SIZE OF PIECES IN DECOMPOSITIONS INTO THE FIRST UNCOUNTABLE CARDINAL MANY PIECES

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Abstract. Within the determinacy setting, $\mathscr{P}(\omega_1)$ is regular (in the sense of cofinality) with respect to many known cardinalities and thus there is substantial evidence to support the conjecture that $\mathscr{P}(\omega_1)$ has globally regular cardinality. However, there is no known information about the regularity of $\mathscr{P}(\omega_2)$. It is not known if $\mathscr{P}(\omega_2)$ is even 2-regular under any determinacy assumptions. The paper will provide the following evidence that $\mathscr{P}(\omega_2)$ may possibly be ω_1-regular: Assume AD^+. If $\langle A_\alpha : \alpha < \omega_1\rangle$ is such that $\mathscr{P}(\omega_2) = \bigcup_{\alpha<\omega_1} A_\alpha$, then there is an $\alpha < \omega_1$ so that $\neg(|A_\alpha| \leq |[\omega_2]^{<\omega_2}|)$.

1. Introduction

A cardinality is an equivalence class under the bijection relation on the class of a sets. The cardinality of X is denoted $|X|$ and consists of all sets in bijection with X. Cardinalities are ordered by the injection comparison relation: $|X| \leq |Y|$ if and only if there is an injection of X into Y. A cardinal is an ordinal which does not inject into any smaller ordinals. Assuming the axiom of choice, every cardinality has a unique cardinal as a member. The axiom of choice will not be assumed here.

If κ is a cardinal, then the classical definition of the cofinality of κ is $\mathrm{cof}(\kappa)$ is the least cardinal δ so that there is an increasing function $\rho : \delta \to \kappa$ so that $\sup(\rho) = \kappa$. An equivalent definition is that it is the least ordinal δ so that for all $\gamma < \delta$ and function $\Phi : \kappa \to \gamma$, there is an $\alpha \in \gamma$ so that $|\Phi^{-1}[\{\alpha\}]| = \kappa$.

In choiceless settings, cardinalities no longer have unique cardinal members since sets may not wellorderable. The collection of cardinalities are also no longer wellordered by the injection comparison relation. In [8], the authors developed a robust notion of regularity and cofinality in the choiceless setting.

Let X be a set and Y be a class. X is said to have Y-regular cardinality if and only if for every function $\Phi : X \to Y$, there is a $y \in Y$ so that $|\Phi^{-1}[\{y\}]| = |X|$. A set X is said to be locally regular if and only if for all sets Y with $|Y| < |X|$, X has Y-regular cardinality. A set X is said to be globally regular if and only if for all sets Y which are surjective images of X and $\neg(|X| \leq |Y|)$, X has Y-regular cardinality.

Since cardinalities are not wellordered under the injection comparison relation, the natural definition of the cofinality of a set is formally a proper class:

- The local cofinality of a set X is the class
$$\mathsf{lcof}(X) = \{Y : (\exists Z)(|Z| = |Y| \wedge Z \subseteq X \wedge \ X \text{ does not have } Y\text{-regular cardinality})\}.$$
- Let $\mathsf{Surj}(X)$ be the class of all sets onto which X surjects. The global cofinality of a set X is the class
$$\mathsf{gcof}(X) = \{Y \in \mathrm{Surj}(X) : X \text{ does not have } Y\text{-regular cardinality}\}.$$

Observe that if X has locally regular cardinality, then $\mathsf{lcof}(X) = |X|$ and if X has globally regular cardinality, then $\mathsf{gcof}(X) = \{Y \in \mathsf{Surj}(X) : |X| \leq |Y|\}$.

The following summarizes some of the results obtained by the authors in [8] concerning regularity and cofinality. If α is an ordinal, then $\mathsf{lcof}(\alpha) = \{X : |\mathrm{cof}(\alpha)| \leq |X| \leq |\alpha|\}$ and $\mathsf{gcof}(\alpha) = \{X \in \mathsf{Surj}(\alpha) : |\mathrm{cof}(\alpha)| \leq |X|\}$. Thus $\mathsf{lcof}(\alpha) = \mathsf{gcof}(\alpha)$. If κ is a regular cardinal, then κ has globally regular cardinality and $\mathsf{lcof}(\kappa) = \mathsf{gcof}(\kappa) = |\kappa|$. Thus the choiceless theory of regularity and cofinality for wellorderable sets has a strong resemblance to the usual theory of cofinality in the choiceful framework.

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Assuming $\mathsf{AC}_\omega^\mathbb{R}$ and all sets of reals have the perfect set property, $\mathbb{R}$ has locally regular cardinality and $\mathsf{lcof}(\mathbb{R}) = |\mathbb{R}|$. Under AD^+, the Woodin's perfect dichotomy ([3], [6]) implies that $\mathbb{R}$ has globally regular cardinality and $\mathsf{gcof}(\mathbb{R}) = \{X \in \mathsf{Surj}(\mathbb{R}) : X \text{ is not wellorderable}\}$.

E_0 is the equivalence relation on ${}^\omega 2$ defined by $x \ E_0 \ y$ if and only if there exists an $m \in \omega$ so that for all $n \in \omega$, if $m \leq n < \omega$, then $x(n) = y(n)$. Under AD^+, the Hjorth's dichotomy ([12]) implies that $\mathbb{R}/E_0$ is globally regular and $\mathsf{gcof}(\mathbb{R}/E_0) = \{X \in \mathsf{Surj}(\mathbb{R}) : X \text{ is not linearly orderable}\}$.

Under $\mathsf{AC}_\omega^\mathbb{R}$ and all subsets of $\mathbb{R}$ have the property of Baire and the perfect set property, $|\mathbb{R}|$ and $|\omega_1|$ are incomparable cardinalities. This can be used to show that $\mathbb{R} \sqcup \omega_1$ does not have 2-regular cardinalities. Thus $\mathsf{gcof}(\mathbb{R} \sqcup \omega_1) = \{X \in \mathsf{Surj}(\mathbb{R}) : |X| \geq 2\}$. Under the same assumptions, $\mathbb{R} \times \omega_1$ does not have $\mathbb{R}$-regular cardinality and does not have ω_1-regular cardinality. Under AD^+, the Woodin perfect set dichotomy will show that $\mathsf{gcof}(\mathbb{R} \times \omega_1) = \{X \in \mathsf{Surj}(\mathbb{R}) : X \text{ is uncountable}\}$.

Martin showed that $\omega_1 \to_* (\omega_1)^{\omega_1}_{<\omega_1}$ and $\omega_2 \to_* (\omega_2)^{<\omega_2}_{<\omega_2}$ under AD. The partition properties on ω_1 can be used to show that for all $\epsilon \leq \omega_1$, $[\omega_1]^\epsilon$ has ω-regular cardinality. If $\epsilon < \omega_1$, then $[\omega_1]^\epsilon$ does not have ω_1-regular cardinality since $[\omega_1]^\epsilon = \bigcup_{\delta < \omega_1} [\delta]^{\omega_1}$ by the regularity of ω_1 and since $|[\delta]^\epsilon| \leq |\mathbb{R}| < |[\omega_1]^\epsilon|$. The partition relation on ω_2 can be used to show that for all $\epsilon < \omega_2$, $[\omega_2]^\epsilon$ has ω_1-regular cardinality. If $\epsilon < \omega_2$, $[\omega_2]^\epsilon = \bigcup_{\delta < \omega_2} [\delta]^\epsilon$ and hence as before, $[\omega_2]^\epsilon$ does not have ω_2-regular cardinality.

The strong partition property $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ can be used to show that for each $\lambda < \omega_1$, $[\omega_1]^{<\omega_1}$ has λ-regular cardinality. $[\omega_1]^{<\omega_1}$ does not have ω_1-regular cardinality since $[\omega_1]^{<\omega_1} = \bigcup_{\epsilon < \omega_1} [\omega_1]^\epsilon$ and $|[\omega_1]^\epsilon| < |[\omega_1]^{<\omega_1}|$ for all $\epsilon < \omega_1$.

At the present time, the regular cardinals, $\mathbb{R}$, and $\mathbb{R}/E_0$ are the only known locally or globally regular cardinalities. $\mathscr{P}(\omega_1)$ is the most natural candidate for another globally regular cardinality. The most important conjecture concerning regularity and cofinality is that $\mathscr{P}(\omega_1)$ has globally regular cardinality. [8] has amassed substantial evidence that $\mathscr{P}(\omega_1)$ should be globally regular under determinacy assumptions. $\mathscr{P}(\omega_1)$ is regular with respect to essentially every set (which does not already have an injective copy of $\mathscr{P}(\omega_1)$) for which one currently has a practical understanding: [5] showed that $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ implies that $\mathscr{P}(\omega_1)$ has ON-regular cardinality. One of the main results of [8] is that $\omega_1 \to_* (\omega_1)^{\omega_1}_{<\omega_1}$ implies that $\mathscr{P}(\omega_1)$ has ${}^{<\omega_1}$ON-regular cardinality. (It is open if the strong partition property $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ implies the very strong partition property $\omega_1 \to_* (\omega_1)^{\omega_1}_{<\omega_1}$; however, the very strong partition property on ω_1 is a consequence of AD.) Assuming AD^+, $\mathscr{P}(\omega_1)$ is regular with respect to quotient of many familiar Borel equivalence relations. If E is an equivalence relation with all classes countable, then $\mathscr{P}(\omega_1)$ has $\mathbb{R}/E$-regular cardinality. If E is E_0, E_1, E_2, a countable Borel equivalence relation, an essentially countable equivalence relation, a hyperfinite equivalence relation, a hypersmooth equivalence relation, or more generally a $\mathbf{\Sigma}^1_1$ equivalence relation which is pinned in any model of ZFC (in the sense of Zapletal [21]), then $\mathscr{P}(\omega_1)$ has $\mathbb{R}/E$-regular cardinality. The Friedman-Stanley jump of $=^+$ is not a pinned equivalence relation. Its quotient ${}^\omega\mathbb{R}/=^+$ is in bijection with $\mathscr{P}_{\omega_1}(\mathbb{R})$, the set of countable subsets of $\mathbb{R}$. One can still show that $\mathscr{P}(\omega_1)$ has $\mathscr{P}_{\omega_1}(\mathbb{R})$-regular cardinality under AD^+.

As mentioned above, $[\omega_2]^{<\omega_2}$ does not have ω_2-regular cardinality. Intuitively, one would expect $[\omega_2]^{<\omega_2}$ to at least have ω_1-regular cardinality. Above, it was remarked that the strong partition property $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ implies $[\omega_1]^{<\omega_1}$ has ω-regular cardinality. However, ω_2 is a weak but non-strong partition cardinal and thus the argument for $[\omega_1]^{<\omega_1}$ does not apply for $[\omega_2]^{<\omega_2}$. Similarly, the intuition is that $\mathscr{P}(\omega_2)$ should be highly regular and perhaps globally regular.

However since ω_2 is weak partition cardinal which is not a strong partition cardinal, $[\omega_2]^{<\omega_2}$ and $\mathscr{P}(\omega_2)$ seems just out of reach of the partition arguments and the Martin's ultrapower analysis of ω_2. (Surprisingly, $[\omega_2]^{<\omega_2}$ and more generally $[\omega_n]^{<\omega_2}$ for $2 \leq n < \omega$ can still be analyzed through the ultrapowers by measures on ω_1 as shown in [8]). Unlike $\mathscr{P}(\omega_1)$, nothing is known about the cofinality of $\mathscr{P}(\omega_2)$. For example, one does not know if $\mathscr{P}(\omega_2)$ even has 2-regular cardinality. The goal of this paper is to produce some evidence that $[\omega_2]^{<\omega_2}$ and $\mathscr{P}(\omega_2)$ could have 2-regular cardinality or more generally could have ω_1-regular cardinality. (In the forthcoming [8], the authors have shown that $[\omega_2]^{<\omega_2}$ and even $[\omega_n]^{<\omega_2}$ are ω_1-regular for all $2 \leq n < \omega$.)

If $[\omega_2]^{<\omega_2}$ does not have ω_1-regular cardinality, then one can decompose $[\omega_2]^{<\omega_2}$ into an ω_1-length sequence of disjoint sets $\langle A_\alpha : \alpha < \omega_1 \rangle$ so that $|A_\alpha| < |[\omega_2]^{<\omega_2}|$. Although the structure of the cardinalities below $[\omega_2]^{<\omega_2}$ is far from understood, perhaps the largest natural cardinality of combinatorial flavor strictly below

$[\omega_2]^{<\omega_2}$ is $[\omega_2]^{\omega_1}$. An instance of ω_1-regularity for $[\omega_2]^{<\omega_2}$ would be to show that $[\omega_2]^{<\omega_2}$ cannot be a union of ω_1-many sets $\langle A_\alpha : \alpha < \omega_1\rangle$ so that each $|A_\alpha| \leq |[\omega_2]^{\omega_1}|$.

Perhaps the largest natural cardinality strictly below $\mathscr{P}(\omega_2)$ is $|[\omega_2]^{<\omega_2}|$. An instance of ω_1-regularity for $\mathscr{P}(\omega_2)$ would be to show that $\mathscr{P}(\omega_2)$ cannot be a union of ω_1-many sets $\langle A_\alpha : \alpha < \omega_1\rangle$ so that each $|A_\alpha| \leq |[\omega_2]^{<\omega_2}|$.

The main results of this paper will verify these two instances of ω_1-regularity:

- (Theorem 3.18) Assume AD^+. If $\langle A_\alpha : \alpha < \omega_1\rangle$ is such that $[\omega_2]^{<\omega_2} = \bigcup_{\alpha<\omega_1} A_\alpha$, then there exists an $\alpha < \omega_1$ so that $\neg(|A_\alpha| \leq |[\omega_2]^{\omega_1}|)$.
- (Theorem 3.19) Assume AD^+. If $\langle A_\alpha : \alpha < \omega_1\rangle$ is such that $\mathscr{P}(\omega_2) = \bigcup_{\alpha<\omega_1} A_\alpha$, then there exists an $\alpha < \omega_1$ so that $\neg(|A_\alpha| \leq |[\omega_2]^{<\omega_2}|)$.

Recently, the authors in [8] have fully verified under AD the conjecture that $[\omega_2]^{<\omega_2}$ is ω_1-regular: For any $\langle A_\alpha : \alpha < \omega_1\rangle$ such that $[\omega_2]^{<\omega_2} = \bigcup_{\alpha<\omega_1} A_\alpha$, there is an $\alpha < \omega_1$ so that $|A_\alpha| = |[\omega_2]^{<\omega_2}|$. (More generally, for all $2 \leq n < \omega$, $[\omega_n]^{<\omega_2}$ is ω_1-regular.) The verification of ω_1-regularity for $[\omega_2]^{<\omega_2}$ (or more generally, $[\omega_n]^{<\omega_2}$ when $2 \leq n < \omega$) uses a very technical analysis of the ultrapower of ω_1 by the club ultrafilter on ω_1 where the type or length of a function into ω_2 represented by a function $f : \omega_1 \to \omega_1$ is not fixed by varies with f. It is still not known if $\mathscr{P}(\omega_2)$ is 2-regular.

For each $1 \leq n < \omega$, the projective ordinal $\boldsymbol{\delta}^1_n$ is the supremum of the length of $\boldsymbol{\Delta}^1_n$ prewellorderings on $\mathbb{R}$. It can be shown that for all $n \in \omega$, $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$. $\boldsymbol{\delta}^1_1 = \omega_1$ and $\boldsymbol{\delta}^1_2 = \omega_2$. Also $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ and $\boldsymbol{\delta}^1_4 = \omega_{\omega+2}$. The last section will show that the results for ω_1 and ω_2 can be generalized to each odd projective ordinal $\boldsymbol{\delta}^1_{2n+1}$ and the next even projective ordinal $\boldsymbol{\delta}^1_{2n+2}$.

- (Theorem 4.39) Assume AD^+. Let $n \in \omega$. If $\langle A_\alpha : \alpha < \boldsymbol{\delta}^1_{2n+1}\rangle$ is such that $\mathscr{P}(\boldsymbol{\delta}^1_{2n+2}) = \bigcup_{\alpha<\boldsymbol{\delta}^1_{2n+1}} A_\alpha$, then there is an $\alpha < \boldsymbol{\delta}^1_{2n+1}$ so that $\neg(|A_\alpha| \leq |[\boldsymbol{\delta}^1_{2n+2}]^{<\boldsymbol{\delta}^1_{2n+2}}|)$.

2. Cardinality of Sets of Functions on Ordinals

ZF will be assumed throughout and all additional assumptions will be made explicit.

Definition 2.1. If X and Y are sets, then let ${}^XY$ be the set of all functions from X to Y.

If δ is a ordinal and X is a set, then let ${}^{<\delta}X = \bigcup_{\epsilon<\delta} {}^{\epsilon}X$.

If δ and λ are ordinals and $X \subseteq \lambda$, then let $[X]^\delta$ be the collection of all increasing functions $f : \delta \to X$. Let $[X]^{<\delta} = \bigcup_{\epsilon<\delta}[X]^\epsilon$.

If δ is a cardinal and X is a set, then let $\mathscr{P}_\delta(X) = \{A \in \mathscr{P}(X) : |A| < \delta\}$.

If $\delta \leq \lambda$ are ordinals, then let $IB(\delta, \lambda) = \{f \in {}^\delta\lambda : (\forall \alpha < \delta)(\sup(f \restriction \alpha) < \lambda)\}$.

This section collects some basic results concerning the cardinality of sets of the form $[\lambda]^\delta$, ${}^\delta\lambda$, and $[\lambda]^{<\delta}$.

Fact 2.2. *Let $\delta \leq \lambda$ be ordinals such that δ is a cardinal. Then $|[\lambda]^{<\delta}| = |\mathscr{P}_\delta(\lambda)| = |{}^{<\delta}\lambda|$.*

Proof. Let $\Phi : [\lambda]^{<\delta} \to \mathscr{P}_\delta(\lambda)$ be defined by $\Phi(f) = \text{rang}(f)$. Φ is a bijection.

Let $\pi : \lambda \times \lambda \to \lambda$ be a bijection. For $f \in {}^{<\delta}\lambda$, let $G_f = \{\pi(\alpha, \beta) : \alpha \in \text{dom}(f) \wedge f(\alpha) = \beta\}$. Note that since $\text{dom}(f) \in \delta$ and δ is a cardinal, $|G_f| < \delta$. Thus $G_f \in \mathscr{P}_\delta(\lambda)$. Define $\Psi : {}^{<\delta}\lambda \to \mathscr{P}_\delta(\lambda)$ by $\Psi(f) = G_f$. Ψ is an injection. The previous paragraph showed there is an bijection of $\mathscr{P}_\delta(\lambda)$ into $[\lambda]^{<\delta}$ and $[\lambda]^{<\delta} \subseteq {}^{<\delta}\lambda$. Thus there is an injection $\Psi : \mathscr{P}_\delta(\lambda) \to {}^{<\delta}\lambda$. By the Cantor-Schröder-Bernstein theorem, $|{}^{<\delta}\lambda| = |\mathscr{P}_\delta(\lambda)| = |[\lambda]^{<\delta}|$. □

Say an ordinal λ is indecomposable if and only if for all $\alpha, \beta < \lambda$, $\alpha + \beta < \lambda$ and $\alpha \cdot \beta < \lambda$.

Fact 2.3. *If $\delta \leq \lambda$ are ordinals and λ is indecomposable, then $|IB(\delta, \lambda)| = |[\lambda]^\delta|$.*

Proof. For $f \in IB(\delta, \lambda)$, define $\Phi(f) \in [\lambda]^\delta$ by recursion as follows. Suppose for all $\beta < \delta$, $\Phi(f) \restriction \beta$ has been defined and for all $\alpha < \beta$, $\Phi(f)(\alpha) \leq \sup(f \restriction \alpha+1)\cdot(\alpha+1) < \lambda$. Then $\sup(\Phi(f) \restriction \beta) \leq \sup(f \restriction \beta)\cdot\beta < \lambda$ since $\sup(f \restriction \beta) < \lambda$ and λ is indecomposable. Let $\Phi(f)(\beta) = \sup(\Phi(f) \restriction \beta) + f(\beta)$ which is less than λ since λ is indecomposable. Then $\Phi(f)(\beta) = \sup(\Phi(f) \restriction \beta) + f(\beta) \leq \sup(f \restriction \beta) \cdot \beta + f(\beta) \leq \sup(f \restriction \beta+1) \cdot (\beta+1) < \lambda$ since λ is indecomposable.

This defines $\Phi : IB(\delta, \lambda) \to [\lambda]^\delta$. Note that for all $\alpha < \delta$, $f(\alpha)$ is the unique ordinal γ so that $\Phi(f)(\alpha) = \sup(\Phi(f) \restriction \alpha) + \gamma$. Thus Φ is an injection. Thus $|IB(\delta, \lambda)| \leq |[\lambda]^\delta|$. Since $[\lambda]^\delta \subseteq IB(\delta, \lambda)$, $|[\lambda]^\delta| \leq |IB(\delta, \delta)|$. By the Cantor-Schröder-Bernstein, $|[\lambda]^\delta| = |IB(\delta, \lambda)|$. □

Fact 2.4. *Let $\delta \leq \lambda$ be ordinals such that λ is indecomposable and $\delta \leq \mathrm{cof}(\lambda)$. Then $|{}^\delta\lambda| = |[\lambda]^\delta|$.*

Proof. Suppose $\delta \leq \mathrm{cof}(\lambda)$. For all $f \in {}^\delta\lambda$ and $\alpha < \delta$, $\sup(f \upharpoonright \alpha) < \lambda$. Thus ${}^\delta\lambda \subseteq B(\delta, \lambda)$. Thus $|{}^\delta\lambda| = |B(\delta,\lambda)| = |[\lambda]^\delta|$ by Fact 2.3. $\square$

Fact 2.5. *Let $\delta \leq \lambda$ be ordinals such that λ is indecomposable,* $\mathrm{cof}(\delta) = \mathrm{cof}(\lambda)$, *and $\delta < \mathrm{cof}(\lambda)^+$. Then $|{}^\delta\lambda| = |[\lambda]^\delta|$.*

Proof. Note that $|{}^\delta\lambda| = |{}^{\mathrm{cof}(\lambda)}\lambda|$ since $|\delta| = |\mathrm{cof}(\delta)|$. By Fact 2.4, $|{}^{\mathrm{cof}(\lambda)}\lambda| = |[\lambda]^{\mathrm{cof}(\lambda)}|$. Thus $|{}^\delta\lambda| = |[\lambda]^{\mathrm{cof}(\lambda)}|$. Thus it suffices to produce an injection of $[\lambda]^{\mathrm{cof}(\lambda)}$ into $[\lambda]^\delta$. Let $\rho : \mathrm{cof}(\lambda) \to \delta$. Since λ is indecomposable, $\delta \cdot \lambda = \lambda$. For each $\alpha < \lambda$, let $\iota(\alpha)$ be the least $\beta < \mathrm{cof}(\lambda)$ so that $\alpha \leq \rho(\beta)$. For $f \in [\lambda]^{\mathrm{cof}(\lambda)}$, let $\Phi(f) : \delta \to \lambda$ be defined by $\Phi(f)(\alpha) = \delta \cdot f(\iota(\alpha)) + \alpha$. One can check that for all $f \in [\lambda]^{\mathrm{cof}(\lambda)}$, $\Phi(f) \in [\lambda]^\delta$ and $\Phi : [\lambda]^{\mathrm{cof}(\lambda)} \to [\lambda]^\delta$ is an injection. $\square$

Fact 2.6. *If κ is a measurable cardinal (has a κ-complete nonprincipal ultrafilter on κ), then for all $\delta < \kappa$, there is no injection of κ into $\mathscr{P}(\delta)$.*

Proof. Suppose $\Phi : \kappa \to \mathscr{P}(\delta)$ is a function. Let μ be a κ-complete nonprincipal ultrafilter on κ. For each $\alpha < \delta$ and $i \in \{0, 1\}$, let $A^i_\alpha = \{\beta < \kappa : \Phi(\beta)(\alpha) = i\}$ (where elements of $\mathscr{P}(\delta)$ are identified with elements of ${}^\delta 2$). For each $\alpha < \delta$, let i_α be the unique $i \in \{0, 1\}$ so that $A^i_\alpha \in \mu$. Since μ is κ-complete, $\bigcap_{\alpha<\delta} A^{i_\alpha}_\alpha \in \mu$. Let $f \in {}^\delta 2$ be defined by $f(\alpha) = i_\alpha$. Since μ is nonprincipal, let $\alpha_1 < \alpha_2 < \delta$ so that $\alpha_1, \alpha_2 \in \bigcap_{\alpha<\delta} A^{i_\alpha}_\alpha$. $\Phi(\alpha_1) = f = \Phi(\alpha_2)$. Thus Φ is not an injection. $\square$

Under AD, ω_1 is a strong partition cardinal and ω_2 is a weak partition cardinal. Thus ω_1 and ω_2 are measurable cardinals. More generally, $\boldsymbol{\delta}^1_{2n+1}$ is a strong partition cardinal and $\boldsymbol{\delta}^1_{2n+2}$ is a weak partition cardinal. (It is known that $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ and $\boldsymbol{\delta}^1_4 = \omega_{\omega+2}$.) (See [6], [18], or [19] for more information concerning partition properties under AD and the associated measures.)

If κ is a cardinal, then one says boldface GCH holds at κ if and only if there is no injection of κ^+ into $\mathscr{P}(\kappa)$. Boldface GCH holds below κ if and only if boldface GCH holds at all $\delta < \kappa$. Fact 2.6 implies the following result.

Fact 2.7. *Assume* AD. *Boldface* GCH *holds at ω and ω_1.*

Recently, the authors ([7]) have shown boldface GCH below ω_ω using purely combinatorial arguments assuming AD (in fact, just assuming $\omega_1 \to (\omega_1)^{\omega_1}_2$ and the ultrapower of ω_1 by the club measure on ω_1 is ω_2). Steel ([26] and [27]) showed that if $L(\mathbb{R}) \models \mathsf{AD}$, then $L(\mathbb{R}) \models$ "boldface GCH holds below Θ" using an inner model theoretic analysis of HOD. Thus by the Moschovakis coding lemma, it is a theorem of AD that boldface GCH holds below $\Theta^{L(\mathbb{R})}$. More generally, Woodin showed that AD^+ implies the boldface GCH holds below Θ.

Fact 2.8. *Suppose λ is cardinal and λ does not inject into $\mathscr{P}(\kappa)$ for any $\kappa < \lambda$. Then $\neg(|[\lambda]^{\mathrm{cof}(\lambda)}| \leq |\bigcup_{\delta\leq\kappa<\lambda}[\kappa]^\delta|)$.*

Proof. Suppose there is an injection $\Phi : [\lambda]^{\mathrm{cof}(\lambda)} \to \bigcup_{\delta\leq\kappa<\lambda}[\kappa]^\delta$. Let $\tilde{\Phi} \subseteq [\lambda]^{\mathrm{cof}(\lambda)} \times \lambda \times \lambda$ be defined by $(f, \alpha, \beta) \in \tilde{\Phi}$ if and only if $\alpha \in \mathrm{dom}(\Phi(f))$ and $\Phi(f)(\alpha) = \beta$. $L[\tilde{\Phi}] \models \mathsf{ZFC}$. In $L[\tilde{\Phi}]$, define $\Psi : [\lambda]^{\mathrm{cof}(\lambda)} \to \bigcup_{\delta\leq\kappa<\lambda}[\kappa]^\delta$ by $\Psi(f)(\alpha) = \beta$ if and only if $\tilde{\Phi}(f, \alpha, \beta)$. Note $\Psi \in L[\tilde{\Phi}]$ and $L[\tilde{\Phi}] \models \Psi : [\lambda]^{\mathrm{cof}(\lambda)} \to \bigcup_{\delta\leq\kappa<\lambda}[\kappa]^\delta$ is an injection. If there are $\delta \leq \kappa < \lambda$ so that $L[\tilde{\Phi}] \models \lambda \leq |[\kappa]^\delta|$, then there is an injection of λ into $[\kappa]^\delta \subseteq \mathscr{P}(\kappa)$ in the real world. This contradicts the assumption that λ does not inject into $\mathscr{P}(\kappa)$ for any $\kappa < \lambda$. Thus $L[\tilde{\Phi}] \models |\bigcup_{\delta\leq\kappa<\lambda}[\kappa]^\delta| = \lambda$. By a theorem of ZFC, $L[\tilde{\Phi}] \models |[\lambda]^{\mathrm{cof}(\lambda)}| \geq \lambda^+$. It is impossible that $L[\tilde{\Phi}] \models \Psi : [\lambda]^{\mathrm{cof}(\lambda)} \to \bigcup_{\delta\leq\kappa<\lambda}[\kappa]^\delta$ is an injection. $\square$

Fact 2.9. *Suppose κ is a regular cardinal and there is no injection of κ into $\mathscr{P}(\delta)$ for any $\delta < \kappa$. Then $|[\kappa]^{<\kappa}| < |\mathscr{P}(\kappa)|$.*

Proof. It is clear that $|[\kappa]^{<\kappa}| \leq |\mathscr{P}(\kappa)|$. Since κ is regular, $[\kappa]^{<\kappa} = \bigcup_{\delta\leq\mu<\kappa}[\mu]^\delta$. By Fact 2.8, $\neg(|\mathscr{P}(\kappa)| = |[\kappa]^\kappa| \leq |\bigcup_{\delta\leq\mu<\kappa}[\mu]^\delta| | = [\kappa]^{<\kappa})$. $\square$

Since Martin showed that $\omega_2 \to (\omega_2)^2_2$ (and in fact, $\omega_2 \to (\omega_2)^\epsilon_2$ for all $\epsilon < \omega_2$), ω_2 is a regular cardinal.

Fact 2.10. *Assume* AD. $|[\omega_2]^{<\omega_2}| < |\mathscr{P}(\omega_2)|$.

Proof. This follows from Fact 2.7 and Fact 2.9. $\square$

Fact 2.11. *Let $\delta \leq \lambda$ be ordinals such that $\mathrm{cof}(\lambda) < \mathrm{cof}(\delta)$ and λ does not inject into $\mathscr{P}(\kappa)$ for all $\kappa < \lambda$. Then $|[\lambda]^\delta| < |{}^\delta\lambda|$.*

Proof. It is clear that $[\lambda]^\delta \subseteq {}^\delta\lambda$. Since $\mathrm{cof}(\delta) \neq \mathrm{cof}(\lambda)$, $[\lambda]^\delta = \bigcup_{\kappa<\lambda}[\kappa]^\delta \subseteq \bigcup_{\mu\leq\kappa<\lambda}[\kappa]^\mu$. Define $\Psi : [\lambda]^{\mathrm{cof}(\lambda)} \to {}^\delta\lambda$ by

$$\Psi(f)(\alpha) = \begin{cases} f(\alpha) & \alpha < \mathrm{cof}(\lambda) \\ 0 & \mathrm{cof}(\lambda) < \alpha \end{cases}.$$

Ψ is an injection. Thus if there was an injection of ${}^\delta\lambda$ into $|[\lambda]^\delta|$, then there would be an injection of $[\lambda]^{\mathrm{cof}(\lambda)}$ into $\bigcup_{\mu\leq\kappa<\lambda}[\kappa]^\mu$ which contradicts Fact 2.8. $\square$

Example 2.12. Assume AD. Recall Steel showed the boldface GCH holds below $\Theta^{L(\mathbb{R})}$ (and one can directly use the analysis of the ultrapower by the finite partition measures on ω_1 to show the boldface GCH below $\omega_{\omega+1}$).

(1) $|[\omega_\omega]^{\omega_1}| < |{}^{\omega_1}\omega_\omega|$. This follows from Fact 2.11. The cardinality of the collection of the increasing sequences can be smaller than the cardinality of the collection of all sequences.
(2) $|IB(\omega_1,\omega_\omega)| = |[\omega_\omega+\omega]^{\omega_1}| < |IB(\omega_1,\omega_\omega+\omega)| = |{}^{\omega_1}(\omega_\omega+\omega)|$. To see this: Note that $[\omega_\omega+\omega]^{\omega_1} = [\omega_\omega]^{\omega_1} \subseteq \bigcup_{\delta\leq\kappa<\omega_\omega}[\kappa]^\delta$. Thus by Fact 2.8, $[\omega_\omega]^{\omega_1}$ does not inject into $\bigcup_{\delta\leq\kappa<\omega_\omega}[\kappa]^\delta$ and thus does not inject into $[\omega_\omega+\omega]^{\omega_1}$. However $[\omega_\omega]^{\omega_1} \subseteq IB(\omega_1,\omega_\omega+\omega)$. This shows that $|[\omega_\omega+\omega]^{\omega_1}| < |IB(\omega_1,\omega_\omega+\omega)|$. Notice that $\omega_\omega+\omega$ is not indecomposable. This shows that the indecomposability assumption of Fact 2.3 is necessary. Also since $[\omega_\omega+\omega]^{\omega_1} = [\omega_\omega]^{\omega_1}$, $|[\omega_\omega+\omega]^{\omega_1}| = |[\omega_\omega]^{\omega_1}| = |IB(\omega_1,\omega_\omega)|$ by Fact 2.3. Note that ${}^{\omega_1}(\omega_\omega) \subseteq IB(\omega_1,\omega_\omega+\omega) \subseteq {}^{\omega_1}(\omega_\omega+\omega)$ and $|{}^{\omega_1}(\omega_\omega+\omega)| = |{}^{\omega_1}\omega_\omega|$. Thus $|{}^{\omega_1}(\omega_\omega+\omega)| = |IB(\omega_1,\omega_\omega+\omega)|$. This shows that $|[\omega_\omega+\omega]^{\omega_1}| < |{}^{\omega_1}(\omega_\omega+\omega)|$.

Fact 2.13.

- *([9]) (AD) $[\omega_1]^{<\omega_1}$ does not inject into ${}^\omega(\omega_\omega)$.*
- *([9]) ($\mathsf{AD}+\mathsf{DC}_\mathbb{R}$). $[\omega_1]^{<\omega_1}$ does not inject into ${}^\omega\mathrm{ON}$, the class of ω-sequences of ordinals.*
- *([10]) More generally, if $\kappa \to (\kappa)^{<\kappa}_2$ (κ is a weak partition cardinal), then $[\kappa]^{<\kappa}$ does not inject into ${}^\lambda\mathrm{ON}$, for all $\lambda < \kappa$.*

Fact 2.14. *Assume* AD. $|[\omega_2]^{\omega_1}| < |[\omega_2]^{<\omega_2}|$.

Proof. Under AD, Martin showed that ω_2 is a weak partition cardinal (that is, satisfies $\omega_2 \to_* (\omega_2)^{<\omega_2}_2$). The result follows from the third point in Fact 2.13. $\square$

Example 2.15. Assume AD. Note that $\neg(|[\omega_\omega]^\omega| \leq |[\omega_\omega]^{\omega_1}|)$.This is because if there was an injection of $[\omega_\omega]^\omega$ into $[\omega_\omega]^{\omega_1}$, then there would be an injection of $[\omega_\omega]^\omega$ into $[\omega_\omega]^{\omega_1} = \bigcup_{\omega_1\leq\kappa<\omega_\omega}[\kappa]^{\omega_1} \subseteq \bigcup_{\delta\leq\kappa<\omega_\omega}[\kappa]^\delta$ (where the first equality follows from the fact that $[\omega_\omega]^{\omega_1}$ consists of increasing ω_1-sequences and $\mathrm{cof}(\omega_\omega) = \omega$) which violates Fact 2.8. Note that $\neg(|[\omega_\omega]^{\omega_1}| \leq |[\omega_\omega]^\omega|)$. This is because $[\omega_1]^{<\omega_1}$ injects into $[\omega_\omega]^{\omega_1}$ and $[\omega_1]^{<\omega_1}$ does not inject into ${}^\omega\mathrm{ON}$ by Fact 2.13. Since $[\omega_\omega]^{\omega_1}$ injects into $[\omega_\omega]^{\omega_1+\omega}$, this shows that $|[\omega_\omega]^\omega| < |[\omega_\omega]^{\omega_1+\omega}|$.

See [4] for more information concerning distinguishing sets of the form $[\kappa]^\delta$ and ${}^\delta\kappa$ for varying $\delta \leq \kappa < \Theta$ under AD^+.

3. Decomposition into ω_1 Many Pieces

Definition 3.1. Fix a bijection $\pi : \omega\times\omega \to \omega$. If $x \in {}^\omega\omega$ and $k \in \omega$, then let $x^{[k]} \in {}^\omega\omega$ be defined by $x^{[k]}(n) = x(\pi(k,n))$.

If $x \in {}^\omega 2$, then define $\mathcal{R}_x \subseteq \omega\times\omega$ by $\mathcal{R}_x(m,n)$ if and only if $x(\pi(m,n)) = 1$. Let $\mathrm{field}(x) = \mathrm{field}(\mathcal{R}_x) = \{m : (\exists n)(\mathcal{R}_x(m,n) \vee \mathcal{R}_x(n,m))\}$.

Let $\mathrm{WO} = \{w \in {}^\omega 2 : \mathcal{R}_w \text{ is a wellordering}\}$. Let $\mathrm{ot} : \mathrm{WO} \to \omega_1$ be defined by $\mathrm{ot}(w)$ is the order type of $(\mathrm{field}(w), \mathcal{R}_w)$. If $\alpha < \omega_1$, then let $\mathrm{WO}_\alpha = \{w \in \mathrm{WO} : \mathrm{ot}(w) = \alpha\}$.

Definition 3.2. Let $\alpha < \omega_1$. For $s \in {}^{<\omega}\alpha$, let $N_s^\alpha = \{f \in {}^\omega\alpha : s \subseteq f\}$. Give ${}^\omega\alpha$ the topology generated by $\{N_s^\alpha : s \in {}^{<\omega}\alpha\}$ as a basis (which is the product of the discrete topology on α). Then ${}^\omega\alpha$ is homeomorphic to ${}^\omega\omega$ with its usual topology.

Under AD, all subsets of ${}^\omega\omega$ have the Baire property and thus well ordered unions of meager subsets of ${}^\omega\omega$ are meager in ${}^\omega\omega$. (For the latter fact: Given a wellordered sequence of meager sets whose union is nonmeager, consider the horizontal and vertical section of the prewellordering induced by the sequence to obtain a contradiction.) Therefore under AD, for all $\alpha < \omega_1$, all subsets of ${}^\omega\alpha$ have the Baire property and wellordered unions of meager subsets of ${}^\omega\alpha$ are meager in ${}^\omega\alpha$.

For $\alpha < \omega_1$, let $\mathsf{surj}_\alpha = \{f \in {}^\omega\omega_1 : f[\omega] = \alpha\}$. For all $\alpha < \omega_1$, surj_α is comeager in ${}^\omega\alpha$.

If $\alpha < \omega_1$, $p \in {}^{<\omega}\alpha$, and φ is a formula, then let $(\forall_p^{*,\alpha} f)\varphi(f)$ be the assertion that for comeagerly many $f \in N_p^\alpha$, $\varphi(f)$ holds.

Definition 3.3. For each $f \in {}^\omega\omega_1$, let $A_f = \{n \in \omega : (\forall m < n)(f(m) \neq f(n))\}$. (Note for all $f \in {}^\omega\omega_1$, $f \restriction A_f : A_f \to f[\omega]$ is a bijection.)

For $f \in {}^\omega\omega_1$, let $\mathfrak{G}(f) \in {}^\omega 2$ be defined by $\mathfrak{G}(f)(\pi(m,n)) = 1$ if and only if $m \in A_f$, $n \in A_f$, and $f(m) < f(n)$. $\mathfrak{G}$ is a simple form of the Kechris-Woodin generic coding function for ω_1 which is developed more generally in [17].

Fact 3.4. *$\mathfrak{G} : {}^\omega\omega_1 \to \mathrm{WO}$ and for all $\alpha < \omega_1$, if $f \in \mathsf{surj}_\alpha$, then $\mathfrak{G}(f) \in \mathrm{WO}_\alpha$.*

Proof. Note that $(\mathrm{field}(\mathfrak{G}(f)), \mathcal{R}_{\mathfrak{G}(f)}) = (A_f, \mathcal{R}_{\mathfrak{G}(f)})$ is order isomorphic to $(f[A_f], <)$ where $<$ is the usual ordering on ω_1. Thus $\mathfrak{G}(f)$ does indeed belong to WO. Also if $f \in \mathsf{surj}_\alpha$, then $f[A_f] = \alpha$ and thus $\mathfrak{G}(f) \in \mathrm{WO}_\alpha$. □

Definition 3.5. Let $\langle \rho_r : r \in \mathbb{R}\rangle$ be some standard coding of strategies $\rho : {}^{<\omega}\omega \to \omega$ on ω by reals. Let $\Xi_r : \mathbb{R} \to \mathbb{R}$ be the Lipschitz continuous function corresponding to the strategy ρ_r. (That is, for each $f \in {}^\omega\omega$, $\Xi_r(f) \in {}^\omega\omega$ is defined by recursion by $\Xi_r(f)(n) = \rho_r(\langle f(0), \Xi_r(f)(0), ..., f(n-1), \Xi_r(f)(n-1), f(n)\rangle)$.) Note that $\langle \Xi_r : r \in \mathbb{R}\rangle$ is a coding of all Lipschitz continuous function by reals.

If $A, B \in \mathbb{R}$, then write $A \leq_L B$ if and only if there is an $r \in \mathbb{R}$ so that $A = \Xi_r^{-1}[B]$. The Wadge lemma under AD asserts that for all $A, B \in \mathscr{P}(\mathbb{R})$, $A \leq_L B$ or $(\mathbb{R} \setminus B) \leq_L A$.

Martin-Monk showed that under AD and $\mathsf{DC}_\mathbb{R}$, $\leq_L$ is a wellfounded relation. For each $A \in \mathscr{P}(\mathbb{R})$, let $\mathrm{rk}_L(A) \in \mathrm{ON}$ be the rank of A in $\leq_L$. Let Θ be the supremum of the ordinals which are surjective images of $\mathbb{R}$. It can be shown that Θ is the length of $\leq_L$ and thus for all $A \in \mathscr{P}(\mathbb{R})$, $\mathrm{rk}_L(A) < \Theta$.

Fact 3.6. *(Moschovakis coding lemma) Assume AD. Let Γ be a pointclass closed under $\exists^\mathbb{R}$, $\wedge$, and continuous preimages. Let $(P, \preceq)$ be a prewellordering in Γ. Let κ be the length of $(P, \preceq)$ and $\varphi : P \to \kappa$ be the associated surjective norm. If $R \subseteq P \times \mathbb{R}$, then there is an $S \in \Gamma$ with the following property.*

- *$S \subseteq R$*
- *For all $\alpha < \kappa$, there exists a $p \in P$ and $x \in \mathbb{R}$ so that $\varphi(p) = \alpha$ and $R(p, x)$ if and only if there exists a $p \in P$ and $x \in \mathbb{R}$ so that $\varphi(p) = \alpha$ and $S(p, x)$.*

The following is a useful coarse consequence of the Moschovakis coding lemma.

Fact 3.7. *If κ is a surjective image of $\mathbb{R}$ (i.e. $\kappa < \Theta$), then $\mathbb{R}$ surjects onto $\mathscr{P}(\kappa)$.*

Fix the following notation which will be used in the discussion that follows: Let X be a surjective image of $\mathbb{R}$. Fix a surjection $\pi : \mathbb{R} \to X$. Let $\delta \leq \lambda < \Theta$. By Fact 3.7, there is a surjection $\varpi : \mathbb{R} \to \mathscr{P}(\lambda)$. If $B \subseteq \mathbb{R}$, let $T_B = \{(x, f) : (\exists z \in B)(x = \pi(z^{[0]}) \wedge f = \varpi(z^{[1]})\}$. Let $\langle A_\alpha : \alpha < \nu\rangle$ be such that for all $\alpha < \nu$, $A_\alpha \subseteq X$. (In this section, ν will either be ω or ω_1.) In the below applications, $|A_\alpha| \leq |{}^{<\delta}\lambda|$ or $|A_\alpha| \leq |{}^\delta\lambda|$ for all $\alpha < \nu$. Elements of ${}^{<\delta}\lambda$ or ${}^\delta\lambda$ can be identified as elements of $\mathscr{P}(\lambda \times \lambda)$ or of $\mathscr{P}(\lambda)$ (after coding pairs). As an example, if $A \subseteq X$ and $\Phi : A \to {}^{<\delta}\lambda$, then the graph of Φ is T_B where $B = \{z \in \mathbb{R} : \Phi(\pi(z^{[0]})) = \varpi(z^{[1]})\}$.

Theorem 3.8. *Assume AD. Suppose X is a surjective image of $\mathbb{R}$. Let $\delta \leq \lambda$ be cardinals so that $1 \leq \delta < \Theta$ and $\omega \leq \lambda < \Theta$. Let $\langle A_n : n \in \omega\rangle$ be a sequence so that for all $n \in \omega$, $A_n \subseteq X$. Assume one of the following three settings.*

(1) $|A_\alpha| \leq |{}^{<\delta}\lambda|$ for all $n \in \omega$.
(2) $|A_\alpha| \leq |{}^\delta\lambda|$ for all $n \in \omega$.

(3) $|A_\alpha| \leq |[\lambda]^\delta|$ *for all* $n \in \omega$.

Assume that there is a $Z \in \mathscr{P}(\mathbb{R})$ *so that for all* $n \in \omega$, *there exists an* $r \in \mathbb{R}$ *so that* $T_{\Xi_r^{-1}[Z]}$ *is a graph of an injection of* A_n *into* ${}^{<\delta}\lambda$ *in (1) (into* ${}^\delta\lambda$ *in (2) or* $[\lambda]^\delta$ *in (3)). Then, respectively, the following hold.*

(1) $|\bigcup_{n\in\omega} A_n| \leq |{}^{<\delta}\lambda|$.
(2) $\bigcup_{n\in\omega} A_n| \leq |{}^\delta\lambda|$.
(3) $|\bigcup_{n\in\omega} A_n| \leq |[\lambda]^\delta|$.

Proof. Assume the setting of (1) that for all $n \in \omega$, $|A_n| \leq |{}^{<\delta}\lambda|$. Let $R \subseteq \omega \times \mathbb{R}$ be defined by $R(n, r)$ if and only if $T_{\Xi_r^{-1}[Z]}$ is the graph of an injection of A_n into ${}^{<\delta}\lambda$. (Recall that $\Xi_r^{-1}[Z]$ is the subset of $\mathbb{R}$ Lipchitz reducible to Z via the Lipschitz continuous function Ξ_r and $T_{\Xi_r^{-1}[Z]}$ was defined before the statement of Theorem 3.8.) By $\mathsf{AC}_\omega^\mathbb{R}$, there is a sequence $\langle r_n : n \in \omega\rangle$ so that for all $n \in \omega$, $R(n, r_n)$. Thus for all $n \in \omega$, $T_{\Xi_{r_n}^{-1}[Z]}$ is the graph of an injection A_n into ${}^{<\delta}\lambda$. Let $\Phi_n : A_n \to {}^{<\delta}\lambda$ be the injection whose graph is $T_{\Xi_{r_n}^{-1}[Z]}$. For each $x \in \bigcup_{n\in\omega} A_n$, let $\iota(x)$ be the least n so that $x \in A_n$. Since $\omega \leq \lambda$, let $\varsigma : \omega \times \lambda \to \lambda$ be a bijection. Define $\Phi : \bigcup_{n\in\omega} A_n \to {}^{<\delta}\lambda$ by letting $\Phi(x) \in [\lambda]^{|\Phi_{\iota(x)}(x)|}$ be defined by $\Phi(x)(\gamma) = \varsigma(\iota(x), \Phi_{\iota(x)}(x)(\gamma))$. Suppose $x \neq y$. If $\iota(x) \neq \iota(y)$, then $\Phi(x) \neq \Phi(y)$ since ς is a bijection. If $\iota(x) = \iota(y)$ with common value $n \in \omega$, then $\Phi_n(x) \neq \Phi_n(y)$ since Φ_n is an injection. Then again $\Phi(x) \neq \Phi(y)$ since ς is an injection. This establishes that Φ is an injection.

In the setting of (2) in which for all $n \in \omega$, $|A_n| \leq |{}^\delta\lambda|$, the proof is essentially the same.

In the setting of (3) in which for all $n \in \omega$, $|A_n| \leq |[\lambda]^\delta|$, observe that the bijection $\varsigma : \omega \times \lambda \to \lambda$ may be chosen with the property that for all $n \in \omega$ and $\alpha < \beta < \lambda$, $\varsigma(n, \alpha) < \varsigma(n, \beta)$. (For instance, ς derived from the Gödel pairing function would have such property.) Then the resulting function $\Phi(x)$ defined as above would belong to $[\lambda]^\delta$. ☐

Theorem 3.9. *Assume* AD. *Suppose* X *is a surjective image of* $\mathbb{R}$. *Let* $\langle A_\alpha : \alpha < \omega_1\rangle$ *be a sequence so that for all* $\alpha < \omega_1$, $A_\alpha \subseteq X$. *Let* δ *and* λ *be cardinals such that* $\omega_1 \leq \delta \leq \lambda < \Theta$. *Assume one of the following three settings.*

(1) $\mathrm{cof}(\delta) \geq \omega_1$ *and for all* $\alpha < \omega_1$, $|A_\alpha| \leq |{}^{<\delta}\lambda|$.
(2) For all $\alpha < \omega_1$, $|A_\alpha| \leq |{}^\delta\lambda|$.
(3) $\mathrm{cof}(\lambda) \geq \omega_1$, *and for all* $\alpha < \omega_1$, $|A_\alpha| \leq |[\lambda]^\delta|$.

Assume that there is a $Z \in \mathscr{P}(\mathbb{R})$ *so that for all* $\alpha < \omega_1$, *there exists an* $r \in \mathbb{R}$ *so that* $T_{\Xi_r^{-1}[Z]}$ *is the graph of an injection of* A_α *into* $[\lambda]^{<\delta}$ *in (1) (into* ${}^\delta\lambda$ *in (2) or into* $[\lambda]^\delta$ *in (3)). Then, respectively, the following hold.*

(1) $|\bigcup_{\alpha<\omega_1} A_\alpha| \leq |{}^{<\delta}\lambda|$.
(2) $|\bigcup_{\alpha<\omega_1} A_\alpha| \leq |{}^\delta\lambda|$.
(3) $|\bigcup_{\alpha<\omega_1} A_\alpha| \leq |[\lambda]^\delta|$.

Proof. Assume the setting of (1) that for all $\alpha < \omega_1$, $|A_\alpha| \leq |{}^{<\delta}\lambda|$ where $\mathrm{cof}(\delta) \geq \omega_1$. Since $|{}^{<\delta}\lambda \setminus \{\emptyset\}| = |{}^{<\delta}\lambda|$, injections from A_α into ${}^{<\delta}\lambda \setminus \{\emptyset\}$ will be considered to simplify notation.

Let $\mathrm{WO} \subseteq \mathbb{R}$ be the Π_1^1 set of reals coding wellorderings and $\mathrm{ot} : \mathrm{WO} \to \omega_1$ be the associated surjective norm given by the order type function. Define $R \subseteq \mathrm{WO} \times \mathbb{R}$ by $R(w, r)$ if and only if $T_{\Xi_r^{-1}[Z]}$ is the graph of an injection of $A_{\mathrm{ot}(w)}$ into ${}^{<\delta}\lambda \setminus \{\emptyset\}$. $(\mathrm{WO}, \mathrm{ot})$ is a prewellordering which belongs to the pointclass $\mathbf{\Sigma}_2^1$ which is closed under continuous preimage, $\wedge$, and $\exists^\mathbb{R}$. By the Moschovakis coding lemma (Fact 3.6), there is a $\mathbf{\Sigma}_2^1$ set $S \subseteq R$ so that for all $\alpha < \omega_1$, there is a $w \in \mathrm{WO}_\alpha$ and $r \in \mathbb{R}$ so that $S(w, r)$. Let $\leq_{\Pi_1^1} \in \Pi_1^1$ and $\leq_{\Sigma_1^1} \in \Sigma_1^1$ be the two norm relations which witness that $(\mathrm{WO}, \mathrm{ot})$ is a Π_1^1-norm. Let $\tilde{S}(w, r)$ if and only if $w \in \mathrm{WO} \wedge (\exists v)(v \leq_{\Sigma_1^1} w \wedge w \leq_{\Sigma_1^1} v \wedge S(v, r))$. $\tilde{S} \in \mathbf{\Sigma}_2^1$ and $\mathrm{dom}(\tilde{S}) = \mathrm{WO}$. Since $\mathbf{\Sigma}_2^1$ has the scale property, let $\Lambda : \mathrm{WO} \to \mathbb{R}$ be a uniformization with the property that for all $w \in \mathrm{WO}$, $\tilde{S}(w, \Lambda(w))$. Thus for all $w \in \mathrm{WO}$, $R(w, \Lambda(w))$. For all $w \in \mathrm{WO}$, $T_{\Xi_{\Lambda(w)}^{-1}[Z]}$ is the graph of an injection of $A_{\mathrm{ot}(w)}$ into ${}^{<\delta}\lambda \setminus \{\emptyset\}$. For each $w \in \mathrm{WO}$, let $\Phi_w : A_{\mathrm{ot}(w)} \to {}^{<\delta}\lambda \setminus \{\emptyset\}$ be the injection whose graph is $T_{\Xi_{\Lambda(w)}^{-1}[Z]}$.

For each $x \in \bigcup_{\alpha<\omega_1} A_\alpha$, let $\iota(x)$ be the least $\alpha < \omega_1$ so that $x \in A_\alpha$. Note that $|{}^{<\omega}\omega_1| = |\omega_1|$. Let $\sigma : \omega_1 \times {}^{<\omega}\omega_1 \times \delta \times \lambda \to \lambda$ be a bijection. Define

$$\Upsilon(x) = \{\sigma(\iota(x), p, \eta, \zeta) : (\exists \epsilon < \delta)(\forall_p^{*,\iota(x)} f)(\epsilon = \mathrm{dom}(\Phi_{\mathfrak{G}(f)}(x)) \wedge \eta < \epsilon \wedge \Phi_{\mathfrak{G}(f)}(x)(\eta) = \zeta)\}.$$

Observe that $\Upsilon(x) \in \mathscr{P}(\lambda)$.

Fix $x \in \bigcup_{\alpha<\omega_1} A_\alpha$. Let $K_x = \{p \in {}^{<\omega}\iota(x) : (\exists \eta, \zeta)(\sigma(\iota(x), p, \eta, \zeta) \in \Upsilon(x)\}$. If $p \in K_x$, then there is a unique $\epsilon < \delta$ so that $(\forall_p^{*,\iota(x)} f)(\mathrm{dom}(\Phi_{\mathfrak{G}(f)}(x)) = \epsilon)$. To see this, suppose $\epsilon, \hat{\epsilon} < \delta$ are such that $(\forall_p^{*,\iota(x)} f)(\mathrm{dom}(\Phi_{\mathfrak{G}(f)}(x)) = \epsilon)$ and $(\forall_p^{*,\iota(x)} f)(\mathrm{dom}(\Phi_{\mathfrak{G}(f)}(x)) = \hat{\epsilon})$. Let $A_0 = \{f \in N_p^{\iota(x)} : \mathrm{dom}(\Phi_{\mathfrak{G}(f)}(x)) = \epsilon\}$ and $A_1 = \{f \in N_p^{\iota(x)} : \mathrm{dom}(\Phi_{\mathfrak{G}(f)}(x)) = \hat{\epsilon}\}$. A_0 and A_1 are comeager subsets of $N_p^{\iota(x)}$. Thus $A_0 \cap A_1 \neq \emptyset$. Let $h \in A_0 \cap A_1$. Then $\epsilon = \mathrm{dom}(\Phi_{\mathfrak{G}(h)}(x)) = \hat{\epsilon}$. Let ϵ_p^x be this unique ϵ associated to x and p. Let $U_{x,p} = \{\eta < \epsilon_p^x : (\exists \zeta)(\sigma(\iota(x), p, \eta, \zeta) \in \Upsilon(x)\}$. Note that $|U_{x,p}| \leq |\epsilon_p^x|$. If $\eta \in U_{x,p}$, there is a unique ζ such that $\sigma(\iota(x), p, \eta, \zeta) \in \Upsilon(x)$. To see this, suppose ζ_1, ζ_2 so that $\sigma(\iota(x), p, \eta, \zeta_1), \sigma(\iota(x), p, \eta, \zeta_2) \in \Upsilon(x)$. Then $B_0 = \{f \in N_p^{\iota(x)} : \Phi_{\mathfrak{G}(f)}(x)(\eta) = \zeta_1\}$ and $B_1 = \{f \in N_p^{\iota(x)} : \Phi_{\mathfrak{G}(f)}(x)(\eta) = \zeta_2\}$ are comeager in $N_p^{\iota(x)}$. $B_0 \cap B_1$ is comeager in $N_p^{\iota(x)}$. Let $h \in B_0 \cap B_1$. Then $\zeta_1 = \Phi_{\mathfrak{G}(h)}(x)(\eta) = \zeta_2$. Let $\zeta_{p,\eta}^x$ be this unique ζ. Thus $\Upsilon(x) = \{\sigma(\iota(x), p, \eta, \zeta_{p,\eta}^x) : p \in K_x \wedge \eta \in U_{x,p}\}$. Thus $|\Upsilon(x)| \leq |\bigcup_{p \in K_x} U_{x,p}| \leq \sup\{|\epsilon_p^x| : p \in K_x\} < \delta$ since $|K_x| \leq |{}^{<\omega}\iota(x)| = \omega$ because $\iota(x) < \omega_1$ and $\mathrm{cof}(\delta) > \omega$. Thus $\Upsilon(x)$ has cardinality less than δ and hence $\Upsilon(x) \in \mathscr{P}_\delta(\lambda)$. It has been shown that $\Upsilon : \bigcup_{\alpha<\omega_1} A_\alpha \to \mathscr{P}_\delta(\lambda)$.

Next, one will show that for all $x \in \bigcup_{\alpha<\omega_1} A_\alpha$, $\Upsilon(x) \neq \emptyset$. Let $\alpha = \iota(x)$. Let $E_1 : \mathsf{surj}_\alpha \to \delta$ be defined by $E_1(f) = \mathrm{dom}(\Phi_{\mathfrak{G}(f)}(x))$. Since wellordered unions of meager subsets of ${}^\omega\alpha$ is a meager subset of ${}^\omega\alpha$ and surj_α is a comeager subset of ${}^\omega\alpha$, there is some $\epsilon < \delta$ so that $E_1^{-1}[\{\epsilon\}]$ is nonmeager. Let $E_2 : E_1^{-1}[\{\epsilon\}] \to \lambda$ be defined by $E_2(f) = \Phi_{\mathfrak{G}(f)}(x)(0)$. Again since $E_1^{-1}[\{\epsilon\}]$ is nonmeager and wellordered unions of meager sets are meager, there is some $\zeta < \lambda$ so that $E_2^{-1}[\{\zeta\}]$ is nonmeager. By the Baire property, there is a $p \in {}^{<\omega}\alpha$ so that $E_2^{-1}[\{\zeta\}]$ is comeager in N_p^α. Then $\sigma(\alpha, p, 0, \zeta) \in \Upsilon(x)$. $\Upsilon(x) \neq \emptyset$.

Next, it will be shown that Υ is an injection. Suppose $x \neq y$. First, suppose $\iota(x) \neq \iota(y)$. Above, it was shown that $\Upsilon(x) \neq \emptyset$. Let $\sigma(\iota(x), p, \eta, \zeta) \in \Upsilon(x)$. Since σ is an injection and all elements of $\Upsilon(y)$ take the form $\sigma(\iota(y), \hat{p}, \hat{\eta}, \hat{\zeta})$, $\Upsilon(x) \neq \Upsilon(y)$. Next, suppose that $\iota(x) = \iota(y)$ and denote this common ordinal by α. Let $D = \{f \in \mathsf{surj}_\alpha : \mathrm{dom}(\Phi_{\mathfrak{G}(f)}(x)) \neq \mathrm{dom}(\Phi_{\mathfrak{G}(f)}(y))\}$. First suppose D is nonmeager. Consider $\varpi : D \to \delta \times \delta$ by $\varpi(f) = (\mathrm{dom}(\Phi_{\mathfrak{G}(f)}(x)), \mathrm{dom}(\Phi_{\mathfrak{G}(f)}(y)))$. Since a wellordered union of meager sets is meager and D is not meager, there is some $\epsilon_1, \epsilon_2 < \delta$ so that $\varpi^{-1}[\{(\epsilon_1, \epsilon_2)\}]$ is nonmeager. Without loss of generality, suppose $\epsilon_1 < \epsilon_2$. Define $\varsigma : \varpi^{-1}[\{(\epsilon_1, \epsilon_2)\}] \to \lambda$ by $\varsigma(f) = \Phi_{\mathfrak{G}(f)}(y)(\epsilon_1)$. Since $\varpi^{-1}[\{(\epsilon_1, \epsilon_2)\}]$ is nonmeager and wellordered union of meager sets is meager, there is a $\zeta \in \lambda$ so that $\varsigma^{-1}[\{\zeta\}]$ is nonmeager. By the Baire property, let $p \in {}^{<\omega}\alpha$ be such that $\varsigma^{-1}[\{\zeta\}]$ is comeager in N_p^α. Then $\sigma(\alpha, p, \epsilon_1, \zeta) \in \Upsilon(y)$. However, $\sigma(\alpha, p, \epsilon_1, \zeta) \notin \Upsilon(x)$ since $(\forall_p^{*,\alpha} f)(\mathrm{dom}(\Phi_{\mathfrak{G}(f)}(x)) = \epsilon_1)$. In this case, $\Upsilon(x) \neq \Upsilon(y)$. Finally, suppose ${}^\omega\alpha \setminus D$ is comeager. Let $\Sigma : {}^\omega\alpha \setminus D \to \delta$ be defined by $\Sigma(f) = \mathrm{dom}(\Phi_{\mathfrak{G}(f)}(x)) = \mathrm{dom}(\Phi_{\mathfrak{G}(f)}(y))$. Since ${}^\omega\alpha \setminus D$ is comeager, there is some $\epsilon < \delta$ so that $\Sigma^{-1}[\{\epsilon\}]$ is nonmeager. Note that since $\Phi_{\mathfrak{G}(f)}$ is an injection for all $f \in \mathsf{surj}(\alpha)$ (because it is the injection whose graph is $T_{\Xi^{-1}_{\Lambda(\mathfrak{G}(f))}}[Z]$), $\Phi_{\mathfrak{G}(f)}(x) \neq \Phi_{\mathfrak{G}(f)}(y)$. Define $\Pi : \Sigma^{-1}[\{\epsilon\}] \to \epsilon$ be defined by $\Pi(f)$ is the least $\eta < \epsilon$ so that $\Phi_{\mathfrak{G}(f)}(x)(\eta) \neq \Phi_{\mathfrak{G}(f)}(y)(\eta)$. Since $\Sigma^{-1}[\{\epsilon\}]$ is nonmeager, there is an $\eta < \epsilon$ so that $\Pi^{-1}[\{\eta\}]$ is nonmeager. Let $\Gamma : \Pi^{-1}[\{\eta\}] \to \lambda \times \lambda$ be defined by $\Gamma(f) = (\Phi_{\mathfrak{G}(f)}(x)(\eta), \Phi_{\mathfrak{G}(f)}(y)(\eta))$. Since $\Pi^{-1}[\{\eta\}]$ is nonmeager, there are $\zeta_1, \zeta_2 \in \lambda$ with $\zeta_1 \neq \zeta_2$ so that $\Gamma^{-1}[\{(\zeta_1, \zeta_2)\}]$ is nonmeager. Since all subsets of ${}^\omega\alpha$ have the Baire property, there is a $p \in {}^{<\omega}\alpha$ so that $\Gamma^{-1}[\{(\zeta_1, \zeta_2)\}]$is comeager in N_p^α. Then $\sigma(\alpha, p, \eta, \zeta_1) \in \Upsilon(x)$ and $\sigma(\alpha, p, \eta, \zeta_1) \notin \Upsilon(y)$. Thus $\Upsilon(x) \neq \Upsilon(y)$. It has been shown that $\Upsilon : \bigcup_{\alpha<\omega_1} A_\alpha \to \mathscr{P}_\delta(\lambda)$ is an injection. Fact 2.2 shows $|{}^{<\delta}\lambda| = |\mathscr{P}_\delta(\lambda)|$.

Next assume the setting of (2). The following will sketch the necessary modifications. By the same argument as above, for each $w \in \mathrm{WO}$, there is an injection $\Phi_w : A_{\mathrm{ot}(w)} \to {}^\delta\lambda$. Let

$$K_x = \{(p, \eta) : p \in {}^{<\omega}\iota(x) \wedge \eta < \delta \wedge (\exists \zeta < \lambda)(\forall_p^{*,\iota(x)} f)(\Phi_{\mathfrak{G}(f)}(x)(\eta) = \zeta)\}$$

For each $(p, \eta) \in K_x$, by the argument provided above, there is a unique ζ so that $(\forall_p^{*,\iota(x)} f)(\Phi_{\mathfrak{G}(f)}(x)(\eta) = \zeta)$. Thus for each $(p, \eta) \in K_x$, let $\zeta_{p,\eta}^x$ be this unique ζ. Note that $K_x \subseteq {}^{<\omega}\iota(x) \times \delta \subseteq {}^{<\omega}\omega_1 \times \delta$. Let

$\tau : {}^{<\omega}\omega_1 \times \delta \to \delta$ be a bijection. Let $\mu : \omega_1 \times \lambda \to \lambda$ be a bijection. Define $\Upsilon : X \to {}^{\delta}\lambda$ by

$$\Upsilon(x)(\alpha) = \begin{cases} \mu(\iota(x), 0) & \tau^{-1}(\alpha) \notin K_x \\ \mu(\iota(x), \zeta^x_{p,\eta}) & \tau^{-1}(\alpha) \in K_x \wedge \tau^{-1}(\alpha) = (p, \eta) \end{cases}.$$

Finally, one will to show Υ is an injection. Suppose $x, y \in \bigcup_{\alpha<\omega_1} A_\alpha$ and $x \neq y$. If $\iota(x) \neq \iota(y)$, then $\Upsilon(x) \neq \Upsilon(y)$ since μ is a bijection. Now suppose $\iota(x) = \iota(y)$ and let α denote this common ordinal. For all $f \in \mathsf{surj}_\alpha$, $\Phi_{\mathfrak{G}(f)}(x) \neq \Phi_{\mathfrak{G}(f)}(y)$. Let $\Sigma : \mathsf{surj}_\alpha \to \delta$ be defined by $\Sigma(f)$ is the least $\eta < \delta$ so that $\Phi_{\mathfrak{G}(f)}(x)(\eta) \neq \Phi_{\mathfrak{G}(f)}(y)(\eta)$. Since surj_α is comeager in ${}^{\omega}\alpha$ and wellordered unions of meager sets are meager, there is an $\eta < \delta$ so that $\Sigma^{-1}[\{\eta\}]$ is nonmeager. Let $\Pi : \Sigma^{-1}[\{\eta\}] \to \lambda \times \lambda$ be defined by $\Pi(f) = (\Phi_{\mathfrak{G}(f)}(x)(\eta), \Phi_{\mathfrak{G}(f)}(y)(\eta))$. Since $\Sigma^{-1}[\{\eta\}]$ is nonmeager, there is some $\zeta_1, \zeta_2 < \lambda$ so that $\zeta_1 \neq \zeta_2$ and $\Pi^{-1}[\{(\zeta_1, \zeta_2)\}]$ is nonmeager. By the Baire property, let $p \in {}^{<\omega}\alpha$ so that $\Pi^{-1}[\{(\zeta_1, \zeta_2)\}]$ is comeager in N^α_p. Let $\beta = \tau(p, \eta)$. Then $\Upsilon(x)(\beta) = \mu(\alpha, \zeta_1) \neq \mu(\alpha, \zeta_2) = \Upsilon(y)(\beta)$. Thus $\Upsilon(x) \neq \Upsilon(y)$. It has been shown that Υ is an injection.

Assume the setting of (3). Let K_x, $\zeta^x_{p,\eta}$, and $\tau : {}^{<\omega}\omega_1 \times \delta \to \delta$ be defined as in (2). The bijection $\mu : \omega_1 \times \lambda \to \lambda$ can be chosen with the property that for all $\nu < \omega_1$ and $\gamma < \lambda$, $\sup\{\mu(\nu, \beta) : \beta < \gamma\} < \lambda$. Let Υ be defined as above in (2). For $x \in X$, $\gamma < \delta$, and $p \in {}^{<\omega}\iota(x)$, let $P^x_{\gamma,p} = \{\eta \in \delta : \tau(p, \eta) < \gamma \wedge \tau(p, \eta) \in K_x\}$. For each $p \in {}^{<\omega}\iota(x)$, let $F^x_{p,\gamma} = \{\zeta^x_{p,\eta} : \eta \in P^x_{\gamma,p}\}$. The claim is that $F^x_{p,\gamma}$ is bounded below λ. To see this, suppose $F^x_{p,\gamma}$ is not bounded below λ. For each $\eta \in P^x_{\gamma,p}$, let $Y^x_{p,\gamma,\eta} = \{f \in N^{\iota(x)}_p : \Phi_{\mathfrak{G}(f)}(x)(\eta) = \zeta^x_{p,\eta}\}$. Each $Y^x_{p,\gamma,\eta}$ is comeager in $N^{\iota(x)}_p$. Since wellordered intersection of comeager subsets of $N^{\iota(x)}_p$ is comeager in $N^{\iota(x)}_p$, $\bigcap_{\eta \in P^x_{\gamma,p}} Y^x_{p,\gamma,\eta}$ is comeager in $N^{\iota(x)}_p$ and is in particular nonempty. Let $f \in \bigcap_{\eta \in P^x_{\gamma,p}} Y^x_{p,\gamma,\eta}$. Then $\sup(\Phi_{\mathfrak{G}(f)}(x) \restriction \gamma) \geq \sup\{\zeta^x_{p,\eta} : \eta \in P^x_{\gamma,p}\} = \sup(F^x_{p,\gamma}) = \lambda$. Then since $\gamma < \delta$, $\Phi_{\mathfrak{G}(f)}(x)(\gamma) \geq \lambda$ and hence $\Phi_{\mathfrak{G}(f)}(x) \notin [\lambda]^\delta$. This is a contradiction. Thus for all $p \in {}^{<\omega}\iota(x)$, $\sup(F^x_{p,\gamma}) < \lambda$. Since $\mathrm{cof}(\lambda) > \omega$ and $|{}^{<\omega}\iota(x)| = \omega$, $\sup\{\sup(F^x_{p,\gamma}) : p \in {}^{<\omega}\iota(x)\} < \lambda$. Note that $\sup(\Upsilon(x) \restriction \gamma) \leq \sup\{\mu(\iota(x), \zeta) : \zeta \in \bigcup_{p \in {}^{<\omega}\iota(x)} F^x_{p,\gamma}\} \leq \sup\{\mu(\iota(x), \zeta) : \zeta < \sup\{\sup(F^x_{p,\gamma}) : p \in {}^{<\omega}\iota(x)\}\} < \lambda$ (by the property of chosen bijection μ). This shows that $\Upsilon : \bigcup_{\alpha<\omega_1} A_\alpha \to IB(\delta, \lambda)$. Υ is an injection by the same argument as in (2). The result now follows from Fact 2.3. $\square$

Theorem 3.10. *Assume* AD*,* $\mathsf{DC}_\mathbb{R}$*, and* $\mathrm{cof}(\Theta) > \omega_1$*. Let X be a surjective image of $\mathbb{R}$. Let $\langle A_\alpha : \alpha < \omega_1\rangle$ be a sequence so that for all $\alpha < \omega_1$, $A_\alpha \subseteq X$. Let δ and λ be cardinals so that $\omega_1 \leq \delta \leq \lambda < \Theta$. Assume one of the following three settings.*

(1) $\mathrm{cof}(\delta) \geq \omega_1$ *and for all* $\alpha < \omega_1$, $|A_\alpha| \leq |{}^{<\delta}\lambda|$.
(2) For all $\alpha < \omega_1$, $|A_\alpha| \leq |{}^{\delta}\lambda|$.
(3) $\mathrm{cof}(\lambda) \geq \omega_1$ *and for all* $\alpha < \omega_1$, $|A_\alpha| \leq |[\lambda]^\delta|$.

Then, respectively, the following hold.

(1) $|\bigcup_{\alpha<\omega_1} A_\alpha| \leq |{}^{<\delta}\lambda|$.
(2) $|\bigcup_{\alpha<\omega_1} A_\alpha| \leq |{}^{\delta}\lambda|$.
(3) $|\bigcup_{\alpha<\omega_1} A_\alpha| \leq |[\lambda]^\delta|$.

Proof. Recall that Martin and Monk showed that AD and $\mathsf{DC}_\mathbb{R}$ prove that strict Lipschitz reduction is wellfounded. For each $\alpha < \omega_1$, let β_α be the least β so that there is some $B \in \mathscr{P}(\mathbb{R})$ with $\mathrm{rk}_L(B) = \beta$ and T_B is the graph of an injection of A_α into ${}^{<\delta}\lambda$. Since $\mathrm{cof}(\Theta) > \omega_1$, $\sup\{\beta_\alpha : \alpha < \omega_1\} < \Theta$. Let $Z \in \mathscr{P}(\mathbb{R})$ so that $\mathrm{rk}_L(Z) = \sup\{\beta_\alpha : \alpha < \omega_1\}$. The result now follows from Theorem 3.9. $\square$

Theorem 3.11. *Assume* AD*,* $\mathsf{DC}_\mathbb{R}$*, and* $\mathrm{cof}(\Theta) > \omega$*. Suppose X is a surjective image of $\mathbb{R}$. Let $1 \leq \delta < \Theta$ and $\omega \leq \lambda < \Theta$. Let $\langle A_n : n \in \omega\rangle$ be a sequence so that for all $n \in \omega$, $A_\alpha \subseteq X$. Assume one of the following three settings.*

(1) $|A_\alpha| \leq |{}^{<\delta}\lambda|$ *for all* $n \in \omega$.
(2) $|A_\alpha| \leq |{}^{\delta}\lambda|$ *for all* $n \in \omega$.
(3) $|A_\alpha| \leq |[\lambda]^\delta|$ *for all* $n \in \omega$

Then, respectively, the following hold.

(1) $|\bigcup_{n\in\omega} A_n| \leq |{}^{<\delta}\lambda|$.

(2) $|\bigcup_{n\in\omega} A_n| \leq |^\delta\lambda|$.
(3) $|\bigcup_{n\in\omega} A_n| \leq |[\lambda]^\delta|$.

Proof. The argument is similar to the proof of Theorem 3.10 using Theorem 3.8. ☐

Woodin defined an extension of AD called AD^+ which includes (1) $\mathsf{DC}_\mathbb{R}$, (2) all sets of reals are ∞-Borel, and (3) ordinal determinacy (For every $\lambda < \Theta$, continuous function $\pi : {}^\omega\lambda \to \mathbb{R}$, and $A \subseteq \mathbb{R}$, the game on λ with payoff $\pi^{-1}[A]$ is determined). It is open whether AD and AD^+ are equivalent. Basic information about aspects of AD^+ can be found in [3], [6], [20], and [18].

Fact 3.12. *(Woodin) Suppose AD^+ and $V = L(\mathscr{P}(\mathbb{R}))$. Then either $\mathsf{AD}_\mathbb{R}$ holds or there is a set of ordinals J so that $V = L(J, \mathbb{R})$.*

Fact 3.13. *If AD^+, $\neg\mathsf{AD}_\mathbb{R}$, and $V = L(\mathscr{P}(\mathbb{R}))$, then Θ is regular.*

Proof. By Fact 3.12, there is a set of ordinals J so that $V = L(J, \mathbb{R})$. All sets in $L(J, \mathbb{R})$ are ordinal definable from J and an $r \in \mathbb{R}$. For each $r \in \mathbb{R}$ and $\alpha < \Theta$, if there is an $\mathrm{OD}_{\{J,r\}}$ surjection $\varpi : \mathbb{R} \to \alpha$, then let $\varpi_{\alpha,r} : \mathbb{R} \to \alpha$ be the least such surjection according to the canonical wellordering of $\mathrm{OD}_{\{J,r\}}$. For each $\alpha < \Theta$, let $\pi_\alpha : \mathbb{R} \to \alpha$ be defined by

$$\pi_\alpha(x) = \begin{cases} \varpi_{x^{[0]}}(x^{[1]}) & \text{if there is an } \mathrm{OD}_{\{J,x^{[0]}\}} \text{ surjection of } \mathbb{R} \text{ onto } \alpha \\ 0 & \text{otherwise.} \end{cases}$$

π_α is a surjection. Thus a sequence $\langle \pi_\alpha : \alpha < \Theta \rangle$ has been defined so that $\pi_\alpha : \mathbb{R} \to \alpha$ is a surjection for each $\alpha < \Theta$. Now suppose $\mathrm{cof}(\Theta) < \Theta$. Let $\tau : \mathbb{R} \to \mathrm{cof}(\Theta)$ be a surjection. Define $\sigma : \mathbb{R} \to \Theta$ by $\sigma(x) = \pi_{\tau(x^{[0]})}(x^{[1]})$. σ is a surjection onto Θ which is impossible. ☐

Let $1 \leq n < \omega$ and $A \subseteq \mathbb{R}^n$ (again $\mathbb{R}$ refers to ${}^\omega\omega$). A is Suslin if and only if there is an ordinal λ and a tree $T \subseteq \omega^n \times \lambda$ so that $A = \{(x_1, ..., x_n) \in \mathbb{R}^n : (\exists f \in {}^\omega\lambda)((x_1, ..., x_n, f) \in [T]\}$. $A \subseteq \mathbb{R}^n$ is coSuslin if and only if $\mathbb{R}^n \setminus A$ is Suslin.

Fact 3.14. *(Woodin) Assume AD^+ and $\mathsf{AD}_\mathbb{R}$. All sets of reals are Suslin.*

A transitive set M is said to be Suslin and coSuslin if and only if there is a surjection $\pi : \mathbb{R} \to M$ so that the equivalence relation $E_\pi \subseteq \mathbb{R} \times \mathbb{R}$ on $\mathbb{R}$ and the relation $F_\pi \subseteq \mathbb{R} \times \mathbb{R}$ defined below are Suslin and coSuslin:

$$x \; E_\pi \; y \Leftrightarrow \pi(x) = \pi(y) \quad \text{and} \quad (x, y) \in F_\pi \Leftrightarrow \pi(x) \in \pi(y).$$

Note that M is in bijection with $\mathbb{R}/E_\pi$. Let $\tilde{F}_\pi \subseteq \mathbb{R}/E_\pi \times \mathbb{R}/E_\pi$ be defined by $([x]_{E_\pi}, [y]_{E_\pi}) \in \tilde{F}_\pi$ if and only if $(x, y) \in F_\pi$. Then $(M, \in)$ is $\in$-isomorphic to $(\mathbb{R}/E_\pi, \tilde{F}_\pi)$. In other words, M is Suslin and CoSuslin if it has a natural coding on $\mathbb{R}$ which is Suslin and coSuslin.

Let $\mathcal{S}$ be the union of the collection of all transitive sets which are Suslin and coSuslin. $(\mathcal{S}, \in)$ is a $\in$-structure. In general, one says a set X is Suslin and coSuslin if and only if $X \in \mathcal{S}$.

Woodin showed that AD^+ implies the following reflection property.

Fact 3.15. *(Woodin; [24]) (Σ_1-reflection into Suslin and coSuslin) Assume AD^+ and $V = L(\mathscr{P}(\mathbb{R}))$. $\mathcal{S} \prec_{\Sigma_1} (V, \in)$. (That is, $\mathcal{S}$ is a Σ_1-elementary substructure of the universe V.)*

Theorem 3.16. *Assume AD^+. Let X be a surjective image of $\mathbb{R}$. Let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a sequence so that for all $\alpha < \omega_1$, $A_\alpha \subseteq X$. Let δ and λ be cardinals so that $\omega_1 \leq \delta \leq \lambda < \Theta$. Assume one of the following three settings.*

(1) $\mathrm{cof}(\delta) \geq \omega_1$ *and for all* $\alpha < \omega_1$, $|A_\alpha| \leq |^{<\delta}\lambda|$.
(2) *For all* $\alpha < \omega_1$, $|A_\alpha| \leq |^\delta\lambda|$.
(3) $\mathrm{cof}(\lambda) \geq \omega_1$ *and for all* $\alpha < \omega_1$, $|A_\alpha| \leq |[\lambda]^\delta|$.

Then, respectively, the following hold.

(1) $|\bigcup_{\alpha<\omega_1} A_\alpha| \leq |^{<\delta}\lambda|$.
(2) $|\bigcup_{\alpha<\omega_1} A_\alpha| \leq |^\delta\lambda|$.
(3) $\bigcup_{\alpha<\omega_1} A_\alpha| \leq |[\lambda]^\delta|$.

Proof. Consider the setting of (1). Let $\varsigma : \mathbb{R} \to X$ be a surjection. Define an equivalence relation E on $\mathbb{R}$ by $x\ E\ y$ if and only if $\varsigma(x) = \varsigma(y)$. Note that X is in bijection with $\mathbb{R}/E$. For each $\alpha < \omega_1$, let $K_\alpha = \varsigma^{-1}[A_\alpha]$ and $E_\alpha = E \upharpoonright K_\alpha$. Then $K_\alpha/E_\alpha \subseteq \mathbb{R}/E$ and A_α is in bijection with K_α/E_α. Injections of A_α into ${}^{<\delta}\lambda$ induce injections of K_α/E_α into ${}^{<\delta}\lambda$. Let $\pi : \mathbb{R} \to \mathbb{R}/E$ be defined by $\pi(x) = [x]_E$. Let $\varpi : \mathbb{R} \to \mathscr{P}(\lambda)$ be a surjection given by Fact 3.7. Then injections between K_α/E_α and $[\lambda]^{<\delta}$ can be coded by sets of reals through the coding $B \mapsto T_B$ described above. This shows that X and $\langle A_\alpha : \alpha < \omega_1\rangle$ with the property stated in setting (1) are in bijection with objects $\mathbb{R}/E$ and $\langle K_\alpha/E_\alpha : \alpha < \omega_1\rangle$ with the properties in setting (1) which belong to $L(\mathscr{P}(\mathbb{R}))$. It suffices to prove the theorem in $L(\mathscr{P}(\mathbb{R}))$.

With this discussion in mind, one will now assume AD^+, $V = L(\mathscr{P}(\mathbb{R}))$, and that X and $\langle A_\alpha : \alpha < \omega_1\rangle$ belong to $L(\mathscr{P}(\mathbb{R}))$ with the properties stated in (1). If $\mathrm{cof}(\Theta) > \omega_1$, then the result follows from Theorem 3.10. Suppose $\mathrm{cof}(\Theta) \leq \omega_1$. Thus Θ is singular and hence $\mathsf{AD}_\mathbb{R}$ holds by Fact 3.13. Assume for the sake of contradiction that there is a set X and a sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ satisfying (1) and $\neg(|\bigcup_{\alpha<\omega_1} A_\alpha| \leq |{}^{<\delta}\lambda|)$. Let $Y = \bigcup_{\alpha<\omega_1} A_\alpha$ and thus $\neg(|Y| \leq |{}^{<\delta}\lambda|)$. Since all sets of reals are Suslin and coSuslin by Fact 3.14 since AD^+ and $\mathsf{AD}_\mathbb{R}$ holds, the sets Y, δ, and λ are Suslin and coSuslin and hence belong to $\mathcal{S}$.

Let ψ be the following sentence with δ, λ, and Y as a parameter: $\delta \leq \lambda < \dot\Theta$ and there exists a sequence $\langle \tilde A_\alpha : \alpha < \omega_1\rangle$ so that $Y = \bigcup_{\alpha<\omega_1} \tilde A_\alpha$ and for all $\alpha < \omega_1$, $|A_\alpha| \leq |{}^{<\delta}\lambda|$. ($\dot\Theta$ is an abbreviation for the ordinal defined as the supremum of the ordinals which are surjective images of $\mathbb{R}$.) Let $\mathfrak{T}$ be some sufficiently strong finite fragment of ZF. Let φ be the following Σ_1-sentence with Y, δ, λ, and $\mathbb{R}$ as parameters: There exists a transitive set $M \models \mathfrak{T} + \mathsf{AD}$ so that $\mathbb{R} \subseteq M$ and $M \models \psi$. Let $\preceq$ be a prewellordering of length λ whose associated norm was used to define the surjection $\varpi : \mathbb{R} \to \mathscr{P}(\lambda)$ which appears in the coding described before Theorem 3.8. Since $L(\mathscr{P}(\mathbb{R})) \models$ "$\mathfrak{T}$, AD, and ψ" and using reflection on the hierarchy $\langle L_\alpha(\mathscr{P}(\mathbb{R})) : \alpha < \mathrm{ON}\rangle$, there is an ordinal $\alpha \geq \Theta$ such that $L_\alpha(\mathscr{P}(\mathbb{R})) \models$ "$\mathfrak{T}$, AD, and ψ". Thus $L(\mathscr{P}(\mathbb{R})) \models \varphi$ as witnessed by $L_\alpha(\mathscr{P}(\mathbb{R}))$. By Σ_1-reflection into Suslin and coSuslin (Fact 3.15), $\mathcal{S} \models \varphi$. Let $M \in \mathcal{S}$ be a transitive set containing $\mathbb{R}$ so that $M \models \psi$. Let $\langle \tilde A_\alpha : \alpha < \omega_1\rangle$ with $Y = \bigcup_{\alpha<\omega_1} \tilde A_\alpha$ witness the existential quantifier in ψ. Since for each $\alpha < \omega_1$, $M \models |\tilde A_\alpha| \leq |{}^{<\delta}\lambda|$, $\mathbb{R} \subseteq M$, satisfies AD, and has the prewellordering $\preceq$ used to code injections of subsets of Y into ${}^{<\delta}\lambda$, there is some $B \in \mathscr{P}(\mathbb{R}) \cap M$ so that T_B codes the graph of an injection of $\tilde A_\alpha$ into ${}^{<\delta}\lambda$. Since $M \in \mathcal{S}$ implies M is a surjective image of $\mathbb{R}$, $\sup\{\mathrm{rk}_L(B) : B \in \mathscr{P}(\mathbb{R}) \cap M\} < \Theta^V$. In the real world, let $Z \in \mathscr{P}(\mathbb{R})$ be such that $\mathrm{rk}_L(Z) \geq \sup\{\mathrm{rk}_L(B) : B \in \mathscr{P}(\mathbb{R}) \cap M\}$. Note that for all $\alpha < \omega_1$, there is an $r \in \mathbb{R}$ so that $T_{\Xi_r^{-1}[Z]}$ codes the graph of an injection of $\tilde A_\alpha$ into $[\lambda]^{<\delta}$. Applying Theorem 3.9 in the real world to $\langle \tilde A_\alpha : \alpha < \omega_1\rangle$, one has that $|Y| = |\bigcup_{\alpha<\omega_1} \tilde A_\alpha| \leq |{}^{<\delta}\lambda|$. This contradicts the assumption that $\neg(|Y| \leq |{}^{<\delta}\lambda|)$. ☐

Theorem 3.17. *Assume* AD^+. *Suppose X is a surjective image of $\mathbb{R}$. Let $1 \leq \delta < \Theta$ and $\omega \leq \lambda < \Theta$. Let $\langle A_n : n \in \omega\rangle$ be a sequence so that for all $n \in \omega$, $A_\alpha \subseteq X$. Assume one of the following three settings.*

(1) $|A_\alpha| \leq |{}^{<\delta}\lambda|$ *for all* $n \in \omega$.
(2) $|A_\alpha| \leq |{}^{\delta}\lambda|$ *for all* $n \in \omega$.
(3) $|A_\alpha| \leq |[\lambda]^\delta|$ *for all* $n \in \omega$

Then, respectively, the following hold.

(1) $|\bigcup_{n\in\omega} A_n| \leq |{}^{<\delta}\lambda|$.
(2) $|\bigcup_{n\in\omega} A_n| \leq |{}^{\delta}\lambda|$.
(3) $|\bigcup_{n\in\omega} A_n| \leq |[\lambda]^\delta|$.

Proof. The proof follows the template of the proof of Theorem 3.16 using Theorem 3.8. ☐

Theorem 3.18. *Assume* AD^+ *(or* AD, $\mathsf{DC}_\mathbb{R}$, *and* $\mathrm{cof}(\Theta) > \omega_1$*). If $\langle A_\alpha : \alpha < \omega_1\rangle$ is a sequence such that $\bigcup_{\alpha<\omega_1} A_\alpha = [\omega_2]^{<\omega_2}$, then there is an $\alpha < \omega_1$ so that $\neg(|A_\alpha| \leq |[\omega_2]^{\omega_1}|)$.*

Proof. Suppose $\langle A_\alpha : \alpha < \omega_1\rangle$ is a sequence such that $[\omega_2]^{<\omega_2} = \bigcup_{\alpha<\omega_1} A_\alpha$. Suppose for the sake of contradiction that for all $\alpha < \omega_1$, $|A_\alpha| \leq |[\omega_2]^{\omega_1}|$. By Theorem 3.16, $|[\omega_2]^{<\omega_2}| \leq |[\omega_2]^{\omega_1}|$ which violates Fact 2.14. ☐

Theorem 3.18 is regarded as partial evidence that $[\omega_2]^{<\omega_2}$ is ω_1-regular which means for any $\langle A_\alpha : \alpha < \omega_1\rangle$ such that $\bigcup_{\alpha<\omega_1} A_\alpha = [\omega_2]^{<\omega_2}$, there is an $\alpha < \omega_1$ so that $|A_\alpha| = |[\omega_2]^{<\omega_2}|$. This conjecture has recently been solved by the authors. The authors in [8] showed that under AD, $[\omega_2]^{<\omega_2}$ has ω_1-regular cardinality. However, it is still not known if $\mathscr{P}(\omega_2)$ is ω_1-regular or even 2-regular. The following is some evidence.

Theorem 3.19. *Assume AD^+ (or AD, $\mathsf{DC}_{\mathbb{R}}$, and $\mathrm{cof}(\Theta) > \omega_1$). If $\langle A_\alpha : \alpha < \omega_1\rangle$ is a sequence such that $\bigcup_{\alpha<\omega_1} A_\alpha = \mathscr{P}(\omega_2)$, then there is an $\alpha < \omega_1$ so that $\neg(|A_\alpha| \leq |[\omega_2]^{<\omega_2}|)$.*

Proof. Suppose $\langle A_\alpha : \alpha < \omega_1\rangle$ is a sequence such that $\mathscr{P}(\omega_2) = \bigcup_{\alpha<\omega_1} A_\alpha$. Suppose for the sake of contradiction that for all $\alpha < \omega_1$, $|A_\alpha| \leq |[\omega_2]^{<\omega_2}|$. By Theorem 3.16, $|\mathscr{P}(\omega_2)| \leq |[\omega_2]^{<\omega_2}|$ which violates Fact 2.10. □

Since under AD, ω_3 is singular with $\mathrm{cof}(\omega_3) = \omega_2$, Fact 2.9 cannot be used to show $[\omega_3]^{<\omega_3}$ or even $[\omega_3]^{\omega_2}$ have smaller cardinality than $\mathscr{P}(\omega_3)$. However [4] shows that $|[\omega_3]^{\omega_2}| < |[\omega_3]^{<\omega_3}| \leq |\mathscr{P}(\omega_3)|$ under AD^+ by the following result.

Fact 3.20. *([4]) Assume AD^+.*

1. *(ABCD Conjecture) Let α, β, γ, and δ be cardinals such that $\omega \leq \alpha \leq \beta < \Theta$ and $\omega \leq \gamma \leq \delta < \Theta$. $|{}^{\alpha}\beta| \leq |{}^{\gamma}\delta|$ if and only if $\alpha \leq \gamma$ and $\beta \leq \delta$.*
2. *If $\kappa < \Theta$ is a cardinal and $\epsilon < \kappa$, then $|{}^{\epsilon}\kappa| < |{}^{<\kappa}\kappa|$.*

It is still open if $|[\omega_3]^{<\omega_3}| < |\mathscr{P}(\omega_3)|$. The following result implies that if one decomposes $[\omega_3]^{<\omega_3}$ or $\mathscr{P}(\omega_3)$ into ω_1-many pieces $\langle A_\alpha : \alpha < \omega_1\rangle$. Then at least one piece A_α does not inject into $[\omega_3]^{\omega_2}$.

Theorem 3.21. *Assume AD^+ (or AD, $\mathsf{DC}_{\mathbb{R}}$, and $\mathrm{cof}(\Theta) > \omega_1$).*

1. *If $\omega_1 \leq \kappa < \Theta$ is a regular cardinal and $\langle A_\alpha : \alpha < \omega_1\rangle$ is a sequence such that $\bigcup_{\alpha<\omega_1} A_\alpha = \mathscr{P}(\kappa)$, then there is an $\alpha < \omega_1$ so that $\neg(|A_\alpha| \leq |[\kappa]^{<\kappa}|)$.*
2. *If $\omega_1 \leq \epsilon < \kappa < \Theta$ and $\langle A_\alpha : \alpha < \omega_1\rangle$ is a sequence such that $\bigcup_{\alpha<\omega_1} A_\alpha = {}^{<\kappa}\kappa$, then there is an $\alpha < \omega_1$ so that $\neg(|A_\alpha| \leq |{}^{\epsilon}\kappa|)$.*
3. *If $\omega_1 \leq \epsilon < \kappa < \Theta$ and $\langle A_\alpha : \alpha < \omega_1\rangle$ is a sequence such that $\bigcup_{\alpha<\omega_1} A_\alpha = \mathscr{P}(\kappa)$, then there is an $\alpha < \omega_1$ so that $\neg(|A_\alpha| \leq |{}^{\epsilon}\kappa|)$.*

Proof. (1) If $|A_\alpha| \leq |[\kappa]^{<\kappa}| = |{}^{<\kappa}\kappa|$, then $|\mathscr{P}(\kappa)| = |{}^{<\kappa}\kappa|$ by Theorem 3.16. Since AD^+ implies boldface GCH below Θ, this would contradict Fact 2.9.

(2) If $|A_\alpha| \leq |{}^{\epsilon}\kappa|$, then $|{}^{<\kappa}\kappa| = |{}^{\epsilon}\kappa|$ by Theorem 3.16. This would contradict Fact 3.20.

The proof of (3) is similar. □

4. Decomposition into a Suslin Cardinal Many Pieces

This section will consider a decomposition of sets into κ many pieces where κ is a Suslin cardinal. Kechris and Woodin ([17]) developed a more general generic coding function on Suslin cardinals (or more generally reliable ordinals). In the previous section, the wellordered additivity of the meager ideal had a prominent role in many arguments. For $\kappa > \omega$, there is no clear analog of this for ${}^{\omega}\kappa$ and its generic coding function. However, if $S \subseteq \kappa$ is a countable set, then ${}^{\omega}S$ is homeomorphic to $\mathbb{R}$ and thus under AD, the meager ideal on ${}^{\omega}S$ (with its usual topology) will satisfy the full wellordered additivity. The idea will be to do an argument similar to the previous section for each countable $S \subseteq \kappa$ and then take an ultrapower by a supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$, the set of all countable subsets of κ. One will need to impose conditions regarding the ultrapower maps of the supercompact measure to successfully generalize these arguments. However, one will still be able establish the analog of the main result of the previous section (concerning decomposition of $\mathscr{P}(\omega_2) = \mathscr{P}(\boldsymbol{\delta}^1_2)$ into $\omega_1 = \boldsymbol{\delta}^1_1$ many pieces) for decomposition of $\mathscr{P}(\omega_{\omega+2}) = \mathscr{P}(\boldsymbol{\delta}^1_4)$ into $\omega_{\omega+1} = \boldsymbol{\delta}^1_3$ many pieces.

Definition 4.1. ([17]) An ordinal λ is reliable if and only if there is a scale $\vec{\varphi} = \langle \varphi_n : n \in \omega\rangle$ on a set $W \subseteq \mathbb{R}$ such that the following holds.

1. For all $n \in \omega$, $\varphi_n : W \to \lambda$ and $\varphi_0 : W \to \lambda$ is a surjection.
2. The relation $S_0(x, y)$ defined by $x, y \in W \wedge \varphi_0(x) \leq \varphi_0(y)$ and $S_1(x, y)$ defined by $x, y \in W \wedge \varphi_0(x) < \varphi_0(y)$ are Suslin subsets of $\mathbb{R}^2$.

$\vec{\varphi}$ with the above property will be called the reliability witness for λ.

If $\sigma \subseteq \lambda$ (which is usually countable) and $\xi \in \sigma$, then σ is said to be ξ-honest (relative to $\vec{\varphi}$) if and only if there is a $w \in W$ so that $\varphi_0(w) = \xi$ and for all $n \in \omega$, $\varphi_n(\xi) \in \sigma$. Such a $w \in W$ will be called a ξ-honest witness for σ (relative to $\vec{\varphi}$). A countable $\sigma \subseteq \lambda$ is honest (relative to $\vec{\varphi}$) if and only if for all $\xi \in \sigma$, σ is ξ-honest.

Fact 4.2. *Suppose λ is a reliable ordinal with reliability witness $\vec{\varphi}$ which is a scale on a set $W \subseteq \mathbb{R}$. For each $\xi < \lambda$, there is a countable set σ so that σ is ξ-honest relative to $\vec{\varphi}$.*

Proof. Let $w \in W$ so that $\varphi_0(w) = \xi$ which is possible since $\varphi_0 : W \to \lambda$ is surjective. Let $\sigma = \{\varphi_n(w) : n \in \omega\}$. σ is ξ-honest with w as its ξ-honest witness. $\square$

It is generally not possible to uniformly associate ξ to a countable ξ-honest set (relative to a reliability witness). However if λ is a reliable ordinal of uncountable cofinality, then one can at least uniformly associate ξ to an ordinal $\xi' < \lambda$ which is ξ-honest which will be sufficient for applications here.

Fact 4.3. *Suppose λ is a reliable ordinal with reliability witness $\vec{\varphi}$ and $\mathrm{cof}(\lambda) > \omega$. For each $\xi < \lambda$, there is a $\xi' < \lambda$ so that for all γ with $\xi' \leq \gamma < \lambda$, γ is ξ-honest relative to $\vec{\varphi}$.*

Proof. By Fact 4.2, there is a countable $\bar{\sigma} \subseteq \lambda$ which is ξ-honest. $\xi' = \sup(\bar{\sigma}) < \lambda$ since $\mathrm{cof}(\lambda) > \omega$. Suppose γ is such that $\xi' \leq \gamma < \kappa$. Since $\bar{\sigma} \subseteq \gamma$, γ is ξ-honest. $\square$

Definition 4.4. Let X be a set. Let $\mathscr{P}_{\omega_1}(X) = \{\sigma \in \mathscr{P}(X) : |\sigma| \leq \omega\}$ (which is the set of countable subsets of X). Let ν be an ultrafilter on $\mathscr{P}_{\omega_1}(X)$. ν is a fine ultrafilter on $\mathscr{P}_{\omega_1}(X)$ if and only if for each $x \in X$, $A_x = \{\sigma \in \mathscr{P}_{\omega_1}(X) : x \in \sigma\} \in \nu$. ν is a normal ultrafilter on $\mathscr{P}_{\omega_1}(X)$ if and only if for every $\Phi : \mathscr{P}_{\omega_1}(X) \to \mathscr{P}_{\omega_1}(X)$ such that $\{\sigma \in \mathscr{P}_{\omega_1}(X) : \emptyset \neq \Phi(\sigma) \subseteq \sigma\} \in \nu$, there is an $x \in X$ so that $\{\sigma \in \mathscr{P}_{\omega_1}(X) : x \in \Phi(\sigma)\} \in \nu$. ν is a supercompact measure on X if and only if ν is a countably complete, fine, and normal measure on $\mathscr{P}_{\omega_1}(X)$.

Fact 4.5. *(Harrington-Kechris; [11]) Assume AD. If κ less than or equal to a Suslin cardinal, then there is a supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$.*

(Woodin; [28]) Assume AD. If κ is less than or equal to a Suslin cardinal, then the supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$ is unique.

Fact 4.6. *Assume AD. Suppose $\vec{\varphi}$ is a sequence of norms on $W \subseteq \mathbb{R}$ which is a reliability witness for λ. Let ν be a countably complete and fine measure on $\mathscr{P}_{\omega_1}(\lambda)$. Let $\xi < \lambda$. Then $\{\sigma \in \mathscr{P}_{\omega_1}(\lambda) : \sigma$ is ξ-honest$\} \in \nu$.*

Proof. Pick any $w \in W$ so that $\varphi_0(w) = \xi$ (which is possible since φ_0 surjects onto λ). By fineness of ν, $A_n = \{\sigma \in \mathscr{P}_{\omega_1}(\lambda) : \varphi_n(w) \in \sigma\} \in \nu$. By countably compleness of ν, $\bigcap_{n\in\omega} A_n \in \nu$. Since ν is a filter, $\bigcap_{n\in\omega} A_n \subseteq \{\sigma \in \mathscr{P}_{\omega_1}(\lambda) : \sigma \text{ is } \xi\text{-honest}\} \in \nu$. $\square$

Fact 4.7. *Assume AD. Suppose $\vec{\varphi}$ is a sequence of norms on $W \subseteq \mathbb{R}$ is a reliability witness for λ. Let ν be a supercompact measure on $\mathscr{P}_{\omega_1}(\lambda)$. Then $A = \{\sigma \in \mathscr{P}_{\omega_1}(\lambda) : \sigma$ is honest$\} \in \nu$.*

Proof. Suppose $A \notin \nu$. Let $\tilde{A} = \mathscr{P}_{\omega_1}(\lambda) \setminus A$. Since ν is an ultrafilter, $\tilde{A} \in \nu$. Let $\Phi : \mathscr{P}_{\omega_1}(\lambda) \to \mathscr{P}_{\omega_1}(\lambda)$ be defined by $\Phi(\sigma) = \{\xi \in \sigma : \sigma \text{ is not } \xi\text{-honest}\}$. Note that for all $\sigma \in \tilde{A}$, $\emptyset \neq \Phi(\sigma) \subseteq \sigma$. So $\tilde{A} \subseteq \{\sigma \in \mathscr{P}_{\omega_1}(\lambda) : \emptyset \neq \Phi(\sigma) \subseteq \sigma\}$ and therefore $\{\sigma \in \mathscr{P}_{\omega_1}(\lambda) : \emptyset \neq \Phi(\sigma) \subseteq \sigma\} \in \nu$. By normality, there is a $\eta \in \lambda$ so that $B = \{\sigma \in \mathscr{P}_{\omega_1}(\lambda) : \eta \in \Phi(\sigma)\} \in \nu$. Pick a $w \in W$ so that $\varphi_0(w) = \eta$. For each $n \in \omega$, $C_n = \{\sigma \in \mathscr{P}_{\omega_1}(\lambda) : \varphi_n(w) \in \sigma\} \in \nu$ by fineness. Then $C = \bigcap_{n\in\omega} C_n \in \nu$ by countably completeness. Then $D = B \cap C \in \nu$. Pick any $\sigma \in D$. w is a η-honest witness for σ since for all $n \in \omega$, $\varphi_n(w) \in \sigma$. Thus σ is η-honest. However, $\eta \in \Phi(\sigma)$ means that σ is not η-honest. This is a contradiction. $\square$

Recall the notation $x^{[n]}$ from Definition 3.1 for $x \in \mathbb{R}$ and $n \in \omega$.

Fact 4.8. *(Kechris-Woodin; [17] Lemma 1.1, [14] Theorem 6.1) Assume AD. Let λ be a reliable ordinal with $\vec{\varphi}$ be a sequence of norms on a set $W \subseteq \mathbb{R}$ being a reliability witness. Then there is a Lipschitz continuous function $\mathfrak{G} : {}^\omega\lambda \to \mathbb{R}$ so that the following holds.*

(1) For all $n \in \omega$ and $f \in {}^\omega\lambda$, $\mathfrak{G}(f)^{[n]} \in W$ and $\varphi_0(\mathfrak{G}(f)^{[n]}) \leq f(n)$.
(2) For all $n \in \omega$ and $f \in {}^\omega\lambda$, if $f[\omega]$ is $f(n)$-honest, then $\varphi_0(\mathfrak{G}(f)^{[n]}) = f(n)$.

Thus if $f[\omega]$ is honest, then for all $n \in \omega$, $\varphi_0(\mathfrak{G}(f)^{[n]}) = f(n)$. For each $n \in \omega$, let $\mathfrak{G}_n : {}^\omega\lambda \to W$ be defined by $\mathfrak{G}_n(f) = \mathfrak{G}(f)^{[n]}$.

A function $\mathfrak{G}$ with the above property is called a generic coding function for λ relative to the reliability witness $\vec{\varphi}$.

Theorem 4.9 will only need the concept of ξ-honest for a particular ordinal $\xi < \lambda$ and will never need full honesty. Thus one will only directly use Fact 4.6 concerning fine and countably complete measures on $\mathscr{P}_{\omega_1}(\lambda)$ rather than Fact 4.7 which involves supercompact measures on $\mathscr{P}_{\omega_1}(\lambda)$. However, it is convenient to use the uniqueness of the supercompact measure (Fact 4.5) to uniformly find long sequences of supercompact measures on various ordinals. Theorem 4.9 will just need codes for $f(0)$ rather than all of f so the function $\mathfrak{G}_0 : {}^\omega\lambda \to W$ will be used directly rather than $\mathfrak{G}$. The full generic coding function will be used later to analyze the ultrapower of the supercompact measure.

Again, use the notation defined before Theorem 3.8: Suppose $\pi : \mathbb{R} \to X$. Let $\delta \leq \lambda < \Theta$ and $\varpi : \mathbb{R} \to \mathscr{P}(\lambda)$. If $B \subseteq \mathbb{R}$, let $T_B = \{(x, f) : (\exists z \in B)(x = \pi(z^{[0]}) \wedge f = \varpi(z^{[1]}))\}$. If $A \subseteq X$ and $\Phi : A \to {}^{<\delta}\lambda$, then there is some $B \in \mathscr{P}(\mathbb{R})$ so that the graph of Φ is T_B.

Theorem 4.9. *Assume AD. Let X be a surjective image of $\mathbb{R}$. Let κ be a reliable cardinal. Let $\kappa \leq \delta \leq \lambda < \Theta$ be a cardinals with $\text{cof}(\delta) > \omega$. For each $\alpha \leq \kappa$, let ν_α be the unique supercompact measure on $\mathscr{P}_{\omega_1}(\alpha)$. Suppose one of the two cases occurs.*

(1) $j_{\nu_\kappa}(\delta) = \delta$ and $j_{\nu_\kappa}(\lambda) = \lambda$.
(2) For all $\alpha < \kappa$, $j_{\nu_\alpha}(\delta) = \delta$ and $j_{\nu_\alpha}(\lambda) = \lambda$.

Let $\langle A_\alpha : \alpha < \kappa\rangle$ be a sequence so that there exists a $Z \in \mathscr{P}(\mathbb{R})$ with the property that for all $\alpha \in \kappa$, $A_\alpha \subseteq X$, $|A_\alpha| \leq |{}^{<\delta}\lambda|$, and there is an $r \in \mathbb{R}$ so that $T_{\Xi_r^{-1}[Z]}$ is the graph of an injection of A_α into ${}^{<\delta}\lambda$. Then $|\bigcup_{\alpha<\kappa} A_\alpha| \leq |{}^{<\delta}\lambda|$.

Proof. Let $\vec{\varphi} = \langle \varphi_n : n \in \omega\rangle$ be a scale on $W \subseteq \mathbb{R}$ which serves as a reliability witness for κ. If case (1) holds, for each $\alpha < \kappa$, let $\xi(\alpha) = \kappa$. If case (2) holds, let $\xi(\alpha)$ be the least ξ which is α-honest relative to $\vec{\varphi}$. Regardless of the case, $j_{\nu_{\xi(\alpha)}}(\delta) = \delta$ and $j_{\xi(\alpha)}(\lambda) = \lambda$ for all $\alpha < \kappa$.

Define $R \subseteq W \times \mathbb{R}$ by $R(w, r)$ if and only if $T_{\Xi_r^{-1}[Z]}$ is the graph of an injection of $A_{\varphi_0(w)}$ into ${}^{<\delta}\lambda \setminus \{\emptyset\}$. Let Γ be a scaled pointclass containing the Suslin relations W and S_0 (from Definition 4.1 for φ_0) and closed under $\exists^\mathbb{R}$ and $\wedge$. By applying the Moschovakis coding lemma to R, φ_0, and Γ, there is a relation $\bar{R} \subseteq W \times \mathbb{R}$ so that $\bar{R} \subseteq R$, $\bar{R} \in \Gamma$, and for all $\alpha < \kappa$, there is a $w \in W$ with $\varphi_0(w) = \alpha$ and $r \in \mathbb{R}$ so that $\bar{R}(w, r)$. Let $\tilde{R} \subseteq W \times \mathbb{R}$ be defined by $\tilde{R}(w, r)$ if and only if $w \in W \wedge (\exists v)(S_0(v, w) \wedge S_0(w, v) \wedge \bar{R}(v, r))$. $\tilde{R} \in \Gamma$ and $\text{dom}(\tilde{R}) = W$. Since Γ is a scaled pointclass, let $\Lambda : W \to \mathbb{R}$ be a uniformization with the property that for all $w \in W$, $\tilde{R}(w, \Lambda(w))$. Thus for all $w \in W$, $R(w, \Lambda(w))$. For all $w \in W$, $T_{\Xi^{-1}_{\Lambda(w)}[Z]}$ is the graph of an injection of $A_{\varphi_0(w)}$ into ${}^{<\delta}\lambda \setminus \{\emptyset\}$. For each $w \in W$, let $\Phi_w : A_{\varphi_0(w)} \to {}^{<\delta}\lambda \setminus \{\emptyset\}$ be the injection whose graph is $T_{\Xi^{-1}_{\Lambda(w)}}[Z]$.

For each $x \in \bigcup_{\alpha<\kappa} A_\alpha$, let $\iota(x)$ be the least $\alpha < \kappa$ so that $x \in A_\alpha$. Let $\tau : {}^{<\omega}\kappa \times \delta \times \lambda \to \lambda$ be a bijection. If σ is a countable set and $p \in {}^{<\omega}\sigma$, then let $N_p^\sigma = \{f \in {}^\omega\sigma : p \subseteq f\}$. ${}^\omega\sigma$ is given the product of the discrete topology on σ which equivalently is generated by $\{N_p^\sigma : p \in {}^{<\omega}\sigma\}$ as a basis. For any countable σ, ${}^\omega\sigma$ is homeomorphic to ${}^\omega\omega$ and has the Baire property for its topology. For $p \in {}^{<\omega}\sigma$ and φ a formula, $(\forall_p^{*,\sigma} f)\varphi(f)$ abbreviates $\{f \in N_p^\sigma : \varphi(f)\}$ is comeager in N_p^σ. For all $x \in \bigcup_{\alpha<\kappa} A_\alpha$ and $\sigma \in \mathscr{P}_{\omega_1}(\xi(\iota(x)))$ with $\iota(x) \in \sigma$, let

$$\Upsilon^x(\sigma) = \{\tau(p, \eta, \zeta) : p \in {}^{<\omega}\sigma \wedge (\exists \epsilon < \delta)(\forall^{*,\sigma}_{\langle\iota(x)\rangle^\frown p} f)(\epsilon = \text{dom}(\Phi_{\mathfrak{G}_0(f)}(x)) \wedge \eta < \epsilon \wedge \Phi_{\mathfrak{G}_0(f)}(x)(\eta) = \zeta)\}.$$

Since τ maps into λ, one has that $\Upsilon^x(\sigma) \in \mathscr{P}(\lambda)$. Thus for each $x \in \bigcup_{\alpha\in\kappa} A_\alpha$, $\Upsilon^x : \mathscr{P}_{\omega_1}(\xi(\iota(x))) \to \mathscr{P}(\lambda)$. Note that the hypothesis that $\prod_{\sigma\in\mathscr{P}_{\omega_1}(\xi(\iota(x)))} \lambda/\nu_{\xi(\iota(x))} = j_{\nu_{\xi(\iota(x))}}(\lambda) = \lambda$ implicitly implies that this ultrapower is wellfounded. Define $\Upsilon(x)$ to be the set of all ordinals γ such that there exist (equivalently, for all) functions $f : \mathscr{P}_{\omega_1}(\xi(\iota(x))) \to \text{ON}$ with $[f]_{\nu_{\xi(\iota(x))}} = \gamma$, $\{\sigma \in \mathscr{P}_{\omega_1}(\xi(\iota(x))) : f(\sigma) \in \Upsilon^x(\sigma)\} \in \nu_{\xi(\iota(x))}$. (Although this ultrapower does not satisfy Łoś' Theorem, Υ is intuitively defined by $\Upsilon(x) = [\Upsilon^x]_{\nu_{\xi(\iota(x))}}$.)

<u>Claim 1</u>: For all $x \in \bigcup_{\alpha<\kappa} A_\alpha$, $\Upsilon(x) \subseteq \lambda$.

To see Claim 1: Suppose $\gamma \in \Upsilon(x)$ and $f : \mathscr{P}_{\omega_1}(\xi(\iota(x))) \to \mathrm{ON}$ with $[f]_{\nu_{\xi(\iota(x))}} = \gamma$. Thus $\{\sigma \in \mathscr{P}_{\omega_1}(\xi(\iota(x))) : f(\sigma) \in \Upsilon^x(\sigma) \subseteq \mathscr{P}(\lambda)\} \in \nu_{\xi(\iota(x))}$. Thus $[f]_{\nu_{\xi(\iota(x))}} < j_{\nu_{\xi(\iota(x))}}(\lambda) = \lambda$. Thus $\gamma < \lambda$. This shows $\gamma \in \lambda$. Claim 1 has been established.

Claim 2: For all $x \in \bigcup_{\alpha<\kappa} A_\alpha$, $\Upsilon(x) \neq \emptyset$.

To see Claim 2: Since $\xi(\iota(x))$ is an $\iota(x)$-honest ordinal, $A = \{\sigma \in \mathscr{P}_{\omega_1}(\xi(\iota(x))) : \sigma \text{ is } \iota(x)\text{-honest}\} \in \nu_{\xi(\iota(x))}$. Pick any $\sigma \in A$. Let $\mathsf{surj}^{\iota(x)}_\sigma = \{f \in {}^\omega\sigma : f[\omega] = \sigma \wedge f(0) = \iota(x)\}$ which is a comeager subset of $N^\sigma_{\langle\iota(x)\rangle}$. For all $f \in \mathrm{surj}^{\iota(x)}_\sigma$, $f[\omega] = \sigma$ is $\iota(x)$-honest or equivalently $f(0)$-honest. By Fact 4.8, $\varphi_0(\mathfrak{G}_0(f)) = \iota(x)$ and therefore, $\Phi_{\mathfrak{G}_0(f)} : A_{\iota(x)} \to {}^{<\delta}\lambda$. For all $\epsilon < \delta$, let $B_\epsilon = \{f \in \mathsf{surj}^{\iota(x)}_\sigma : \mathrm{dom}(\Phi_{\mathfrak{G}_0(f)}(x)) = \epsilon\}$. One has that $\mathrm{surj}^{\iota(x)}_\sigma = \bigcup_{\epsilon<\delta} B_\epsilon$. Since wellordered union of meager sets is meager and $\mathrm{surj}^{\iota(x)}_\sigma$ is a comeager subset of $N^\sigma_{\langle\iota(x)\rangle}$, there is some $\bar\epsilon$ so that $B_{\bar\epsilon}$ is nonmeager. (Note that $\bar\epsilon > 0$ since $\Phi_{\mathfrak{G}_0(f)} : A_{\iota(x)} \to {}^{<\delta}\lambda \setminus \{\emptyset\}$.) For each $\zeta < \lambda$, let $C_\zeta = \{f \in B_{\bar\epsilon} : \Phi_{\mathfrak{G}_0(f)}(x)(0) = \zeta\}$. $B_{\bar\epsilon} = \bigcup_{\zeta<\lambda} C_\zeta$. Again since wellordered union of meager subsets of ${}^\omega\sigma$ are meager and $B_{\bar\epsilon}$ is nonmeager, there is $\bar\zeta$ so that $C_{\bar\zeta}$ is nonmeager. Since ${}^\omega\sigma$ has the Baire property, there is a $\bar p \in {}^{<\omega}\sigma$ so that $B_{\bar\epsilon}$ is comeager in $N^\sigma_{\langle\iota(x)\rangle\hat{}p}$. Then $\tau(\bar p, 0, \bar\zeta) \in \Upsilon^x(\sigma)$. This shows that for all $\sigma \in A$, $\Upsilon^x(\sigma) \neq \emptyset$. Let $h : A \to \lambda$ be defined by $h(\sigma) = \min(\Upsilon^x(\sigma))$. Then $[h]_{\nu_{\xi(\iota(x))}} \in \Upsilon(x)$. This establishes Claim 2.

Claim 3: For all $x \in \bigcup_{\alpha<\kappa} A_\alpha$ and $\sigma \in \mathscr{P}_{\omega_1}(\xi(\iota(x)))$, $|\Upsilon^x(\sigma)| < \delta$.

To see Claim 3: Let $B = \{p \in {}^{<\omega}\sigma : (\exists\epsilon)(\forall^{*,\sigma}_{\langle\iota(x)\rangle\hat{}p} f)(\epsilon = \mathrm{dom}(\Phi_{\mathcal{G}_0(f)}(x)))\}$. For each $p \in B$, there is a unique $\epsilon_p < \delta$ so that $(\forall^{*,\sigma}_{\langle\iota(x)\rangle\hat{}p} f)(\epsilon_p = \mathrm{dom}(\Phi_{\mathfrak{G}_0(f)}))$. Thus ϵ_p surjects onto $K^\sigma_p = \{\tau(p,\eta,\zeta) : \tau(p,\eta,\zeta) \in \Upsilon^x(\sigma)\}$ since if $\tau(p,\eta,\zeta) \in K^\sigma_p$, then $\eta < \epsilon_p$ and ζ is uniquely determined from p and η. Hence $|K^\sigma_p| \leq |\epsilon_p| < \delta$. Since $B \subseteq {}^{<\omega}\sigma$ is countable, $\Upsilon^x(\sigma) = \bigcup_{p\in B} K^\sigma_p$, and $\mathrm{cof}(\delta) > \omega$, one has that $|\Upsilon^x(\sigma)| < \delta$.

Claim 4: For all $x \in \bigcup_{\alpha<\kappa} A_\alpha$, $|\Upsilon(x)| < \delta$ and thus $\Upsilon(x) \in \mathscr{P}_\delta(\lambda)$.

To see Claim 4: Suppose $\gamma \in \Upsilon(x)$ and $[f]_{\nu_{\xi(\iota(x))}} = \gamma$. For each $\sigma \in \mathscr{P}_{\omega_1}(\xi(\iota(x)))$, let $h_f(\sigma)$ be the ordertype of $f(\sigma)$ in $\Upsilon^x(\sigma)$. By Claim 3, $h_f : \mathscr{P}_{\omega_1}(\xi(\iota(x))) \to \delta$. Let $\Sigma^x(\gamma) = [h_f]_{\nu_{\xi(\iota(x))}}$ and note that $\Sigma^x(\gamma)$ is independent of the choice of representative f. Let $g^x : \mathscr{P}_{\omega_1}(\xi(\iota(x))) \to \delta$ be defined by $g^x(\sigma) = \mathrm{ot}(\Upsilon^x(\sigma))$. Note that $g^x(\sigma) < \delta$ by Claim 3. Thus $\Sigma^x(\gamma) = [h_f]_{\nu_{\xi(\iota(x))}} < [g^x]_{\nu_{\xi(\iota(x))}} < j_{\nu_{\xi(\iota(x))}}(\delta) = \delta$. Thus $\Sigma^x : \Upsilon(x) \to [g^x]_{\nu_{\xi(\iota(x))}}$ where $[g^x]_{\nu_{\xi(\iota(x))}} < \delta$. Now suppose $\gamma_0 < \gamma_1$ and $\gamma_0, \gamma_1 \in \Upsilon(x)$. Let f_0 and f_1 be such that $[f_0]_{\nu_{\xi(\iota(x))}} = \gamma_0$ and $[f_1]_{\nu_{\xi(\iota(x))}} = \gamma_1$. Thus $\{\sigma \in \mathscr{P}_{\omega_1}(\xi(\iota(x))) : f_0(\sigma) < f_1(\sigma)\} \in \nu_{\xi(\iota(x))}$. Thus $\Sigma^x(\gamma_0) = [h_{f_0}]_{\nu_{\xi(\iota(x))}} < [h_{f_1}]_{\nu_{\xi(\iota(x))}} = \Sigma^x(\gamma_1)$. Thus $\Sigma^x : \Upsilon(x) \to [g^x]_{\nu_{\xi\iota(x)}}$ is an order-preserving map. Thus $|\Upsilon(x)| < \delta$ and hence $\Upsilon(x) \in \mathscr{P}_\delta(\lambda)$. This shows Claim 4.

Define $\chi : \bigcup_{\alpha<\kappa} A_\alpha \to \kappa \times \mathscr{P}_\delta(\lambda)$ by $\chi(x) = (\iota(x), \Upsilon(x))$.

Claim 5: $\chi : \bigcup_{\alpha<\kappa} A_\alpha \to \kappa \times \mathscr{P}_\delta(\lambda)$ is an injection.

To see Claim 5: Suppose $x_0, x_1 \in \bigcup_{\alpha<\kappa} A_\alpha$ and $x_0 \neq x_1$. First suppose $\iota(x_0) \neq \iota(x_1)$. Then $\chi(x_0) = (\iota(x_0), \Upsilon(x_0)) \neq (\iota(x_1), \Upsilon(x_1)) = \chi(x_1)$. Now assume $\iota(x_0) = \iota(x_1)$ and let α be this common ordinal. Let $A = \{\sigma \in \mathscr{P}_{\omega_1}(\xi(\alpha)) : \sigma \text{ is } \alpha\text{-honest}\}$ and note that $A \in \nu_{\xi(\alpha)}$. Let A_0 be the set of $\sigma \in A$ so that $E^\alpha_\sigma = \{f \in \mathsf{surj}^\alpha_\sigma : \mathrm{dom}(\Phi_{\mathfrak{G}_0(f)}(x_0)) = \mathrm{dom}(\Phi_{\mathfrak{G}_0(f)}(x_1))\}$ is nonmeager in ${}^\omega\sigma$. Let $A_1 = \mathsf{surj}^\alpha_\sigma \setminus A_0$. Since $A = A_0 \cup A_1$ and $A \in \nu_{\xi(\alpha)}$, exactly one of $A_0 \in \nu_{\xi(\alpha)}$ or $A_1 \in \nu_{\xi(\alpha)}$. Suppose $A_0 \in \nu_{\xi(\alpha)}$. Fix $\sigma \in A_0$ so E^α_σ is nonmeager. Let $F^\sigma_\epsilon = \{f \in E^\alpha_\sigma : \mathrm{dom}(\Phi_{\mathfrak{G}_0(f)}(x_0)) = \epsilon = \mathrm{dom}(\Phi_{\mathfrak{G}_0(f)}(x_1))\}$. Since $E^\alpha_\sigma = \bigcup_{\epsilon<\delta} F^\alpha_\sigma$ and E^α_σ is nonmeager in ${}^\omega\sigma$, let $\bar\epsilon_\sigma < \delta$ be the least ϵ so that F^σ_ϵ is nonmeager. Since for all $f \in F^\sigma_{\bar\epsilon_\sigma}$, $\Phi_{\mathfrak{G}_0(f)} : A_\alpha \to {}^{<\delta}\lambda$ is an injection, $\Phi_{\mathfrak{G}_0(f)}(x_0) \neq \Phi_{\mathfrak{G}_0(f)}(x_1)$. For each $\eta < \bar\epsilon_\sigma$, let H^σ_η be the set of $f \in F^\sigma_{\bar\epsilon_\sigma}$ so that η is least η' so that $\Phi_{\mathfrak{G}_0(f)}(x_0)(\eta') \neq \Phi_{\mathfrak{G}_0(f)}(x_1)(\eta')$. Since $F^\sigma_{\bar\epsilon_\sigma} = \bigcup_{\eta<\bar\epsilon_\sigma} H^\sigma_\eta$, let $\bar\eta_\sigma$ be the least η so that H^σ_η is nonmeager. For each pair (ζ_0, ζ_1) of distinct ordinals in λ, let $K^\sigma_{\zeta_0,\zeta_1}$ be the set of $f \in H^\sigma_{\bar\eta_\sigma}$ so that $\Phi_{\mathfrak{G}_0(f)}(x_0)(\bar\eta_\sigma) = \zeta_0$ and $\Phi_{\mathfrak{G}_0(f)}(x_1)(\bar\eta_\sigma) = \zeta_1$. Since $H^\sigma_{\bar\eta_\sigma} = \bigcup\{K^\sigma_{\zeta_0,\zeta_1} : \zeta_0, \zeta \in \lambda \wedge \zeta_0 \neq \zeta_1\}$, let $(\bar\zeta^\sigma_0, \bar\zeta^\sigma_1)$ be least pair (ζ_0, ζ_1) so that $K^\sigma_{\zeta_0,\zeta_1}$ is nonmeager. Since ${}^\omega\sigma$ has the Baire property, let $\bar p_\sigma$ be the least $p \in {}^{<\omega}\sigma$ (under a uniformly defined wellordering of ${}^{<\omega}\sigma$) so that $K^\sigma_{\bar\zeta^\sigma_0,\bar\zeta^\sigma_1}$ is comeager in N^σ_p. Then $\tau(\bar p_\sigma, \bar\eta_\sigma, \bar\zeta^\sigma_0) \in \Upsilon^{x_0}(\sigma)$ but $\tau(\bar p_\sigma, \bar\eta_\sigma, \bar\zeta^\sigma_0) \notin \Upsilon^{x_1}(\sigma)$. Let $h(\sigma) = \tau(\bar p_\sigma, \bar\eta_\sigma, \bar\zeta^\sigma_0)$. Then $h(\sigma) \in \Upsilon^{x_0}(\sigma)$ but $h(\sigma) \notin \Upsilon^{x_1}(\sigma)$ for all $\sigma \in A_0$. Then $[h]_{\nu_{\xi(\alpha)}} \in \Upsilon(x_0)$ but $[h]_{\nu_{\xi(\alpha)}} \notin \Upsilon(x_1)$. So $\Upsilon(x_0) \neq \Upsilon(x_1)$. Hence $\chi(x_0) = (\alpha, \Upsilon(x_0)) \neq (\alpha, \Upsilon(x_1)) = \chi(x_1)$. Now suppose $A_1 \in \nu_{\xi(\alpha)}$. Let $\sigma \in A_1$. Then E^α_σ is meager in ${}^\omega\sigma$. Let $I^\alpha_\sigma = \mathsf{surj}^\alpha_\sigma \setminus E^\alpha_\sigma$ which is comeager in ${}^\omega\sigma$. For each pair of $\epsilon_0 \neq \epsilon_1$ less than δ, let $J^\sigma_{\epsilon_0,\epsilon_1}$ be the set of $f \in I^\alpha_\sigma$ so that $\mathrm{dom}(\Phi_{\mathfrak{G}_0(f)}(x_0)) = \epsilon_0$ and $\mathrm{dom}(\Phi_{\mathfrak{G}_0(f)}(x_1)) = \epsilon_1$. Then $I^\alpha_\sigma = \bigcup\{J^\sigma_{\epsilon_0,\epsilon_1} : \epsilon_0, \epsilon_1 < \delta \wedge \epsilon_0 \neq \epsilon_1\}$. Let $(\bar\epsilon^\sigma_0, \bar\epsilon^\sigma_1)$ be the least pair (ϵ_0, ϵ_1) with $\epsilon_0 \neq \epsilon_1$ so that $J^\sigma_{\epsilon_0,\epsilon_1}$ is nonmeager. Without loss of generality, suppose

$\bar{\epsilon}_0^\sigma < \bar{\epsilon}_1^\sigma$. For each $\zeta < \lambda$, let $Q_\zeta^\sigma = \{f \in J^\sigma_{\bar{\epsilon}_0^\sigma,\bar{\epsilon}_1^\sigma} : \Phi_{\mathfrak{G}_0(f)}(x_1)(\bar{\epsilon}_0) = \zeta\}$. $J^\sigma_{\bar{\epsilon}_0^\sigma,\bar{\epsilon}_1^\sigma} = \bigcup_{\zeta<\lambda} Q_\zeta^\sigma$. Let $\bar{\zeta}_\sigma$ be least ζ so that Q_ζ^σ is nonmeager. Since ${}^\omega\sigma$ has the Baire property, let $\bar{p}_\sigma$ be the least $p \in {}^{<\omega}\sigma$ so that $Q^\sigma_{\bar{\zeta}_\sigma}$ is comeager in N_p^σ. Let $h(\sigma) = \tau(\bar{p}_\sigma, \bar{\epsilon}_0^\sigma, \bar{\zeta}_\sigma)$. For all $\sigma \in A_1$, $h(\sigma) \in \Upsilon^{x_1}(\sigma)$ however $h(\sigma) \notin \Upsilon^{x_0}(\sigma)$. Thus $[h]_{\nu_{\xi(\alpha)}} \in \Upsilon(x_1)$ and $[h]_{\nu_{\xi(\alpha)}} \notin \Upsilon(x_0)$. So $\Upsilon(x_0) \neq \Upsilon(x_1)$. Therefore, $\chi(x_0) = (\alpha, \Upsilon(x_0)) \neq (\alpha, \Upsilon(x_1)) = \chi(x_1)$. Claim 5 has been established.

Since $|\mathscr{P}_\delta(\lambda)| = |{}^{<\delta}\lambda|$ by Fact 2.2 and $|\mathscr{P}_\delta(\lambda)| = |\kappa \times \mathscr{P}_\delta(\lambda)|$, one has that there is an injection of $\bigcup_{\alpha<\kappa} A_\alpha$ into ${}^{<\delta}\lambda$. $\square$

Theorem 4.10. *Assume* AD *and* $\mathsf{DC}_\mathbb{R}$. *Suppose X is a surjective image of $\mathbb{R}$. Let κ be a reliable cardinal. Assume* $\mathrm{cof}(\Theta) > \kappa$. *Let δ and λ be cardinals such that $\kappa \leq \delta \leq \lambda < \Theta$ and $\mathrm{cof}(\delta) > \omega$. For each $\alpha \leq \kappa$, let ν_α be the unique supercompact measure on $\mathscr{P}_{\omega_1}(\alpha)$. Suppose one of the two cases occurs.*

(1) $j_{\nu_\kappa}(\delta) = \delta$ and $j_{\nu_\kappa}(\lambda) = \lambda$.

(2) For all $\alpha < \kappa$, $j_{\nu_\alpha}(\delta) = \delta$ and $j_{\nu_\alpha}(\lambda) = \lambda$.

Let $\langle A_\alpha : \alpha < \kappa\rangle$ be a sequence so that for all $\alpha \in \kappa$, $A_\alpha \subseteq X$, and $|A_\alpha| \leq |{}^{<\delta}\lambda|$. Then $|\bigcup_{\alpha<\kappa} A_\alpha| \leq |{}^{<\delta}\lambda|$.

Proof. The proof follows from Theorem 4.9 in a manner similar to how Theorem 3.10 follows from Theorem 3.9. $\square$

Theorem 4.11. *Assume* AD^+. *Suppose X is a surjective image of $\mathbb{R}$. Let κ be a reliable cardinal which is below a Suslin cardinal. Let $\kappa \leq \delta \leq \lambda < \Theta$ be cardinals with $\mathrm{cof}(\delta) > \omega$. For each $\alpha \leq \kappa$, let ν_α be the unique supercompact measure on $\mathscr{P}_{\omega_1}(\alpha)$. Suppose one of the cases occurs.*

(1) $j_{\nu_\kappa}(\delta) = \delta$ and $j_{\nu_\kappa}(\lambda) = \lambda$.

(2) For all $\alpha < \kappa$, $j_{\nu_\alpha}(\delta) = \delta$ and $j_{\nu_\alpha}(\lambda) = \lambda$.

Let $\langle A_\alpha : \alpha < \kappa\rangle$ be a sequence so that for all $\alpha \in \kappa$, $A_\alpha \subseteq X$, and $|A_\alpha| \leq |{}^{<\delta}\lambda|$. Then $|\bigcup_{\alpha<\kappa} A_\alpha| \leq |{}^{<\delta}\lambda|$.

Proof. This result follows from Theorem 4.9 and Theorem 4.10 as in the proof of Theorem 3.16. $\square$

It is implicit in the assumption that $j_{\nu_\alpha}(\lambda) = \lambda$ that the ultrapower $\prod_{\mathscr{P}_{\omega_1}(\alpha)} \lambda/\nu_\alpha$ is wellfounded. This is addressed in Fact 4.22. Then next few results will work toward showing $j_{\nu_\alpha}(\boldsymbol{\delta}_4^1) = \boldsymbol{\delta}_4^1$ which is due to Becker [1] Theorem 4.2. One will need an explicit characterization of the supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$ when κ is a reliable ordinal. Various constructions of a supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$ can be found in Solovay [23], Harrrington-Kechris [11], and Becker [1]. By Woodin's result [28] concerning the uniqueness of the supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$, they all define the same measure. Here, one will use a construction of the supercompact measure from generic codings presented in [14]. However, one uses the "ordinal determinacy" clause of AD^+ to get the necessary determinacy of certain games with moves on the ordinal. Many results below have AD^+ as a hypothesis but had previously been proved under AD using the determinacy of certain real games given by [11] Harrington-Kechris. The generic coding methods seems more suitable for generalization as Becker-Jackson [2] and Jackson [13] showed certain cardinals (for instance the projective ordinals $\boldsymbol{\delta}_n^1$) have higher degree of supercompactness (i.e. are $\boldsymbol{\delta}_1^2$-supercompact).

Fact 4.12. *Let κ be an ordinal, ν be a supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$, and $f : {}^{<\omega}\kappa \to \kappa$ be a function. Then $\{\sigma \in \mathscr{P}_{\omega_1}(\kappa) : f[{}^{<\omega}\sigma] \subseteq \sigma\} \in \nu$.*

Proof. Let $A = \{\sigma \in \mathscr{P}_{\omega_1}(\kappa) : f[{}^{<\omega}\sigma] \subseteq \sigma\}$. For the sake of contradiction, suppose $A \notin \nu$. Let $\tilde{A} = \mathscr{P}_{\omega_1}(\kappa) \setminus A$ and note that $\tilde{A} \in \nu$ since ν is an ultrafilter. Fix a wellordering $\prec$ of ${}^{<\omega}\kappa$. If $\sigma \in \tilde{A}$, then there is a $p \in {}^{<\omega}\kappa$ so that $f(p) \notin \sigma$. Let p_σ be the least such p according to $\prec$. By the countably additivity of ν, there is an $\bar{n}$ so that $B = \{\sigma \in \tilde{A} : |p_\sigma| = n\} \in \nu$. If $\bar{n} = 0$, then $p_\sigma = \emptyset$ for all $\sigma \in B$. By fineness, $C = \{\sigma \in B : f(\emptyset) \in \sigma\} \in \nu$. For all $\sigma \in C$, $f(p_\sigma) = f(\emptyset) \in \sigma$ which contradicts the definition of p_σ. Now suppose $\bar{n} > 0$. For each $k < \bar{n}$, let $\Phi_k : B \to \mathscr{P}_{\omega_1}(\kappa)$ be defined by $\Phi_k(\sigma) = \{p_\sigma(k)\}$. For all $k < \bar{n}$, $\{\sigma \in B : \emptyset \neq \Phi_k(\sigma) \subseteq \sigma\} \in \nu$. By normality, there is an $\alpha_k \in \kappa$ so that $D_k = \{\sigma \in B : \alpha_k \in \Phi_k(\sigma)\} \in \nu$. Let $\bar{p} \in {}^{\bar{n}}\kappa$ be defined by $\bar{p}(k) = \alpha_k$. Thus $E = \{\sigma \in B : p_\sigma = \bar{p}\} = \bigcap_{k<\bar{n}} D_k \in \nu$ by the countably completeness of ν. By fineness, $F = \{\sigma \in D : f(\bar{p}) \in \sigma\} \in \nu$. For all $\sigma \in F$, $f(p_\sigma) = f(\bar{p}) \in \sigma$ which contradicts the definition of p_σ. This completes the proof. $\square$

Definition 4.13. Formally a strategy on κ is a function $\rho : {}^{<\omega}\kappa \to \kappa$. If ρ_0 and ρ_1 are two strategies, then $\rho_0 * \rho_1 \in {}^{\omega}\kappa$ is defined by recursion as follows: If n is even, then $(\rho_0 * \rho_1)(n) = \rho_0(\rho_0 * \rho_1 \restriction n)$. If n is odd, then $(\rho_0 * \rho_1)(n) = \rho_1(\rho_0 * \rho_1 \restriction n)$. If $f \in {}^{\omega}\kappa$, then let ρ^1_f be the strategy defined by $\rho^1_f(2n) = f(n)$ and $\rho^1_f(2n+1) = 0$ for all $n \in \omega$. If $f \in {}^{\omega}\kappa$, then let ρ^2_f be the strategy defined by $\rho^2_f(2n) = 0$ and $\rho^2_f(2n+1) = f(n)$. If $f \in {}^{\omega}\kappa$, let $f_{\text{even}} \in {}^{\omega}\kappa$ and $f_{\text{odd}} \in {}^{\omega}\kappa$ be defined by $f_{\text{even}}(n) = f(2n)$ and $f_{\text{odd}}(n) = f(2n+1)$. If ρ is a strategy, then let $\Xi^1_\rho, \Xi^2_\rho : {}^{\omega}\kappa \to {}^{\omega}\kappa$ be defined by $\Xi^1_\rho(f) = (\rho * \rho^2_f)_{\text{even}}$ and $\Xi^2_\rho(f) = (\rho^1_f * \rho)_{\text{odd}}$.

Fix a bijection $\pi^{\kappa,2} : \kappa \to \kappa \times \kappa$. Let $\pi^{\kappa,2}_0, \pi^{\kappa,2}_1 : \kappa \to \kappa$ be defined by $\pi^{\kappa,2}_0(\alpha) = \beta$ and $\pi^{\kappa,2}_1(\alpha) = \gamma$ where $\pi^{\kappa,2}(\alpha) = (\beta, \gamma)$. If ρ is a strategy on κ, let $\chi^\kappa_\rho = \pi^{\kappa,2}_0 \circ \rho$ and $\tau^\kappa_\rho = \pi^{\kappa,2}_1 \circ \rho$.

Definition 4.14. Let κ be a reliable ordinal with reliability witness $\vec{\varphi}$ which is a scale on $W \subseteq \mathbb{R}$. Let $\rho : {}^{<\omega}\kappa \to \kappa$ be a strategy on κ. Let K_ρ be the set of $\sigma \in \mathscr{P}_{\omega_1}(\kappa)$ so that σ is honest relative to the reliability witness $\vec{\varphi}$ and $\rho[{}^{<\omega}\sigma] \subseteq \sigma$.

Fact 4.15. *Let κ be a reliable ordinal with reliability witness $\vec{\varphi}$ which is a scale on $W \subseteq \mathbb{R}$. Let $\rho : {}^{<\omega}\kappa \to \kappa$ be a strategy on κ. Then $K_\rho \in \nu_\kappa$.*

Proof. This follows from Fact 4.7 and Fact 4.12. ☐

Generic coding can be used to define the unique supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$ when κ is a reliable ordinal. The game will be provided next and used to show that sets of the form K_ρ for strategies ρ on κ form a basis for the supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$.

Fact 4.16. *Assume* AD^+. *Let κ be a reliable ordinal with reliability witness $\vec{\varphi}$ which is a scale on $W \subseteq \mathbb{R}$. Let ν_κ be the unique supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$. Let $A \subseteq \mathscr{P}_{\omega_1}(\kappa)$. $A \in \nu_\kappa$ if and only if there is a strategy $\rho : {}^{<\omega}\kappa \to \kappa$ so that $K_\rho \subseteq A$.*

Proof. Fix $A \subseteq \mathscr{P}_{\omega_1}(\kappa)$. Define the game G_A on κ as following.

$$\begin{array}{c|cccccccc}
 & \text{I} & \alpha_0 & & \alpha_2 & & \alpha_4 & & \cdots & \\
G_A & & & & & & & & & f \\
 & \text{II} & & \alpha_1 & & \alpha_3 & & \alpha_5 & \cdots &
\end{array}$$

Player 1 and 2 alternate playing ordinals from κ. Player 1 plays the ordinals α_{2n} and Player 2 plays the ordinals α_{2n+1} for all $n \in \omega$. Player 1 wins G_A if and only if $\{\varphi_0(\mathfrak{G}_n(f)) : n \in \omega\} \in A$. Let ν^*_κ be the set of all $A \subseteq \mathscr{P}_{\omega_1}(\kappa)$ so that Player 1 has a winning strategy in G_A. Let $B \subseteq {}^{\omega}\omega$ be $B = \{r \in {}^{\omega}\omega : (\forall n)(r^{[n]} \in W) \wedge \{\varphi_0(r^{[n]}) : n \in \omega\} \in A\}$. The payoff set for G_A is $\mathfrak{G}^{-1}[B]$. Since $\mathfrak{G} : {}^{\omega}\kappa \to {}^{\omega}\omega$ is continuous, the "ordinal determinacy" clause of AD^+ implies that G_A is determined. It can be shown that ν^*_κ is a supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$. (Thus one can define the unique supercompact measure ν_κ on $\mathscr{P}_{\omega_1}(\kappa)$ to be ν^*_κ.)

If there is strategy ρ on κ so that $K_\rho \subseteq A$, then $A \in \nu_\kappa$ since $K_\rho \in \nu_\kappa$ by Fact 4.15. Now suppose $A \in \nu_\kappa = \nu^*_\kappa$. Let ρ be a Player 1 winning strategy in G_A. Let $\sigma \in K_\rho$ which means that σ is honest and $\rho[{}^{<\omega}\sigma] \subseteq \sigma$. Let $g : \omega \to \sigma$ be a surjection. Let $f = \rho * \rho^2_g$ be the run of player 1 playing the terms of g against Player 1 using ρ. Since $\rho[{}^{<\omega}\sigma] \subseteq \sigma$ and $g[\omega] = \sigma$, one has that $f[\omega] = \sigma$. Since $f[\omega] = \sigma$ is honest, by the properties of the generic coding function (Fact 4.8), $\varphi_0(\mathfrak{G}_n(f)) = f(n)$. Thus $\{\varphi_0(\mathfrak{G}_n(f)) : n \in \omega\} = \sigma$. Since ρ is a Player 1 winning strategy, $\sigma = \{\varphi_0(\mathfrak{G}_n(f)) : n \in \omega\} \in A$. Since $\sigma \in K_\rho$ was arbitrary, $K_\rho \subseteq A$. ☐

Fact 4.17. *Suppose κ be an ordinal, $\lambda < \kappa$, and ν is a supercompact measure on κ. Let $\Pi : \mathscr{P}_{\omega_1}(\kappa) \to \mathscr{P}_{\omega_1}(\lambda)$ be defined by $\Pi(\sigma) = \sigma \cap \lambda$. Then the Rudin-Keisler pushforward $\mu = \Pi_*\nu$ defined by $A \in \mu$ if and only if $\Pi^{-1}[A] \in \nu$ is a supercompact measure on $\mathscr{P}_{\omega_1}(\lambda)$.*

Proof. It is straightforward to see that μ is an ultrafilter and countably complete. Suppose $\alpha \in \lambda$. Let $A = \{\tau \in \mathscr{P}_{\omega_1}(\lambda) : \alpha \in \tau\}$. By the fineness of ν, $B = \{\sigma \in \mathscr{P}_{\omega_1}(\kappa) : \alpha \in \kappa\} \in \nu$. Note that $B = \Pi^{-1}[A]$. By definition $A \in \mu$. Thus μ is fine. Let $\Phi : \mathscr{P}_{\omega_1}(\lambda) \to \mathscr{P}_{\omega_1}(\lambda)$ be such that $C = \{\tau \in \mathscr{P}_{\omega_1}(\lambda) : \emptyset \neq \Phi(\tau) \subseteq \tau\} \in \mu$. Define $\Psi : \mathscr{P}_{\omega_1}(\kappa) \to \mathscr{P}_{\omega_1}(\kappa)$ by $\Psi(\sigma) = \Phi(\sigma \cap \lambda)$ and note that Ψ actually maps into $\mathscr{P}_{\omega_1}(\lambda)$. Let $D = \{\sigma \in \mathscr{P}_{\omega_1}(\kappa) : \emptyset \neq \Psi(\sigma \cap \lambda) \subseteq \sigma\}$. Note that $D = \Pi^{-1}[C]$. Thus $D \in \nu$ since $C \in \mu = \Pi_*\nu$. By the normality of ν, there is an $\alpha \in \kappa$ so that $E = \{\sigma \in \mathscr{P}_{\omega_1}(\kappa) : \alpha \in \Psi(\sigma)\} \in \nu$. Note that $\alpha \in \lambda$. Let $F = \{\tau \in \mathscr{P}_{\omega_1}(\lambda) : \alpha \in \Phi(\tau)\}$. Note that $E = \Pi^{-1}[F]$ and hence $F \in \mu$. This shows that μ is normal. ☐

Using the proof of Fact 4.17, one can provide an explicit characterization of the supercompact measure on $\mathscr{P}_{\omega_1}(\lambda)$ when λ less than or equal to a Suslin cardinal using the generic coding on a reliable ordinal greater than or equal to λ.

Fact 4.18. *Assume* AD^+. *Let λ be less than or equal to a Suslin cardinal and let κ be any reliable cardinal greater than or equal to λ. Let $\vec{\varphi}$ be a reliability witness for κ. For any strategy ρ on κ, let $K^\lambda_\rho = \{\sigma \cap \lambda : \sigma \in K_\rho\}$. For any $A \subseteq \mathscr{P}_{\omega_1}(\lambda)$, $A \in \nu_\lambda$ if and only if there is a strategy ρ on κ so that $K^\lambda_\rho \subseteq A$.*

Proof. Let $\Pi : \mathscr{P}_{\omega_1}(\kappa) \to \mathscr{P}_{\omega_1}(\lambda)$ be defined by $\Pi(\sigma) = \sigma \cap \lambda$. By Fact 4.17 and the uniqueness of the supercompact measure on $\mathscr{P}_{\omega_1}(\lambda)$, one has that $\nu_\lambda = \Pi_*\nu_\kappa$. Suppose $A \in \nu_\lambda$. Then $\Pi^{-1}[A] \in \nu_\kappa$. By Fact 4.16, there is a strategy ρ on κ so that $K_\rho \subseteq \Pi^{-1}[A]$. Thus $K^\lambda_\rho = \{\sigma \cap \lambda : \sigma \in K_\rho\} = \{\Pi(\sigma) : \sigma \in K_\rho\} \subseteq A$. Now suppose there is a strategy ρ so that $K^\lambda_\rho \subseteq A$. Since $\Pi^{-1}[K^\lambda_\rho] \supseteq K_\rho$, , $\Pi^{-1}[K^\lambda_\rho] \in \nu_\kappa$. So $K^\lambda_\rho \in \nu_\lambda$. Thus $A \in \nu_\lambda$. $\square$

The following is straightforward.

Fact 4.19. *Suppose κ is an ordinal, $|\kappa| \leq \lambda < \kappa^+$, and ν is a supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$. Let $\pi : \kappa \to \lambda$ be a bijection. Let $\Pi : \mathscr{P}_{\omega_1}(\kappa) \to \mathscr{P}_{\omega_1}(\lambda)$ be defined by $\Pi(\sigma) = \pi[\sigma]$. Then the Rudin-Keisler pushforward $\mu = \Pi_*\nu$ defined by $A \in \mu$ if and only if $\Pi^{-1}[A] \in \nu$ is a supercompact measure on $\mathscr{P}_{\omega_1}(\lambda)$.*

Fact 4.20. *Assume* AD *and* $\mathsf{DC}_\mathbb{R}$. *For any κ less than or equal to a Suslin cardinal, let ν_κ denote the unique supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$. If $\lambda < \kappa^+$, then ν_λ is Rudin-Keisler reducible to ν_κ.*

Proof. If $\lambda < \kappa$, then Fact 4.17 defines a supercompact measure on $\mathscr{P}_{\omega_1}(\lambda)$ which is Rudin-Keisler reducible to ν_κ. By Woodin uniqueness of the supercompact measure on $\mathscr{P}_{\omega_1}(\lambda)$, this measure must be ν_λ. Similarly, if $\kappa \leq \lambda < \kappa^+$, then Fact 4.19 defines a supercompact measure on $\mathscr{P}_{\omega_1}(\lambda)$ which is Rudin-Keiser below ν_κ. Again by uniqueness, this must be ν_λ. $\square$

Fact 4.21. *Assume* AD. *The Suslin cardinals are unbounded below their supremum.*

Proof. Let λ be the supremum of the Suslin cardinals. If $\lambda = \Theta$, then the result is clear. Suppose $\lambda < \Theta$ but λ is not a limit of Suslin cardinals. This means λ is the largest Suslin cardinal. Let S_λ be the set of λ-Suslin sets which is equivalently the set of all Suslin sets. First, the claim is that S_λ is closed under $\forall^\mathbb{R}$. Suppose S_λ is not closed under $\forall^\mathbb{R}$. There is a set $A \subseteq \mathbb{R} \times \mathbb{R}$ so that $\forall^\mathbb{R} A \notin S_\lambda$. However, the proof of the second periodicity theorem of Moschovakis [22] shows that a Suslin representation for A can be used to create a Suslin representation for $\forall^\mathbb{R} A$. Thus $\forall^\mathbb{R} A$ is a Suslin set which does not belong to S_λ which is the set of all Suslin sets. This is a contradiction. [14] Lemma 3.6 states that if S_λ is closed under $\forall^\mathbb{R}$, then λ is a limit of Suslin cardinals. This completes the proof. $\square$

Using this explicit characterization of the supercompact measure, it will be shown next that the ultrapower ordinals below Θ by the supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$ when κ is below a Suslin cardinal is wellfounded under AD^+.

Fact 4.22. *Assume* AD^+. *Let κ less than or equal to a Suslin cardinal. Let ν_κ be the unique supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$. Let $(\nu_\kappa)^{L(\mathscr{P}(\mathbb{R}))}$ be the unique supercompact measure on $\mathscr{P}_{\omega_1}(\kappa)$ in $L(\mathscr{P}(\mathbb{R}))$. Let $\lambda < \Theta$. Then $\nu_\kappa = (\nu_\kappa)^{L(\mathscr{P}(\mathbb{R}))}$, $\prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda/\nu_\kappa = \left(\prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda/\nu_\kappa\right)^{L(\mathscr{P}(\mathbb{R}))}$, and $\prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda/\nu_\kappa$ is wellfounded.*

Proof. Since κ and λ are less than Θ, there are surjections $\pi_0 : \mathbb{R} \to \kappa$ and $\pi_1 : \mathbb{R} \to \lambda$. Thus $\pi_2 : \mathbb{R} \to \mathscr{P}_{\omega_1}(\kappa)$ defined by $\pi_2(r) = \{\pi_0(r^{[n]}) : n \in \omega\}$ is a surjection. For each $A \subseteq \mathbb{R}$, let $C_A = \{\pi_2(r) : r \in A\}$. For any $X \subseteq \mathscr{P}_{\omega_1}(\kappa)$, there is an $A \in \mathscr{P}(\mathbb{R})$ so that $C_A = X$. Let $\pi_3 : \mathbb{R} \to \mathscr{P}_{\omega_1}(\kappa) \times \lambda$ be defined by $\pi_3(r) = (\pi_2(r^{[0]}), \pi_1(r^{[1]}))$. π_3 is a surjection. For any $A \in \mathscr{P}(\mathbb{R})$, let $D_A = \{\pi_3(r) : r \in A\}$. Thus for any $f : \mathscr{P}_{\omega_1}(\kappa) \to \lambda$, there is an $A \in \mathscr{P}(\mathbb{R})$ so that D_A is the graph of f. The prewellorderings corresponding to π_0 and π_1 are subsets of $\mathbb{R}$. Thus $L(\mathscr{P}(\mathbb{R}))$ can recover C_A and D_A from $A \in \mathscr{P}(\mathbb{R})$. This shows that $\mathscr{P}_{\omega_1}(\kappa) = (\mathscr{P}_{\omega_1}(\kappa))^{L(\mathscr{P}(\mathbb{R}))}$ and $\prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda = \left(\prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda\right)^{L(\mathscr{P}(\mathbb{R}))}$.

The real world and $L(\mathscr{P}(\mathbb{R}))$ have the same Suslin cardinals since the real world and $L(\mathscr{P}(\mathbb{R}))$ have the same trees on ordinals below $\Theta = \Theta^{L(\mathscr{P}(\mathbb{R}))}$ using the Moschovakis coding lemma. Note that since κ is less than or equal to a Suslin cardinal in the real world, κ is still less than or equal to a Suslin cardinal in

$L(\mathscr{P}(\mathbb{R}))$. Since the Suslin cardinals are unbounded below the supremum of the Suslin cardinals by Fact 4.21, there is a reliable ordinal (even a Suslin cardinal) $\bar{\kappa} \geq \kappa$. Since $\bar{\kappa}$ is a reliable ordinal, fix a reliability witness $\vec{\varphi}$ on $W \subseteq \mathbb{R}$. Since $\vec{\varphi} = \langle \varphi_n : n \in \omega \rangle$ is a scale, $\vec{\varphi} \in L(\mathscr{P}(\mathbb{R}))$. For any strategy ρ on κ, let K_ρ be the set of $\sigma \in \mathscr{P}_\omega(\bar{\kappa})$ such that $\rho[{}^{<\omega}\sigma] \subseteq \sigma$ and σ is honest relative to $\vec{\varphi}$. Let $K^\kappa_\rho = \{\sigma \cap \kappa : \sigma \in K_\rho\}$. By Fact 4.18, $A \in \nu_\kappa$ if and only if there is a strategy τ on $\bar{\kappa}$ so that $K^\kappa_\tau \subseteq A$. Strategies on $\bar{\kappa}$ are essentially subsets of $\bar{\kappa}$. By using the Moschovakis coding lemma applied in $L(\mathscr{P}(\mathbb{R}))$ using a surjection of $\mathbb{R}$ onto $\bar{\kappa}$ in $L(\mathscr{P}(\mathbb{R}))$ (for instance φ_0), one can show that the real world and $L(\mathscr{P}(\mathbb{R}))$ have the same set of strategies on $\bar{\kappa}$. Note also that for any strategy ρ on $\bar{\kappa}$, $K^\kappa_\rho = (K^\kappa_\rho)^{L(\mathscr{P}(\mathbb{R}))}$ since the notion of honesty is absolute. Using the explicit definition of ν_κ (having sets of the form K^κ_ρ as a basis) applied in the real world or $L(\mathscr{P}(\mathbb{R}))$, one has that $\nu_\kappa = (\nu_\kappa)^{L(\mathscr{P}(\mathbb{R}))}$. This with the previous observation that $\prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda = \left(\prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda\right)^{L(\mathscr{P}(\mathbb{R}))}$ implies that $\prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda/\nu_\kappa = \left(\prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda/\nu_\kappa\right)^{L(\mathscr{P}(\mathbb{R}))}$.

Since AD^+ holds in the real world, $L(\mathscr{P}(\mathbb{R})) \models \mathsf{AD}^+$. By the above, $\prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda/\nu_\kappa$ is wellfounded if and only if $\left(\prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda/\nu_\kappa\right)^{L(\mathscr{P}(\mathbb{R}))}$ is wellfounded. So work inside $L(\mathscr{P}(\mathbb{R}))$ and assume for the sake of contradiction that there is some κ less than or equal to a Suslin cardinal and ordinal $\lambda < \Theta$ so that $\prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda/\nu_\kappa$ is illfounded. For each $\alpha \leq \Theta$, let $\mathcal{W}_\alpha$ be the set of reals of Wadge rank less than α. Let φ be the sentence "there exist ordinals α and β so that $L_\alpha(\mathcal{W}_\beta) \models (\exists \kappa, \lambda)(\kappa$ is less than or equal to a Suslin cardinal $\wedge\, \lambda < \Theta \wedge \prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda/\nu_\kappa$ is illfounded)". By the reflection theorem and since $\mathscr{P}(\mathbb{R}) = \mathcal{W}_\Theta$, there is some α so that $L_\alpha(\mathcal{W}_\Theta) \models (\exists \kappa, \lambda)(\kappa$ is less than or equal to a Suslin cardinal $\wedge\, \lambda < \Theta \wedge \prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda/\nu_\kappa$ is illfounded). Thus $L(\mathscr{P}(\mathbb{R})) \models \varphi$ with witnesses α as above and $\beta = \Theta$. By the Σ_1-reflection into Suslin-coSuslin (Fact 3.15), $S \prec_{\Sigma_1} L(\mathscr{P}(\mathbb{R}))$. There exists $\alpha < \mathcal{S}$ and $\beta \in \mathcal{S}$ so that

$$L_\alpha(\mathcal{W}_\beta) \models (\exists \kappa, \lambda)(\kappa \text{ is less than or equal to a Suslin cardinal } \wedge \lambda < \Theta \wedge \prod_{\mathscr{P}_{\omega_1}(\kappa)} \lambda/\nu_\kappa \text{ is illfounded}).$$

Since $\alpha, \beta < \Theta$, $L_\alpha(\mathcal{W}_\beta)$ is a surjective image of $\mathbb{R}$. Working in $L(\mathscr{P}(\mathbb{R})) \models \mathsf{DC}_\mathbb{R}$, one can find $\langle f_n : n \in \omega \rangle$ so that $f_n \in L_\alpha(\mathcal{W}_\beta)$, $f_n : \mathscr{P}_{\omega_1}(\kappa) \to \lambda$, and $L_\alpha(\mathcal{W}_\beta) \models [f_{n+1}]_{\nu_\kappa} < [f_n]_{\nu_\kappa}$ for each $n \in \omega$. For each $n \in \omega$, $L_\alpha(\mathcal{W}_\beta) \models A_n = \{\sigma \in \mathscr{P}_{\omega_1}(\kappa) : f_{n+1}(\sigma) < f_n(\sigma)\} \in \nu_\kappa$. Note $L_\alpha(\mathcal{W}_\beta) \models \kappa$ is less than or equal to a Suslin cardinal. Thus $L_\alpha(\mathcal{W}_\beta)$ has a reliable ordinal $\bar{\kappa} \geq \kappa$. Pick a reliability witness $\vec{\varphi}$ for $\bar{\kappa}$ in $L_\alpha(\mathcal{W}_\beta)$ and note that it is a reliability witness for $\bar{\kappa}$ in $L(\mathscr{P}(\mathbb{R}))$. For any strategy ρ on $\bar{\kappa}$, define K^κ_ρ relative to this reliability witness $\vec{\varphi}$. By applying the explicit definition of the supercompact measure on κ within $L_\alpha(\mathcal{W}_\beta)$, for each $n \in \omega$, there is a strategy ρ on $\bar{\kappa}$ so that $K^\kappa_\rho \subseteq A_n$. Again since there is surjection of $\mathbb{R}$ onto $L_\alpha(\mathcal{W}_\beta)$ in $L(\mathscr{P}(\mathbb{R}))$, one can use $\mathsf{AC}^\mathbb{R}_\omega$ in $L(\mathscr{P}(\mathbb{R}))$ to find a sequence $\langle \rho_n : n \in \omega \rangle$ so that for each $n \in \omega$, $\rho_n \in L_\alpha(\mathcal{W}_\beta)$ is a strategy on $\bar{\kappa}$, and $K^\kappa_{\rho_n} \subseteq A_n$. Note for all $n \in \omega$, $K^\kappa_{\rho_n} \in \nu_\kappa$. Since $L(\mathscr{P}(\mathbb{R})) \models \nu_\kappa$ is countably compete, $\bigcap_{n \in \omega} K^\kappa_{\rho_n} \neq \emptyset$. Let $\sigma \in \bigcap_{n \in \omega} K^\kappa_{\rho_n} \subseteq \bigcap_{n \in \omega} A_n$. Then in $L(\mathscr{P}(\mathbb{R}))$, $\langle f_n(\sigma) : n \in \omega \rangle$ is an infinite descending sequence of ordinals below λ. This is a contradiction. $\square$

Fact 4.23. *(Almost everywhere honest-enumeration uniformization) Assume* AD^+. *Let κ be a reliable ordinal with reliability witness $\vec{\varphi}$ which is a scale on a set $W \subseteq \mathbb{R}$. Let $R \subseteq \mathscr{P}_{\omega_1}(\kappa) \times {}^\omega\omega$ be such that* $\mathrm{dom}(R) = \mathscr{P}_{\omega_1}(\kappa)$. *There is a strategy ρ on κ with the following properties.*

(1) For all $s \in {}^{<\omega}\kappa$ with $|s|$ odd, $\tau^\kappa_\rho(s) \in \omega$.
(2) For all $f \in {}^\omega\kappa$ such that $f[\omega] \in K_{\chi^\kappa_\rho}$, $R(f[\omega], \Xi^2_{\tau^\kappa_\rho}(f))$.

Proof. Consider the game H_R on κ defined as follows.

$$\begin{array}{cccccccccc} & \mathrm{I} & \alpha_0 & & \alpha_2 & & \alpha_4 & & \cdots & & \\ H_R & & & & & & & & & g & f, x \\ & \mathrm{II} & & \beta_1 & & \beta_3 & & \beta_5 & \cdots & & \end{array}$$

$$\pi^{\kappa,2}(\alpha_1, x_0) \quad \pi^{\kappa,2}(\alpha_3, x_1) \quad \pi^{\kappa,2}(\alpha_5, x_2)$$

Player 1 and Player 2 alternate playing ordinals from κ. Player 1 plays α_{2n} and Player 2 plays β_{2n+1} as in the picture above for each $n \in \omega$. Practically, one should regard Player 2 as playing a pair $\alpha_{2n+1} \in \kappa$ and $x_n \in \omega$ such that $\pi^{\kappa,2}(\alpha_{2n+1}, x_n) = \beta_{2n+1}$. Let $g = \langle \alpha_0, \beta_1, \alpha_2, \beta_3, ...\rangle$. Let $f = \langle \alpha_n : n \in \omega\rangle$ and $x = \langle x_n : n \in \omega\rangle$. Player 2 wins if and only if the conjunction of the following holds.

- For all $n \in \omega$, $x_n \in \omega$.
- $R(\{\varphi_0(\mathfrak{G}_n(f)) : n \in \omega\}, x)$.

This game is determined by AD^+.

The claim is that Player 2 has a winning strategy in H_R. For the sake of contradiction, suppose ρ is a winning strategy for Player 1 in H_R. Let $\sigma \in \mathscr{P}_{\omega_1}(\kappa)$ have the following two properties.

(1) σ is honest relative to the reliability witness $\vec{\varphi}$.
(2) $\rho(\emptyset) \in \sigma$. For all $k \in \omega$, $\gamma_0, ..., \gamma_{2k+1} \in \sigma$, $n_0, ..., n_k \in \omega$,

$$\rho(\langle \gamma_0, \pi^{\kappa,2}(\gamma_1, n_0), \gamma_2, \pi^{\kappa,2}(\gamma_3, n_1), ..., \pi^{\kappa,2}(\gamma_{2k+1}, n_k)\rangle) \in \sigma.$$

Let $x \in {}^\omega\omega$ be such that $R(\sigma, x)$. Let $h : \omega \to \sigma$ be a surjection onto σ. Let $\tilde{h} : \omega \to \kappa$ be defined by $\tilde{h}(n) = \pi^{\kappa,2}(h(n), x(n))$. Consider the run of H_R where Player 1 uses ρ and Player 2 uses $\rho^2_{\tilde{h}}$. Let $g = \rho * \rho^2_{\tilde{h}}$. Let $f(2n) = g(2n)$ and $f(2n+1) = \pi_0^{\kappa,2}(g(2n+1)) = h(n)$. By (2), for all $n \in \omega$, $f(2n) \in \sigma$. Since for all $n \in \omega$, $f(2n+1) = h(n)$ and $h : \omega \to \sigma$ is a surjection, $f[\omega] = \sigma$. By (1), $f[\omega]$ is honest. By the properties of the generic coding function $\mathfrak{G}$ (Fact 4.8), $\varphi_0(\mathfrak{G}_n(f)) = f(n)$. Thus $\sigma = \{\varphi_0(\mathfrak{G}_n(f)) : n \in \omega\}$. Note that $x(n) = \pi_1^{\kappa,2}(g(2n+1))$ and $R(\sigma, x)$. This shows that Player 2 has won this run of H_R which contradicts ρ being a winning strategy for Player 1.

Thus by the determinacy of H_R, Player 2 has a winning strategy $\bar{\rho}$. By the first condition for Player 2 winning, condition (1) must hold for $\bar{\rho}$. Now suppose $h \in \mathscr{P}_{\omega_1}(\kappa)$ is such that $h[\omega] \in K_{\chi^\kappa_{\bar{\rho}}}$. Consider the run of H_R where Player 1 plays by ρ^1_h and Player 2 plays by $\bar{\rho}$. Let $g = \rho^1_h * \bar{\rho}$. Let $f : \omega \to \kappa$ be defined by $f(2n) = g(2n)$ and $f(2n+1) = \pi_0^{\kappa,2}(g(2n+1))$. By the hypothesis that $h[\omega] \in K_{\chi^\kappa_{\bar{\rho}}}$, $f(2n+1) = \pi^{\kappa,0}(g(2n+1)) \in h[\omega]$. Thus $f[\omega] = \{f(n) : n \in \omega\} = h[\omega]$ which is an honest set by the hypothesis that $h[\omega] \in K_{\chi^\kappa_{\bar{\rho}}}$. By the properties of the generic coding function, $\varphi_0(\mathfrak{G}_n(f)) = f(n)$. Thus $h[\omega] = \{\varphi_0(\mathfrak{G}_n(f)) : n \in \omega\}$. Let $x \in {}^\omega\omega$ be defined by $x(n) = \pi_1^{\kappa,2}(g(2n+1))$. Since $\bar{\rho}$ is a Player 2 winning strategy, $R(\{\varphi_0(\mathfrak{G}_n(f)) : n \in \omega\}, x)$ holds or equivalently $R(h[\omega], x)$. Since $x = \Xi^2_{\tau^\kappa_{\bar{\rho}}}(h)$, one has that $R(h[\omega], \Xi^2_{\tau^\kappa_{\bar{\rho}}}(h))$. This completes the proof. ☐

In the following, one will focus on the supercompact measure on $\mathscr{P}_{\omega_1}(\omega_\omega)$. One will develop first a coding of strategies on ω_ω. The following objects will be fixed for the rest of the discussion concerning ω_ω.

Definition 4.24. Fix a $\mathbf{\Pi}^1_2$ set W and a $\mathbf{\Delta}^1_3$ scale $\vec{\varphi}$ on W of length ω_ω which witnesses the reliability of ω_ω. (This can be obtained by applying the scale property for $\mathbf{\Pi}^1_3$ on some complete $\mathbf{\Pi}^1_2$ set. More explicity, one can let $W = \{x^\sharp : x \in \mathbb{R}\}$ and let $\vec{\varphi}$ be a modification of the sharp scale so that $\varphi_0 : W \to \omega_\omega$ is a surjection.) Let $\prec_n$ denote the prewellordering on W induced by $\varphi_n : W \to \omega_\omega$. Note that $\prec_n \in \mathbf{\Delta}^1_3$ for all $n \in \omega$. Fix a bijection $\pi^{\omega_\omega,<\omega} : \omega_\omega \to {}^{<\omega}(\omega_\omega)$. Fix $U \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ which is universal for $\mathbf{\Sigma}^1_3$ subsets of $\mathbb{R}^2$.

Let scode be the set of $x \in \mathbb{R}$ so that the following holds.

(1) For all $s \in {}^{<\omega}\omega_\omega$, there exist $y, v \in \mathbb{R}$ such that $y \in W$, $\pi^{\omega_\omega,<\omega}(\varphi_0(y)) = s$, and $U(x, y, v)$.
(2) For all $y, z \in W$, for all $v, w \in \mathbb{R}$, if $\varphi_0(y) = \varphi_0(z)$, $U(x, y, v)$, and $U(x, z, w)$, then $v, w \in W$ and $\varphi_0(v) = \varphi_0(w)$.

For any $x \in \mathsf{scode}$, $s \in {}^{<\omega}(\omega_\omega)$, and $\alpha \in \omega_\omega$, let $\rho_x(s) = \alpha$ if and only if there is a $y \in W$ and $v \in W$ so that $\pi^{\omega_\omega,<\omega}(\varphi_0(y)) = s$, $\varphi_0(v) = \alpha$, and $U(x, y, v)$. By the two properties of $x \in \mathsf{scode}$, ρ_x is a well-defined function from ${}^{<\omega}(\omega_\omega)$ into ω_ω (that is, ρ_x is a strategy on ω_ω).

Let scode^* be the set of $x \in \mathbb{R}$ so that the following holds.

(a) $x \in \mathsf{scode}$.
(b) For all $s \in {}^{<\omega}(\omega_\omega)$ so that $|s|$ is odd, for all $v \in \mathbb{R}$, if $U(x, y, v)$, then $\pi_1^{\omega_\omega,2}(\varphi_0(v)) \in \omega$.

Note that if $x \in \mathsf{scode}^*$, then $\Xi^2_{\tau^\kappa_\rho} : {}^\omega\kappa \to {}^\omega\omega$.

Fact 4.25. *For all strategies $\rho : {}^{<\omega}(\omega_\omega) \to \omega_\omega$, there is an $x \in \mathsf{scode}$ so that $\rho = \rho_x$.*

Proof. Define $R \subseteq W \times W$ by $R(y,v)$ if and only if $\rho(\pi^{\omega_\omega,<\omega}(\varphi_0(y))) = \varphi_0(v)$. Applying the Moschovakis coding lemma to the pointclass $\mathbf{\Sigma}^1_3$ with the prewellordering φ_0, there is an $S \subseteq R$ with $S \in \mathbf{\Sigma}^1_3$ so that for all $\beta \in \omega_\omega$, there exists a $y \in W$ with $\varphi_0(y) = \beta$ and $v \in \mathbb{R}$ so that $S(y,v)$. Since $\pi^{\omega_\omega,<\omega} : \omega_\omega \to {}^{<\omega}(\omega_\omega)$ is a bijection, this can be expressed also as: for all $s \in {}^{<\omega}(\omega_\omega)$, there exist $y \in W$ and $v \in \mathbb{R}$ so that $\pi^{\omega_\omega,<\omega}(\varphi_0(y)) = s$, $S(y,v)$. Since $U \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is universal for $\mathbf{\Sigma}^1_3$ subsets of $\mathbb{R}^2$, there is some $x \in \mathbb{R}$ so that $U_x = S$. By the previous observation and the fact that $U_x = S \subseteq R$, one has properties (1) and (2) of Definition 4.24 and that $\rho_x = \rho$. ☐

One will need to make several complexity computations in order to use the Kunen-Martin theorem to bound the ultrapower $j_{\nu_{\omega_\omega}}$. The closure of $\mathbf{\Delta}^1_4$, $\mathbf{\Sigma}^1_4$, and $\mathbf{\Pi}^1_4$ under ω_ω-length unions will be helpful in making several complexity computations. This result, due to Harrington and Kechris, has analogs for other scaled pointclasses. For the results here, one can make even better complexity calculations using the Kechris-Martin theorem ([15] Corollary 1.6) to show $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_3$ are closed under ω_ω-length unions and intersections. Jackson has extended the Kechris-Martin theorem throughout the projective hierarchy using the description theory ([14] Section 4.4). However, these arguments are not known to generalize much further.

Fact 4.26. *(Harrington-Kechris;* [11] *Corollary 2.2) Assume* AD*. For all $n \in \omega$, for all $\kappa < \boldsymbol{\delta}^1_n$, $\mathbf{\Pi}^1_{n+1}$, $\mathbf{\Sigma}^1_{n+1}$, and $\mathbf{\Delta}^1_{n+1}$ are closed under κ-length union. In particular, $\mathbf{\Pi}^1_4$, $\mathbf{\Sigma}^1_4$, and $\mathbf{\Delta}^1_4$ are closed under ω_ω-length unions.*

Proof. The last statement follows from the first using $n = 3$ and the fact that $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$. ☐

Fact 4.27. *(Martin, Moschovakis;* [16] *Theorem 8.4) Assume* AD*. For all $n \in \omega$, $\mathbf{\Delta}^1_{2n+1}$ is closed under κ-length unions and intersections for all $\kappa < \boldsymbol{\delta}^1_{2n+1}$. In particular, $\mathbf{\Delta}^1_3$ is closed under ω_ω-length unions and intersections.*

Fact 4.28. *Assume* AD*.* scode *and* scode^* *are $\mathbf{\Delta}^1_4$.*

Proof. For each $s \in {}^{<\omega}(\omega_\omega)$, let A_s be the set $x \in \mathbb{R}$ so that there exist $y, v \in \mathbb{R}$ so that $y \in W$, $\varphi_0(y) = (\pi^{\omega_\omega,<\omega})^{-1}(s)$, and $U(x,y,v)$. Note that A_s is $\mathbf{\Sigma}^1_3$ since W is $\mathbf{\Pi}^1_2$, φ_0 is a Δ^1_3-norm, and U is $\mathbf{\Sigma}^1_3$. In particular, A_s is $\mathbf{\Delta}^1_4$. Let $A = \bigcap\{A_s : s \in {}^{<\omega}(\omega_\omega)\}$ which is $\mathbf{\Delta}^1_4$ since $\mathbf{\Delta}^1_4$ is closed under ω_ω-length intersection by Fact 4.26. (A is actually $\mathbf{\Sigma}^1_3$ since $\mathbf{\Sigma}^1_3$ is closed under ω_ω-length intersections by the Kechris-Martin theorem.) Note that A is the set of $x \in \mathbb{R}$ which satisfies Definition 4.24 property (1). Let B be the set of x which satisfies Definition 4.24 property (2). Since $W \in \mathbf{\Pi}^1_2$, $U \in \mathbf{\Sigma}^1_3$, and φ_0 is a Δ^1_3 norm, one has that B is $\mathbf{\Pi}^1_3$. Since $\mathsf{scode} = A \cap B$, $\mathsf{scode} \in \mathbf{\Delta}^1_4$.

Let $X = \{\alpha \in \omega_\omega : \pi^{\omega_\omega,2}_1(\alpha) \in \omega\}$. For each $\alpha \in X$ and $s \in {}^{<\omega}(\omega_\omega)$ with $|s|$ odd, let $C_{\alpha,s}$ be the set of x so that for all $y, v \in \mathbb{R}$, if $v \in W$, $\varphi_0(y) = (\pi^{\omega_\omega,<\omega})^{-1}(s)$, and $U(x,y,v)$, then $\varphi_0(v) = \alpha$. Note that $C_{\alpha,s}$ is $\mathbf{\Pi}^1_3$. Let $C = \bigcap\{\bigcup\{C_{\alpha,s} : \alpha \in X\} : s \in {}^{<\omega}(\omega_\omega) \wedge |s| \text{ is odd}\}$. Since $\mathbf{\Delta}^1_4$ is closed under ω_ω-length intersections and unions, $C \in \mathbf{\Delta}^1_4$. Since $\mathsf{scode}^* = \mathsf{scode} \cap C$, scode^* is $\mathbf{\Delta}^1_4$. ☐

Lemma 4.29. *Assume* AD*.*

(1) *Let* $\mathsf{String} \subseteq \omega \times \mathbb{R} \times \mathbb{R}$ *be defined by* $\mathsf{String}(n,r,y)$ *if and only if $y \in W$, for all $m < n$, $r^{[m]} \in W$, and $\pi^{\omega_\omega,<\omega}(\varphi_0(y)) = \langle \varphi_0(r^{[0]}), ..., \varphi_0(r^{[n-1]})\rangle$ (that is, $\pi^{\omega_\omega,<\omega}(\varphi_0(y))$ is the length n-string $\langle \varphi_0(r^{[0]}), ..., \varphi_0(r^{[n-1]})\rangle$).* String *is $\mathbf{\Delta}^1_3$.*

(2) *Let* $\mathsf{IntPart} \subseteq \mathbb{R} \times \omega$ *be defined by* $\mathsf{IntPart}(v,n)$ *if and only if $v \in W$ and $\pi^{\omega_\omega,2}_1(\varphi_0(v)) = n$.* $\mathsf{IntPart} \in \mathbf{\Delta}^1_3$.

(3) *Let* $\mathsf{ONPart} \subseteq \mathbb{R} \times \mathbb{R}$ *be defined by* $\mathsf{ONPart}(v,w)$ *if and only if $v \in W$ and $\pi^{\omega_\omega,2}_0(\varphi_0(v)) = \varphi_0(w)$.* $\mathsf{ONPart} \in \mathbf{\Delta}^1_3$.

(4) *There is a Δ^1_3 relation* $\mathsf{NormCompare} \subseteq \omega \times \omega \times \mathbb{R} \times \mathbb{R}$ *so that for all $m, n \in \omega$ and $v, w \in \mathbb{R}$,* $\mathsf{NormCompare}(m,n,v,w)$ *if and only if $v, w \in W$ and $\varphi_m(v) = \varphi_n(w)$ (where $\vec{\varphi} = \langle \varphi_n : n \in \omega\rangle$ come from the fixed reliability witness).*

(5) *There is a $\mathbf{\Sigma}^1_3$ set* $\mathsf{Honest} \subseteq \mathbb{R}$ *so that* $\mathsf{Honest}(r)$ *if and only if for all $n \in \omega$, $r^{[n]} \in W$ and $\{\varphi_0(r^{[n]}) : n \in \omega\}$ is honest relative to the reliability witness $\vec{\varphi}$.*

(6) *There is a $\mathbf{\Sigma}^1_3$ relation* $\mathsf{Run}_{\mathbf{\Sigma}^1_3} \subseteq \mathbb{R} \times \mathbb{R}$ *and a $\mathbf{\Pi}^1_3$ relation* $\mathsf{Run}_{\mathbf{\Pi}^1_3}$ *so that if $x \in$* scode*, then* $\mathsf{Run}_{\mathbf{\Sigma}^1_3}(x,r)$ *if and only if* $\mathsf{Run}_{\mathbf{\Pi}^1_3}(x,r)$ *if and only if $\langle \varphi_0(r^{[n]}) : n \in \omega\rangle$ is a run according to ρ_x used as a strategy for Player 2.*

(7) *There is a* $\mathbf{\Sigma}^1_3$ *relation* $\mathsf{Closed}_{\mathbf{\Sigma}^1_3} \subseteq \mathbb{R}\times\mathbb{R}$ *and* $\mathbf{\Pi}^1_3$ *relation* $\mathsf{Closed}_{\mathbf{\Pi}^1_3} \subseteq \mathbb{R}\times\mathbb{R}$ *with the property that whenever* $x \in \mathsf{scode}$, $\mathsf{Closed}_{\mathbf{\Sigma}^1_3}(x,r)$ *if and only if* $\mathsf{Closed}_{\mathbf{\Pi}^1_3}(x,r)$ *if and only if for all* $n\in\omega$, $r^{[n]}\in W$ *and for all for all* $s \in {}^{<\omega}(\{\varphi_0(r^{[n]}) : n\in\omega\})$, $\rho_x(s) \in \{\varphi_0(r^{[n]}) : n \in \omega\}$.

(8) *There is a* $\mathbf{\Sigma}^1_3$ *relation* $\mathsf{fClosed}_{\mathbf{\Sigma}^1_3} \subseteq \mathbb{R}\times\mathbb{R}$ *and* $\mathbf{\Pi}^1_3$ *relation* $\mathsf{fClosure}_{\mathbf{\Pi}^1_3} \subseteq \mathbb{R}\times\mathbb{R}$ *with the property that whenever* $x\in\mathsf{scode}$, $\mathsf{fClosed}_{\mathbf{\Sigma}^1_3}(x,r)$ *if and only if* $\mathsf{fClosed}_{\mathbf{\Pi}^1_3}(x,r)$ *if and only if for all* $n\in\omega$, $r^{[n]}\in W$ *and for all* $s\in {}^{<\omega}(\{\varphi_0(r^{[n]}) : n\in\omega\})$, $\chi^{\omega_\omega}_{\rho_x}(s) \in \{\varphi_0(r^{[n]}) : n\in\omega\}$.

Proof.

(1) For each $s\in {}^{<\omega}(\omega_\omega)$, let A_s be the set of $(|s|, r, y)$ such that $y\in W$, $\varphi_0(y) = (\pi^{\omega_\omega,<\omega})^{-1}(s)$, and for all $m<n$, $r^{[m]}\in W$ and $\varphi_0(r^{[m]}) = s(m)$. Note that $A_s \in \mathbf{\Delta}^1_3$ and $\mathsf{String} = \bigcup\{A_s : s \in {}^{<\omega}(\omega_\omega)\}$. $\mathsf{String}\in\mathbf{\Delta}^1_3$ since $\mathbf{\Delta}^1_3$ is closed under ω_ω-length unions by Fact 4.27.

(2) For each $\alpha\in\omega_\omega$ and $n\in\omega$, let $V_{\alpha,n} = \{(v,n) : v\in W \wedge \varphi_0(v) = (\pi^{\omega_\omega,2})^{-1}((\alpha,n))\}$. Since φ_0 is a $\mathbf{\Delta}^1_3$-norm, $V_{\alpha,n}\in\mathbf{\Delta}^1_3$. Then $\mathsf{IntPart} = \bigcup\{V_{\alpha,n} : \alpha\in\omega_\omega\wedge n\in\omega\}$ which is $\mathbf{\Delta}^1_3$ since $\mathbf{\Delta}^1_3$ is closed under ω_ω-length unions.

(3) For each $\alpha,\beta<\omega_\omega$, let $(v,w)\in A_{\alpha,\beta}$ if and only if $\varphi_0(v) = \pi^{\omega_\omega,2}(\alpha,\beta)$ and $\beta = \varphi_0(w)$. $A_{\alpha,\beta}$ is Δ^1_3. $\mathsf{ONPart} = \bigcup\{A_{\alpha,\beta} : \alpha,\beta<\omega_\omega\}$ which is $\mathbf{\Delta}^1_3$ since $\mathbf{\Delta}^1_3$ is closed under ω_ω-length unions.

(4) Let $m,n\in\omega$ and $\alpha<\omega_\omega$. If α is greater than or equal to the rank of either φ_m or φ_n, then let $A_{m,n,\alpha} = \emptyset$. If α less than the rank of both φ_m and φ_n, then let $A_{m,n,\alpha} = \{(m,n,v,w) : \varphi_m(v) = \alpha\wedge\varphi_n(w) = \alpha\}$. $A_{m,n,\alpha}$ is $\mathbf{\Delta}^1_3$ since all the norms in $\vec{\varphi}$ are $\mathbf{\Delta}^1_3$ norms. Then $\mathsf{NormCompare} = \bigcup\{A_{m,n,\alpha} : m,n\in\omega\wedge\alpha<\omega_\omega\}$ which is $\mathbf{\Delta}^1_3$ since $\mathbf{\Delta}^1_3$ is closed under ω_ω-length unions.

(5) Note that $r\in\mathsf{Honest}$ if and only if for all $n\in\omega$, there exists $w\in W$ so that $\varphi_0(w) = \varphi_0(r^{[n]})$ and for all $k\in\omega$, there exists $j\in\omega$ such that $\mathsf{NormCompare}(0,k,r^{[j]},w)$. Since $\mathsf{NormCompare}$ is $\mathbf{\Delta}^1_3$, Honest is $\mathbf{\Sigma}^1_3$.

(6) Let $\mathsf{Run}_{\mathbf{\Sigma}^1_3}(x,r)$ if and only if for all $n\in\omega$, $r^{[n]}\in W$ and there exist $y,v\in\mathbb{R}$ so that $\mathsf{String}(2n{+}1,r,y)$, $U(x,y,v)$, and $\varphi_0(v) = \varphi_0(r^{[2n+1]})$. $\mathsf{Run}_{\mathbf{\Sigma}^1_3}$ is $\mathbf{\Sigma}^1_3$ and if $x\in\mathsf{scode}$, then $\mathsf{Run}_{\mathbf{\Sigma}^1_3}(x,r)$ has the intended meaning stated above.

Let $\mathsf{Run}_{\mathbf{\Pi}^1_3}(x,r)$ if and only if for all $n\in\omega$, $r^{[n]}\in W$ and for all $y,v\in\mathbb{R}$, if $\mathsf{String}(2n+1,r,y)$ and $U(x,y,v)$, then $\varphi_0(v) = \varphi_0(r^{[2n+1]})$. $\mathsf{Run}_{\mathbf{\Pi}^1_3}$ is $\mathbf{\Pi}^1_3$ and if $x\in\mathsf{scode}$, then $\mathsf{Run}_{\mathbf{\Pi}^1_3}(x,r)$ has the intended meaning.

(7) This is a similar and simpler than the argument shown next for (8).

(8) Define $\mathsf{fClosed}_{\mathbf{\Pi}^1_3}(x,r)$ if and only if the conjunction of the following holds.
- For all $n\in\omega$, $r^{[n]}\in W$.
- For all $n\in\omega$, for all $t,y,v,v_0\in\mathbb{R}$, if the conjunction of the following holds:
  - For all $k<n$, there exists $i\in\omega$, $\varphi_0(t^{[k]}) = \varphi_0(r^{[i]})$
  - $\mathsf{String}(n,t,y)$.
  - $U(x,y,v)$
  - $\mathsf{ONPart}(v,v_0)$.

  then there exists a $j\in\omega$, $\varphi_0(v_0) = \varphi_0(r^{[j]})$.

Note that $\mathsf{fClosed}_{\mathbf{\Pi}^1_3}\in\mathbf{\Pi}^1_3$.

Define $\mathsf{fClosed}_{\mathbf{\Sigma}^1_3}(x,r)$ if and only if the conjunction of the following holds.
- For all $n\in\omega$, $r^{[n]}\in W$.
- For all $n\in\omega$ and function $\ell : n\to\omega$, there exist $j\in\omega$ and $t,y,v,v_0\in\mathbb{R}$ so that the conjunction of the following holds.
  - For all $k<n$, $t^{[k]} = r^{[\ell(k)]}$.
  - $\mathsf{String}(n,t,y)$.
  - $U(x,y,v)$
  - $\mathsf{ONPart}(v,v_0)$.
  - $\varphi_0(v_0) = \varphi_0(r^{[j]})$.

Note that $\mathrm{fClosed}_{\mathbf{\Sigma}^1_3}$ is $\mathbf{\Sigma}^1_3$.

If $x\in\mathsf{scode}$, then $\mathsf{fClosed}_{\mathbf{\Sigma}^1_3}$ and $\mathsf{fClosed}_{\mathbf{\Pi}^1_3}$ have the intended meanings.

□

Fact 4.30. *Assume* AD*. Suppose* $x \in \mathsf{scode}^*$*. Let* A *be the set of* $f \in {}^\omega(\omega_\omega)$ *so that* $f[\omega] \in K_{\chi^{\omega_\omega}_{\rho_x}}$*. Then* $\Xi^2_{\tau^{\omega_\omega}_{\rho_x}}[A]$ *is* $\mathbf{\Sigma}^1_3$ *(note that since* $x \in \mathsf{scode}^*$*,* $\Xi^2_{\tau^{\omega_\omega}_{\rho_x}}[A]$ *is a set of reals).*

Proof. Observe that $u \in \Xi^2_{\tau^{\omega_\omega}_{\rho_x}}[A]$ if and only if there exist $r, t \in \mathbb{R}$ so that the conjunction of the following holds

- $\mathsf{fClosed}_{\mathbf{\Sigma}^1_3}(x, r)$
- $\mathsf{Honest}(r)$.
- For all $n \in \omega$, $t^{[2n]} = r^{[n]}$.
- $\mathsf{Run}_{\mathbf{\Sigma}^1_3}(x, t)$.
- For all $n \in \omega$, $\mathsf{IntPart}(t^{[2n+1]}, u(n))$.

The above expression is $\mathbf{\Sigma}^1_3$ and it works because $x \in \mathsf{scode}^*$ (and note that $\mathsf{scode}^* \subseteq \mathsf{scode}$). $\square$

Fact 4.31. *(Steel; [25], [14] Theorem 2.28) Assume* AD *and* $\mathsf{DC}_{\mathbb{R}}$*. If* $\kappa < \Theta$ *is a limit ordinal, then there is a surjective norm* $\psi : P \to \kappa$ *which is* δ*-Suslin bounded for all* $\delta < \mathrm{cof}(\kappa)$*, which means that for all* $A \subseteq P$ *that are* δ*-Suslin,* $\sup(\varphi[A]) < \kappa$*.*

Fact 4.32. *Assume* AD^+*. Let* $\kappa < \Theta$ *with* $\mathrm{cof}(\kappa) > \omega_\omega$*. Let* $\Phi : \mathscr{P}_{\omega_1}(\omega_\omega) \to \kappa$*. Then there is an* $A \in \nu_{\omega_\omega}$ *so that* $\sup(\Phi[A]) < \kappa$*.*

Proof. Fix $\kappa < \Theta$ with $\mathrm{cof}(\kappa) > \omega_\omega$. By Fact 4.31, let $\psi : P \to \kappa$ be a surjective ω_ω-Suslin bounded prewellordering. Fix $\Phi : \mathscr{P}_{\omega_1}(\omega_\omega) \to \kappa$. Let $R \subseteq \mathscr{P}_{\omega_1}(\omega_\omega) \times \mathbb{R}$ be defined by $R(\sigma, p)$ if and only if $\Phi(\sigma) = \psi(p)$. Applying Fact 4.23, there is a strategy ρ so that the following holds:

1. For all odd length $s \in {}^{<\omega}(\omega_\omega)$, $\tau^{\omega_\omega}_\rho(s) \in \omega$.
2. For all $f \in {}^\omega(\omega_\omega)$ so that $f[\omega] \in K_{\chi^{\omega_\omega}_\rho}$,$R(f[\omega], \Xi^2_{\tau^{\omega_\omega}_\rho}(f))$.

By Fact 4.25, there is an $x \in \mathsf{scode}$ so that $\rho_x = \rho$. Moreover, $x \in \mathsf{scode}^*$ by condition (1) above. Let B be the set of $f \in {}^\omega(\omega_\omega)$ so that $f[\omega] \in K_{\chi^{\omega_\omega}_{\rho_x}}$. By condition (2), for any $f \in B$, $R(f[\omega], \Xi^2_{\tau^{\omega_\omega}_{\rho_x}}(f))$ and thus $\Xi^2_{\tau^{\omega_\omega}_{\rho_x}}(f) \in P$ by the definition of R. Thus $\Xi^2_{\tau^{\omega_\omega}_{\rho_x}}[B] \subseteq P$ and $\Xi^2_{\tau^{\omega_\omega}_{\rho_x}}[B]$ is $\mathbf{\Sigma}^1_3$ (and hence ω_ω-Suslin) by Fact 4.30. Since ψ is a ω_ω-Suslin bounded norm, there is a $\delta < \kappa$ so that $\psi[\Xi^2_{\rho_x,1}[B]] \subseteq \delta$. $K_{\chi^{\omega_\omega}_{\rho_x}} \in \nu_{\omega_\omega}$ by Fact 4.15. Let $\sigma \in K_{\chi^{\omega_\omega}_{\rho_x}}$. Let $f : \omega \to \sigma$ be any surjection and thus $f[\omega] = \sigma$. Note that $f \in B$. Therefore by (2), $R(\sigma, \Xi^2_{\tau^{\omega_\omega}_{\rho_x}}(f))$. This means $\Phi(\sigma) = \psi(\Xi^2_{\tau^{\omega_\omega}_{\rho_x}}(f))$. Since $\psi(\Xi^2_{\tau^{\omega_\omega}_{\rho_x}}(f)) \in \Xi^2_{\tau^{\omega_\omega}_{\rho_x}}[B]$, one has that $\psi(\Xi^2_{\tau^{\omega_\omega}_{\rho_x}}(f)) < \delta$. So $\Phi(\sigma) < \delta$. This shows that $\sup(\Phi[K_{\chi^{\omega_\omega}_{\rho_x}}]) \leq \delta < \kappa$. $\square$

Definition 4.33. Let scode^+ consists of those $x \in \mathbb{R}$ so that the following hold.

1. $x \in \mathsf{scode}^*$.
2. For all $f \in {}^\omega(\omega_\omega)$ so that $f[\omega] \in K_{\chi^{\omega_\omega}_{\rho_x}}$, $\Xi^2_{\tau^{\omega_\omega}_{\rho_x}}(f) \in W$ (where recall W is the underlying set of norms that form the reliability witness $\vec{\varphi}$).
3. For all $f_0, f_1 \in {}^\omega(\omega_\omega)$ so that $f_0[\omega], f_1[\omega] \in K_{\chi^{\omega_\omega}_{\rho_x}}$ and $f_0[\omega] = f_1[\omega]$, then $\varphi_0(\Xi^2_{\tau^{\omega_\omega}_{\rho_x}}(f_0)) = \varphi_0(\Xi^2_{\tau^{\omega_\omega}_{\rho_x}}(f_1))$.

If $x \in \mathsf{scode}^+$, then let $\Phi_x : K_{\chi^{\omega_\omega}_{\rho_x}} \to \omega_\omega$ be defined by $\Phi_x(\sigma) = \varphi_0(\Xi^2_{\tau^{\omega_\omega}_{\rho_x}}(f))$ for any $f : \omega \to \sigma$ which is a surjection. The conditions of the definition of scode^+ imply that Φ_x is a well-defined function independent of the choice of f which surjects onto σ.

Fact 4.34. *Assume* AD^+*. For any* $\Phi : \mathscr{P}_{\omega_1}(\omega_\omega) \to \omega_\omega$*, there is an* $x \in \mathsf{scode}^+$ *so that* $[\Phi]_{\nu_{\omega_\omega}} = [\Phi_x]_{\nu_{\omega_\omega}}$*.*

Proof. This was shown in the proof of Fact 4.32. (Replace the $\psi : P \to \kappa$ of the proof of Fact 4.32 with $\varphi_0 : W \to \omega_\omega$.) (Moreover, if one inspects the payoff set for Player 2 in the game H_R for the relevant relation R from Fact 4.32, one can even strengthen Definition 4.33 condition (2) to say that for all $f \in {}^\omega(\omega_\omega)$, $\Xi^2_{\tau^{\omega_\omega}_{\rho_x}}(f) \in W$.) $\square$

Fact 4.35. *Assume* AD*.* scode^+ *is* $\mathbf{\Delta}^1_4$*.*

Proof. Note that $x \in \mathsf{scode}^+$ if and only if the conjunction of the following hold.

- $x \in \mathsf{scode}^*$.
- For all $r, t, u \in \mathbb{R}$, if the conjunction of the following hold:

– $\mathsf{Honest}(r)$.
– $\mathsf{fClosed}_{\mathbf{\Sigma}^1_3}(x, r)$.
– For all $n \in \omega$, $t^{[2n]} = r^{[n]}$.
– For all $n \in \omega$, $\mathsf{IntPart}(t^{[2n+1]}, u(n))$
– $\mathsf{Run}_{\mathbf{\Sigma}^1_3}(x, t)$,

then $u \in W$.

- For all $r_0, t_0, u_0, r_1, t_1, u_1 \in \mathbb{R}$, if the conjunction of the following hold:
  – $\mathsf{Honest}(r_0)$ and $\mathsf{Honest}(r_1)$.
  – $\mathsf{fClosed}_{\mathbf{\Sigma}^1_3}(x, r_0)$. $\mathsf{fClosed}_{\mathbf{\Sigma}^1_3}(x, r_1)$.
  – For all $n \in \omega$, $(t_0)^{[2n]} = (r_0)^{[n]}$ and $(t_1)^{[2n]} = (r_1)^{[n]}$.
  – For all $n \in \omega$, $\mathsf{IntPart}((t_0)^{[2n+1]}, u_0(n))$ and $\mathsf{IntPart}((t_0)^{[2n+1]}, u_0(n))$.
  – $\mathsf{Run}_{\mathbf{\Sigma}^1_3}(x, t_0)$ and $\mathsf{Run}_{\mathbf{\Sigma}^1_3}(x, t_1)$,
  – For all $m \in \omega$, there exists $n \in \omega$ so that $\varphi_0((r_0)^{[m]}) = \varphi_0((r_1)^{[n]})$. For all $m \in \omega$, there exists $n \in \omega$ so that $\varphi_0((r_1)^{[m]}) = \varphi_0((r_0)^{[n]})$.

  then $\varphi_0(u_0) = \varphi_0(u_1)$.

The first point is $\mathbf{\Delta}^1_4$ since $\mathsf{scode}^* \in \mathbf{\Delta}^1_4$. The second and third points are $\mathbf{\Pi}^1_3$. The entire expression is $\mathbf{\Delta}^1_4$. □

Fact 4.36. *(Kunen-Martin Theorem) Assume $\mathsf{AC}^{\mathbb{R}}_\omega$. Every κ-Suslin wellfounded relation on $\mathbb{R}$ has length less than κ^+.*

Fact 4.37. *(Becker; [1] Theorem 4.2) Assume AD^+. Let $\alpha < \boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ and ν_α be the unique supercompact measure on $\mathscr{P}_{\omega_1}(\alpha)$. Then $j_{\nu_\alpha}(\boldsymbol{\delta}^1_4) = j_{\nu_\alpha}(\omega_{\omega+2}) = \boldsymbol{\delta}^1_4 = \omega_{\omega+2}$.*

Proof. Note that these ultrapowers are wellfounded by Fact 4.22. For all $\alpha < \boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, ν_α is Rudin-Keisler reducible to ν_{ω_ω} by Fact 4.20 and therefore $j_{\nu_\alpha}(\boldsymbol{\delta}^1_4) \leq j_{\nu_{\omega_\omega}}(\boldsymbol{\delta}^1_4)$. Thus it suffices to show that $j_{\nu_{\omega_\omega}}(\boldsymbol{\delta}^1_4) = \boldsymbol{\delta}^1_4$.

The representatives of ordinals below $j_{\nu_{\omega_\omega}}(\boldsymbol{\delta}^1_4)$ are functions of the form $\Phi : \mathscr{P}_{\omega_1}(\omega_\omega) \to \boldsymbol{\delta}^1_4$. Since $\boldsymbol{\delta}^1_4$ is regular, Fact 4.32 implies that Φ is ν_{ω_ω}-almost equal to a function which is strictly bounded below $\boldsymbol{\delta}^1_4$. Thus $j_{\nu_{\omega_\omega}}(\boldsymbol{\delta}^1_4) = \sup\{j_{\nu_{\omega_\omega}}(\beta) : \beta < \boldsymbol{\delta}^1_4\}$. To prove the theorem, it suffices to show that $j_{\nu_{\omega_\omega}}(\beta) < \boldsymbol{\delta}^1_4$ for all $\beta < \boldsymbol{\delta}^1_4$.

Let $\beta < \boldsymbol{\delta}^1_4 = \omega_{\omega+2}$. Since $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, let $\psi_\beta : \boldsymbol{\delta}^1_3 \to \beta$ be a surjection. For each $\Phi : \mathscr{P}_{\omega_1}(\omega_\omega) \to \boldsymbol{\delta}^1_3$, let $\tilde{\Phi} : \mathscr{P}_{\omega_1}(\omega_\omega) \to \beta$ be defined by $\tilde{\Phi}(\sigma) = \psi(\Phi(\sigma))$. For every $\Upsilon : \mathscr{P}_{\omega_1}(\omega_\omega) \to \beta$, there is a $\Phi : \mathscr{P}_{\omega_1}(\omega_\omega) \to \boldsymbol{\delta}^1_3$ so that $\tilde{\Phi} = \Upsilon$. Thus $\Psi : j_{\nu_{\omega_\omega}}(\boldsymbol{\delta}^1_3) \to j_{\nu_{\omega_\omega}}(\beta)$ defined by $\Psi([\Phi]_{\nu_{\omega_\omega}}) = [\tilde{\Phi}]_{\nu_{\omega_\omega}}$ for any $\Phi : \mathscr{P}_{\omega_1}(\omega_\omega) \to \boldsymbol{\delta}^1_3$ is a well-defined surjection. Since $\boldsymbol{\delta}^1_4$ is a cardinal, it suffices to show that $j_{\nu_{\omega_\omega}}(\boldsymbol{\delta}^1_3) < \boldsymbol{\delta}^1_4$.

Since $\boldsymbol{\delta}^1_3$ is regular, Fact 4.32 again implies $j_{\nu_{\omega_\omega}}(\boldsymbol{\delta}^1_3) = \sup\{j_{\nu_{\omega_\omega}}(\gamma) : \gamma < \boldsymbol{\delta}^1_3\}$. Since $\boldsymbol{\delta}^1_4$ is regular, it suffices to show that $j_{\nu_{\omega_\omega}}(\gamma) < \boldsymbol{\delta}^1_4$ for all $\gamma < \boldsymbol{\delta}^1_3$. Since $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, the same argument from the previous paragraph shows that $j_{\nu_{\omega_\omega}}(\omega_\omega)$ surjects onto $j_{\nu_{\omega_\omega}}(\gamma)$ for all $\gamma < \boldsymbol{\delta}^1_3$. Finally, it has been shown that to prove the theorem it suffices to show $j_{\nu_{\omega_\omega}}(\omega_\omega) < \boldsymbol{\delta}^1_4$.

Define a relation $\mathsf{compare} \subseteq \mathbb{R} \times \mathbb{R}$ as follows: $\mathsf{compare}(x, y)$ if and only there exists a $z \in \mathbb{R}$ such that the conjunction of the following hold.

(1) $x, y \in \mathsf{scode}^+$ and $z \in \mathsf{scode}$.
(2) For all $r, t_0, t_1, u_0, u_1 \in \mathbb{R}$, if the conjunction of the following hold:
  - $\mathsf{Honest}(r)$.
  - $\mathsf{Closed}_{\mathbf{\Sigma}^1_3}(z, r)$, $\mathsf{fClosed}_{\mathbf{\Sigma}^1_3}(x, r)$, and $\mathsf{fClosed}_{\mathbf{\Sigma}^1_3}(y, r)$.
  - For all $n \in \omega$, $(t_0)^{[2n]} = (t_1)^{[2n]} = r^{[n]}$.
  - For all $n \in \omega$, $\mathsf{IntPart}((t_0)^{[2n+1]}, u_0(n))$ and $\mathsf{IntPart}((t_1)^{[2n+1]}, u_1(n))$.
  - $\mathsf{Run}_{\mathbf{\Sigma}^1_3}(x, t_0)$ and $\mathsf{Run}_{\mathbf{\Sigma}^1_3}(y, t_1)$.

  then $\varphi_0(u_0) < \varphi_0(u_1)$.

Observe that (1) is $\mathbf{\Delta}^1_4$ and (2) is $\mathbf{\Pi}^1_3$. Thus $\mathsf{compare}$ is $\mathbf{\Sigma}^1_4$.

Claim 1: $\mathsf{compare}(x, y)$ if and only if $x, y \in \mathsf{scode}^+$ and $[\Phi_x]_{\nu_{\omega_\omega}} < [\Phi_y]_{\nu_{\omega_\omega}}$.

To see Claim: ($\Rightarrow$) Let z witness the existential quantifier in $\mathsf{compare}(x, y)$. Note $K_{\chi^{\omega_\omega}_{\rho_x}} \cap K_{\chi^{\omega_\omega}_{\rho_y}} \cap K_{\rho_z} \in \nu_{\omega_\omega}$. Let $\sigma \in K_{\chi^{\omega_\omega}_{\rho_x}} \cap K_{\chi^{\omega_\omega}_{\rho_y}} \cap K_{\rho_z}$. By definition, this means that σ is honest and closed under $\chi^{\omega_\omega}_{\rho_x}$, $\chi^{\omega_\omega}_{\rho_x}$, and ρ_z.

Let $f : \omega \to \sigma$ be any surjection. Let $g_x = \rho_f^1 * \rho_x$ and $g_y = \rho_f^1 * \rho_y$. Let r, t_0, t_1 be such that for all $n \in \omega$, $\varphi_0(r^{[n]}) = f(n)$, $r^{[n]} = (t_0)^{[2n]}$, $r^{[n]} = (t_1)^{[2n]}$, $\varphi_0((t_0)^{[n]}) = g_x(n)$, and $\varphi_0((t_1)^{[n]}) = g_y(n)$. For all $n \in \omega$, let $u_0(n) = \pi_1^{\omega_\omega,2}(\varphi_0((t_0)^{[2n+1]}))$ and $u_1(n) = \pi_1^{\omega_\omega,2}(\varphi_0((t_1)^{[2n+1]}))$. r, t_0, t_1, u_0, u_1 satisfy the hypothesis of the conditional in statement (2). Thus $\varphi_0(u_0) < \varphi_0(u_1)$. Since $u_0 = \Xi^2_{\tau_{\rho_x}^{\omega_\omega}}(f)$ and $u_1 = \Xi^2_{\tau_{\rho_y}^{\omega_\omega}}(f)$, one has that $\Phi_x(\sigma) = \varphi_0(u_0) < \varphi_0(u_1) = \Phi_y(\sigma)$ by definition. Since $\sigma \in K_{\chi_{\rho_x}^{\omega_\omega}} \cap K_{\chi_{\rho_y}^{\omega_\omega}} \cap K_{\rho_z} \in \nu_{\omega_\omega}$ was arbitrary, this shows that $[\Phi_x]_{\nu_{\omega_\omega}} < [\Phi_y]_{\nu_{\omega_\omega}}$.

($\Leftarrow$) Suppose $[\Phi_x]_{\nu_{\omega_\omega}} < [\Phi_y]_{\nu_{\omega_\omega}}$. The set $A = \{\sigma \in \mathscr{P}_{\omega_1}(\omega_\omega) : \Phi_x(\sigma) < \Phi_y(\sigma)\} \in \nu_{\omega_\omega}$. By Fact 4.16, there is a strategy ρ so that $K_\rho \subseteq A$. By Fact 4.25, there is a $z \in \mathsf{scode}$ so that $\rho_z = \rho$. By much of the same argument as before, z witnesses the existential to show that $\mathsf{compare}(x, y)$ holds. This establishes the claim.

Define an equivalence relation $\sim$ on scode^+ by $x \sim y$ if and only if $[\Phi_x]_{\nu_{\omega_\omega}} = [\Phi_y]_{\nu_{\omega_\omega}}$. Let $H = \mathsf{scode}^+/\sim$ be the set of equivalence classes of $\sim$. For $X, Y \in H$, define $X < Y$ if and only if for any $x \in X$ and $y \in Y$, $[\Phi_x]_{\nu_{\omega_\omega}} < [\Phi_y]_{\nu_{\omega_\omega}}$. Observe that $(H, <)$ order embeds into $j_{\nu_{\omega_\omega}}(\omega_\omega)$ by the well-defined map $\Lambda(X) = [\Phi_x]_{\nu_{\omega_\omega}}$ for any $x \in X$. This shows that $(H, <)$ is a wellordering. Hence by using the claim, $\mathsf{compare}$ is a wellfounded relation whose length corresponds to the ordertype of $(H, <)$. By Fact 4.34, every $\Phi : \mathscr{P}_{\omega_1}(\omega_\omega) \to (\omega_\omega)$ has an $x \in \mathsf{scode}^+$ so that $[\Phi]_{\nu_{\omega_\omega}} = [\Phi_x]_{\nu_{\omega_\omega}}$. This shows that the ordertype of $(H, <)$ is exactly $j_{\nu_{\omega_\omega}}(\omega_\omega)$. Hence the length of $\mathsf{compare}$ is exactly $j_{\nu_{\omega_\omega}}(\omega_\omega)$. Since $\mathsf{compare}$ is a wellfounded $\mathbf{\Sigma}^1_4$ and hence $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ Suslin relation, the Kunen-Martin theorem states that the length of $\mathsf{compare}$ is less than $(\boldsymbol{\delta}^1_3)^+ = (\omega_{\omega+1})^+ = \omega_{\omega+2} = \boldsymbol{\delta}^1_4$. Thus $j_{\nu_{\omega_\omega}}(\omega_\omega) < \boldsymbol{\delta}^1_4$. This completes the proof. $\square$

Theorem 4.38. *Assume* AD^+. *Let* $\langle A_\alpha : \alpha < \boldsymbol{\delta}^1_3\rangle$ *be such that* $\bigcup_{\alpha<\boldsymbol{\delta}^1_3} A_\alpha = \mathscr{P}(\boldsymbol{\delta}^1_4)$. *Then there is an* $\alpha < \boldsymbol{\delta}^1_3$ *so that* $\neg(|A_\alpha| \leq |^{<\boldsymbol{\delta}^1_4}\boldsymbol{\delta}^1_4|)$.

Proof. Suppose $\mathscr{P}(\boldsymbol{\delta}^1_4) = \bigcup_{\alpha<\boldsymbol{\delta}^1_3} A_\alpha$ and $|A_\alpha| \leq |^{<\boldsymbol{\delta}^1_4}\boldsymbol{\delta}^1_4|$ for all $\alpha < \boldsymbol{\delta}^1_3$. $\boldsymbol{\delta}^1_3$ is a Suslin cardinal and hence reliable. By Fact 4.37, the hypothesis of Theorem 4.11 holds. Thus $|\mathscr{P}(\boldsymbol{\delta}^1_4)| = |\bigcup_{\alpha<\boldsymbol{\delta}^1_3} A_\alpha| \leq |^{<\boldsymbol{\delta}^1_4}\boldsymbol{\delta}^1_4|$. $\boldsymbol{\delta}^1_4$ is a weak partition cardinal and hence a measurable cardinal. Thus $\boldsymbol{\delta}^1_4$ does not inject into $\mathscr{P}(\gamma)$ for any $\gamma < \boldsymbol{\delta}^1_4$. So $|^{<\boldsymbol{\delta}^1_4}\boldsymbol{\delta}^1_4| < |\mathscr{P}(\boldsymbol{\delta}^1_4)|$ by Fact 2.9. This is a contradiction. $\square$

This argument can be generalized to the suitable analog at higher projective ordinals.

Theorem 4.39. *Assume* AD^+. *Let* $n \in \omega$. *Let* $\langle A_\alpha : \alpha < \boldsymbol{\delta}^1_{2n+1}\rangle$ *be such that* $\bigcup_{\alpha<\boldsymbol{\delta}^1_{2n+1}} A_\alpha = \mathscr{P}(\boldsymbol{\delta}^1_{2n+2})$. *Then there is an* $\alpha < \boldsymbol{\delta}^1_{2n+1}$ *so that* $\neg(|A_\alpha| \leq |^{<\boldsymbol{\delta}^1_{2n+2}}\boldsymbol{\delta}^1_{2n+2}|)$.

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