsf{surj}_{\omega+\beta}$, $K^\beta_{h(\beta)}(x, \mathfrak{G}(f))$ holds. Since $\mathsf{surj}_{\omega+\beta}$ is a comeager subset of ${}^\omega(\omega+\beta)$, for comeagerly many $f \in {}^\omega(\omega+\beta)$, $K^\beta_{h(\beta)}(x, \mathfrak{G}(f))$. Thus $x \in \mathsf{GC}^*_{\beta,h(\beta)}$. So $\mathsf{decode}^*(x) = h$. $\square$

<u>Claim 3</u>: For $\beta < \omega_1$, let $\mathsf{GC}^*_\beta = \bigcup_{\gamma<\omega_1} \mathsf{GC}^*_{\beta,\gamma}$. If $A \subseteq \mathsf{GC}^*_\beta$ is $\mathbf{\Sigma}^1_1$, then there is a $\delta < \omega_1$ so that $A \subseteq \bigcup_{\gamma<\delta} \mathsf{GC}^*_{\beta,\gamma}$.

Proof. Since $A \subseteq \mathsf{GC}^*_\beta$, for each $x \in A$, let γ_x be the unique γ so that $x \in \mathsf{GC}^*_{\beta,\gamma}$. Let $E_x = \{f \in {}^\omega(\omega+\beta) : K^\beta_{\gamma_x}(x, \mathfrak{G}(f))\}$ which is a comeager set. For each $f \in E_x$, let n^f_x be such that

$$(\Xi^2_{\tau_x}(v)^{[n^f_x]})^{[0]} \in \mathrm{WO}_\beta \wedge (\forall^\omega m < n^f_x)((\Xi^2_{\tau_x}(v)^{[m]})^{[0]} \notin \mathrm{WO}_\beta)$$

Note that for all $f \in E_x$, $(\Xi^2_{\tau_x}(\mathfrak{G})^{[n_x]})^{[1]} \in \mathrm{WO}_{\gamma_x}$.

Define $I \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by $I(x, y, v)$ if and only if there exist n_0, $n_1 \in \omega$ so that the conjunction of the following holds.

(1) $x, y \in A$
(2) $(\Xi^2_{\tau_x}(v)^{[n_0]})^{[0]} \in \mathrm{WO}_\beta \wedge (\forall m < n_0)((\Xi^2_{\tau_x}(v)^{[m]})^{[0]} \notin \mathrm{WO}_\beta)$
(3) $(\Xi^2_{\tau_y}(v)^{[n_1]})^{[0]} \in \mathrm{WO}_\beta \wedge (\forall m < n_1)((\Xi^2_{\tau_y}(v)^{[m]})^{[0]} \notin \mathrm{WO}_\beta)$
(4) $(\Xi^2_{\tau_x}(v)^{[n_0]})^{[1]} <^{\mathrm{ot}}_{\boldsymbol{\Sigma}^1_1} (\Xi^2_{\tau_y}(v)^{[n_1]})^{[1]}$.

Observe that I is $\boldsymbol{\Sigma}^1_1$. Define $J \subseteq \mathbb{R} \times \mathbb{R}$ by $J(x, y)$ if and only if $(\forall^*_{\omega+\beta} f) I(x, y, \mathfrak{G}(f))$. As before, $J(x, y)$ if and only if $(\forall^*_\omega r) I(x, y, \mathfrak{G}^\beta(r))$ and the latter is $\boldsymbol{\Sigma}^1_1$ since the pointclass $\boldsymbol{\Sigma}^1_1$ is closed under continuous substitution and the universal category quantifier by Fact 7.14. Since J is $\boldsymbol{\Sigma}^1_1$, J is ω-Suslin.

One seeks to show that $J(x, y)$ if and only if $\gamma_x < \gamma_y$. Suppose $J(x, y)$ holds. Let $E = \{f \in {}^\omega(\omega+\beta) : I(x, y, \mathfrak{G}(f))\}$ which is comeager. Since $E \cap E_x \cap E_y$ is a comeager subsets of ${}^\omega(\omega+\beta)$, there exists an $f \in E \cap E_x \cap E_y$. Then $\mathrm{ot}((\Xi^2_{\tau_x}(\mathfrak{G}(f))^{[n_x]})^{[1]}) = \gamma_x$ and $\mathrm{ot}((\Xi^2_{\tau_y}(\mathfrak{G}(f))^{[n_y]})^{[1]}) = \gamma_y$. So condition (4) in the definition of I implies that $\gamma_x < \gamma_y$. Conversely, suppose that $\gamma_x < \gamma_y$. Then for all $f \in E_x \cap E_y$, $(\Xi^2_{\tau_x}(\mathfrak{G}(f))^{[n_x]})^{[1]} \in \mathrm{WO}_{\gamma_x}$ and $(\Xi^2_{\tau_y}(\mathfrak{G}(f))^{[n_y]})^{[1]} \in \mathrm{WO}_{\gamma_y}$. So $(\Xi^2_{\tau_x}(\mathfrak{G}(f))^{[n_x]})^{[1]} <^{\mathrm{ot}}_{\boldsymbol{\Sigma}^1_1} (\Xi^2_{\tau_y}(\mathfrak{G}(f))^{[n_y]})^{[1]}$. This shows that n_x and n_y witness the number quantifiers in $I(x, y, \mathfrak{G}(f))$. It has been shown that for all f in the comeager set $E_x \cap E_y$, $I(x, y, \mathfrak{G}(f))$. Thus $J(x, y)$ holds.

Thus J is a wellfounded ω-Suslin relation. By the Kunen-Martin theorem (Fact 7.11), the length of J is less than ω_1. Thus $\{\gamma_x : x \in A\}$ has ordertype less than ω_1 and since ω_1 is regular, there is a $\delta < \omega_1$ so that $\{\gamma_x : x \in A\} \subseteq \delta$. This implies that $A \subseteq \bigcup_{\gamma<\delta} \mathsf{GC}^*_{\beta,\gamma}$. □

It has been verified that these objects form a good coding system for ${}^{\omega_1}\omega_1$. Next, it will be checked that this is a long-uniformly good coding system for ${}^{\omega_1}\omega_1$.

Let $\mathcal{G} : {}^\omega\omega \times {}^\omega\omega \to {}^\omega\omega$ be the function from Fact 7.16. There is a Δ^1_1 function $H : \mathbb{R} \to \mathbb{R}$ so that if $v \in \mathrm{WO}$, then $H(v) \in \mathrm{WO}_{\omega+\mathrm{ot}(v)}$.

Define $S \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times {}^\omega\omega$ by $S(x, v, w, r)$ if and only if there exists an $n \in \omega$ so that the conjunction of the following holds

(1) $(\Xi^2_{\tau_x}(\mathcal{G}(H(v), r))^{[n]})^{[0]} \leq^{\mathrm{ot}}_{\Sigma^1_1} v \wedge v \leq^{\mathrm{ot}}_{\Sigma^1_1} (\Xi^2_{\tau_x}(\mathcal{G}(H(v), r))^{[n]})^{[0]}$.
(2) $(\forall^\omega m < n)[\neg((\Xi^2_{\tau_x}(\mathcal{G}(H(v), r))^{[m]})^{[0]} \leq^{\mathrm{ot}}_{\Pi^1_1} v) \vee \neg(v \leq^{\mathrm{ot}}_{\Pi^1_1} (\Xi^2_{\tau_x}(\mathcal{G}(H(v), r))^{[m]})^{[0]})]$
(3) $(\Xi^2_{\tau_x}(\mathcal{G}(H(v), r))^{[n]})^{[1]} \leq^{\mathrm{ot}}_{\Sigma^1_1} w \wedge w \leq^{\mathrm{ot}}_{\Sigma^1_1} (\Xi^2_{\tau_x}(\mathcal{G}(H(v), r))^{[n]})^{[1]}$.

Observe that S is Σ^1_1.

Define $P \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times {}^\omega\omega$ by $P(x, v, w, r)$ if and only if there exists an $n \in \omega$ so that the conjunction of the following holds.

(1) $(\Xi^2_{\tau_x}(\mathcal{G}(H(v), r))^{[n]})^{[0]} \leq^{\mathrm{ot}}_{\Pi^1_1} v \wedge v \leq^{\mathrm{ot}}_{\Pi^1_1} (\Xi^2_{\tau_x}(\mathcal{G}(H(v), r))^{[n]})^{[0]}$.
(2) $(\forall^\omega m < n)[\neg((\Xi^2_{\tau_x}(\mathcal{G}(H(v), r))^{[m]})^{[0]} \leq^{\mathrm{ot}}_{\Sigma^1_1} v) \vee \neg(v \leq^{\mathrm{ot}}_{\Sigma^1_1} (\Xi^2_{\tau_x}(\mathcal{G}(H(v), r))^{[m]})^{[0]})]$
(3) $(\Xi^2_{\tau_x}(\mathcal{G}(H(v), r))^{[n]})^{[1]} \leq^{\mathrm{ot}}_{\Pi^1_1} w \wedge w \leq^{\mathrm{ot}}_{\Pi^1_1} (\Xi^2_{\tau_x}(\mathcal{G}(H(v), r))^{[n]})^{[1]}$.

Observe that P is Π^1_1.

Note that for all $\beta, \gamma < \omega_1$, $v \in \mathrm{WO}_\beta$, $w \in \mathrm{WO}_\gamma$, and $x \in \mathbb{R}$,

$$x \in \mathsf{GC}^*_{\beta,\gamma} \Leftrightarrow (\forall^*_\omega r)(S(x, v, w, r)) \Leftrightarrow (\forall^*_\omega r)(P(x, v, w, r)).$$

Define $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1}$ and $\mathsf{uGC}_{\boldsymbol{\Pi}^1_1}$ by $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1}(x, v, w) \Leftrightarrow (\forall^*_\omega r)(S(x, v, w, r))$ and $\mathsf{uGC}_{\boldsymbol{\Pi}^1_1}(x, v, w) \Leftrightarrow (\forall^*_\omega r)(P(x, v, w, r))$. By Fact 7.14, $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1}$ is $\boldsymbol{\Sigma}^1_1$ and $\mathsf{uGC}_{\boldsymbol{\Pi}^1_1}$ is $\boldsymbol{\Pi}^1_1$. These two relations witness that the above good coding system is long-uniformly good. □

Fact 7.19. *(Moschovakis) Suppose $n \in \omega$. A $\boldsymbol{\delta}^1_{2n+1}$-length union of $\boldsymbol{\Sigma}^1_{2n+1}$ sets is $\boldsymbol{\Sigma}^1_{2n+2}$.*

Proof. Let $\langle A_\alpha : \alpha \in \boldsymbol{\delta}^1_{2n+1}\rangle$ be a sequence of sets in $\boldsymbol{\Sigma}^1_{2n+1}$. Let $U \subseteq \mathbb{R}^2$ be a $\boldsymbol{\Sigma}^1_{2n+1}$ subset of $\mathbb{R}^2$ which is universal for the $\boldsymbol{\Sigma}^1_{2n+1}$ subsets of $\mathbb{R}$. If $A \in \boldsymbol{\Sigma}^1_{2n+1}$, then say that $e \in \mathbb{R}$ is a code for A if and only if $U_e = A$. Let $P \in \boldsymbol{\Pi}^1_{2n+1}$ be a $\boldsymbol{\Pi}^1_{2n+1}$-complete set and $\pi : P \to \boldsymbol{\delta}^1_{2n+1}$ be a $\boldsymbol{\Pi}^1_{2n+1}$-norm on P. Define $R \subseteq P \times \mathbb{R}$ by $R(x,e)$ if and only if e is a code for $A_{\pi(x)}$. By the Moschovakis coding lemma, there is a $\boldsymbol{\Sigma}^1_{2n+2}$ relation $S \subseteq R$ so that for all $\alpha < \boldsymbol{\delta}^1_{2n+1}$, there is a $x \in P$ and $e \in \mathbb{R}$ so that $\pi(x) = \alpha$ and $S(x,e)$. Then $z \in \bigcup_{\alpha < \boldsymbol{\delta}^1_{2n+1}} A_\alpha$ if and only if $(\exists x)(\exists e)(S(x,e) \wedge U_e(z))$. Thus $\bigcup_{\alpha<\boldsymbol{\delta}^1_{2n+1}} A_\alpha$ is $\boldsymbol{\Sigma}^1_{2n+2}$. ☐

Remark 7.20. The good coding system $(\boldsymbol{\Pi}^1_1, \mathsf{decode}^*, \mathsf{GC}^*_{\beta,\gamma} : \beta,\gamma < \omega_1)$ produced in Fact 7.18 also has the feature that for all $\beta < \omega_1$, $\mathsf{GC}^*_\beta = \bigcup_{\gamma<\omega_1} \mathsf{GC}_{\beta,\gamma}$ is $\boldsymbol{\Pi}^1_1$. To see this, note that $x \in \mathsf{GC}^*_\beta$ if and only if

$$(\forall^*_{\omega+\beta} f)(\exists^\omega n)\Big((\Xi^2_{\tau_x}(\mathfrak{G}(f))^{[n]})^{[0]} \in \mathrm{WO}_\beta \wedge (\forall^\omega m < n)((\Xi^2_{\tau_x}(\mathfrak{G}(f))^{[m]})^{[0]} \notin \mathrm{WO}_\beta) \wedge (\Xi^2_{\tau_x}(\mathfrak{G}(f))^{[n]})^{[1]} \in \mathrm{WO}\Big).$$

The latter expression is $\boldsymbol{\Pi}^1_1$. Since $\mathsf{GC}^* = \bigcap_{\beta<\omega_1} \mathsf{GC}^*_\beta$ which is an $\omega_1 = \boldsymbol{\delta}^1_1$ length intersection of $\boldsymbol{\Pi}^1_1$ sets, Fact 7.19 can also be used to show that GC^* is $\boldsymbol{\Pi}^1_2$.

Definition 7.21. $\langle(\boldsymbol{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta,\gamma < \omega_1), \mathsf{BS}, \mathsf{seq}, \mathsf{nGC}, \mathsf{merge}\rangle$ is a uniformly good coding family for ω_1 if and only if

1. $(\boldsymbol{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta,\gamma < \omega_1)$ is a long-uniformly good coding system for ${}^{\omega_1}\omega_1$
2. BS is $\boldsymbol{\Pi}^1_1$.
3. nGC is $\boldsymbol{\Pi}^1_2$.
4. There is a $\boldsymbol{\Delta}^1_1$ function $\mathsf{length} : \mathbb{R} \to \mathbb{R}$ so that whenever $u \in \mathsf{BS}$, $\mathsf{length}(u) \in \mathrm{WO}$ and $\mathrm{ot}(\mathsf{length}(u)) = |\mathsf{seq}(u)|$.
5. There is a $\boldsymbol{\Sigma}^1_1$ relation $\mathsf{uBS}_{\boldsymbol{\Sigma}^1_1} \subseteq \mathbb{R}^3$ so that for all $u \in \mathsf{BS}$ and $v,w \in \mathbb{R}$, $\mathsf{uBS}_{\boldsymbol{\Sigma}^1_1}(u,v,w)$ if and only if $v \in \mathrm{WO} \wedge w \in \mathrm{WO} \wedge \mathrm{ot}(v) < |\mathsf{seq}(u)| \wedge \mathsf{seq}(u)(\mathrm{ot}(v)) = \mathrm{ot}(w)$.
6. There is a $\boldsymbol{\Pi}^1_1$ relation $\mathsf{uBS}_{\boldsymbol{\Pi}^1_1} \subseteq \mathbb{R}^3$ so that for all $u \in \mathsf{BS}$ and $v,w \in \mathbb{R}$, $\mathsf{uBS}_{\boldsymbol{\Pi}^1_1}(u,v,w)$ if and only if $v \in \mathrm{WO} \wedge w \in \mathrm{WO} \wedge \mathrm{ot}(v) < |\mathsf{seq}(u)| \wedge \mathsf{seq}(u)(\mathrm{ot}(v)) = \mathrm{ot}(w)$.

Fact 7.22. *Suppose* $\langle(\boldsymbol{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta,\gamma \leq \omega_1), \mathsf{BS}, \mathsf{seq}, \mathsf{nGC}, \mathsf{merge}\rangle$ *is a uniformly good coding family for* ω_1.

1. *There is a* $\boldsymbol{\Pi}^1_1$ *relation* $\mathsf{INC}^{\mathsf{BS}} \subseteq \mathbb{R}^2$ *so that for all* $z \in \mathsf{clubcode}$, $\mathsf{INC}^{\mathsf{BS}}(z,u)$ *if and only if* $u \in \mathsf{BS}$ *and* $\mathsf{seq}(u) \in [\mathfrak{C}_z]^{<\omega_1}$.
2. *For any* $A \subseteq \omega_1$, *define* $\mathsf{Inc}^{\mathsf{BS}}(A) = \{u \in \mathsf{BS} : \mathsf{seq}(u) \in [A]^{<\omega_1}\}$. *If* $z \in \mathsf{clubcode}$, *then* $\mathsf{Inc}^{\mathsf{BS}}(\mathfrak{C}_z)$ *is* $\boldsymbol{\Pi}^1_1$.
3. *There is a* $\boldsymbol{\Pi}^1_1$ *relation* ${}^\omega\mathsf{BS}$ *such that* ${}^\omega\mathsf{BS}(u)$ *if and only if* $u \in \mathsf{BS}$ *and there is an* $\epsilon \leq |\mathsf{seq}(u)|$ *so that* $|\mathsf{seq}(u)| = \omega \cdot \epsilon$.
4. *There is a* $\boldsymbol{\Pi}^1_1$ *relation* $\mathsf{initSeg}$ *so that* $\mathsf{initSeg}(u,v,\bar{u})$ *if and only if* $u,\bar{u} \in \mathsf{BS}$, $v \in \mathrm{WO}$, $\mathrm{ot}(v) \leq \mathsf{length}(u)$, $\mathsf{length}(\bar{u}) = v$, $\mathsf{seq}(\bar{u}) = \mathsf{seq}(u) \restriction \mathrm{ot}(v)$.
5. *There is a* $\boldsymbol{\Pi}^1_1$ *relation* $\mathsf{sameBlock}^{\mathsf{BS}}$ *so that* $\mathsf{sameBlock}^{\mathsf{BS}}(u_0,u_1)$ *holds if and only if*
   (a) $u_0 \in \mathsf{BS}$ *and* $u_1 \in \mathsf{BS}$.
   (b) $|\mathsf{seq}(u_0)| = |\mathsf{seq}(u_1)|$.
   (c) ${}^\omega\mathsf{BS}(u_0)$.
   (d) $\mathsf{block}(\mathsf{seq}(u_0)) = \mathsf{block}(\mathsf{seq}(u_1))$.
6. *There is a* $\boldsymbol{\Pi}^1_1$ *relation* $\mathsf{acceptableMerge}$ *so that whenever* $e \in \mathsf{GC}$, $\mathsf{acceptableMerge}(u,e)$ *if and only if* ${}^\omega\mathsf{BS}(u)$ *and* $\sup(\mathsf{seq}(u)) < \mathsf{decode}(e)(|\mathsf{seq}(u)|)$.

Proof. Define $\mathsf{INC}^{\mathsf{BS}}$ by $\mathsf{INC}^{\mathsf{BS}}(z,u)$ if and only if

1. $u \in \mathsf{BS}$
2. $(\forall v)(\forall w)(\mathsf{uBS}_{\boldsymbol{\Sigma}^1_1}(u,v,w) \Rightarrow \mathsf{inClub}(z,w))$.
3. $(\forall v_0)(\forall v_1)(\forall w_0)(\forall w_1)\{[\mathsf{uBS}_{\boldsymbol{\Sigma}^1_1}(u,v_0,w_0) \wedge \mathsf{uBS}_{\boldsymbol{\Sigma}^1_1}(u,v_1,w_1) \wedge v_0 <^{\mathrm{ot}}_{\Sigma^1_1} v_1] \Rightarrow w_0 <^{\mathrm{ot}}_{\Pi^1_1} w_1\}$.

This expression is $\boldsymbol{\Pi}^1_1$.

If $z \in \mathsf{clubcode}$, then $u \in \mathsf{Inc}^{\mathsf{BS}}(\mathfrak{C}_z)$ if and only if $\mathsf{INC}^{\mathsf{BS}}(z,u)$. Thus $\mathsf{Inc}^{\mathsf{BS}}(\mathfrak{C}_z)$ is $\boldsymbol{\Pi}^1_1$.

There is a Δ^1_1 function $S : \mathbb{R} \to \mathbb{R}$ so that for all $w \in \mathrm{WO}$, $\mathrm{ot}(S(w)) = \mathrm{ot}(w) + 1$. There is a Δ^1_1 function $H : \mathbb{R} \to \mathbb{R}$ so that whenever $w \in \mathrm{WO}$, $\mathrm{ot}(H(w)) = \omega \cdot \mathrm{ot}(w)$. There is also a Δ^1_1 funtion $T : \mathbb{R} \to \mathbb{R}$ so that whenever $w \in \mathrm{WO}$, $\mathrm{ot}(T(w)) = \omega \cdot \mathrm{ot}(w) + \omega$, namely $H \circ S$.

Define ${}^{\omega}\mathsf{BS}$ by ${}^{\omega}\mathsf{BS}(u)$ if and only

(1) $u \in \mathsf{BS}$.
(2) The disjunction of the following:
 (a) $\mathsf{length}(u) \leq^{\mathrm{ot}}_{\Pi^1_1} H(\mathsf{length}(u)) \wedge H(\mathsf{length}(u)) \leq^{\mathrm{ot}}_{\Pi^1_1} \mathsf{length}(u)$.
 (b) $(\exists^n \omega)(\mathsf{length}(u) \leq^{\mathrm{ot}}_{\Pi^1_1} H(\mathsf{rest}(\mathsf{length}(u), n)) \wedge H(\mathsf{rest}(\mathsf{length}(u), n)) \leq_{\Pi^1_1} \mathsf{length}(u))$.

(2a) asserts that $|\mathsf{seq}(u)| = \omega \cdot |\mathsf{seq}(u)|$. (2b) asserts that there is an $\epsilon < |\mathsf{seq}(u)|$ so that $|\mathsf{seq}(u)| = \omega \cdot \epsilon$.

Define $\mathsf{initSeg}(u, v, \bar{u})$ if and only if

(1) $u \in \mathsf{BS}$, $\bar{u} \in \mathsf{BS}$.
(2) $v \leq^{\mathrm{ot}}_{\Pi^1_1} \mathsf{length}(u)$
(3) $v \leq^{\mathrm{ot}}_{\Pi^1_1} \mathsf{length}(\bar{u}) \wedge \mathsf{length}(\bar{u}) \leq^{\mathrm{ot}}_{\Pi^1_1} v$.
(4) $(\forall^{\omega} n)(\forall^{\mathbb{R}} w, \bar{w})\Big((\mathsf{uBS}_{\mathbf{\Sigma}^1_1}(u, \mathsf{rest}(v, n), w) \wedge \mathsf{uBS}_{\mathbf{\Sigma}^1_1}(\bar{u}, \mathsf{rest}(v, n), \bar{w}) \Rightarrow (w \leq^{\mathrm{ot}}_{\Pi^1_1} \bar{w} \wedge \bar{w} \leq^{\mathrm{ot}}_{\Pi^1_1} w)\Big)$

Define $\mathsf{sameBlock}^{\mathsf{BS}}$ by $\mathsf{sameBlock}^{\mathsf{BS}}(u_0, u_1)$ if and only if the conjunction of the following holds.

(1) $u_0 \in {}^{\omega}\mathsf{BS} \wedge u_1 \in {}^{\omega}\mathsf{BS}$.
(2) $\mathsf{length}(u_0) \leq^{\mathrm{ot}}_{\Pi^1_1} \mathsf{length}(u_1) \wedge \mathsf{length}(u_1) \leq^{\mathrm{ot}}_{\Pi^1_1} \mathsf{length}(u_0)$.
(3) $(\exists^{\omega} \iota)[E(u_0, \iota) \wedge (\forall^{\omega} \ell)(\forall^{\omega} m_0)(\forall w_0)\{A(u_0, \iota, \ell, m_0, w_0) \Rightarrow (\exists^{\omega} m_1)(\forall^{\mathbb{R}} w_1) B(u_0, u_1, \iota, \ell, m_0, w_0, m_1, w_1)\}]$
(4) $(\exists^{\omega} \iota)[E(u_1, \iota) \wedge (\forall^{\omega} \ell)(\forall^{\omega} m_0)(\forall w_0)\{A(u_1, \iota, \ell, , m_0, w_0) \Rightarrow (\exists^{\omega} m_1)(\forall^{\mathbb{R}} w_1) B(u_1, u_0, \iota, \ell, m_0, w_0, m_1, w_1)\}]$

where

- $E(a, \iota)$ asserts $H(\mathsf{rest}(S(\mathsf{length}(a)), \iota)) \leq^{\mathrm{ot}}_{\Sigma^1_1} \mathsf{length}(a) \wedge \mathsf{length}(a) \leq^{\mathrm{ot}}_{\Sigma^1_1} H(\mathsf{rest}(S(\mathsf{length}(a)), \iota))$
- $A(a, \iota, \ell, m_0, w_0)$ asserts $\mathsf{uBS}_{\mathbf{\Sigma}^1_1}(a, \mathsf{rest}(T(\mathsf{rest}(\mathsf{rest}(S(\mathsf{length}(a)), \iota), \ell), m_0)), w_0)$.
- $B(a, b, \iota, \ell, m_0, w_0, m_1, w_1)$ asserts the conjunction of
 (1) $\mathsf{uBS}_{\Pi^1_1}(b, \mathsf{rest}(T(\mathsf{rest}(\mathsf{rest}(S(\mathsf{length}(a)), \iota), \ell), m_1)), w_1)$.
 (2) $w_0 <^{\mathrm{ot}}_{\Pi^1_1} w_1$.

To understand (3): Let $\zeta = |\mathsf{seq}(u_0)| = |\mathsf{seq}(u_1)|$. $\mathrm{ot}(S(\mathsf{length}(u_0))) = \zeta + 1$. Thus if $\epsilon = \mathrm{ot}(\mathsf{rest}(S(\mathsf{length}(u_0)), \iota))$, then $\epsilon < \zeta + 1$. $E(u_0, \iota)$ asserts that $\omega \cdot \epsilon = \zeta$. Now $A(u_0, \iota, \ell, m_0, w_0)$ means the following. Let $\alpha = \mathrm{ot}(\mathsf{rest}(\mathsf{rest}(S(\mathsf{length}(u_0)), \iota), \ell))$ which represents an ordinal $\alpha < \epsilon$. Then $\mathrm{ot}(T(\mathsf{rest}(\mathsf{rest}(S(\mathsf{length}(u_0)), \iota), \ell))) = \omega \cdot \alpha + \omega$. $\gamma = \mathsf{rest}(T(\mathsf{rest}(\mathsf{rest}(S(\mathsf{length}(u_0)), \iota), \ell)), m_0)$ represents an ordinal $\gamma < \omega \cdot \alpha + \omega$. Then $\mathrm{ot}(w_0) = \mathsf{seq}(u_0, \gamma)$. Within the expession $B(u_0, u_1, \iota, \ell, m_0, w_0, m_1, w_1)$, $\nu = \mathrm{ot}(\mathsf{rest}(T(\mathsf{rest}(\mathsf{rest}(S(\mathsf{length}(u_0)), \iota), \ell)), m_1)$ represents an ordinal $\nu < \omega \cdot \alpha + \omega$ so that $w_1 = \mathsf{seq}(u_1)(\nu)$ and $\mathsf{seq}(u_0)(\gamma) < \mathsf{seq}(u_1)(\nu)$. (4) has the same meaning as (3) with the role of u_0 and u_1 reversed. Together this gives the intended meaning of $\mathsf{sameBlock}^{\mathsf{BS}}$. Note that E and A are $\mathbf{\Sigma}^1_1$ and B is $\mathbf{\Pi}^1_1$. Thus both (3) and (4) are $\mathbf{\Pi}^1_1$ and hence $\mathsf{sameBlock}$ is $\mathbf{\Pi}^1_1$.

Define $\mathsf{acceptableMerge}$ by $\mathsf{acceptableMerge}(u, e)$ if and only if

(1) $u \in {}^{\omega}\mathsf{BS}$
(2) $(\exists^{\omega} n)(\forall^{\mathbb{R}} v, w, y)\{[v <^{\mathrm{ot}}_{\Sigma^1_1} \mathsf{length}(u) \wedge \mathsf{uBS}_{\mathbf{\Sigma}^1_1}(u, v, w) \wedge \mathsf{uGC}_{\mathbf{\Sigma}^1_1}(e, \mathsf{length}(u), y)] \Rightarrow w \leq^{\mathrm{ot}}_{\Pi^1_1} \mathsf{rest}(y, n)\}$.

This expression is $\mathbf{\Pi}^1_1$. ☐

Theorem 7.23. *If there exists a long-uniformly good coding system $(\mathbf{\Pi}^1_1, \mathsf{decode}^*, \mathsf{GC}^*_{\beta,\gamma} : \beta, \gamma < \omega_1)$ for ${}^{\omega_1}\omega_1$, then there exists a uniformly good coding family $\langle(\mathbf{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \omega_1), \mathsf{BS}, \mathsf{seq}, \mathsf{merge}\rangle$ for ω_1.*

Proof. Let $(\mathbf{\Pi}^1_1, \mathsf{decode}^*, \mathsf{GC}^*_{\beta,\gamma} : \beta, \gamma < \omega_1)$ be a fixed long-uniformly good coding system for ${}^{\omega_1}\omega_1$. Let $\mathsf{uGC}^*_{\mathbf{\Sigma}^1_1}$ and $\mathsf{uGC}^*_{\mathbf{\Pi}^1_1}$ be the relations witnessing the long-uniform goodness.

Define BS by $u \in \mathsf{BS}$ if and only if $u^{[0]} \in \mathrm{WO}$ and for all $m \in \mathsf{field}(u^{[0]})$, $(u^{[1]})^{[m]} \in \mathrm{WO}$. Note that BS is Π^1_1. Let $\mathsf{length} : \mathbb{R} \to \mathbb{R}$ by $\mathsf{length}(u) = u^{[0]}$ which is a Δ^1_1 function. Let $\mathsf{seq} : \mathsf{BS} \to {}^{<\omega_1}\omega_1$ be defined by $\mathsf{seq}(u) \in {}^{\mathrm{ot}(\mathsf{length}(u))}\omega_1$ and for all $\alpha < \mathsf{length}(u)$, $\mathsf{seq}(u)(\alpha) = \mathrm{ot}((u^{[1]})^{[\mathsf{num}(\mathsf{length}(u), \alpha)]})$. It is clear that $\mathsf{seq} : \mathsf{BS} \to {}^{<\omega_1}\omega_1$ is a surjection.

For $\beta < \omega_1$, let $u \in D_\beta$ if and only if $\mathsf{length}(u) \in \mathsf{LO} \wedge (\exists^{\omega} n)(\mathsf{rest}(\mathsf{length}(u), n) \in \mathrm{WO}_\beta)$. Note that D_β is $\mathbf{\Delta}^1_1$. For $\beta, \gamma < \omega_1$, let $u \in \mathsf{BS}_{\beta,\gamma}$ if and only if

$$\mathsf{length}(u) \in \mathsf{LO} \wedge (\exists^{\omega} n)(\mathsf{rest}(\mathsf{length}(u), n) \in \mathrm{WO}_\beta \wedge (u^{[1]})^{[n]} \in \mathrm{WO}_\gamma).$$

Note that $\mathsf{BS}_{\beta,\gamma}$ is $\mathbf{\Delta}_1^1$.

Now for $\beta, \gamma < \omega_1$, define $\mathsf{GC}_{\beta,\gamma}$ by $(u, e) \in \mathsf{GC}_{\beta,\gamma}$ if and only if

$$(u \in D_\beta \Rightarrow u \in \mathsf{BS}_{\beta,\gamma}) \wedge (u \notin D_\beta \Rightarrow e \in \mathsf{GC}^*_{\beta,\gamma}).$$

Note $\mathsf{GC}^*_{\beta,\gamma} \in \mathbf{\Delta}_1^1$ since it comes from the fixed good coding system for ${}^{\omega_1}\omega_1$. From this and the above observations, $\mathsf{GC}_{\beta,\gamma}$ is $\mathbf{\Delta}_1^1$.

Define $\mathsf{decode} : \mathbb{R} \to \mathscr{P}(\omega_1 \times \omega_1)$ by $\mathsf{decode}(u, e)(\beta, \gamma)$ if and only if $(u, e) \in \mathsf{GC}_{\beta,\gamma}$. One can check that $\mathsf{decode}(u, e)(\beta, \gamma)$ holds if and only if $(u, e) \in \mathsf{GC}_{\beta,\gamma} \wedge (\forall \gamma' < \omega_1)(\mathsf{decode}(u, e)(\beta, \gamma') \Rightarrow (\gamma = \gamma'))$ by using the definition of $\mathsf{BS}_{\beta,\gamma}$ and the definition of $\mathsf{GC}^*_{\beta,\gamma}$ which comes from the fixed good coding system for ${}^{\omega_1}\omega_1$.

Note that $(u, e) \in \mathsf{GC}$ does not imply that $u \in \mathsf{BS}$ since it may be that $\mathsf{length}(u) \in \mathsf{LO} \setminus \mathrm{WO}$. Intuitively if $(u, e) \in \mathsf{GC}$, then $\mathsf{decode}(u, e)$ is the function with the following properties.

- If β is an ordinal less than the ordertype of the wellfounded part of $\mathsf{length}(u) \in \mathsf{LO}$ and $n \in \omega$ is such that $\mathrm{ot}(\mathsf{rest}(u^{[0]}, n)) = \beta$, then $\mathsf{decode}(u, e)(\beta) = \mathrm{ot}((u^{[1]})^{[n]})$.
- If β is greater than or equal to the ordertype of the wellfounded part of $\mathsf{length}(u) \in \mathsf{LO}$, then $\mathsf{decode}(u, e)(\beta) = \mathsf{decode}^*(e)(\beta)$.

Thus one has defined $(\mathbf{\Pi}_1^1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \omega_1)$. Now suppose $A \subseteq \mathsf{GC}_\beta$ and A is $\mathbf{\Sigma}_1^1$. Let $A_0 = \{(u, e) : (u, e) \in A \wedge u \in D_\beta\}$. Let $A_1 = \{(u, e) : (u, e) \in A \wedge u \notin D_\beta\}$. Note that since D_β is $\mathbf{\Delta}_1^1$, A_0 and A_1 are $\mathbf{\Sigma}_1^1$.

Let B_0 be defined by $v \in B_0$ if and only if

$$(\exists^{\mathbb{R}} u, e)(\exists^\omega n)((u, e) \in A_0 \wedge \mathsf{rest}(x, n) \in \mathrm{WO}_\beta \wedge v = (u^{[1]})^{[n]}).$$

Note that $(u, e) \in \mathsf{GC}_\beta$ and $u \in D_\beta$ imply that there is a $\gamma < \omega_1$ such that $u \in \mathsf{BS}_{\beta,\gamma}$. Thus B_0 is $\mathbf{\Sigma}_1^1$ and $B_0 \subseteq \mathrm{WO}$. By boundedness (Fact 4.5), there is a $\delta_0 < \omega_1$ so that $\mathrm{ot}(v) < \delta_0$ for all $v \in B_0$. This shows that $A_0 \subseteq \bigcup_{\gamma < \delta_0} \mathsf{GC}_{\beta,\gamma}$.

Let B_1 be defined by $v \in B_1$ if and only if $(\exists^{\mathbb{R}} u, e)((u, e) \in A_1 \wedge v = e)$. By definition of A_1, B_1 is $\mathbf{\Sigma}_1^1$ and $B_1 \subseteq \mathsf{GC}^*_\beta$. Using the boundedness property of the original fixed good coding system for ${}^{\omega_1}\omega_1$, there is a $\delta_1 < \omega_1$ so that $B_1 \subseteq \bigcup_{\gamma < \delta_1} \mathsf{GC}^*_{\beta,\gamma}$. Then this implies that $A_1 \subseteq \bigcup_{\gamma < \delta_1} \mathsf{GC}_{\beta,\gamma}$.

Let $\delta = \sup\{\delta_0, \delta_1\}$. It has been established that $A \subseteq \bigcup_{\gamma < \delta} \mathsf{GC}_{\beta,\gamma}$. This completes the argument that $(\mathbf{\Pi}_1^1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \omega_1)$ is also a good coding system for ${}^{\omega_1}\omega_1$.

Define merge by $\mathsf{merge}(u, (u', e)) = (u, e)$. Note that merge is continuous. Fix a $u^* \in \mathsf{BS}$ so that $\mathsf{seq}(u^*) = \emptyset$. Let $\mathsf{nGC} = \{(u^*, e) : e \in \mathsf{GC}^*\}$. One can verify that merge and nGC satisfies the last condition in Definition 5.10, that is, $\mathsf{merge}(u, (u^*, e)) = (u, e)$ has the intended meaning whenever $u \in \mathsf{BS}$ and $(u^*, e) \in \mathsf{nGC}$. (Note that if $(v, e) \notin \mathsf{nGC}$ with $\mathsf{seq}(v) \neq \emptyset$ and $u \in \mathsf{BS}$ with $|\mathsf{seq}(u)| < |\mathsf{seq}(v)|$, then $\mathsf{merge}(u, (v, e)) = (u, e)$ may fail to have the intended meaning.)

Now pick $\epsilon < \omega_1$, $\ell \in [\omega_1]^\epsilon_*$, and $\sigma \subseteq \sup(\ell)$ so that for each $\alpha < \epsilon$, $\ell(\alpha)$ is a limit point of σ. Let $A = \{u : \mathsf{length}(u) \in \mathrm{WO}_{\omega \cdot \epsilon}\}$ which belongs to $\mathbf{\Delta}_1^1$. Let $B = \{u \in A : (\forall \alpha < \omega \cdot \epsilon)(\exists \xi \in \sigma)((u^{[1]})^{[\mathsf{num}(\mathsf{length}(u), \alpha)]} \in \mathrm{WO}_\xi)\}$. For each $\alpha < \omega \cdot \epsilon$, define $T_\alpha \subseteq \mathbb{R} \times \omega$ by $T_\alpha(u, n)$ if and only if $\mathsf{rest}(\mathsf{length}(u), n) \in \mathrm{WO}_\alpha$ which is a $\mathbf{\Delta}_1^1$ set. Observe that $u \in B$ if and only if $u \in A \wedge (\forall \alpha < \omega_1)(\exists \xi \in \sigma)(\exists n)(T_\alpha(u, n) \wedge (u^{[1]})^{[n]} \in \mathrm{WO}_\xi)$. The set B is $\mathbf{\Delta}_1^1$ since σ is countable and the pointclass of $\mathbf{\Delta}_1^1$ sets is closed under countable unions and intersection. For all $u \in B$, $\mathsf{seq}(u) \in {}^{\omega \cdot \epsilon}\sigma$. Let $C = \{u \in B : (\forall^\omega m, n)(\mathcal{R}_{\mathsf{length}(u)}(m, n) \Rightarrow \mathrm{ot}((u^{[1]})^{[m]}) < \mathrm{ot}((u^{[1]})^{[n]}))\}$. Note that $C \in \mathbf{\Delta}_1^1$ since ot is a Π_1^1 norm and therefore it is $\mathbf{\Delta}_1^1$ when restricted to $\mathrm{WO}_{<\sup(\sigma)}$. For all $u \in C$, $\mathsf{seq}(u)$ is increasing and therefore, $\mathsf{seq}(u) \in [\sigma]^{\omega \cdot \epsilon}$.

For each $\alpha < \epsilon$ and $\nu < \ell(\alpha)$, let

$$E_{\alpha,\nu} = \{u \in C : (\exists^\omega n)(\nu < \mathrm{ot}((u^{[1]})^{[\mathsf{num}(\mathsf{length}(u), \alpha + n)]}) < \ell(\alpha))\}.$$

Observe $E_{\alpha,\nu}$ is $\mathbf{\Delta}_1^1$ for each $\alpha < \epsilon$ and $\nu < \ell(\alpha)$. Let $E_\alpha = \bigcap_{\nu < \ell(\alpha)} E_{\alpha,\nu}$ which is also $\mathbf{\Delta}_1^1$. For any $u \in E_\alpha$, $\mathsf{block}(\mathsf{seq}(u))(\alpha) = \ell(\alpha)$. Thus $\mathsf{BSS}^{\ell,\sigma} = \bigcap_{\alpha < \epsilon} E_\alpha$ which is $\mathbf{\Delta}_1^1$.

This shows that $\langle (\mathbf{\Pi}_1^1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \omega_1), \mathsf{BS}, \mathsf{seq}, \mathsf{nGC}, \mathsf{merge} \rangle$ is a good coding family for ω_1. It remains to show that this is a uniformly good coding family for ω_1.

Let $\mathsf{D}_{\mathbf{\Sigma}_1^1} \subseteq \mathbb{R} \times \mathbb{R}$ by $\mathsf{D}_{\mathbf{\Sigma}_1^1}(u, v)$ if and only if

$$\mathsf{length}(u) \in \mathsf{LO} \wedge (\exists^\omega n)(\mathsf{rest}(\mathsf{length}(u), n) \leq^{\mathrm{ot}}_{\Sigma_1^1} v \wedge v \leq^{\mathrm{ot}}_{\Sigma_1^1} \mathsf{rest}(\mathsf{length}(u), n)).$$

Let $\mathsf{D}_{\boldsymbol{\Pi}^1_1} \subseteq \mathbb{R} \times \mathbb{R}$ by $\mathsf{D}_{\boldsymbol{\Pi}^1_1}(u, v)$ if and only if

$$\mathsf{length}(u) \in \mathsf{LO} \wedge (\exists^\omega n)(\mathsf{rest}(\mathsf{length}(u), n) \leq^{\mathrm{ot}}_{\Pi^1_1} v \wedge v \leq^{\mathrm{ot}}_{\Pi^1_1} \mathsf{rest}(\mathsf{length}(u), n)).$$

Note that $\mathsf{D}_{\boldsymbol{\Sigma}^1_1}$ is $\boldsymbol{\Sigma}^1_1$, $\mathsf{D}_{\boldsymbol{\Pi}^1_1}$ is $\boldsymbol{\Pi}^1_1$, and if $v \in \mathrm{WO}$ and $\beta = \mathrm{ot}(v)$, then

$$u \in D_\beta \Leftrightarrow \mathsf{D}_{\boldsymbol{\Sigma}^1_1}(u, v) \Leftrightarrow \mathsf{D}_{\boldsymbol{\Pi}^1_1}(u, v).$$

Let $\mathsf{B}_{\boldsymbol{\Sigma}^1_1} \subseteq \mathbb{R}^3$ be define by $\mathsf{B}_{\boldsymbol{\Sigma}^1_1}(u, v, w)$ if and only if the conjunction of the following holds.

- $\mathsf{length}(u) \in \mathsf{LO}$
- $(\exists^\omega n)(\mathsf{rest}(\mathsf{length}(u), n) \leq^{\mathrm{ot}}_{\Sigma^1_1} v \wedge v \leq^{\mathrm{ot}}_{\Sigma^1_1} \mathsf{rest}(\mathsf{length}(u), n) \wedge (u^{[1]})^{[n]} \leq^{\mathrm{ot}}_{\Sigma^1_1} w \wedge w \leq^{\mathrm{ot}}_{\Sigma^1_1} (u^{[1]})^{[n]})$

Let $\mathsf{B}_{\boldsymbol{\Pi}^1_1} \subseteq \mathbb{R}^3$ be define by $\mathsf{B}_{\boldsymbol{\Pi}^1_1}(u, v, w)$ if and only if the conjunction of the following holds.

- $\mathsf{length}(u) \in \mathsf{LO}$
- $(\exists^\omega n)(\mathsf{rest}(\mathsf{length}(u), n) \leq^{\mathrm{ot}}_{\Pi^1_1} v \wedge v \leq^{\mathrm{ot}}_{\Pi^1_1} \mathsf{rest}(\mathsf{length}(u), n) \wedge (u^{[1]})^{[n]} \leq^{\mathrm{ot}}_{\Pi^1_1} w \wedge w \leq^{\mathrm{ot}}_{\Pi^1_1} (u^{[1]})^{[n]})$

Note that $\mathsf{B}_{\boldsymbol{\Sigma}^1_1}$ is $\boldsymbol{\Sigma}^1_1$, $\mathsf{B}_{\boldsymbol{\Pi}^1_1}$ is $\boldsymbol{\Pi}^1_1$, and if $v, w \in \mathrm{WO}$ with $\beta = \mathrm{ot}(v)$ and $\gamma = \mathrm{ot}(w)$, then

$$u \in \mathsf{BS}_{\beta,\gamma} \Leftrightarrow \mathsf{B}_{\boldsymbol{\Sigma}^1_1}(u, v, w) \Leftrightarrow \mathsf{B}_{\boldsymbol{\Pi}^1_1}(u, v, w).$$

Define $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1} \subseteq \mathbb{R}^3$ by $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1}((u, e), v, w)$ if and only if the conjunction of the following holds.

$$[\mathsf{D}_{\boldsymbol{\Pi}^1_1}(u, v, w) \Rightarrow \mathsf{B}_{\boldsymbol{\Sigma}^1_1}(u, v, w)] \wedge [\neg \mathsf{D}_{\boldsymbol{\Sigma}^1_1}(u, v, w) \Rightarrow \mathsf{uGC}_{\boldsymbol{\Sigma}^1_1}(e, v, w)].$$

Define $\mathsf{uGC}_{\boldsymbol{\Pi}^1_1} \subseteq \mathbb{R}^3$ by $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1}((u, e), v, w)$ if and only if the conjunction of the following holds.

$$[\mathsf{D}_{\boldsymbol{\Sigma}^1_1}(u, v, w) \Rightarrow \mathsf{B}_{\boldsymbol{\Pi}^1_1}(u, v, w)] \wedge [\neg \mathsf{D}_{\boldsymbol{\Pi}^1_1}(u, v, w) \Rightarrow \mathsf{uGC}_{\boldsymbol{\Pi}^1_1}(e, v, w)].$$

Note that $\mathsf{uGC}_{\boldsymbol{\Sigma}^1_1}$ is $\boldsymbol{\Sigma}^1_1$ and $\mathsf{uGC}_{\boldsymbol{\Pi}^1_1}$ is $\boldsymbol{\Pi}^1_1$. These two relations witness that the new good coding system $(\boldsymbol{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \omega_1)$ for ${}^{\omega_1}\omega_1$ is long-uniformly good.

It has already been shown BS is $\boldsymbol{\Pi}^1_1$. The $\boldsymbol{\Delta}^1_1$ function length has already been defined above.

Define $\mathsf{uBS}_{\boldsymbol{\Sigma}^1_1}$ by $\mathsf{uBS}_{\boldsymbol{\Sigma}^1_1}(u, v, w)$ if and only if

$$v <^{\mathrm{ot}}_{\Sigma^1_1} \mathsf{length}(u) \wedge (\exists^\omega n)(\mathsf{rest}(\mathsf{length}(u), n) \leq^{\mathrm{ot}}_{\Sigma^1_1} v \wedge v \leq^{\mathrm{ot}}_{\Sigma^1_1} \mathsf{rest}(\mathsf{length}(u), n) \wedge (u^{[1]})^{[n]} \leq^{\mathrm{ot}}_{\Sigma^1_1} w \wedge w \leq^{\mathrm{ot}}_{\Sigma^1_1} (u^{[1]})^{[n]}).$$

$\mathsf{uBS}_{\boldsymbol{\Sigma}^1_1}$ is $\boldsymbol{\Sigma}^1_1$.

Define $\mathsf{uBS}_{\boldsymbol{\Pi}^1_1}$ by $\mathsf{uBS}_{\boldsymbol{\Pi}^1_1}(u, v, w)$ if and only if

$$v <^{\mathrm{ot}}_{\Pi^1_1} \mathsf{length}(u) \wedge (\exists^\omega n)(\mathsf{rest}(\mathsf{length}(u), n) \leq^{\mathrm{ot}}_{\Pi^1_1} v \wedge v \leq^{\mathrm{ot}}_{\Pi^1_1} \mathsf{rest}(\mathsf{length}(u), n) \wedge (u^{[1]})^{[n]} \leq^{\mathrm{ot}}_{\Pi^1_1} w \wedge w \leq^{\mathrm{ot}}_{\Pi^1_1} (u^{[1]})^{[n]}).$$

$\mathsf{uBS}_{\boldsymbol{\Pi}^1_1}$ is $\boldsymbol{\Pi}^1_1$.

These objects verify that this is a uniformly good coding family for ω_1. ☐

Theorem 7.24. *There exists a uniformly good coding family for ω_1.*

Proof. This follows from Fact 7.18 and Theorem 7.23. ☐

Theorem 7.25. *([5] Theorem 3.10) (Strong almost everywhere short length club uniformization at ω_1) For every relation $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed in the club_{ω_1}-coordinate, there is a club $C \subseteq \omega_1$ so that for all $\ell \in [C]^{<\omega_1}_*$, $R(\ell, C \setminus (\sup(\ell) + 1))$.*

Proof. This follows from Theorem 7.24 and Theorem 6.2. ☐

8. Almost Everywhere Behavior of Functions

If δ is a δ-reasonable cardinal, then μ^δ_ϵ is δ-complete for all $\epsilon \leq \delta$ by Fact 5.5. This implies that for every function $\Phi : [\delta]^\epsilon \to \kappa$ where $\kappa < \delta$, there is a club $C \subseteq \delta$ and an $\alpha < \kappa$ so that $\Phi[[C]^\delta_*] = \{\alpha\}$. Thus μ^δ_ϵ-almost everywhere, $\Phi : [\delta]^\epsilon \to \kappa$ is a constant function if $\kappa < \delta$.

Next, one will investigate the $\mu^{\omega_1}_\epsilon$-almost everywhere behavior of functions from $\Phi : [\omega_1]^\epsilon \to \kappa$ when $\kappa > \omega_1$ is a regular cardinal and $\epsilon \leq \omega_1$.

Fact 8.1. *(Steel, [13] Theorem 2.28) Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. Let κ be a limit ordinal. Then there is a $P \subseteq \mathbb{R}$ and a surjective norm $\pi : P \to \kappa$ which is δ-Suslin bounded for all $\delta < \mathrm{cof}(\kappa)$: meaning whenever $Q \subseteq P$ is δ-Suslin, there is a $\zeta < \kappa$ so that $\pi[Q] \subseteq \zeta$.*

Theorem 8.2. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$*. Let* $\epsilon < \omega_1$ *and* $\kappa < \Theta$ *with* $\mathrm{cof}(\kappa) > \omega_1$*. Suppose* $\Phi : [\omega_1]^\epsilon \to \kappa$*. Then there is a* $\zeta < \kappa$ *and a club* $D \subseteq \omega_1$ *so that* $\Phi[[D]^\epsilon_*] \subseteq \zeta$*.*

Proof. Let $(\mathbf{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \omega \cdot \epsilon, \gamma < \omega_1)$ be a short-uniformly good coding system for ${}^{\omega\cdot\epsilon}\omega_1$ (for instance from Example 7.7). By Fact 8.1, let $P \subseteq \mathbb{R}$ and $\pi : P \to \kappa$ be a surjective norm which is ω_1-Suslin bounded. Define a relation $R \subseteq [\omega_1]^\epsilon_* \times P$ by $R(f,x)$ if and only if $\Phi(f) = \pi(x)$. By Theorem 5.9, there is a club $C \subseteq \omega_1$ and a Lipschitz function $\Xi : \mathbb{R} \to \mathbb{R}$ so that for all $x \in \mathsf{GC}$ with $\mathsf{decode}(x) \in [C]^{\omega\cdot\epsilon}$, $R(\mathsf{block}(\mathsf{decode}(x)), \Xi(x))$. Let $z \in \mathsf{clubcode}$ be such that $\mathfrak{C}_z \subseteq C$. By Fact 7.5, $\mathsf{Inc}(\mathfrak{C}_z)$ is $\mathbf{\Pi}^1_1$. Therefore, $\Xi[\mathsf{Inc}(\mathfrak{C}_z)]$ is $\mathbf{\Sigma}^1_2$. Note that $\Xi[\mathsf{Inc}(\mathfrak{C}_z)] \subseteq P$. A $\mathbf{\Sigma}^1_2$ set is ω_1-Suslin. Since π satisfies ω_1-Suslin bounding, there is some $\zeta < \kappa$ so that $\pi[\Xi[\mathsf{Inc}(\mathfrak{C}_z)]] \subseteq \zeta$. Let $D \subseteq \mathfrak{C}_z$ be the club of limit points of $\mathfrak{C}_z$. For any $f \in [D]^\epsilon_*$, there is a $g \in [\mathfrak{C}_z]^{\omega\cdot\epsilon}$ so that $\mathsf{block}(g) = f$. There is an $x \in \mathsf{Inc}(\mathfrak{C}_z)$ so that $\mathsf{decode}(x) = g$. Thus $f = \mathsf{block}(\mathsf{decode}(x))$ and $\Phi(f) = \pi(\Xi(x)) < \zeta$. This completes the proof. ☐

The following is an important instance of Theorem 8.2 (which does not require $\mathsf{DC}_{\mathbb{R}}$ since $L(\mathbb{R}) \models \mathsf{AD}{+}\mathsf{DC}$.)

Corollary 8.3. *Assume* $\mathsf{ZF}{+}\mathsf{AD}$*. Let* $\epsilon < \omega_1$*. If* $\kappa < \Theta^{L(\mathbb{R})}$ *and* $\mathrm{cof}(\kappa) > \omega_1$ *(for instance* ω_2 *or* $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$*), then any function* $\Phi : [\omega_1]^\epsilon \to \kappa$*, there is a club* $C \subseteq \omega_1$ *and a* $\zeta < \kappa$ *so that* $\Phi[[C]^\epsilon_*] \subseteq \delta$*.*

When $\kappa = \omega_1$ and $\epsilon < \omega_1$, one has the following continuity properties.

Fact 8.4. *([8]) Assume* $\mathsf{ZF} + \mathsf{AD}$*. Let* $\epsilon < \omega_1$ *and* $\Phi : [\omega_1]^\epsilon \to \omega_1$*.*

- *(Almost everywhere short function continuity) There is a club* $C \subseteq \omega_1$ *and a* $\delta < \epsilon$ *so that for all* $f, g \in [C]^\epsilon_*$*, if* $f \restriction \delta = g \restriction \delta$ *and* $\sup(f) = \sup(g)$*, then* $\Phi(f) = \Phi(g)$*.*
- *(Almost everywhere strong short function continuity) There is a club* $C \subseteq \omega_1$ *and a finitely many ordinals* $\beta_0 < \beta_1 < ... < \beta_n \leq \epsilon$ *so that for all* $f, g \in [C]^\epsilon_*$*, if for all* $0 \leq i \leq n$*,* $\sup(f \restriction \beta_i) = \sup(g \restriction \beta_i)$*, then* $\Phi(f) = \Phi(g)$*.*

Fact 8.4 has a relatively simple proof under AD using Kunen functions for ω_1 (see [9] Fact 2.5). However a stronger result can be shown by purely combinatorial arguments using suitable partition properties.

Fact 8.5. *([6]) Assume* κ *is a cardinal,* $\epsilon < \kappa$*, and* $\kappa \to (\kappa)^{\epsilon\cdot\epsilon}_2$ *holds. Let* $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$*.*

- *(Almost everywhere short function continuity) If* $\mathrm{cof}(\epsilon) = \omega$*, then there is a club* $C \subseteq \kappa$ *and a* $\delta < \epsilon$ *so that for all* $f, g \in [C]^\epsilon_*$*, if* $f \restriction \delta = g \restriction \delta$ *and* $\sup(f) = \sup(g)$*, then* $\Phi(f) = \Phi(g)$*.*
- *(Almost everywhere strong short function continuity) If* $\epsilon < \omega_1$*, there is a club* $C \subseteq \kappa$ *and finitely many ordinals* $\beta_0 < \beta_1 < ... < \beta_n \leq \epsilon$ *so that for all* $f, g \in [C]^\epsilon_*$*, if for all* $0 \leq i \leq n$*,* $\sup(f \restriction \beta_i) = \sup(g \restriction \beta_i)$*, then* $\Phi(f) = \Phi(g)$*.*

The restriction that $\mathrm{cof}(\epsilon) < \omega_1$ for the almost everywhere short function continuity and that $\epsilon < \omega_1$ for the almost everywhere strong short function continuity are generally necessary. Under AD, counterexamples can be found using a function $\Phi : [\omega_2]^{\omega_1} \to \omega_3$. However, under AD, if one considers functions $\Phi : [\omega_2]^\epsilon \to \omega_2$ for $\epsilon < \omega_2$, then the strong short function continuity can hold when ϵ can hold even if $\epsilon \geq \omega_1$ and $\mathrm{cof}(\epsilon) = \omega_1$.

Fact 8.6. *([8]) Assume* $\mathsf{ZF} + \mathsf{AD}$*. Suppose* $\epsilon < \omega_2$ *and* $\Phi : [\omega_2]^\epsilon \to \omega_1$*.*

- *(Almost everywhere short function continuity) There is a club* $C \subseteq \omega_2$ *and a club* $\delta < \omega_2$ *so that for all* $f, g \in [C]^\epsilon_*$*, if* $f \restriction \delta = g \restriction \delta$ *and* $\sup(f) = \sup(g)$*, then* $\Phi(f) = \Phi(g)$*.*
- *(Almost everywhere strong short function continuity) There is a club* $C \subseteq \omega_2$ *and finitely many ordinals* $\beta_0 < \beta_1 < ... < \beta_n \leq \epsilon$ *so that for all* $f, g \in [C]^\epsilon_*$*, if for all* $0 \leq i \leq n$*,* $\sup(f \restriction \beta_i) = \sup(g \restriction \beta_i)$*, then* $\Phi(f) = \Phi(g)$*.*

Theorem 8.2 essentially states that if $\mathrm{cof}(\kappa) > \omega_1$, then every function $\Phi : [\omega_1]^\epsilon \to \kappa$ is $\mu^{\omega_1}_\epsilon$-almost everywhere a bounded function when $\epsilon < \omega_1$. Next, one will attempt to copy the argument of Theorem 8.2 when $\epsilon = \omega_1$. However, in this case, the definability estimate only allows this argument to work for regular cardinals greater than $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$. Fact 8.15 will show that this argument necessarily cannot handle ω_2.

Theorem 8.7. *Assume* $\mathsf{ZF}{+}\mathsf{DC}_{\mathbb{R}}{+}\mathsf{AD}$*. Suppose* $\kappa < \Theta$ *and* $\mathrm{cof}(\kappa) \geq (\omega_\omega)^+ = \omega_{\omega+1} = \boldsymbol{\delta}^1_3$*. For any function* $\Phi : [\omega_1]^{\omega_1} \to \kappa$*, there exists a* $\zeta < \kappa$ *and a club* $D \subseteq \omega_1$ *so that* $\Phi[[D]^{\omega_1}_*] \subseteq \zeta$*.*

Proof. Let $(\mathbf{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta,\gamma < \omega_1)$ be a long-uniformly good coding system for ${}^{\omega_1}\omega_1$ (for example, as exposited in Fact 7.18). By Fact 8.1, let $P \subseteq \mathbb{R}$ and $\pi : P \to \kappa$ be a surjective norm which satisfies ω_ω-Suslin bounding. Define a relation $R \subseteq [\omega_1]^{\omega_1}_* \times P$ by $R(f,x)$ if and only if $\Phi(f) = \pi(x)$. By Theorem 5.9, there is a club $C \subseteq \omega_1$ and a Lipschitz function $\Xi : \mathbb{R} \to \mathbb{R}$ so that for all $x \in \mathsf{GC}$ with $\mathsf{decode}(x) \in [C]^{\omega_1}$, $R(\mathsf{block}(\mathsf{decode}(x)), \Xi(x))$. Pick a $z \in \mathsf{clubcode}$ such that $\mathfrak{C}_z \subseteq C$. By Fact 7.9, $\mathsf{Inc}(\mathfrak{C}_z)$ is $\mathbf{\Pi}^1_2$. Thus $\Xi[\mathsf{Inc}(\mathfrak{C}_z)] \subseteq P$ is a $\mathbf{\Sigma}^1_3$ set. Under AD, Martin showed that $\mathbf{\Sigma}^1_3$ sets are ω_ω-Suslin. (See the proof of [14] Theorem 6.5.) Since π satisfies ω_ω-Suslin bounding, there is a $\zeta < \kappa$ so that $\pi[\Xi[\mathsf{Inc}(\mathfrak{C}_z)]] \subseteq \zeta$. Let $D \subseteq \mathfrak{C}_z$ be the set of limit points of $\mathfrak{C}_z$. For any $f \in [D]^{\omega_1}_*$, there is a $g \in [\mathfrak{C}_z]^{\omega_1}$ so that $\mathsf{block}(g) = f$. Then there is an $x \in \mathsf{Inc}(\mathfrak{C}_z)$ so that $\mathsf{decode}(x) = g$. Thus $f = \mathsf{block}(\mathsf{decode}(x))$ and $\Phi(f) = \pi(\Xi(x)) < \zeta$. This finishes the proof. ☐

Corollary 8.8. *Assume* AD. *For every function* $\Phi : [\omega_1]^{\omega_1} \to \omega_{\omega+1}$, *there is a* $\zeta < \omega_{\omega+1}$ *and a club* $C \subseteq \omega_1$ *so that* $\Phi[[C]^{\omega_1}_*] \subseteq \zeta$.

See Theorem 11.13 for the analogs of Theorem 8.2 and Theorem 8.7 for $\boldsymbol{\delta}^1_{2n+1}$ for all $n \in \omega$.

Theorem 8.9. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$. *Suppose* $\delta < \Theta^{L(\mathbb{R})}$ *is a* δ*-reasonable cardinal within* $L(\mathbb{R})$. *Let* κ *be such that* $\mathrm{cof}(\kappa) > \Theta^{L(\mathbb{R})}$. *Then for every function* $\Phi : [\delta]^\delta \to \kappa$, *there exists a club* $C \subseteq \delta$ *and a* $\zeta < \kappa$ *so that* $\Phi[[C]^\delta_*] \subseteq \zeta$.

Proof. Fix a good coding system for ${}^\delta\delta$ in $L(\mathbb{R})$. First suppose $\mathscr{P}(\mathbb{R}) = (\mathscr{P}(\mathbb{R}))^{L(\mathbb{R})}$. Then $\Theta = \Theta^{L(\mathbb{R})}$. Suppose $\Phi[[\delta]^\delta]$ has ordertype greater than $\mathrm{cof}(\kappa) \geq \Theta$. By the Moschovakis coding lemma, there is a surjection of $\mathbb{R}$ onto $[\delta]^\delta$. Thus, there is a surjection of $\mathbb{R}$ onto an ordinal greater than or equal to Θ which is impossible.

So suppose that $(\mathscr{P}(\mathbb{R}))^{L(\mathbb{R})} \subsetneq \mathscr{P}(\mathbb{R})$. Then $\mathbb{R}^\sharp$ exists and Solovay showed that every set of reals in $L(\mathbb{R})$ has a $\Theta^{L(\mathbb{R})}$-scale with each norm in $L(\mathbb{R})$. (See the discussion before [23] Theorem 8.) Also using an argument involving the Moschovakis coding lemma and the fact that the norms belong to $L(\mathbb{R})$, one can show that $\mathrm{cof}(\Theta^{L(\mathbb{R})}) = \omega$ in the model of AD with a set of reals that does not belong to $L(\mathbb{R})$.

By Fact 8.1, pick a $\pi : P \to \kappa$ which is $\Theta^{L(\mathbb{R})}$-Suslin bounded. Let $R \subseteq [\delta]^\delta \times \mathbb{R}$ by defined by $R(f,x)$ if and only if $\Phi(f) = \pi(x)$. Let $\Xi : \mathbb{R} \to \mathbb{R}$ and club $C \subseteq \delta$ be the objects given by Theorem 5.9. Let $z \in \mathsf{clubcode}^\varphi_\delta$ (where $\varphi \in L(\mathbb{R})$ is a norm used to code club subsets of δ) so that $\mathfrak{C}^{\varphi,\delta}_z \subseteq C$. Since the good coding system belongs $L(\mathbb{R})$, $\mathsf{Inc}(\mathfrak{C}^{\varphi,\delta}_z)$ is a set of reals in $L(\mathbb{R})$. Since Ξ is a Lipschitz function (essentially coded by a real), $\Xi[\mathsf{Inc}(\mathfrak{C}^{\varphi,\kappa}_z)]$ is a set of reals in $L(\mathbb{R})$ which is a subset of P. Since it belongs to $L(\mathbb{R})$, the result of Solovay mentioned above implies that $\Xi[\mathsf{Inc}(\mathfrak{C}^{\varphi,\delta}_z)]$ is $\Theta^{L(\mathbb{R})}$-Suslin. Since π is $\Theta^{L(\mathbb{R})}$-Suslin bounded, the result follows as in Theorem 8.7. ☐

Example 8.10. The result in Theorem 8.9 is generally the best possible. For instance, $\delta_\emptyset = (\delta^2_1)^{L(\mathbb{R})}$ is $\delta_\emptyset$-reasonable and, in fact, very reasonable by Example 5.14. Woodin ([22] Theorem 4.12) shows that for any $\lambda < \Theta^{L(\mathbb{R})}$, there is a normal measure μ (coming from the various reflection filters) so that $\kappa = \mathrm{ot}({}^{\delta_\emptyset}\delta_\emptyset / \mu) > \lambda$. κ is a regular cardinal since Martin showed that an ultrapower of a strong partition cardinal by a normal measure is always a regular cardinal. (See [2] Fact 5.4.) Define $\Phi_\mu : [\delta_\emptyset]^{\delta_\emptyset} \to \kappa$ by $\Phi_\mu(f) = [f]_\mu$. Then for any club $C \subseteq \delta_\emptyset$, $\Phi[[C]^{\delta_\emptyset}_*]$ is unbounded in κ.

When $\kappa = \omega_1$ and $\epsilon = \omega_1$, one has the following continuity property which is a main application of the almost everywhere short length club uniformization.

Fact 8.11. *(*[5] *Theorem 4.5) Assume* AD. *Let* $\Phi : [\omega_1]^{\omega_1} \to \omega_1$. *Then there is a club* $C \subseteq \omega_1$ *so that for all* $f \in [C]^{\omega_1}_*$, *there is an* $\alpha < \omega_1$ *so that for all* $g \in [C]^{\omega_1}_*$, *if* $f \restriction \alpha = g \restriction \alpha$, *then* $\Phi(f) = \Phi(g)$.

For the analysis of the ultrapower of ω_1 by the strong partition measure on ω_1, the following finer continuity will be necessary for coding and complexity calculations.

Theorem 8.12. *Assume* AD. *Let* $\Phi : [\omega_1]^{\omega_1} \to \omega_1$. *There is a club* $C \subseteq \omega_1$ *with the following properties.*

(1) For each $f \in [C]^{\omega_1}_*$, *if* β_f *is the unique* β *so that* $\sup(f \restriction \beta) \leq \Phi(f) < f(\beta)$, *then* $f \restriction \beta_f$ *is a minimal continuity point for* Φ *relative to* C.

(2) For any $\sigma \in [C]^{<\omega_1}_*$, *if there is a* $g \in [C]^{\omega_1}_*$ *so that* $\sup(\sigma) < g(0)$ *and* $\Phi(\sigma\hat{\ }g) < g(0)$, *then* σ *is a continuity point relative to* C.

Proof. Note that ω_1 is a very reasonable cardinal (by Fact 7.24). Theorem 7.25 implies that the almost everywhere short length club uniformization holds for ω_1. The result now follows from Theorem 3.2. $\square$

Theorem 8.12 is used in [6] to show that every function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ satisfies a finite form of continuity as long as closure points are maintained. If $f \in [\omega_1]^{\omega_1}_*$ and $\alpha \in \mathrm{ON}$, then α is a closure point of f if and only if $\sup(f \restriction \alpha) = \alpha$. Let $\mathfrak{Z}_f = \{\alpha \in \omega_1 : \sup(f \restriction \alpha) = \alpha\}$ be the club of closure points for f.

Fact 8.13. *([6]) Assume* AD. *Let* $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$. *There is a club* $C \subseteq \omega_1$ *and finitely many functions* $\Upsilon_0, ..., \Upsilon_{n-1} : [C]^{\omega_1}_* \to \omega_1$ *so that for all* $f \in [C]^{\omega_1}_*$, *for all* $g \in [C]^{\omega_1}_*$, *if* $\mathfrak{Z}_g = \mathfrak{Z}_f$ *and for all* $i < n$, $\sup(g \restriction \Upsilon_i(f)) = \sup(f \restriction \Upsilon_i(f))$, *then* $\Phi(f) = \Phi(g)$.

Theorem 8.7 implies that if $\mathrm{cof}(\kappa) > \omega_\omega$, then every function $\Phi : [\omega_1]^{\omega_1} \to \kappa$ is $\mu^{\omega_1}_{\omega_1}$-almost everywhere a bounded function. The next example implies that functions $\Phi : [\omega_1]^{\omega_1} \to \omega_2$ cannot be $\mu^{\omega_1}_{\omega_1}$-almost everywhere a bounded functions. By Fact 8.15, it also implies that for any good coding system for ${}^{\omega_1}\omega_1$, $\mathsf{Inc}(\mathfrak{C}_z)$ cannot be $\mathbf{\Sigma}^1_2$.

Example 8.14. The continuity property for function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$expressed in Fact 8.11 cannot hold for functions of the form $\Psi : [\omega_1]^{\omega_1}_* \to \omega_2$. Let $\mu^{\omega_1}_1$ denote the club measure on ω_1. Martin showed that the ultrapower ${}^{\omega_1}\omega_1/\mu^{\omega_1}_1$ has ordertype ω_2. Let $\Psi^* : [\omega_1]^{\omega_1}_* \to \omega_2$ be defined by $\Psi^*(f) = [f]_{\mu^{\omega_1}_1}$. For any club $C \subseteq \omega_1$ and $\sigma \in [\omega_1]^{<\omega_1}_*$, for any $g \in [C]^{\omega_1}_*$, one has that $\Psi^*(\sigma\hat{\ }g) = [\sigma\hat{\ }g]_{\mu^{\omega_1}_1} = [g]_{\mu^{\omega_1}_1} = \Psi^*(g)$. Thus Ψ^* cannot be constant on sets of the form $\{\sigma\hat{\ }g : g \in [C]^{\omega_1}_*\}$ since Ψ is unbounded through ω_2 on this set. In fact, this shows that for any club $C \subseteq \omega_1$, Ψ^* must be unbounded through ω_2 on $[C]^{\omega_1}_*$. The assumption that $\mathrm{cof}(\kappa) \geq \omega_{\omega+1}$ is necessary in Theorem 8.7.

In the proof of Theorem 8.7, one used a long-uniformly good coding system $(\mathbf{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \omega_1)$. The key definability limitation comes from the fact that $\mathsf{Inc}(\mathfrak{C}_z)$ is $\mathbf{\Pi}^1_2$. One may ask if there could be a good coding system $(\mathbf{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \omega_1)$ so that the analogous set of codes $\mathsf{Inc}(\mathfrak{C}_z)$ is $\mathbf{\Sigma}^1_2$. The function Ψ^* of Example 8.14 shows this is impossibles.

Fact 8.15. *Suppose* $(\mathbf{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \omega_1)$ *is an arbitrary good coding system for* ${}^{\omega_1}\omega_1$. *Then* $\mathsf{Inc}(\mathfrak{C}_z)$ *(which is defined relative to this good coding system) cannot be* $\mathbf{\Sigma}^1_2$. *Therefore by Wadge's lemma, if* GC *is* $\mathbf{\Pi}^1_2$, *then* GC *is* $\mathbf{\Pi}^1_2$*-complete.*

Proof. Consider the function $\Psi^* : [\omega_1]^{\omega_1}_* \to \omega_2$ defined by $\Psi^*(f) = [f]_{\mu^{\omega_1}_1}$. By Fact 8.1, there is a set $P \subseteq \mathbb{R}$ and $\pi : P \to \omega_2$ which satisfies ω_1-Suslin bounding. If $\mathsf{Inc}(\mathfrak{C}_z)$ was $\mathbf{\Sigma}^1_2$, then $\Xi[\mathsf{Inc}(\mathfrak{C}_z)]$ would also be $\mathbf{\Sigma}^1_2$, where Ξ is the suitable analog of the function from the proof of Theorem 8.7. The argument of Theorem 8.7 would imply that there is a club $C \subseteq \omega_1$ and a $\delta < \omega_2$ so that $\Psi^*[[C]^{\omega_1}_*] \subseteq \delta$. This is impossible since $[C]^{\omega_1}_*/\mu^{\omega_1}_1$ is unbounded in ${}^{\omega_1}\omega_1/\mu^{\omega_1}_1 = \omega_2$. $\square$

The fact that for any $z \in \mathsf{clubcode}$, $\mathsf{Inc}(\mathfrak{C}_z)$ is not $\mathbf{\Sigma}^1_2$ is the main obstacle that make many questions concerning the strong partition measure $\mu^{\omega_1}_{\omega_1}$ substantially more difficult than the corresponding combinatorial questions for the shorter partition measures $\mu^{\omega_1}_\epsilon$ when $\epsilon < \omega_1$. The concepts of the very reasonable cardinal and the good coding family were devised to avoid quantification over GC (which is necessary very complicated for $\epsilon = \omega_1$) in the proof of the almost everywhere short length club uniformization.

9. Ultrapower by the Partition Measures

Definition 9.1. Let $\dot{\in}$ be a binary relation symbol and $\mathscr{L} = \{\dot{\in}\}$. Suppose X is a set, μ is a countably complete measure on X, and $\langle Y_x : x \in X\rangle$ is a sequence of sets (which can be considered $\mathscr{L}$-structures with $\dot{\in}$ interpreted as usual set membership $\in$). Let $\prod_{x\in X} Y_x$ be the collection of functions f with $\mathrm{dom}(f) = X$ and for all $x \in X$, $f(x) \in Y_x$. Define an equivalence relation $\sim_\mu$ on $\prod_{x\in X} Y_x$ by $f \sim_\mu g$ if and only if $\{x \in X : f(x) = g(x)\} \in \mu$. Let $[f]_\mu$ be the equivalence class of f under $\sim_\mu$. Let $\prod_{x\in X} Y_x/\mu$ be the collection of all $\sim_\mu$-equivalence classes. If $f, g \in \prod_{x\in X} Y_x$, then define $f \in_\mu g$ if and only if $\{x \in X : f(x) \in g(x)\}$. Define $[f]_\mu \bar{\in} [g]_\mu$ if and only if $f \in_\mu g$. $(\prod_{x\in X} Y_x/\mu, \bar{\in})$ is called the ultraproduct of $\langle Y_x : x \in X\rangle$ by μ. If there is a Y so that for all $x \in X$, $Y_x = Y$, then one will write $\prod_X Y$ for $\prod_{x\in X} Y = \prod_{x\in X} Y_x$ and $\prod_X Y/\mu$ for $\prod_{x\in X} Y_x/Y = \prod_{x\in X} Y/\mu$.

Frequently, $\langle Y_x : x \in X\rangle$ is a sequence of ordinals. In this case, if $f, g \in \prod_{x\in X} Y_x$, one will write $f <_\mu g$ if and only if $\{x \in X : f(x) < g(x)\} \in \mu$. Also one writes $[f]_\mu < [g]_\mu$ if and only if $f <_\mu g$. $(\prod_{x\in X} Y_x/\mu, <)$ is a linear ordering.

If κ is an ordinal and μ is a countably complete measures on a set X, then $\prod_X \kappa/\mu$ is a wellordering using DC. Under AD, if $\mathbb{R}$ surjects onto $\prod_{x\in X}\kappa$, then just $\mathsf{DC}_\mathbb{R}$ is enough to show $\prod_{x\in X}\kappa/\mu$ is a wellordering. Here, one will be most interested in $\prod_{[\omega_1]^\epsilon_*}\kappa/\mu^{\omega_1}_\epsilon$ when $\epsilon \le \omega_1$ and $\kappa < \Theta^{L(\mathbb{R})}$. These ultrapowers can be shown to be wellfounded in just AD (without $\mathsf{DC}_\mathbb{R}$). For instance, using the Kunen tree and the normality of $\mu^{\omega_1}_1$, [2] Fact 5.8 shows that $\prod_{\omega_1}\omega_1/\mu^{\omega_1}_1$ is wellfounded (directly without appealing to the Moschovakis coding lemma or the result of Kechris that $L(\mathbb{R}) \models \mathsf{DC}$). After $\prod_{\omega_1}\omega_1/\mu^{\omega_1}_1$ is known to be wellfounded, a result of Martin ([2] Fact 5.2) shows that $\mathrm{ot}(\prod_{\omega_1}\omega_1/\mu^{\omega_1}_1)$ is a regular cardinal and an analysis involving Kunen trees shows it is actually ω_2. Using the almost everywhere good code uniformization and the result of Kechris that AD implies $L(\mathbb{R}) \models \mathsf{DC}$, many ultrapowers of ordinals by the partition measures of reasonable cardinals are wellfounded in just AD.

Fact 9.2. *Suppose $\delta < \Theta^{L(\mathbb{R})}$, $\kappa < \Theta^{L(\mathbb{R})}$, $\epsilon \le \delta$, and δ is an $\omega\cdot\epsilon$-reasonable cardinal with a good coding system for ${}^{\omega\cdot\epsilon}\delta$ inside $L(\mathbb{R})$. Then $\prod_{[\delta]^\epsilon_*}\kappa/\mu^\delta_\epsilon$ is wellfounded. In particular, for any $\kappa < \Theta^{L(\mathbb{R})}$, $\prod_{[\omega_1]^\epsilon_*}\kappa/\mu^{\omega_1}_\epsilon$ is wellfounded for all $\epsilon \le \omega_1$.*

Proof. Fix a good coding system $(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \omega\cdot\epsilon, \gamma < \delta)$ for ${}^{\omega\cdot\epsilon}\delta$ which belongs to $L(\mathbb{R})$. Let $P, \varphi \in L(\mathbb{R})$ be such that P is a Γ-complete set and $\varphi : P \to \delta$ is a Γ-norm on P. Using φ and the Moschovakis coding lemma, one has that $\mathscr{P}(\delta) = (\mathscr{P}(\delta))^{L(\mathbb{R})}$. Since being a club is absolute, one can show that $\mu^\delta_\epsilon \in L(\mathbb{R})$. Let $\pi : \mathbb{R} \to \kappa$ be a surjection with $\pi \in L(\mathbb{R})$.

Now let $\Phi : [\delta]^\epsilon_* \to \kappa$ be a function (in the real world satisfying determinacy). Define a relation $R \subseteq [\delta]^\epsilon_* \times \mathbb{R}$ by $R(f, x)$ if and only if $\Phi(f) = \pi(x)$. By Theorem 5.9, let $C \subseteq \delta$ be a club and $\Xi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with the property that whenever $x \in \mathsf{GC}$ and $\mathsf{decode}(x) \in [C]^{\omega\cdot\epsilon}$, $R(\mathsf{block}(\mathsf{decode}(x)), \Xi(x))$. Note that $C \in L(\mathbb{R})$ since $\mathscr{P}(\delta) = (\mathscr{P}(\delta))^{L(\mathbb{R})}$. Because Ξ is a Lipschitz function which can be coded by a real, one also has $\Xi \in L(\mathbb{R})$.

Let D be the set of limit points of C (which also belongs to $L(\mathbb{R})$). Define $\Phi^* : [D]^\epsilon_* \to \kappa$ by $\Phi^*(f) = \zeta$ if and only if there exists a $x \in \mathsf{GC}$ with $\mathsf{decode}(x) \in [C]^{\omega\cdot\epsilon}$, $\mathsf{block}(\mathsf{decode}(x)) = f$, and $\pi(\Xi(x)) = \zeta$. This is well defined since if $x_0, x_1 \in \mathsf{GC}$ with $\mathsf{block}(\mathsf{decode}(x_0)) = f = \mathsf{block}(\mathsf{decode}(x_1))$, then $\pi(\Xi(x_0) = \pi(\Xi(x_1))$. Note that this definition of Φ^* can be made in $L(\mathbb{R})$ since $\pi \in L(\mathbb{R})$. So $\Phi^* \in L(\mathbb{R})$. By construction, $\Phi \restriction [D]^\epsilon_* = \Phi^* \restriction [D]^\epsilon_*$. It has been shown that every $\Phi : [\delta]^\epsilon_* \to \delta$ in the real world is μ^δ_ϵ-almost equal to a function Φ^* which belongs to $L(\mathbb{R})$. Thus $\prod_{[\delta]^\epsilon_*}\kappa/\mu^\delta_\epsilon = (\prod_{[\delta]^\epsilon_*}\kappa/\mu^\delta_\epsilon)^{L(\mathbb{R})}$. The latter is wellfounded since $L(\mathbb{R}) \models \mathsf{DC}$. ☐

Martin and Paris computed the ordertype of the ultrapower of ω_1 by the finite partition measures on ω_1.

Fact 9.3. *(Martin-Paris) For each $1 \le n < \omega_1$, $\mathrm{ot}(\prod_{[\omega_1]^n_*}\omega_1/\mu^{\omega_1}_n) = \omega_{n+1}$ and $\mathrm{cof}(\omega_{n+1}) = \omega_2$.*

Thus for any $\omega \le \epsilon \le \omega_1$, $\omega_\omega \le \mathrm{ot}(\prod_{[\omega_1]^\epsilon_*}\omega_1/\mu^{\omega_1}_\epsilon)$. Goldberg asked the very natural question: Is $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*}\omega_1/\mu^{\omega_1}_\epsilon) < \omega_{\omega+1} = \boldsymbol{\delta}^1_3$ for all $\epsilon \le \omega_1$? One can also ask if $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*}\omega_{\omega+1}/\mu^{\omega_1}_\epsilon) = \mathrm{ot}(\prod_{[\omega_1]^\epsilon_*}\boldsymbol{\delta}^1_3/\mu^{\omega_1}_\epsilon) = \boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ for all $\epsilon \le \omega_1$?

Fact 9.4. *For all $1 \le \epsilon < \omega_1$, $\mathrm{cof}(\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*}\omega_1/\mu^{\omega_1}_\epsilon)) = \omega_2$.*

Proof. Fix $\epsilon < \omega_1$. For each $\Psi : \omega_1 \to \omega_1$, let $\Lambda(\Psi) : [\omega_1]^\epsilon_* \to \omega_1$ be defined by $\Lambda(\Psi)(f) = \Psi(\sup(f))$. Now suppose $\Psi_0 \sim_{\mu^{\omega_1}_1} \Psi_1$. There is a club $C \subseteq \omega_1$ so that $\Psi_0(\alpha) = \Psi_1(\alpha)$ for all $\alpha \in C$. For all $f \in [C]^\epsilon_*$, $\sup(f) \in C$ and therefore $\Lambda(\Psi_0)(f) = \Psi_0(\sup(f)) = \Psi_1(\sup(f)) = \Lambda(\Psi_1)(f)$. Hence $\Lambda(\Psi_0) \sim_{\mu^{\omega_1}_\epsilon} \Lambda(\Psi_1)$. This shows that $\Gamma : \prod_{\omega_1}\omega_1/\mu^{\omega_1}_1 \to \prod_{[\omega_1]^\epsilon_*}\omega_1/\mu^{\omega_1}_\epsilon$ defined by $\Gamma([\Psi]_{\mu^{\omega_1}_1}) = [\Lambda(\Psi)]_{\mu^{\omega_1}_\epsilon}$ is a well defined order preserving map. Since Fact 9.3 implies that $\mathrm{ot}(\prod_{\omega_1}\omega_1/\mu^{\omega_1}_1) = \omega_2$, it remains to show that Γ is cofinal.

Let $\Phi : [\omega_1]^\epsilon_* \to \omega_1$. Let $P : [\omega_1]^{\epsilon+1}_* \to 2$ by $P(h) = 0$ if and only if $\Phi(h \restriction \epsilon) < h(\epsilon)$. By $\omega_1 \to_* (\omega_1)^{\epsilon+1}_2$, there is a club $C \subseteq \omega_1$ which is homogeneous for P. Take any $f \in [C]^\epsilon_*$. Let $\gamma \in C$ be such that $\gamma > \Phi(f)$. Let $h \in [C]^{\epsilon+1}_*$ be such that $h \restriction \epsilon = f$ and $h(\epsilon) = \gamma$. Then $P(h) = 0$. Thus C must be homogeneous for P taking value 0. Now take any $f \in [C]^\epsilon_*$. Let $h \in [C]^{\epsilon+1}_*$ be defined by $h \restriction \epsilon = f$ and $h(\epsilon) = \mathsf{next}_C(\sup(f))$.

$P(h) = 0$ implies that $\Phi(f) < \mathsf{next}_C(\sup(f))$. It has been shown that $\Gamma([\mathsf{next}_C]_{\mu_1^{\omega_1}}) > [\Phi]_{\mu_\epsilon^{\omega_1}}$. Thus Γ is cofinal. $\square$

Recall that every Lipschitz continuous function Ξ takes the form Ξ_ρ^2 for a strategy ρ. Lipschitz continuous function will be coded by reals through strategies. From this point on, Ξ_ρ will denote Ξ^2_ρ which is the Lipschitz continuous function coded by the real ρ.

Definition 9.5. Let $\epsilon \leq \omega_1$. Let $(\mathbf{\Pi}_1^1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \omega \cdot \epsilon, \gamma < \omega_1)$ be a short uniformly good coding system for ${}^{\omega\cdot\epsilon}\omega_1$ if $\epsilon < \omega_1$ or a long-uniformly good coding system for ${}^{\omega_1}\omega_1$ if $\epsilon = \omega_1$.

Let $\mathsf{Fcode}_\epsilon \subseteq \mathbb{R}^2$ consists of pairs (ρ, z) with the following properties.

(1) $z \in \mathsf{clubcode}$.
(2) $(\forall^{\mathbb{R}} x)(\mathsf{INC}(z,x) \Rightarrow \Xi_\rho(x) \in \mathrm{WO})$
(3) $(\forall^{\mathbb{R}} x_0)(\forall^{\mathbb{R}} x_1)\{(\mathsf{INC}(z,x_0) \wedge \mathsf{INC}(z,x_1) \wedge \mathsf{sameBlock}(x_0,x_1)) \Rightarrow [\Xi_\rho(x_0) \leq^{\mathrm{ot}}_{\Pi_1^1} \Xi_\rho(x_1) \wedge \Xi_\rho(x_1) \leq^{\mathrm{ot}}_{\Pi_1^1} \Xi_\rho(x_0)]\}$.

Note that Fcode_ϵ is defined relative to a fixed uniformly good coding system for ${}^{\omega\cdot\epsilon}\omega_1$ (short-uniformly good if $\epsilon < \omega_1$ and long-uniformly good if $\epsilon = \omega_1$).

Intuitively, $(\rho, z) \in \mathsf{Fcode}_\epsilon$ means the following. ρ is a real coding a strategy which represents the Lipschitz function Ξ_ρ. z codes a club subset of ω_1. (ρ, z) has the property that whenever x is a good code such that $\mathsf{decode}(x) \in [\mathfrak{C}_z]^{\omega\cdot\epsilon}$, $\Xi_\rho(x)$ will return an element of WO. Moreover, if x_0, x_1 are two such good codes with $\mathsf{decode}(x_0), \mathsf{decode}(x_1) \in [\mathfrak{C}_z]^{\omega\cdot\epsilon}$ and $\mathsf{block}(\mathsf{decode}(x_0)) = \mathsf{block}(\mathsf{decode}(x_1))$, then $\Xi_\rho(x_0)$ and $\Xi_\rho(x_1)$ code isomorphic wellorderings.

Suppose $\epsilon < \omega_1$. By Fact 4.11, $\mathsf{clubcode}$ is $\mathbf{\Pi}_2^1$ and therefore (1) is $\mathbf{\Pi}^1_2$. INC and $\mathsf{sameBlock}$ are $\mathbf{\Pi}_1^1$ by Fact 7.5. Thus (2) and (3) are $\mathbf{\Pi}^1_2$. Therefore (3) is also $\mathbf{\Pi}^1_2$. This shows Fcode_ϵ is $\mathbf{\Pi}^1_2$.

Suppose $\epsilon = \omega_1$. (1) is still $\mathbf{\Pi}^1_2$ as before. By Fact 7.9, INC and $\mathsf{sameBlock}$ are $\mathbf{\Pi}^1_2$. Thus (2) and (3) are $\mathbf{\Pi}^1_3$. $\mathsf{Fcode}_{\omega_1}$ is $\mathbf{\Pi}^1_3$.

Suppose $(\rho, z) \in \mathsf{Fcode}_\epsilon$. Let $\mathfrak{D}_z$ denote the limit points of $\mathfrak{C}_z$. Define $\Phi^{(\rho,z)} : [\mathfrak{D}_z]^\epsilon_* \to \omega_1$ by $\Phi^{(\rho,z)}(f) = \gamma$ if and only if there exists an $x \in \mathsf{Inc}(\mathfrak{C}_z)$ so that

(1) $\mathsf{decode}(x) \in [\mathfrak{C}_z]^{\omega\cdot\epsilon}$.
(2) $\mathsf{block}(\mathsf{decode}(x)) = f$
(3) $\mathrm{ot}(\Xi_\rho(x)) = \gamma$.

Fact 9.6. *If $(\rho, z) \in \mathsf{Fcode}_\epsilon$, then $\Phi^{(\rho,z)} : [\mathfrak{D}_z]^\epsilon_* \to \omega_1$ is well defined.*

Proof. Suppose $f \in [\mathfrak{D}_z]^\epsilon_*$. Let $F : \omega_1 \times \omega \to \omega_1$ be a function witnessing that f has uniform cofinality ω. For each $\alpha < \epsilon$, by the discontinuity of f, $\sup(f \restriction \alpha) < f(\alpha) = \sup\{F(\alpha, n) : n < \omega\}$. Let $g(\omega \cdot \alpha) = \mathsf{next}_{\mathfrak{C}_z}(\max\{\sup(f \restriction \alpha), F(\alpha, 0)\}) < f(\alpha)$ since $f(\alpha)$ is a limit point of $\mathfrak{C}_z$. Suppose $g(\omega \cdot \alpha + n)$ has been defined. Let $g(\omega \cdot \alpha + n + 1) = \mathsf{next}_{\mathfrak{C}_z}(\max\{g(\omega \cdot \alpha + n), F(\alpha, n+1)\})$. Then $g \in [\mathfrak{C}_z]^{\omega\cdot\epsilon}_*$ and $\mathsf{block}(g) = f$. Let $x \in \mathsf{Inc}(\mathfrak{C}_z)$ so that $\mathsf{decode}(x) = g$. This shows that there is an $x \in \mathsf{Inc}(\mathfrak{C}_z)$ so that $\mathsf{block}(\mathsf{decode}(x)) = f$. By Definition 9.5 (2), $\Xi_\rho(x) \in \mathrm{WO}$. By (3), if $x' \in \mathsf{Inc}(\mathfrak{C}_z)$ and $\mathsf{block}(\mathsf{decode}(x')) = f$, then $\Xi_\rho(x')$ is order-isomorphic to $\Xi_\rho(x)$. Thus $\Phi^{(\rho,z)}(f) = \mathrm{ot}(\Xi_\rho(x))$ and this is well defined. $\square$

Fact 9.7. *Suppose $\Phi : [\omega_1]^\epsilon_* \to \omega_1$. There is a $(\rho, z) \in \mathsf{Fcode}_\epsilon$ so that $[\Phi]_{\mu_\epsilon^{\omega_1}} = [\Phi^{(\rho,z)}]_{\mu_\epsilon^{\omega_1}}$.*

Proof. Let $\Phi : [\omega_1]^\epsilon_* \to \omega_1$. Define $R \subseteq [\omega_1]^\epsilon_* \times \mathrm{WO}$ by $R(f, w)$ if and only if $\Phi(f) = \mathrm{ot}(w)$. The almost everywhere good code uniformization (Fact 5.9) applied to R gives a Lipschitz continuous function Ξ and a club $C \subseteq \omega_1$ so that for all $x \in \mathsf{GC}$ with $\mathsf{decode}(x) \in [C]^{\omega\cdot\epsilon}$, $R(\mathsf{block}(\mathsf{decode}(x)), \Xi(x))$. This implies that $\Phi(\mathsf{block}(\mathsf{decode}(x))) = \mathrm{ot}(\Xi(x))$. Let ρ be a strategy so that $\Xi_\rho = \Xi$ and let $z \in \mathsf{clubcode}$ so that $\mathfrak{C}_z \subseteq C$. Then $(\rho, z) \in \mathsf{Fcode}_\epsilon$ and $\Phi(f) = \Phi^{(\rho,z)}(f)$ for all $f \in [\mathfrak{D}_z]^\epsilon_*$. Thus $[\Phi]_{\mu_\epsilon^{\omega_1}} = [\Phi^{(\rho,z)}]_{\mu_\epsilon^{\omega_1}}$. $\square$

Fact 9.8. *(With Jackson) Let $\epsilon \leq \omega_1$. Suppose $\langle F_\alpha : \alpha < \omega_1 \rangle$ is a sequence of elements of $\prod_{[\omega_1]^\epsilon_*} \omega_1 / \mu_\epsilon^{\omega_1}$. Then there is a sequence $\langle \Phi_\alpha : \alpha < \omega_1 \rangle$ such that for each $\alpha < \omega_1$, $\Phi_\alpha : [\omega_1]^\epsilon_* \to \omega_1$ and $F_\alpha < [\Phi_\alpha]_{\mu_\epsilon^{\omega_1}}$.*

Proof. Define $S \subseteq \mathrm{WO} \times \mathbb{R}^2$ by $S(w, (\rho, z))$ if and only if $(\rho, z) \in \mathsf{Fcode}_\epsilon$ and $[\Phi^{(\rho,z)}]_{\mu_\epsilon^{\omega_1}} = F_{\mathrm{ot}(w)}$. Note that $\mathrm{dom}(S) = \mathrm{WO}$ by Fact 9.7. By the uniform coding lemma (Fact 4.14), there is an $e \in \mathbb{R}$ so that for all $\alpha < \omega_1$, $U^{(2)}_e(\mathrm{WO}_{<\alpha}, \mathrm{WO}_\alpha) \subseteq S \cap (\mathrm{WO}_\alpha \times \mathbb{R})$ and $U^{(2)}_e(\mathrm{WO}_{<\alpha}, \mathrm{WO}_\alpha) \neq \emptyset$.

Let $\varpi_1, \varpi_2 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $\varpi_1(w, (\rho, z)) = \rho$ and $\varpi_2(w, (\rho, z)) = z$. Now fix an $\alpha < \omega_1$. Since $U_e^{(2)}(\mathrm{WO}_{<\alpha}, \mathrm{WO}_\alpha) \subseteq S \cap (\mathrm{WO}_\alpha \times \mathbb{R})$ is $\mathbf{\Sigma}_1^1$, $\varpi_2[U_e^{(2)}(\mathrm{WO}_{<\alpha}, \mathrm{WO}_\alpha)] \subseteq \mathsf{clubcode}$ is $\mathbf{\Sigma}_1^1$. By Fact 4.12, there is a club C_α (produced uniformly from $\varpi_2[U_e^{(2)}(\mathrm{WO}_{<\alpha}, \mathrm{WO}_\alpha)]$ as a set) with the property that for all $z \in \varpi_2[U_e^{(2)}(\mathrm{WO}_{<\alpha}, \mathrm{WO}_\alpha)]$, $C_\alpha \subseteq \mathfrak{C}_z$. Let D_α be the set of limit points of C_α.

Define a function $\Phi_\alpha : [D_\alpha]^\epsilon_* \to \omega_1$ as follows: Let $f \in [D_\alpha]^\epsilon_*$. Pick any $x \in \mathsf{GC}$ so that $\mathsf{decode}(x) \in [C_\alpha]^{\omega \cdot \epsilon}$ and $\mathsf{block}(\mathsf{decode}(x)) = f$. Let $Z_\alpha^x = \{w \in \mathrm{WO} : (\exists \rho)(\rho \in \varpi_1[U_e^{(2)}(\mathrm{WO}_{<\alpha}, \mathrm{WO}_\alpha)] \wedge w = \Xi_\rho(x))\}$ which is a $\mathbf{\Sigma}_1^1$ subset of WO. By boundedness, let $\Phi_\alpha(f)$ be the least δ so that for all $w \in Z_\alpha^x$, $\mathrm{ot}(w) < \delta$. Note that by Definition 9.5 (3), $\Phi_\alpha(f)$ is independent of the choice of x so that $\mathsf{decode}(x) \in [C_\alpha]^{\omega\cdot\epsilon}$ and $\mathsf{block}(\mathsf{decode}(x)) = f$. Therefore $\Phi_\alpha(f)$ is well defined.

Finally the claim is that $F_\alpha < [\Phi_\alpha]_{\mu_\epsilon^{\omega_1}}$. Let $\Phi : [\omega_1]^\epsilon_* \to \omega_1$ be such that $[\Phi]_{\mu_\epsilon^{\omega_1}} = F_\alpha$. Pick any $(\rho, z) \in \varpi_2[U_e^{(2)}(\mathrm{WO}_{<\alpha}, \mathrm{WO}_\alpha)]$. Since $\Phi^{(\rho,z)}$ is $\mu_\epsilon^{\omega_1}$-almost equal to Φ, there is a club $D' \subseteq D_\alpha$ so that for all $f \in D'$, $\Phi(f) = \Phi^{(\rho,z)}(f)$. Pick an $f \in [D']^\epsilon_*$ and an $x \in \mathsf{GC}$ so that $\mathsf{decode}(x) \in [C_\alpha]^{\omega\cdot\epsilon}$ and $\mathsf{block}(\mathsf{decode}(x)) = f$. Note that $\Xi_\rho(x) \in Z_f^x$ and $\mathrm{ot}(\Xi_\rho(x)) = \Phi(f)$. By definition of Φ_α, one has that $\Phi_\alpha(f) > \mathrm{ot}(\Xi_\rho(x)) = \Phi^{(\rho,z)}(f) = \Phi(f)$. Since $f \in [D']^\epsilon_*$ was arbitrary, it has been shown that $F_\alpha = [\Phi]_{\mu_\epsilon^{\omega_1}} < [\Phi_\alpha]_{\mu_\alpha^{\omega_1}}$. This completes the argument. ☐

Fact 9.4 shows that when $\epsilon < \omega_1$, $\mathrm{cof}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1}) = \omega_2$. The next result provides a lower bound for $\mathrm{cof}(\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu_{\omega_1}^{\omega_1})$.

Fact 9.9. *(With Jackson)* $\mathrm{cof}(\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu_{\omega_1}^{\omega_1})) \geq \omega_2$.

Proof. First, let $\langle F_n : n \in \omega\rangle$ be an ω-sequence in $\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu_{\omega_1}^{\omega_1}$. Let $R \subseteq \omega \times \mathbb{R}$ by $R(n, (\rho, z))$ if and only if $(\rho, z) \in \mathsf{Fcode}_{\omega_1}$ and $[\Phi^{(\rho,z)}]_{\mu_{\omega_1}^{\omega_1}} = F_\alpha$. Applying $\mathsf{AC}_\omega^\mathbb{R}$ to R, one can obtain a sequence $\langle \Phi_n : n \in \omega\rangle$ such that for all $n \in \omega$, $\Phi_n : [\omega_1]^{\omega_1}_* \to \omega_1$ and $[\Phi_n]_{\mu_{\omega_1}^{\omega_1}} = F_n$. Let $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ be defined by $\Phi(f) = \sup\{\Phi_n(f) + 1 : n \in \omega\}$. Then for all $n \in \omega$ and for all $f \in [\omega_1]^{\omega_1}_*$, $\Phi_n(f) < \Phi(f)$ and thus $F_n < [\Phi]_{\mu_{\omega_1}^{\omega_1}}$. $\langle F_n : n \in \omega\rangle$ cannot be cofinal through $\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu_{\omega_1}^{\omega_1}$.

Next, let $\langle F_\alpha : \alpha < \omega_1\rangle$ be an increasing cofinal sequence. By Fact 9.8, let $\langle \Phi_\alpha : \alpha < \omega_1\rangle$ be a sequence such that for all $\alpha \in \omega_1$, $\Phi_\alpha : [\omega_1]^{\omega_1}_* \to \omega_1$ and $[\Phi_\alpha]_{\mu_{\omega_1}^{\omega_1}} > F_\alpha$. Define $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ by $\Phi(f) = \sup\{\Phi_\alpha(f) + 1 : \alpha < f(0)\}$. Fix an $\alpha < \omega_1$. For any $f \in [\omega_1]^{\omega_1}_*$ so that $f(0) > \alpha$, one has that $\Phi_\alpha(f) < \Phi(f)$. This shows that $F_\alpha < [\Phi_\alpha]_{\mu_{\omega_1}^{\omega_1}} < [\Phi]_{\mu_{\omega_1}^{\omega_1}}$. Since α was arbitrary, $\langle F_\alpha : \alpha < \omega_1\rangle$ is not cofinal through $\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu_{\omega_1}^{\omega_1}$.

This shows the cofinality must be greater than or equal to ω_2. (Note this argument works to show $\mathrm{cof}(\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1})) \geq \omega_2$ for any $\epsilon \leq \omega_1$; however, one already knows the cofinality is ω_2 from Fact 9.4 when $\epsilon < \omega_1$.) ☐

Definition 9.10. Let $\epsilon \leq \omega_1$. Define $\mathfrak{F}_\epsilon$ by $\mathfrak{F}_\epsilon((\rho_0, z_0), (\rho_1, z_1))$ if and only if the conjunction of the following holds.

1. $(\rho_0, z_0) \in \mathsf{Fcode}_\epsilon \wedge (\rho_1, z_1) \in \mathsf{Fcode}_\epsilon$.
2. There exists a real z with the following properties
   - (a) $z \in \mathsf{clubcode}$
   - (b) $\mathsf{clubSubset}(z, z_0) \wedge \mathsf{clubSubset}(z, z_1)$.
   - (c) $(\forall^\mathbb{R} x)(\mathsf{INC}(z, x) \Rightarrow \Xi_{\rho_0}(x) <^{\mathrm{ot}}_{\Pi_1^1} \Xi_{\rho_1}(x))$.

Intuitively, $\mathfrak{F}_\epsilon((\rho_0, z_0), (\rho_1, z_1))$ states that (ρ_0, z_0) and (ρ_1, z_1) belong to Fcode_ϵ and there exists a club $\mathfrak{C}_z$ such that $\mathfrak{C}_z \subseteq \mathfrak{C}_{z_0}$, $\mathfrak{C}_z \subseteq \mathfrak{C}_{z_1}$, and for all $x \in \mathsf{Inc}(\mathfrak{C}_z)$, $\mathrm{ot}(\Xi_{\rho_0}(x)) < \mathrm{ot}(\Xi_{\rho_1}(x))$. Therefore, for any $f \in [\mathfrak{D}_z]^\epsilon_*$, $\Phi^{(\rho_0,z_0)}(f) < \Phi^{(\rho_1,z_1)}(f)$ and hence $[\Phi^{(\rho_0,z_0)}]_{\mu_\epsilon^{\omega_1}} < [\Phi^{(\rho_1,z_1)}]_{\mu_\epsilon^{\omega_1}}$.

Lemma 9.11. *For all $\epsilon \leq \omega_1$, $\mathfrak{F}_\epsilon$ is a wellfounded relation and $\mathsf{rk}(\mathfrak{F}_\epsilon) = \mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1})$.*

Proof. Define $\Upsilon : \mathsf{Fcode}_\epsilon \to \prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_*^\epsilon$ by $\Upsilon(\rho, z) = [\Phi^{(\rho,z)}]_{\mu_\epsilon^{\omega_1}}$. By the above discussion, $\mathfrak{F}_\epsilon((\rho_0, z_0), (\rho_1, z_1))$ if and only if $[\Phi^{(\rho_0,z_0)}]_{\mu_\epsilon^{\omega_1}} < [\Phi^{(\rho_1,z_1)}]_{\mu_\epsilon^{\omega_1}}$. Since $\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1}$ is a wellordering by Fact 9.2, $\mathfrak{F}_\epsilon$ is a wellfounded relation. This also shows that $\mathsf{rk}(\mathfrak{F}_\epsilon) \leq \mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1})$.

Define $\Sigma : \mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1}) \to \mathsf{rk}(\mathfrak{F}_\epsilon)$ as follows: Suppose $F \in \prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1}$. By Fact 9.7, there is $(\rho^*, z^*) \in \mathsf{Fcode}_\epsilon$ so that for all $[\Phi^{(\rho^*,z^*)}]_{\mu_\epsilon^{\omega_1}} = F$. Let $\Sigma(F) = \mathsf{rk}_{\mathfrak{F}_\epsilon}(\rho, z)$. Note that for all (ρ_0, z_0) and

(ρ_1, z_1) so that $[\Phi^{(\rho_0,z_0)}]_{\mu_\epsilon^{\omega_1}} = [\Phi^{(\rho_1,z_1)}]_{\mu_\epsilon^{\omega_1}}$, $\{(\rho, z) : \mathfrak{F}_\epsilon((\rho, z), (\rho_0, z_0))\} = \{(\rho, z) : \mathfrak{F}_\epsilon((\rho, z), (\rho_1, z_1))\}$. This shows that the definition of $\Sigma(F)$ is independent of the choice of (ρ^*, z^*). Suppose $F_0, F_1 \in \prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1}$ with $F_0 < F_1$. Let $(\rho_0, z_0), (\rho_1, z_1) \in \mathsf{Fcode}_\epsilon$ so that $[\Phi^{(\rho_0,z_0)}]_{\mu_\epsilon^{\omega_1}} = F_0$ and $[\Phi^{(\rho_1,z_1)}]_{\mu_\epsilon^{\omega_1}} = F_1$. Then $\mathfrak{F}_\epsilon((\rho_0, z_0), (\rho_1, z_1))$ and thus $\Sigma(F_0) < \Sigma(F_1)$. This shows $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1} \leq \mathsf{rk}(\mathfrak{F}_\epsilon)$. ☐

Theorem 9.12. *(With Jackson) Let $\epsilon < \omega_1$. $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1}) < \omega_{\omega+1} = \boldsymbol{\delta}^1_3$. Thus for $\omega \leq \epsilon < \omega_1$, $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1})$ is a non-cardinal ordinal strictly between ω_ω and $\omega_{\omega+1}$ with cofinality ω_2.*

Proof. From Definition 9.5, Fcode_ϵ is $\boldsymbol{\Pi}^1_2$ and so Definition 9.10 (1) is $\boldsymbol{\Pi}^1_2$. By Fact 4.11, $\mathsf{clubcode}$ and $\mathsf{clubSubset}$ are $\boldsymbol{\Pi}^1_2$. By Fact 7.5, INC is $\boldsymbol{\Pi}^1_1$. Thus Definition 9.10 (2a), (2b), and (2c) are $\boldsymbol{\Pi}^1_2$. Therefore Definition 9.5 (2) is $\boldsymbol{\Sigma}^1_3$. Thus $\mathfrak{F}_\epsilon$ is $\boldsymbol{\Sigma}^1_3$ and wellfounded by Lemma 9.11. Martin showed that $\boldsymbol{\Sigma}^1_3$ relations are ω_ω-Suslin. By the Kunen-Martin theorem (Fact 7.11), $\mathsf{rk}(\mathfrak{F}_\epsilon) < (\omega_\omega)^+ = \omega_{\omega+1} = \boldsymbol{\delta}^1_3$. By Lemma 9.11, $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1}) < \omega_{\omega+1}$.

Now suppose $\omega \leq \epsilon < \omega_1$. By Fact 9.3, for all $n \in \omega$, $\omega_{n+1} = \mathrm{ot}(\prod_{[\omega_1]^n_*} \omega_1/\mu_n^{\omega_1}) < \mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1})$. Thus $\omega_\omega \leq \mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1})$. Since $\mathrm{cof}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1}) = \omega_2$ by Fact 9.4, one must have that $\omega_\omega < \mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu_\epsilon^{\omega_1}) < \omega_{\omega+1}$. Hence the ultrapower is a non-cardinal ordinal. ☐

Theorem 9.13. *Let $\epsilon < \omega_1$. For $1 \leq n < \omega_1$, $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_n/\mu_\epsilon^{\omega_1}) < \omega_{\omega+1}$. $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_\omega/\mu_\epsilon^{\omega_1}) < \omega_{\omega+1}$.*

Proof. The case $n = 1$ has been handled by Theorem 9.12. So suppose $2 \leq n < \omega$ and the result holds for $n - 1$. Then $\mathrm{cof}(\omega_n) = \omega_2$ by Fact 9.3. Suppose $\Phi : [\omega_1]^\epsilon_* \to \omega_n$. Theorem 8.2 implies there is a $\zeta < \omega_n$ and a club C so that $\Phi : [C]^\epsilon_* \to \zeta$. Hence $\prod_{[\omega_1]^\epsilon_*} \omega_n/\mu_\epsilon^{\omega_1} = \bigcup_{\zeta<\omega_n} \prod_{[\omega_1]^\epsilon_*} \zeta/\mu_\epsilon^{\omega_1}$.

Note that if ζ_0 and ζ_1 are ordinals and ζ_0 is in bijection with ζ_1, then $\prod_{[\omega_1]^\epsilon_*} \delta_0/\mu_\epsilon^{\omega_1}$ is in bijection with $\prod_{[\omega_1]^\epsilon_*} \delta_1/\mu_\epsilon^{\omega_1}$. Thus by the induction hypothesis, one has that $\prod_{[\omega_1]^\epsilon_*} \omega_n/\mu_\epsilon^{\omega_1}$ is an ω_n size union of ordinals less than $\omega_{\omega+1}$. Since $\omega_{\omega+1}$ is regular and $\mathrm{cof}(\omega_n) = \omega_2$, one must have that $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_n/\mu_\epsilon^{\omega_1}) < \omega_{\omega+1}$.

For any function $\Phi : [\omega_1]^\epsilon_* \to \omega_\omega$, the countable completeness of $\mu_\epsilon^{\omega_1}$ implies that there is a $n < \omega$ and a club $C \subseteq \omega_1$ so that $\Phi : [C]^\epsilon_* \to \omega_n$. Thus $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_\omega/\mu_\epsilon^{\omega_1}) = \bigcup_{n\in\omega} \mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_n/\mu_\epsilon^{\omega_1})$. By the previous result, this shows that $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_\omega/\mu_\epsilon^{\omega_1})$ is an ω-union of ordinals less than $\omega_{\omega+1}$. Again using the regularity of $\omega_{\omega+1}$, $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_\omega/\mu_\epsilon^{\omega_1}) < \omega_{\omega+1}$. ☐

Theorem 9.14. *Let $\epsilon < \omega_1$. $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_{\omega+1}/\mu_\epsilon^{\omega_1}) = \mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \boldsymbol{\delta}^1_3/\mu_\epsilon^{\omega_1}) = \boldsymbol{\delta}^1_3 = \omega_{\omega+1}$.*

Proof. By Theorem 8.2, for every function $\Phi : [\omega_1]^\epsilon \to \omega_{\omega+1}$, there is a $\zeta < \omega_{\omega+1}$ and club $C \subseteq \omega_1$ so that $\Phi : [C]^\epsilon_* \to \delta$. This implies that $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_{\omega+1}/\mu_\epsilon^{\omega_1}) = \bigcup_{\zeta<\omega_{\omega+1}} \mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \zeta/\mu_\epsilon^{\omega_1})$. By Theorem 9.13, this is an $\omega_{\omega+1}$-length increasing union of ordinals below $\omega_{\omega+1}$. Thus $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_{\omega+1}/\mu_\epsilon^{\omega_1}) = \omega_{\omega+1}$. ☐

Theorem 9.15. *(With Jackson) $\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu_{\omega_1}^{\omega_1}) < \omega_{\omega+2} = \boldsymbol{\delta}^1_4$ and has cofinality greater than or equal to ω_2.*

Proof. From Definition 9.5, $\mathsf{Fcode}_{\omega_1}$ is $\boldsymbol{\Pi}^1_3$ and hence Definition 9.10 (1) is $\boldsymbol{\Pi}^1_3$. By Fact 4.11, $\mathsf{clubcode}$ and $\mathsf{subsetClub}^{\mathrm{ot}}_{\omega_1}$ are $\boldsymbol{\Pi}^1_2$ and hence Definition 9.10 (2a) and (2b) are $\boldsymbol{\Pi}^1_2$. By Fact 7.9, INC is $\boldsymbol{\Pi}^1_2$. Thus Definition 9.10 (2c) is $\boldsymbol{\Pi}^1_3$. This implies that (2) is $\boldsymbol{\Sigma}^1_4$. Moschovakis ([14] Theorem 3.11) showed that every $\boldsymbol{\Sigma}^1_4$ sets is $\boldsymbol{\delta}^1_3$-Suslin and hence $\omega_{\omega+1}$-Suslin since $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$. Hence $\mathfrak{F}_{\omega_1}$ is a wellfounded $\omega_{\omega+1}$-Suslin relation by Lemma 9.11. The Kunen-Martin theorem (Fact 7.11) and Lemma 9.11 imply $\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu_{\omega_1}^{\omega_1}) = \mathsf{rk}(\mathfrak{F}_{\omega_1}) < (\omega_{\omega+1})^+ = \omega_{\omega+2} = \boldsymbol{\delta}^1_4$. The cofinality of this ultrapower is greater than or equal to ω_2 by Fact 9.9. ☐

The results of the subsequent sections will improve this upper bound.

10. Local Bounds on the Ultrapower by the Strong Partition Measure

For any long-uniformly good coding system for ${}^{\omega_1}\omega_1$, the collection of good codes, GC or $\mathsf{INC}(\mathfrak{C}_z)$ are $\boldsymbol{\Pi}^1_2$ (and even $\boldsymbol{\Pi}^1_2$-complete by Fact 8.15). Definition 9.10 (2c) involves a universal quantification over GC. This causes $\mathfrak{F}_{\omega_1}$ to be $\boldsymbol{\Sigma}^1_4$. Definition 9.5 (2) and (3) also requires a universal quantification over GC. This causes $\mathsf{Fcode}_{\omega_1}$ to be $\boldsymbol{\Pi}^1_3$.

To improve the upper bound, one needs to code the ultrapower of ω_1 by its strong partition measure by a simpler relation. To do this, the universal quantification over GC needs to be eliminated. The main idea is to use continuity for functions of the form $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ to quantify over the continuity points which are coded by elements of the $\mathbf{\Pi}^1_1$ set BS. The good coding family with the function merge will provide the mechanism to quantify over BS by merging into a fixed good code for a long function. A priori, recognizing an element of BS is a continuity point for Φ seems to require a quantification over good codes (in order to assert all extensions of a short sequence to a long sequence take the same value). This quantification can be avoided by using the finer continuity property of Theorem 8.12 (2) which recognizes continuity points relative to a club C as those $\sigma \in [C]^{<\omega_1}_*$ so that there exists a $g \in [C]^{\omega_1}_*$ with $\sup(\sigma) < g(0)$ and $\Phi(\sigma\hat{\ }g) < \sup(\sigma)$. (Of course, $\Phi(\sigma\hat{\ }g) < g(0)$ or even $\Phi(\sigma\hat{\ }g) \leq \sup(\sigma)$ would also imply σ is a continuity point for Φ relative to C. In the coding mechanism, one always speaks of $f \in [\omega_1]^{\omega_1}_*$ indirectly through an auxillary function $h \in [\omega_1]^{\omega_1}$ so that $\mathsf{block}(h) = f$. Because of this, using $\Phi(\sigma\hat{\ }g) < g(0)$ to recognizes continuity points makes the exposition slightly more complicated.)

This finer continuity motivates the definition of a pseudo-continuity point for the triple (ρ, z, e). Throughout this entire section, fix a uniformly good coding family $\langle(\mathbf{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \omega_1), \mathsf{BS}, \mathsf{seq}, \mathsf{nGC}, \mathsf{merge}\rangle$ for ω_1.

Definition 10.1. Define PCP by $\mathsf{PCP}(u, \rho, z, e)$ if and only if

(1) $\mathsf{INC}^{\mathsf{BS}}(z, u)$.
(2) $\mathsf{acceptableMerge}(u, e)$
(3) $\Xi_\rho(\mathsf{merge}(u, e)) \in \mathrm{WO}$
(4) $(\exists^\omega m)(\forall^{\mathbb{R}} w)(\mathsf{uBS}_{\mathbf{\Sigma}^1_1}(u, \mathsf{rest}(\mathsf{length}(u), m), w) \Rightarrow \Xi_\rho(\mathsf{merge}(u, e)) <^{\mathrm{ot}}_{\mathbf{\Pi}^1_1} w)$.

Observe that PCP is $\mathbf{\Pi}^1_1$. To understand $\mathsf{PCP}(u, \rho, z, e)$: In the intended setting, $z \in \mathsf{clubcode}$ and $e \in \mathsf{GC}$. Let $\tau = \mathsf{seq}(u)$. There is an ϵ such that $|\tau| = \omega\cdot\epsilon$. Let $\sigma = \mathsf{block}(\tau)$ which has length ϵ. Let $h = \mathsf{decode}(e)$. (1) asserts $\sigma \in [\mathfrak{C}_z]^{<\omega_1}_*$. (2) asserts that $\sup(\sigma) = \sup(\tau) < h(\epsilon)$. (3) and (4) assert that $\Xi_\rho(\mathsf{merge}(u, e)) \in \mathrm{WO}$ and $\mathrm{ot}(\Xi_\rho(\mathsf{merge}(u, e)) < \sup(\sigma) = \sup(\tau)$.

Definition 10.2. Define $\mathsf{PFcode} \subseteq \mathbb{R}^3$ by $\mathsf{PFcode}(\rho, z, e)$ if and only if the following holds.

(1) $z \in \mathsf{clubcode}$.
(2) $\mathsf{INC}(z, e)$.
(3) $(\forall^{\mathbb{R}} u, \bar{u}, \bar{u}_0)$

$$\Big([\mathsf{PCP}(u, \rho, z, e) \wedge \mathsf{INC}^{\mathsf{BS}}(z, \bar{u}) \wedge \mathsf{acceptableMerge}(\bar{u}, e) \wedge \mathsf{length}(u) \leq^{\mathrm{ot}}_{\Pi^1_1} \mathsf{length}(\bar{u}) \wedge$$
$$\mathsf{initSeg}(\bar{u}, \mathsf{length}(\bar{u}_0), \bar{u}_0) \wedge \mathsf{sameBlock}^{\mathsf{BS}}(u, \bar{u}_0)] \Rightarrow$$
$$[\Xi_\rho(\mathsf{merge}(u, e)) \leq^{\mathrm{ot}}_{\Pi^1_1} \Xi_\rho(\mathsf{merge}(\bar{u}, e)) \wedge \Xi_\rho(\mathsf{merge}(\bar{u}, e)) \leq^{\mathrm{ot}}_{\Pi^1_1} \Xi_\rho(\mathsf{merge}(u, e))]\Big).$$

(1) is $\mathbf{\Pi}^1_2$. (2) is $\mathbf{\Pi}^1_2$. (3) is $\mathbf{\Pi}^1_2$ since PCP, $\mathsf{INC}^{\mathsf{BS}}$, $\mathsf{acceptableMerge}$, $\mathsf{initSeg}$, and $\mathsf{sameBlock}^{\mathsf{BS}}$ are $\mathbf{\Pi}^1_1$. Thus PFcode is $\mathbf{\Pi}^1_2$. Elements (ρ, z, e) of PFcode are called pseudo-function codes.

The intuition for $\mathsf{PFcode}(\rho, z, e)$ is:

(i) (1) asserts z codes a club $C = \mathfrak{C}_z$.
(ii) (2) asserts e is a good code with $h = \mathsf{decode}(e) \in [C]^{\omega_1}$.
(iii) (3) asserts the following: Suppose $u \in \mathsf{BS}$ codes $\tau = \mathsf{seq}(u) \in [C]^{<\omega_1}_*$, there is an ϵ so that $|\tau| = \omega\cdot\epsilon$, and $\mathsf{PCP}(u, \rho, z, e)$. Suppose $\bar{u} \in \mathsf{BS}$ codes $\bar{\tau} = \mathsf{seq}(\bar{u}) \in [C]^{<\omega_1}_*$ with $|\bar{\tau}| \geq |\tau|$ and $\sup(\bar{\tau}) < \mathsf{decode}(e)(|\bar{\tau}|)$. Let $\bar{\tau}_0 = \mathsf{seq}(\bar{u}_0)$. $\bar{\tau}_0$ is the initial segment of $\bar{\tau}$ of length $\omega\cdot\epsilon$ and $\mathsf{block}(\tau) = \mathsf{block}(\bar{\tau}_0)$. Then $\Xi_\rho(\mathsf{merge}(u), e)$ and $\Xi_\rho(\mathsf{merge}(\bar{u}), e)$ code the same countable ordinal. (In other words, if $\mathsf{PCP}(u, \rho, z, e)$ and $\mathsf{block}(\mathsf{seq}(u)) \subseteq \mathsf{block}(\mathsf{seq}(\bar{u}))$, then $\Xi_\rho(\mathsf{merge}(u, e))$ and $\Xi_\rho(\mathsf{merge}(\bar{u}, e))$ codes the same ordinal.) (Note this also implies that $\mathsf{PCP}(\bar{u}, \rho, z, e)$, i.e. $\bar{u}$ is a pseudo-continuity point for (ρ, z, e).)

Definition 10.3. Suppose $\mathsf{PFcode}(\rho, z, e)$. Let $\mathfrak{D}_z$ be the limit points of $\mathfrak{C}_z$. Define a partial function $\Phi^{(\rho,z,e)}$ on $[\mathfrak{D}_z]^{\omega_1}_*$ as follows: Suppose $f \in [\mathfrak{D}_z]^{\omega_1}_*$. Say that $f \in \mathrm{dom}(\Phi^{(\rho,z,e)})$ if and only if there exists a $u \in \mathsf{BS}$ so that $\mathsf{PCP}(u, \rho, z, e)$ and $\mathsf{block}(\mathsf{seq}(u))$ is an initial segment of f. If $f \in \mathrm{dom}(\Phi^{(\rho,z,e)})$, then define $\Phi^{(\rho,z,e)}(f) = \mathrm{ot}(\Xi_\rho(\mathsf{merge}(u, e)))$ where $u \in \mathsf{BS}$ is such that $\mathsf{PCP}(u, \rho, z, e)$, $\mathsf{seq}(u) \in [\mathfrak{C}_z]^{<\omega_1}$, and

$\mathsf{block}(\mathsf{seq}(u))$ is an initial segment of f. By Definition 10.2 (3), this is well defined independent of the choice of initial segment of f and $u \in \mathsf{BS}$ so that $\mathsf{block}(u)$ is this initial segment.

Let $\mathsf{tPFcode}(\rho, z, e)$ if and only if $\mathrm{dom}(\Phi^{(\rho,z,e)}) \in \mu^{\omega_1}_{\omega_1}$. (In other words, there is a club $D \subseteq \omega_1$ so that for all $f \in [D]^{\omega_1}_*$, $\Phi^{(\rho,z,e)}(f)$ is defined.) Elements of $\mathsf{tPFcode}$ are called true pseudo-function codes because (ρ, z, e) successfully codes a function $\Phi^{(\rho,z,e)}$ which is defined $\mu^{\omega_1}_{\omega_1}$-almost everywhere.

Remark 10.4. Although PFcode is $\mathbf{\Pi}^1_2$, one must restrict to $\mathsf{tPFcode}$ in order to establish the wellfoundedness of relations coding subsets of the ultrapower of ω_1 by the strong partition measure. A priori, $\mathsf{tPFcode}$ seems very complicated. For instance, $\mathsf{tPFcode}(\rho, z, e)$ if and only if

$$\mathsf{PFcode}(\rho, z, e) \wedge (\exists^{\mathbb{R}} \bar{z})\{\bar{z} \in \mathsf{clubcode} \wedge \mathsf{clubSubset}(\bar{z}, z) \wedge (\forall^{\mathbb{R}} \bar{e})(\mathsf{INC}(\bar{z}, \bar{e}) \Rightarrow$$
$$(\exists^{\mathbb{R}} u)[\mathsf{PCP}(u, \rho, z, e) \wedge \mathsf{block}(\mathsf{seq}(u)) \subseteq \mathsf{block}(\mathsf{decode}(\bar{e}))])\}.$$

This expresses that (ρ, z, e) is a pseudo-function code and there exists a club $\mathfrak{C}_{\bar{z}}$ so that $\mathfrak{C}_{\bar{z}} \subseteq \mathfrak{C}_z$ and for all good code $\bar{e}$ coding an increasing function through $\mathfrak{C}_{\bar{z}}$, there exists a $u \in \mathsf{BS}$ which codes a pseudo continuity point for (ρ, z, e) and $\mathsf{block}(\mathsf{seq}(u))$ is an initial segment of $\mathsf{block}(\mathsf{decode}(\bar{e}))$. After formally expressing $\mathsf{block}(\mathsf{seq}(u)) \subseteq \mathsf{block}(\mathsf{decode}(\bar{e}))$, one can check that this expression is $\mathbf{\Sigma}^1_4$. The complexity again arises from the universal quantification over good codes.

Fact 10.5. *Let $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$. There exists a $(\rho, z, e) \in \mathsf{tPFcode}$ so that $[\Phi^{(\rho,z,e)}]_{\mu^{\omega_1}_{\omega_1}} = [\Phi]_{\mu^{\omega_1}_{\omega_1}}$.*

Proof. By the argument in Fact 9.7, there is a Lipschitz continuous function Ξ and a club C_0 so that for all $x \in \mathsf{GC}$ with $\mathsf{decode}(x) \in [C_0]^{\omega_1}$, $\Phi(\mathsf{block}(\mathsf{decode}(x))) = \mathrm{ot}(\Xi(x))$. There is a club C_1 so that Φ satisfies the fine continuity property relative to C_1 in the sense of Theorem 8.12. Let $\rho, z, e \in \mathbb{R}$ be such that $\Xi_\rho = \Xi$, $z \in \mathsf{clubcode}$ with $\mathfrak{C}_z \subseteq C_0 \cap C_1$, and $e \in \mathsf{Inc}(\mathfrak{C}_z)$. Let $h = \mathsf{decode}(e)$. Let $g = \mathsf{block}(h)$. Let $\mathfrak{D}_z$ be the limit points of $\mathfrak{C}_z$. Note that the argument in Fact 9.7 shows $(\rho, z) \in \mathsf{Fcode}_{\omega_1}$ with $\Phi(f) = \Phi^{(\rho,z)}(f)$ for all $f \in [\mathfrak{D}_z]^{\omega_1}_*$.

Suppose $u \in \mathsf{BS}$ and let $\tau = \mathsf{seq}(u)$. Suppose $\mathsf{PCP}(u, \rho, z, e)$. This implies that $|\tau| = \omega \cdot \epsilon$ for some $\epsilon < \omega_1$ and $\sup(\tau) < h(\omega \cdot \epsilon)$. Let $\sigma = \mathsf{block}(\tau)$. Note $\mathsf{PCP}(u, \rho, z, e)$ implies

$$\Phi(\sigma\hat{\ }\mathsf{drop}(g, \epsilon)) = \Phi^{(\rho,z)}(\sigma\hat{\ }\mathsf{drop}(g, \epsilon)) = \Phi^{(\rho,z)}(\mathsf{block}(\mathsf{decode}(\mathsf{merge}(u, e)))) = \mathrm{ot}(\Xi_\rho(\mathsf{merge}(u, e))) < \sup(\sigma).$$

Since Φ satisfies the fine continuity relative to $\mathfrak{C}_z$, σ is a continuity point for Φ relative to $\mathfrak{D}_z$. Suppose $\bar{u} \in \mathsf{BS}$ and let $\bar{\tau} = \mathsf{seq}(\bar{u})$. Suppose $\bar{\tau} \in [\mathfrak{C}_z]^{<\omega_1}$ and there is an $\bar{\epsilon} < \omega_1$ so that $|\bar{\tau}| = \omega \cdot \bar{\epsilon}$ and $\sup(\bar{\tau}) < h(\omega \cdot \bar{\epsilon})$. Let $\bar{\sigma} = \mathsf{block}(\tau)$. Suppose that $|\tau| \leq |\bar{\tau}|$ and $\sigma \subseteq \bar{\sigma}$. Hence $\bar{\sigma}$ is also a continuity point for Φ relative to $\mathfrak{D}_z$. Therefore

$$\mathrm{ot}(\Xi_\rho(\mathsf{merge}(\bar{u}, e))) = \Phi^{(\rho,z)}(\bar{\sigma}\hat{\ }\mathsf{drop}(g, \bar{\epsilon})) = \Phi(\sigma\hat{\ }\mathsf{drop}(g, \epsilon)) = \mathrm{ot}(\Xi_\rho(\mathsf{merge}(u, e))).$$

This shows that $(\rho, z, e) \in \mathsf{PFcode}$.

Let $f \in [\mathfrak{D}_z]^{\omega_1}_*$. Pick an α_0 so that $\Phi(f) < \sup(f \restriction \alpha_0)$. Since Φ satisfies the fine continuity property relative to $\mathfrak{D}_z$, $f \restriction \alpha_0$ is a continuity point for Φ relative to $\mathfrak{D}_z$. Find an $\alpha_1 > \alpha_0$ so that $\omega \cdot \alpha_1 = \alpha_1$ and $\sup(f \restriction \alpha_1) \leq h(\alpha_1)$. Since $f \restriction \alpha_0 \subseteq f \restriction \alpha_1$, $f \restriction \alpha_1$ is also a continuity point for Φ relative to $\mathfrak{D}_z$. Pick an $\ell \in [\mathfrak{C}_z]^{\alpha_1}$ so that $\mathsf{block}(\ell) = f \restriction \alpha_1$. Pick a $u \in \mathsf{BS}$ so that $\mathsf{seq}(u) = \ell$. Since $(\rho, z) \in \mathsf{Fcode}$, $\Xi_\rho(\mathsf{merge}(u, e)) \in \mathrm{WO}$ and moreover

$$\mathrm{ot}(\Xi_\rho(\mathsf{merge}(u, e))) = \Phi^{(\rho,e)}(\mathsf{block}(\ell\hat{\ }\mathsf{drop}(h, \alpha_1))) = \Phi^{(\rho,e)}(f \restriction \alpha_1\hat{\ }\mathsf{drop}(g, \alpha_1))$$
$$= \Phi(f \restriction \alpha_1\hat{\ }\mathsf{drop}(g, \alpha_1)) = \Phi(f) < \sup(f \restriction \alpha_0) \leq \sup(f \restriction \alpha_1) = \sup(\ell) = \sup(\mathsf{seq}(u)).$$

This shows that $\mathsf{PCP}(u, \rho, z, e)$ with $\mathsf{block}(\mathsf{seq}(u)) = \mathsf{block}(\ell) = f \restriction \alpha_1$ being an initial segment of f. Hence $f \in \mathrm{dom}(\Phi^{(\rho,z,e)})$. Since $f \in [\mathfrak{D}_z]^{\omega_1}_*$ was arbitrary, $[D_z]^{\omega_1}_* \subseteq \mathrm{dom}(\Phi^{(\rho,z,e)})$ and hence $(\rho, z, e) \in \mathsf{tPFcode}$. The argument also shows that for all $f \in [\mathfrak{D}_z]^{\omega_1}_*$, $\Phi^{(\rho,z,e)}(f) = \Phi^{(\rho,z)}(f) = \Phi(f)$. ☐

Definition 10.6. Define $\mathsf{compare}$ by $\mathsf{compare}(u, z^*, \rho, z, e, \bar{\rho}, \bar{z}, \bar{e})$ if and only if the conjunction of the following holds.

(1) $z^*, z, \bar{z} \in \mathsf{clubcode} \wedge \mathsf{INC}^{\mathsf{BS}}(u, z^*)$
(2) $\mathsf{clubSubset}(z^*, z) \wedge \mathsf{clubSubset}(z^*, \bar{z})$.
(3) $\mathsf{acceptableMerge}(u, e) \wedge \mathsf{acceptableMerge}(u, \bar{e})$.
(4) $\Xi_\rho(\mathsf{merge}(u, e)) <^{\mathrm{ot}}_{\Pi^1_1} \Xi_{\bar{\rho}}(\mathsf{merge}(u, \bar{u}))$.

Note that $\mathsf{compare}$ is $\mathbf{\Pi}^1_2$.

Definition 10.7. If $A \subseteq \mathsf{tPFcode}$, then let $\mathfrak{A}(A) \subseteq \prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1}$ consists of those F so that there is a $(\rho, z, e) \in A$ such that $F = [\Phi^{(\rho,z,e)}]_{\mu^{\omega_1}_{\omega_1}}$.

Define $\mathfrak{P}_A$ by $\mathfrak{P}_A((\rho, z, e), (\bar{\rho}, \bar{z}, \bar{e}))$ if and only if the conjunction of the following holds.

(1) $(\rho, z, e) \in A$.
(2) $(\bar{\rho}, \bar{z}, \bar{e}) \in A$.
(3) $(\exists^{\mathbb{R}} z^*)(\forall^{\mathbb{R}} u)([\mathsf{PCP}(u, \rho, z, e) \wedge \mathsf{PCP}(u, \bar{\rho}, \bar{z}, \bar{e})] \Rightarrow \mathsf{compare}(u, z^*, \rho, z, e, \bar{\rho}, \bar{z}, \bar{e}))$

Lemma 10.8. *Let $A \subseteq \mathsf{tPFcode}$ and $(\rho_0, z_0, e_0), (\rho_1, z_1, e_1) \in A$. $\mathfrak{P}_A((\rho_0, z_0, e_0), (\rho_1, z_1, e_1))$ if and only if $[\Phi^{(\rho_0,z_0,e_0)}]_{\mu_{\omega_1}} < [\Phi^{(\rho_1,z_1,e_1)}]_{\mu_{\omega_1}}$.*

Proof. Observe $\mathrm{dom}(\Phi^{(\rho_0,z_0,e_0)}), \mathrm{dom}(\Phi^{(\rho_1,e_1,z_1)}) \in \mu^{\omega_1}_{\omega_1}$. Let $z^* \in \mathsf{clubcode}$ witness the existential quantifier in the definition of $\mathfrak{P}_A((\rho_0, z_0, e_0), (\rho_1, z_1, e_1))$. Let $C \subseteq \mathfrak{C}_{z^*}$ be such that if D is the club of limit points of C, then $[D]^{\omega_1}_* \subseteq \mathrm{dom}(\Phi^{(\rho_0,z_0,e_0)}) \cap \mathrm{dom}(\Phi^{(\rho_1,z_1,e_1)})$. Pick any $f \in [D]^{\omega_1}_*$. For each $i \in 2$, there is an $\alpha_i < \omega_1$ so that there exists a $u_i \in \mathsf{BS}$ with $\mathsf{seq}(u_i) \in [C]^{\omega \cdot \alpha_i}$, $\mathsf{block}(\mathsf{seq}(u_i)) = f \restriction \alpha_i$, and $\mathsf{PCP}(u_i, \rho_i, z_i, e_i)$. Pick any $\alpha^* > \max\{\alpha_0, \alpha_1\}$ so that $\sup(f \restriction \alpha^*) \leq \mathsf{decode}(e_0)(\omega \cdot \alpha^*)$ and $\sup(f \restriction \alpha^*) \leq \mathsf{decode}(e_1)(\omega \cdot \alpha^*)$. Let $u^* \in \mathsf{BS}$ with $\mathsf{seq}(u^*) \in [C]^{\omega \cdot \alpha^*}$ and $\mathsf{block}(u^*) = f \restriction \alpha^*$. Note that $\mathsf{PCP}(u^*, \rho_0, z_0, e_0)$ and $\mathsf{PCP}(u^*, \rho_1, z_1, e_1)$. Then $\mathsf{compare}(u^*, z^*, \rho_0, z_0, e_0, \rho_1, z_1, e_1)$ implies that

$$\Phi^{(\rho_0,z_0,e_0)}(f) = \mathrm{ot}(\Xi_{\rho_0}(\mathsf{merge}(u^*, e_0))) < \mathrm{ot}(\Xi_{\rho_1}(\mathsf{merge}(u^*, e_1))) = \Phi^{(\rho_1,z_1,e_1)}(f).$$

Therefore, $[\Phi^{(\rho_0,z_0,e_0)}]_{\mu^{\omega_1}_{\omega_1}} < [\Phi^{(\rho_1,z_1,e_1)}]_{\mu^{\omega_1}_{\omega_1}}$.

Suppose that $[\Phi^{(\rho_0,z_0,e_0)}]_{\mu^{\omega_1}_{\omega_1}} < [\Phi^{(\rho_1,z_1,e_1)}]_{\mu^{\omega_1}_{\omega_1}}$. Let $C \subseteq \omega_1$ be a club so that $C \subseteq \mathfrak{C}_{z_0} \cap \mathfrak{C}_{z_1}$ and if D is the set of limit points of C, then $[D]^{\omega_1}_* \subseteq \mathrm{dom}(\Phi^{(\rho_0,z_0,e_0)}) \cap \mathrm{dom}(\Phi^{(\rho_1,z_1,e_1)})$ and for all $f \in [D]^{\omega_1}_*$, $\Phi^{(\rho_0,z_0,e_0)}(f) < \Phi^{(\rho_1,z_1,e_1)}(f)$. Pick $z^* \in \mathsf{clubcode}$ with $\mathfrak{C}_{z^*} \subseteq C$. Suppose $u \in \mathsf{BS}$ with $\mathsf{seq}(u) \in [\mathfrak{C}_z]^{<\omega_1}$, $\mathsf{PCP}(u, \rho_0, z_0, e_0)$, and $\mathsf{PCP}(u, \rho_1, z_1, e_1)$. Pick $g \in [D]^{\omega_1}_*$ with $\sup(\mathsf{seq}(u)) < g(0)$. Let $f^* = \mathsf{block}(\mathsf{seq}(u))\hat{\ }g$.

$$\mathrm{ot}(\Xi_{\rho_0}(\mathsf{merge}(u, e_0))) = \Phi^{(\rho_0,z_0,e_0)}(f^*) < \Phi^{(\rho_1,z_1,e_1)}(f^*) = \mathrm{ot}(\Xi_{\rho_1}(\mathsf{merge}(u, e_1))).$$

This shows $\mathfrak{P}_A((\rho_0, z_0, e_0), (\rho_1, z_1, e_1))$. $\square$

Theorem 10.9. *If $A \subseteq \mathsf{tPFcode}$ and A is $\mathbf{\Sigma}^1_3$, then $\mathfrak{P}_A$ is a $\mathbf{\Sigma}^1_3$ wellfounded relation, $\mathsf{rk}(\mathfrak{P}_A) = \mathrm{ot}(\mathfrak{A}(A))$, and $\mathrm{ot}(\mathfrak{A}(A)) < \boldsymbol{\delta}^1_3 = \omega_{\omega+1}$.*

Proof. Since $\mathsf{compare}$ is $\mathbf{\Pi}^1_2$, A is $\mathbf{\Sigma}^1_3$, and PCP is $\mathbf{\Pi}^1_1$, $\mathfrak{P}_A$ is $\mathbf{\Sigma}^1_3$.

Define $\Upsilon : A \to \mathfrak{A}(A)$ by $\Upsilon(\rho, z, e) = [\Phi^{(\rho,z,e)}]_{\mu^{\omega_1}_{\omega_1}}$. This is well defined since $(\rho, z, e) \in \mathsf{tPFcode}$. By Lemma 10.8, $\mathfrak{P}_A((\rho, z, e), (\bar{\rho}, \bar{z}, \bar{e}))$ if and only if $[\Phi^{(\rho,z,e)}]_{\mu^{\omega_1}_{\omega_1}} < [\Phi^{(\bar{\rho},\bar{z},\bar{e})}]_{\mu^{\omega_1}_{\omega_1}}$. Since $\mathfrak{A}(A) \subseteq \prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1}$ and $\prod_{[\omega_1]^{\omega_1}_{\omega_1}} \omega_1/\mu^{\omega_1}_{\omega_1}$ is wellfounded by 9.2, $\mathfrak{P}_A$ is a wellfounded relation and $\mathsf{rk}(\mathfrak{P}_A) \leq \mathrm{ot}(\mathfrak{A}(A))$.

Define $\Sigma : \mathrm{ot}(\mathfrak{A}(A)) \to \mathsf{rk}(\mathfrak{P}_A)$ as follows: Suppose $F \in \mathfrak{A}(A)$. Pick a $(\rho, z, e) \in A$ so that $F = [\Phi^{(\rho,z,e)}]_{\mu^{\omega_1}_{\omega_1}}$. Let $\Sigma(F) = \mathsf{rk}_{\mathfrak{P}_A}(\rho, z, e)$. For any (ρ, z, e) and $(\bar{\rho}, \bar{z}, \bar{e})$ so that $[\Phi^{(\rho,z,e)}]_{\mu^{\omega_1}_{\omega_1}} = [\Phi^{(\bar{\rho},\bar{z},\bar{e})}]_{\mu^{\omega_1}_{\omega_1}}$, $\{(\rho', z', e') \in A : \mathfrak{P}_A((\rho', z', e'), (\rho, z, e,))\} = \{(\rho', z', e') \in A : \mathfrak{P}_A((\rho', z', e'), (\bar{\rho}, \bar{z}, \bar{e}))\}$. This shows that Σ is well defined independent of the choice of (ρ, z, e). If $F_1 < F_1$ belong to $\mathfrak{A}(A)$, then $\Sigma(F_1) < \Sigma(F_2)$. Thus $\mathrm{ot}(\mathfrak{A}(A)) \leq \mathsf{rk}(\mathfrak{P}_A)$.

Since $\mathfrak{P}_A$ is a $\mathbf{\Sigma}^1_3$ and hence ω_ω-Suslin wellfounded relation, the Kunen-Martin theorem (Fact 7.11) implies $\mathrm{ot}(\mathfrak{A}(A)) = \mathsf{rk}(\mathfrak{P}_A) < (\omega_\omega)^+ = \omega_{\omega+1} = \boldsymbol{\delta}^1_3$. $\square$

Definition 10.10. Suppose $(\hat{\rho}, \hat{z}) \in \mathsf{Fcode}_{\omega_1}$ and $\hat{e} \in \mathsf{Inc}(\mathfrak{C}_z)$. Assume that $\Phi^{(\hat{\rho},\hat{z})}$ satisfies the fine continuity property of Definition 8.12 relative to $\mathfrak{D}_{\hat{z}}$, which is the collection of limit points of $\mathfrak{C}_z$. Let $A_{(\hat{\rho},\hat{z}),\hat{e}}$ be the collection of (ρ, z, e) such that

(1) $(\rho, z, e) \in \mathsf{PFcode}$.
(2) $\mathsf{clubSubset}(z, \hat{z})$.
(3) $(\forall^{\mathbb{R}} u)\Big([\mathsf{INC}^{\mathsf{BS}}(z, u) \wedge \mathsf{PCP}(u, (\hat{\rho}, \hat{z}, \hat{e})) \wedge \mathsf{acceptableMerge}(u, e)] \Rightarrow \mathsf{compare}(u, z, \rho, z, e, \hat{\rho}, \hat{z}, \hat{e})\Big)$

$A_{(\hat{\rho},\hat{z}),\hat{e}}$ is $\mathbf{\Pi}^1_2$.

The intuition is the following: $(\hat{\rho},\hat{z}) \in \mathsf{Fcode}_{\omega_1}$ so it codes a function $\Phi^{(\hat{\rho},\hat{z})}$ in the more natural or direct sense of Definition 9.5. Thus $(\hat{\rho},\hat{z},\hat{e}) \in \mathsf{tPFcode}$, i.e. is a true pseudo-function code with $\Phi^{(\hat{\rho},\hat{z})}(f) = \Phi^{(\hat{\rho},\hat{z},\hat{e})}(f)$ for all $f \in [\mathfrak{D}_z]^{\omega_1}_*$. Condition (3) roughly states that any pseudo-continuity point for $(\hat{\rho},\hat{z},\hat{e})$ is a pseudo-continuity point for $(\rho,z,e) \in A_{(\hat{\rho},\hat{z}),\hat{e}}$. Since $(\hat{\rho},\hat{z},\hat{e}) \in \mathsf{tPFcode}$, every $f \in [\mathfrak{D}_{\hat{z}}]^{\omega_1}_*$ has an initial segment $f \restriction \alpha$ which possesses a $u \in \mathsf{BS}$ with $\mathsf{PCP}(u,\hat{\rho},\hat{z},\hat{e})$ and $\mathsf{seq}(u) = f \restriction \alpha$. Thus if $(\rho,z,e) \in A_{(\hat{\rho},\hat{z}),\hat{e}}$, then any $f \in [\mathfrak{D}_z]^{\omega_1}_* \subseteq [\mathfrak{D}_{\hat{z}}]^{\omega_1}_*$ has an initial segment coded by a $u \in \mathsf{BS}$ with $\mathsf{PCP}(u,\rho,z,e)$. This implies that $\mathrm{dom}(\Phi^{(\rho,z,e)}) \in \mu^{\omega_1}_{\omega_1}$ and thus $(\rho,z,e) \in \mathsf{tPFcode}$. The details are in the next lemma.

Lemma 10.11. *Suppose $(\hat{\rho},\hat{z})$ and $\hat{e}$ satisfy the conditions from Definition 10.10. $A_{(\hat{\rho},\hat{z}),\hat{e}}$ is $\mathbf{\Pi}^1_2$. $A_{(\hat{\rho},\hat{z}),\hat{e}} \subseteq \mathsf{tPFcode}$. For all $(\rho,z,e) \in A_{(\hat{\rho},\hat{z}),\hat{e}}$, $[\Phi^{(\rho,z,e)}]_{\mu^{\omega_1}_{\omega_1}} < [\Phi^{(\hat{\rho},\hat{z})}]_{\mu^{\omega_1}_{\omega_1}}$. For all $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ with $[\Phi]_{\mu^{\omega_1}_{\omega_1}} < [\Phi^{(\hat{\rho},\hat{z},\hat{e})}]_{\mu^{\omega_1}_{\omega_1}}$, there is a $(\rho,z,e) \in A_{(\hat{\rho},\hat{z}),\hat{e}}$ so that $[\Phi]_{\mu^{\omega_1}_{\omega_1}} = [\Phi^{(\rho,z,e)}]_{\mu^{\omega_1}_{\omega_1}}$.*

Proof. Fix $(\rho,z,e) \in A_{(\hat{\rho},\hat{z}),\hat{e}}$. Let $C = \mathfrak{C}_z$ and $D = \mathfrak{D}_z$. Let $\hat{h} = \mathsf{decode}(\hat{e})$ and $h = \mathsf{decode}(e)$. Let $f \in [D]^{\omega_1}_*$. Let α_0 be such that $\Phi^{(\hat{\rho},\hat{z})}(f) < \sup(f \restriction \alpha_0)$. So $f \restriction \alpha_0$ is a continuity point for $\Phi^{(\hat{\rho},\hat{z})}$ relative to D due to the fine continuity property. Let $\alpha_1 > \alpha$ be so that $\sup(f \restriction \alpha_1) \leq \hat{h}(\omega \cdot \alpha_1)$ and $\sup(f \restriction \alpha_1) \leq h(\omega \cdot \alpha_1)$. Let $u \in \mathsf{BS}$ with $\mathsf{seq}(u) \in [C]^{\omega \cdot \alpha_1}$ and $\mathsf{block}(\mathsf{seq}(u)) = f \restriction \alpha_1$. Observe that $\mathsf{acceptableMerge}(u,e)$ and $\mathsf{PCP}(u,\hat{\rho},\hat{z},\hat{e})$. Then $\mathsf{compare}(u,z,\rho,z,e,\hat{\rho},\hat{z},\hat{e})$ implies

$$\mathrm{ot}(\Xi_\rho(\mathsf{merge}(u,e))) < \mathrm{ot}(\Xi_{\hat{\rho}}(\mathsf{merge}(u,\hat{e}))) = \Phi^{(\hat{\rho},\hat{z})}(f) < \sup(f \restriction \alpha_0) \leq \sup(f \restriction \alpha_1) = \sup(\mathsf{seq}(u)).$$

Thus $\mathsf{PCP}(u,\rho,z,e)$. This implies $f \in \mathrm{dom}(\Phi^{(\rho,z,e)})$. Since $f \in [D]^{\omega_1}_*$ was arbitrary, $\mathrm{dom}(\Phi^{(\rho,z,e)}) \in \mu^{\omega_1}_{\omega_1}$ and hence $(\rho,z,e) \in \mathsf{tPFcode}$. The above equation also shows that $\Phi^{(\rho,z,e)}(f) = \mathrm{ot}(\Xi_\rho(\mathsf{merge}(u,e))) < \Phi^{(\hat{\rho},\hat{z})}(f)$. Thus $[\Phi^{(\rho,z,e)}]_{\mu^{\omega_1}_{\omega_1}} < [\Phi^{(\hat{\rho},\hat{z})}]_{\mu^{\omega_1}_{\omega_1}}$.

Now suppose $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ with $[\Phi]_{\mu^{\omega_1}_{\omega_1}} < [\Phi^{(\hat{\rho},\hat{z})}]_{\mu^{\omega_1}_{\omega_1}}$. Let $D \subseteq \omega_1$ be such that Φ satisfies the fine continuity property relative to D and for all $f \in [D]^{\omega_1}_*$, $\Phi(f) < \Phi^{(\hat{\rho},\hat{z})}(f)$. By Fact 9.7, there is a $(\rho,z) \in \mathsf{Fcode}_{\omega_1}$ so that $\mathfrak{C}_z \subseteq \mathfrak{C}_{\hat{z}}$ and $\mathfrak{D}_z \subseteq D$ and for all $f \in [\mathfrak{D}_z]^{\omega_1}_*$, $\Phi(f) = \Phi^{(\rho,z)}(f)$. Pick any $e \in \mathsf{GC}$ with $\mathsf{decode}(e) \in [\mathfrak{C}_z]^{\omega_1}$. Observe $(\rho,z,e) \in \mathsf{tPFcode}$ and for all $f \in [\mathfrak{D}_z]^{\omega_1}_*$, $\Phi^{(\rho,z,e)}(f) = \Phi(f)$. Now suppose $u \in \mathsf{BS}$ is such $\mathsf{seq}(u) \in [\mathfrak{C}_z]^{<\omega_1}$, $\mathsf{PCP}(u,\hat{\rho},\hat{z},\hat{e})$, and $\mathsf{acceptableMerge}(u,e)$. Let $\hat{h} = \mathsf{decode}(\hat{e})$ and $h = \mathsf{decode}(e)$. Let $\tau = \mathsf{seq}(u)$ and ϵ be such that $|\tau| = \omega \cdot \epsilon$. Let $\sigma = \mathsf{block}(\mathsf{seq}(\tau))$. $\mathsf{PCP}(u,\hat{\rho},\hat{z},\hat{e})$ implies that σ is a continuity point for $\Phi^{(\hat{\rho},\hat{z})}$ relative to $\mathfrak{D}_z$. Let $g = \mathsf{block}(\mathsf{seq}(\tau)\hat{\ }\mathsf{drop}(h,\omega \cdot \epsilon))$.

$$\mathrm{ot}(\Xi_\rho(\mathsf{merge}(u,e))) = \Phi^{(\rho,z)}(g) = \Phi(g) < \Phi^{(\hat{\rho},\hat{e})}(g) = \mathrm{ot}(\Xi_{\hat{\rho}}(\mathsf{merge}(u,\hat{e}))).$$

This implies $\mathsf{compare}(u,z,\rho,z,e,\hat{\rho},\hat{z},\hat{e})$. Thus $(\rho,z,e) \in A_{(\hat{\rho},\hat{z}),e}$ and represents $[\Phi]_{\mu^{\omega_1}_{\omega_1}}$. ◻

Theorem 10.12. *Suppose $\Psi : [\omega_1]^{\omega_1}_* \to \omega_1$. Then $\mathrm{ot}(\prod_{f \in [\omega_1]^{\omega_1}_*} \Psi(f)/\mu^{\omega_1}_{\omega_1}) < \omega_{\omega+1} = \boldsymbol{\delta}^1_3$.*

Proof. Pick a $(\hat{\rho},\hat{z})$ and an $\hat{e} \in \mathsf{Inc}(\mathfrak{C}_z)$ so that $\Phi^{(\hat{\rho},\hat{z})}$ satisfies the fine continuity property relative to $\mathfrak{C}_z$. By Lemma 10.11 $A_{(\hat{\rho},\hat{z}),\hat{e}}$ is $\mathbf{\Sigma}^1_3$ (in fact, $\mathbf{\Pi}^1_2$) and $\mathfrak{A}(A_{(\hat{\rho},\hat{z}),\hat{e}}) = \prod_{f \in [\omega_1]^{\omega_1}_*} \Psi(f)/\mu^{\omega_1}_{\omega_1}$. Theorem 10.9 implies $\mathrm{ot}(\prod_{f \in [\omega_1]^{\omega_1}_*} \Psi(f)/\mu^{\omega_1}_{\omega_1}) < \omega_{\omega+1} = \boldsymbol{\delta}^1_3$. ◻

Theorem 10.13. $\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1}) \leq \omega_{\omega+1} = \boldsymbol{\delta}^1_3$.

Proof. This follows from Theorem 10.12. ◻

11. Local Bounds on the Ultrapower by the Partition Measures on Higher Odd Projective Ordinals

Much of the analysis of $\omega_1 = \boldsymbol{\delta}^1_1$ and its ultrapowers by its partition measures in the previous section has only used properties of $\boldsymbol{\delta}^1_1$ which are known to holds for the higher odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$ using Jackson's theory of descriptions. This section will present a sketch of how to adapt the methods for ω_1 to establish the almost everywhere short length club uniformization at $\boldsymbol{\delta}^1_{2n+1}$, bounds on the ultrapower of $\boldsymbol{\delta}^1_{2n+1}$ by its short partition measures, and a local bound on the ultrapower of $\boldsymbol{\delta}^1_{2n+1}$ by its strong partition measures. Relevant properties of $\boldsymbol{\delta}^1_{2n+1}$ will be mentioned; see [12] and [13] for Jackson's description analysis.

The reader interested in the full solution to Goldberg's question by bounding the ultrapower of ω_1 by its strong partition measure $\mu^{\omega_1}_{\omega_1}$ stictly below $\omega_{\omega+1}$ can safely skip this section. The global bound on the ultrapower of ω_1 by $\mu^{\omega_1}_{\omega_1}$ will however use techniques which are specific to ω_1.

Fact 11.1. *(Kechris; [14] Theorem 3.20) For each $n \in \omega$, there is a cardinal $\boldsymbol{\kappa}^1_{2n+1}$ with $\mathrm{cof}(\boldsymbol{\kappa}^1_{2n+1}) = \omega$ and $\boldsymbol{\delta}^1_{2n+1} = (\boldsymbol{\kappa}^1_{2n+1})^+$.*

For instance $\boldsymbol{\kappa}^1_1 = \omega$, $\boldsymbol{\delta}^1_1 = \omega_1$, $\boldsymbol{\kappa}^1_3 = \omega_\omega$, and $\boldsymbol{\delta}^1_{2n+1} = \omega_{\omega+1}$. An important aspect of Jackson's argument to establish the $\boldsymbol{\delta}^1_{2n+1}$-reasonableness of $\boldsymbol{\delta}^1_{2n+1}$ and hence the strong partition property of $\boldsymbol{\delta}^1_{2n+1}$ is a suitable coding of the subsets of $\boldsymbol{\kappa}^1_{2n+1}$. Kunen ([29]) produced such a coding for $\boldsymbol{\kappa}^1_3 = \omega_\omega$ using indiscernibles and Jackson ([12] and [13]) produced a more general coding for all $\boldsymbol{\kappa}^1_{2n+1}$ using descriptions. The following definition isolates the essential features of this coding that will be needed here.

Definition 11.2. Let κ be a cardinal. A suitable coding for $\mathscr{P}(\kappa)$ consists of $(\Gamma, \mathfrak{S}, \mathsf{M}_\beta : \beta < \kappa)$ with the following properties.

(1) Γ is a pointclass with $\delta(\Gamma) = \kappa^+$.
(2) $\Delta = \Gamma \cap \check{\Gamma}$ is closed under κ-length wellorderable unions and intersections.
(3) $\mathfrak{S} : \mathbb{R} \to \mathscr{P}(\kappa)$ is a surjection.
(4) For all $\beta < \kappa$, $\mathsf{M}_\beta = \{z \in \mathbb{R} : \beta \in \mathfrak{S}(z)\}$ and $\mathsf{M}_\beta \in \Delta$.

Definition 11.3. Fix a suitable coding $(\Gamma, \mathfrak{S}, \mathsf{M}_\beta : \beta < \kappa)$ for $\mathscr{P}(\kappa)$. Let $\pi : \kappa \times \kappa \to \kappa$ be a bijection.

For $z \in \mathbb{R}$, let $\mathcal{R}_z \subseteq \kappa \times \kappa$ be defined by $\mathcal{R}_z(\alpha, \beta)$ if and only if $z \in \mathsf{M}_{\pi(\alpha,\beta)}$. Let $\mathsf{field}(z) = \mathsf{field}(\mathcal{R}_z) = \{\alpha < \kappa : (\exists \beta)(\mathcal{R}_z(\alpha, \beta) \vee \mathcal{R}_z(\beta, \alpha))\}$.

Let LO^κ be the collection of $z \in \mathbb{R}$ so that $(\mathsf{field}(z), \mathcal{R}_z)$ is a linear ordering. Let $\mathrm{ot}(z) < \kappa^+$ be the ordertype of $(\mathsf{field}(z), \mathcal{R}_z)$. If $\alpha < \kappa$, then let $\mathrm{ot}(z, \alpha)$ be the ordertype of α in $\mathcal{R}_z$. Let WO^κ consists of $z \in \mathrm{LO}^\kappa$ so that $(\mathsf{field}(z), \mathcal{R}_z)$ is a wellordering. For each $\beta < \kappa^+$, let $\mathrm{WO}^\kappa_\beta = \{z \in \mathrm{WO}^\kappa : \mathrm{ot}(z) = \beta\}$. If $z \in \mathrm{WO}_\beta$ and $\alpha < \beta$, then let $\mathsf{num}(z, \alpha)$ be the unique $\gamma < \kappa$ so that $\mathrm{ot}(z, \gamma) = \alpha$.

For each $z \in \mathbb{R}$ and $\nu < \kappa$, let $\mathcal{S}^\nu_z \subseteq \kappa \times \kappa$ be defined by $\mathcal{S}^\nu_z(\alpha, \beta)$ if and only if $\mathcal{R}_z(\pi(\nu, \pi(\alpha, \beta)))$. Let $\mathcal{L} = \{z \in \mathbb{R} : (\forall \nu < \kappa)(\mathcal{S}^\nu_z \text{ is a linear ordering})\}$. If $z \in \mathcal{L}$, then z codes a sequence $\langle \mathcal{S}^\nu_z : \nu < \kappa\rangle$ of linear orderings on κ. Let $\mathrm{sot}(z, \nu)$ be the ordertype of $\mathcal{S}^\nu_z$. For each $\nu < \kappa$ and $\gamma < \kappa^+$, let $\mathcal{V}^\nu_\gamma = \{z \in \mathcal{L} : \mathrm{sot}(z, \nu) = \gamma\}$.

Fact 11.4. *Assume the setting of Definition 11.3. For any $\alpha, \beta < \kappa$, the relation $\mathcal{R}_z(\alpha, \beta)$ as a relation on the variable z is Δ. For each $\alpha < \kappa$, the relation "$\alpha \in \mathsf{field}(z)$" is Δ as a relation in the variable z. LO^κ is Δ. For all $\beta < \kappa^+$, $\mathrm{WO}^\kappa_\beta \in \Delta$. For each $\nu < \kappa^+$, let T^ν be defined by $T^\nu(x, \alpha)$ if and only if $x \in \mathsf{LO}^\kappa$, $\alpha \in \mathsf{field}(x)$, and $\mathrm{ot}(x, \alpha) = \nu$. T^ν is Δ. $\mathcal{L} \in \Delta$. For all $\nu < \kappa$ and $\gamma < \kappa^+$, $\mathcal{V}^\nu_\gamma \in \Delta$.*

Proof. For any fixed $\alpha, \beta < \kappa$, $\mathcal{R}_z(\alpha, \beta)$ if only if $\mathsf{M}_{\pi(\alpha,\beta)}$. The latter is Δ by the definition of the suitable coding of $\mathscr{P}(\kappa)$.

For a fixed α, "$\alpha \in \mathsf{field}(z)$" if and only if $(\forall \beta < \kappa)(\mathcal{R}_z(\alpha, \beta) \vee \mathcal{R}_z(\beta, \alpha))$ if and only if $z \in \bigcup_{\beta < \kappa} \mathsf{M}_{\pi(\alpha,\beta)} \cup \mathsf{M}_{\pi(\beta,\alpha)}$. The latter is Δ since Δ is closed under κ-length wellordered unions.

Similar arguments will show LO^κ and $\mathcal{L}$ are Δ.

For each $\alpha < \kappa$ and $\gamma < \kappa^+$, let $C^\alpha_\gamma = \{z \in \mathsf{LO}^\kappa : \alpha \in \mathsf{field}(z) \wedge \mathrm{ot}(z, \alpha) = \gamma\}$. For all $\alpha < \kappa$, it will be shown by induction on γ that $C^\alpha_\gamma \in \Delta$.

$z \in C^\alpha_0$ if and only if $\alpha \in \mathsf{field}(\mathcal{R}_z) \wedge (\forall \beta < \kappa)(\neg \mathcal{R}_z(\beta, \alpha))$.

Suppose $\gamma < \kappa^+$ and for all $\beta < \kappa$ and $\xi < \gamma$, C^β_ξ has been shown to belong to Δ. Then $z \in C^\alpha_\gamma$ if and only if

$$\alpha \in \mathsf{field}(z) \wedge (\forall \beta < \kappa)(\mathcal{R}_z(\beta, \alpha) \Rightarrow (\exists \xi < \gamma)C^\beta_\xi(z)) \wedge (\forall \xi < \gamma)(\exists \beta < \kappa)(\mathcal{R}_z(\beta, \alpha)) \wedge C^\beta_\xi(z)).$$

C^α_γ is in Δ by closure under κ-length unions and intersections.

For each $\gamma < \kappa^+$, $z \in \mathrm{WO}^\kappa_\gamma$ if and only if

$$(\forall \alpha < \kappa)(\alpha \in \mathsf{field}(z) \Rightarrow (\exists \xi < \gamma)C^\alpha_\xi(x)) \wedge (\forall \xi < \gamma)(\exists \alpha < \kappa)(C^\alpha_\xi(z)).$$

This is Δ by closure under κ-length unions and intersections.

$T^\nu \in \Delta$ and $\mathcal{V}^\nu_\gamma \in \Delta$ are shown by similar arguments. ☐

Fact 11.5. *([12], [13] Theorem 4.33, [29] Theorem 3.8) Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$. For all $n \in \omega$, there is a suitable coding for $\mathscr{P}(\boldsymbol{\kappa}^1_{2n+1})$ of the form $(\boldsymbol{\Pi}^1_{2n+1}, \mathfrak{S}, \mathsf{M}_\beta : \beta < \boldsymbol{\kappa}^1_{2n+1})$.*

Definition 11.6. For $n \in \omega$ and $\epsilon < \boldsymbol{\delta}^1_{2n+1}$, $(\boldsymbol{\Pi}^1_{2n+1}, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \boldsymbol{\delta}^1_{2n+1})$ is a short-uniformly good coding system for ${}^{\epsilon}\boldsymbol{\delta}^1_{2n+1}$ if and only Definition 7.3 holds with the following modifications: $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$ should be replaced everywhere with $\boldsymbol{\Sigma}^1_{2n+1}$ and $\boldsymbol{\Pi}^1_{2n+1}$. W should now be an element of $\mathrm{WO}_{\epsilon}^{\boldsymbol{\kappa}^1_{2n+1}}$. WO is replaced with $\mathrm{WO}^{\boldsymbol{\kappa}^1_{2n+1}}$. All quantifications of $n \in \omega$ should be replaced by quantification of $\alpha \in \boldsymbol{\kappa}^1_{2n+1}$.

Fact 11.7. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. *For each $n \in \omega$ and $\epsilon < \boldsymbol{\delta}^1_{2n+1}$, there is a short-uniformly good coding system* $(\boldsymbol{\Pi}^1_{2n+1}, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta < \epsilon, \gamma < \boldsymbol{\delta}^1_{2n+1})$.

Proof. [12], [13], and [29] established the existence of such good coding system while proving the weak partition property. For instance: Fix $W \in \mathrm{WO}_{\epsilon}^{\boldsymbol{\kappa}^1_{2n+1}}$. Let $\mathsf{GC}_{\beta,\gamma} = \mathcal{V}_{\gamma}^{\mathsf{num}(W,\beta)}$. The remaining details and the uniform goodness follow the template of Example 7.7. ☐

Theorem 11.8. *Suppose $\kappa < \delta$ are cardinals so that $\delta = \kappa^+$. Suppose δ is δ-reasonable with a good coding system $(\Gamma, \mathsf{decode}^*, \mathsf{GC}^*_{\beta,\gamma} : \beta, \gamma < \delta)$ for ${}^{\delta}\delta$ and there is a suitable coding of $\mathscr{P}(\kappa)$ of the form $(\Gamma, \mathfrak{S}, \mathsf{M}_\beta : \beta < \kappa)$. Then this good coding system for ${}^{\delta}\delta$ can be modified into a good coding family $\langle(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \delta), \mathsf{BS}, \mathsf{nGC}, \mathsf{merge}\rangle$ for δ.*

Proof. The main idea is that the notion of suitability is sufficient for the construction in Theorem 7.23. The closure of Δ by κ-length unions and intersections are important for complexity computations. Some details will be provided below.

Define BS by $u \in \mathsf{BS}$ if and only if $u^{[0]} \in \mathrm{WO}^{\kappa}$, $u^{[1]} \in \mathcal{L}$, and for all $\nu \in \mathsf{field}(u^{[0]})$, $\mathcal{S}^{\nu}_{u^{[1]}}$ is a wellordering. Let $\mathsf{length} : \mathbb{R} \to \mathbb{R}$ be defined by $\mathsf{length}(u) = u^{[0]}$. Let $\mathsf{seq} : \mathsf{BS} \to {}^{<\delta}\delta$ be defined by $\mathsf{seq}(u) \in {}^{\mathrm{ot}(\mathsf{length}(u))}\delta$ and for all $\alpha < \mathsf{length}(u)$, $\mathsf{seq}(u)(\alpha) = \mathrm{sot}(u^{[1]}, \mathsf{num}(u^{[0]}, \alpha))$. Note that $\mathsf{seq} : \mathsf{BS} \to {}^{<\delta}\delta$ is a surjection.

For $\beta < \delta$, let $u \in D_\beta$ if and only if $\mathsf{length}(u) \in \mathsf{LO}^{\kappa} \wedge (\exists \alpha < \kappa)(\mathrm{ot}(\mathsf{length}(u), \alpha) = \beta)$ which belong to Δ using closure under κ-length union and Fact 11.4. For $\beta, \gamma < \delta$, let $u \in \mathsf{BS}_{\beta,\gamma}$ if and only if

$$\mathsf{length}(u) \in \mathsf{LO}^{\kappa} \wedge (\exists \alpha < \kappa)(\mathrm{ot}(\mathsf{length}(u), \alpha) = \eta \wedge u^{[1]} \in \mathcal{V}^{\alpha}_{\gamma}).$$

For $\beta, \gamma < \delta$, define $\mathsf{GC}_{\beta,\gamma}$ by $(u, e) \in \mathsf{GC}_{\beta,\gamma}$ if and only if

$$(u \in D_\beta \Rightarrow u \in \mathsf{BS}_{\beta,\gamma}) \wedge (u \notin D_\beta \Rightarrow e \in \mathsf{GC}^*_{\beta,\gamma}).$$

$\mathsf{GC}_{\beta,\gamma} \in \Delta$. Define $\mathsf{decode} : \mathbb{R} \to \mathscr{P}(\delta \times \delta)$ by $\mathsf{decode}(u, e)(\beta, \gamma)$ if and only if $(u, e) \in \mathsf{GC}_{\beta,\gamma}$. As in Theorem 7.23, $(\boldsymbol{\Pi}^1_1, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \delta)$ is a good coding system for ${}^{\delta}\delta$.

Define $\mathsf{merge}(u, (u', e)) = (u, e)$. Fix a $u^* \in \mathsf{BS}$ so that $\mathsf{seq}(u^*) = \emptyset$. Let $\mathsf{nGC} = \{(u^*, e) : e \in \mathsf{GC}^*\}$. As in Theorem 7.23, $\langle(\Gamma, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \delta), \mathsf{BS}, \mathsf{seq}, \mathsf{merge}\rangle$ is a good coding family for ${}^{\delta}\delta$. ☐

Definition 11.9. Let $n \in \omega$. Say that $(\boldsymbol{\Pi}^1_{2n+1}, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \boldsymbol{\delta}^1_{2n+1})$ is a long-uniformly good coding system for ${}^{\boldsymbol{\delta}^1_{2n+1}}\boldsymbol{\delta}^1_{2n+1}$ if and only if it satisfies Definition 7.8 with $\boldsymbol{\Sigma}^1_1$, $\boldsymbol{\Pi}^1_1$, and WO replaced with $\boldsymbol{\Sigma}^1_{2n+1}$, $\boldsymbol{\Pi}^1_{2n+1}$, and $\mathrm{WO}^{\boldsymbol{\kappa}^1_{2n+1}}$, respectively

$\langle(\boldsymbol{\Pi}^1_{2n+1}, \mathsf{decode}, \mathsf{GC}_{\beta,\gamma} : \beta, \gamma < \boldsymbol{\delta}^1_{2n+1}), \mathsf{BS}, \mathsf{seq}, \mathsf{nGC}, \mathsf{merge}\rangle$ is a uniformly good coding family for $\boldsymbol{\delta}^1_{2n+1}$ if and only if satisfies Definition 7.21 holds with $\boldsymbol{\Sigma}^1_1$, $\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Pi}^1_2$, and WO replaced with $\boldsymbol{\Sigma}^1_{2n+1}$, $\boldsymbol{\Pi}^1_{2n+1}$, $\boldsymbol{\Pi}^1_{2n+2}$, and $\mathrm{WO}^{\boldsymbol{\kappa}^1_{2n+1}}$, respectively.

Theorem 11.10. $(\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}})$ *Let $n \in \omega$. There is a long-uniformly good coding system for ${}^{\boldsymbol{\delta}^1_{2n+1}}\boldsymbol{\delta}^1_{2n+1}$. There is a uniformly good coding family for $\boldsymbol{\delta}^1_{2n+1}$.*

Proof. Jackson ([12], [13]) showed there is a long-uniformly good coding system $(\boldsymbol{\Pi}^1_{2n+1}, \mathsf{decode}^*, \mathsf{GC}^*_{\beta,\gamma} : \beta, \gamma < \boldsymbol{\delta}^1_{2n+1})$ for ${}^{\boldsymbol{\delta}^1_{2n+1}}\boldsymbol{\delta}^1_{2n+1}$. The good coding family constructed in Theorem 11.8 using this long-uniformly good coding system for ${}^{\boldsymbol{\delta}^1_{2n+1}}\boldsymbol{\delta}^1_{2n+1}$ and the suitable coding for $\mathscr{P}(\boldsymbol{\kappa}^1_{2n+1})$ from Fact 11.5 will be a uniformly good coding family for $\boldsymbol{\delta}^1_{2n+1}$ using argument similar to Theorem 7.23 for ω_1. ☐

With the existence of a uniformly good coding family for $\boldsymbol{\delta}^1_{2n+1}$, many of the results established in the previous section for $\omega_1 = \boldsymbol{\delta}^1_1$ can be adapted for $\boldsymbol{\delta}^1_{2n+1}$.

Theorem 11.11. $(\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}})$ *The almost everywhere short length club uniformization principle holds for $\boldsymbol{\delta}^1_{2n+1}$ for all $n \in \omega$.*

Theorem 11.12. $(\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}})$ *Let $n\in\omega$ and $\Phi:[\boldsymbol{\delta}^1_{2n+1}]^{\boldsymbol{\delta}^1_{2n+1}}_* \to \boldsymbol{\delta}^1_{2n+1}$. There is a club $C\subseteq \boldsymbol{\delta}^1_{2n+1}$ with the following properties.*

(1) *For each $f\in[C]^{\boldsymbol{\delta}^1_{2n+1}}_*$, if β_f is the unique β so that $\sup(f\restriction\beta)\leq\Phi(f)<f(\beta)$, then $f\restriction\beta_f$ is a minimal continuity point for Φ relative to C.*

(2) *For any $\sigma\in[C]^{<\boldsymbol{\delta}^1_{2n+1}}_*$, if there is a $g\in[C]^{\boldsymbol{\delta}^1_{2n+1}}_*$ so that $\sup(\sigma)<g(0)$ and $\Phi(\sigma\hat{\ }g)<g(0)$, then σ is a continuity point relative to C.*

Theorem 11.13. $(\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}})$. *Let $n\in\omega$.*

(1) *Let $\epsilon<\boldsymbol{\delta}^1_{2n+1}$ and $\lambda<\Theta$ with $\mathrm{cof}(\lambda)>\boldsymbol{\delta}^1_{2n+1}$. For every function $\Phi:[\boldsymbol{\delta}^1_{2n+1}]^\epsilon_*\to\lambda$, there is a $\xi<\lambda$ and a club $C\subseteq\boldsymbol{\delta}^1_{2n+1}$ so that $\Phi[[C]^\epsilon_*]\subseteq\xi$.*

(2) *Let $\lambda<\Theta$ with $\mathrm{cof}(\lambda)>\boldsymbol{\kappa}^1_{2n+3}$. For every function $\Phi:[\boldsymbol{\delta}^1_{2n+1}]^{\boldsymbol{\delta}^1_{2n+1}}_*\to\lambda$, there is a $\xi<\lambda$ and a club $C\subseteq\boldsymbol{\delta}^1_{2n+1}$ so that $\Phi[[C]^{\boldsymbol{\delta}^1_{2n+1}}_*]\subseteq\xi$.*

Proof. (1) and (2) are analogs of Theorem 8.2 and Theorem 8.7. A sketch of the proof of (2) will be given below.

By Fact 8.1, let $\pi:P\to\lambda$ be a norm which is $\boldsymbol{\kappa}^1_{2n+3}$-Suslin bounding. Let $(\boldsymbol{\Pi}^1_{2n+1},\mathsf{decode},\mathsf{GC}_{\beta,\gamma}:\beta,\gamma<\boldsymbol{\delta}^1_{2n+1})$ be a long-uniformly good coding system for $\boldsymbol{\delta}^1_{2n+1}$. Define a relation $R\subseteq[\boldsymbol{\delta}^1_{2n+}]^{\boldsymbol{\delta}^1_{2n+1}}\times\mathbb{R}$ by $R(f,x)$ if and only if $\Phi(f)=\pi(x)$. By Theorem 5.9, let $\Xi:\mathbb{R}\to\mathbb{R}$ and the club $C\subseteq\boldsymbol{\delta}^1_{2n+1}$ be as given by the theorem. Let $z\in\mathsf{clubcode}^{\varphi}_{\boldsymbol{\delta}^1_{2n+1}}$ (where φ is a fixed $\boldsymbol{\Pi}^1_{2n+1}$-norm on a complete $\boldsymbol{\Pi}^1_{2n+1}$ set). Since $\mathsf{Inc}(\mathfrak{C}^{\varphi,\boldsymbol{\delta}^1_{2n+1}}_z)$ is $\boldsymbol{\Pi}^1_{2n+2}$ (which follows from the long-uniform goodness), one has that $\Xi[\mathsf{Inc}(\mathfrak{C}^{\varphi,\boldsymbol{\delta}^1_{2n+1}}_z)]\subseteq P$ is $\boldsymbol{\Sigma}^1_{2n+3}$ which is $\boldsymbol{\kappa}^1_{2n+3}$-Suslin (Moschovakis; [14] Theorem 3.11). Now the result is finished much like in Theorem 8.7. $\square$

Theorem 11.14. $(\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}})$ *For all $n\in\omega$ and $\epsilon<\boldsymbol{\delta}^1_{2n+1}$,*

$$\mathrm{ot}\left(\prod_{[\boldsymbol{\delta}^1_{2n+1}]^\epsilon_*}\boldsymbol{\delta}^1_{2n+1}/\mu_\epsilon^{\boldsymbol{\delta}^1_{2n+1}}\right)<\boldsymbol{\delta}^1_{2n+3}\quad\textit{and}\quad\mathrm{ot}\left(\prod_{[\boldsymbol{\delta}^1_{2n+1}]^\epsilon_*}\boldsymbol{\delta}^1_{2n+3}/\mu_\epsilon^{\boldsymbol{\delta}^1_{2n+1}}\right)=\boldsymbol{\delta}^1_{2n+3}.$$

Theorem 11.15. $(\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}})$ *For all $n\in\omega$,*

$$\mathrm{ot}\left(\prod_{[\boldsymbol{\delta}^1_{2n}]^{\boldsymbol{\delta}^1_{2n+1}}_*}\boldsymbol{\delta}^1_{2n+1}/\mu^{\boldsymbol{\delta}^1_{2n+1}}_{\boldsymbol{\delta}^1_{2n+1}}\right)\leq\boldsymbol{\delta}^1_{2n+3}.$$

12. Iterable Structures

This section will provide a minimal review of some aspects of iterable inner models and make some complexity calculations which will be necessary to study the ultrapower of ω_1 by the strong partition measure on ω_1. Here, mice are certain iterable structures with an internal sequence of normal measures whose critical points form an increasing discontinuous sequence of ordinals whose supremum possesses an external normal measure. (The term mouse is used differently in different context in set theory. The precise definition used here will be give below.)

Definition 12.1. Let $\mathscr{L}_M$ be the language consisting of the following objects.

- $\dot\in$ is a binary relation.
- $\dot\kappa$, $\dot{\bar\mu}$, and $\dot A$ are constant symbols.
- $\dot U$ is a unary relation symbol.

Let $\mathscr{L}_S=\{\dot\in,\dot\kappa,\dot{\bar\mu},\dot A\}$.

Let $\mathcal{T}_M$ be the $\mathscr{L}_M$-theory including the following schemes.

(1) $\mathsf{ZFC}-\mathsf{Powerset}+V=L[\dot A,\dot{\bar\mu}]$.
(2) $\dot A\subseteq\omega$.
(3) $\dot\kappa$ is the largest cardinal.

(4) Σ_0-comprehension (in $\mathscr{L}_M$)
(5) $\dot{\bar{\mu}}$ is a $\dot{\kappa}$-length sequence of normal measures whose critical points form an increasing discontinuous sequence cofinal through $\dot{\kappa}$.
(6) $\dot{U}$ is a nonprincipal normal $\dot{\kappa}$-complete ultrafilter on $\dot{\kappa}$.

$\mathcal{T}_M$ is called the theory of premice. Let $\dot{\bar{\delta}}$ be a defined function symbol with the property that if $\mathcal{M} \models \mathcal{T}_M$, then $\mathcal{M} \models (\forall \iota < \dot{\kappa})(\dot{\bar{\delta}}(\iota) = \text{crit}(\dot{\bar{\mu}}(\iota))$. In other words, $\dot{\bar{\delta}}$ is the sequence of critical points of $\dot{\bar{\mu}}$. $\mathcal{T}_M$ asserts that $\dot{\bar{\delta}}$ is an increasing discontinuous sequence cofinal through $\dot{\kappa}$

A transitive model of $\mathcal{T}_M$ is called a premouse. Premice take the form $(L_\lambda[a, \bar{\mu}], \in, \kappa, a, \bar{\mu}, U)$ where $\kappa < \lambda$, $a \subseteq \omega$, $\bar{\mu} = \langle \mu_\alpha : \alpha < \kappa \rangle$ is a sequence of normal measures with a corresponding increasing discontinuous sequence $\langle \delta_\alpha : \alpha < \kappa \rangle$ of ordinals below κ so that μ_α is a δ_α-complete normal measure on δ_α, and U is a normal measure on κ. In this case, the premouse is said to be on a and $\dot{\bar{\mu}}$.

Linear iterations involving both internal and external ultrapowers will be used. The following is the internal ultrapower by the ι^{th}-measure, $\dot{\bar{\mu}}(\iota)$, where $\iota < \dot{\kappa}$.

Definition 12.2. Suppose $\mathcal{M} = (L_\lambda[a, \bar{\mu}], \in, \kappa, a, \bar{\mu}, U)$ is a premouse. Suppose $\iota < \kappa$ and $\delta_\iota = \text{crit}(\bar{\mu}(\iota))$. Define an equivalence relation $\sim$ on the collection of functions $f, g \in M$ such that $\text{dom}(f) = \delta_\iota = \text{dom}(g)$ by $f \sim g$ if and only if $\{\alpha \in \delta_\iota : f(\alpha) = g(\alpha)\} \in \bar{\mu}(\iota)$. Let $[f]_{\bar{\mu}(\iota)}$ denote the equivalence class of f (formally, Scott's trick should be used here to avoid proper classes of $\mathcal{M}$). Let D be the collection of all $\sim$-equivalence classes. Define $[f]_{\bar{\mu}(\iota)}\ E\ [g]_{\bar{\mu}(\iota)}$ if and only if $\{\alpha < \delta_\iota : f(\alpha) \in g(\alpha)\} \in \bar{\mu}(\iota)$. For each $x \in L_\lambda[a, \bar{\mu}]$, let $c_x : \delta_\iota \to \{x\}$ be the constant function $c_x(\alpha) = x$. Let $k = [c_\kappa]$ and $b = [c_a]_{\bar{\mu}(\iota)}$. Define $\bar{\nu} = [c_{\bar{\mu}}]_{\bar{\mu}(\iota)}$. Let V be the collection of $[f]_{\bar{\mu}(\iota)}$ such that $\{\alpha \in \kappa : f(\alpha) \in U\} \in \bar{\mu}(\iota)$. Define an $\mathscr{L}_M$-structure $\mathcal{P}$ by letting the domain be D, $\dot{\in}^{\mathcal{P}} = E$, $\dot{\kappa}^{\mathcal{P}} = k$, $\dot{a}^{\mathcal{P}} = b$, $\dot{\bar{\mu}}^{\mathcal{P}} = \dot{\nu}$, and $\dot{U}^{\mathcal{P}} = V$. $\mathcal{P} \models \mathcal{T}_M$ and will be denoted $\text{ult}(\mathcal{M}; \iota)$. If it is wellfounded, then $\text{ult}(\mathcal{M}; \iota)$ will be identified with its Mostowski collapse and hence will be a premouse. The map $\pi^{\mathcal{M};\iota} : \mathcal{M} \to \text{ult}(\mathcal{M}; \iota)$ defined by $\pi^{\mathcal{M};\iota}(x) = [c_x]_{\bar{\mu}(\iota)}$ is a Σ_ω elementary map in the language $\mathscr{L}_M$.

The following is the external Σ_0-ultrapower by $\dot{U}$.

Definition 12.3. Suppose $\mathcal{M} = (L_\lambda[a, \bar{\mu}], \in, \kappa, a, \bar{\mu}, U)$ is a premouse. Define an equivalence relation $\sim$ on the collection of functions $f, g \in M$ such that $\text{dom}(f) = \kappa = \text{dom}(g)$ by $f \sim g$ if and only if $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \in U$. Let $[f]_U$ denote the equivalence class of f. Let D be the collection of all $\sim$-equivalence classes. Define $[f]_U\ E\ [g]_U$ if and only if $\{\alpha < \kappa : f(\alpha) \in g(\alpha)\} \in U$. For each $x \in L_\lambda[a, \bar{\mu}]_U$, let $c_x : \kappa \to \{x\}$ be the constant function $c_x(\alpha) = x$. Let $k = [c_\kappa]_U$ and $b = [c_a]_U$. Define $\bar{\nu} = [c_{\bar{\mu}}]_U$. Let V be the collection of $[f]_U$ such that $\{\alpha \in \kappa : f(\alpha) \in U\} \in U$. Define an $\mathscr{L}_M$-structure $\mathcal{P}$ by letting the domain by D, $\dot{\in}^{\mathcal{P}} = E$, $\dot{\kappa}^{\mathcal{P}} = k$, $\dot{a}^{\mathcal{P}} = b$, $\dot{\bar{\mu}} = \dot{\nu}$, and $\dot{U} = V$. $\mathcal{P} \models \mathcal{T}_M$ and will be denoted $\text{ult}_{\Sigma_0}(\mathcal{M})$. If it is wellfounded, then $\text{ult}_{\Sigma_0}(\mathcal{M})$ will be identified with its Mostowski collapse and hence will be a premouse. Define $\varsigma^{\mathcal{M}} : \mathcal{M} \to \text{ult}_{\Sigma_0}(\mathcal{M})$ by $\varsigma^{\mathcal{M}}(x) = [c_x]_U$. $\varsigma^{\mathcal{M}}$ is a Σ_0 (in fact, Σ_1) elementary map in the language $\mathscr{L}_M$ and is a Σ_ω elementary map in $\mathscr{L}_S$.

Only a very restrictive linear iteration of premice will be needed.

Definition 12.4. An iteration instruction is a function $\mathfrak{t} : \epsilon \to \text{ON}$ with the following properties
(1) $\epsilon > 0$ is an ordinal
(2) $\mathfrak{t}$ is non-decreasing: for all $\alpha \leq \beta < \epsilon$, $\mathfrak{t}(\alpha) \leq \mathfrak{t}(\beta)$.
(3) $\text{rang}(\mathfrak{t})$ is an ordinal. (In other words, if $\gamma \in \text{rang}(\mathfrak{t})$, then every $\delta < \gamma$ belongs to $\text{rang}(\mathfrak{t})$.)

Let $\mathcal{M}$ be a premouse. A putative iteration of $\mathcal{M}$ with instruction $\mathfrak{t}$ is a sequence $\langle \mathcal{M}^{\mathfrak{t} \upharpoonright \alpha}, \pi_{\alpha,\beta} : \alpha \leq \beta < \epsilon \rangle$ with the following properties.
(1) $\mathcal{M}^{\emptyset} = \mathcal{M}$. If $\alpha + 1 < \epsilon$, then $\mathcal{M}^{t \upharpoonright \alpha}$ is a premouse.
(2) For each $\alpha \leq \beta < \epsilon$, $\pi_{\alpha,\beta} : \mathcal{M}^{t \upharpoonright \alpha} \to \mathcal{M}^{t \upharpoonright \beta}$ is a Σ_1-elementary map in the language $\mathscr{L}_M$ and Σ_ω-elementary in $\mathscr{L}_S$. For all $\alpha \leq \beta \leq \gamma < \epsilon$, $\pi_{\alpha,\gamma} = \pi_{\beta,\gamma} \circ \pi_{\alpha,\beta}$.
(3) Suppose $\gamma < \epsilon$ is a limit ordinal. Then let $\mathcal{M}^{\mathfrak{t} \upharpoonright \gamma}$ be the direct limit of $\langle \mathcal{M}^{\mathfrak{t} \upharpoonright \alpha}, \pi_{\alpha,\beta} : \alpha \leq \beta < \gamma \rangle$. For each $\alpha < \gamma$, let $\pi_{\alpha,\gamma}$ be the direct limit map.
(4) Suppose $\beta < \epsilon$ and $\beta = \alpha + 1$.
    (a) Suppose $\mathfrak{t}(\alpha) = \mathfrak{t}(\gamma)$ for some $\gamma < \alpha$. Then let $\mathcal{M}^{\mathfrak{t} \upharpoonright \beta} = \text{ult}(\mathcal{M}^{\mathfrak{t} \upharpoonright \alpha}; \mathfrak{t}(\alpha))$. Let $\pi_{\alpha,\beta} = \pi^{\mathcal{M}^{\mathfrak{t} \upharpoonright \alpha}, \mathfrak{t}(\alpha)}$.

(b) Suppose $\mathfrak{t}(\alpha) \neq \mathfrak{t}(\gamma)$ for all $\gamma < \alpha$. Then let $\mathcal{M}^{t\restriction\beta} = \mathrm{ult}(\mathrm{ult}_{\Sigma_0}(\mathcal{M}^{\mathfrak{t}\restriction\alpha}); \mathfrak{t}(\alpha))$. Let $\pi_{\alpha,\beta} = \pi^{\mathrm{ult}_{\Sigma_0}(\mathcal{M}^{\mathfrak{t}\restriction\alpha}); \mathfrak{t}(\alpha)} \circ \varsigma^{\mathcal{M}^{\mathfrak{t}\restriction\alpha}}$.

The putative iteration is an iteration if and only if ϵ is a limit ordinal or ϵ is a successor ordinal and the last model $\mathcal{M}^{\mathfrak{t}\restriction\epsilon} = \mathcal{M}^{\mathfrak{t}}$ is wellfounded.

Suppose $\epsilon_0 \leq \epsilon_1$ and $\mathfrak{t}_0 : \epsilon_0 \to \mathrm{ON}$ and $\mathfrak{t}_1 : \epsilon_1 \to \mathrm{ON}$ are two iteration instructions with $\mathfrak{t}_0 \subseteq \mathfrak{t}_1$ (thus $\mathfrak{t}_1 \restriction \epsilon_0 = \mathfrak{t}_0$). Let $\langle \mathcal{M}^{\mathfrak{t}\restriction\alpha}, \pi_{\alpha,\beta} : \alpha \leq \beta < \epsilon_0 \rangle$ and $\langle \mathcal{N}^{\mathfrak{t}\restriction\alpha}, \tau_{\alpha,\beta} : \alpha \leq \beta < \epsilon_1 \rangle$ be the associated putative iteration of $\mathcal{M}$. Then for any α with $\alpha + 1 < \epsilon_0$, $\mathcal{M}^{\mathfrak{t}_0\restriction\alpha} = \mathcal{N}^{\mathfrak{t}_1\restriction\alpha}$. Thus if $\mathfrak{t}$ is an iteration, then let $\mathrm{Ult}^{\mathfrak{t}}(\mathcal{M}) = \mathcal{M}^{\mathfrak{t}}$ where $\langle \mathcal{M}^{t\restriction\alpha}, \pi_{\alpha,\beta} : \alpha \leq \beta < \epsilon \rangle$ is any iteration of $\mathcal{M}$ with instruction $\mathfrak{t}$.

A premouse $\mathcal{M}$ is said to be a mouse if and only if for every iteration instruction $\mathfrak{t}$, the putative iteration of $\mathcal{M}$ with instruction $\mathfrak{t}$ is an iteration. A premouse $\mathcal{M}$ is iterable by countable instructions if and only if for every countable instruction $\mathfrak{t}$ (thus $|\mathfrak{t}| < \omega_1$ and for every $\alpha < |\mathfrak{t}|$, $\mathfrak{t}(\alpha) < \omega_1$), the putative iteration with instruction $\mathfrak{t}$ is an iteration.

Remark 12.5. Condition 4a asserts that a measure index at $\mathfrak{t}(\alpha)$ has already been used at an earlier stage γ. In this case, 4a asserts that at the α^{th} stage, $\mathcal{M}^{t\restriction\alpha+1}$ is just the internal ultrapower of $\mathcal{M}^{\mathfrak{t}\restriction\alpha}$ by the measure index at $\mathfrak{t}(\alpha)$, namely $\dot{\bar{\mu}}^{\mathcal{M}^{\mathfrak{t}\restriction\alpha}}(\mathfrak{t}(\alpha))$.

Condition 4b asserts a measure index at $\mathfrak{t}(\alpha)$ has never been used at an earlier stage. Note that $\mathcal{M}^{\mathfrak{t}\restriction\alpha}$ may not have a measure with index $\mathfrak{t}(\alpha)$. This can only occur if $\mathfrak{t}(\alpha) = \dot{\kappa}^{\mathcal{M}^{\mathfrak{t}\restriction\alpha}}$. In case 4b, regardless of whether the measure with index $\mathfrak{t}(\alpha)$ exists, the external Σ_0-ultrapower of $\mathcal{M}^{\mathfrak{t}\restriction\alpha}$ by the external measure $\dot{U}^{\mathcal{M}^{\mathfrak{t}\restriction\alpha}}$ is taken first. Now $\dot{\kappa}^{\mathrm{ult}_{\Sigma_0}(\mathcal{M}^{\mathfrak{t}\restriction\alpha})} > \mathfrak{t}(\alpha)$ in either case. Hence $\mathrm{ult}_0(\mathcal{M}^{\mathfrak{t}\restriction\alpha})$ does have a measure index by $\mathfrak{t}(\alpha)$. $\mathcal{M}^{\mathfrak{t}\restriction\alpha+1}$ is then defined to be the internal ultrapower of $\mathrm{ult}_0(\mathcal{M}^{\mathfrak{t}\restriction\alpha})$ by its measure index at $\mathfrak{t}(\alpha)$, namely $\dot{\bar{\mu}}^{\mathrm{ult}_0(\mathcal{M}^{\mathfrak{t}\restriction\alpha})}(\mathfrak{t}(\alpha))$.

Note that iteration instructions described above do not account for all possible linear iterations of a premouse. For the purpose of this paper, only these iterations will be needed. The notation is simplified by restricting to only such iterations.

Fact 12.6. *(Jensen)* (ZF) *Suppose $\mathcal{M}$ is a countable premouse. If $\mathcal{M}$ is iterable by countable instructions, then $\mathcal{M}$ is a mouse.*

Proof. Suppose $\mathcal{M}$ is a premouse which is iterable by countable instructions but is not a premouse. There is an iteration instruction $\mathfrak{t}$ of length $\epsilon \geq \omega_1$ so that the putative iteration $\mathcal{I} = \langle \mathcal{M}^{\mathfrak{t}\restriction\alpha}, \pi_{\alpha,\beta} : \alpha \leq \beta < \epsilon \rangle$ by $\mathfrak{t}$ is not an iteration. Thus for all $\alpha < \epsilon$, $\mathcal{M}^{\mathfrak{t}\restriction\alpha}$ is wellfounded but $\mathcal{M}^{\mathfrak{t}}$ is illfounded. By absoluteness, $L[\mathcal{M}, \mathfrak{t}]$ sees that $\mathcal{I}$ is a putative iteration which is not an iteration. Let $\theta = (|\epsilon|^+)^{L[\mathcal{M},\mathfrak{t}]}$. $H_\theta^{L[\mathcal{M},\mathfrak{t}]}$ also sees that $\mathcal{I}$ is a putative iteration which is not an iteration. Let $Y \prec_{\Sigma_\omega} H_\theta^{L[\mathcal{M},\mathfrak{t}]}$ be a countable elementary substructure of $H_\theta^{L[\mathcal{M},\mathfrak{t}]}$ with $\mathfrak{t}, \mathcal{M} \in Y$. Let $\mathfrak{m} : Y \to X$ be the Mostowski collapse of Y where X is transitive. Since $\mathcal{M}$ is countable, $\mathfrak{m}(\mathcal{M}) = \mathcal{M}$ and $\mathfrak{p} = \mathfrak{m}(\mathfrak{t})$ is a countable iteration instruction with countable length $\zeta = \mathfrak{m}(\epsilon)$. Let $\mathcal{J} = \langle \mathcal{M}^{\mathfrak{p}\restriction\alpha}, \sigma_{\alpha,\beta} : \alpha \leq \beta < \zeta \rangle$. Since $\mathfrak{m}(\mathcal{I}) = \mathcal{J}$, elementarity implies that $X \models \mathcal{J}$ is a putative iteration which is not an iteration. By absoluteness, $\mathcal{J}$ is a putative iteration of $\mathcal{M}$ with a countable iteration instruction which is not an iteration. This violates the assumption that $\mathcal{M}$ is iterable by countable instructions. ☐

Fact 12.7. (ZF). *Suppose $\mathcal{M}$ is a countable mouse, φ is a Σ^1_2 formula, and $x \in \mathbb{R}^{\mathcal{M}}$. Then $\mathcal{M} \models \varphi(x) \Leftrightarrow V \models \varphi(x)$.*

Proof. Let $\mathfrak{t}$ be any iteration instruction with $|\mathfrak{t}| \geq \omega_1^V$. Let $\pi_{\mathfrak{t}} : \mathcal{M} \to \mathrm{Ult}^{\mathfrak{t}}(\mathcal{M})$ be the associated iterated ultrapower map. Observe that $\pi_{\mathfrak{t}}(x) = x$ and thus $x \in \mathrm{Ult}^{\mathfrak{t}}(\mathcal{M})$. Note that $\omega_1^V \subseteq \mathrm{Ult}^{\mathfrak{t}}(\mathcal{M})$. By Schoenfield absoluteness, $\mathrm{Ult}^{\mathfrak{t}}(\mathcal{M}) \models \varphi(x) \Leftrightarrow V \models \varphi(x)$. By elementarity, $\mathcal{M} \models \varphi(x) \Leftrightarrow \mathrm{Ult}^{\mathfrak{t}}(\mathcal{M}) \models \varphi(x)$. Thus $\mathcal{M} \models \varphi(x) \Leftrightarrow V \models \varphi(x)$. ☐

Definition 12.8. $\mathscr{L}_M$-structure on ω can be coded by elements of $\mathbb{R}$. If $x \in \mathbb{R}$, then let $\mathcal{K}^x$ denote the $\mathscr{L}_M$-structure coded by x. There is a Δ^1_1 set $\mathrm{Mod}_M \subseteq \mathbb{R}$ consisting of those $x \in \mathbb{R}$ so that $\mathcal{K}^x \models \mathcal{T}_M$ and $\omega^{\mathcal{K}^x} = \omega$. (This is the set of reals coding ω-models of $\mathcal{T}_M$.)

Given an $x \in \mathsf{Mod}_M$, the internal and external ultrapower maps are effective. For instance, there is a Δ^1_1-partial function so that given $x \in \mathsf{Mod}_M$ and $w \in \mathrm{WO}$ with $\mathrm{ot}(w) < \dot{\kappa}^{\mathcal{K}^x}$, the function will output a real $z \in \mathsf{Mod}_M$ so that $\mathcal{K}^z$ is isomorphic to the internal ultrapower of $\mathcal{K}^x$ by the $\mathrm{ot}(w)^{\text{th}}$ measure on

$\dot{\mu}^{\mathcal{K}^x}$. Similarly, there is also a partial Δ^1_1 function which given $x \in \mathsf{Mod}_M$ will return a real z so that $\mathcal{K}^z$ is isomorphic to the external Σ_0-ultrapower of $\mathcal{K}^x$. The direct limit construction is also effective in a similar sense.

Let Instr be the Π^1_1 set of reals coding countable iteration instructions. More explicitly, $I \in \mathsf{Instr}$ if and only if

(1) $I^{[0]} \in \mathrm{WO}$
(2) For all $n \in \mathsf{field}(I)$, $(I^{[1]})^{[n]} \in \mathrm{WO}$.
(3) The function $\mathfrak{t} : \mathrm{ot}(I^{[0]}) \to \omega_1$ defined by $\mathfrak{t}(\alpha) = \mathrm{ot}((I^{[1]})^{\mathsf{num}(I^{[0]},\alpha)})$ is an iteration instruction.

There is also a Π^1_1 relation $\mathsf{PIter} \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ so that $\mathsf{Piter}(x, I, z)$ if and only if

(1) $x \in \mathsf{Mod}_M$.
(2) $I \in \mathsf{Instr}$ and I codes an instruction $\mathfrak{t} : \epsilon \to \omega_1$ where $\epsilon < \omega_1$.
(3) z codes a sequence $\langle \mathcal{M}^{\mathfrak{t}\restriction\alpha}, \pi_{\alpha,\beta} : \alpha \leq \beta < \epsilon\rangle$ which is a putative iteration of $\mathcal{K}^x$ with instruction $\mathfrak{t}$.

To show PIter is Π^1_1 requires the effectiveness of the internal ultrapower, the external ultrapower, and the direct limit construction. The Π^1_1-complexity comes from Instr being Π^1_1 and to say a sequence is a putative iteration requires asserting that if $\alpha + 1 < \epsilon$, then $\mathcal{M}^{t\restriction\alpha}$ is wellfounded.

Fact 12.9. *Let* $\mathsf{Mouse} \subseteq \mathbb{R}$ *be the set of* $x \in \mathsf{Mod}_M$ *so that* $\mathcal{K}^x$ *is a mouse.* Mouse *is* Π^1_2.

Proof. By Fact 12.6, $x \in \mathsf{Mouse}$ if and only if $\mathcal{K}^x$ is iterable by countable instructions. By induction, the latter statement can be expressed as follows.

For all $I \in \mathbb{R}$, for all $z \in \mathbb{R}$, if $\mathsf{Piter}(x, I, z)$, then the conjunction of the following holds.

(1) Let $\epsilon = \mathrm{ot}(I^{[0]})$. Let $\mathfrak{t} : \epsilon \to \mathrm{ON}$ be the iteration instruction coded by I.
(2) Let $\langle \mathcal{M}^{\mathfrak{t}\restriction\alpha}, \pi_{\alpha,\beta} : \alpha \leq \beta < \epsilon\rangle$ be the putative iteration of $\mathcal{K}^x$ with the instruction $\mathfrak{t}$ coded by I.
(3) If ϵ is a limit ordinal, then the associated direct limit is wellfounded.
(4) If ϵ is a successor, then the last model $\mathcal{M}^{\mathfrak{t}}$ is wellfounded.

These four conditions can be expressed as a Π^1_1 statement using Δ^1_1 functions coding the internal ultrapower, the external ultrapower, and the direct limit construction. Since PIter is Π^1_1, then entire expression is Π^1_2. ☐

Definition 12.10. Suppose $\ell : \epsilon \to \mathrm{ON}$ is an increasing discontinuous sequence of nonzero indecomposable ordinals. Define $\mathfrak{t}_\ell : \sup(\ell) \to \sup(\ell)$ as follows: Let $\alpha < \sup(\ell)$. Let ν be such that $\sup(\ell \restriction \nu) \leq \alpha < \ell(\nu)$. Then let $\mathfrak{t}_\ell(\alpha) = \nu$.

Note that for all $\nu < \epsilon$, the ordertype of $\{\alpha : \sup(\ell \restriction \nu) \leq \alpha < \ell(\nu)\}$ is $\ell(\nu)$ since $\ell(\nu)$ is indecomposable. A putative iteration using the iteration instruction $\mathfrak{t}_\ell$ will use the measure indexed at ν for $\ell(\nu)$-many stages.

Fact 12.11. *Let* $\mathcal{M}$ *be a countable mouse. There is a club* $C \subseteq \omega_1$ *so that for all* $\alpha \in C$*, if* $\mathfrak{t}$ *is an iteration instruction with* $|\mathfrak{t}| < \alpha$*, then* $\dot{\kappa}^{\mathcal{M}^{\mathfrak{t}}} < \alpha$.

Proof. Let $x \in \mathsf{Mouse}$ be such that $\mathcal{K}^x = \mathcal{M}$. For any $\alpha < \omega_1$, let $\mathsf{Instr}^\alpha = \{I \in \mathsf{Instr}_M : \mathrm{ot}(I^{[0]}) < \alpha\}$. This is a $\boldsymbol{\Delta}^1_1$ set. Knowing that $x \in \mathsf{Mouse}$ and using the effectiveness of the ultrapower maps and the direct system construction, there is a $\boldsymbol{\Delta}^1_1$ function F so that for all $I \in \mathsf{Inst}^\alpha$, if I codes the iteration instruction $\mathfrak{t} : \epsilon \to \mathrm{ON}$ and $\langle \mathcal{M}^{\mathfrak{t}\restriction\alpha}, \pi_{\alpha,\beta} : \alpha \leq \beta < \epsilon\rangle$ is the associated putative iteration of $\mathcal{K}^x$ with instruction $\mathfrak{t}$, then $F(I) \in \mathrm{WO}$ and $\mathrm{ot}(F(I)) = \dot{\kappa}^{\mathcal{M}^{\mathfrak{t}}}$. Let $A_\alpha = \{z \in \mathbb{R} : (\exists I)(I \in \mathsf{Instr}^\alpha \wedge F(I) = z)\}$. Then $A_\alpha \subseteq \mathrm{WO}$ is a $\boldsymbol{\Sigma}^1_1$ subset of WO. By $\boldsymbol{\Sigma}^1_1$-boundedness, there is an ordinal $\delta < \omega_1$ so that $\mathrm{ot}[A_\alpha] \subseteq \delta$.

For each $\alpha < \omega_1$, let $\Phi : \omega_1 \to \omega_1$ be defined by $\Phi(\alpha)$ is the least $\delta < \omega_1$ so that $\mathrm{ot}[A_\alpha] \subseteq \delta$. Observe that $\Phi(\alpha) \geq \alpha$ and for all $\alpha \leq \beta < \omega_1$, $\Phi(\alpha) \leq \Phi(\beta)$. Let $C = \{\eta < \omega_1 : (\forall \alpha < \eta)(\Phi(\alpha) < \eta)\}$. C is a club subset of ω_1 and C has the desired properties. ☐

Fact 12.12. *Let* $\mathcal{M}$ *be a countable mouse. Then there is a club* $C \subseteq \omega_1$ *consisting entirely of indecomposable ordinals so that for all* $\ell \in [C]^{\leq\omega_1}_*$, $\dot{\vec{\delta}}^{\mathrm{Ult}^{\mathfrak{t}_\ell}(\mathcal{M})} = \ell$. *(The sequence of critical points of the sequence of internal measures of* $\mathrm{Ult}^{\mathfrak{t}_\ell}(\mathcal{M})$ *obtained by iterating* $\mathcal{M}$ *according to* $\mathfrak{t}_\ell$ *is* ℓ*.)*

In particular, for all $f \in [C]^{\omega_1}_*$, $f = \dot{\vec{\delta}}^{\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M})}$. *(The sequence of critical points of the sequence of internal measures of* $\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M})$ *obtained by iterating* $\mathcal{M}$ *with instruction* $\mathfrak{t}_f$ *is* f*.)*

Proof. Let C be the club from Fact 12.11. Take ℓ as above. The following is the main idea of the inductive argument: For each $\alpha < |\ell|$, after the iterations at all stages ν with $\sup(\ell \upharpoonright \alpha) \leq \nu < \ell(\alpha)$, the critical point of the α^{th} measure of $\mathcal{M}^{\mathfrak{t}_\ell \upharpoonright \ell(\alpha)}$ has been pushed up to exactly $\ell(\alpha)$. The later iteration of $\mathfrak{t}_\ell$ will not move the critical point of the α^{th} measure so the α^{th} critical point will remain $\ell(\alpha)$ in all later stages. By Fact 12.11 applied to $\mathfrak{t}_\ell \upharpoonright \ell(\alpha)$ whose length is $|\ell(\alpha)| < \ell(\alpha+1) \in C$, one knows that $\dot{\kappa}$ of $\mathcal{M}^{\mathfrak{t}_\ell \upharpoonright \ell(\alpha)}$ and hence the $(\alpha+1)^{\text{th}}$ critical point is less than $\ell(\alpha+1)$. This allows one to proceed in the same manner to push up the $(\alpha+1)^{\text{th}}$ measure to $\ell(\alpha+1)$. In the end, $\mathcal{M}^{\mathfrak{t}_\ell}$ has all its critical points aligned with ℓ. ☐

Fact 12.13. *Suppose $x \in \mathbb{R}$. If there exists a proper class inner model of* ZFC *containing x with a measurable cardinal κ which is a limit of measurable cardinals, then there exists a countable mouse containing x.*

Woodin [22] showed that assuming AD, for any $x \in \mathbb{R}$, HOD_x possesses a Woodin cardinal. This implies for any real x, there are very complicated iterable structures containing x. In particular, for any $x \in \mathbb{R}$, there is a proper class inner model of ZFC containing x with a measurable limit of measurable cardinals. By Fact 12.13, there is a countable mouse containing x.

The next result will show that the existence of a strong partition cardinal possessing the almost everywhere short length club uniformization at κ is sufficient to produce inner models of ZFC with a measurable limit of measurable cardinals and hence countable mice. In fact, it will be shown that for μ^κ_κ-almost all f, $L[f]$ has such an inner model. In fact, there is a close connection between this inner model and the original model, $L[f]$. This result introduces methods from Prikry forcing and iterability into the study of combinatorics of the strong partition measure. These ideas will be explored further in forthcoming work with Jackson and Trang.

Theorem 12.14. *(With Jackson and Trang)* (ZF) *Suppose κ is an uncountable cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$ and the almost everywhere short length club uniformization at κ. Then for all $x \in \mathbb{R}$ and for μ^κ_κ-almost all f, $L[\mu^\kappa_1, x, f]$ has an inner model containing x in which κ is measurable and a limit of a κ-length increasing discontinuous sequence of critical points of normal measures.*

Proof. Let $\mu = \mu^\kappa_1$. Fix an $x \in \mathbb{R}$ and $f \in [\kappa]^\kappa_*$. The following definitions are made in $L[\mu, x, f]$. If $\ell_0, \ell_1 \in [\kappa]^\omega_*$, then let $\ell_0 \ E_{\text{tail}} \ \ell_1$ if and only if there exist $m_0, m_1 \in \omega$ so that for all $k \in \omega$, $\ell_0(m_0 + k) = \ell_1(m_1 + k)$. For each $\ell \in [\kappa]^\omega_*$, let $\mathfrak{c}_\ell = [\ell]_{E_{\text{tail}}}$ be the E_{tail}-equivalence class of ℓ. Let $\delta_\ell = \sup(\ell)$ and note that δ_ℓ depends only on $\mathfrak{c}_\ell$. If $f \in [\kappa]^\kappa_*$, then for each $\alpha < \kappa$, let $f^\alpha \in [\kappa]^\omega_*$ be defined by $f^\alpha(n) = f^\alpha(\omega \cdot \alpha + n)$, which is the α^{th} ω-block of f. Let $\vec{\mathfrak{c}}_f = \langle \mathfrak{c}_{f^\alpha} : \alpha < \kappa \rangle$, which is the sequence of the tail class of each ω-block of f. Let ν^f_α be $\{A \in \mathscr{P}(\delta_{f^\alpha})^{L[\mu,x,f]} : L[\mu, x, f] \models (\exists \iota \in \mathfrak{c}_{f^\alpha})(\iota[\omega] \subseteq A)\} \cap \text{HOD}^{L[\mu,x,f]}_{\mu,x,\vec{\mathfrak{c}}_f}$ which is the tail filter of $\mathfrak{c}_{f^\alpha}$ within $\text{HOD}^{L[\mu,x,f]}_{\mu,x,\vec{\mathfrak{c}}_f}$. Observe that $\nu^f_\alpha \in \text{HOD}^{L[\mu,x,f]}_{\mu,x,\vec{\mathfrak{c}}_f}$ and depends only on μ, x, and $\vec{\mathfrak{c}}_f$.

Claim 1: For any $f \in [\kappa]^\kappa_*$, κ is measurable in $\text{HOD}^{L[\mu,x,f]}_{\mu,x,\vec{\mathfrak{c}}_f}$.

To see Claim 1: Let $\tau = \mu \cap \text{HOD}^{L[\mu,x,f]}_{\mu,x,\vec{\mathfrak{c}}_f}$ and note that $\tau \in \text{HOD}^{L[\mu,x,f]}_{\mu,x,\vec{\mathfrak{c}}_f}$. Since $\mu = \mu^\kappa_1$ is a normal ultrafilter on κ (Fact 2.13), by absoluteness, τ is a normal ultrafilter in $\text{HOD}^{L[\mu,x,f]}_{\mu,x,\vec{\mathfrak{c}}_f}$.

Claim 2: If $\sigma \in [\kappa]^{<\kappa}_*$ and there exists an $\epsilon \leq |\sigma|$ so that $|\sigma| = \omega \cdot \epsilon$, then there is a club $C \subseteq \kappa$ so that for all $g \in [C]^\kappa_*$, $\text{HOD}^{L[\mu,x,\sigma\hat{}g]}_{\mu,x,\vec{\mathfrak{c}}_{\sigma\hat{}g}} \models \nu^{\sigma\hat{}g}_\epsilon$ is an ultrafilter on $\delta_{g^0} = \delta_{(\sigma\hat{}g)^\epsilon}$.

To see Claim 2: Fix a $\sigma \in [\kappa]^{<\kappa}_*$ and define $P^0 : [\kappa]^\kappa_* \to 2$ by $P^0(g) = 0$ if and only if the property of g and σ asserted in Claim 2 holds. By $\kappa \to_* (\kappa)^\kappa_2$, let $C_0 \subseteq \kappa$ be a club homogeneous for P^0. Suppose for the sake of contradiction that C_0 was homogeneous for P^0 taking value 1. Thus for each $g \in [C_0]^\kappa_*$, $\text{HOD}^{L[\mu,x,\sigma\hat{}g]}_{\mu,x,\vec{\mathfrak{c}}_{\sigma\hat{}g}}$ does not think that $\nu^{\sigma\hat{}g}_\epsilon$ is an ultrafilter. Let $(\varphi^g, \zeta^g, \bar{\alpha}^g)$ be the least triple (according to some fixed wellordering of suitable triples) so that φ^g is a formula, $\zeta^g \in \text{ON}$, and $\bar{\alpha}^g$ is a finite tuple of ordinals so that if $L[\mu, x, \sigma\hat{}g]$ thinks that if A^g is the unique solution to w in $V^{L[\mu,x,\sigma\hat{}g]}_{\zeta^g} \models \varphi^g(\mu, x, \vec{\mathfrak{c}}_{\sigma\hat{}g}, \bar{\alpha}^g, w)$, then A^g is a subsets of $\delta_{g^0} = \delta_{(\sigma\hat{}g)^\epsilon}$ such that $A^g \notin \nu^{\sigma\hat{}g}_\epsilon$ and $\delta_{g^0} \setminus A^g \notin \nu^{\sigma\hat{}g}_\epsilon$. Note that if $g \in [C]^\kappa_*$ and $m \in \omega$, then $L[\mu, x, g] = L[\mu, x, \mathsf{drop}(g, m)]$, $\vec{\mathfrak{c}}_{\sigma\hat{}g} = \vec{\mathfrak{c}}_{\sigma\hat{}\mathsf{drop}(g,m)}$, $\text{HOD}^{L[\mu,x,\sigma\hat{}g]}_{\mu,x,\vec{\mathfrak{c}}_{\sigma\hat{}g}} = \text{HOD}^{L[\mu,x,\sigma\hat{}\mathsf{drop}(g,m)]}_{\mu,x,\vec{\mathfrak{c}}_{\sigma\hat{}\mathsf{drop}(g,m)}}$, and $A^g = A^{\mathsf{drop}(g,m)}$.

Define $P^1 : [C]^\kappa \to_* 2$ by $P^1(g) = 0$ if and only if $g(0) \in A^g$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for P^1. Without loss of generality, suppose C_1 is homogeneous for P^1 taking value 0 (if it was homogeneous for 1, then replace all sets with their complements). Observe that for any $g \in [C_1]^\kappa_*$, for each $n \in \omega$, $P^1(\mathsf{drop}(g,n)) = 0$ implies that $g(n) = \mathsf{drop}(g,n)(0) \in A^{\mathsf{drop}(g,n)} = A^g$. This implies that for all $g \in [C_1]^\kappa_*$, $L[\mu, x, \sigma\hat{\ }g] \models g^0 \in \mathfrak{c}_{(\sigma\hat{\ }g)^\epsilon}$ and $g^0[\omega] \subseteq A^g$. Therefore, $A^g \in \nu_\epsilon^{\sigma\hat{\ }g}$, which contradicts the choice of A^g.

It has been shown that C_0 must be homogenous for P^0 taking value 0 and thus C_0 is the desired club in Claim 2.

Claim 3: If $\sigma \in [\kappa]^{<\kappa}_*$ and there exists an $\epsilon \leq |\sigma|$ so that $|\sigma| = \omega \cdot \epsilon$, then there is a club $C \subseteq \kappa$ so that for all $g \in [C]^\kappa_*$, $\mathrm{HOD}^{L[\mu,x,\sigma\hat{\ }g]}_{\mu,x,\vec{\mathfrak{c}}_{\sigma\hat{\ }g}} \models \nu_\epsilon^{\sigma\hat{\ }g}$ is a normal ultrafilter on δ_{g^0}.

To see Claim 3: Fix $\sigma \in [\kappa]^{<\kappa}_*$. Let $Q^0 : [\kappa]^\kappa_* \to 2$ be defined by $Q^0(g)$ if and only if the property of g and σ asserted in Claim 3 holds. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_0 \subseteq \kappa$ which is homogeneous for Q^0. Suppose for the sake of contradiction that C_0 was homogeneous for Q^0 taking value 1. Therefore for each $g \in [C_0]^\kappa_*$, $Q^0(g)$ implies that there is a least triple $(\varphi^g, \zeta^g, \bar{a}^g)$ so that if F^g is the unique solution to w in $V_{\zeta^g}^{L[\mu,x,\sigma\hat{\ }g]} \models \varphi^g(\mu, x, \vec{\mathfrak{c}}_g, \bar{\alpha}^g, w)$, then F^g is a function from δ_{g^0} into δ_{g^0} so that $B^g = \{\alpha \in \delta_{g^0} : F^g(\alpha) < \alpha\} \in \nu_\epsilon^{\sigma\hat{\ }g}$ and for all $\gamma < \delta_{g^0}$, $E^g_\gamma = \{\alpha < \delta_{g^0} : F^g(\alpha) = \gamma\} \notin \nu_\epsilon^{\sigma\hat{\ }g}$. As before, for all $g \in [C_0]^\kappa_*$ and $m \in \omega$, $F^g = F^{\mathsf{drop}(g,m)}$, $B^g = B^{\mathsf{drop}(g,m)}$, and for all $\gamma < \delta_{g^0} = \delta_{(\mathsf{drop}(g,m))^0}$, $E^g_\gamma = E^{\mathsf{drop}(g,m)}_\gamma$.

Define $Q^1 : [C_0]^\kappa_* \to 2$ by $Q^1(g) = 0$ if and only if $g(0) \in B^g$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for Q^1. Pick a $g \in [C_1]^\kappa_*$. $B^g \in \nu_\epsilon^{\sigma\hat{\ }g}$ and therefore there is an $m \in \omega$ so that for all $n \geq m$, $g(n) \in B^g$. Thus $\mathsf{drop}(g,m)(0) = g(m) \in B^g = B^{\mathsf{drop}(g,m)}$. Since $\mathsf{drop}(g,m) \in [C_1]^\kappa_*$ and $Q^1(\mathsf{drop}(g,m)) = 0$, C_1 must be homogeneous for Q^1 taking value 0. Define $\Sigma : [C_1]^\kappa_* \to \kappa$ by $\Sigma(g) = F^g(g(0))$. Note that for all $g \in [C_1]^\kappa$, $\Sigma(g) = F^g(g(0)) < g(0)$ since $g(0) \in B^g$ because $Q^1(g) = 0$. By Fact 2.12, there is a club $C_2 \subseteq C_1$ and $\gamma \in \kappa$ so that for all $g \in [C_2]^\kappa_*$, $\Sigma(g) = \gamma$. Fix a $g \in [C_2]^\kappa_*$. For all $m \in \omega$, $F^g(g(m)) = F^{\mathsf{drop}(g,m)}(\mathsf{drop}(g,m)(0)) = \Sigma(\mathsf{drop}(g,m)) = \gamma$. Thus $g[\omega] \subseteq E^g_\gamma$ and therefore, $E^g_\gamma \in \nu_\epsilon^{\sigma\hat{\ }g}$ which contradicts the assumption on E^g_γ.

Thus C_0 must have been homogeneous for Q^0 taking value 0 and therefore, C_0 is the desired club satisfying Claim 3.

Claim 4: There is a club $C \subseteq \kappa$ so that for all $f \in [C]^\kappa_*$,

$$\mathrm{HOD}^{L[\mu,x,f]}_{\vec{\mathfrak{c}}_f} \models (\forall \epsilon < \kappa)(\nu_\epsilon^f \text{ is a normal ultrafilter on } \delta_{f^\epsilon}).$$

To see Claim 4: Define $R \subseteq [\kappa]^{<\kappa}_* \times \mathsf{club}_\kappa$ by $R(\sigma, D)$ if and only if there is an $\epsilon \leq |\sigma|$ and for all $g \in [D]^\kappa_*$, $\mathrm{HOD}^{L[\mu,x,\sigma\hat{\ }g]}_{\vec{\mathfrak{c}}_f} \models \nu_\epsilon^{\sigma\hat{\ }g}$ is a normal ultrafilter on δ_{g^0}. By Claim 2 and Claim 3, $\mathrm{dom}(R)$ consists of all σ so that there exists an $\epsilon \leq |\sigma|$ with $|\sigma| = \omega \cdot \epsilon$. Since κ satisfies the almost everywhere short length club uniformization at κ, Fact 2.27 implies there is a club $C \subseteq \kappa$ so for all $\sigma \in [C]^{<\kappa}_* \cap \mathrm{dom}(R)$, $R(\sigma, C \setminus \sup(\sigma))$.

Suppose $f \in [C]^\kappa_*$. Let $\epsilon < \kappa$. Let $\sigma = f \upharpoonright \omega \cdot \epsilon$. Since $\mathsf{drop}(f, \omega \cdot \epsilon) \in [C \setminus \sup(\sigma)]^\kappa_*$, $R(\sigma, C \setminus \sup(\sigma))$ implies that $\mathrm{HOD}^{L[\mu,x,f]}_{\mu,x,\vec{\mathfrak{c}}_f} \models \nu_\epsilon^f$ is a normal ultrafilter. Since ϵ was arbitrary, this shows that for all $\epsilon < \omega_1$, $\mathrm{HOD}^{L[\mu,x,f]}_{\vec{\mathfrak{c}}_f} \models (\forall \epsilon < \omega_1)(\nu_\epsilon^f$ is a normal ultrafilter). C is the desired club satisfying Claim 4. This completes the proof of the theorem. ☐

Theorem 12.15. (ZF) *Suppose κ is an uncountable cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$ and the almost everywhere short length club uniformization. Then for any $x \in \mathbb{R}$, there is a countable mouse containing x.*

Proof. This follows from Theorem 12.14 and Fact 12.13. ☐

13. Global Bound on the Ultrapower by the Strong Partition Measure

Note that PFcode and the associated notions were defined relative to a fixed uniformly good coding family for ω_1. If one used the good coding family derived from Fact 7.18, one can check the associated descriptive set theoretic computation of all the associated notions belong to the lightface version of the pointclass. Thus these notions can be interpreted in any mouse. (If there were additional reals used as parameters, then in the following discussion, one would use mice that contain the necessary real parameters.)

Theorem 13.1. $\mathsf{tPFcode}$ *is* $\boldsymbol{\Delta}^1_3$.

Proof. If $\zeta \in \mathrm{ON}$, let $\mathrm{Coll}(\omega, \zeta)$ be the forcing of functions $p : n \to \zeta$ with $n \in \omega$. $p \leq_{\mathrm{Coll}(\omega,\zeta)} q$ if and only if $q \subseteq p$. Generic filters for $\mathrm{Coll}(\omega, \zeta)$ add surjections of ω onto ζ.

Let $\varphi(\rho, z, e)$ be the $\mathscr{L}_M$ formula asserting:

(1) $\mathsf{PFcode}(\rho, z, e)$
(2) There exists an $\xi < \dot{\kappa}$ so that if $\zeta = \sup(\dot{\bar{\delta}} \restriction \xi)$ and $1_{\mathrm{Coll}(\omega,\zeta)} \Vdash_{\mathrm{Coll}(\omega,\zeta)}$ "There is a u such that $\mathsf{INC}^{\mathsf{BS}}(z, u)$ and $\mathsf{block}(\mathsf{seq}(u)) = \check{\dot{\bar{\delta}}} \restriction \check{\xi}$ and $\mathsf{PCP}(u, \check{\rho}, \check{z}, \check{e})$".

Intuitively, $\varphi(\rho, z, e)$ states that (ρ, z, e) is a pseudo-function code and there is an initial segment ℓ of $\dot{\bar{\delta}}$, the sequence of critical points of $\dot{\bar{\mu}}$, $1_{\mathrm{Coll}(\omega,\sup(\dot{\bar{\delta}}\restriction\xi))}$ forces that there is a $u \in \mathsf{BS}$ such that $\mathsf{seq}(u) \in [\mathfrak{C}_z]_*^{<\omega_1}$ with $\mathsf{block}(\mathsf{seq}(u)) = \ell$ and ℓ is a pseudo-continuity point for (ρ, z, e).

Claim 1: $\mathsf{tPFcode}(\rho, z, e)$ if and only if there is a countable mouse $\mathcal{M}$ so that $(\rho, z, e) \in \mathcal{M}$ and $\mathcal{M} \models \varphi(\rho, z, e)$.

First, suppose $\mathsf{tPFcode}(\rho, z, e)$. Thus $\mathrm{dom}(\Phi^{(\rho,z,e)}) \in \mu_{\omega_1}^{\omega_1}$. There is a club $C_0 \subseteq \omega_1$ with $[C_0]_*^{\omega_1} \subseteq \mathrm{dom}(\Phi^{(\rho,z,e)})$. This means that for all $f \in [C_0]_*^{\omega_1}$, there is a $u \in \mathsf{BS}$ so that $\mathsf{seq}(u) \in [\mathfrak{C}_z]^{<\omega_1}$, $\mathsf{PCP}(u, \rho, z, e)$, and $\mathsf{block}(\mathsf{seq}(u))$ is an initial segment of f. By Theorem 12.15, there is a mouse $\mathcal{M}$ containing (ρ, z, e). (In fact, the argument below will work for any mouse containing (ρ, z, e).) Let C_1 be the club satisfying Fact 12.12 for the mouse $\mathcal{M}$. Let $C_2 = C_1 \cap C_2$. $\mathcal{M} \models \mathsf{PFcode}(\rho, z, e)$ since PFcode is Π^1_2 and by Fact 12.7. Pick any $f \in [C_2]_*^{\omega_1}$. By Fact 12.12, $\dot{\bar{\delta}}^{\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M})} = f$. By the property of C_0, there is a $u \in \mathsf{BS}$ so that $\mathsf{seq}(u) \in [\mathfrak{C}_z]^{<\omega_1}$, $\mathsf{block}(\mathsf{seq}(u))$ is an initial segment of f, and $\mathsf{PCP}(u, \rho, z, e)$. Let $\tau = \mathsf{seq}(u)$ and $\zeta = \sup(\mathsf{seq}(u)) = \sup(\tau)$. Let $G \subseteq \mathrm{Coll}(\omega, \zeta)$ be a $\mathrm{Coll}(\omega, \zeta)$-generic filter over $\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M})$. (Note that since $\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M}) \models \mathsf{AC}$ and $\xi < \omega_1^V$, $\mathscr{P}(\zeta) \cap \mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M})$ is countable in the real universe V satisfying AD and thus such generic filters do exist in V.) In $\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M})[G]$, there is a $v \in \mathsf{BS}$ so that $\mathsf{seq}(v) = \tau$. (Note that $v \in V$ because $G \in V$.) In V, since $\mathsf{PFcode}(\rho, z, e)$, $\mathsf{PCP}(u, \rho, z, e)$, and $\mathsf{seq}(u) = \tau = \mathsf{seq}(v)$, one also has $\mathsf{PCP}(v, \rho, z, e)$ (see (iii) in Definition 10.2). Since PCP is Σ^1_2 and $\omega_1 \subseteq \mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M})[G]$, Schoenfield absoluteness implies $\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M})[G] \models \mathsf{PCP}(v, \rho, z, e)$. Then by the forcing theorem and the homogeneity of $\mathrm{Coll}(\omega, \zeta)$, $\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M}) \models 1_{\mathrm{Coll}(\omega,\zeta)} \Vdash_{\mathrm{Coll}(\omega,\zeta)}$ "there exists a v so that $\mathsf{INC}(z, v)$ and $\mathsf{block}(\mathsf{seq}(v)) = \check{\dot{\bar{\zeta}}} \restriction \check{\xi}$ and $\mathsf{PCP}(v, \check{\rho}, \check{z}, \check{e})$". Thus $\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M}) \models \varphi(\rho, z, e)$. By elementarity, $\mathcal{M} \models \varphi(\rho, z, e)$.

Now suppose $\mathcal{M}$ is a mouse containing (ρ, z, e) and $\mathcal{M} \models \varphi$. Let $C_0 \subseteq \omega_1$ be the club given by Fact 12.12 for $\mathcal{M}$. Let $C_1 = C_0 \cap \mathfrak{D}_z$ where $\mathfrak{D}_z$ is the collection of limit point for $\mathfrak{C}_z$. Let $f \in [C_1]_*^{\omega_1}$. By Fact 12.12, $\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M}) \models f = \dot{\bar{\delta}}$. By elementarity, $\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M}) \models \varphi$. There is $\xi < \omega_1 = \dot{\kappa}^{\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M})}$ so that if $\zeta = \sup(f \restriction \xi)$, then $1_{\mathrm{Coll}(\omega,\zeta)}$ forces that there is a u with $\mathsf{INC}^{\mathsf{BS}}(z, u)$ and $\mathsf{block}(\mathsf{seq}(u)) = \check{f} \restriction \xi$. As before, using the fact that $\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M}) \models \mathsf{AC}$, there is a G (in the real world satisfying AD) which is a $\mathrm{Coll}(\omega, \zeta)$-generic filter over $\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M})$. Then $\mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M})[G]$ thinks there is a $v \in \mathsf{BS}$ so that $\mathsf{INC}^{\mathsf{BS}}(z, v)$, $\mathsf{block}(\mathsf{seq}(v)) = f \restriction \xi$, and $\mathsf{PCP}(v, \rho, z, e)$. Since PCP is Σ^1_2 and $\omega_1 \subseteq \mathrm{Ult}^{\mathfrak{t}_f}(\mathcal{M})$, Schoenfield absoluteness implies that in the real world, $\mathsf{PCP}(v, \rho, z, e)$ and $\mathsf{block}(\mathsf{seq}(v))$ is an initial segment of f. Hence $f \in \mathrm{dom}(\Phi^{(\rho,z,e)})$. This shows that $[C_1]_*^{\omega_1} \subseteq \mathrm{dom}(\Phi^{(\rho,z,e)})$ and thus $\mathrm{dom}(\Phi^{(\rho,z,e)}) \in \mu_{\omega_1}^{\omega_1}$. By definition, $(\rho, z, e) \in \mathsf{tPFcode}$.

Claim 2: $\mathsf{tPFcode}$ is $\boldsymbol{\Sigma}^1_3$.

By Claim 1 and the existence of a mice containing (ρ, z, e), $(\rho, z, e) \in \mathsf{tPFcode}$ if and only if $(\exists^{\mathbb{R}} x)(x \in \mathsf{Mouse} \wedge (\rho, z, e) \in \mathcal{K}^x \wedge \mathcal{K}^x \models \varphi(\rho, z, e))$. By Fact 12.9, Mouse is $\boldsymbol{\Pi}^1_2$. Membership of a real in ω-models of $\mathcal{T}_M$ and the satisfaction relation among ω-models of $\mathcal{T}_M$ are $\boldsymbol{\Delta}^1_1$. Thus the latter expression is $\boldsymbol{\Sigma}^1_3$. (Claim 2 is already enough for Theorem 13.2.)

Claim 3: $\mathsf{tPFcode}$ is $\boldsymbol{\Pi}^1_3$.

Since mice containing (ρ, z, e) do exist and since the argument in Claim 1 holds for any mice containing (ρ, z, e), one also has that $(\rho, z, e) \in \mathsf{tPFcode}$ if and only if $(\forall^{\mathbb{R}})([x \in \mathsf{Mouse} \wedge (\rho, z, e) \in \mathcal{K}^x] \Rightarrow \mathcal{K}^x \models \varphi(\rho, z, e))$. The latter expression is $\boldsymbol{\Pi}^1_3$. ☐

Theorem 13.2. $\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_{\omega_1}} \omega_1/\mu^{\omega_1}_{\omega_1}) < \omega_{\omega+1} = \boldsymbol{\delta}^1_3$ *and* $\mathrm{cof}(\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1})) = \omega_2$.

Proof. By Fact 10.5, $\mathfrak{A}(\mathsf{tPFcode}) = \prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1}$. Theorem 10.9 and Theorem 13.1 imply that $\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1}) < \omega_{\omega+1} = \boldsymbol{\delta}^1_3$. Since the only regular cardinals below $\omega_{\omega+1}$ are ω, ω_1, and ω_2, Fact 9.9 implies that $\mathrm{cof}(\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1})) = \omega_2$. ☐

14. Concluding Remarks

Theorem 9.14 asserts that $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \boldsymbol{\delta}^1_3/\mu^{\omega_1}_\epsilon) = \boldsymbol{\delta}^1_3$ for each $\epsilon < \omega_1$. This implies that for all $\zeta < \boldsymbol{\delta}^1_3$, $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \zeta/\mu^{\omega_1}_\epsilon) < \boldsymbol{\delta}^1_3$. Theorem 9.14 is ultimately derived from $\mathrm{ot}(\prod_{[\omega_1]^\epsilon_*} \omega_1/\mu^{\omega_1}_\epsilon) < \boldsymbol{\delta}^1_3$ (for $\epsilon < \omega_1$) (from

Theorem 9.12) by Theorem 8.2 which asserts that every $\Phi : [\omega_1]^\epsilon_* \to \omega_n$ for $2 \leq n < \omega_1$ is $\mu^{\omega_1}_\epsilon$-almost everywhere bounded below ω_n.

For the strong partition measure $\mu^{\omega_1}_{\omega_1}$, this fact is not true since by Example 8.14, the ultrapower map $\Psi : [\omega_1]^{\omega_1}_* \to \omega_2$ defined by $\Psi(f) = [f]_{\mu^{\omega_1}_1}$ cannot be $\mu^{\omega_1}_{\omega_1}$-almost everywhere bounded below ω_2. Therefore, Theorem 13.2 cannot be used to show $\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_*} \omega_2/\mu^{\omega_1}_{\omega_1}) < \boldsymbol{\delta}^1_3$ in the same manner as for the short partition measures $\mu^{\omega_1}_\epsilon$ ($\epsilon < \omega_1$) in Theorem 9.14. However, the ultrapower by the function Ψ from above can be shown to be less than $\boldsymbol{\delta}^1_3$.

Fact 14.1. *Let $\Psi : [\omega_1]^{\omega_1}_* \to \omega_2$ be defined by $\Psi(f) = [f]_{\mu^{\omega_1}_1}$. $\mathrm{ot}(\prod_{f\in[\omega_1]^{\omega_1}_*} \Psi(f)/\mu^{\omega_1}_{\omega_1}) = \mathrm{ot}(\prod_{f\in[\omega_1]^{\omega_1}_*} [f]_{\mu^{\omega_1}_1}/\mu^{\omega_1}_{\omega_1}) < \boldsymbol{\delta}^1_3 = \omega_{\omega+1}$.*

Proof. If $F \in [\omega_1]^{\omega_1}_*$ and $n \in \omega$, let $F^n \in [\omega_1]^{\omega_1}_*$ be defined by $F^n(\alpha) = F(\omega\cdot\alpha + n)$. Let $F^\omega \in [\omega_1]^{\omega_1}_*$ be defined by $F^\omega(\alpha) = \sup\{F(\omega\cdot\alpha+n) : n \in \omega\}$. Note that for all $F \in [\omega_1]^{\omega_1}_*$, $\sup\{[F^n]_{\mu^{\omega_1}_1} : n \in \omega\} = [F^\omega]_{\mu^{\omega_1}_1}$. For each $\delta < \omega_2$, let $c_\delta : [\omega_1]^{\omega_1}_* \to \{\delta\}$ be the constant function.

Claim 1: If $\Phi : [\omega_1]^{\omega_1}_* \to \omega_2$ and $[\Phi]_{\mu^{\omega_1}_{\omega_1}} < [\Psi]_{\mu^{\omega_1}_{\omega_1}}$, then there is a $\delta < \omega_2$ so that $[\Phi]_{\mu^{\omega_1}_{\omega_1}} < [c_\delta]_{\mu^{\omega_1}_{\omega_1}}$.

To see Claim 1: Since $[\Phi]_{\mu^{\omega_1}_{\omega_1}} < [\Psi]_{\mu^{\omega_1}_{\omega_1}}$, let $C_0 \subseteq \omega_1$ be a club so that for all $f \in [C_0]^{\omega_1}_*$, $\Phi(f) < \Psi(f) = [f]_{\mu^{\omega_1}_{\omega_1}}$. Define $P : [C_0]^{\omega_1}_* \to 2$ by $P(F) = 0$ if and only if $\Phi(F^\omega) < [F^0]_{\mu^{\omega_1}_1}$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for P. Pick an $F \in [C_1]^{\omega_1}_*$. Since $\Phi(F^\omega) < [F^\omega]_{\mu^{\omega_1}_1}$ and $\sup\{[F^n]_{\mu^{\omega_1}_1} : n \in \omega\} = [F^\omega]_{\mu^{\omega_1}_1}$, there is an $m \in \omega$ so that $\Phi(F^\omega) < [F^m]_{\mu^{\omega_1}_1}$. Define $G \in [C_1]^{\omega_1}_*$ by $G(\omega\cdot\alpha + n) = F(\omega\cdot\alpha + m + n)$. Since $G^0 = F^m$ and $G^\omega = F^\omega$, one has that $\Phi(G^\omega) < [G^0]_{\mu^{\omega_1}_1}$ and thus $P(G) = 0$. It has been shown that C_1 is homogeneous for P taking value 0.

Let $\delta = \min\{[f]_{\mu^{\omega_1}_1} : f \in [C_1]^{\omega_1}_*\}$. Let $C_2 = \{\alpha \in C_1 : \mathsf{enum}_{C_1}(\alpha) = \alpha\}$. Fix an $f \in [C_2]^{\omega_1}_*$, $[f]_{\mu^{\omega_1}_1}$ has cofinality ω (using the fact that f has uniform cofinality ω). Pick an increasing cofinal sequence $\langle \delta_n : n \in \omega\rangle$ through $[f]_{\mu^{\omega_1}_*}$ so that $\delta_0 = \delta$ and for all $n \in \omega$, $\mathrm{cof}(\delta_n) = \omega$. Using a body of combinatorial techniques for handling ultrapowers known as sliding arguments (see [2] Section 5), there is an $F \in [C_1]^{\omega_1}_*$ so that for all $n \in \omega$, $[F^n]_{\mu^{\omega_1}_n} = \delta_n$ and $[F^\omega]_{\mu^{\omega_1}_1} = [f]_{\mu^{\omega_1}_1}$. $P(F) = 0$ implies that $\Phi(f) = \Phi(F^\omega) < [F^0]_{\mu^{\omega_1}_1} = \delta = c_\delta(f)$. Since $f \in [C_2]^{\omega_1}_*$ was arbitrary, this shows that $[\Phi]_{\mu^{\omega_1}_{\omega_1}} < [c_\delta]_{\mu^{\omega_1}_{\omega_1}}$ which completes the proof of Claim 1.

Claim 1 implies that $\mathrm{cof}([\Psi]_{\mu^{\omega_1}_{\omega_1}}) = \omega_2$ with $\langle [c_\delta]_{\mu^{\omega_1}_{\omega_1}} : \delta < \omega_2\rangle$ being an increasing cofinal sequence through $[\Psi]_{\mu^{\omega_1}_{\omega_1}}$. Note that if $\omega_1 \leq \delta < \omega_1$, $[c_\delta]_{\mu^{\omega_1}_{\omega_1}} = \prod_{[\omega_1]^{\omega_1}_*} \delta/\mu^{\omega_1}_{\omega_1}$ is in bijection with $\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1}$. The latter is less than $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ by Theorem 13.2 and therefore, $\mathrm{ot}([c_\delta]_{\mu^{\omega_1}_{\omega_1}}) = \mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_{\omega_1}} \delta/\mu^{\omega_1}_{\omega_1}) < \omega_{\omega+1} = \boldsymbol{\delta}^1_3$. Since $\boldsymbol{\delta}^1_3$ is regular and $[\Psi]_{\mu^{\omega_1}_{\omega_1}} = \sup\{[c_\delta]_{\mu^{\omega_1}_{\omega_1}} : \delta < \omega_2\}$, $[\Psi]_{\mu^{\omega_1}_{\omega_1}} < \boldsymbol{\delta}^1_3$. ☐

Question 14.2. For all $2 \leq n \leq n$, is $\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_*} \omega_n/\mu^{\omega_1}_{\omega_1}) < \omega_{\omega+1} = \boldsymbol{\delta}^1_3$? (It may be easier to produce local bounds first and show $\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_*} \omega_n/\mu^{\omega_1}_{\omega_1}) \leq \boldsymbol{\delta}^1_3$.) Is $\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_*} \boldsymbol{\delta}^1_3/\mu^{\omega_1}_{\omega_1}) = \boldsymbol{\delta}^1_3$?

Although functions $\Phi : [\omega_1]^{\omega_1}_* \to \omega_2$ do not satisfy an initial segment continuity property $\mu^{\omega_1}_{\omega_1}$-almost everywhere, perhaps all such functions Φ may resemble the ultrapower function Ψ from above in some manner $\mu^{\omega_1}_{\omega_1}$-almost everywhere. A deeper analysis of the $\mu^{\omega_1}_{\omega_1}$-everywhere behavior of $\Phi : [\omega_1]^{\omega_1}_* \to \omega_2$ will be useful for understanding the ultrapower and developing simple coding mechanisms.

Question 14.3. Describe the $\mu^{\omega_1}_{\omega_1}$-almost everywhere behavior of functions $\Phi : [\omega_1]^{\omega_1}_* \to \omega_n$ for all $2 \leq n < \omega$. Develop a coding mechanism which interacts with this $\mu^{\omega_1}_{\omega_1}$-almost everywhere behavior in order to answer Question 14.2.

The methods in showing that $\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1}) \leq \boldsymbol{\delta}^1_3$ use certain features of the good coding family for ω_1 which good coding families for the odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$ also possess using the description analysis of Jackson. The fine continuity property for functions $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ and the good coding family for ω_1 reduce the analysis of the ultrapower of the strong partition measure on ω_1 to certain hereditary countable objects and therefore, the methods of iterable inner models and forcing can be used to show $\mathrm{ot}(\prod_{[\omega_1]^{\omega_1}_{\omega_1}} \omega_1/\mu^{\omega_1}_{\omega_1}) < \boldsymbol{\delta}^1_3$. Although functions $\Phi : [\boldsymbol{\delta}^1_{2n+1}]^{\boldsymbol{\delta}^1_{2n+1}}_* \to \boldsymbol{\delta}^1_{2n+1}$ also satisfies the fine continuity property, initial segments are now no long hereditarily countable objects so the techniques from Section 13 are not applicable.

Question 14.4. Assuming $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$, is

$$\mathrm{ot}\left(\prod_{[\boldsymbol{\delta}^1_{2n+1}]^{\boldsymbol{\delta}^1_{2n+1}}_*} \boldsymbol{\delta}^1_{2n+1}/\mu^{\boldsymbol{\delta}^1_{2n+1}}_{\boldsymbol{\delta}^1_{2n+1}}\right) < \boldsymbol{\delta}^1_{2n+3} \quad \text{or} \quad \mathrm{ot}\left(\prod_{[\boldsymbol{\delta}^1_{2n+1}]^{\boldsymbol{\delta}^1_{2n+1}}_*} \boldsymbol{\delta}^1_{2n+3}/\mu^{\boldsymbol{\delta}^1_{2n+1}}_{\boldsymbol{\delta}^1_{2n+1}}\right) = \boldsymbol{\delta}^1_{2n+3}$$

for all $n \in \omega$?

There are other measures on $[\omega_1]^{\omega_1}_*$ that are closely related to the strong partition measures.

Definition 14.5. Let $\mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}$ be the tensor product of the strong partition measure on ω_1 with itself which is a measure on $[\omega_1]^{\omega_1}_* \times [\omega_1]^{\omega_1}_*$ defined by $A \in \mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}$ if and only if $\{\alpha \in \omega_1 : \{\beta \in \omega_1 : (\alpha, \beta) \in A\} \in \mu^{\omega_1}_{\omega_1}\} \in \mu^{\omega_1}_{\omega_1}$.

Question 14.6. Investigate the ultrapower of ω_1 by $\mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}$. If the ultrapower is wellfounded (which it is under $\mathsf{DC}_{\mathbb{R}}$), is its ordertype less than $\boldsymbol{\delta}^1_3$?

Fact 14.7. *$\prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1}$ is the proper initial segment of $\prod_{[\omega_1]^{\omega_1}_* \times [\omega_1]^{\omega_1}_*} \omega_1/(\mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1})$ represented by the function $\Sigma : [\omega_1]^{\omega_1}_* \times [\omega_1]^{\omega_1}_* \to \omega_1$ defined by $\Sigma(f, g) = g(0)$.*

Proof. If $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$, then define $\hat{\Phi} : [\omega_1]^{\omega_1}_* \times [\omega_1]^{\omega_1}_* \to \omega_1$ by $\hat{\Phi}(f, g) = \Phi(f)$. Define $\Lambda : \prod_{[\omega_1]^{\omega_1}_*} \omega_1/\mu^{\omega_1}_{\omega_1} \to \prod_{[\omega_1]^{\omega_1}_* \times [\omega_1]^{\omega_1}_*} \omega_1/(\mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1})$ by $\Lambda([\Phi]_{\mu^{\omega_1}_{\omega_1}}) = [\hat{\Phi}]_{\mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}}$. Λ is well defined independent of the choice of representative for $[\Phi]_{\mu^{\omega_1}_{\omega_1}}$. Λ is also order preserving.

For any $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$, $\{(f, g) : \hat{\Phi}(f, g) < \Sigma(f, g)\} = \{(f, g) : \Phi(f) < g(0)\} \in \mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}$ since for each $f \in [\omega_1]^{\omega_1}_*$, $\{g \in [\omega_1]^{\omega_1}_* : \Phi(f) < g(0)\} \in \mu^{\omega_1}_{\omega_1}$. Thus $\Lambda([\Phi]_{\mu^{\omega_1}_{\omega_1}}) < [\Sigma]_{\mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}}$. Λ maps into $[\Sigma]_{\mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}}$.

Suppose $\Psi : [\omega_1]^{\omega_1}_* \times [\omega_1]^{\omega_1}_* \to \omega_1$ and $[\Psi]_{\mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}} < [\Sigma]_{\mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}}$. Thus $E = \{(f, g) : \Psi(f, g) < \Sigma(f, g)\} \in \mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}$. For each $f \in [\omega_1]^{\omega_1}_*$, let $B_f = \{g \in [\omega_1]^{\omega_1}_* : \Psi(f, g) < \Sigma(f, g) = g(0)\}$. Let $A = \{f \in [\omega_1]^{\omega_1}_* : B_f \in \mu^{\omega_1}_{\omega_1}\}$. Since $E \in \mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}$, $A \in \mu^{\omega_1}_{\omega_1}$. For each $f \in A$, let $\Upsilon_f : B_f \to \omega_1$ by $\Upsilon_f(g) = \Psi(f, g)$. For all $g \in B_f$, $\Upsilon_f(g) < g(0)$. By Fact 2.12, there is a unique γ_f so that $B'_f = \{g \in [\omega_1]^{\omega_1}_* : \Upsilon_f(g) = \gamma_f\} \in \mu^{\omega_1}_{\omega_1}$. Define $\Phi : A \to \omega_1$ by $\Phi(f) = \gamma_f$. Note $L = \{(f, g) : f \in A \wedge g \in B'_f\} \in \mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}$. Also $L = \{(f, g) : \hat{\Phi}(f, g) = \Psi(f, g)\}$. Thus $\Lambda([\Phi]_{\mu^{\omega_1}_{\omega_1}}) = [\Psi]_{\mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}}$. This shows that Λ is an order preserving bijection into $[\Sigma]_{\mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}}$. ☐

The almost everywhere short length club uniformization at ω_1 and its consequences were important for understanding the ultrapower of ω_1 by the strong partition on ω_1. It seems that the following may be potentially useful for understanding the ultrapower of the tensor product of the strong partition measure on ω_1 and is a naturally interesting generalization.

Question 14.8. Does the almost everywhere long length club uniformization at ω_1 hold under AD? That is, if $R \subseteq [\omega_1]^{\omega_1}_* \times \mathsf{club}_{\omega_1}$ is $\subseteq$-downward closed in the club_{ω_1}-coordinate, then is there a club $C \subseteq \omega_1$ and a function $\Lambda : ([C]^{\omega_1}_* \cap \mathrm{dom}(R)) \to \mathsf{club}_{\omega_1}$ so that for all $f \in [C]^{\omega_1}_* \cap \mathrm{dom}(R)$, $R(f, \Lambda(f))$.

In order to bound the ultrapower of ω_1 by $\mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}$ using the Kunen-Martin theorem, one needs a reasonable coding of the representatives of each equivalence class almost everywhere with respect to $\mu^{\omega_1}_{\omega_1} \otimes \mu^{\omega_1}_{\omega_1}$ by reals. Progress on Question 14.8 might elucidate how this can be done. The main obstacle is again the $\boldsymbol{\Pi}^1_2$-complexity of GC for any good coding system for ${}^{\omega_1}\omega_1$.

In this paper, the main concern has been combinatorics with respect to the partition measures μ^κ_ϵ for some partition cardinals κ as well as the ultrapowers by partition measures. Thus almost everywhere club uniformization results are sufficient for all results here. However, everywhere short length club uniformization was in fact discovered first in [5] from strong determinacy hypothesis and the almost everywhere version is derived from a careful inspection of the proof.

Let $\mathsf{Uniformization}$ be the the assertion that every relation $R \subseteq \mathbb{R} \times \mathbb{R}$, there is a $\Lambda : \mathrm{dom}(R) \to \mathbb{R}$ so that for all $r \in \mathrm{dom}(R)$, $R(r, \Lambda(r))$. $\mathsf{Uniformization}$ follows from $\mathsf{AD}_{\mathbb{R}}$ but is not known to be equivalent. Let $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ be the determinacy of real games where one of the two players must always make moves from ω. Kechris [15] showed that over AD, $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ is equivalent to $\mathsf{Uniformization}$.

Fact 14.9. *([5] Theorem 3.7) (Everywhere Countable Length Uniformization for ω_1) Assume* $\mathsf{AD}+\mathsf{Uniformization}$*. For every $R \subseteq {}^{<\omega_1}\omega_1 \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed in the club_{ω_1}-coordinate, there is a function $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_{\omega_1}$ so that for all $\ell \in \mathrm{dom}(R)$, $R(\ell, \Lambda(\ell))$.*

Assuming a potentially stronger determinacy assumption, this can be generalized to larger uncountable cardinals.

Fact 14.10. *([7] Theorem 3.8) Assume* AD *and all sets of reals are Suslin. Let Γ be a boldface pointclass closed under $\wedge$, $\vee$, $\forall^{\mathbb{R}}$, and has the scale property. Let $\kappa = \delta(\Gamma)$ be the supremum of the prewellordering on $\mathbb{R}$ belonging to $\Delta = \Gamma \cap \check{\Gamma}$. For every relation $R \subseteq {}^{<\omega_1}\kappa \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed in the club_κ-coordinate, there is a function $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_\kappa$ so that for all $\ell \in \mathrm{dom}(R)$, $R(\ell, \Lambda(\ell))$.*

In particular under these assumptions, the everywhere countable length club uniformization holds for $\boldsymbol{\delta}^1_{2n+1}$ for all $n \in \omega$.

Question 14.11. Under any strong determinacy hypothesis, can the everywhere ω_1-length club uniformization for ω_1 hold? That is, if $R \subseteq {}^{\omega_1}\omega_1 \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed in the club_{ω_1}-coordinate, is there a function $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_{\omega_1}$ so that for all $f \in \mathrm{dom}(R)$, $R(f, \Lambda(f))$?

Under any strong determinacy hypthosis, is any instance of an everywhere uncountable length club uniformization? That is, is there $\omega_1 \leq \lambda \leq \kappa$ so that for all $R \subseteq {}^{\lambda}\kappa \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed in the club_κ-coordinate, there exists a $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_\kappa$ so that for all $\ell \in \mathrm{dom}(R)$, $R(\ell, \Lambda(\ell))$?

Further investigations into appropriate codings for uncountable objects may help understand various properties of the cardinalty of $\mathscr{P}(\omega_1)$ under AD.

Question 14.12. Is $\mathscr{P}(\omega_1)$ a regular cardinality under AD or an extension of AD? That is, if $X \subseteq \mathscr{P}(\omega_1)$ with $|X| < |\mathscr{P}(\omega_1)|$ and $\langle Y_x : x \in X\rangle$ is a sequence with $Y_x \subseteq \mathscr{P}(\omega_1)$ for all $x \in X$ and $\mathscr{P}(\omega_1) = \bigcup_{x \in X} Y_x$, then must there be an $x \in X$ with $|Y_x| = |\mathscr{P}(\omega_1)|$? (Note that Theorem 2.17 asserts that $\mathscr{P}(\omega_1)$ satisfies a wellordered regularity.)

Question 14.13. The structure of the cardinalities below $|[\omega_1]^{<\omega_1}|$ appears to be very complicated and not fully understood. Woodin [30] isolated several cardinalities below $[\omega_1]^{<\omega_1}$ and esablished some important structural result under $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$.

Since $|\mathscr{P}(\omega_1)|$ involves sets which are not hereditarily countable, the investigation below $|\mathscr{P}(\omega_1)|$ is much more difficult. [3] showed that under $\mathsf{AD} + \mathrm{V} = \mathrm{L}(\mathbb{R})$ or, more generally, $\mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}} + \mathrm{V} = \mathrm{L}(\mathscr{P}(\mathbb{R}))$, there are several cardinalities strictly between $|[\omega_1]^{<\omega_1}|$ and $|\mathscr{P}(\omega_1)|$. However, all such examples collapse below $|[\omega_1]^{<\omega_1}|$ under $\mathsf{AD} + \mathsf{Uniformization}$. This suggests the following question.

Can one show under $\mathsf{AD}_{\mathbb{R}}$ (or an extension) that for all $X \subseteq \mathscr{P}(\omega_1)$, $|X| \leq |[\omega_1]^{<\omega_1}|$ or $|\mathscr{P}(\omega_1)| \leq |X|$?

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