A WADGE DETERMINACY PRINCIPLE EQUIVALENT TO WEAK KÖNIG LEMMA

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Abstract. $\mathsf{RCA_0}$ is a weak subsystem of second order arithmetic which will serve as the base theory. $\mathsf{WKL_0}$ is the subsystem of second order arithmetic corresponding to the Weak König Lemma which states that every infinite tree on $\{0,1\}$ has an infinite path. $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-$\mathsf{WDET}^2$ is the principle asserting that one of the two players has a winning strategy in all Wadge game with moves from $\{0,1\}$ of the form such that Player 1 owns a set which is the intersection of an open and a closed subset of ${}^{\mathbb{N}}2$ in its natural topology and Player 2 owns a clopen subset of ${}^{\mathbb{N}}2$. It will be shown that $\mathsf{RCA_0}$ proves that $\mathsf{WKL_0}$ is equivalent to $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-$\mathsf{WDET}^2$.

This article will be concerned with certain two player games. A simple form of these games is the following. Let A be a subset of ${}^\omega\omega$, which is the set of functions $f : \omega \rightarrow \omega$. Consider the game G_A^ω defined as in the following diagram.

$$\begin{array}{cccccccccccc} & \mathrm{I} & a_0 & & a_2 & & a_4 & & a_6 & & \cdots & \\ G_A^\omega & & & & & & & & & & & a \\ & \mathrm{II} & & a_1 & & a_3 & & a_5 & & a_7 & \cdots & \end{array}$$

Player 1 and 2 take turns playing elements of ω with Player 1 making the even moves a_{2n} and Player 2 making the odd moves a_{2n+1}. Let $a = \langle a_n : n \in \omega \rangle$ be the sequence of moves played jointly by the two players. Player 1 wins if and only if $a \in A$. G_A^ω is determined if and only if one of the two players has a winning strategy. More precisely, a strategy is a function $\rho : {}^{<\omega}\omega \rightarrow \omega$ which looks upon the partial play and determines what natural number should be played next. Note that if one uses ρ as a Player 1's (Player 2's) strategy, then only the output of ρ on the even (odd, respectively) length strings are relevant.

The axiom of determinacy, AD, asserts that for all $A \subseteq {}^\omega\omega$, G_A^ω is determined. AD is incompatible with the axiom of choice, AC, but it is an alternative axiom system for set theory which is often a more suitable framework for studying definable mathematics in the sense of recursion theory or descriptive set theory. AD and extensions of AD give a rigid and nice structure to the sets which are surjective image of $\mathbb{R}$. AD implies all sets of reals are Lebesgue measurable, have the Baire property, and have the perfect set property. AD also implies many cardinals which are surjective images of $\mathbb{R}$ have interesting combinatorial properties such as the partition properties (see [6] and [1]). Definability often appears in the study of combinatorics under AD. For instance, combinatorial questions surrounding ω_1 under AD involve the pointclass of analytic ($\boldsymbol{\Sigma}_1^1$) set and its boundedness property. (See [1], [2], and [3] for more recent results concerning ω_1 under AD using these ideas.)

AD has interesting implications concerning definability through its interaction with boldface pointclasses. For instance, given two subsets A and B of ${}^\omega\omega$, A is Wadge reducible to B, denoted $A \leq_W B$, if and only if there is a continuous function $\Phi : {}^\omega\omega \rightarrow {}^\omega\omega$ so that for all $x \in {}^\omega\omega$, $x \in A \Leftrightarrow \Phi(x) \in B$. (That is, $\Phi^{-1}[B] = A$.) Similarly, A is Lipschitz reducible to B, denoted $A \leq_L B$, if and only there is a Lipschitz function $\Phi : {}^\omega\omega \rightarrow {}^\omega\omega$ so that for all $x \in {}^\omega\omega$, $x \in X \Leftrightarrow \Phi(x) \in B$, where Φ is Lipschitz means that for all $n \in \omega$, for all $f, g \in {}^\omega\omega$, if $f \upharpoonright n = g \upharpoonright n$, then $\Phi(f) \upharpoonright n = \Phi(g) \upharpoonright n$. A boldface pointclass is a collection Γ of subsets of ${}^k\omega \times {}^j({}^\omega\omega)$ for various $k, j \in \omega$ with $j \geq 1$ which is closed under Wadge reduction, that is, if $A \leq_W B$ and $B \in \Gamma$, then $A \in \Gamma$.

The relevance of games comes from the following observations. If $\rho : {}^{<\omega}\omega \rightarrow \omega$ is considered as a Player 2 strategy, then define $\Xi_\rho^2 : {}^\omega\omega \rightarrow {}^\omega\omega$ by $\Xi_\rho^2(x) = y$ if and only if y is the sequence of Player 2 moves according to ρ when Player 1 simply plays the bits of x. Ξ_ρ^2 is a Lipschitz function and in fact, all Lipschitz functions come from Player 2 strategies. (One can define an analogous Lipschitz function $\Xi_\rho^1 : {}^\omega\omega \rightarrow {}^\omega\omega$ using ρ as a

May 5, 2020. The author was supported by NSF grant DMS-1703708.

Player 1 strategy.) This suggests the following concept of the Wadge game. Let A and B be subsets of ${}^\omega\omega$. Consider the game $W^\omega_{A,B}$ defined as follows.

$$W^\omega_{A,B} \quad \begin{array}{cccccccccccc} \mathrm{I} & a_0 & & a_1 & & a_2 & & a_3 & & \cdots & a \\ \mathrm{II} & & b_0 & & b_1 & & b_2 & & b_2 & \cdots & b \end{array}$$

Player 1 plays integers $a = \langle a_n : n \in \omega \rangle$ and Player 2 plays integers $b = \langle b_n : n \in \omega \rangle$. Player 2 wins $W^\omega_{A,B}$ if and only if $a \in A \Leftrightarrow b \in B$. Thus if $\rho : {}^{<\omega}\omega \to \omega$ is a Player 2 winning strategy in $W^\omega_{A,B}$, then Ξ^2_ρ is a Lipschitz function witnessing $A \leq_L B$. If Player 1 has a winning strategy ρ, then Ξ^1_ρ is a Lipschitz function witnessing $B \leq_L {}^\omega\omega \setminus A$. Under AD, $A \leq_L B$ or $B \leq_L {}^\omega\omega \setminus A$, and thus also this holds for $\leq_W$. This observation is known as Wadge's lemma which has various remarkable consequences. For instance, any set A and B can be compared via Lipschitz or Wadge reductions (after taking into account the nonselfdual sets): that is $(A \leq_L B \vee A \leq_L {}^\omega\omega \setminus B) \vee (B \leq_L A \vee B \leq_L {}^\omega\omega \setminus A)$. If Γ is a nonselfdual boldface pointclass and $\check{\Gamma}$ is the dual pointclass, then any set in $\Gamma \setminus \check{\Gamma}$ is complete (for Wadge reduction) and even Lipschitz-complete. One can define the notion of a Lipschitz degree of a set A to be $[A]_L = \{B : A \leq_L B \wedge B \leq_L A\}$. One can define an ordering on Lipschitz degrees by setting $\mathbf{a} \leq \mathbf{b}$ if and only if there exists an $A \in \mathbf{a}$ and a $B \in \mathbf{b}$ so that $A \leq_L B$. If $A \subseteq {}^\omega\omega$, let $[A]^*_L = [A]_L \cup [{}^\omega\omega \setminus A]_L$ be the modified Lipschitz degree of A. The Wadge lemma implies that the modified Lipschitz degree with its natural ordering is a linear ordering. (All the same concepts and properties apply to the Wadge reduction and Wadge degree.)

Since Wadge games appear frequently in the study of pointclasses, let the axiom of Wadge determinacy assert that for all $A, B \subseteq {}^\omega\omega$, $W^\omega_{A,B}$ is determined. AD implies Wadge determinacy. Under ZF, $\mathsf{DC}_\mathbb{R}$ (dependent choice for $\mathbb{R}$), all sets of reals have the Baire property, and Wadge determinacy, Martin and Monk showed that the modified Lipschitz degrees and modified Wadge degrees are wellorderings under their natural orderings. Thus every $A \subseteq \mathbb{R}$ can be assigned two ordinals: the Lipschitz rank of $[A]^*_L$ and the Wadge rank of $[A]^*_W$. Every nonselfdual Lipschitz degree is a nonselfdual Wadge degree. The collection of Wadge degrees alternate between selfdual and nonselfdual Wadge degrees. The next ω_1 Lipschitz degrees after a given Lipschitz degree are selfdual. Wadge or Lipschitz degrees of limit rank are selfdual if and only if its rank has countable cofinality. The use of Wadge rank and game arguments have applications to the study of the reduction, separation, and prewellordering properties of pointclasses. (See [17], [18], and [20] for more information.)

Much work has been done to study the relation between the consistency strength of determinacy. Woodin showed that the consistency strength of AD is the existence of ω Woodin cardinals. Although the full axiom of determinacy is inconsistent with AC, local versions of determinacy restricted to specific pointclasses such as $\boldsymbol{\Delta}^1_1$, $\boldsymbol{\Sigma}^1_1$, and the projective sets can hold in ZFC assuming large cardinals. Martin and Steel studied the consistency strength of projective determinacy from Woodin cardinals. (See [10], [7] and [13] for more information.)

Closer to the concerns of this article is the investigation of the strength of game principles in terms of equivalence with other principles. The simplest question of this form is an open question of whether AD and Wadge determinacy are equivalent over ZF. Results of Martin and Harrington showed the following are equivalent over ZFC

- $\boldsymbol{\Sigma}^1_1$ determinacy.
- For each $x \in \mathbb{R}$, $x^\sharp$ exists or equivalently there is a nontrivial elementary embedding $j : L[x] \to L[x]$.
- (Wadge's lemma for $\boldsymbol{\Sigma}^1_1$) Any $A \in \boldsymbol{\Sigma}^1_1 \setminus \boldsymbol{\Pi}^1_1$ is $\boldsymbol{\Sigma}^1_1$-complete.

There are many other interesting equivalences of $\boldsymbol{\Sigma}^1_1$ determinacy involving admissible ordinals, iterable objects (mice), and covering lemmas.

Subsystems of second order arithmetic provides a rich setting for studying weaker determinacy principles. (See [15] and [5] for more on subsystems of second order arithmetic, which will be briefly reviewed in the next section.) Friedman [4] showed that the principle Σ^0_5-DET^ω (corresponding to the determinacy of $\boldsymbol{\Sigma}^0_5$ subsets of ${}^\omega\omega$) is not provable in second order arithmetic. Martin showed that even Σ^0_4-DET^ω is not provable in second order arithmetic. Over RCA_0 which is the weak base system for second order arithmetic,

- (Steel [15] Theorem V.8.7) Σ^0_1-DET^ω, Δ^0_1-DET^ω, and ATR_0 are equivalent.

- (Tanaka [19] or [15] Theorem VI.5.4) $(\Sigma_1^0 \wedge \Pi_1^0)$-$\mathsf{DET}^\omega$ (which corresponds to the determinacy of games with payoff sets which are intersections of an open and closed subset of ${}^\omega\omega$) and Π_1^1-CA is equivalent.

Montalbán and Shored studied in detail the limits of determinacy given by Friedman and Martin. Let n-Π_3^0 denote boolean combinations of n-many Π_3^0 sets. They [11] showed that Π_{n+2}-CA proves n-Π_3^0-DET^ω but Δ_{n+2}-CA cannot prove n-Π_3^0-DET^ω. Montalbán and Shore [12] showed that these principles actually have different consistency strength.

To obtain determinacy principles equivalent to subsystems weaker than ATR_0, one needs to consider determinacy principles on ${}^\omega 2$. It is shown in [14] that over RCA_0,

- WKL_0, Σ_1^0-DET^2 (open determinacy on ${}^\omega 2$), and Δ_1^0-DET^2 are equivalent.
- ACA_0 and $(\Sigma_1^0 \wedge \Pi_1^0)$-$\mathsf{DET}^2$ are equivalent.

See [21] for a more detail account of determinacy in second order arithmetic.

A natural question early in the study of determinacy was whether Borel determinacy is needed to establish Wadge's lemma within the Borel sets. Louveau and Saint-Raymond [9] showed that Borel Wadge determinacy on ${}^\omega\omega$ can be proved in second order arithmetic. Since Friedman showed Borel determinacy cannot be prove in second order arithmetic, this answers the above question negatively. The result of Louveau and Saint-Raymond suggests a more detail analysis of Borel Wadge determinacy within subsystems of second order arithmetic would be interesting and different than the analysis of ordinary determinacy mentioned above.

The main result of the paper is that over RCA_0, WKL_0 and $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-$\mathsf{WDET}^2$ are equivalent. WKL_0 is a subsystem that includes the statement of weak König lemma which asserts infinite binary trees (on $\{0,1\}$) have infinite paths. $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-$\mathsf{WDET}^2$ is the principle corresponding to determinacy for Wadge games on ${}^\omega 2$ where Player 1 owns a set which is an intersection an open and closed subset of ${}^\omega 2$ and Player 2 owns a clopen subset of ${}^\omega 2$.

This article is a polished version of a draft produced in 2011-2012 and largely forgotten by the author. More recently, slides [8] by Loureiro announced some work by Cordón-Franco, Lara-Martín, and Loureiro on the strength of Wadge determinacy and the Wadge's lemma within second order arithmetic. For instance, they announced that over RCA_0, ACA_0 is equivalent to $(\Sigma_1^0 \wedge \Pi_1^0, \Sigma_1^0 \wedge \Pi_1^0)$-$\mathsf{WDET}^2$. In the slides, Loureiro stated that it is open what Wadge determinacy principle could be equivalent to WKL_0. The solution to their question provided by this article combined with their results gives an interesting observation that $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-$\mathsf{WDET}^2$ is weaker than $(\Sigma_1^0 \wedge \Pi_1^0, \Sigma_1^0 \wedge \Pi_1^0)$-$\mathsf{WDET}^2$.

The work in this paper followed a topic course by Montalbán covering the result of Steel and Tanaka and a reading course with Montalbán covering portions of [14] at the University of Chicago in 2011. The author seems to recall that the suggestion to investigate Wadge determinacy in second order arithmetic came from Takako Nemoto. The author would especially like to thank Antonio Montalbán for introducing the author to the determinacy axioms.

1. Basics

The reader should consult [15] and [5] for more information about the syntax and semantics of the first order theory of second order arithmetic.

Let $\mathscr{L}_2 = \{\mathbb{N}, \mathscr{P}(\mathbb{N}), +, \cdot, <, 0, 1, \in\}$ denote the first order language of second order arithmetic. $\mathbb{N}$ and $\mathscr{P}(\mathbb{N})$ are unary relation symbols. The arity and type of the other symbols should be clear from their frequent usage in mathematics. Let P denote the basic $\mathscr{L}_2$ axioms for second order arithmetics. These include all the usual axioms of arithmetic and order. One often writes $x \in \mathbb{N}$ and $x \in \mathscr{P}(\mathbb{N})$ rather than $\mathbb{N}(x)$ and $\mathscr{P}(\mathbb{N})(x)$. P should also include sentences which assert there are two types: $\mathbb{N}$ is the first order object called the numbers. $\mathscr{P}(\mathbb{N})$ are the second order objects representing subsets of the numbers. Some examples of sentences asserting these type-axioms may include the following.

$$(\forall x)(x \in \mathbb{N} \vee x \in \mathscr{P}(\mathbb{N}))$$

$$(\forall x)\neg(x \in \mathbb{N} \wedge x \in \mathscr{P}(\mathbb{N}))$$

$$(\forall x)(\forall y)(x \in y \Rightarrow (x \in \mathbb{N} \wedge y \in \mathscr{P}(\mathbb{N}))).$$

Note that $\mathbb{N}$ refers to the "number" part of an arbitrary model of Z_2, and ω will be used to denote the standard natural numbers.

Finally P contains full set induction

$$(\forall X)[\{(0 \in X \wedge (\forall n)(n \in X \Rightarrow n+1 \in X)\} \Rightarrow (\forall n)(n \in X))].$$

Suppose $\varphi(v)$ is a formula with one free variable v (and assume that n and X do not occur free in $\varphi(v)$), then φ-comprehension is that sentence

$$(\exists X)(\forall n)(n \in X \Leftrightarrow \varphi(n)).$$

Let Z_2 denote the axiom system consisting of P and φ-comprehension for all suitable formula φ. Z_2 is the $\mathscr{L}_2$-theory of second order arithmetic.

Suppose Γ is a collection of formulas $\varphi(v)$ as above, then Γ-comprehension (denoted Γ-CA) is the collection of sentences assert φ-comprehension for each $\varphi \in \Gamma$.

A formula is $\Delta^0_0 = \Sigma^0_0 = \Pi^0_0$ if and only if if it has only bounded quantifiers. For $k \geq 1$, a formula φ is Σ^0_n if and only if it takes the form $(\exists n_1)(\forall n_2)...(Qn_k)\phi$ where $Q = \exists$ if k is odd, $Q = \forall$ if k is even, and ϕ is a formula with only bounded quantifiers. A formula is Π^0_n if and only if it takes the form $(\exists n_1)(\forall n_2)...(Qn_k)\phi$ where $Q = \forall$ if k is odd, $Q = \exists$ if k is even, and ϕ is a formula with only bounded quantifiers. Over P, negations of Σ^0_n formulas are equivalent to Π^0_n formulas and negations of Π^0_n are equivalent to Σ^0_n formulas. For instance, one will often say that if φ is Σ^0_1, then $\neg\varphi$ is Π^0_1 although formally it is not. A formula is arithmetic if it has no bounded set quantifier. Thus a formula is arithmetic if and only if there is an $n \in \omega$ so that the formula is Σ^0_n. A formula φ is $\Sigma^0_n \wedge \Pi^0_n$ if and only if there a formula $\varphi_0 \in \Sigma^0_n$ and $\varphi_1 \in \Pi^0_n$ so that φ is $\varphi_0 \wedge \varphi_1$.

ACA_0 is the axiom system including P along with φ-comprehension for every arithmetic formula φ.

For each formula $\varphi(v)$ with one free variable v. φ-induction is the following sentence

$$[\varphi(0) \wedge (\forall n)(\varphi(n) \Rightarrow \varphi(n+1))] \Rightarrow (\forall n)\varphi(n).$$

If Γ is a collection of formulas, then Γ-IND is the collection of sentences stating φ-induction for each $\varphi \in \Gamma$.

Roughly a formula is Δ^0_n if and only if it is "equivalent" to a Σ^0_n formula and to a Π^0_n formula, but this is not precise since two formulas being equivalent is not syntactical. The concept of a Δ^0_n formula usually requires an equivalence of Σ^0_n and Π^0_n formula over some theory or inside a particular model. Concepts involving "Δ^0_n formulas" needs to be defined carefully. For each $n \in \omega$, one will say that Δ^0_n-comprehension, denoted Δ^0_n-CA, is the following collection of sentences: For any formulas $\varphi_0 \in \Sigma^0_n$ and $\varphi_1 \in \Pi^0_n$, one has

$$(\forall n)(\varphi_0(n) \Leftrightarrow \varphi_1(n)) \Rightarrow (\exists X)(\forall n)(n \in X \Leftrightarrow \varphi_0(n)).$$

If $\varphi(v)$ is a formula with one free variable v, then bounded φ-comprension is the following sentence

$$(\forall j)(\exists X)(\forall n)(n \in X \Leftrightarrow [n < j \wedge \varphi(n)]).$$

If Γ is a collection of formulas, then bounded Γ-CA is the axiom scheme consisting of bounded φ-comprehension for each $\varphi \in \Gamma$.

Let RCA_0 consists of P, Δ^0_1-CA, and Σ^0_1-IND. RCA_0 will serve as the weak base system. [15] Theorem II.3.9 shows that RCA_0 proves bounded Σ^0_1-CA. Thus RCA_0 can prove bounded Π^0_1-CA and bounded $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{CA}$.

Let REC be the following $\mathscr{L}_2$-structure: Let $\mathcal{C}$ consists of the recursive subsets of ω. The domain or universe of REC is $\omega \cup \mathcal{C}$. $\mathbb{N}^{\mathsf{REC}} = \omega$. $\mathscr{P}(\mathbb{N})^{\mathsf{REC}} = \mathcal{C}$. The other symbols are interpreted as their usual arithmetic operations on ω. [15] Corollary II.18 shows that $\mathsf{REC} \models \mathsf{RCA}_0$ and in fact, for any $\mathcal{M} \models \mathsf{RCA}_0$ such that $\mathbb{N}^{\mathcal{M}} = \omega$, one has that $\mathscr{P}(\mathbb{N})^{\mathsf{REC}} \subseteq \mathscr{P}(\mathbb{N})^{\mathcal{M}}$.

RCA_0 can formalize basic notions concerning arithmetic and sequence of numbers. For instance, RCA_0 implies for every number n, there is a k so that $n = 2k$ or $n = 2k+1$. ${}^{<\mathbb{N}}\mathbb{N}$ will refer to the collection of numbers used to code finite sequences of numbers although in practice, one will consider this the set of such finite sequences. If $K \subseteq \mathbb{N}$, then ${}^{<\mathbb{N}}K$ will refer to the set of sequences that only take values from K. For instance, ${}^{<\mathbb{N}}2$ is the collection of binary strings (taking value 0 or 1). There are functions that determines the length of strings and the values that appear along the string. Let ${}^{\mathbb{N}}\mathbb{N}$ refer to the class of infinite strings. If $f \in {}^{\mathbb{N}}\mathbb{N}$ and $n \in \omega$, then $f \restriction n \in {}^{<\mathbb{N}}\mathbb{N}$ is the initial segment of f of length n. See [15] Chapter II for details concerning sequences and their coding.

Next, one will show some basic facts in RCA_0 that will be needed.

Fact 1.1. RCA_0 *proves the following statements.*

(1) *For all $f:\mathbb{N}\to\mathbb{N}$, if f is an increasing function (meaning if $m<n$, then $f(m)<f(n)$), then for all n, $f(n)\geq n$. If $f:\mathbb{N}\to\mathbb{N}$ is injective, then for all n, there exists a k so that $f(k)>n$.*
(2) *For all $f:\mathbb{N}\to\mathbb{N}$, if f is an increasing function, then $\mathrm{rang}(f)$ exists.*
(3) *If $f:\mathbb{N}\to\mathbb{N}$ is injective, then there is an infinite set X so that $(\forall n)(n\in X\to(\exists m)(f(m)=n))$.*
(4) *If $\varphi(v)$ is a Σ^0_1 formula (in one distinguished free variable v and possible has paramters), then either there is a finite set F so that $(\forall n)(n\in F\leftrightarrow\varphi(n))$ or there is an injective function $f:\mathbb{N}\to\mathbb{N}$ so that $(\forall n)(\varphi(n)\Leftrightarrow(\exists m)(f(m)=n))$.*
(5) *If $\varphi(v)$ is a Σ^0_1 formula (possibly with parameters), then*

$$(\forall m)(\exists n>m)\varphi(n)\Rightarrow(\exists X)(\forall k)(k\in X\Rightarrow\varphi(k)).$$

In other words, if there are infinitely many n so that $\varphi(n)$, then there is an infinite set X consisting of solutions to φ.
(6) *There is no function $f:\mathbb{N}\to\mathbb{N}$ so that $(\forall n)(f(n+1)<f(n))$.*
(7) *Suppose $f:\mathbb{N}\to\mathbb{N}$ is an injective function. Consider the Π^0_1 formula $\varphi(n)$ defined by $(\forall m>n)(f(m)>f(n))$. Then there are infinitely many n so that $\varphi(n)$.*
(8) *Suppose $f:\mathbb{N}\to\mathbb{N}$ is an injective function. For any $M\in\mathbb{N}$, there is an $N\in\mathbb{N}$ so that for all $n>N$, $f(n)>M$.*

Proof. (1) This can be shown by Σ^0_1-IND.

(2) Define a Σ^0_0 formula $\varphi(n)$ by $(\exists m\leq n)(f(m)=n)$. By Δ^0_1-CA, there is a set X such that for all n, $n\in X$ if and only if $\varphi(n)$. By (1), X is the range of f.

(3) By using (1) and an application of minimization ([15] Theorem II.3.5), there is a function $g_0:\mathbb{N}\to\mathbb{N}$ so that $g_0(n)$ is the least k so that $f(k)>f(n)$. By primitive recursion ([15] Theorem II.3.4), there is a function g_1 such that $g_1(n+1)=g_0(g_1(n))$. Let $g_2:\mathbb{N}\to\mathbb{N}$ be defined by $g_2(n)=f(g_1(n))$. g_2 is an increasing function. By (2), $\mathrm{rang}(g_2)$ exists, is infinite, and consists only of elements which are in the "range of f" (although the range of f is not assumed to be a set).

(4) This is [15] Lemma II.3.7.

(5) This follows from (3) and (4).

(6) Suppose there is an $f:\mathbb{N}\to\mathbb{N}$ so that $f(n+1)<f(n)$ for all $n\in\mathbb{N}$. Consider the Π^0_1 formula $\varphi(n)$ defined as $(\forall m)(f(m)\neq n)$. Note that $\varphi(0)$ holds since if there was an m so that $f(m)=0$, then $f(m+1)<f(m)=0$ which is impossible. Suppose $\varphi(k)$ holds for all $k\leq n$. Suppose there is an m so that $f(m)=n+1$. Then $f(m+1)\leq n$. By the induction hypothesis, one has $\varphi(f(m+1))$ but by definition, one has $\neg\varphi(f(n+1))$. Since $\mathsf{RCA_0}$ proves Π^0_1-IND by [15] Corollary II.3.10, one has $(\forall n)\varphi(n)$. This is impossible since $\neg\varphi(f(0))$ holds.

(7) Consider the Π^0_1 formula $\varphi(n)$ defined by $(\forall m>n)(f(m)>f(n))$. Suppose there exists an N so that for all n, $\varphi(n)$ implies $n<N$. Thus for all $n\geq N$, there exists an $m>n$ so that $f(m)\leq f(n)$ and since f is injective, one actually has $f(m)<f(n)$. Using minimization, there is a function $g_0:\mathbb{N}\to\mathbb{N}$ with the property that $g_0(n)$ is the least $m>n$ so that $f(m+N)<f(n+N)$. By primitive recursion, let $g_1:\mathbb{N}\to\mathbb{N}$ have the property that $g_1(0)=0$ and $g_1(n+1)=g_0(g_1(n))$ for all $n\in\mathbb{N}$. Let $h:\mathbb{N}\to\mathbb{N}$ be defined by $h(n)=f(g_1(n)+N)$. Note that h has the property that $h(n+1)<h(n)$ for all $n\in\mathbb{N}$ which contradicts (6).

(8) Let φ be the Π^0_1 formula for f given in (7). By (7), there is some $K\in\mathbb{N}$ so that $F=\{n<K:\varphi(k)\}$ (which is a set under $\mathsf{RCA_0}$ by bounded Π^0_1-CA) has size $M+2$. Let $r:M+2\to F$ be the increasing enumeration of F. Let $s:M+2\to\mathbb{N}$ be defined by $s(n)=f(s(n))$. By definition of φ, s is a finite increasing function. Using (1), it can be shown that $s(M+1)>M$. By definition of $\varphi(r(M+1))$, one has that for all $n>r(M+1)$, $f(n)>f(r(M+1))=s(m+1)>M$. Thus $N=r(M+1)$ suffices. ☐

Fact 1.2. $\mathsf{RCA_0}$ *proves the following are equivalent.*

(1) $\mathsf{ACA_0}$.
(2) Σ^0_1-CA.
(3) *For every injective function $f:\mathbb{N}\to\mathbb{N}$, there exists a set X so that $(\forall n)(n\in X\Leftrightarrow(\exists m)(f(m)=n))$. (That is, $\mathrm{rang}(f)$ exists as a set.)*
(4) *For every Π^0_1 formula $\varphi(v)$ with one free variable v (and possibly has parameters),*

$$(\forall m)(\exists n>m)\varphi(n)\Rightarrow(\exists X)[(\forall m)(\exists n>m)(n\in X)\wedge(\forall k)(k\in X\Rightarrow\varphi(k))].$$

In other words, if there are infinitely many n so that $\varphi(n)$, then there is an infinite set X consisting of solutions to φ.

Proof. The equivalence of (1), (2), and (3) is [15] Lemma III.1.3.

(1) $\Rightarrow$ (4) Fix such a Π^0_1 formulas. By $\mathsf{ACA_0}$, there is an X so that $(\forall n)(n \in X \Leftrightarrow \varphi(n))$. This set X satisfies the condition of (4).

(4) $\Rightarrow$ (3) Let $f : \mathbb{N} \to \mathbb{N}$ be an injective function. Consider the Π^0_1 formula $\varphi(n)$ defined by $(\forall m > n)(f(m) > f(n))$. By Fact 1.1 (7), one must have that for all N, there exists an $n > N$ so that $\varphi(n)$. Thus (4) applies to give an infinite set X with the property that $(\forall k)(k \in X \Rightarrow \varphi(k))$. Using minimization and primitive recursion, let $r : \mathbb{N} \to X$ be the increasing enumeration of X. Let $h = f \circ r$ which is an increasing function by the property of X and φ. Thus by Fact 1.1 (1), one has that $h(n) \geq n$. Let ψ be defined by $\psi(n)$ if and only if $(\exists m \leq r(n))(f(m) = n)$. By the property of X and φ and the fact that $h(n) \geq n$ for all $n \in \mathbb{N}$, one has that $(\exists m)(f(m) = n)$ if and only if $\psi(n)$. Since ψ is Σ^0_0 and using Δ^0_1-CA, let Y be a set so that $(\forall n)(n \in Y \Leftrightarrow \psi(n))$. Thus Y is $\mathrm{rang}(f)$ and hence $\mathrm{rang}(f)$ exists as a set. $\square$

A tree on $2 = \{0, 1\}$ is a nonempty set $T \subseteq {}^{<\mathbb{N}}2$ which is closed under the string extension relation $\subseteq$. That is, if $t \in T$ and $s \subseteq t$, then $s \in T$. If $f \in {}^{\mathbb{N}}2$, then one writes $f \in [T]$ if and only if for all $n \in \mathbb{N}$, $f \restriction n \in T$, and one will say that f is a path through T. Weak König lemma, abbreviated WKL, is the statement that every infinite tree T on 2 has a path. $\mathsf{WKL_0}$ is the axiom system consisting of $\mathsf{RCA_0}$ with WKL.

Let $T \subseteq {}^{<\mathbb{N}}2$. Using $\mathsf{ACA_0}$, one can define $\hat{T} = \{s \in T : (\forall n)(\exists t)(s \subseteq t \wedge t \in T \wedge |t| > n\}$ which is an infinite subtree tree of T with no dead branches. By primitive recursion, one can define the usual leftmost branch of $\hat{T}$. Thus $\mathsf{ACA_0}$ can prove $\mathsf{WKL_0}$.

The next result shows $\mathsf{WKL_0}$ can prove that ${}^{\mathbb{N}}2$ is compact.

Fact 1.3. $\mathsf{WKL_0}$ *can prove the following: Suppose U is a set so that for all $f : \mathbb{N} \to 2$, there exists an $n \in \mathbb{N}$ so that $f \restriction n \in U$. Then there is a finite set $F \subseteq U$ so that for all $f : \mathbb{N} \to 2$, there is an $n \in \mathbb{N}$ so that $f \restriction n \in F$.*

Proof. Let U be a set as above. By Δ^0_1-CA, define a tree $T = \{s \in {}^{<\mathbb{N}}2 : (\forall n \leq |s|)(s \restriction n \notin U)\}$. By the property of U, for all $f : \mathbb{N} \to 2$, there is an n so that $f \restriction n \notin T$. Thus there are no paths through T. $\mathsf{WKL_0}$ implies that T must be finite. Let $n \in \mathbb{N}$ be such that ${}^{n}2 \cap T = \emptyset$, i.e. no string of length n belongs to T. Thus for all $t \in {}^{n}2$, there is an initial segment in U. Using Δ^0_1-CA, define a finite set $F \subseteq {}^{<\mathbb{N}}2$ by $s \in F \Leftrightarrow (s \in U \wedge |s| \leq n)$. For all $f : \mathbb{N} \to 2$, there an $m \leq n$ so that $f \restriction m \in F$. $\square$

Now suppose $\varphi(X)$ is a Σ^0_1 formula with one free set variable. Identifying $\mathscr{P}(\mathbb{N})$ and ${}^{\mathbb{N}}2$, $\mathsf{RCA_0}$ can prove that the solutions to the Σ^0_1 formula $\varphi(X)$ is an open set. Precisely, [15] Theorem II.2.7 shows that in $\mathsf{RCA_0}$, if $\varphi(X)$ is a Σ^0_1 formula with free set variable X, then there is a Σ^0_0 formula $\phi(v)$ with number variable v so that for all $f : \mathbb{N} \to 2$, $\varphi(f)$ if and only if $(\exists n)\phi(f \restriction n)$, where ${}^{<\mathbb{N}}2$ is coded using $\mathbb{N}$. By Δ^0_1-CA, there is a set $U \subseteq {}^{<\mathbb{N}}2$ so that $(\forall s \in {}^{<\mathbb{N}}2)(s \in U \Leftrightarrow \phi(s))$. It has been shown that if $\varphi(f)$ is Σ^0_1 with one free set variable f, then there is set $U \subseteq {}^{<\mathbb{N}}2$ so that $(\forall f)(\varphi(f) \Leftrightarrow (\exists n)(f \restriction n \in U))$.

Thus in $\mathsf{RCA_0}$, if $\psi(f)$ is Π^0_1 in one free set variable f, then there is a set $U \subseteq {}^{<\mathbb{N}}2$ so that $(\forall f)(\psi(f) \Leftrightarrow (\forall n)(f \restriction n \notin U))$. Let $T = \{s \in {}^{<\mathbb{N}}2 : (\forall n \leq |s|)(s \restriction n \notin U)\}$ which is a set by Δ^0_1-CA and is a tree. Note that $\psi(f)$ if and only if $f \in [T]$. It has been shown in $\mathsf{RCA_0}$ that if $\psi(f)$ is a Π^0_1-formula in one free set variable f, then there is a tree T on 2 so that $(\forall f)(\psi(f) \Leftrightarrow f \in [T])$.

Fact 1.4. $\mathsf{RCA_0}$ *proves the following are equivalent.*

(1) $\mathsf{WKL_0}$.
(2) *For all injective functions $g_0, g_1 : \mathbb{N} \to \mathbb{N}$ with the property that $(\forall m)(\forall n)(m \neq n \Rightarrow g_0(m) \neq g_1(n))$, there exists a set X so that $(\forall n)(g_0(n) \notin X \wedge g_1(n) \in X)$.*
(3) *Suppose $\phi(f)$ is a Σ^0_1 formula and $\psi(f)$ is a Π^0_1 formula in the free set variable f. If $(\forall f)(\phi(f) \Leftrightarrow \psi(f))$, then there exists a finite set $F \subseteq {}^{<\mathbb{N}}2$ so that for all $f : \mathbb{N} \to 2$, $\phi(f) \Leftrightarrow (\exists n)(f \restriction n \in F)$.*

Proof. The equivalence of (1) and (2) is shown in [15] Lemma IV.4.4.

(1) $\Rightarrow$ (3). Using the comments above, since $\phi(f)$ is Σ^0_1, there is a set $U_0 \subseteq {}^{<\mathbb{N}}2$ so that $(\forall f)(\phi(f) \Leftrightarrow (\exists n)(f \restriction n \in U_0))$. Since $\neg\psi(f)$ is Σ^0_1, there is also a $U_1 \subseteq {}^{<\mathbb{N}}2$ so that $(\forall f)(\neg\psi(f) \Leftrightarrow (\exists n)(f \restriction n \in U_1))$.

Since $\neg\phi \Leftrightarrow \neg\psi$, one has that for all $(\forall f)(\exists n)(f \upharpoonright n \in U_0 \cup U_1)$ and $U_0 \cap U_1 = \emptyset$. By Fact 1.3, there is a finite $E \subseteq U_0 \cup U_1$ so that $(\forall f)(\exists n)(f \upharpoonright n \in E)$. Let $F = E \cap U_0$ which is a finite set. Then one has $(\forall f)(\phi(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in F))$.

(3) $\Rightarrow$ (2). Suppose (2) fails. Let $g_0, g_1 : \mathbb{N} \to \mathbb{N}$ be two injective functions so that for all $m \neq n$, $g_0(m) \neq g_1(m)$ and there is no separating set X with the property that $(\forall n)(g_0(n) \notin X \wedge g_1(n) \in X)$.

If $f : \mathbb{N} \to 2$, then f naturally is the characteristic function of a set $X_f = \{n \in \mathbb{N} : f(n) = 1\}$. Say that f fails to be a separation first for g_0 if and only if $\phi_0(f)$ holds, where $\phi_0(f)$ is defined by

$$(\exists n)[(\forall m < n)(f(g_0(m)) = 0 \wedge f(g_1(m)) = 1) \wedge f(g_0(n)) = 1 \wedge f(g_1(n)) = 1].$$

Intuitively, the witness n to the existential quantifier in $\phi_0(f)$ states that X_f fails to be a separation for g_0 and g_1 with n being the first error in the sense that $g_0(n) \in X_f$ and n does not cause an error for g_1 in the sense that $g_1(n) \in X_f$. Note that ϕ_0 is Σ^0_1.

Say that f fails to be a separation first for g_1 or fails for both g_0 and g_1 at the same time if and only if $\phi_1(f)$ holds, where $\phi_1(f)$ is defined by

$$(\exists n)[(\forall m < n)(f(g_0(m)) = 0 \wedge f(g_1(m)) = 1) \wedge f(g_1(n)) = 0].$$

Intuitively, the witness n to the existential quantifier in $\phi_1(f)$ states that X_f fails to be a separation for g_0 and g_1 first at n in the sense that $g_1(n) \notin X_f$ (and it is possible that at this n, f also has an error for g_0 in the sense that $g_0(n) \in X_f$). Note that ϕ_1 is Σ^0_1.

Observe that if $\neg\phi_0(f) \wedge \neg\phi_1(f)$, then one would have that $X_f = \{n \in \mathbb{N} : f(n) = 1\}$ is a separation for g_0 and g_1. Since one has assumed that no such separation exists, one has $(\forall f)(\phi_0(f) \vee \phi_1(f))$. Moreover, one also has $(\forall f)(\neg\phi_0(f) \vee \neg\phi_1(f))$. This shows that $(\forall f)(\phi_0(f) \Leftrightarrow \neg\phi_1(f))$.

Now let $F \subseteq {}^{<\mathbb{N}}2$ be a finite set. Let $N_0 \in \mathbb{N}$ be the length of the longest string in F. Fact 1.1 (8) implies there is an $N_1 \in \mathbb{N}$ so that for all $n > N_1$, $g_0(n) > N_0$ and $g_1(n) > N_0$. Define $s : N_0 \to 2$ by

$$s(n) = \begin{cases} 1 & (\exists t \leq N_1)(g_1(t) = n) \\ 0 & \text{otherwise} \end{cases}.$$

Note that s exists by Δ^0_1-CA. The main property of s is that for any $f : \mathbb{N} \to 2$ extending s, the least n for which X_f fails to be a separation in the sense that either $f(g_0(n)) = 1$ or $f(g_1(n)) = 0$ must have the property that $g_0(n) \geq N_0$ or $g_1(n) \geq N_0$.

(Case I) Suppose there is an initial segment of s which belongs to F. Using Δ^0_1-CA, let $f_0 : \mathbb{N} \to 2$ be defined by $f_0 = s\hat{\ }\bar{0}$ which is the function with s as its initial segment followed by the constant 0 sequence. As noted above, the least n such $f_0(g_0(n)) = 1$ or $f_0(g_1(n)) = 0$ must have the property that $g_0(n) \geq N_0$ or $g_1(n) \geq N_0$. Since $f_0(k) = 0$ for all $k \geq N_0$, the latter must occur. This means $\phi_1(f_0)$ holds and therefore $\neg\phi_0(f_0)$. It has been shown that there is a function f_0 so that an initial segment of f_0 belongs to F but $\neg\phi_0$.

(Case II) There is no initial segment of s in F. Let $f_1 : \mathbb{N} \to 2$ be defined by $f_1 = s\hat{\ }\bar{1}$, i.e. the function with s as an initial segment followed by the constant 1 function. As noted above, the least n so that $f_1(g_0(n)) = 1$ or $f_1(g_1(n)) = 0$ must have the property that $g_0(n) \geq N_0$ or $g_1(n) \geq N_0$. Since $f_1(k)$ takes only value 1 at $k \geq N_0$, the former must occur. Thus one has $\phi_0(f)$. Note that no initial segment of f_1 belongs to F since $f_1 \upharpoonright N_0 = s$ and N_0 is the length of the longest string in F. It has been shown that there is an f_1 so that no initial segment of f_1 belongs to F but $\phi_0(f)$.

The conclusions from Case I and II show that F does not witness the condition in (3) for the formula ϕ_0. Since $F \subseteq {}^{<\mathbb{N}}2$ was an arbitrary finite set, (3) fails. $\square$

The following are the basic notions of determinacy. See [15] V.8 for more information.

Definition 1.5. (RCA_0) Let $K = \mathbb{N}$ or $K \in \mathbb{N}$ (thinking of $K = \{n \in \mathbb{N} : n < K\}$). (For example, $K = 2 = \{0, 1\}$.) A stategy in K is a function $\rho : {}^{<\mathbb{N}}K \to K$.

Now suppose ρ_0 and ρ_1 are two strategies in K. By primitive recursion in RCA_0 ([15] Theorem II.3.4), let $\rho_0 * \rho_1 \in {}^{\mathbb{N}}K$ be the unique function $f \in {}^{\mathbb{N}}K$ so that for all $k \in \mathbb{N}$, $f(2k) = \rho_0(f \upharpoonright 2k)$ and $f(2k+1) = \rho_1(f \upharpoonright 2k+1)$.

If $\varphi(X)$ is an $\mathscr{L}_2$-formula with one free "set" variable X. One says that the game G^K_φ is determined if and only if one of the following holds.

(1) There exists a strategy (in K) ρ_0 so that for all strategies ρ_1 (in K), $\varphi(\rho_0 * \rho_1)$.

(2) There exists a strategy (in K) ρ_1 so that for all strategies ρ_0 (in K), $\neg\varphi(\rho_0 * \rho_1)$.

The following diagram illustrates the game G_φ^K. Here ρ_0 acts as Player 1's strategy, and ρ_1 acts as Player 2's strategy. $\rho_0 * \rho_1$ is the infinite run of the corresponding play.

$$
\begin{array}{llllllll}
 & \rho_0 \ \mathrm{I} & a_0 & & a_2 & & a_4 & \cdots & \\
G_\varphi^K & & & & & & & & a = \rho_0 * \rho_1 \\
 & \rho_1 \ \mathrm{II} & & a_1 & & a_3 & & a_5 \quad \cdots &
\end{array}
$$

In the above diagram, Player 1 produces the even moves $a_{2k} \in K$ and Player 2 produces the odd moves a_{2k+1}. Together they produce an infinite sequence $a \in {}^{\mathbb{N}}K$. Player 1 is said to win G_φ^K if and only if $\varphi(a)$ holds. In the definition of the determinacy of G_φ^K, the first case indicates ρ_0 is a Player 1 winning strategy. The second case indicates that ρ_1 is a Player 2 winning strategy.

If Γ is a collection of formulas with one free set variable, then $\Gamma\text{-}\mathsf{DET}^K$ is the statement that for all $\varphi \in \Gamma$, the game G_φ^K is determined.

This article will be mostly concerned with Wadge games.

Definition 1.6. (RCA_0) If $f \in {}^{\mathbb{N}}\mathbb{N}$, then let $f_{\mathrm{even}} \in {}^{\mathbb{N}}\mathbb{N}$ be defined by $f_{\mathrm{even}}(n) = f(2n)$ and $f_{\mathrm{odd}} \in {}^{\mathbb{N}}\mathbb{N}$ be defined by $f_{\mathrm{odd}}(n) = f(2n+1)$. Similarly, if $s \in {}^{<\mathbb{N}}\mathbb{N}$, let $s_{\mathrm{even}}(n) = s(2n)$ for all n such that $2n < |s|$ and $s_{\mathrm{odd}}(n) = s(2n+1)$ for all n such that $2n+1 < |s|$.

Let $K = \mathbb{N}$ or $K \in \mathbb{N}$. Let $\varphi_0(X)$ and $\varphi_1(X)$ be two formulas with X as its only free set variable. One says that the Wadge game $W_{\varphi_0,\varphi_1}^K$ is determined if and only if one of the following holds.

(1) There exists a strategy ρ_0 (in K) so that for all strategies ρ_1 (in K),

$$\varphi_0((\rho_0 * \rho_1)_{\mathrm{even}}) \Leftrightarrow \neg\varphi_1((\rho_0 * \rho_1)_{\mathrm{odd}}).$$

(2) There exists a strategy ρ_1 (in K) so that for all strategies ρ_0 (in K),

$$\varphi_0((\rho_0 * \rho_1)_{\mathrm{even}}) \Leftrightarrow \varphi_1((\rho_0 * \rho_1)_{\mathrm{odd}}).$$

The following diagram illustrates the game $W_{\varphi_0,\varphi_1}^K$.

$$
\begin{array}{llllllll}
 & \rho_0 \ \mathrm{I} & a_0 & a_1 & a_2 & & \cdots & a = (\rho_0 * \rho_1)_{\mathrm{even}} \\
W_{\varphi_0,\varphi_1}^K & & & & & & & \\
 & \rho_1 \ \mathrm{II} & b_0 & b_1 & b_2 & & \cdots & b = (\rho_0 * \rho_1)_{\mathrm{even}}
\end{array}
$$

Player 1 and Player 2 take turns making moves from K. Player 1 plays $a_n \in K$ and Player 2 plays $b_n \in K$ for all $n \in \omega$. Player 1 independently produces $a \in {}^{\mathbb{N}}K$ and Player 2 independently produces $b \in {}^{\mathbb{N}}K$. Player 2 wins the game $W_{\varphi_0,\varphi_1}^K$ if and only if $\varphi_0(a) \Leftrightarrow \varphi_1(b)$. In the definition of the determinacy of the Wadge game $W_{\varphi_0,\varphi_1}^K$, the first case indicates Player 1 has a winning strategy ρ_0 and the second case indicates that Player 2 has a winning strategy ρ_1.

Suppose Γ_0 and Γ_1 are two collections of formulas with one free set variable. Then $(\Gamma_0, \Gamma_1)\text{-}\mathsf{WDET}^K$ is the statement that for all $\varphi_0(X) \in \Gamma_0$ and $\varphi_1(X) \in \Gamma_1$, the Wadge game $W_{\varphi_0,\varphi_1}^K$ is determined.

Again Wadge determinacy principles involving the classes Δ_n^0 require more care to define. The main Wadge determinacy principle of concern in this article is $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)\text{-}\mathsf{WDET}^2$. For explicitness, this is the following axiom schema: For each formula $\psi \in \Sigma_1^0 \wedge \Pi_1^0$ (i.e. there is a formula $\zeta_0 \in \Sigma_1^0$ and a formula $\zeta_1 \in \Pi_1^0$ so that $\psi = \zeta_0 \wedge \zeta_1$) and any pair of formulas $\varphi_0 \in \Sigma_1^0$ and $\varphi_1 \in \Pi_1^0$, one has the following statement: if $(\forall n)(\varphi_0(n) \Leftrightarrow \varphi_1(n))$, then the Wadge game (on $2 = \{0, 1\}$) W_{ψ,φ_0}^2 is determined.

Suppose that if φ_0 and φ_1 are two formulas. Note that ρ_0 is a Player 1 winning strategy in $W_{\varphi_0,\varphi_1}^2$ if and only if ρ_0 is a Player 1 winning strategy in $W_{\neg\varphi_0,\neg\varphi_1}^2$. Similarly, ρ_1 is a Player 2 winning strategy in $W_{\varphi_0,\varphi_1}^2$ if and only if ρ_1 is a Player 2 winning strategy in $W_{\neg\varphi_0,\neg\varphi_1}^2$. Thus $W_{\varphi_0,\varphi_1}^2$ is determined if and only if $W_{\neg\varphi_0,\neg\varphi_1}^2$ is determined. In particular, this can be used to show that $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)\text{-}\mathsf{WDET}^2$ is equivalent to $(\Sigma_1^0 \vee \Pi_1^0, \Delta_1^0)\text{-}\mathsf{WDET}^2$.

2. Wadge Determinacy and $\mathsf{WKL_0}$

First one will show that $\mathsf{RCA_0}$ cannot prove $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-$\mathsf{WDET}^2$. This will be done by showing that $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-$\mathsf{WDET}^2$ fails in the model $\mathsf{REC} \models \mathsf{RCA_0}$. This result is important as it motivates the framework of the main theorems.

Theorem 2.1. $\mathsf{REC} \not\models (\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-$\mathsf{WDET}^2$. *Hence* $\mathsf{RCA_0}$ *does not prove* $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-$\mathsf{WDET}^2$.

Proof. Recall that lightface Δ_1^0 subset of ω in the sense of descriptive set theory or recursion theory is equivalently a recursive subset of ω. A lightface Σ_1^0 subset of ω is a recursively enumerable subset of ω which is also the range of an injective recursive function. Let $\mathcal{C}$ denote the collection of recursive subsets of ω. Recall that REC is the $\mathscr{L}_2$ structure where $\mathbb{N}^{\mathsf{REC}} = \omega$ and $\mathscr{P}(\mathbb{N})^{\mathsf{REC}} = \mathcal{C}$.

Let $\langle \Phi_e : e \in \omega\rangle$ denote a recursive enumeration of all partial recursive functions. Let $B_0 = \{e \in \omega : \Phi_e(e) = 0\}$ and $B_1 = \{e \in \omega : \Phi_e(e) = 1\}$. One can show that B_0 and B_1 are recursively enumerable disjoints sets which which are recursively inseparable, meaning that there is no recursive set $X \subseteq \omega$ so that $B_1 \subseteq X$ and $B_0 \cap X = \emptyset$. (Note B_0 and B_1 are not computable so they do not belong to $\mathscr{P}(\mathbb{N})^{\mathsf{REC}} = \mathcal{C}$.) Thus there are recursive functions $g_0 : \omega \to \omega$ and $g_1 : \omega \to \omega$ so that $g_0[\omega] = B_0$ and $g_1[\omega] = B_1$ (in the real world). Note g_0 and g_1 are functions that belong to REC. Let $\varphi_0(f)$ be

$$(\exists n)[(\forall m < n)(f(g_0(m)) = 0 \wedge f(g_1(m)) = 1) \wedge f(g_0(n)) = 1 \wedge f(g_1(n)) = 1].$$

Let $\varphi_1(f)$ be

$$(\exists n)[(\forall m < n)(f(g_0(m)) = 0 \wedge f(g_1(m)) = 1) \wedge f(g_1(n)) = 0].$$

As argued in Fact 1.4 using the assumption that REC has no separation for g_0 and g_1, one has that $\mathsf{REC} \models (\forall f)(\varphi_0(f) \vee \varphi_1(f))$ and $\mathsf{REC} \models (\forall f)(\neg\varphi_0(f) \vee \neg\varphi_1(f))$. Thus $\mathsf{REC} \models (\forall f)(\varphi_0(f) \Leftrightarrow \neg\varphi_1(f))$ (although in the real world this equivalence does not hold). As argued in Fact 1.4, there are no finite sets F so that $(\forall f)(\varphi_0(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in F))$. Since φ_0 and φ_1 are both lightface Σ_1^0 subsets of ${}^\omega 2$, there are computable sets $U_0, U_1 \subseteq {}^{<\omega}2$ (and hence U_0 and U_1 are sets in REC) so that for all f, $\varphi_0(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in U_0)$ and $\varphi_1(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in U_1)$. Note that $U_0 \cap U_1 = \emptyset$. Let $\varpi(s)$ be the following Σ_1^0 formula

$$(\exists t_0 \in U_0)(\exists t_1 \in U_1)(s \subseteq t_0 \wedge s \subseteq t_1)\}.$$

The set $A = \{s \in {}^{<\omega}2 : \varpi(s)\}$ is a recursively enumerable set (which may not belong to REC). Intuitively, $\varpi(s)$ means that s has not decided φ_0 in the sense there are extensions of s which belong to U_0 and there are extensions of s which belong U_1. A is infinite and in fact, REC proves that ϖ has infinitely many solutions. To see this, suppose that REC thinks there is an $N \in \omega$ so that for all $s \in {}^{<\omega}2$, $\varpi(s)$ implies $|s| < N$. Then since $(\forall f)(\varphi_0(f) \vee \varphi_1(f))$, one has that for all $s \in {}^N 2$, there is an $i \in \{0, 1\}$ so that for all $f \supseteq s$, f has an initial segment in U_i and therefore $\varphi_i(f)$. (That is, all extension of f have been decided in the same way.) Let F be the finite set of $s \in {}^N 2$ so that there exists a $t \in U_0$ so that $t \subseteq s$ or $s \subseteq t$. Since F is finite, F is recursive and therefore belongs to REC. Note that for all $f : \omega \to 2$, $\varphi_0(f)$ if and $(\exists n)(f \upharpoonright U_0)$ if and only if $f \upharpoonright N \in F$ if and only if $(\exists n)(f \upharpoonright n \in F)$. However, REC does not think that such a finite set exists for φ_0. It has been shown that A is an infinite recursively enumerable set and since every infinite recursively enumerable set has an infinite computable subset, let $Y \subseteq A$ be an infinite computable set. Note that Y is a set belonging to REC. Also REC thinks Y is infinite and for all $n \in Y$, $\varpi(n)$ holds.

Let S be a recursively enumerable set whose complement does not have any infinite recursively enumerable subsets. (S is called a simple set and $\omega \setminus S$ is a Π_1^0 immune set. Post ([16] Theorem 5.2.3) showed simple sets exist.) Let ζ be a Σ_1^0 formula so that $n \in S$ if and only if $\zeta(n)$.

Define a formula $\psi(f)$ in the free set variable f as follows.

$$(\exists n)[(f(n) = 1) \wedge \zeta(n) \wedge (\forall m < n)(f(m) = 0)] \vee (\forall n)(f(n) = 0).$$

Intuitively, one can think of $f : \omega \to 2$ as the characteristic function of the set $X_f = \{n \in \omega : f(n) = 1\}$. $\psi(f)$ holds means either $X_f = \emptyset$ or the least element of X_f belongs to the simple set S. Note that ψ is a $\Sigma_1^0 \vee \Pi_1^0$ formula.

Now one will consider the Wadge game W^2_{ψ,φ_0}.

(Case I) Suppose Player 2 has a winning strategy ρ_1^* for W^2_{ψ,φ_0} in REC.

Let $\tilde{\rho}_0 : {}^{<\omega}2 \to 2$ be the constant 0 function. (That is, $\tilde{\rho}_0$ has Player 1 put down 0 regardless of Player 2's move.) $\tilde{\rho}_0$ is recursive and therefore $\tilde{\rho}_0$ belongs to REC. $(\tilde{\rho}_0 * \rho_1^*)_{\text{even}} = \bar{0}$, the constant 0 sequence. Thus

$\psi((\tilde{\rho}_0 * \rho_1^*)_{\text{even}})$ by the second disjunct in the definition of ψ. Since ρ_1^* is a Player 2 winning strategy in W^2_{ψ,φ_0}, one must have $\varphi_0((\tilde{\rho}_0 * \rho_1^*)_{\text{odd}})$. Thus there is some $n \in \omega$ so that $(\tilde{\rho}_0 * \rho_1^*)_{\text{odd}} \upharpoonright n \in U_0$. Since S is a simple set, $\omega \setminus S$ is an immune set and hence infinite. Thus there exists some $m > n$ so that $\neg\zeta(m)$. Let $\hat{\rho}_0 : {}^{<\omega}2 \to 2$ be the Player 1's strategy that put down 1 on Player 1's $(m+1)^{\text{st}}$ move and 0 for its other moves. That is

$$\hat{\rho}_0(s) = \begin{cases} 1 & |s| = 2m \\ 0 & \text{otherwise} \end{cases}.$$

$\hat{\rho}_0$ is recursive and thus belongs to REC. By definition of $\hat{\rho}_0$,

$$(\hat{\rho}_0 * \rho_1^*)_{\text{even}}(k) = \begin{cases} 1 & k = m \\ 0 & \text{otherwise} \end{cases}$$

So m is the least k so that $(\hat{\rho}_0 * \rho_1^*)_{\text{even}}(k) = 1$. Since $\neg\zeta(m)$ holds, one has that $\neg\psi((\hat{\rho}_0 * \rho_1^*)_{\text{even}})$ holds. Since $m > n$, one has that $\tilde{\rho}_0$ and $\hat{\rho}_0$ agree on all strings of length less than n. Thus $(\hat{\rho}_0 * \rho_1^*)_{\text{odd}} \upharpoonright n = (\tilde{\rho}_0 * \rho_1^*)_{\text{odd}} \upharpoonright n \in U_0$. Hence $\varphi_0((\hat{\rho}_0 * \rho_1^*)_{\text{odd}})$ holds. Player 1 using $\hat{\rho}_0$ defeats Player 2 using ρ_1^*. This contradicts ρ_1^* being a Player 2 winning strategy in W^2_{ψ,φ_0}.

(Case II) Suppose Player 1 has a winning strategy ρ_0^* for W^2_{ψ,φ_0} in REC.

Since ρ_0^* belongs to REC, ρ_0^* is a recursive function. Since Y is an infinite computable subset of the recursively enumerable set A (defined by ϖ), one has that for each $s \in Y$, there exists some $t_0 \in U_0$ so that $s \subseteq t_0$. Define $\Psi : Y \to U_0$ by $\Psi(s)$ is the least $t \in U_0$ so that $s \subseteq t$ (sequences can be coded by natural numbers so "least" refers to the ordering on ω). Because for each $s \in Y$, there does exists such a t, this implies that Ψ is well defined on X and Ψ is a recursive function. Hence Ψ belongs to REC. $\Psi[Y] = \{t : (\exists s)(s \in Y \wedge \Psi(s) = t\}$ is an infinite recursively enumerable set since Y is infinite and recursive. Thus there is an infinite recursive set $Z \subseteq \Psi[Y]$ and thus Z is a set in REC.

Next, one will define a recursive family of Player 2 strategies $\langle \rho_1^u : u \in Z \rangle$ so that for each $u \in Z$, ρ_1^u is the player 2 strategy that simply puts down u and then 0 forever afterward. More precisely, define a function $\Theta : {}^{<\omega}2 \times {}^{<\omega}2 \to 2$ as follows.

$$\Theta(u, s) = \begin{cases} u(k) & u \in Z \wedge |s| = 2k+1 \wedge k < |u| \\ 0 & \text{otherwise} \end{cases}$$

Note that Θ is a computable function. Θ defines a Z-index family of Player 2 strategies as follows. For each $u \in Z$, let $\rho_1^u : {}^{<\omega}2 \to 2$ be defined by $\rho_1^u(s) = \Theta(u, s)$. Let $\varsigma(n)$ be the following formula.

$$(\exists u \in Z)[(\rho_0^* * \rho_1^u)_{\text{even}}(n) = 1 \wedge (\forall m < n)((\rho_0^* * \rho_1^u)_{\text{even}}(m) = 0)]$$

Note that $\varsigma(n)$ is a Σ_1^0 formula involving the computable function Θ. Intuitively $\varsigma(n)$ holds if and only if there is some $u \in Z$ so that n is the least k so that $(\rho_0^* * \rho_1^u)_{\text{even}}(k) = 1$. Let $L = \{n \in \omega : \varsigma(n)\}$. L is a recursively enumerable set.

First, one seeks to show that $L \subseteq \omega \setminus S$. Suppose $n \in L$. Let $u \in Z$ be such that n is the least natural number k so that $(\rho_0^* * \rho_1^u)_{\text{even}}(k) = 1$. If $n \in S$, then $\zeta(n)$ holds. Thus by definition of ψ, $\psi((\rho_0^* * \rho_1^u)_{\text{even}})$ holds. However, $u \in Z$ implies that $u \in \Psi[Y] \subseteq U_0$. By definition of ρ_1^u, $(\rho_0^* * \rho_1^u)_{\text{odd}} = u\hat{\ }\bar{0}$. Since $u \subset (\rho_0^* * \rho_1^u)_{\text{odd}}$ and $u \in U_0$, one has that $\varphi_0((\rho_0^* * \rho_1^u)_{\text{odd}})$. Thus Player 2 has won. Contradiction. This shows that $n \in L$ implies that $n \notin S$.

Next, one seeks to show that L is infinite. Suppose L is finite. Then there is a $P \in \omega$ so that for all $n \in \omega$, $n \in L$ implies $n < P$. Let $G = \{s \in Y : |s| \leq P\}$ which is a finite set. Thus $\Psi[G] = \{\Psi(s) : s \in G\}$ is a finite set. Since Z is infinite, $Z \setminus \Psi[G]$ is nonempty. Let $u \in Z \setminus \Psi[G]$. There is some $v \in Y$ so that $\Psi(v) = u$ and one must have that $|v| > P$. Since $u \in U_0$, $u \subseteq u\hat{\ }\bar{0}$, and $(\rho_0^* * \rho_1^u)_{\text{odd}} = u\hat{\ }\bar{0}$, one has $\varphi_0((\rho_0^* * \rho_1^u)_{\text{odd}})$. Since ρ_0^* is a Player 1 winning strategy in W^2_{ψ,φ_0}, one must have that $\neg\psi((\rho_0^* * \rho_1^u)_{\text{even}})$. First this means that there exists an n so that $(\rho_0^* * \rho_1^u)_{\text{even}}(n) = 1$. Let n^* be the least such n. By definition of L, $n^* \in L$. Since one was assuming that L is bounded by P, one must have that $n^* < P$. Also note that $\neg\psi((\rho_0^* * \rho_1^u)_{\text{even}})$ implies that $\neg\zeta(n^*)$. Since $v \in Y$ and hence $\varpi(v)$, one has that there exists $\hat{u} \in U_1$ so that $v \subset \hat{u}$. Since $|v| > P$, $|\hat{u}| > P$. Let $\hat{\rho}_1 : {}^{<\omega}2 \to 2$ be the Player 2 strategy which simply has Player 2 put down $\hat{u}$ and then

0 forever. Precisely

$$\hat{\rho}_1(s) = \begin{cases} \hat{u}(k) & |s| = 2k+1 \wedge k < |\hat{u}| \\ 0 & \text{otherwise} \end{cases}$$

Note that $\hat{\rho}_1$ is recursive and hence belongs to REC. Now since $v \subseteq \hat{u}$ and $|v| > P$, one has that ρ_1^u and $\hat{\rho}_1$ agree on all strings of length less than P. Thus $(\rho_0^* * \hat{\rho}_1)_{\text{even}} \upharpoonright P = (\rho_0^* * \rho_1^u)_{\text{even}} \upharpoonright P$. Since $n^* < P$, n^* is also the least n so that $(\rho_0^* * \hat{\rho}_1)_{\text{even}}(n) = 1$. Since it was noted above that $\neg\zeta(n^*)$, one must have that $\neg\psi((\rho_0^* * \hat{\rho}_1)_{\text{even}})$. Also by definition of $\hat{\rho}_1$, one has that $(\rho_0^* * \hat{\rho}_1)_{\text{odd}} = \hat{u}\hat{\ }\bar{0}$. Since $\hat{u} \in U_1$ and $\hat{u} \subseteq (\rho_0^* * \hat{\rho}_1)_{\text{odd}}$, one has that $\varphi_1((\rho_0^* * \hat{\rho}_1)_{\text{odd}})$. Thus $\neg\varphi_0((\rho_0^* * \hat{\rho}_1)_{\text{odd}})$. This implies that Player 2 using $\hat{\rho}_1$ has defeated Player 1 using ρ_0^*. This contradicts that ρ_0^* was a Player 1 winning strategy in W^2_{ψ,φ_0}. Thus this shows that L is infinite.

It has been shown that L is an infinite recursively enumerable subset of $\omega \setminus S$. Since S is a simple set, $\omega \setminus S$ is immune meaning $\omega \setminus S$ cannot have an infinite recursively enumerable subset. Contradiction.

Since neither Case I nor Case II can occur, one has that W^2_{ψ,φ_0} is not determined in REC. ψ is $\Sigma^0_1 \vee \Pi^0_1$ so $\neg\psi$ is $\Sigma^0_1 \wedge \Pi^0_1$. Since the determinacy of W^2_{ψ,φ_0} is equivalent to the determinacy of $W^2_{\neg\psi,\neg\varphi_0}$, one has that $W^2_{\neg\psi,\neg\varphi_0}$ is not determined. Thus $(\Sigma^0_1 \wedge \Pi^0_1, \Delta^0_1)$-$\mathsf{WDET}^2$ fails in REC. $\square$

Theorem 2.1 motivates why Fact 1.4 (3) is the preferred form of $\mathsf{WKL_0}$. The argument for Theorem 2.1 suggests an abstract argument to show $(\Sigma^0_1 \wedge \Pi^0_1, \Delta^0_1)$-$\mathsf{WDET}^2$ implies $\mathsf{WKL_0}$. Note if one assumes the failure of WKL and replacing the Π^0_1 immune set with an arbitrary Π^0_1-formula with infinitely many solutions, the argument gives a procedure for producing an infinite set of solutions to the Π^0_1 formula. However Fact 1.2 implies this is equivalent to $\mathsf{ACA_0}$. This is impossible since $\mathsf{ACA_0}$ can prove $\mathsf{WKL_0}$. The formal details is given in the next theorem whose proof is very similar to the argument in Theorem 2.1.

Theorem 2.2. $\mathsf{RCA_0}$ *proves that* $(\Sigma^0_1 \wedge \Pi^0_1, \Delta^0_1)$-$\mathsf{WDET}^2$ *implies* $\mathsf{WKL_0}$.

Proof. Suppose $\mathsf{WKL_0}$ fails. By Fact 1.4 (3), there exist Σ^0_1-formulas $\varphi_0(f)$ and $\varphi_1(f)$ in one free set variable f so that $(\forall f)(\varphi_0(f) \Leftrightarrow \neg\varphi_1(f))$ and there is no finite set F with the property that $(\forall f)(\varphi_0(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in F))$. As mentioned above, $\mathsf{RCA_0}$ proves there are quantifier free formulas θ_0 and θ_1 so that $(\forall f)(\varphi_0(f) \Leftrightarrow (\exists n)\theta_0(f \upharpoonright n))$ and $(\forall f)(\varphi_1(f) \Leftrightarrow (\exists n)\theta_1(f \upharpoonright n))$. By Δ^0_1-CA, let $U_0 = \{s \in {}^{<\mathbb{N}}2 : \theta_0(s)\}$ and $U_1 = \{s \in {}^{<\mathbb{N}}2 : \theta_1(s)\}$. Thus $(\forall f)(\varphi_0(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in U_0)$ and $(\forall f)(\varphi_1(f) \Leftrightarrow (\exists n)(f \upharpoonright n \in U_1)$. Since $(\forall f)(\varphi_0(f) \Leftrightarrow \neg\varphi_1(f))$, one has that $U_0 \cap U_1 = \emptyset$. Let $\varpi(s)$ be the following Σ^0_1 formula

$$(\exists t_0 \in U_0)(\exists t_1 \in U_1)(s \subseteq t_0 \wedge s \subseteq t_1).$$

$\varpi(s)$ means that s has not decided φ_0 in the sense that s has an extension that belongs to U_0 and has an extension that belongs to U_1. Note that ϖ has infinitely many solution. To see this, suppose ϖ has only finitely many solutions. Then there is an $N \in \mathbb{N}$ so that for all $s \in {}^{<\mathbb{N}}2$, $\varpi(s)$ implies $|s| < N$. Thus if $|s| = N$, then one must have that there is an $i \in \{0,1\}$ so that for all $f_0, f_1 \supset s$, $\varphi_i(f_0)$ and $\varphi_i(f_1)$. To see this, suppose there are $f_0, f_1 \supseteq s$ so that $\varphi_0(f_0)$ and $\varphi_1(f_1)$. Then there exists $n_0, n_1 > |s|$ so that $f_0 \upharpoonright n_0 \in U_0$ and $f_1 \upharpoonright n_1 \in U_1$. Thus $\varpi(s)$ and $|s| = N$ which contradiction the assumption that all solutions to $\varpi(s)$ have length less than N. It has been shown that if $|s| = N$, then all extensions $f \supset s$ decides φ_0 in the same way. Using bounded Σ^0_1-CA which holds in $\mathsf{RCA_0}$, let $F = \{s \in {}^{N}2 : (\exists t)(t \in U_0 \wedge t \subseteq s \vee s \subseteq t)\}$ which is a finite set. By the observation above that all s of length N has the property that all extensions of s decides membership in φ_0 in the same way, one has that if $s \in F$, then any $f \supset s$, one must have $\varphi_0(f)$. Thus one has that $\varphi_0(f)$ if and only if $f \upharpoonright N \in F$ if and only if $(\exists n)(f \upharpoonright n \in F)$. This contradicts the assumptions that no such finite set F exists for φ_0. This completes the argument that ϖ has infinitely many solutions. Since ϖ is a Σ^0_1 formula with infinitely many solutions, Fact 1.1 (5) implies that there is an infinite set Y so that $(\forall n)(n \in Y \Rightarrow \varpi(n))$.

Since one is assuming $\mathsf{WKL_0}$ fails and $\mathsf{ACA_0}$ proves $\mathsf{WKL_0}$, one must also have that $\mathsf{ACA_0}$ fails. By Fact 1.2 (4), there is a Σ^0_1-formula ζ so that $\neg\zeta$ has infinitely many solutions and there is no infinite set X so that $(\forall n)(n \in X \Rightarrow \neg\zeta(n))$, i.e. no infinite set of solutions for $\neg\zeta$.

Define the formula $\psi(f)$ in the free set variable f by

$$(\exists n)((f(n) = 1) \wedge \zeta(n) \wedge (\forall m < n)(f(m) = 0)) \vee (\forall n)(f(n) = 0).$$

Thus $\psi(f)$ means that either $f = \bar{0}$, the constant 0 function, or the first n so that $f(n) = 1$ satisfies $\zeta(n)$.

Now one will consider the Wadge game W^2_{ψ,φ_0}.

(Case I) Suppose Player 2 has a winning strategy ρ_1^* for $W^2_{\psi,v\varphi_0}$.

Let $\tilde{\rho}_0 : {}^{<\mathbb{N}}2 \to 2$ be the constant 0 function. Since $(\tilde{\rho}_0 * \rho_1^*)_{\text{even}} = \bar{0}$, one has that $\psi((\tilde{\rho}_0 * \rho_1^*)_{\text{even}})$ holds. Since ρ_1^* is a Player 2 winning strategy in W^2_{ψ,φ_0}, one must have that $\varphi_0((\tilde{\rho}_0 * \rho_1^*)_{\text{odd}})$. So there is some $n \in \omega$ so that $(\tilde{\rho}_0 * \rho_1^*)_{\text{odd}} \upharpoonright n \in U_0$. Since $\neg\zeta$ has infinitely many solutions, there is an $m > n$ so that $\neg\zeta(m)$. Define $\hat{\rho}_0 : {}^{<\mathbb{N}}2 \to 2$ by

$$\hat{\rho}_0(s) = \begin{cases} 1 & |s| = 2m \\ 0 & \text{otherwise} \end{cases}.$$

Thus $\hat{\rho}_0$ is the strategy that plays 1 on the $(m+1)^{\text{st}}$-move and 0 on all other moves. Thus m is the least k so that $(\hat{\rho}_0 * \rho_1^*)_{\text{even}}(k) = 1$. Since $\neg\zeta(m)$ holds, one has that $\neg\psi((\hat{\rho}_0 * \rho_1^*)_{\text{even}})$. Since $m > n$, $\tilde{\rho}_0$ and $\hat{\rho}_0$ agree on all strings of length less than n. Thus $(\hat{\rho}_0 * \rho_1^*)_{\text{odd}} \upharpoonright n = (\tilde{\rho}_0 * \rho_1^*)_{\text{odd}} \upharpoonright n \in U_0$. Thus $\varphi_0((\hat{\rho}_0 * \rho_1^*)_{\text{odd}})$. Player 1 using $\hat{\rho}_0$ has defeated Player 2 using ρ_1^*. Contradiction.

(Case II) Suppose Player 1 has a winning strategy ρ_0^* for W^2_{ψ,φ_0}.

By definition of Y, which is an infinite set of solutions for ϖ, for each $s \in Y$, there exists a $t \in U_0$ so that $s \subseteq t$. Thinking of ${}^{<\mathbb{N}}2$ as coded by $\mathbb{N}$, minimization ([15] Theorem II.3.5) can be used to show there is a function $\Psi : Y \to U_0$ so that $\Psi(s)$ is the least $t \in U_0$ so that $s \subseteq t$. Define a Σ^0_1 formula $\xi(t)$ by $(\exists s)(s \in Y \wedge \Psi(s) = y)$. Since Y is infinite, for each n, there is an $s \in Y$ so that $|s| > n$. Hence $|\Psi(s)| > n$ and $\xi(\Psi(s))$. Thus ξ is a Σ^0_1 formula with infinitely many solutions. By Fact 1.1 (5), there is an infinite set $Z \subseteq {}^{<\mathbb{N}}2$ so that $(\forall s)(s \in Z \Rightarrow \xi(s))$.

Using Δ^0_1-CA, define $\Theta : {}^{<\mathbb{N}}2 \times {}^{<\mathbb{N}}2 \to 2$ by

$$\Theta(u, s) = \begin{cases} u(k) & u \in Z \wedge |s| = 2k+1 \wedge k < |u| \\ 0 & \text{otherwise} \end{cases}$$

Define $\rho_1^u : {}^{<\mathbb{N}}2 \to 2$ by $\rho_1^u(s) = \Theta(u, s)$. ρ_1^u are considered Player 2 strategies that simply put downs the bits of u and then plays 0 forever. Define the formula $\varsigma(n)$ by

$$(\exists u \in Z)[(\rho_0^* * \rho_1^u)_{\text{even}}(n) = 1 \wedge (\forall m < n)((\rho_0^* * \rho_1^u)_{\text{even}}(m) = 0)].$$

$\varsigma(n)$ is a Σ^0_1 formula expressed using Θ.

First, one seeks to show that $(\forall n)(\varsigma(n) \Rightarrow \neg\zeta(n))$. Suppose $\varsigma(n)$. There is a $u \in Z$ so that n is the least k so that $(\rho_0^* * \rho_1^u)_{\text{even}}(k) = 1$. Suppose that $\zeta(n)$ holds. Then $\psi((\rho_0^* * \rho_1^u)_{\text{even}})$ holds. However, since $u \in Z$, one has $\xi(u)$ holds. This means there is some $v \in X$ so that $\Psi(v) = u$. Since Ψ maps into U_0, one has that $u \in U_0$. Since $(\rho_0^* * \rho_1^u)_{\text{odd}} = u\hat{\ }\bar{0}$ and $u \subseteq (\rho_0^* * \rho_1^u)_{\text{odd}}$, one has that $\varphi_0((\rho_0^* * \rho_1^u)_{\text{odd}})$. This shows that Player 2 using ρ_1^u defeated the Player 1 winning strategy ρ_0^*. Contradiction. Thus one must have $\neg\zeta(n)$ holds.

Next, one will show that ς has infinitely many solutions. Suppose there is a P so that $(\forall n)(\varsigma(n) \Rightarrow n < P)$. Let $G = \{s \in Y : |s| \leq P\}$ and let $\Psi[G] = \{\Psi(s) : s \in G\}$ which are both finite sets. Since Z is infinite, $Z \setminus \Psi[G]$ is nonempty so fix $u \in Z \setminus \Psi[G]$. Since $\xi(u)$ holds, there is a $v \in Y$ so that $u = \Psi(v)$. One must have that $|v| > P$. Since Ψ maps into U_0, one has that $u \in U_0$. Since $(\rho_0^* * \rho_1^u)_{\text{odd}} = u\hat{\ }\bar{0}$ and $u \in U_0$, one has that $\varphi_0((\rho_0^* * \rho_1^u)_{\text{odd}})$. Because ρ_0^* is a Player 1 winning strategy in W^2_{ψ,φ_0}, one has $\neg\psi((\rho_0^* * \rho_1^u)_{\text{even}})$. Hence letting n^* be the least n so that $(\rho_0^* * \rho_1^u)_{\text{even}}(n^*) = 1$, one has that $\varsigma(n^*)$ and $\neg\zeta(n^*)$. Since P bounds the solutions to ς, one has $n^* < P$. Since $v \in Y$ which means $\varpi(v)$, there exists a $\hat{u} \in U_1$ so that $s \subseteq \hat{u}$. Using Δ^0_1-CA, define $\hat{\rho}_1 : {}^{<\mathbb{N}}2 \to 2$ by

$$\hat{\rho}_1(s) = \begin{cases} \hat{u}(k) & |s| = 2k+1 \wedge k < |\hat{u}| \\ 0 & \text{otherwise} \end{cases}.$$

Note that $\hat{\rho}_1$ is the Player 2 strategy that puts down $\hat{u}$ and then plays 0 forever. Since $|v| > P$, one has $|\hat{u}| > P$. Thus ρ_1^u and $\hat{\rho}_1$ agree on all strings of length less than P. Thus $(\rho_0^* * \hat{\rho}_1)_{\text{even}} \upharpoonright P = (\rho_0^* * \rho_1^u)_{\text{even}} \upharpoonright P$. This implies n^* is also the least k so that $(\rho_0^* * \hat{\rho}_1)_{\text{even}}(k) = 1$. Since $\neg\zeta(n^*)$, one has that $\neg\psi((\rho_0^* * \hat{\rho}_1)_{\text{even}})$. Since $(\rho_0^* * \hat{\rho}_1)_{\text{odd}} = \hat{u}\hat{\ }\bar{0}$ and $\hat{u} \in U_1$, one has $\varphi_1((\rho_0^* * \hat{\rho})_{\text{odd}})$ and therefore $\neg\varphi_0((\rho_0^* * \hat{\rho}_1)_{\text{odd}})$. This shows that Player 2 using $\hat{\rho}_1$ has defeated ρ_0^*. This contradicts ρ_0^* being a Player 1 winning strategy. This concludes the argument showing that ς has infinitely many solutions.

Since ς is an infinite Σ^0_1 set, Fact 1.1 (5) implies that there is an infinite set K so that $(\forall n)(n \in K \Rightarrow \varsigma(k))$. However, one has already shown that $(\forall n)(\varsigma(n) \Rightarrow \neg\zeta(n))$. Thus $(\forall n)(n \in K \Rightarrow \neg\zeta(n))$. However, $\neg\zeta$ was taken to be a formula with infinitely many solution but no infinite set of solutions. Contradiction.

Since neither Case I nor Case II can occur, one has shown that W^2_{ψ,φ_0} is an undetermined game. Since the determinacy of W^2_{ψ,φ_0} is equivalent to the determinacy of $W^2_{\neg\psi,\neg\varphi_0}$, one has that $W^2_{\neg\psi,\neg\varphi_0}$ is undetermined. Since ψ is $\Sigma^0_1 \vee \Pi^0_1$ formula, $\neg\psi$ is a $\Sigma^0_1 \wedge \Pi^0_1$ formula. Hence $(\Sigma^0_1 \wedge \Pi^0_1, \Delta^0_1)$-$\mathsf{WDET}^2$ fails. ☐

Theorem 2.3. WKL_0 *proves* $(\Sigma^0_1 \wedge \Pi^0_1, \Delta^0_1)$-$\mathsf{DET}^2$*.*

Proof. Suppose $\psi(f)$ is a $\Sigma^0_1 \wedge \Pi^0_1$ formula in one free set variable f. Let $\varphi_0(f)$ and $\varphi_1(f)$ be Σ^0_1 formulas such that $(\forall f)(\varphi_0(f) \Leftrightarrow \neg\varphi_1(f))$. One will show the determinacy of the Wadge game W^2_{ψ,φ_0}.

By Fact 1.4 (3) applied to φ_0 and φ_1, there are finite sets $F_0 \subseteq {}^{<\mathbb{N}}2$ and $F_1 \subseteq {}^{<\mathbb{N}}2$ with the property that $(\forall f)(\varphi_0(f) \Leftrightarrow (\exists n)(f \restriction n \in F_0))$ and $(\forall f)(\varphi_1(f) \Leftrightarrow (\exists n)(f \restriction n \in F_1))$. Let

$$A = \{s \in {}^{<\mathbb{N}}2 : (\exists t_0 \in F_0)(\exists t_1 \in F_1)(s \subseteq t_0 \wedge s \subseteq t_1)\}$$

which exists by Δ^0_1-CA and is a finite set since all strings in A are less than the length of the longest string in the finite set $F_0 \cup F_1$. Intuitively, the elements of A consists of those strings s which are undecided for φ_0 meaning there are extensions of s belonging to F_0 and extensions of s belonging to F_1. Since A is finite, let M be the length of the longest string in A. Thus any string s of length greater than M has been decided for φ_0 meaning that there is an $i \in \{0,1\}$ so that for all $f : \mathbb{N} \to 2$ with $s \subset f$, $\varphi_i(f)$. In particular, if $|s|$ has length greater than M, then if $i \in \{0,1\}$ is such that s has an initial segment in F_i or s has an extension in F_i, then for all $f \supset s$, $\varphi_i(f)$. Using Δ^0_1-CA, define the following two finite sets.

$$K_0 = \{s \in {}^{M+1}2 : (\exists t \in F_0)(t \subseteq s \vee s \subseteq t)\} \quad K_1 = \{s \in {}^{M+1}2 : (\exists t \in F_1)(t \subseteq s \vee s \subseteq t)\}.$$

Note that $K_0 \cap K_1 = \emptyset$, $K_0 \cup K_1 = {}^{M+1}2$, and for all $i \in \{0,1\}$, forall all $s \in {}^{M+1}2$, and for all $f : \mathbb{N} \to 2$, if $s \in K_i$ and $f \supset s$, then $\varphi_i(f)$.

Consider the sentence θ asserting

$$(\exists s)[s \in {}^{<\mathbb{N}}2 \wedge |s| > M \wedge (\exists f_0)(\exists f_1)(\neg\psi(f_0) \wedge \psi(f_1) \wedge s \subset f_0 \wedge s \subset f_1)].$$

(Case I) θ holds.

Let u, g_0, and g_1 witness the existential quantifiers on the variables s, f_0, and f_1 respectively in the sentence θ. Intuitively, this means the string u (which has length longer than M) is undecided for ψ specifically because $g_0, g_1 : \mathbb{N} \to 2$ extend u but $\neg\psi(g_0)$ and $\psi(g_1)$. Player 1 wins W^2_{ψ,φ_0} using the following idea for a winning strategy ρ^*_0. The Player 1 strategy ρ^*_0 will play the bits of u for its first $|u|$ moves. After Player 2 moves, let t be the $|u|$ string produced thus far by Player 2. Since $|t| = |u| > M$, t has decided φ_0 or $\neg\varphi_0$ in the sense that $t \restriction M+1$ belongs to K_0 or K_1. However, u has not yet decided to satisfy ψ or $\neg\psi$ since Player 1 still has the option to continue to play g_0 or g_1. If $t \restriction M+1 \in K_0$, then ρ^*_0 will continue by playing the bits of g_0 regardless of Player 2's moves. If $t \restriction M+1 \in K_1$, then ρ^*_0 will continue to play the bits of g_1 regardless of Player 2's moves. Let $h : \mathbb{N} \to 2$ be Player 2's infinite sequence of moves. In the first case, since $t \subset h$ and $t \restriction M+1 \in K_0$, one has that $\varphi_0(h)$ holds. However Player 1 using ρ^*_0 produced g_0 as its infinite sequence of moves. Thus $\neg\psi(g_0)$. Player 1 wins. Similarly, in the second case, since $t \subseteq h$ and $t \restriction M+1 \in K_1$, one has that $\varphi_1(h)$ and hence $\neg\varphi_0(h)$. However Player 1 using ρ^*_0 produced g_1 as its infinite sequence of moves. Thus $\psi(g_1)$. Player 1 wins. Thus ρ^*_0 is a Player 1 winning strategy.

The formal details are as follows. Using Δ^0_1-CA, define $\rho^*_0 : {}^{<\mathbb{N}}2 \to 2$ by

$$\rho^*_0(s) = \begin{cases} u(k) & |s| = 2k \wedge k < |u| \\ g_0(k) & |s| = 2k \wedge k \geq |u| \wedge s_{\mathrm{odd}} \restriction M+1 \in K_0 \\ g_1(k) & |s| = 2k \wedge k \geq |u| \wedge s_{\mathrm{odd}} \restriction M+1 \in K_1 \\ 0 & |s| = 2k+1 \text{ (irrelevant if used as a Player 1's strategy)} \end{cases}.$$

Now suppose ρ_1 is a Player 2 strategy. Let $\rho^*_0 * \rho_1$ be the joint run. Suppose $(\rho^*_0 * \rho_1)_{\mathrm{odd}} \restriction M+1 \in K_0$ and therefore $\varphi_0((\rho^*_0 * \rho_1)_{\mathrm{odd}})$. Now since $|u| \geq M+1$, $u \subset g_0$, and by the definition of ρ^*_0, one has that $(\rho^*_0 * \rho_1)_{\mathrm{even}} = g_0$. Since $\neg\psi(g_0)$, one has $\neg\psi((\rho^*_0 * \rho_1)_{\mathrm{odd}})$. Player 1 has won this run of W^2_{ψ,φ_0}. Suppose $(\rho^*_0 * \rho_1)_{\mathrm{odd}} \in K_1$ and hence $\neg\varphi_0((\rho^*_0 * \rho_1)_{\mathrm{odd}})$. Again by definition of ρ^*_0, one has that $(\rho^*_0 * \rho_1)_{\mathrm{even}} = g_1$.

Since $\psi(g_1)$, one has that $\psi((\rho_0^* * \rho_1)_{\text{even}})$. Player 1 has won. This shows that ρ_0^* is a Player 1 winning strategy.

(Case II) $\neg\theta$ holds.

This implies that for all strings s with $|s| = M+1$, s has decided ψ in the sense that every f_0 and f_1 extending s, either $\psi(f_0) \wedge \psi(f_1)$ or $\neg\psi(f_0) \wedge \neg\psi(f_1)$. For each $s \in {}^{<\omega}2$, let $s\hat{}\bar{0}$ be the sequence which has s as its initial segment followed by the constant 0 function. Using bounded $(\Sigma_1^0 \wedge \Pi_1^0)$-$\mathsf{CA}$ and bounded $(\Sigma_1^0 \vee \Pi_1^0)$-$\mathsf{CA}$ (which both follow from bounded Σ_1^0-CA of $\mathsf{RCA_0}$), let

$$H_0 = \{s \in {}^{M+1}2 : \neg\psi(s\hat{}\bar{0})\} \quad H_1 = \{s \in {}^{M+1}2 : \psi(s\hat{}\bar{0}\}.$$

Note that $H_0 \cap H_1 = \emptyset$ and $H_0 \cup H_1 = {}^{M+1}2$. If $s \in H_0$, then for all $f \supset s$, $\neg\psi(f)$, and if $s \in H_1$, then for all $f \supset s$, $\psi(f)$. Since M is the length of the longest string in A, let $u \in U$ be a string of length M and note that u has the property there is an $i_0 \in \{0,1\}$ so that $u\hat{}i_0 \in K_0$ and $u\hat{}i_1 \in K_1$ where $i_1 = 1 - i_0$.

The following is the intuitive idea of how to construct a Player 2 winning strategy ρ_1^*. For the first $|u| = M$ many moves, ρ_1^* will just put down the bits of u. After Player 1 makes the next move, let t be the length $M+1$ string consisting of Player 1's moves thus far. Since $H_0 \cup H_1 = {}^{M+1}2$, either $t \in H_0$ or $t \in H_1$. If $t \in H_0$, then t decides ψ in the sense that all extensions $f \supset t$ satisfy $\neg\psi(f)$. Thus ρ_1^* will have Player 2 play i_1 forever afterward. Let h be Player 1's infinite sequence of moves. Player 2 following ρ_1 will have $u\hat{}\bar{i_1}$ as its infinite sequence of moves. Since $t \subset h$ and $t \in H_0$, one has that $\neg\psi_0(h)$ holds. Since $u\hat{}i_1 \subset u\hat{}\bar{i_1}$ and $u\hat{}i_1 \in K_1$, one has $\varphi_1(u\hat{}\bar{i_1})$ and thus $\neg\varphi_0(u\hat{}\bar{i_1})$. Hence Player 2 has won. Similarly if $t \in H_1$, then t decides ψ in the sense that all extensions $f \supset t$ satisfy $\psi(f)$. Thus ρ_1^* will have Player 2 play i_0 forever. Let h be Player 1's infinite sequence of moves. Player 2 following ρ_1^* will have $u\hat{}\bar{i_0}$ as its infinite sequence of moves. Since $t \subseteq h$ and $t \in H_1$, one has that $\psi(h)$ holds. Since $u\hat{}i_0 \in K_0$ and $u\hat{}i_0 \subseteq u\hat{}\bar{i_0}$, one has that $\varphi_0(u\hat{}\bar{i_0})$. Thus Player 2 has won. This shows ρ_1^* is a Player 2 winning strategy.

The formal details are as follows. Using Δ_1^0-CA, define $\rho_1^* : {}^{<\mathbb{N}}2 \to 2$ by

$$\rho_1(s) = \begin{cases} u(k) & |s| = 2k+1 \wedge k < M \\ i_1 & |s| = 2k+1 \wedge k \geq M \wedge s_{\text{even}} \upharpoonright M+1 \in H_0 \\ i_0 & |s| = 2k+1 \wedge k \geq M \wedge s_{\text{even}} \upharpoonright M+1 \in H_1 \\ 0 & |s| = 2k \text{ (irrelevant if used as a Player 2's strategy)} \end{cases}.$$

Let ρ_0 be a Player 1 strategy. Let $\rho_0 * \rho_1^*$ be the joint run. Suppose $\psi((\rho_0 * \rho_1^*)_{\text{even}})$. Thus $\rho_0 * \rho_1^* \upharpoonright M+1 \in H_1$ and hence $(\rho_0 * \rho_1^*)_{\text{odd}} = u\hat{}\bar{i_0}$. Since $u\hat{}i_0 \subseteq (\rho_0 * \rho_1^*)_{\text{odd}}$ and $u\hat{}i_0 \in K_0$, one has that $\varphi_0((\rho_0 * \rho_1^*)_{\text{odd}})$. Thus Player 2 has won. Suppose $\neg\psi((\rho_0 * \rho_1^*)_{\text{even}})$. Thus $(\rho_0 * \rho_1^*)_{\text{even}} \upharpoonright M+1 \in H_0$ and hence $(\rho_0 * \rho_1^*)_{\text{odd}} = u\hat{}\bar{i_1}$. Since $u\hat{}i_1 \subseteq (\rho_0 * \rho_1^*)_{\text{odd}}$ and $u\hat{}i_1 \in K_1$, one has that $\varphi_1((\rho_0 * \rho_1^*)_{\text{odd}})$ and thus $\neg\varphi_0((\rho_0 * \rho_1^*)_{\text{odd}})$. Thus Player 2 has won. It has been shown that ρ_1^* is a Player 2 winning strategy for W^2_{ψ,φ_0}.

The game W^2_{ψ,φ_0} is determined. ☐

Theorem 2.4. $\mathsf{RCA_0}$ *proves that* $(\Sigma_1^0 \wedge \Pi_1^0, \Delta_1^0)$-$\mathsf{WDET}^2$ *is equivalent to* $\mathsf{WKL_0}$.

Proof. This follows from Theorem 2.2 and Theorem 2.3. ☐

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