The Hausdorff Dimension of Limit Sets of Well-distributed Schottky Groups

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by

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CERTIFICATION OF APPROVAL

I certify that I have read *The Hausdorff Dimension of Limit Sets of Well-distributed Schottky Groups* by William Huanshan Chuangand that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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Let a finitely generated Schottky group G be given, and $\mathbb{B}^2$ be a Poincaré disk model of two-dimensional hyperbolic space. An unsolved problem is: How to find an exact value of Hausdorff dimension of the limit set $L(G)$ when $L(G) \neq \partial\mathbb{B}^2$?

To solve one specific case of this problem, a well-distributed Schottky group $\Gamma = \langle T_1, T_2, ..., T_m \rangle$ is defined. Our main theorem gives sharp bounds on the critical exponent of Poincaré series, and the theorem was proved based on: properties of Poincaré series, isometric circles, and some nice properties come with the definition of well-distributed Schottky group, especially that we found a way to reconstruct the orbit $\Gamma(0)$ by using only two operators T and R for any $m \in \mathbb{Z}^+$.

Following the proof of the main theorem, for the first time, an exact form of Hausdorff dimension for all possible well-distributed Schottky groups of rank two was conjectured and used to generate results against the best approximation derived using McMullen's algorithm.

I certify that the Abstract is a correct representation of the content of this thesis.

Chun-Kit Lai, PhD, Chair of the Thesis Committee Date

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Chapter 1

Introduction

The purpose of this chapter is to establish the historical context and motivation, as well as to describe the format of this paper.

To begin, consider the following set of points on the complex plane.

Definition 1.1. The set of all complex numbers with modulus less than one is called *the unit disk* and is denoted by $\mathbb{B}^2$; that is,

$$\mathbb{B}^2 = \{z \in \mathbb{C} : |z| < 1\},$$

and define the unit circle as its boundary, denoted by $\partial\mathbb{B}^2$.

Secondly, consider a set of functions $\{T_1, ..., T_m, m \geq 2, m \in \mathbb{Z}\}$, where each $T_i : \mathbb{B}^2 \to \mathbb{B}^2$, and consider that there are $2m$ disjoint closed disks associated to the set:

$$K(T_1), K(T_1^{-1}), K(T_2), K(T_2^{-1}), \ldots, K(T_m), K(T_m^{-1})$$

which boundaries are all orthogonal to the boundary of the unit disk $\partial\mathbb{B}^2$. Furthermore, we are taking intersections to make partial closed disk $D(T_i)$ and $D(T_i^{-1})$; that is,

$$D(T_i) = K(T_i) \cap \mathbb{B}^2,$$

and

$$D(T_i^{-1}) = K(T_i^{-1}) \cap \mathbb{B}^2.$$

If we have each T_i taking $\mathbb{B}^2 \setminus D(T_i^{-1})$ onto $\mathrm{int}(D(T_i))$, and T_i^{-1} taking $\mathbb{B}^2 \setminus D(T_i)$ onto $\mathrm{int}(D(T_i^{-1}))$, then the group generated by these conformal mappings is a finitely generated Schottky group denoted by $G = \langle T_1, T_2, ..., T_m \rangle$ (see section 3.1).

Then the associated Poincaré series (see section 3.2) of G is:

$$\mathcal{P}(G,t) := \sum_{g_i \in G} \exp\left\{-t\rho_{\mathbb{B}^2}(z, g_i(z))\right\}$$

where $\delta(G) := \inf\{t \in \mathbb{R} : \mathcal{P}(G,t) < \infty\} = \sup\{t \in \mathbb{R} : \mathcal{P}(G,t) = \infty\}$ is called the exponent of convergence of the Poincaré series, or the critical exponent, and $z, g_i(z) \in \mathbb{B}^2$. Furthermore, the series is independent of z.

The following are five top historical reasons that highlight the need of creating a systematic method for computing the critical exponent of the Poincaré series, albeit the technical specifics will not be discussed in the paper. Let G be a finitely generated Schottky group.

- In the prime geodesic theorem[11], $\delta(G)$ plays a role $z = \delta(G) + it$ which is similar to the line $z = 1$ in the prime number theorem where the Riemann zeta function has its only singularity at the point $z = 1$ and no zeros on this line as proved by von Mangoldt in 1895.

- In the spectral gap problem of hyperbolic surfaces, a hyperbolic surface has an essential spectral gap if it can meet some conditions in the theorem they proved[13], meaning that there are finitely many zeros within the region $\Re(s) > \frac{1}{2} - \beta = \delta(G)$.
- In $\text{AdS}_3/\text{CFT}_2$, $\delta(G)$ is a criterion for the critical exponent of a scalar field to have stable solutions in 3d quantum gravity[16].
- It might also be viewed as an illustration of the philosophy of Langlands program that Dirichlet characters, algebraic curves, and automorphic forms are all interrelated[14, 19, 20, 21, 24] because of the following three reasons:
  - (i) $\delta(G)$ uniquely determines the first resonance of the corresponding Selberg zeta function of G, and the first resonance is the lowest eigenvalue of eigenfunctions of the Laplacian operated on a Riemann surface (algebraic curve) $\mathbb{H}^2/G$,
  - (ii) $\delta(G)$ is also the convergence exponent of Poincaré series of G, and this series can be written into a Dirichlet $L-$function, and
  - (iii) eigenfunctions of $\mathbb{H}^2/G$ are automorphic forms.

Hence, an exact connection between number theory and harmonic analysis might possibly be established.

- Furthermore, since the Patterson-Sullivan measure μ is a constant multiple of the Hausdorff measure $H^{\delta}|_{L(G)}$, by Sullivan's theorem[31], we can have the Hausdorff dimension of the limit set of G equals the critical exponent, i.e. $\dim_H L(G) = \delta(G)$.

Thus, the development of a technique to precisely compute the critical exponent $\delta(G)$ of a Poincaré series of a Schottky group G has a long history. However, except for the case when $\delta(G) = 1$, so far there is no explicit formula to describe $\delta(G)$. All we have are approximations. The link between a specific given finitely generated Schottky group G to the exact value of its critical exponent $\delta(G)$ has been missing since the notions were defined.

The earliest modern investigations of the behavior of the exponent of convergent Poincaré series were done by Akaza and Beardon independently[4, 5, 7, 8, 9]. Their studies focused on determining the range of $\delta(G)$ of some specific type of discrete group G. In [23], McMullen constructed an algorithm to estimate $\delta(G)$ numerically. However, its accuracy is up to $O(N)$, i.e. given N digits, the accuracy can be approximately accurate after up to $C \cdot N$ iterations of applying McMullen's algorithm, where C is a constant. Furthermore, in implementing the algorithm, one extra assumption was added in [23] which is it assumed that at each iteration all perpendicular bisectors are assumed to be having an identical diameter. This assumption is another source of inaccuracies in addition to the systematic error generated due to the not large enough N and the finite digit of floating point operations, and eventually accumulated in each floating point operation on a computer system.

In this paper, a geometric method was developed to conjecture $\delta(G)$ exactly, where G are some Schottky groups that satisfy a predefined set of conditions. Then, this might be the first time to derive an exact value of $\lambda_1 := \delta(G)(\delta(G) - 1)$ of the lowest eigenvalue of the Laplace equation of the corresponding hyperbolic surface $\mathbb{H}^2/G$.

Chapters 2 and 3 provide the background for much of the paper.

In Chapter 2, we show how to build Rieman metric to construct the Poincaré disk model from the only required background in complex analysis, as well as how to build automorphisms specified on this model. The methods of isometric circles and perpendicular bisectors, which correspond to functions defined on the Poincaré disk, are also described in Chapter 2 as our tools for building concepts in the next chapters.

Following the introduction of functions defined on the Poincaré disk, Chapter 3 describes a specific collection of a certain type of functions called Schottky groups, which is a discrete subgroup of the projective special unitary group $\mathrm{PSU}(1,1)$. Chapter 3 also investigates the critical exponent of Poincaré series and Sullivan's theorem that proved that the critical exponent of Poincaré series of every Schottky group equals the Hausdorff dimension of the limit set of the Schottky group. In addition, in Chapter 3, we looked at certain aspects of the critical exponents, such as how it is an invariant under a conformal map and how it is independent of the point chosen to be the input of the Poincaré series.

Chapter 4 studies a special case of Schottky groups that can meet our well-distributed configuration requirement. In this chapter, our main theorem establishes tight bounds on rank two well-distributed Schottky groups. Based on our understanding, this might be the first time in history that we have this constraints for the critical exponent of Poincaré series of limit sets of Schottky groups other than the case when the limit sets coincide the boundary of the unit circle $\partial\mathbb{B}^2$. Based on our derivations of the main theorem in this chapter, we also provide a conjecture for the exact form of the Hausdorff dimension of the limit set of a rank-2 well-distributed Schottky group.

Chapter 5 provides an introduction of McMullen's algorithm which is by far the most accurate algorithm to approximate the Hausdorff dimension of limit sets of Schottky groups numerically. However, because the original source code used to get the results was never disclosed, we re-implemented the method in C to reproduce the results to ensure that our understanding of Patterson-Sullivan theory and McMullen's technique were correct. Furthermore, based on McMullen's derivation, for small angles, we derived a result that coincides to the result generated by our conjecture is also presented in this chapter.

Chapter 2

The Geometry of Isometry Groups

The purpose of this chapter is to demonstrate how to construct hyperbolic spaces in the Poincaré disk model, as well as the automorphisms given on this model. The methods of isometric circles (see Definition 2.10) and perpendicular bisectors (see Definition 2.9), which relate to functions defined on the Poincaré disk, are also explained in Chapter 2 as tools for developing notions in the following chapters.

2.1 Conformal Automorphisms

The goal of this part is to bridge the reader from complex analysis to the Poincaré disk model and hyperbolic space. We assume that the reader is familiar with complex analysis, particularly conformal mappings[1, 30], thus only required concepts and characteristics utilized in developing our main theorem, such as the Riemann metric[30, Chpater 8], geodesic, and length of geodesics, are introduced. This shall pave the way for anyone with a background in complex analysis to intuitively derive the Poincaré disk model.

Recall that one of the most important questions based on the definition of holomorphic function is: does there exist a holomorphic bijection between two open sets U and V in $\mathbb{C}$?

This question leads to the definition of conformal maps:

Definition 2.1. Let U and V be two open sets in $\mathbb{C}$ and f a complex-valued function on U. *A conformal map or biholomorphism* is a bijective holomorphic function $f : U \to V$.

If there exists such a mapping f, then U and V are called *conformally equivalent*.

The most important example is the conformal equivalence between the unit disk and the upper half-plane. The unit disk has been defined in Chapter 1. Now, recall the definition of the upper half-plane from complex analysis:

Definition 2.2. The set of all complex numbers with positive imaginary part is called *the upper half-plane* and is denoted by $\mathbb{H}^2$; that is,

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \Im(z) > 0\}.$$

One of the conformal mappings between the unbounded set $\mathbb{H}^2$ and the unit disk $\mathbb{B}^2$ is denoted by $\Psi : \mathbb{H}^2 \to \mathbb{B}^2$:

$$\Psi(z) = i\frac{z-i}{z+i}$$

which maps 0 to $-i$, 1 to 1, and -1 to $-i$. The point i on the circle corresponds to the point at infinity of the upper half-plane.

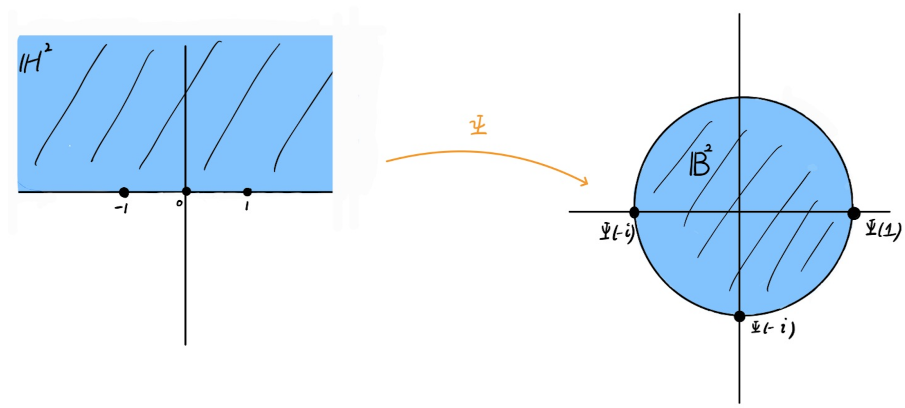

Figure 2.1: The conformal mapping $\Psi : \mathbb{H}^2 \to \mathbb{B}^2$.

Let $\Omega \subseteq \mathbb{C}$ be a region. A conformal mapping from Ω to Ω is *a conformal automorphism of* Ω. The group of conformal automorphisms of Ω is denoted by $\mathrm{Aut}(\Omega)$.

The automorphisms of the unit disk and the upper half-plane are two notable examples of conformal mappings in the aforementioned form.

Assuming the reader has a background in complex analysis, the most natural approach to grasp the Poincaré disk model is through the derivation of the automorphism of the upper half-plane - which is why the upper half-plane is presented even though it is not directly used in the demonstration of our main theorem.

However, a natural way to derive the automorphism of the upper half-plane is based on the automorphism of the unit disk - this can also demonstrate the usage of conformal mappings; that is, a difficult problem can generally have a simple solution provided a proper conformal mapping.

It can be proved by using Schwarz lemma[30] that if f is an automorphism of the unit disk, then there exist $\theta \in \mathbb{R}$ and $a \in \mathbb{B}^2$ such that

$$f(z) = e^{i\theta} \frac{a - z}{1 - \overline{a}z}.$$

Definition 2.3. *The special unitary group* $SU(1,1)$ is the set of all matrices of the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

that satisfy the following three conditions[30]:

(i) a, b, c, and d are complex numbers,

(ii) $\det(M) = 1$,

(iii) M preserves the following hermitian form on $\mathbb{C}^2 \times \mathbb{C}^2$:

$$\langle X, Y \rangle = x_1\overline{y}_1 - x_2\overline{y}_2,$$

where $X = (x_1, x_2)$ and $Y = (y_1, y_2)$; that is, for all $X, Y \in \mathbb{C}^2$

$$\langle MX, MY \rangle = \langle X, Y \rangle \, .$$

It can be showed that

$$\mathrm{SU}(1,1) = \{M \in \mathrm{U}(1,1) : \det(M) = 1\} = \left\{ \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} : |\alpha|^2 - |\gamma|^2 = 1 \right\},$$

where $\alpha \in \mathbb{C}$ and $\gamma \in \mathbb{C}$.

Definition 2.4. Let z, a, b, c, and d be complex numbers. Then, a mapping with of the following fractional form

$$z \longmapsto \frac{az + b}{cz + d},$$

is called *fractional linear transformations* or *linear fractional transformations.*

Since for every matrix M in $\mathrm{SU}(1,1)$ we can associate a fractional linear transformation

$$f_M(z) = \frac{\alpha z + \overline{\gamma}}{\gamma z + \overline{\alpha}},$$

and then we have $f_M = f_{-M}$ since

$$\frac{\alpha z + \overline{\gamma}}{\gamma z + \overline{\alpha}} = \frac{-\alpha z - \overline{\gamma}}{-\gamma z - \overline{\alpha}},$$

so we would like to take the quotient $\mathrm{SU}(1,1)/\{\pm\,\mathrm{id}\}$. This $\{\pm\,\mathrm{id}\}$ is proved to be the center of $\mathrm{SU}(1,1)$ denoted by $Z(\mathrm{SU}(1,1))$ in the following.

To find the center $Z(\mathrm{SU}(1,1))$, consider that suppose $A \in Z(\mathrm{SU}(1,1))$, then we have $AB = BA$ for all $B \in \mathrm{SU}(1,1)$. Then, A can be solved by firstly taking

$$B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and this gives

$$A = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}$$

where $|a|^2 = 1$ and $a \in \mathbb{C}$, since $A \in \mathrm{SU}(1,1)$. Secondly, we can let

$$B = \begin{pmatrix} \sqrt{2} & -i \\ i & \sqrt{2} \end{pmatrix},$$

then this gives $a = \pm 1$. Therefore, we can obtain the center of $\mathrm{SU}(1,1)$ as follows

$$Z(\mathrm{SU}(1,1)) = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

It can also be proved that $\mathrm{SU}(1,1)/Z(\mathrm{SU}(1,1))$ is isomorphic to the group of automorphisms of $\mathbb{B}^2$ by using the following map[30]:

$$e^{2i\theta}\frac{z-\alpha}{1-\overline{\alpha}z} \longmapsto \begin{pmatrix} \frac{e^{i\theta}}{\sqrt{1-|\alpha|^2}} & -\frac{\alpha e^{i\theta}}{\sqrt{1-|\alpha|^2}} \\ -\frac{\alpha e^{-i\theta}}{\sqrt{1-|\alpha|^2}} & \frac{e^{-i\theta}}{\sqrt{1-|\alpha|^2}} \end{pmatrix}.$$

This quotient group is called the projective special unitary group denoted by $\mathrm{PSU}(1,1) \simeq \mathrm{Aut}(\mathbb{B}^2)$.

Furthermore, in general, for all $n \geq 2$ dimensions, we can define a norm on the general linear group $\mathrm{GL}(n, \mathbb{C})$:

$$|A| := \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}.$$

Thus, a metric can be defined for this space:

$$d(A, B) := |A - B|.$$

Then, for subspace topology, or quotient topology, in general we have

Theorem 2.1 (Theorem 5.1.2 in [27])**.** *If N is a normal subgroup of a topological group G, then G/N is a topological group with the quotient topology.*

Since $\mathrm{SU}(1,1) \subseteq \mathrm{GL}(2, \mathbb{C})$, it has a subspace topology. Then, we take $G = \mathrm{SU}(1,1)$ and $N = Z(\mathrm{SU}(1,1))$ in the theorem to conclude that $\mathrm{PSU}(1,1)$ is a topological group.

The next goal is to derive the group of automorphisms of the upper half-plane. Denote the group of automorphisms of the upper half-plane by $\mathrm{Aut}(\mathbb{H}^2)$ and let $\phi \in \mathrm{Aut}(\mathbb{H}^2)$. Recall the conformal mapping $\Psi : \mathbb{H}^2 \to \mathbb{B}^2$, then by taking the conjugation by Ψ we have

$$\Phi : \phi \longmapsto \Psi^{-1} \circ \phi \circ \Psi.$$

Since it can also be verified that $\Phi(\varphi_1 \circ \varphi_2) = \Phi(\varphi_1) \circ \Phi(\varphi_2)$, and Φ is a composition of bijections, Φ defines an isomorphism between $\mathrm{PSU}(1,1)$ and $\mathrm{Aut}(\mathbb{H}^2)$.

After applying the isomorphism Φ to $\mathrm{PSU}(1,1)$ we can show that every automorphism of the upper half-plane takes the form

$$f_M(z) = \frac{az+b}{cz+d}$$

where $ad - bc = 1$ and $a, b, c, d \in \mathbb{R}$, and $f_M(z)$ is corresponding to a matrix M in the special linear group

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } \det(M) = 1 \right\}.$$

Then in [30], we have the converse result: every map of the form $f_M(z)$ is an automorphism of $\mathbb{H}^2$.

Since f_M is a fractional linear transformation, we have $f_M = f_{-M}$, and hence we should mod out

$$\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\};$$

that is, the center of $\mathrm{SL}(2, \mathbb{R})$.

Thus, we can derive the group of automorphisms of $\mathbb{H}^2$:

$$\mathrm{Aut}(\mathbb{H}^2) \cong \mathrm{SL}(2, \mathbb{R}) / \{\pm \mathrm{id}\}.$$

This group is also called the projective special linear group denoted by $\mathrm{PSL}(2, \mathbb{R})$.

2.2 Basics of Hyperbolic Geometry

The Poincaré disk model of the two-dimensional hyperbolic space will be constructed based on the upper half-plane model. Hence, we introduce the upper half-plane model first.

2.2.1 The Upper Half-plane $\mathbb{H}^2$ Model

To introduce the upper half-plane model, recall that in Euclidean geometry on $\mathbb{R}^2$, by collecting all possible rigid motions: identity map, translation, rotation about a pint, reflection in a line, and glide reflection, we have the Euclidean group $\mathrm{E}(2)$. Furthermore, the length of the line segment is an invariant under these types of mappings and the Euclidean metric that can be checked by using the line integral defined on the 2-dimensional Euclidean space $(\mathbb{R}^2, d_{\mathrm{E}})$.

Point sets	$\mathbb{R}^2$	$\mathbb{H}^2$	$\mathbb{B}^2$
Line elements ds	$\|dz\|$	$\frac{\|dz\|}{\Im z}$	$\frac{2\|dz\|}{1-\|z\|^2}$
Isometry subgroups	E(2)	$\mathrm{PSL}(2,\mathbb{R})$	$\mathrm{PSU}(1,1)$
Geodesics	Lines	Vertical lines, Semicircles	Partial circles, Diameters

Table 2.1: Isometry Groups: A comparison table for the spaces: $(\mathbb{R}^2, d_{\mathrm{E}}), (\mathbb{H}^2, d_{\mathbb{H}^2})$, and $(\mathbb{B}^2, \rho_{\mathbb{B}^2})$.

Similarly, the idea is that we want to define a notion of length, i.e. a metric on $\mathbb{H}^2$, that is invariant under $\mathrm{PSL}(2,\mathbb{R})$. Suppose now we have the group of automorphisms of $\mathbb{H}^2$ which is $\mathrm{PSL}(2,\mathbb{R})$, and we want to use elements in $\mathrm{PSL}(2,\mathbb{R})$ to move the line segment. Recall also that each element in $\mathrm{PSL}(2,\mathbb{R})$ is a conformal mapping. Since the line segment is a geometric object, we would like to preserve its length by using the automorphism group $\mathrm{PSL}(2,\mathbb{R})$.

To achieve this goal, let us recall the line integrals in complex analysis[30]: Given a continuous function g on $\mathbb{C}$, for some piecewise-smooth path $\gamma : [t_i, t_f] \to \mathbb{C}$, we define the line integral as:

$$\int_\gamma g(z)|dz| = \int_{t_i}^{t_f} g(\gamma(t))|\gamma'(t)|dt.$$

Likewise, on $\mathbb{H}^2$ we define a more general notion in the following:

Definition 2.5. Let g be a continuous nonzero function on $\mathbb{H}^2$, then *the line integral is invariant under PSL*$(2,\mathbb{R})$ if for any piecewise smooth path $\gamma : [t_i, t_f] \to \mathbb{H}^2$ and any $h \in \mathrm{PSL}(2,\mathbb{R})$, we have

$$\int_{\gamma} g(z)|dz| = \int_{h(\gamma)} g(z)|dz|.$$

Remark: By using the parametric expression:

$$\int_{\gamma} g(z)|dz| = \int_{t_i}^{t_f} g(\gamma(t))|\gamma'(t)|dt,$$

the above definition is equivalent to say if

$$\int_{t_i}^{t_f} g(\gamma(t))|\gamma'(t)|dt = \int_{t_i}^{t_f} g(h(\gamma(t)))|h'(\gamma(t))|dt,$$

for any $h \in \mathrm{PSL}(2,\mathbb{R})$, then *the line integral is invariant under* $PSL(2,\mathbb{R})$.

Furthermore, for every $T \in \mathrm{PSL}(2,\mathbb{R})$, $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have a corresponding fractional linear map[1]

$$S_T(z) = \frac{az+b}{cz+d},$$

and recall that in complex analysis, when we proved the fractional linear transformations that are corresponding to $\mathrm{PSL}(2,\mathbb{R})$ are automorphisms of $\mathbb{H}^2$, we used an important identity[30][Chapter 8, Theorem 2.4]:

[1]Starting from the next section, we will use the same symbol T to represent the fractional linear map, $z \mapsto \frac{az+b}{cz+d}$, and the corresponding matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\frac{1}{\Im\left(S_T(z)\right)}\left|\frac{dS_T}{dz}\right| = \frac{1}{\Im(z)}.$$

Then, it can be showed that for every positive real number k, the following line element of the line integral on $\mathbb{H}^2$ is invariant under $\mathrm{PSL}(2,\mathbb{R})$:

$$ds = \frac{k}{\Im(z)}|dz|.$$

We are going to take $k = 1$ in the following derivation. Since our goal is to define a way to estimate the arc length of a line segment such that it is invariant under $\mathrm{PSL}(2,\mathbb{R})$, the above observation implies that if we define length with the following line element:

$$ds = \frac{|dz|}{\Im(z)},$$

then we have

$$\text{Arc length}_{\mathbb{H}^2}(f) = \int_f \frac{1}{\Im(z)}|dz|,$$

where $f : [t_i, t_f] \to \mathbb{H}^2$ is a piecewise-smooth path. This length is invariant under $\mathrm{PSL}(2,\mathbb{R})$, since if we take $T \in \mathrm{PSL}(2,\mathbb{R})$, then for $T(f)$, by using the above identity, we have

$$\text{Arc length}_{\mathbb{H}^2}(f) = \int_f \frac{1}{\Im(z)}|dz| = \int_{t_i}^{t_f} \frac{1}{\Im(f(t))}|f'(t)|dt$$

$$= \int_{T(t_i)}^{T(t_f)} \frac{1}{\Im(T(f))}|T'(f)|df = \text{Arc length}_{\mathbb{H}^2}(T \circ f).$$

For any two points z_1, z_2, let $K(z_1, z_2)$ be the set of all piecewise-smooth paths connecting z_1 and z_2. Then consider the function:

$$d_{\mathbb{H}^2} : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}$$

defined by:

$$d_{\mathbb{H}^2} = \inf \left\{ \text{Arc length}_{\mathbb{H}^2}(f) : f \in K(z_1, z_2) \right\}.$$

We immediately derive the invariance of $d_{\mathbb{H}^2}$, namely

$$d_{\mathbb{H}^2}(Tz_1, Tz_2) = d_{\mathbb{H}^2}(z_1, z_2)$$

and this proves that each such $T \in \mathrm{PSL}(2, \mathbb{R})$ is an isometry of $(\mathbb{H}^2, d_{\mathbb{H}^2})$.

Recall the definitions of the isometry:

Definition 2.6. Let (X, d) be a metric space. If f is on X that preserves distance, i.e. for a pair of points $z_1, z_2 \in X$:

$$d(z_1, z_2) = d(f(z_1), f(z_2)),$$

then f is called *an isometry of the metric space* (X, d).

It can be proved that the set of isometries of the metric space of $(\mathbb{H}^2, d_{\mathbb{H}^2})$ is the projective special linear group $\mathrm{PSL}(2, \mathbb{R})$.

Given two points to find the equation that describes the path that gives the infimum with respect to the metric $d_{\mathbb{H}^2}$, all it takes is only the calculus level math. However, because this shortest path is an important building block in this paper, it is worthwhile to provide the derivation in order to construct a strong bridge for every reader with a background earned from undergraduate and graduate level courses to the development of our main theorem.

Definition 2.7. Let $z_1, z_2 \in \mathbb{H}^2$ be distinct. The smooth shortest path on $\mathbb{H}^2$ connects z_1 and z_2 is called *the geodesic* denoted by $[z_1, z_2]$.

The method to derive the geodesic $[z_1, z_2]$ is called the calculus of variations[22]. Now, let us use Cartesian coordinates and denote $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. The goal is to determine the path function $y(x)$ such that the integral of the arc length

$$J := \int_{x_1}^{x_2} f\left(y(x), y'(x); x\right) dx$$

is an extremum, where $y' := \frac{dy}{dx}$, and the semicolon in f separates the independent variable x from the dependent variable $y(x)$ and its derivative $y'(x)$.

The integral J is determined by the path function $y(x)$, and the path endpoints x_1 and x_2 are fixed. After then, the function $y(x)$ is to be modified until an extreme value of J is obtained.

A parametric representation to all possible functions y is defined in the form $y = y(\alpha, x)$, such that for $\alpha = 0$, so we now have the function $y = y(0, x) = y(x)$

produces an extremum for J:

$$y(\alpha, x) = y(0, x) + \alpha\eta(x)$$

where η has a continuous first derivative and disappears at x_1 and x_2, i.e. $\eta(x_1) = \eta(x_2) = 0$.

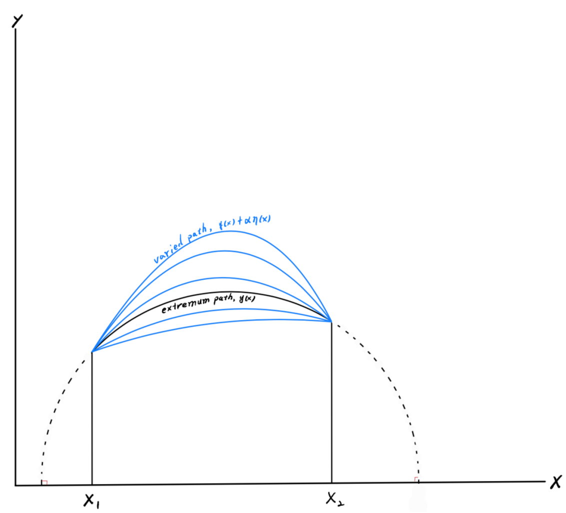

Figure 2.2: The path that makes the functional $J(\alpha)$ an extremum is the function $y(x)$. The adjacent function $y(x) + \alpha\eta(x)$ vanishes and can be near to $y(x)$, but they are not the extremum.

The integral J is redefined as a functional of the parametric α:

$$J(\alpha) = \int_{x_1}^{x_2} f\left\{y(\alpha, x), y'(\alpha, x); x\right\} dx.$$

The necessary condition that the integral have an extremum occurs when $\frac{\partial J}{\partial \alpha} = 0$ as $\alpha = 0$. Hence, to solve it, we apply the Leibniz integral rule as follows

$$\frac{\partial J(\alpha)}{\partial \alpha} = f\frac{dx_2}{d\alpha} - f\frac{dx_1}{d\alpha} + \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha} f\left\{y(\alpha, x), y'(\alpha, x); x\right\} dx.$$

Since x_1 and x_2 are constant,

$$\begin{aligned}\frac{\partial J(\alpha)}{\partial \alpha} &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right) dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta}{dx}\right) dx.\end{aligned}$$

The second term can be integrated by parts, and using $\eta(x_1) = \eta(x_2) = 0$, then we have

$$\frac{\partial J(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) \eta(x) dx.$$

Since $\eta(x)$ is arbitrary chosen, to have $\frac{\partial J(\alpha)}{\partial \alpha} = 0$ as $\alpha = 0$, we have the following Euler-Lagrange equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.$$

Since

$$ds = \frac{|dz|}{\Im(z)} = \frac{\sqrt{(dx)^2 + (dy)^2}}{y} = \frac{\sqrt{1 + y'^2}}{y} dx,$$

we have

$$f(x) = \frac{\sqrt{1 + y'^2}}{y}.$$

Then,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = -y^{-2}\left(1+y'^2\right)^{\frac{1}{2}} - \frac{yy'' - (y')^2(1+y'^2)}{y^2(1+y'^2)^{\frac{3}{2}}} = 0.$$

This Euler-Lagrange equation can be simplified to

$$yy'' + y'^2 + 1 = 0.$$

To solve the differential equation, firstly, let $u(y) = \frac{dy}{dx}$. Then we have

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = u\frac{du}{dy}.$$

Hence, we solve the following differential equation first:

$$u^2 + yu\frac{du}{dy} + 1 = 0.$$

The differential equation is separable,

$$-\int \frac{u}{1+u^2}du = \int \frac{dy}{y},$$

$$u = \frac{\pm\sqrt{c_1 - y^2}}{y}.$$

Since $u(y) = \frac{dy}{dx}$, we solve the following separable differential equation:

$$\frac{dy}{dx} = \frac{\pm\sqrt{c_1 - y^2}}{y},$$

and this gives

$$y = \pm\sqrt{c_1^2 - (x - c_2)^2}.$$

Since in $\mathbb{H}^2$, $y > 0$, we discard the negative. Then, it is a semicircle centered at the real axis at $(c_2, 0)$ and radius c_1; that is,

$$(x - c_2)^2 + y^2 = c_1^2,$$

where $y > 0$.

Furthermore, since $(x - c_2)^2 + y^2 = c_1^2 \Rightarrow \frac{x^2 - 2c_1x + y^2}{c_1} = 0$, if we take the radius c_1 to infinity

$$\lim_{c_1 \to \infty} \left(\frac{x^2 + y^2}{c_1} - 2x \right) = 0 \Rightarrow -2x = 0,$$

then after dividing both sides by -2 we have

$$x = 0$$

which is a vertical line. Since it can be translated by linear fractional maps in $\mathrm{PSL}(2, \mathbb{R})$, in $\mathbb{H}^2$, geodesics are vertical lines and semi-circles that are centered on real axis.

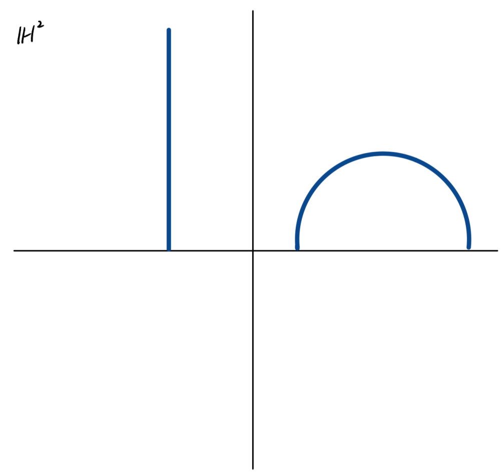

Figure 2.3: Geodesics in $\mathbb{H}^2$ are vertical lines or semicircles that are orthogonal to the boundary $\partial\mathbb{B}^2$.

2.2.2 The Poincaré Disk $\mathbb{B}^2$ Model

Now, we are ready to construct the Poincaré disk model of the two-dimensional hyperbolic space.

Recall the definition of $\mathbb{B}^2$:

$$\mathbb{B}^2 = \{z \in \mathbb{C} : |z| < 1\}.$$

The map $\Psi(z) = i\frac{z-i}{z+i}$ is a 1-1 map of $\mathbb{H}^2$ onto $\mathbb{B}^2$, hence $\rho_{\mathbb{B}^2}$ given by

$$\rho_{\mathbb{B}^2}(z,w) = d_{\mathbb{H}^2}(\Psi^{-1}z, \Psi^{-1}w),$$

where $z, w \in \mathbb{B}^2$. Then, $\rho_{\mathbb{B}^2}(z,w)$ is a metric on $\mathbb{B}^2$, and the arc length between z and w measured by the metric $\rho_{\mathbb{B}^2}(z,w)$ is also called *the hyperbolic length between z and w*. However, since

$$\frac{1}{\Im(z)} = \left|\frac{d\Psi^{-1}}{dz}\right| \frac{1}{\Im\left(\Psi^{-1}\left(z\right)\right)} = \frac{|-2|}{|z-i|^2}\frac{|z-i|^2}{1-|z|^2} = \frac{2}{1-|z|^2},$$

and with the condition $\rho_{\mathbb{B}^2}(z,w) = d_{\mathbb{H}^2}(\Psi^{-1}z, \Psi^{-1}w)$, define the line element as follows[3, 30]:

$$ds = \frac{2|dz|}{1-|z|^2}.$$

Then, to construct a metric space, we use the line element to define a metric in the following.

Firstly, for any two points in $\mathbb{B}^2$, every path function γ has an invariant length

$$\int_\gamma \frac{2|dz|}{1-|z|^2}.$$

Definition 2.8. *An isometry group* that acts on $\mathbb{B}^2$ is a collection that each of its element is a transformation denoted by $T \in \mathrm{PSU}(1,1)$ such that $\rho_{\mathbb{B}^2}(a,b) = \rho_{\mathbb{B}^2}(T(a), T(b))$.

The shortest arc, i.e. the geodesic, can be attained by firstly mapping the two distinct endpoints $z_1, z_2 \in \mathbb{B}^2$ to $\mathbb{H}^2$ using Ψ^{-1}. Then we can find the unique geodesic connects $\Psi^{-1}(z_1)$ and $\Psi^{-1}(z_2)$ by using the solutions we derived previously; that is, either a vertical line, or a semicircle $y = \sqrt{c_1^2 - (x - c_2)^2}$ which orthogonal to the real axis.

Next, recall that in complex analysis[3, 30] we have an important result for the conformal inversion map $z \longmapsto \frac{1}{z}$; that is, it maps a circle or a line into either a circle or a line. Since for any fractional linear transformation $T(z) = \frac{az+b}{cz+d}$, we can rewrite it as

$$T(z) = \frac{az + b}{cz + d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz + d},$$

if $c \neq 0$. Then, $u = Tz$ is a sequence of compositions of the following maps:

$$z \mapsto u_1 = cz + d \mapsto u_2 = \frac{1}{u_1} \mapsto u = \frac{a}{c} + \left(b - \frac{ad}{c}\right) u_2.$$

Hence, except $u_2 = \frac{1}{u_2}$, all other maps are affine maps that also map circles and lines into circles and lines.

Since the conformal mapping we used is Ψ, and it can be decomposed into the above maps, it also carries vertical lines and semi-circles on $\mathbb{H}^2$, i.e. geodesics on $\mathbb{H}^2$, to diameters (radial lines) or partial circles that are orthogonal to the boundary $\partial\mathbb{B}^2$.

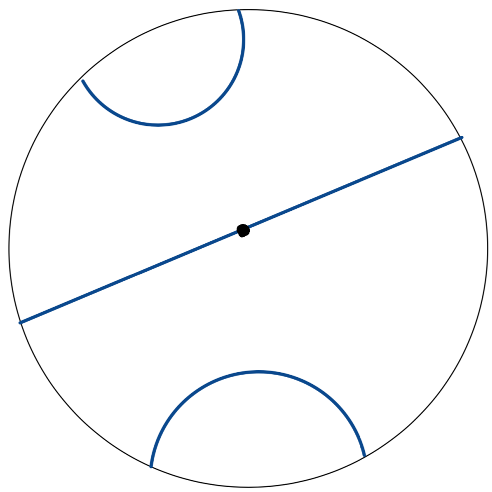

Figure 2.4: Geodesics in $\mathbb{B}^2$ are diameters (radial lines) or partial circles that are orthogonal to the boundary $\partial\mathbb{B}^2$.

That is, geodesics are partial circles or diameters orthogonal to $|z| = 1$. The hyperbolic distance measured by using the metric $\rho_{\mathbb{B}^2}(0, r)$ from 0 to $r > 0$ along a diameter γ of the unit disk is

$$\int_0^r \frac{2dr}{1 - r^2} = \ln \frac{1 + r}{1 - r}. \tag{2.1}$$

Now, to define a metric such that $\mathbb{B}^2$ becomes a metric space:

$$\rho_{\mathbb{B}^2}(a, b) := \int_\gamma \frac{2|dz|}{1 - |z|^2}$$

where $a = \gamma(0)$, $b = \gamma(1)$, and γ is a geodesic on $\mathbb{B}^2$ parameterized by $[0, 1]$.

As a result of the definition of the metric $\rho_{\mathbb{B}^2}$, the projective special unitary group $\mathrm{PSU}(1,1)$ is the isometry group of the Poincaré disk $\mathbb{B}^2$.

Let $g \in \mathrm{PSU}(1,1)$, then the following identity can be derived[10]:

$$\sinh^2\left(\frac{1}{2}\rho_{\mathbb{B}^2}(z,w)\right) = \frac{|z-w|^2}{(1-|z|^2)\,(1-|w|^2)}, \tag{2.2}$$

Remark: This equation is going to be used to show that isometric circles can coincide to perpendicular bisectors in Section 2.3.

Definition 2.9. *A perpendicular bisector of a geodesic* $[v,w]$ in $\mathbb{B}^2$ is a geodesic

$$\left\{z \in \mathbb{B}^2 : \rho_{\mathbb{B}^2}(z,v) = \rho_{\mathbb{B}^2}(z,w)\right\}.$$

Let $g \in \mathrm{PSU}(1,1)$ and $z_0 \in \mathbb{B}^2$. Define the notation:

(i) The perpendicular bisector of the geodesic between z_0 and gz_0, (z_0, gz_0), is denoted be $M_{z_0}(g)$,

(ii) $M_{z_0}(g^{-1})$ is the perpendicular bisector of the geodesic between z_0 and $g^{-1}0$, $(z_0, g^{-1}z_0)$,

(iii) the closed half-space in $\mathbb{B}^2$ bounded by $M_{z_0}(g)$ is denoted by $D_{z_0}(g)$, and

(iv) $D_{z_0}(g^{-1})$ is the closed half-space in $\mathbb{H}^2$ bounded by $M_{z_0}(g^{-1})$.

Remarks:

(a) Both $M_{z_0}(g)$ and $M_{z_0}(g^{-1})$ are also geodesics in $\mathbb{B}^2$.

(b) The reason to call $D_{z_0}(g)$ and $D_{z_0}(g^{-1})$ as half-spaces is that both of them are conformal images of the space $\mathbb{B}^2$.

Lemma 2.2. *If $g \in PSU(1,1) \setminus \{\mathrm{id}\}$, and $z_0 \in \mathbb{B}^2$ which is not fixed by g, then $g(M_{z_0}(g^{-1})) = M_{z_0}(g)$, and $g(int(D_{z_0}(g^{-1}))) = \mathbb{B}^2 \setminus D_{z_0}(g)$.*

Proof. Let $L_1 := [z_0, gz_0]$ denote the geodesic between z_0 and gz_0, and $L_2 := [z_0, g^{-1}z_0]$ be the geodesic between z_0 and $g^{-1}z_0$, and we also denote $M_1 := M_{z_0}(g)$ and $M_2 := M_{z_0}(g^{-1})$. Additionally, denote the intersection between L_i and M_i by m_i.

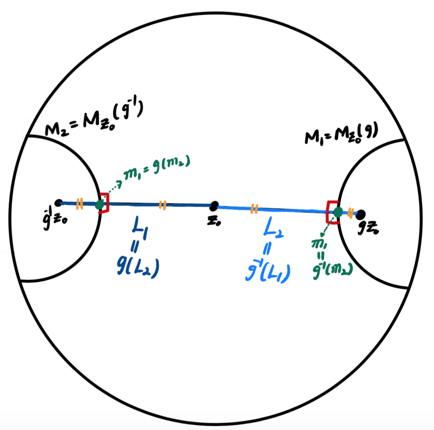

Figure 2.5: By applying the property of isometry of PSU$(1,1)$, we have $g(\rho_{\mathbb{B}^2}(g^{-1}z_0, m_2) = \rho_{\mathbb{B}^2}(z_0, gm_2)$ and $g\left(\rho_{\mathbb{B}^2}(m_2, z_0)\right) = \rho_{\mathbb{B}^2}(gm_2, gz_0)$. Since $g(L_2) = L_1$, this implies $gm_2 = m_1$.

(i) Firstly, it can be checked that

$$g(L_2) = g([z_0, g^{-1}z_0]) = [gz_0, z_0] = L_1,$$

and

$$g^{-1}(L_1) = g^{-1}([z_0, gz_0]) = [g^{-1}z_0, z_0] = L_2.$$

Furthermore, based on the above result and the property of isometry, since $m_i \in L_i$, and

$$\rho^2_{\mathbb{B}}(g^{-1}z_0, m_2) = \rho^2_{\mathbb{B}}(z_0, gm_2) = \rho^2_{\mathbb{B}}(z_0, gm_2),$$

and

$$\rho^2_{\mathbb{B}}(m_2, z_0) = \rho^2_{\mathbb{B}}(gm_2, gz_0),$$

these two conditions uniquely determine

$$gm_2 = m_1.$$

Likewise,

$$g^{-1}m_1 = m_2.$$

(ii) Since g is conformal, so the angle between L_i and M_i will be preserved. Since $M_{z_0}(g^{-1})$ is the perpendicular bisector of points z_0 and $g^{-1}(z_0)$ and $M_{z_0}(g)$ is the perpendicular bisector of points z_0 and $g(z_0)$, $M_{z_0}(g^{-1})$ orthogonally intersects L_2

at the midpoint m_2 of L_2 with angle ϑ_2, and $M_{z_0}(g)$ orthogonally intersects L_1 at the midpoint m_1 of L_1 with angle ϑ_1.

Recall that:

(1) a geodesic will be mapped into another geodesic by a non-identity isometry,

(2) angles are preserved: $\vartheta_1 = \vartheta_2 = \frac{\pi}{2}$,

(3) intersections are mapped to each other: $g(m_1) = m_2$ and $g^{-1}(m_2) = m_1$,

(4) geodesic is unique at each point $m_i \in \mathbb{B}^2$ if a tangent vector at that point m_i is given, and

(5) it can be checked that each M_i is a geodesic of $\mathbb{B}^2$.

These conditions uniquely determine the image of M_1 mapped by g is M_2, and the image of M_2 mapped by g^{-1} is M_1.

That is,

$$g^{-1}(M_1) = M_2,$$

and

$$g(M_2) = M_1.$$

(iv) To check the interior, we can verify it by using the point z_0 in the region $\mathbb{B}^2 \setminus D(g^{-1})$ and check that it will be mapped into the interior of $M_{z_0}(g)$ by definition of the perpendicular bisector $M_{z_0}(g)$. Since g is a conformally (i.e. a bijection) continuous open map, by the open map theorem[30], we have

$$g(\mathrm{int}(D_{z_0}(g^{-1}))) = \mathbb{B}^2 \setminus D_{z_0}(g).$$

□

2.3 Isometric Circles

The purpose of this section is to explain what isometric circles are. Isometric circles are easier to find than perpendicular bisectors wherever they are defined.

Definition 2.10. *The isometric circle I_T of $T = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} \in PSU(1,1)$ is defined by*

$$I_g := \left\{ z \in \mathbb{B}^2 : |\gamma z + \overline{\alpha}| = 1 \right\};$$

that is with radius $\frac{1}{|\gamma|}$ and centered at $-\frac{\overline{\alpha}}{\gamma}$.

Remark: The radius of I_g is $\frac{1}{|\gamma|}$ and it is measuring by taking the modulus of complex numbers, hence I_g is an Euclidean circle, not a hyperbolic circle. Furthermore, by considering $g(z) = \frac{\alpha z + \overline{\gamma}}{\gamma z + \overline{\alpha}}$, the meaning of this definition $|\gamma z + \overline{\alpha}| = |g'| = 1$ is that isometric circles locally fixes Euclidean length. A natural question to ask is: why isometric circles and why $\mathbb{B}^2$ with $\mathrm{PSU}(1,1)$? Because perpendicular bisector is formidable to find, but based on Lemma 2.2, it will be the building block for constructing Schottky groups. Then, since in $\mathbb{B}^2$, by choosing $z = 0$, we have

$$I_g = M_0(g^{-1}), I_g^{-1} = M_0(g),$$

and isometric circles are easy to find whenever g is given and $\gamma \neq 0$ (γ must be nonzero to make g be hyperbolic).

To see this, let $g = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}$, and using the equation 2.2, we can find z is on the perpendicular bisector of $[0, g0]$ if and only if

$$\frac{|z|^2}{(1-|z|^2)(1-|0|^2)} = \frac{|z-g0|^2}{(1-|z|^2)(1-|g0|^2)} \Rightarrow |\overline{\alpha}z - \overline{\gamma}|^2 = |z|^2.$$

Then, since $|\overline{\alpha}z - \overline{\gamma}|^2 - |\gamma z - \alpha|^2 = |z|^2 - 1$, we have

$$|\overline{\alpha}z - \overline{\gamma}|^2 - |z|^2 = |\gamma z - \alpha|^2 - 1 = 0$$

for the same $z \in \mathbb{B}^2$. Hence, the perpendicular bisector of $[0, g0]$ is the isometric circle of g^{-1}.

Furthermore, by definition, we can see that the isometric circles I_g and $I_{g^{-1}}$ have an identical radius $\frac{1}{|\gamma|}$.

Definition 2.11. Let $g \in \mathrm{PSU}(1,1)$ and $g = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}$, where $|\alpha|^2 - |\gamma|^2 = 1$, then $gz = z$ is a quadratic equation, and by using its discriminant $\mathrm{Tr}(g)^2 - 4$, g can be classified into three types:

- The operator g is *elliptic*, if $0 \leq \mathrm{Tr}^2(g) < 4$;
- The operator g is *parabolic*, if $\mathrm{Tr}^2(g) = 4$;
- The operator g is *hyperbolic*, if $\mathrm{Tr}^2(g) > 4$.

Furthermore, fixed points of g are solutions of the equation $gz = z$, and their existence is guaranteed by Brouwer fixed point theorem[17, 25, 29].

Lemma 2.3. *Let $T \in PSU(1,1) \setminus \{id\}$ and $w \in \mathbb{B}^2$ which is not fixed by T. Then, we can also have the following classification[10, 15]:*

- *The perpendicular bisector $M_w(T)$ and $M_w(T^{-1})$ intersect in $\mathbb{B}^2$ if and only if T is elliptic.*
- *The perpendicular bisector $M_w(T)$ and $M_w(T^{-1})$ are disjoint in $\overline{\mathbb{B}^2}$ if and only if T is hyperbolic.*
- *The perpendicular bisector $M_w(T)$ and $M_w(T^{-1})$ coincide at exactly one point and that point is also the fixed point of T if and only if T is parabolic.*

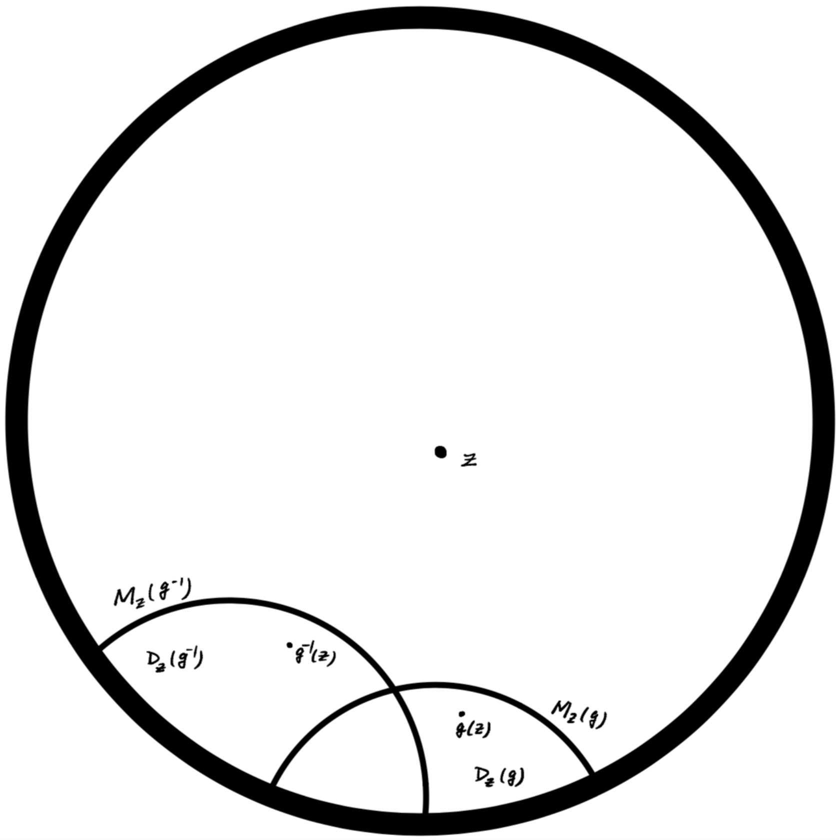

Figure 2.6: The figure demonstrates the two perpendicular bisectors $M_z(g)$ and $M_z(g^{-1})$ which shows g is elliptic.

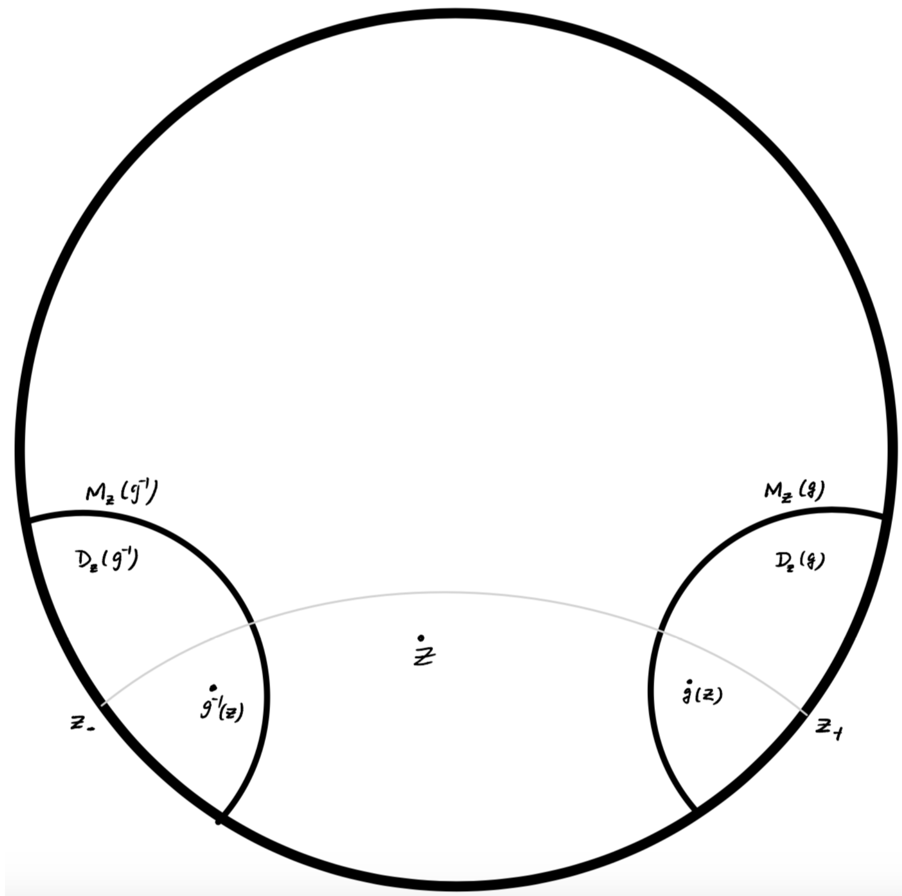

Figure 2.7: The figure demonstrates the two perpendicular bisectors $M_z(g)$ and $M_z(g^{-1})$ which shows g is hyperbolic.

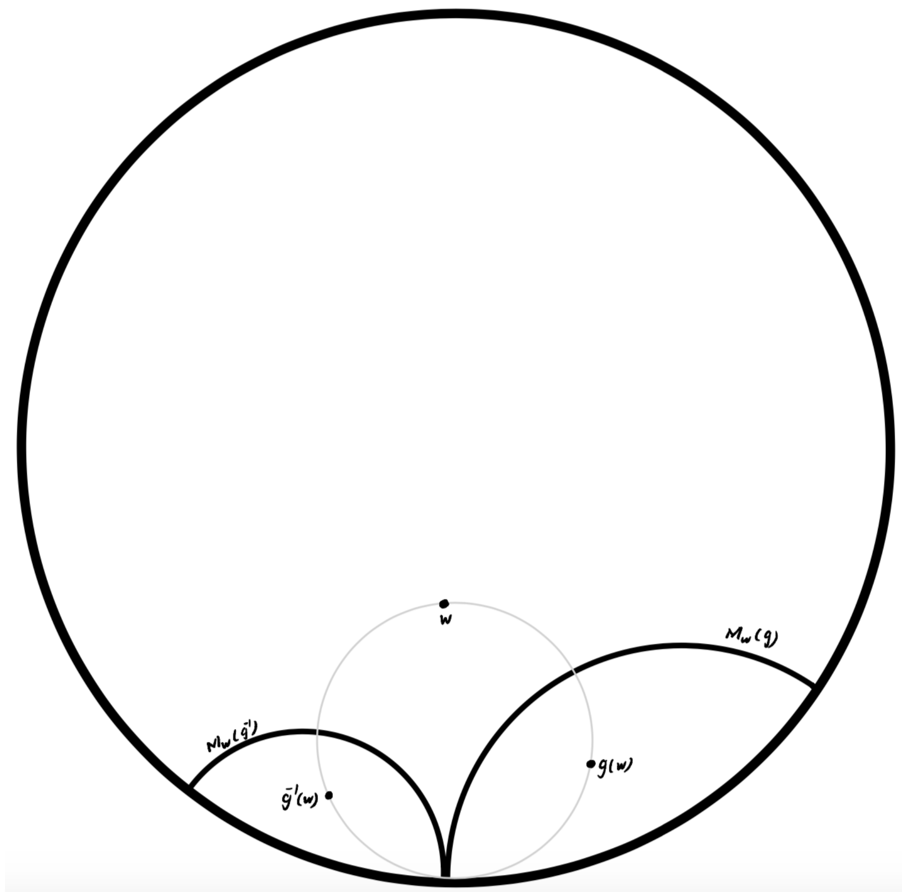

Figure 2.8: The figure demonstrates the two perpendicular bisectors $M_w(g)$ and $M_w(g^{-1})$ which shows g is parabolic.

Assume $z \in \overline{\mathbb{B}^2}$, and

$$gz = \frac{\alpha z + \overline{\gamma}}{\gamma z + \overline{\alpha}},$$

where $g \in \mathrm{PSU}(1,1)$ is non-elliptic.

Let $Tw = \Psi g^{-1}\Psi^{-1} := \frac{aw+b}{cw+d}$, where $T \in \mathrm{PSL}(2,\mathbb{R})$ and $w \in \overline{\mathbb{H}^2}$.

To find fixed points of T, we solve the equation: $Tw = w$.

If $c = 0$, then the equation $Tw = w$ is linear, and $w \in \{0, \infty\}$. If $c = b = 0$, then we have fixed points at the origin and infinity. If $c = 0$, and $b \neq 0$, then the only fixed point is at infinity.

Please notice that $\Psi 0 = -i$, and $\Psi\infty = i$.

If $c \neq 0$, then $Tw = w$ is quadratic. Then, by solving the quadratic equation $Tw = w$, and using the condition $ad - bc = 1$, we have the solutions of this equation as follows

$$w_1 := \frac{(a-d) + \sqrt{(\mathrm{Tr}(T))^2 - 4}}{2c},$$

and

$$w_2 := \frac{(a-d) - \sqrt{(\mathrm{Tr}(T))^2 - 4}}{2c}.$$

Furthermore, these fixed points are mapped to $\overline{\mathbb{B}^2}$ by Ψ, so we have:

$$z_1 := \Psi w_1,$$

and

$$z_2 = \Psi w_2.$$

The pair of fixed points z_1 and z_2 then are fixed simultaneously by g and g^{-1}.

Lemma 2.4. *Let $g \in PSU(1,1)$ be non-elliptic. Then, the limit $\lim_{n\to\infty} g^n(w)$ and*

$\lim_{n\to\infty} g^{-n}(w)$ *exist and converge to fixed points of* g *in* $\partial\mathbb{B}^2$*, where* w *is a point in* $\mathbb{B}^2$*.*

Proof. Firstly, denote two fixed points of g by z_1 and z_2, then by induction we have

$$g^{\pm n}(z_1) = z_1, g^{\pm n}(z_2) = z_2,$$

thus we have

$$\lim_{n\to\infty} g^{\pm n} z_1 = z_1,$$

and

$$\lim_{n\to\infty} g^{\pm n} z_2 = z_2.$$

Let $\epsilon > 0$ be given, and fixed an index i. Recall Lemma 2.2, result from the region enclosed by perpendicular bisector (which can be checked that its equation is an Euclidean circle) and they are nested, thus there exists $N = i + k, k \in \mathbb{Z}^+$ such that $2r_{i+k} < \epsilon$ where r_{i+k} is the radius of the perpendicular bisector of $i + k$-level. Then there exists a point z_a in the intersection of all half-space $D_w(g^n)$; that is,

$$z_a = \bigcap_{n=1}^{\infty} D_w(g^n).$$

Further, we also have

$$g(z_a) = g\left(\lim_{n\to\infty} g^n z_a\right).$$

Hence, we can obtain: $|g^{i+k}w - g^{i+k}z_a| = |g^{i+k}w - z_a| < \epsilon$.

Similarly, the above approach can be applied for proving the case of the other fixed point z_b when we consider the case of $|g^{-i-k}(w)-g^{-i-k}z_b| = |g^{-i-k}(w)-z_b| < \epsilon$, where $a \neq b$.

Since we know that for non-elliptic g, we only could have at most two distinct fixed points, and from the above we know that if we pick $w \in \mathbb{B}^2$, then

$$\lim_{n\to\infty} g^n w \in \{z_a, z_b\} \subset \partial\mathbb{B}^2,$$

and thus we must have

$$\{z_a, z_b\} = \{z_1, z_2\}.$$

(If g is parabolic, then we have $z_1 = z_2 = z_a = z_b$.)

Furthermore, it is due to Lemma 2.2, the radii of perpendicular bisectors converge to zero, and the half-spaces $D_w(g^n)$ and $D_w(g^{-n})$ converge to points on the the boundary of the unit disk, thus we have

$$\{z_a, z_b\} = \{z_1, z_2\} \subset \partial\mathbb{B}^2.$$

$\square$

In general, we denote z_+ to be the attractive fixed point, and z_- to be the repelling fixed point and they are defined as follows:

1. the attracting fixed point $z_+ = \lim_{n\to\infty} g^n(w)$ for any $w \in \mathbb{B}^2$, and
2. the repelling fixed point $z_- = \lim_{n\to\infty} g^{-n}(w)$ for any $w \in \mathbb{B}^2$.

Remark: If $T = \Psi g \Psi^{-1}$ where g is defined as above, and all entries of the matrix T are all positive or negative, then we have $z_+ = z_1$ and $z_- = z_2$. Otherwise, if $T = \Psi g^{-1} \Psi^{-1}$ where g^{-1} is defined as above, and all entries of the matrix T are all positive or negative, then we have $z_- = z_1$ and $z_+ = z_2$.

Chapter 3

Dynamics of Schottky Groups

After the introduction of functions on the Poincaré disk, the purpose of this chapter is to describe a special type of group, which is a set of non-elliptic isometries known as Schottky groups.

3.1 Schottky Groups

In the previous chapter, we have introduced the notions of isometric circles of $T_i \in \mathrm{PSU}(1,1)$ and perpendicular bisectors (which are also circles) of $[0, T_i 0]$. One nice property is for each $T_i \in \mathrm{PSU}(1,1)$, if we choose z_0 to be zero, then the perpendicular bisector of $[0, T_i 0]$ will be the isometric circle of T_i^{-1}. This can make the problem of finding the perpendicular bisector of each g_i become very easy whenever T_i is known.

In this section, we are going to use those isometric circles to be the boundary of closed disk $D_{z_0}(T_i)$ to generate Schottky groups. If T_i is an isometry of $\mathbb{B}^2$ which does not fix 0 and we let $z_0 = 0$, then the set $D_0(T_i) = D(T_i)$, and in the following each $D(T_i)$ will be further abbreviated to D_i.

To continue, we define some useful notations as follows.

Definition 3.1. Fix a finitely generated group $G \subset \mathrm{PSU}(1,1)$, where $G = \langle T_1, ..., T_m\rangle$, i.e. each T_i is a generator of the group. The *alphabet* of this group is the set $\mathcal{A} = \left\{T_1^{\pm}, T_2^{\pm}, \dots, T_m^{\pm}\right\}$.

Then, we define three notions: letter, word, and word length:

Definition 3.2. For each j such that $T_{i_j} \in \mathcal{A}$, T_{i_j} is called a *letter*. If $n > 1$, a product of n letters $T_{i_1}T_{i_2}\dots T_{i_n}$, for each $T_{i_j} \in \mathcal{A}$, $T_{i_1}T_{i_2}\dots T_{i_n}$ is called a *word* of $G\left\langle T_1, T_2, \dots, T_m\right\rangle$.

Example: Let $\Gamma = \langle A, B, C, D, E\rangle$ be a Schottky group. Then, each A, B, C, D, E, A^{-1}, B^{-1}, C^{-1}, D^{-1}, E^{-1} are all possible letters of this group. Operators such as ABC, CAB, $ABABAB$,..., and so on are words.

Now, let us define the notion of reduced word:

Definition 3.3. A product of n letters $T_{i_1}T_{i_2}\dots T_{i_n}$ in $\mathcal{A}$ is named a *reduced word* of $\Gamma(T_1, T_2, \dots, T_m)$, if $n > 1$ and for each letter in the reduced word we have $T_{i_j} \neq T_{i_{j+1}}^{-1}$ for all $1 \leq j \leq n-1$. The integer n is named the *length* of the reduced word

$$T_{i_1}T_{i_2}\dots T_{i_n} \in \Gamma(T_1, T_2, \dots, T_m)$$

denoted by $\mathrm{len}(T_{i_1}\dots T_{i_n}) = n$.

Each T_{i_j} is taken from the set $\mathcal{A} = \left\{T_j^k : 1 \leq j \leq m, k = \pm 1\right\}$, i.e. the set of all generators of Γ and their inverses which equals the alphabet of Γ.

Now, we can define our dictionary and a cross product abbreviation for each reduced word $T_{i_1}T_{i_2}\ldots T_{i_n}$:

Definition 3.4. Let $\Gamma(T_1, T_2, \ldots, T_m)$ be a finitely generated discrete subgroup of $\mathrm{PSU}(1,1)$ and $\mathcal{W}_n$ be the set of all possible reduced words $T = T_{i_1}T_{i_2}\ldots T_{i_n}$, where each $i_j \in \{1, 2, \ldots, 2m\}$ i.e. $\mathrm{len}(T) = n$. Then, *the dictionary of* Γ is

$$\bigcup_{n\geq 1} W_n,$$

and *the cross product abbreviation of* $T_{i_1}T_{i_2}...T_{i_n}$ is by defining the index I as a cross product of n alphabets: $I \in \mathcal{A}^n$ and each $i_j \in \mathcal{A}$, where $\mathcal{A}^n$ is a direct product of n many copies of $\mathcal{A}$. Therefore, we can denote

$$T_I := T_{i_1}T_{i_2}\ldots T_{i_n},$$

where

$$I = (i_1, i_2, \cdots, i_n).$$

We will write $T_I = T_{i_1}T_{i_2}\ldots T_{i_n}$ and $T_I 0 = T_I(0)$. Furthermore, we have $\#\mathcal{W}_n = m(2m-1)^{n-1}$.

Let $T_I \in \mathcal{W}_n$ and define T_I^- be the word for which the last letter of T_I is deleted.

In other words, $T_I = T_{i_1}T_{i_2}\dots T_{i_n} \in \mathcal{W}_n$ and define $T_I^- = T_{i_1}T_{i_2}...T_{i_{n-1}}$ be the word for which the last letter of T_I is deleted.

Now, we can define Schottky groups that have group actions on $\mathbb{B}^2$:

Definition 3.5. Given $D_1, \dots, D_{2m}$ disjoint closed disks which intersect the boundary of the unit disk $\mathbb{B}^2$ orthogonally where $m \geq 2$, we let non-elliptic $T_i \in \mathrm{PSU}(1,1)$, identified as the fractional linear transformation, be the mapping such that

1. $T_i(\mathbb{B}^2 \setminus D^\circ_{i+m}) = D_i$
2. $T_i^{-1}(\mathbb{B}^2 \setminus D^\circ_i) = D_{i+m}$.

Then, *a Schottky group* Γ *of rank* m is finitely generated by $T_1, \dots, T_m$. We will denote it by

$$\Gamma = \langle T_1, \dots, T_m \rangle.$$

We denote a cyclic notation for the group generators by $T_{i+m} = T_i^{-1}$.

Remark: For the inverses, by Definition 3.5, each element T_j in the set $\mathcal{A}$ is labeled by an integer i_j, and the inverse of T_j is labeled by i_{j+m}, i.e. we have the label $T_k^{-1} = T_{k+m}$.

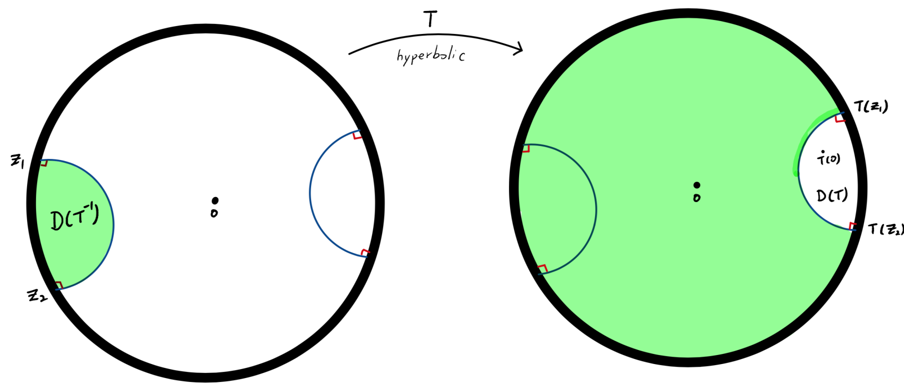

Figure 3.1: Let T be a hyperbolic isometry in PSU$(1,1)$. Then $T(D(T^{-1})) = \mathbb{B}^2 \setminus D(T)$ and $D(T) \cap D(T^{-1}) = \emptyset$.

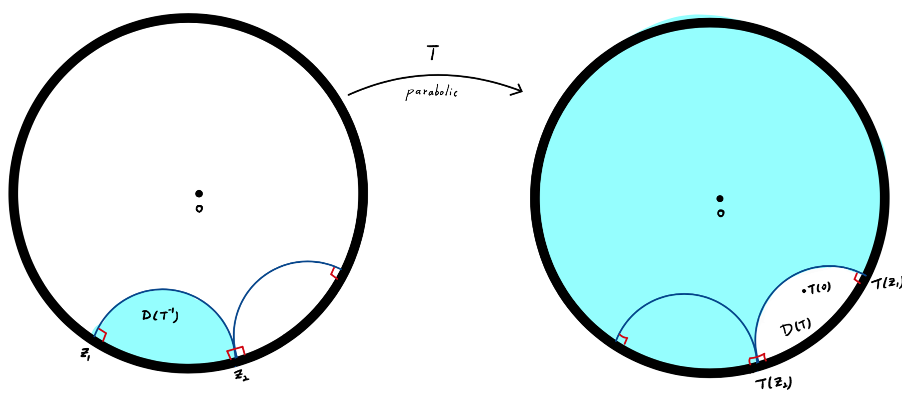

Figure 3.2: Let T be a parabolic isometry in PSU$(1,1)$. Then $T(D(T^{-1})) = \mathbb{B}^2 \setminus D(T)$ and $D(T) \cap D(T^{-1}) = z_2$.

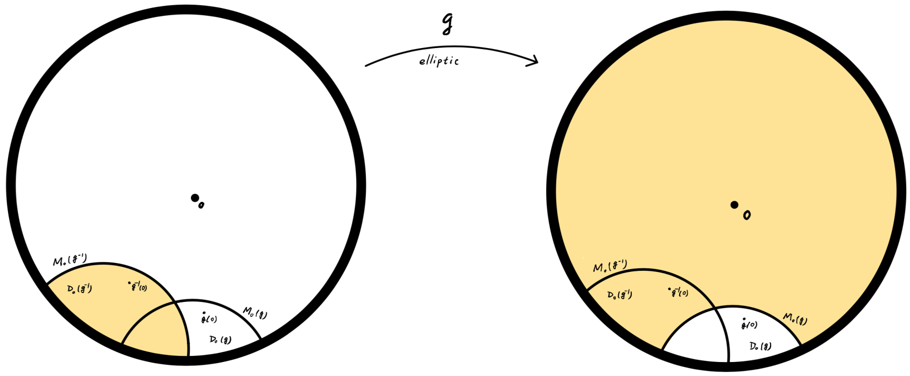

Figure 3.3: Let T be a parabolic isometry in PSU$(1,1)$. Then $T(D(T^{-1})) = \mathbb{B}^2 \setminus D(T)$ and the cardinality of $D(T) \cap D(T^{-}1)$ is greater than one.

Now that we have known several useful properties of isometric circles, we can have the following theorem for Schottky group (originally it is a definition of a restricted type of Schottky group that has several useful properties due to the way Γ is generated in [15]):

Theorem 3.1. *Let $\{T_i : 1 \leq i \leq m, m \geq 2\}$ be a set of non-elliptic isometries of $PSU(1,1)$ and each T_i and T_j, $i \neq j$, do not have a common fixed point. Take $z_0 \in \mathbb{B}^2$ be a point that is not fixed by each T_i. Then, if for each T_i we denote the half-spaces $D_0(T_i^{\pm}) := D(T_i^{\pm})$, for $i = 1, ..., m$ satisfy $\left(\overline{D(T_i) \cup D(T_i^{-1})}\right) \cap \left(\overline{D(T_j) \cup D(T_j^{-1})}\right) = \emptyset, \forall i \neq j \in \{1, \ldots, m\}$, then $\langle T_i, 1 \leq i \leq m \rangle$ is a Schottky group.*

Proof. By lemma 2.2, we have $T_i(\mathbb{B}^2 \setminus D_{i+m}^{\circ}) = D_i$, and because $\left(\overline{D(T_i) \cup D(T_i^{-1})}\right) \cap \left(\overline{D(T_j) \cup D(T_j^{-1})}\right) = \emptyset$, we have $T_i^{-1}(\mathbb{B}^2 \setminus D_i^{\circ}) = D_{i+m}$. $\square$

Remark: The converse is not true. Considering we are in $\mathbb{B}^2$, the perpendicular bisector that bounds $D_0(T_i^{-1})$ is actually the isometric circle of T_i[10, section 7.22]. Furthermore, if we let $T_i = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}$, then the radius of the perpendicular bisector (isometric circle) of T_i is $\frac{1}{|\gamma|}$ which is the same as the radius of the perpendicular bisector (isometric circle) of T_i^{-1}, i.e. the radii of ∂D_i and ∂D_{i+m} are identical. However, in the definition of Schottky group, ∂D_i and ∂D_{i+m} can have distinct radii which results in a configuration that does not have the symmetry that Theorem 3.1 has.

With the same conditions stated in Theorem 3.1, we have the following lemma.

Lemma 3.2. *Let $\{T_i : 1 \leq i \leq m, m \geq 2\}$ be a set of non-elliptic isometries of $PSU(1,1)$ and each T_i and T_j, $i \neq j$, do not have a common fixed point. Take $z_0 \in \mathbb{B}^2$ be a point that is not fixed by each T_i. Then, the center of each isometric circle of T_i must not be on the unit circle $\partial\mathbb{B}^2$.*

Proof. Since for T_i we have $|\alpha|^2 - |\gamma|^2 = 1$, hence $|\alpha| > |\gamma|$, and this implies

$$\left|\frac{\alpha}{\gamma}\right| > 1.$$

Recall $\left|\frac{\alpha}{\gamma}\right|$ is the modulus of the center of the isometric circle of T_i and T_i^{-1}. $\square$

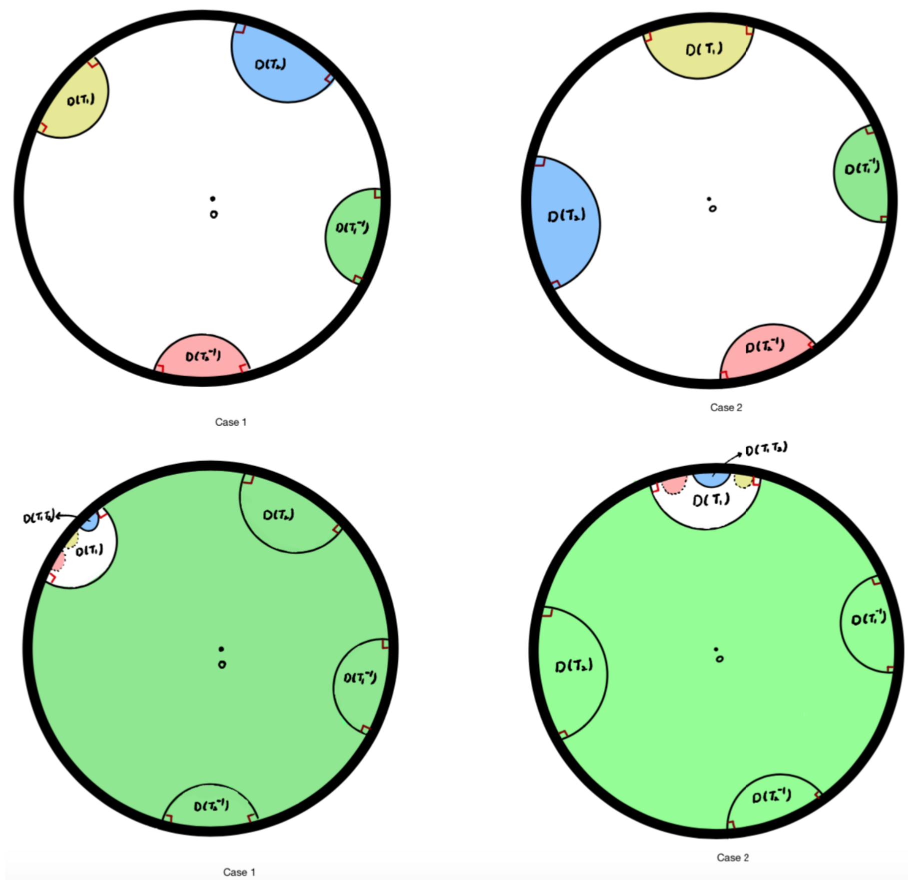

Figure 3.4: Let $G = \langle T_1, T_2 \rangle$ be a Schottky group. The figure demonstrates the half-space $D(T_1T_2) = D_0(T_1T_2)$ in four possible cases.

The following definitions and theorems can be used to demonstrate that a finitely generated Schottky group is discrete and acts discontinuous on $\mathbb{B}^2$.

Since $\mathrm{PSU}(1,1)$ is equipped with the metric induced by the norm $|A|$ defined as above, $\mathrm{PSU}(1,1)$ is a topological space, and $\mathrm{PSU}(1,1)$ is also called topological group.

Definition 3.6 (Discontinuous.)**.** Let G be a topological group. *The group actions of G are discontinuous* on a metric space X if and only if for all orbits $G(x)$, for all $x \in X$, each $G(x)$ is locally finite.

Since Schottky groups are subgroups of $\mathrm{PSU}(1,1)$, Schottky groups are topological. Hence, Schottky groups are examples of the above definition:

Lemma 3.3. *If G is a Schottky group, then the group actions of G are discontinuous.*

Proof. This result directly follows Theorem 3.1, since D_i are disjoint, and by lemma 2.2, this structure preserved level by level after we repeated apply the group generator on those D_i. To show it is locally finite, take $z_0 \in \mathbb{B}^2$ to make an orbit $G(z_0)$, and take $z \in\in \mathbb{B}^2$ which is in the open disk $B(z, \epsilon)$, then we can choose ϵ to be less than the minimum distance to the nearest boundary of the half-spaces, then there will be at most one orbit point in $B(z, \epsilon)$, hence $G(x)$ is locally finite, and thus the group actions of G are discontinuous. $\square$

The next goal is to prove finitely generated Schottky groups are discrete.

Definition 3.7 (Discrete.)**.** A subgroup G of a topological group $\mathrm{PSU}(1,1)$ is *discrete* if G has no accumulation points in $\mathrm{PSU}(1,1)$.

Remark: This is equivalent to say, there exists an open neighborhood U_{id} of $\mathrm{id} \in G$ such that $G \bigcap U_{\mathrm{id}} = \{\mathrm{id}\}$. Or, there exists an open neighborhood U_{id} of id such that for each $A \in \mathrm{PSU}(1,1)$, the cardinality of the set $G \bigcap g(U_{\mathrm{id}})$ is less than 2.

First of all, other than Schottky groups, for some simpler cases, if we are given a finitely generated subgroup of $\mathrm{PSU}(1,1)$ that is generated by diagonal matrices,we can use the following lemma.

Lemma 3.4. *If there exist g_i and g_j, $i \neq j$, in a given generator set of a group G, and there exists an isomorphism between the subgroup $\langle g_i, g_j\rangle \subseteq G$ and $\langle a^m, b^n\rangle$ where $a, b \in \mathbb{R}$ and $\frac{m}{n} \notin \mathbb{Q}$, then G is not discrete.*

The above result is easy to apply to the cases when off-diagonal entries are zeros. However, in general this is not the case. Take Schottky groups as examples, we have to introduce a more powerful result.

Since Schottky groups are subgroups of $\mathrm{PSU}(1,1)$, it also equipped the same norm $|A| := \left(\sum\limits_{i,j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$ for all $A \in \mathrm{PSU}(1,1)$.

Then, a powerful theorem[27, See Theorem 5.3.3] can be applied to check whether Schottky groups are discrete.

Theorem 3.5. *Let $SL(2,\mathbb{C})$ be the group of complex 2×2 matrices whose determinant is ± 1. A subgroup G of $SL(2,\mathbb{C})$ is discrete if and only if for each $r > 0$, the set $\{A \in G : |A| \leq r\}$ is finite.*

Let $G \subset \mathrm{PSU}(1,1)$ be a Schottky group. Then, we can use the conjugation map $\Psi G \Psi^{-1}$ to transform G to become a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Since the group generators of finitely generated Schottky groups are non-elliptic, so one of the diagonal entries is increasing exponentially, by using the norm $|A| := \left(\sum_{i,j=1}^{2} |a_{ij}|^2\right)^{\frac{1}{2}}$ and Theorem 3.5, we have the following result:

Lemma 3.6. *Every finitely generated Schottky group is discrete.*

Remark: Furthermore, let G be a subgroup of $\mathrm{PSU}(1,1)$. It can be proved that G is discrete if and only if G acts discontinuous on $\mathbb{B}^2$[27].

Definition 3.8 (Limit Point and Limit Set)**.** [2] Let G be a discrete subgroup of the $\mathrm{PSU}(1,1)$. The *limit set* of G is the set of all accumulation points in the intersection of all orbits $G(x)$ for all $x \in \mathbb{B}^2 \cup \partial\mathbb{B}^2$, and the *limit set* of G is denoted by $L(G)$.

The set of all accumulation points of $G(a_i), a_i \in \overline{\mathbb{B}^2}$ is denoted by L_{a_i}. If we take the union of all points $a_i \in \overline{\mathbb{B}^2}$, then $L(G) = \bigcup_{a_i} L_{a_i}$. The following theorem shows that it only takes one point to obtain the whole limit set $L(G)$, and it does not depend on the choice of the point (hence $L(G)$ can be the group invariant). $L(G)$ is a closed set, since each set has all of their limit points by definition.

Theorem 3.7. *[2, P.80] Let G be non-elementary. Then, $L(G) = L_{z_i}$ for all $z_i \in \overline{\mathbb{B}^n}$.*

Proof of Theorem 3.7.[2, 15, 27]. Case (i). This first case follows that fact the if two points z_1, z_2 are in $\mathbb{B}^2$, then $L_{z_1} = L_{z_2}$. In other words, there exists a sequence $\{g_n(z_1)\}_{n\geq 1}, z_1 \in \mathbb{H}^n$, converges to a point z_2 in $\partial\mathbb{B}^2\,\{\infty\}$, and this point does not depend on z_1. A contradiction can be generated by assuming the opposite, and $z_2 = -i$, and let $g \in G$. Since g is an isometry, $d(gz_1, gz_2) = d(z_1, z_2)$ is fixed and for all $z_2 \in \mathbb{B}^2$. Thus $g_n(z_2)$ is a hyperbolic circle centered at $g_n(z_1)$ with ray $d(z_1, z_2)$. The Euclidean length of the diameter of this circle tends to be zero as $g_n(z_1) \to 0$, therefore, $g_n(z_2) \to 0$ as well.

Case (ii). If $z_i \in \partial\mathbb{B}^n$, and assume $G(z_i) \neq \{z_i\}$, z_i is not a fixed point of all elements in G. There exists $z_2 = g_0 z_1 \neq a_1$, $g_0 \in G$. To show $L_O \subset L_{z_1}$. Let $c \in (z_1, z_2)_h$. By case (i), since $L_c = L_O$, for every $e \in L_O$ there exists a sequence $\{g_n\}_{n \geq 1} \subset G$ such that $g_n z \to e \in L_O$.

Consider two subsequences $g_n z_1 \to z_1'$ and $g_n z_2 \to z_2'$. (a) If $z_1' = z_2'$, then by lemma 2.2, $e = z_1' = z_2'$. If $z_1' \neq z_2'$, then again by lemma 2.2, $g_n c$ converges either to z_1' or z_2'. Since $z_2 = g_0 z_1$, thus $z_1' \in \lambda_{z_1} \subset G(z_1)$ and $z_2' \in \Lambda_{z_1} \subset G(z_1)$. Therefore, $g_n c$ converges to a point e in the accumulation set $\Lambda_{z_1} \subset G(z_1)$.

For the opposite direction, take $z_3 \in \Lambda_{z_1}$ and a sequence such that $g_n z_1 \to z_3$, $g_n O \to e$, $g_n^{-1} O \to e'$. Consider the region divided by isometric circle I_g that if $|g(z)|' = 1$ then $g(z) \in I_g$; if $|g(z)|' < 1$ then $g(z)$ is in the exterior of I_g; and if $|g(z)|' > 1$ then $g(z)$ is in the interior of I_g, then since O is in the exterior of I_{g_n}, its image $g_n O$ lies in the interior of $I_{g_n^{-1}}$. Similarly, $g_n^{-1} O$ lies in the interior of I_{g_n}. Therefore, the interior of $I_{g_n^{-1}}$ shrinks to e and the interior of I_{g_n} shrinks to e'. If $z_1 \neq e'$, then z_1 is in the exterior of I_{g_n} for large n, and since $g_n z_1$ is in the exterior of $I_{g_n}^{-1}$ which shrinks to e, hence $g_n z_1 \to e$, i.e. $z_3 = e \in L_O$. On the other hand, if $z_1 = e' \in \Lambda_O$, then $g_n z_1 \in \Lambda_O$ which is due to $g_n(\Lambda_O) = \Lambda_O$. Since Λ_O is a limit set which is closed, hence $z_3 \in \Lambda_O$. Because z_3 was arbitrarily chosen from Λ_{z_1}, thus $\Lambda_{z_1} \subset \Lambda_O$.

□

Proposition 3.8. *If G is a Schottky group, then for each point z of $\mathbb{B}^2$, we have $L(G) = \overline{G(z)} \cap \partial\mathbb{B}^2$.*

Proof. This is equivalent to show that the limit set of a Schottky group is a subset of the boundary of the unit disk $\mathbb{B}^2$, and this result directly follows the definition of Schottky groups, lemma 2.2, and lemma 2.4. Since all isometric circles corresponding to the orbit $G(z)$ has a nested structure and the radius is converging to zero for each infinite sequence of applying the group generators recursively; that is, eventually, each sequence of nested isometric circles converges to a limit point, and each limit point has a sequence of converging nested isometric circles. Since all sequences of convergent nested isometric circles are converging to $\partial\mathbb{B}^2$, hence $L(G) \subset \partial\mathbb{B}^2$. □

In general, we can have a fundamental result.

Theorem 3.9. *If $G \subset PSU(1,1)$ is a discrete group, and $a \in \mathbb{B}^2$, then the orbit $G(a)$ can only accumulate at the boundary $\partial\mathbb{B}^2$.*

The proof in[2, p.79] is an application of Theorem 3.7 and isometric circles. Furthermore, this theorem can also be proved by using Lemma 3.11.

Definition 3.9. The complement $\partial\mathbb{B}^2 \setminus L(G)$ is called *the ordinary set* denoted by $O(G)$.

We may define the following using the notion of ordinary set:

Definition 3.10. If G is a discrete subgroup of PSU$(1,1)$, and $O(G) = \emptyset$, then G is of *the first kind*. Otherwise, G is of *the second kind*. Furthermore, if $|L(G))|$, then G is *elementary*; otherwise, G is *non-elementary*.

Examples: The most well-known example of the first kind is the modular group PSL$(2,\mathbb{Z})$, and of the second kind is the Schottky group[15].

Fix a Schottky group $G(g_1, g_2)$ of rank 2. We define the following notations for half-spaces. When the length of a reduced word is $n = 1$, then we denote it as before $D(g_i) = D_0(g_i)$. For $n \geq 1$, the half-space $D(g_I)$ is defined by

$$D(g_I) = D_0(g_{i_1}g_{i_2}\dots g_{i_n}) = g_{i_1}g_{i_2}\dots g_{i_{n-1}}D(g_{i_n}).$$

Proposition 3.10. *Let $G(g_1, g_2)$ be a Schotkky group, generated by two hyperbolic isometries g_1, g_2 from PSU$(1,1)$. Recall the notation that $g_I = g_{i_1}g_{i_2}\dots g_{i_n}$ is a reduced word. Then we have the following properties[15]:*

(i) $g_I(\mathbb{B}^2 \setminus int(g_{i_n}^{-1})) \subset D(g_{i_1})$.

(ii) If $n \geq 2$, $D(g_I) \subset D(g_I g_n^{-1})$.

(iii) If g_I and g_I' are two distinct reduced words, then $D(g_I)$ and $D(g_I')$ are disjoint.

What is the topological structure of the limit set of a finitely generated Schottky group? To start answering this question, we begin with the following lemma[15, 18]:

Lemma 3.11. *Let $G = \langle g_1, g_2 \rangle$ be a Schottky group. Given a sequence $(g_i)_{\geq 1}$ in $\mathcal{A}$ satisfying $g_{i+1} \neq g_i^{-1}$, the sequence of Euclidean diameters $(r_i)_{\geq 1}$ of the set $D(g_I)$ where $len(g_I) = n$ converges to zero.*

This lemma directly follows Lemma 2.4 and Lemma 2.2. Alternatively, it can be proved by contradiction (see [15, Chapter II, lemma 1.10]).

Remark: Since each r_i is measured by taking the modulus of complex numbers, it is a Euclidean length.

3.2 Poincaré Series

The goal of this section is to study the critical exponent of Poincaré series and Sullivan's theorem, which proves that the critical exponent of Poincaré series of every Schottky group equals the Hausdorff dimension of limit sets of Schottky groups. In addition, this chapter investigates certain properties of critical exponents, such as how it is an invariant under a conformal transformation and how it is independent of the point z selected to be the input of the Poincaré series.

Let $g_1, ..., g_m$ be isometries of $\mathrm{PSU}(1, 1)$ on $\mathbb{B}^2$ such that $G = \langle g_1, ..., g_m \rangle$ generates a Schottky group of rank m.

Our goal is to compute the exponent of convergence of the following Poincaré series of the Schottky group G:

$$\mathcal{P}(G,t,z,w) = \sum_{g\in G} e^{-t\rho_{\mathbb{B}^2}(z,gw)}.$$

where z and w are in $\mathbb{B}^2$. Here, the exponent of convergence is defined to be

$$\delta(G) = \inf\{t > 0 : P(G,t) < \infty\} = \sup\{t > 0 : P(G,t) = \infty\}.$$

To show $\delta(G)$ does not depend on the choice of z and w, let us recall triangular inequalities, reverse triangular inequalities, and temporarily write Poincaré series as a series that has four arguments G, t, z and w: $\mathcal{P}(G,t,z,w)$. Hence,

$$\rho_{\mathbb{B}^2}(z,gw) \le \rho_{\mathbb{B}^2}(z,w) + \rho_{\mathbb{B}^2}(w,gw)$$

and

$$\rho_{\mathbb{B}^2}(z,gw) \ge \rho_{\mathbb{B}^2}(w,gw) - \rho_{\mathbb{B}^2}(z,w)$$

gives

$$e^{-t\rho_{\mathbb{B}^2}(z,w)}\mathcal{P}(G,t,z,z) \le \mathcal{P}(G,t,z,w) \le e^{t\rho_{\mathbb{B}^2}(z,w)}\mathcal{P}(G,t,z,z).$$

It follows that $\delta(G)$ does not depend on z or w, because the factors $e^{t\rho_{\mathbb{B}^2}(z,w)}$ and $e^{-t\rho_{\mathbb{B}^2}(z,w)}$ do not affect the convergent behavior of $\mathcal{P}(G,t)$, and it is bounded by $\mathcal{P}(G,t,z,z)$. Thus, from now on, we let $z = w = 0$. The problem turns into estimating $\mathcal{P}(G,t) := \sum\limits_{g_i \in G} e^{-t\rho_{\mathbb{B}^2}(0,g_i 0)}$, where $0 \neq g_i 0 \in \mathbb{B}^2$. The corner case is when there is a generator $g_i \in G$ which is elliptic and fixed 0. However, this can be modified to $0 + a$, $a \in \mathbb{B}^2$ and $a \neq 0$ by applying a conjugation on G. Thus, we can assume the group does not fix 0, since if it does we can use conjugation to modify it.

Our next goal is to show the proof ideas of a well-known result[31] that the Hausdorff dimension of the limit set of a Schottky group is the critical exponent of the Poincaré series.

Firstly, we have to define Hausdorff measure:

Definition 3.11. For $A \subset \partial\mathbb{B}^2$, *the s-dimensional Hausdorff measure* is

$$H^s(A) := \lim_{\epsilon \to 0} \inf \left\{ \sum_j |I_j|^s : A \subset \cup_j I_j, |I_j| < \epsilon \right\}.$$

where $|\cdot|$ denotes the Euclidean arc length in $\partial\mathbb{B}^2$.

For some threshold α,

$$H^s(A) = \begin{cases} 0, & \text{if } s < \alpha, \\ \infty, & \text{if } s > \alpha. \end{cases}$$

and *the Hausdorff dimension* is

$$\dim_H A := \alpha.$$

Secondly, we define a probability measure:

$$\mu^{(s)} := \frac{\sum\limits_{T\in G} e^{-s\rho_{\mathbb{B}^2}(0,T0)}\nu_{T0}}{\sum\limits_{T\in G} e^{-s\rho_{\mathbb{B}^2}(0,T0)}}$$

where ν_z is the point measure at $z \in \mathbb{B}^2$ with total mass one.

Recall the Banach-Alaoglu theorem, i.e. the closed unit ball of the dual space (the space of Borel measures on $\overline{\mathbb{B}^2}$ with norm given by the total mass) of a normed vector space is compact in the weak$-^*$topology, we can have the limiting measure as follows:

Definition 3.12. There exists a sequence $s_j \to \delta(G)$ such that $\mu^{(s_j)}$ converge weakly to a limiting measure, and this measure is called *the Patterson-Sullivan measure associated to* G:

$$\mu := \lim_{s_j\to\delta(G)} \mu^{(s_j)}.$$

Patterson[26] proves that

$$\mathcal{P}(G,t) > \frac{C}{t-\delta(G)}$$

where C is a constant, meaning that the Poincaré series diverges at $t = \delta(G)$.

By Theorem 1 in [31]:

Theorem 3.12. *Let G be a finitely generated Schottky group. The Patterson-Sullivan measure μ is a constant multiple of the Hausdorff measure $H^{\delta(G)}|_{\Lambda(G)}$. Fur-*

thermore, $\dim_H \Lambda(G) = \delta(G)$.

Proof idea:

For $q \in L(G)$, let I_q denote an interval in $\partial\mathbb{B}^2$ centered at q. Based on Sullivan's shadow lemma[11][Lemma 14.12], it can be showed that if there exists $\epsilon > 0$ such that for any $|I_q| < \epsilon$, we have

$$\mu\left(I_q\right) \asymp |I_q|^{\delta}(G),$$

uniformly in q. If $T \in G$, the the pullback measure $T^*\mu$ is defined by $T^*\mu(E) := \mu(TE)$. Then, on one hand, since for a Möbius transformation $T \in G$ the local distortion of Euclidean length is given by $|T'|$, thus $T^*H^s = |T'|^s H^s$. On the other hand, for Patterson-Sullivan measure, for $T \in G$, we have $T^*\mu = |T'|^{\delta(G)}\mu$ [11][Lemma 14.2]. This hints that Patterson-Sullivan measure μ transforms under the action of G like the Hausdorff measure of dimension $\delta(G)$. Since μ is absolutely continuous with respect to $H^{\delta(G)}$, we have $d\mu = kdH^{\delta(G)}$ for some function k on $L(G)$. By ergodicity [11][Corollary 14.11], k is constant. Recall the definition of Hausdorff measure,

$$H^s\left(A\right) := \lim_{\epsilon\to 0}\inf\left\{\sum_j |I_j|^s : A \subset_j I_j, |I_j| < \epsilon\right\}$$

where $A \subset L(G)$.

Furthermore, we know that when $s \to \delta(G)$, we have $\mu(L(G)) = 1$. Hence, we have

$$\int_{L(G)} d\mu = \lim_{s \to \delta(G)} \mu^{(s)}(L(G)) = \mu(L(G)) = 1 = k \int_{L(G)} dH^{\delta(G)}.$$

If $s > \delta(G)$, then we have $\sum |I_q|^s \to \infty$, so $H^s\left(L(G)\right) \to \infty$; and if $s < \delta(G)$, we have $\sum |I_q|^s \to 0$, thus $H^s\left(L(G)\right) \to 0$. Therefore, $\delta(G)$ is the Hausdorff dimension of $L(G)$.

Chapter 4

Bounds of the Critical Exponents

The purpose of this chapter is to investigate a specific example of Schottky groups that can satisfy our well-distributed configuration definition. Our main theorem in this chapter sets tight constraints on rank-2 well-distributed Schottky groups. According to our knowledge, this is maybe the first time in history that we have such bounds for Schottky groups other than the case when the Hausdorff dimension approaches the unit one. We also present a conjecture for the exact form of a general formula for the Hausdorff dimension of the limit set of a rank-2 well-distributed Schottky group in this chapter, hinted by our derivations of the main theorem.

4.1 Lower Bound for Poincaré series

Using the following proposition, we aim to prove that there exists a lower bound of the Hausdorff dimension of limit sets of Schottky groups.

Proposition 4.1. *Let* Γ *be a Schottky group, and* $T_i \in \Gamma$*. Denote* $d_{\max} := \max_{i=1,\dots,m} \{\rho_{\mathbb{B}^2}(0, T_i(0))\}$*. Then*

$$\delta(\Gamma) \geq \frac{\ln(2m-1)}{d_{\max}}.$$

Proof. We know that all Schottky groups are free groups. Therefore, the group Γ are all the reduced words generated by T_i, and Γ_n denotes the set of all reduced words of length n. We can find that $\#\Gamma_n \asymp (2m-1)^n$ by writing out the tree of the orbit $\Gamma(0)$, and see the pattern[1] that there must be $(2m-1)^n$ for each n. We also notice that if $T \in \Gamma_n$, then we can write $T = T_{i_1}....T_{i_n}$ where $T_{i_j} \neq T_{i_{j+1}}^{-1}$ and $T_{i_n} \in \{T_1, ..., T_m, T_1^{-1}, ..., T_m^{-1}\}$. We have

$$\begin{aligned} \rho_{\mathbb{B}^2}(0, T(0)) \leq \rho_{\mathbb{B}^2}(0, T_{i_1}(0)) + \rho_{\mathbb{B}^2}(T_{i_1}(0), T_{i_1}T_{i_2}(0)) + ... \\ + \rho_{\mathbb{B}^2}(T_{i_1}...T_{i_{n-1}}(0), T_{i_1}...T_{i_n}(0)) \leq n d_{\max} \end{aligned} \tag{4.1}$$

by the fact that T_i are isometries in the hyperbolic space. Hence, combining the above information,

$$\begin{aligned} \mathcal{P}(\Gamma, t) &= \sum_{n=1}^{\infty} \sum_{T \in \Gamma_n} e^{-t\rho_{\mathbb{B}^2}(0, T0)} \\ &\geq c \cdot \sum_{n=1}^{\infty} (2m-1)^n e^{-tnd_{\max}}. \end{aligned} \tag{4.2}$$

The above sum diverges if $(2m-1)e^{-td_{\max}} \geq 1$, which means that $t \leq \frac{\ln(2m-1)}{d_{\max}}$. As $\delta(\Gamma) \geq t$, our desired result follows by taking $t \to \frac{\ln(2m-1)}{d_{\max}}$. The proof is complete. $\square$

[1] The $(2m-1)$ factor is resulting from the inverse of each parent node.

Now we have a lower bound

$$\delta'(\Gamma) = \frac{\ln(2m-1)}{d_{\max}} \leq \delta(\Gamma) = \dim_H(L(\Gamma)).$$

4.2 The Least Upper Bound for Poincaré series

We need a more rigorous set-up for $T_i(0)$ to obtain an upper bound for the $\delta(\Gamma)$. The following is our definition.

Definition 4.1. We say that Γ is *a well-distributed Schottky group of rank* m with generators $T_1, ..., T_m$, if the following holds.

(1) $\rho_{\mathbb{B}^2}(0, T_i 0)$ are equal to each other for all $i = 1, ..., m$.

(2) Let $r = \rho_{\mathbb{B}^2}(0, T_i 0)$ be the number defined in (1). The argument of the complex numbers $\{T_i(0) : i = 1, ..., 2m\}$, arranged in ascending order, are equal angles apart in the circle $|z| = r$.

(3) Half-spaces $D_0(T_i)$ are closed disjoint disks.

Let us now define some trigonometric notations.

1. $[A, B]$ denotes the geodesic joining the point A, B.

2. $\triangle[A, B, C]$ denotes the hyperbolic triangle with vertices A, B, C.

3. $\angle[A,\ B,\ C]$ denotes the angle between the geodesic $[A, B]$ and $[B, C]$.

Lemma 4.2. *Let* T_I *and* T_I^- *be given. Then, for any* T_{i_j} *in the generator set* $\{T_1, ..., T_m\}$ *such that* $T_{i_{j+1}} \neq i_j^{-1}$*, and* $T_{i_{n+1}} \neq T_{i_n}^{-1}$*, we have*

$$\angle[T_{I^-}0,\ T_I 0,\ T_I T_{i_{n+1}} 0] = \angle[T_{i_n}^{-1} 0,\ 0,\ T_{i_{n+1}} 0].$$

In particular, this angle is equal to $\frac{j\pi}{m}$ *for some* $j = 1, 2, ..., m-1$ *if* Γ *is a well-distributed Schottky group.*

Proof. Apply T_I^{-1} to $\triangle[T_{I^-}0,\ T_I 0,\ T_I T_{i_{n+1}} 0]$. Its image will be $\triangle[T_{i_n}^{-1} 0,\ 0,\ T_{i_{n+1}} 0]$. However, given T_I^{-1} is a conformal map, the angle must be preserved. The second statement follows immediately from the second condition of a well-distributed Schottky group. $\square$

Lemma 4.3. *Let* $T_I \in \mathcal{W}_n$ *and* Γ *is a well-distributed Schottky group. Then*

$$0 < \angle[0,\ T_I 0,\ T_{I^-} 0] < \frac{\pi}{m}.$$

Proof. Apply T_I^{-1} to $\triangle[0,\ T_I 0,\ T_{I^-} 0]$, its image will be $\triangle[T_{i_n}^{-1} T_{I^-}^{-1} 0,\ 0,\ T_{i_n}^{-1} 0]$. But T_I^{-1} is a conformal map, the angle

$$\angle[0, T_I 0, T_{I^-} 0] = \angle[T_{i_n}^{-1} T_{I^-}^{-1} 0,\ 0,\ T_{i_n}^{-1} 0].$$

As $T_{i_n}^{-1}T_{I^-}^{-1}0$ and $T_{i_n}^{-1}0$ are contained in the closed disk $D_{T_{i_n}^{-1}}$ and the geodesic

$$[0, T_{i_n}^{-1}T_{I^-}^{-1}0], \text{ and } [0, T_{i_n}^{-1}0]$$

are straight line, the angle must be less than all $\angle[T_{i_j}0,\ 0,\ T_{i_k}0]$ for all $i_j \neq i_k = 1, ..., 2m$, i.e. T_{i_j} and T_{i_k} are group generators or their inverses. Therefore, by the fact that Γ is well-distributed, $\angle[0,\ T_I0,\ T_{I^-}0] < \pi/m$. It is positive because $T_{i_n}^{-1}T_{I^-}^{-1}0 \neq T_{i_n}^{-1}0$. □

Lemma 4.4. *Let $T_I \in \mathcal{W}_n$ and Γ is a well-distributed Schottky group. Let $x_I = \rho_{\mathbb{B}^2}(0, T_I0)$,*

$$\ell_I = \min\{\rho_{\mathbb{B}^2}(0, T_IT_{i_{n+1}}0) : T_IT_{i_{n+1}} \in \mathcal{W}_{n+1}\}$$

and $\theta_I = \angle[0,\ T_I0,\ T_{I^-}0]$. Then

$$\cosh \ell_I = \cosh x_I \cosh r - \sinh x_I \sinh r \cos\left(\frac{\pi}{m} - \theta_I\right).$$

Proof. Considering the triangle $\triangle[0,\ T_IT_{i_{n+1}}0,\ T_I0]$, and applying the hyperbolic cosine law to this triangle, we obtained

$$\cosh x_{I,T_{i_{n+1}}} = \cosh x_I \cosh r - \sinh x_I \sinh r \cos \alpha_{Ii_{n+1}}.$$

where

$$\begin{aligned}\alpha_{Ii_{n+1}} =&\angle[0,\ T_I 0,\ T_{Ii_{n+1}}0]\\ =&\angle[T_{I^-}0,\ T_I 0,\ T_{Ii_{n+1}}0] - \angle[0,\ T_I 0,\ T_{I^-}0]\\ =&\frac{j\pi}{m} - \theta_I\end{aligned}$$

for some $j = 1, 2, ..., 2m-1$ using Lemma 4.2 and Lemma 4.3. The lemma now follows by noticing that $\cos\alpha_{Ii_{n+1}}$ achieves its maximum when $j = 1$. $\square$

The next step is to illustrate several examples of this sort of Schottky group by charting their $\Gamma(0)$ orbit.

Let Γ be a subgroup of $\mathrm{PSU}(1,1)$ and be a well-distributed Schottky group with m generators. Then we would like to demonstrate that by using either Method 1 or Method 2 introduced in the following, the orbit $\Gamma(0)$ can be constructed only using two operators T, and R.

Since on $\mathbb{B}^2$, to be an automorphism, we know T must be in $\mathrm{PSU}(1,1)$. Thus, assume

$$T = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}.$$

Then, for $0 < \theta < \frac{\pi}{4}, \forall m \in \mathbb{Z}^+$, we can have

$$T0 = \cos\left(\frac{\theta}{2}\right).$$

Thus

$$|\gamma|^2 = |\alpha|^2 \cos^2\left(\frac{\theta}{2}\right).$$

Since $T \in \mathrm{PSU}(1,1)$, by definition of $\mathrm{PSU}(1,1)$, we can obtain

$$|\alpha|^2 - |\gamma|^2 = 1.$$

Hence,

$$|\alpha|^2 \sin^2\left(\frac{\theta}{2}\right) = 1.$$

That is

$$\left(\sin^2\left(\frac{\theta}{2}\right)\right)^{-1} = |\alpha|^2.$$

Without loss the generality, assume $\alpha = \left(\sin\left(\frac{\theta}{2}\right)\right)^{-1}$, and $\gamma = \cot\left(\frac{\theta}{2}\right)$ to satisfy the identity.

Recall Definition 2.10, the definition of isometric circles, with the above information, once we know the center and radius of an isometric circle, we can derive γ and α immediately and derive T as follows

$$T = \begin{pmatrix} \frac{1}{\sin\left(\frac{\theta}{2}\right)} & \cot\left(\frac{\theta}{2}\right) \\ \cot\left(\frac{\theta}{2}\right) & \frac{1}{\sin\left(\frac{\theta}{2}\right)} \end{pmatrix}.$$

In essence, since Γ is a well-distributed Schottky group, assume $\Gamma = \langle T_1, ..., T_m \rangle$. Denote $T_1^{-1} = T_{m+1}, ..., T_m^{-1} = T_{2m}$. Assume for $\mathrm{len}(T_I) = 1$, assume the half-space $D_0(T_i)$ is surrounding by $D_0(T_j)$ and $D_0(T_k)$, and the half-space $D_0(T_a)$ is surrounding by $D_0(T_b)$ and $D_0(T_c)$, where $a \neq i$. Then since Γ is well-distributed, we must have not only

$$\angle[0,\ T_i0,\ T_j0] = \angle[0,\ T_i0,\ T_k0] = \angle[0,\ T_a0,\ T_b0] = \angle[0,\ T_a0,\ T_c0],$$

but also

$$\rho_{\mathbb{B}^2}(0, T_i0) = \rho_{\mathbb{B}^2}(0, T_i^{-1}0) = \rho_{\mathbb{B}^2}(0, T_a0) = \rho_{\mathbb{B}^2}(0, T_a^{-1}0),$$

for all $i \neq a$.

Denote the angle $\angle[0,\ T_i0,\ T_j0] := \phi$.

Then, we can construct the rotation operator R. We would like it to be the function $R(z) = e^{i\phi}z$ which is a counter-clockwise rotating operation, where $\phi := \frac{\pi}{m} \geq \theta$.

Then, we can define R as follows

$$R := \begin{pmatrix} e^{\frac{i\phi}{2}} & 0 \\ 0 & e^{\frac{-i\phi}{2}} \end{pmatrix} \in \mathrm{PSU}(1,1)$$

which fixes 0, and its upper half-plane conjugate $\Psi^{-1}R\Psi$ fixes $+i$ in the upper half-plane.

Then, we can have two methods to plot $\Gamma(0)$. Both of them give the same output.

Method 1: Let Γ be well-distributed. Recall that $T_{i_j}0 \in D_0\left(T_{i_j}\right)$. Without loss of generality, we can assume $T_{i_j}(0)$ is on the positive real axis of the complex plane. In addition, since Γ is a well-distributed Schottky group, there exists T_{i_k} such that $T_{i_k}0 = RT_j0$. Furthermore, for each $k \in \{1, 2, 3, ..., 2m\}$, we all have

$$T_{i_k}0 = R^l T_j 0$$

for some $l \in \{0, 1, 2, 3, ..., 2m-1\}$.

The goal is: Instead of using T_{i_k} to derive $T_{i_k}0$, we want to use T_{i_j}.

Then assume $z \in D_0\left(T_{i_k}\right)$.

Firstly we apply R^{-l} to z to rotate z such that

$$R^{-l}z \in D_0\left(T_{i_j}\right).$$

Then we apply T_{i_j} to $R^{-l}z$ to translate it to $T_{i_j}R^{-l}z \in D_0\left(T_{i_j}\right)$.

Finally, we apply R^l to $T_{i_j}R^{-l}z$ to rotate it back to $D_0\left(T_{i_k}\right)$. Then, since Γ is distributed, so in general we have

$$R^l T_{i_j} R^{-l} z = T_{i_k} z, z \in \mathbb{B}^2.$$

Thus, instead of considering the generators $T_1, .., T_m, T_1^{-1}, ..., T_m^{-1}$, we can derive:

$$\Gamma = \left\langle T_{i_j}, RT_{i_j}R^{-1}, ...R^{m-1}T_{i_j}R^{-(m-1)} \right\rangle,$$

and for simplicity we can let $T = T_{i_j}$, then we can obtain

$$G' := \left\langle T, RTR^{-1}, ...R^{m-1}TR^{-(m-1)} \right\rangle$$

Then, we can use $G'(0) \cap \Gamma(0)$ to reconstruct $\Gamma(0)$.

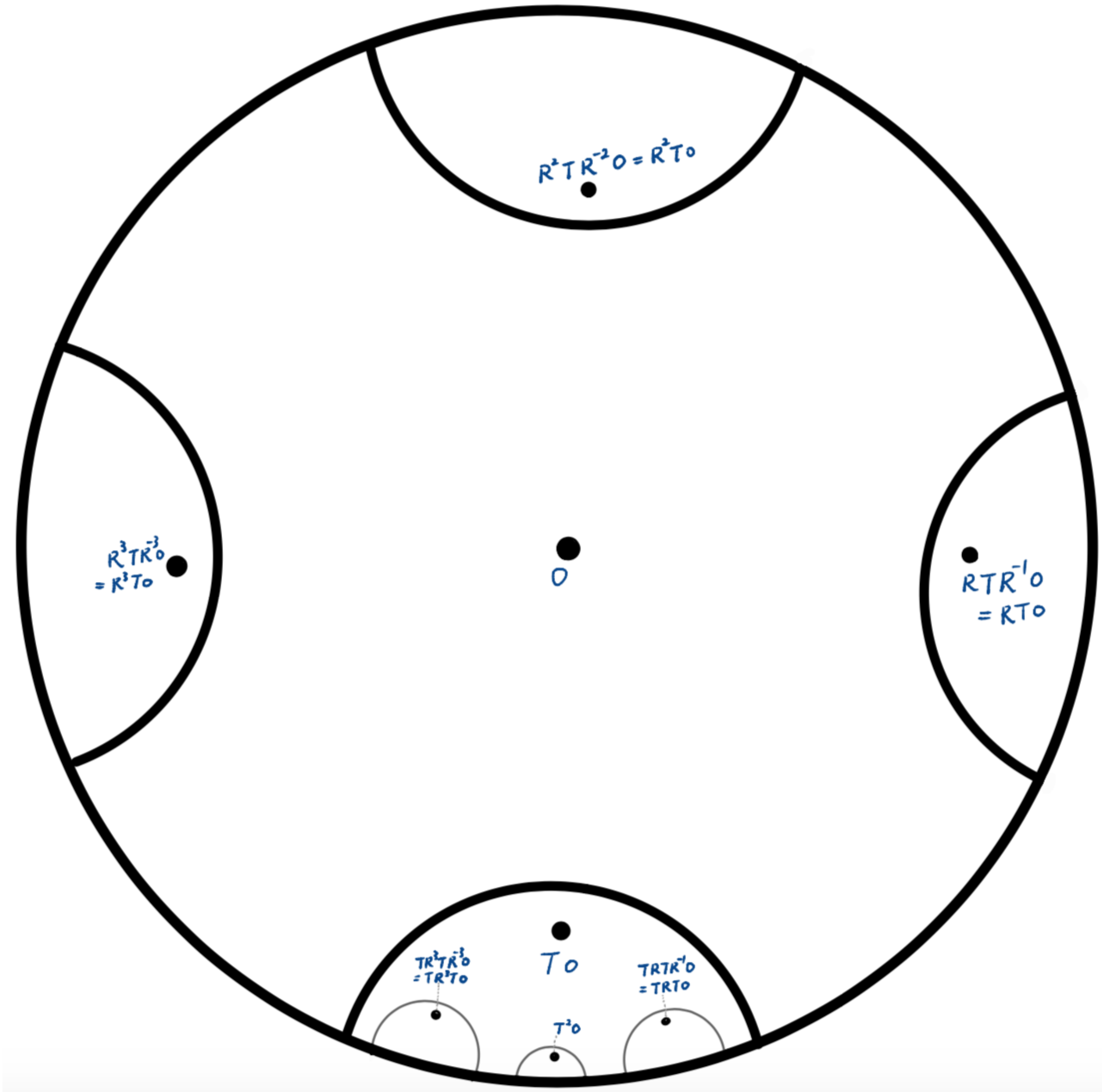

Figure 4.1: Let $G' = \langle T, RTR^{-1}\rangle$ be a well-distributed Schottky group. The figure demonstrates the first level of all four half-spaces: $D(T0)$, $D(RT0) = D(RTR^{-1}0)$, $D(R^2T0) = D(R^2TR^{-2}0)$, $D(R^3T0) = D(R^3TR^{-3}0)$; and the second level half-space within $D(T0)$: $D(T^20)$, $D(TRT0) = D(TRTR^{-1}0)$, and $D(TR^3T0) = D(TR^3TR^{-3}0)$.

Method 2: Since the original goal is to reconstruct $\Gamma(0)$. Hence, method 1 can be simplified by only considering the operation

$$R^l T_{i_j} z = T_{i_k} z, z \in \mathbb{B}^2.$$

The idea of this method: Instead of starting from picking a point in $D_0 T_{i_k}$, since $\Gamma(0)$ is already given (and our goal is to reconstruct it by only using two operators), we can find a point $T_{i_j} z$ such that after a certain rotation R^l, we can rotate $T_{i_j} z$ to $T_{i_k} z$.

The details of the reconstruction of the whole $\Gamma(0)$ is the following.

At the highest level, Γ is reconstructed counter-clockwisely and level by level. Starting from $N = 1$, we start with $T_{i_j} 0$, and map it to $RT_{i_j} 0$, $R^2 T_{i_j} 0$,..., $R^k T_{i_j} 0$, ..., $R^{2m-1} T_{i_j} 0$.

For $N = 2$, apply T_{i_j} at each $T_I 0 = R^k T_{i_j} 0$ where $\text{len}(T_I) = 1, T_I \neq T_{i_j}$, then we can determine each node in the second level in $D_0 \left(T_{i_j}\right)$. Next, use R^k to map all these nodes to $D_0 \left(T_{i_a}\right), a \neq j$, $a \in \{1, ..., 2m\}$.

Assume at $N = K$, all nodes in $D_0 \left(T_{i_j}\right)$ and the remaining $D_0 \left(T_{i_a}\right)$ are given.

For $N = K + 1$ level, we use $\Gamma(0)$ to see where each $T_{i_j} T_I 0$ goes, when $\text{len}(T_I) = K$. Next, we use the R^k to map those $(2m - 1)^{K+1}$ nodes represented only by operators R and T_{i_j} in $D_0 \left(T_{i_j}\right)$ to $D_0 \left(T_{i_a}\right)$.

Either way, since T fixes $\pm i$, we start to plot $\Gamma(0)$ from the sector that is in the lower half-plane for all m. We would like to find the angle between the intercepts of the half-space $D_0(T)$ and the unit circle $\partial\mathbb{B}^2$.

In the following, we utilize method 2 to plot the scenario where $m = 2$ to $N = 14$, i.e. each shortened word has a word length of at most 14.

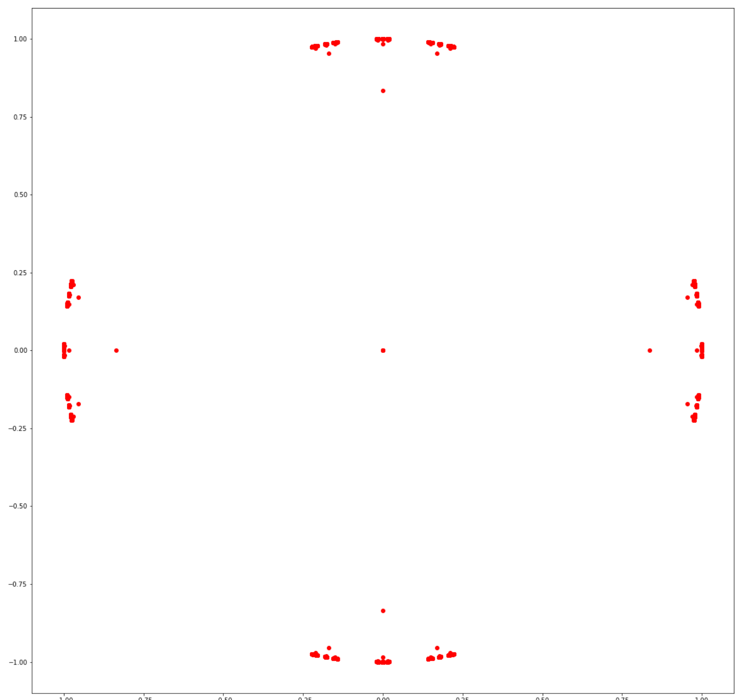

Figure 4.2: $m = 2$, $\frac{\theta}{2} \approx 33.398473447277695°$, Level 14 ($N = 14$).

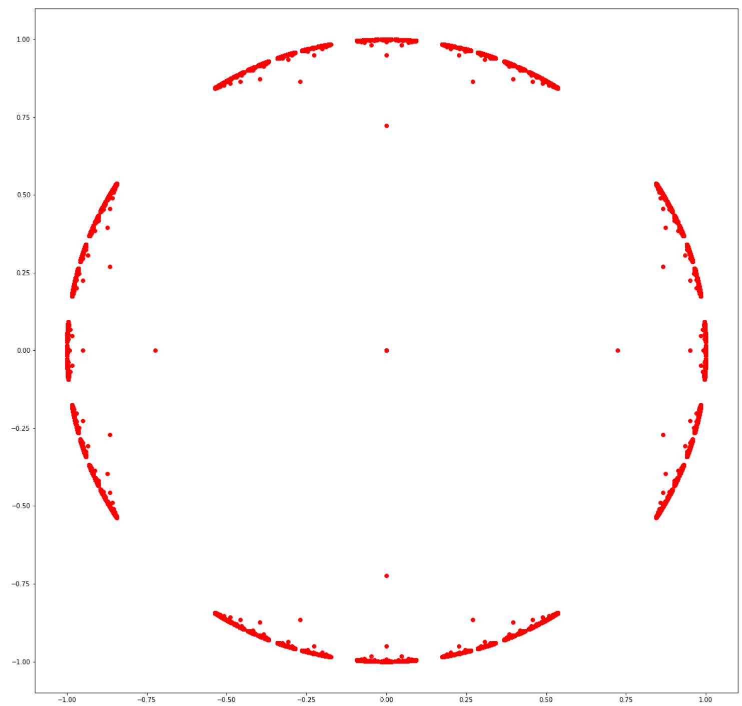

Figure 4.3: $m = 2$, $\frac{\theta}{2} \approx 43.602794482778144°$, Level 14 ($N = 14$).

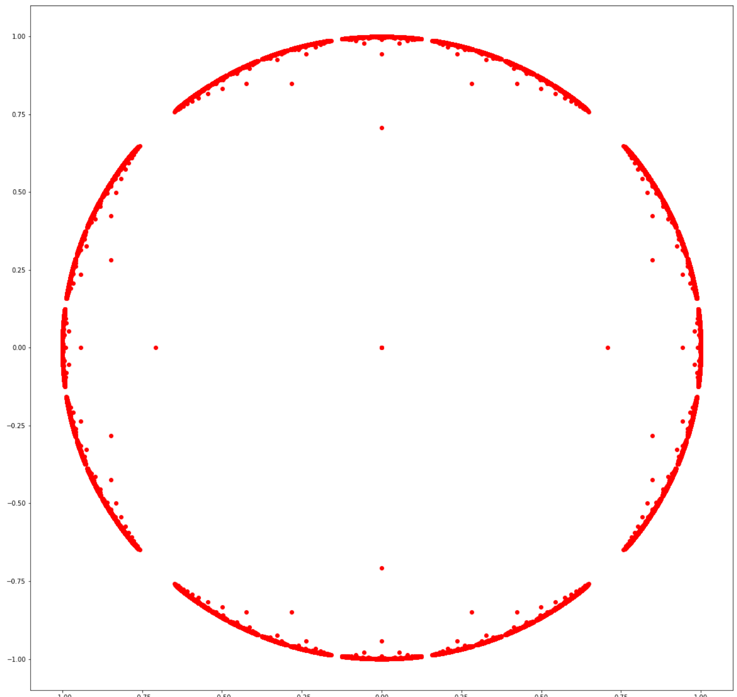

Figure 4.4: $m = 2$, $\frac{\theta}{2} \to 45°$, Level 14 ($N = 14$).

Our computer system has a limit on the number of digits in a floating point number, but an irrational number $\frac{\pi}{2}$ has an endless number of digits after the decimal point.

That is, our computer system can only handle rational numbers, i.e., it can only employ rational numbers to approximate irrational numbers. As a result, in this picture, we have not obtained $\delta(G) = 1$.

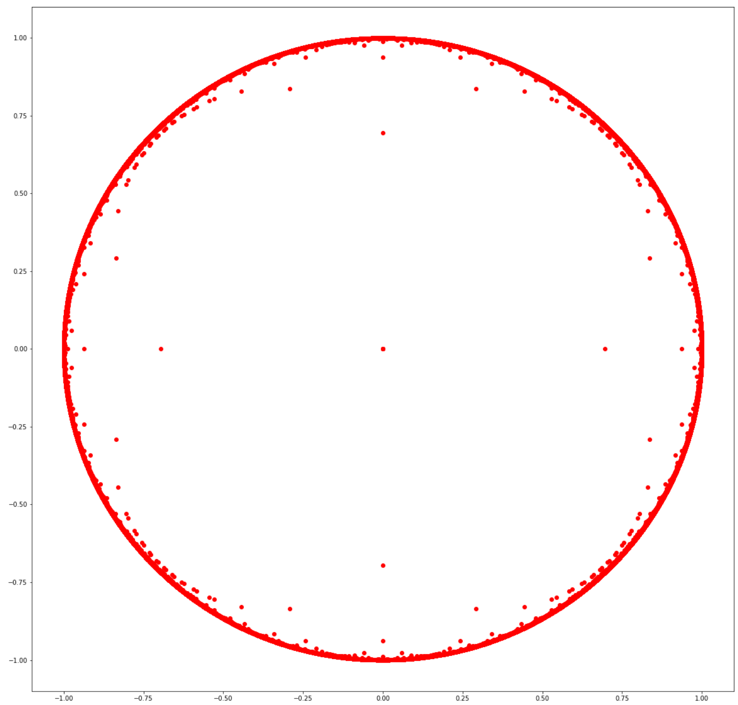

Figure 4.5: $m = 2$, $\frac{\theta}{2} \approx 45.9746105715017°$, Level 14 ($N = 14$).

To reach $\delta(G) = 1$, we added a small number 0.01 to let Λ pass the number $\frac{\sqrt{2}}{2+\sqrt{2}}$, or equivalently to let θ passes $\frac{\pi}{2}$.

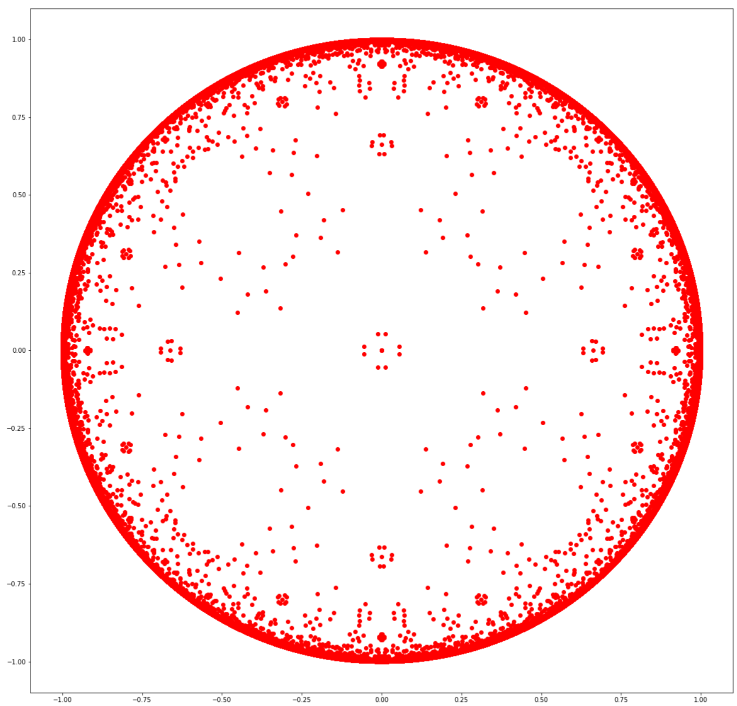

Figure 4.6: $m = 2$, $\frac{\theta}{2} \approx 48.455517824418166°$, Level 14 ($N = 14$).

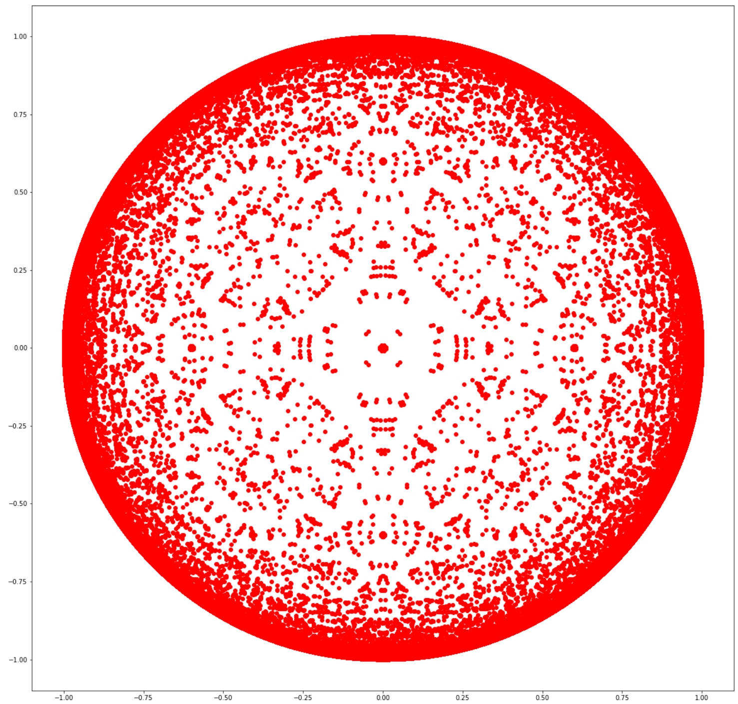

Figure 4.7: $m = 2$, $\frac{\theta}{2} \approx 53.13010858755201°$, Level 14 ($N = 14$).

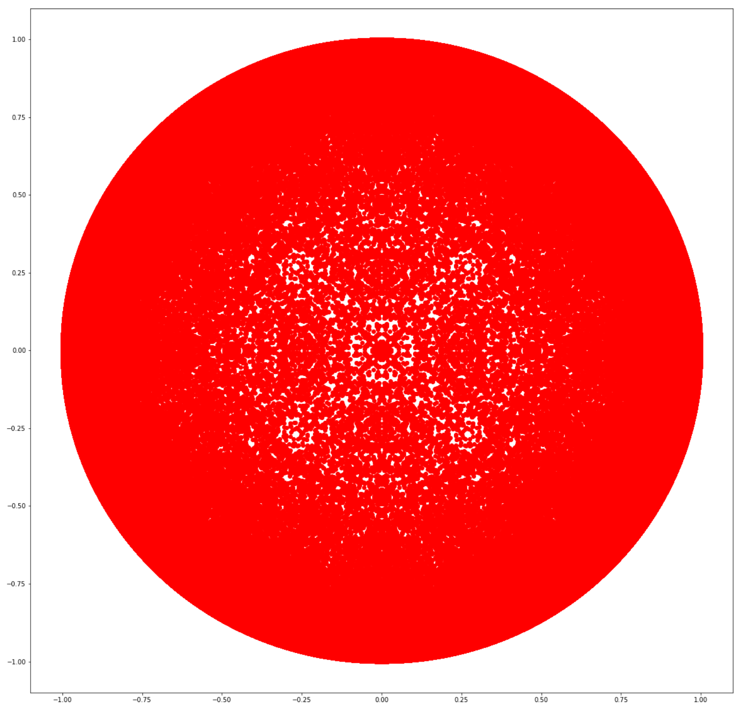

Figure 4.8: $m = 2$, $\frac{\theta}{2} \approx 61.927531757108724°$, Level 14 ($N = 14$).

Lemma 4.5. *Let* $\Gamma = \langle T_1, \ldots, T_m \rangle$ *be well-distributed Schottky group, and let*

$$X_{I_n} = \rho_{\mathbb{B}^2}\left(0, T_I 0\right).$$

Consider the infinite index of a reduced word $T_{I_n} = T_{i_1} T_{i_2} \ldots T_{i_n} \ldots \in \mathcal{A} \times \mathcal{A} \cdots \times$

$\mathcal{A}$, where each $\mathcal{A}$ is an alphabet, and $\mathcal{W}_n = \left\{T_{I_n} : T_{i_j} \in \mathcal{A}\right\}$. Then from hyperbolic cosine law, we can have

$$\cosh\left(X_{I_n}\right) = \cosh^n\left(r\right) \prod_{j=1}^{n-1} \left(1 - \tanh\left(X_{I_{j+1}}\right) \tanh\left(r\right) \cos\left(\frac{i_{j+1}\pi}{m} - \theta_{I_j}\right)\right)$$

Proof. After applying $T^{-1}_{I_n-1}$ and using the property of isometry, we can derive the following

$$\angle\left(0, T_{I_{n-1}}0, T_{I_n}0\right) = \angle\left(T_{I_{n-2}}, T_{I_{n-1}}0, T_{I_n}0\right) - \angle\left(T_{I_{n-2}}, T_{I_{n-1}}0, 0\right)$$

$$= \angle\left(T^{-1}_{i_{n-1}}, 0, T_{i_n}0\right) - \angle\left(T^{-1}_{i_{n-1}}0, 0, T^{-1}_{I_n-1}0\right)$$

$$= \frac{i_n\pi}{m} - \theta_{I_{n-1}}.$$

Thus, the hyperbolic cosine law can be rewritten as follows

$$\cosh\left(X_{I_n}\right) = \cosh\left(X_{I_{n-1}}\right)\cosh\left(r\right) - \sinh\left(X_{I_{n-1}}\right)\sinh\left(r\right)\cos\left(\angle\left(0, T_{I_{n-1}}0, T_{I_n}0\right)\right)$$

$$= \cosh\left(X_{I_{n-1}}\right)\cosh\left(r\right)\left(1 - \tanh\left(X_{I_{n-1}}\right)\tanh\left(r\right)\cos\left(\frac{i_n\pi}{m} - \theta_{I_{n-1}}\right)\right)$$

$$= \cosh^2\left(r\right)\cosh\left(X_{I_{n-2}}\right)\left(1 - \tanh\left(X_{I_{n-2}}\right)\tanh\left(r\right)\cos\left(\frac{i_{n-1}\pi}{m} - \theta_{I_{n-2}}\right)\right)$$

$$\times\left(1 - \tanh\left(X_{I_{n-1}}\right)\tanh\left(r\right)\cos\left(\frac{i_n\pi}{m} - \theta_{I_{n-1}}\right)\right).$$

By induction, we derived the following result

$$\cosh\left(X_{I_n}\right)$$

$$= \cosh^{n-1}(r)\cosh\left(X_{I_1}\right)\prod_{j=1}^{n-1}\left(1-\tanh\left(X_{I_{j+1}}\right)\tanh(r)\cos\left(\frac{i_{n-1}\pi}{m}-\theta_{I_{n-2}}\right)\right)$$

$$= \cosh^{n}(r)\prod_{j=1}^{n-1}\left(1-\tanh\left(X_{I_{j+1}}\right)\tanh(r)\cos\left(\frac{i_{j+1}\pi}{m}-\theta_{I_j}\right)\right)$$

$\square$

Definition 4.2. Define the symbol for the previous lemma's outcome as follows.

$$Y_{I_n} := \cosh^{n}(r)\prod_{j=1}^{n-1}\left(1-\tanh\left(X_{I_{j+1}}\right)\tanh(r)\cos\left(\frac{i_{j+1}\pi}{m}-\theta_{I_j}\right)\right)$$

and let

$$P_n := \sum_{T\in\mathcal{W}_n}\frac{1}{Y_{I_n}^{t}}.$$

Lemma 4.6. *The Poincaré series can be rewritten using the above definition*

$$\mathcal{P}(\Gamma,t) \asymp \sum_{n=1}^{\infty} P_n.$$

Proof. Recall $\rho_{\mathbb{B}^2} = \cosh^{-1}\left(X_{I_n}\right) = \ln\left(Y_{I_n} + \sqrt{Y_{I_n}^2 - 1}\right)$.

$$\mathcal{P}\left(\Gamma, t\right) = \sum_{T\in\Gamma} e^{-t\rho_{\mathbb{B}^2}(0,T0)} \sum_{n=1}^{\infty} \sum_{T\in\mathcal{W}_n} e^{t\rho_{\mathbb{B}^2}(0,T0)} = \sum_{n=1}^{\infty} \sum_{T\in\mathcal{W}_n} \left(\frac{1}{Y_{I_n} + \sqrt{Y_{I_n}^2 - 1}}\right)^t$$

$$\asymp \sum_{n=1}^{\infty} \sum_{T\in\mathcal{W}_n} \frac{1}{Y_{I_n}^t} = \sum_{n=1}^{\infty} P_n.$$

$\square$

Theorem 4.7. *Suppose Γ is a well-distributed Schottky group of order 2. Then,*

$$\frac{\ln\left(2\right)}{\ln\left(\cosh\left(r\right)\right)} \leq \delta\left(\Gamma\right) \leq \min\left\{\frac{\ln\left(4\right)}{\ln\left(\cosh\left(r\right)\right)}, 1\right\}.$$

Proof. Recall definition 4.2,

$$Y_{I_n} = Y_{I_{n-1}i_n} = Y_{I_{n-1}} Y_{i_n}$$

$$= \cosh\left(X_{I_{n-1}}\right)\cosh\left(r\right)\left(1 - \tanh\left(X_{I_{n-1}}\right)\tanh\left(r\right)\cos\left(\frac{i_n\pi}{m} - \theta_{I_{n-1}}\right)\right).$$

Hence,

$$Y_{I_{n+1}} = Y_{I_n} Y_{i_{n+1}} = \cosh\left(X_{I_n}\right)\cosh\left(r\right)\left(1 - \tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n}\right)\right).$$

Thus, for the $n+1$ term in $\mathcal{P}$, we can derive the following

$$P_{n+1} = \sum_{T \in \mathcal{W}_{n+1}} \frac{1}{Y_{I_{n+1}}^t} = \sum_{T \in \mathcal{W}_n} \sum_{i_{n+1}=1}^{2m-1} \frac{1}{Y_{I_n i_{n+1}}^t}$$

$$= \sum_{T \in \mathcal{W}_n} \frac{1}{(\cosh(X_{I_n}))^t} \frac{1}{(\cosh(r))^t} \sum_{i_{n+1}=1}^{2m-1} \left(1 - \tanh(X_{I_n}) \tanh(r) \cos\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n}\right)\right)^{-t}.$$

Recall Newton's generalized binomial theorem: Firstly, for $k \geq 1, k \in \mathbb{Z}$, we can have

$$\binom{-t}{k} := \frac{(-t-0)(-t-1)\cdots(-t-(k-1))}{k!}$$
$$= \frac{t(t+1)\cdots(t+(k-1))}{k!}(-1)^k, \forall k \in \mathbb{Z}^+,$$

and secondly for $k = 0$, we can obtain

$$\binom{-t}{0} := 1.$$

Hence,

$$(x+y)^{-t} = \sum_{k=0}^{\infty} \binom{-t}{k} x^{-t-k} y^k.$$

Now, let $x = 1$, and replace y with $-y$.

$$(1-y)^{-t} = \sum_{k=0}^{\infty} \binom{-t}{k} (-1)^k y^k$$

$$= \sum_{k=0}^{\infty} \frac{t\,(t+1)\cdots(t+(k-1))}{k!}\,(-1)^{2k}\,y^k$$

$$= \sum_{k=0}^{\infty} \frac{t\,(t+1)\cdots(t+(k-1))}{k!} y^k$$

$$= 1 + ty + \sum_{k=2}^{\infty} \frac{t\,(t+1)\cdots(t+(k-1))}{k!} y^k.$$

Thus,

$$\sum_{i_{n+1}=1}^{2m-1} \left(1 - \tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n}\right)\right)^{-t}$$

$$= \sum_{i_{n+1}=1}^{2m-1} \left(1 + t\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n}\right)\right.$$

$$\left. + \sum_{k=2}^{\infty} \frac{t\,(t+1)\cdots(t+(k-1))}{k!} \tanh^k\left(X_{I_n}\right)\tanh^k\left(r\right)\cos^k\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n}\right)\right)$$

$$= 2m - 1 + t\tanh\left(X_{I_n}\right)\tanh\left(r\right)\left(\sum_{i_{n+1}=1}^{2m-1} \cos\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n}\right)\right)$$

$$+ \sum_{k=2}^{\infty} \frac{t\,(t+1)\cdots(t+(k-1))}{k!} \left(\sum_{i_{n+1}=1}^{2m-1} \tanh^k\left(X_{I_n}\right)\tanh^k\left(r\right)\cos^k\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n}\right)\right)$$

Let $m = 2$. Hence,

$$\sum_{i_{n+1}=1}^{3} \cos^k\left(\frac{i_{n+1}\pi}{2} - \theta_{I_n}\right) = \cos^k\left(\frac{\pi}{2} - \theta_{I_n}\right) + \cos^k\left(\pi - \theta_{I_n}\right) + \cos^k\left(\frac{3\pi}{2} - \theta_{I_n}\right)$$

$$= \begin{cases} -\cos^k(\theta_{I_n}) & k \text{ is odd} \\ 2\sin^k(\theta_{I_n}) + \cos^k(\theta_{I_n}) & k \text{ is even.} \end{cases}$$

The estimation becomes the following

$$\sum_{i_{n+1}=1}^{3} \left(1 - \tanh(X_{I_n}) \tanh(r) \cos\left(\frac{i_{n+1}\pi}{2} - \theta_{I_n}\right)\right)^{-t}$$

$$= 3 - t \tanh(X_{I_n}) \tanh(r) \cos(\theta_{I_n})$$

$$- \sum_{l=1}^{\infty} \frac{t(t+1)\cdots(t+((2l+1)-1))}{(2l+1)!} \left(\tanh^{2l+1}(X_{I_n}) \tanh^{2l+1}(r) \cos^{2l+1}(\theta_{I_n})\right)$$

$$+ \sum_{l=1}^{\infty} \frac{t(t+1)\cdots(t+((2l)-1))}{(2l)!} \left(\tanh^{2l}(X_{I_n}) \tanh^{2l}(r) \left(2\sin^{2l}(\theta_{I_n}) + \cos^{2l}(\theta_{I_n})\right)\right)$$

$$= 3 - t \tanh(X_{I_n}) \tanh(r) \cos(\theta_{I_n})$$

$$+ \sum_{l=2}^{\infty} (-1)^l \frac{t(t+1)\cdots(t+((l)-1))}{(l)!} \left(\tanh^{2l+1}(X_{I_n}) \tanh^{2l+1}(r) \cos^{l}(\theta_{I_n})\right)$$

$$+ \sum_{l=1}^{\infty} \frac{t(t+1)\cdots(t+((2l)-1))}{(2l)!} \left(\tanh^{2l}(X_{I_n}) \tanh^{2l}(r)\, 2\sin^{2l}(\theta_{I_n})\right).$$

Notice $0 < \theta_{I_n} < \frac{\pi}{4}$, $0 < t < 1$,

$$\sum_{l=1}^{\infty} \frac{t(t+1)\cdots(t+((2l)-1))}{(2l)!} \left(\tanh^{2l}(X_{I_n}) \tanh^{2l}(r)\, 2\sin^{2l}(\theta_{I_n})\right)$$

$$\leq \sum_{l=1}^{\infty} \frac{1 \cdot 2 \cdot \cdots \cdot 2l}{(2l)!} 2 \cdot \frac{1}{\sqrt{2}^{2k}} = 2 \cdot \sum_{k=2}^{\infty} \frac{1}{2^k} = 1.$$

Hence,

$$\sum_{i_{n+1}=1}^{3}\left(1-\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\frac{i_{n+1}\pi}{2}-\theta_{I_n}\right)\right)^{-t}\leq 3+$$

$$\left(1+\sum_{l=2}^{\infty}(-1)^{l}\frac{t\left(t+1\right)\cdots\left(t+\left(\left(l\right)-1\right)\right)}{\left(l\right)!}\left(\tanh^{2l+1}\left(X_{I_n}\right)\tanh^{2l+1}\left(r\right)\cos^{l}\left(\theta_{I_n}\right)\right)\right)$$

$$=3+\left(1+\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\theta_{I_n}\right)\right)^{-t}.$$

Since

$$\left(1+y\right)^{-t}=1+\sum_{k=1}^{\infty}\frac{(-t)(-t-1)\cdots(-t-(k-1))}{k!}y^{k}$$

$$=1+\sum_{k=1}^{\infty}\frac{t(t+1)\cdots(t+k-1)(-1)^{k}y^{k}}{k!},$$

and $\frac{1}{(1+\tanh(X_{I_n})\tanh(r)\cos(\theta_{I_n}))^{t}}\leq 1$,

$$P_{n+1}\leq\sum_{T\in\mathcal{W}_n}\frac{1}{\left(\cosh\left(X_{I_n}\right)\right)^{t}\left(\cosh\left(r\right)\right)^{t}}\left(3+\frac{1}{\left(1+\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\theta_{I_n}\right)\right)^{t}}\right)$$

$$\leq\frac{4}{\cosh^{t}\left(r\right)}P_n.$$

Thus,

$$\mathcal{P}\left(\Gamma,t\right)\leq P_1\sum_{n=1}^{\infty}\left(\frac{4}{\left(\cosh\left(r\right)\right)^{t}}\right)^{n}.$$

This implies that

$$\delta\left(\Gamma\right)\leq\frac{\ln\left(4\right)}{\ln\left(\cosh\left(r\right)\right)}.$$

To obtain a lower bound, recall

$$\sum_{i_{n+1}=1}^{3}\left(1-\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\frac{i_{n+1}\pi}{2}-\theta_{I_n}\right)\right)^{-t}$$

$$=3-t\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\theta_{I_n}\right)$$

$$+\sum_{l=2}^{\infty}(-1)^{l}\frac{t\left(t+1\right)\cdots\left(t+\left(\left(l\right)-1\right)\right)}{\left(l\right)!}\left(\tanh^{2l+1}\left(X_{I_n}\right)\tanh^{2l+1}\left(r\right)\cos^{l}\left(\theta_{I_n}\right)\right)$$

$$+\sum_{l=1}^{\infty}\frac{t\left(t+1\right)\cdots\left(t+\left(\left(2l\right)-1\right)\right)}{\left(2l\right)!}\left(\tanh^{2l}\left(X_{I_n}\right)\tanh^{2l}\left(r\right)2\sin^{2l}\left(\theta_{I_n}\right)\right)$$

$$\geq 2+\frac{1}{\left(1+\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\theta_{I_n}\right)\right)^{t}}.$$

Implying that

$$P_{n+1}\geq\frac{2}{\cosh^{t}\left(r\right)}P_n,$$

and hence

$$\mathcal{P}\left(\Gamma,t\right)\geq P_1\sum_{n=1}^{\infty}\left(\frac{2}{\cosh^{t}\left(r\right)}\right)^{n}.$$

Therefore,

$$\delta\left(\Gamma\right)\geq\frac{\ln\left(2\right)}{\ln\left(\cosh\left(r\right)\right)}.$$

□

The above bounds can be further sharpen into the following:

Theorem 4.8. *Suppose Γ is a well-distributed Schottky group of order 2. Then,*

$$\frac{\ln\left(2+\frac{1}{1+2\sqrt{2}}\right)}{\ln\left(\cosh\left(r\right)\right)} \leq \delta\left(\Gamma\right) \leq \min\left\{\frac{\ln\left(4-\frac{2}{1+2\sqrt{2}}\right)}{\ln\left(\cosh\left(r\right)\right)}, 1\right\}.$$

Proof. Firstly, since $\tanh\left(X_{I_n}\right) \geq \tanh\left(r\right)$, and $0 < \alpha \leq \frac{\pi}{4}$. Thus,

$$\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\theta_{I_n}\right) > \tanh^2\left(r\right)\cos\left(\theta_{I_n}\right) > \frac{1}{\sqrt{2}}\tanh^2\left(r\right).$$

Secondly, the choice of the operator T is as follows:

$$T = \begin{pmatrix} \frac{1}{\sin\left(\frac{\alpha}{2}\right)} & \cot\left(\frac{\alpha}{2}\right) \\ \cot\left(\frac{\alpha}{2}\right) & \frac{1}{\sin\left(\frac{\alpha}{2}\right)} \end{pmatrix}.$$

Hence,

$$|T0| = \cos\left(\frac{\alpha}{2}\right).$$

Then,

$$\tanh\left(r\right) = \tanh\left(\ln\left(\frac{1+|T0|}{1-|T0|}\right)\right) = \frac{2}{\cos\left(\frac{\alpha}{2}\right)}.$$

Therefore,

$$\frac{1}{\left(1+\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\theta_{I_n}\right)\right)^t} \geq \frac{1}{\left(1+2\sqrt{2}\right)^t} \geq \frac{1}{1+2\sqrt{2}}.$$

On the other hand, since

$$t\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\theta_{I_n}\right)\leq\frac{2}{\cos\left(\frac{\alpha}{2}\right)},$$

and recall that

$$\sum_{i_{n+1}=1}^{3}\left(1-\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\frac{i_{n+1}\pi}{2}-\theta_{I_n}\right)\right)^{-t}$$

$$=3-t\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\theta_{I_n}\right)$$

$$-\sum_{l=1}^{\infty}\frac{t\left(t+1\right)\cdots\left(t+\left(\left(2l+1\right)-1\right)\right)}{\left(2l+1\right)!}\left(\tanh^{2l+1}\left(X_{I_n}\right)\tanh^{2l+1}\left(r\right)\cos^{2l+1}\left(\theta_{I_n}\right)\right)$$

$$+\sum_{l=1}^{\infty}\frac{t\left(t+1\right)\cdots\left(t+\left(\left(2l\right)-1\right)\right)}{\left(2l\right)!}\left(\tanh^{2l}\left(X_{I_n}\right)\tanh^{2l}\left(r\right)\left(2\sin^{2l}\left(\theta_{I_n}\right)+\cos^{2l}\left(\theta_{I_n}\right)\right)\right).$$

Instead of taking the minimum of the term $-t\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\theta_{I_n}\right)$ by zero, now we use a sharper bound $\frac{2}{\cos\left(\frac{\alpha}{2}\right)}$, and this implies

$$\sum_{i_{n+1}=1}^{3}\left(1-\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\frac{i_{n+1}\pi}{2}-\theta_{I_n}\right)\right)^{-t}$$

$$\leq 3-\left(\frac{2}{\cos\left(\frac{\alpha}{2}\right)}\right)+\frac{1}{\left(1+\frac{2}{\cos\left(\frac{\alpha}{2}\right)}\right)^{t}}.$$

The proof is complete after inserting the aforementioned revised bounds into the prior theorem's proof. $\square$

Based on the above theorems, we conjecture

$$\delta\left(\Gamma\right)=\frac{\ln\left(3\right)}{\ln\left(\cosh\left(r\right)\right)}.$$

To see why this conjecture may be true, we consider $\alpha=\frac{\pi}{4}$. Then,

$$T0=\cos\left(\frac{\pi}{4}\right),$$

and

$$r=\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

Then,

$$\cosh\left(r\right)=\frac{e^{r}+e^{-r}}{2}=\frac{\frac{\sqrt{2}+1}{\sqrt{2}-1}+\frac{\sqrt{2}-1}{\sqrt{2}+1}}{2}=3.$$

Therefore,

$$\frac{\ln\left(3\right)}{\ln\left(\cosh\left(r\right)\right)}=1.$$

Furthermore, towards the conclusion of the next chapter, this conjecture was analytically tested using an approximation based on McMullen's technique.

Chapter 5

McMullen's Algorithm

The purpose of this chapter is to introduce McMullen's algorithm, which is by far the most accurate approach for numerically approximating the Hausdorff dimension of limit sets of Schottky groups. However, the original source code used to get the findings was never published[23], therefore we re-implemented the algorithm in C and reproduced the results to check that our knowledge and implementation of Patterson-Sullivan theory and McMullen's algorithm were valid. Furthermore, based on McMullen's derivation, we obtained an approximation for small angles that corresponds to the result produced by our conjecture, which is also included in this chapter.

In 1998, McMullen[23] proposed an algorithm for estimating the Hausdorff dimension of a set associated with a conformal dynamical system (as Julia sets or limit set of a geometrically finite Kleinian groups). The dimension of the related Patterson-Sullivan measures coincides with the Hausdorff dimension of its limit set, and McMullen's approach works to estimate the Hausdorff dimension of the Schottky group limit set.

In [23], some definitions were used for proving a theorem that supports McMullen's algorithm that were used in our set-up of well-distributed group:

Definition 5.1. *A $G-$invariant density of dimension* δ is a finite positive measure μ on $\mathbb{S}^1$ such that

$$\mu\left(T\left(E\right)\right)=\int_E |T'(z)|^\delta d\mu$$

whenever $T|_E$ is injective, $E \subset \partial\mathbb{B}^2 \cup \mathbb{B}^2$ is a Borel set and $T \in G$.

To describe the theorem and McMullen's algorithm, we need to define Markov partitions and their refinements.

Definition 5.2. Let G be a Schottky group. *A Markov partition* for (G, μ) is a nonempty collection $\mathcal{I}\left\langle (I_i, T_i)\right\rangle$ of connected compact intervals $I_i \subset \mathbb{S}^1$, $T_i \in G$ meet the following requirements:

- $T_i(I_i) \supset \cup_{i\mapsto j} I_j$, where the relation $i \mapsto j$ means $\mu\left(T_i(I_i) \cap I_j\right) > 0$.
- T_i is a homeomorphism on a neighborhood of $I_i \cap T_i^{-1}(I_j)$, when $i \mapsto j$;
- $\mu(I_i) > 0$;
- $\mu(I_i \cap I_j) = 0$ if $i \neq j$; and
- $\mu(T_i(I_i))\mu(\cup_{i\mapsto j} I_j) = \sum_{i\mapsto j} \mu(I_j)$.

Definition 5.3. *A refinement* of a Markov partition $\mathcal{I}$ is a new Markov partition:

$$\mathcal{R}(\mathcal{I}) = \langle (R_{ij}, T_i) : i \mapsto j \rangle$$

where $R_{ij} := T_i^{-1}(I_j) \cap I_i$.

Definition 5.4. If there exists a smooth conformal metric ρ on $\mathbb{S}^1$ and a constant ν such that

$$|T_i'(x)|_\rho > \nu > 1,$$

whenever $x \in I_i$ and $T_i(x) \in I_j$ for some j, then the Markov partition is *expanding*.

To outline the algorithm's five primary phases for computing the dimension δ of the density μ, firstly, let us consider a Schottky group G, a Markov partition $\mathcal{I} = \langle (I_i, T_i) \rangle$, and sample points $x_i \in I_i$ (in our implementation, the midpoint of each interval I_i was chosen to be x_i). The algorithm computes a sequence of estimates from $\alpha\left(\mathcal{R}^n(I)\right)$ to δ and then proceeds as follows:

- Step 1. For each $i \mapsto j$, solve for $y_{ij} \in I_i$ such that $T_i(y_{ij}) = x_j$.

- Step 2. Compute the transition matrix

$$T_{ij} = \begin{cases} |T_i'(y_{ij})|^{-1} & \text{if } i \mapsto j, \\ 0 & \text{otherwise.} \end{cases}$$

- Step 3. Find $\alpha(\mathcal{I}) \geq 0$ such that the spectral radius (the greatest eigenvalue of the matrix T_{ij}^{δ}, i.e. the transition matrix raised to the power α) equals unity.
- Step 4. Return $\alpha(\mathcal{I})$ as an approximation to δ of this iteration.
- Step 5. Refine the Markov partition, define new sample points $x_{ij} = y_{ij} \in R_{ij}$, and start afresh from Step 1.

The algorithm is supported by the following theorem[23]:

Theorem 5.1 (McMullen)**.** *Let $\mathcal{I}$ be an expanding Markov partition for a Schottky group G with invariant density μ of dimension δ. Then*

$$\alpha(\mathcal{R}^n(\mathcal{I})) \to \delta$$

as $n \to \infty$.

Hence, the dimension of the conformal measure, δ, is approximated by the power α.

The worst case scenario of this theorem could be the case when angle $\theta \to \frac{2\pi}{3}$ in the symmetric pairs of pants example in [23]. In this case, it must take at least more than one iteration to obtain an accurate digit for the first figure which is unit one.

Please keep in mind that in McMullen's article [23], this technique was developed in such a way that it was changed to adaptively refine only those partition blocks that exceeded a certain dimension r.

According to [23], the best possible results are produced when all of the blocks are about the same size, because if this extra assumption that all of the blocks are about the same size is not added, then a precise computation that only follows the theorem will be ruined by the inaccuracies introduced by larger blocks of that level.

Additionally, supercomputers in 1990s were about 5 to 10 GFLOPS[1] and 2010s laptops could at least have ranging from 100 to 400 GFLOPS, and laptops of the 2010s could be 5 times as much memory, and access at least 4 times as fast.

In our implementation so far, we have not added this extra assumption into our implementation in C:

`https://github.com/williamchuang/well-distributed-schottky-groups/tree/main/code`

Furthermore, for solving eigenvalues, it usually has $O(n^3)$ time complexity, and $O(n\ln(n))$ complexity for memory allocation. This method of the computation of Hausdorff dimension using eigenvalue algorithm is algebraic and numerical. In this paper we tried to develop a geometric and analytic method (resulted in our main theorems).

Our goals of studying McMullen's algorithm include:

(i) to reproduce some results of McMullen's paper[23] to show our understanding and implementation of this algorithm were correct, and

(ii) to use an approximation (an example is demonstrated in [23][Theorem 3.5])

[1] A computer system with 1 GFLOPS can do one billion (10^9) floating-point calculations per second.

based on the first-level approximation of this algorithm to check our conjecture in the next chapter.

Since in [23], the explicit forms of three generators of the Schottky group $\Gamma = \langle g_1, g_2, g_3 \rangle$ were omitted, to reproduce some results in this paper, we must reconstruct the group.

Hinted by each g_i is an inversion of a circle that intersect the unit circle orthogonally, and based on [10][7.21 The Perpendicular Bisector of a Segment], we can reconstruct Γ by using the idea of isometric circle.

Let $T \in \mathrm{PSU}(1,1)$, T is hyperbolic, and $T = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}$, where $\alpha \in \mathbb{C}$, $\gamma \in \mathbb{C}$, $\alpha + \overline{\alpha} > 2$, and $|\alpha|^2 - |\gamma|^2 = 1$. The perpendicular bisector of $[0, T0]$ is the isometric circle of T^{-1}.

Then the isometric circle of T^{-1} is $|\gamma z + \overline{\alpha}| = 1$, denoted by $I_{T^{-1}}$. Hence, the center of $I_{T^{-1}}$ is $\frac{-\overline{\alpha}}{\gamma}$ denoted by $z_{I_{T^{-1}}}$, and the radius of $I_{T^{-1}}$ is $\frac{1}{|r|}$.

Since $|T0| < 1$, if $|\alpha| \neq 0$, then

$$\frac{\overline{\gamma}}{\overline{\alpha}} < 1 \Rightarrow |\alpha| > |\gamma|.$$

Then the modulus of the center of $I_{T^{-1}}$ is

$$|z_{I_{T^{-1}}}| = \left|\frac{-\overline{\alpha}}{\gamma}\right| = \left|\frac{\alpha}{\gamma}\right| > 1,$$

if $|\gamma| \neq 0$.

Thus, we can have the following result:

Lemma 5.2. *If $I_{T^{-1}}$ intercepts $\partial\mathbb{B}^2$ orthogonally, then the Euclidean center of $I_{T^{-1}}$, i.e. $z_{I_{T^{-1}}}$, is always located outside of $\overline{\mathbb{B}^2}$.*

Next, since we know each conformal isometry in PSU$(1,1)$ can be decomposed into a product of either two reflections, two inversions, or a composition of one reflection and one inversion, we assume each g_i in McMullen's symmetric pairs of pants is composed by a reflection in a line and an inversion in a circle.

Recall that a reflection in a circle centered at q with radius R is a map:

$$I(z) = \frac{R^2}{\overline{z} - \overline{q}} + q,$$

and a reflection in a line $(ax + by = c)$ is a map with the following form:

$$h(z) = \frac{2ic + (b - ai)\overline{z}}{b + ai}.$$

Notice that $I(z)$ and $h(z)$ are non-analytic if we only have one of them alone, but any composition of even numbers of them is analytic. Hence, we assumed each $g_i(z)$ is decomposed into the following:

$$g_i(z) = I_i(h_i(z)).$$

Therefore, we have the following lemma of half-spaces

Lemma 5.3. *For each g_i,*

$$D_0(g_i) = D_0(g_i^{-1}).$$

Proof. Since for each g_i,

$$g_i = g_i^{-1}.$$

□

This setting is different compared to group generators of Schottky groups in other places such as [11, 12, 13, 15]. The reason is that according to the set-up of Schottky groups in these papers and books, we have $D_0(T_i) \cap D_0(T_i^{-1}) = \emptyset$.

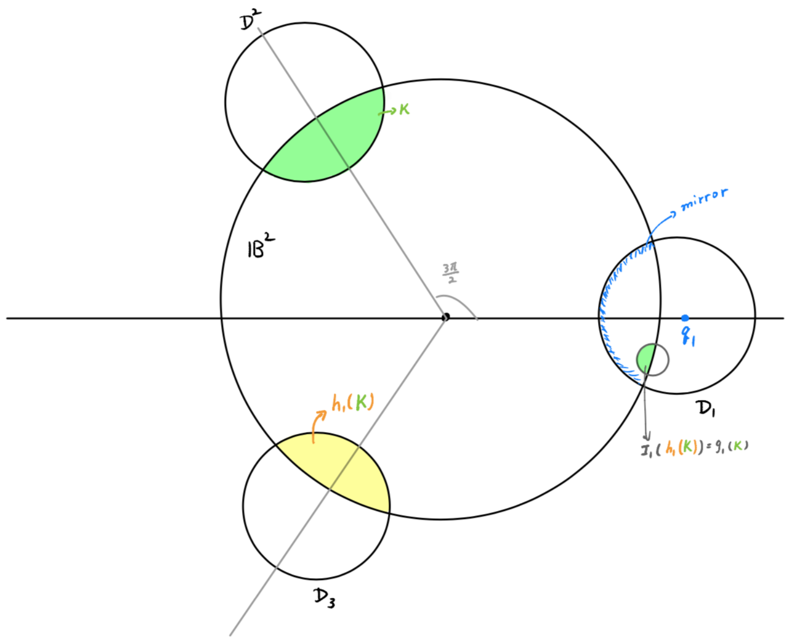

Figure 5.1: Please keep in mind that $I_i(z)$ and $h_i(z)$ are non-analytic if we only have one of them alone, but any composition of even numbers of them is analytic.

The next step is to find each g_i explicitly. According to the figure of the example of symmetric pairs of pants in [23], assume the three reflection lines are

$$L_1 : y = 0,$$

$$L_2 : \sqrt{3}x + y = 0, \text{ and}$$

$$L_3 : \sqrt{3}x - y = 0$$

corresponding to g_1, g_2, and g_3 that generate Γ. Therefore, for g_1, we have $h_1(z) = \overline{z}$, $I_1(z) = \frac{R^2}{\overline{z}-\overline{q_1}} + q_1$. Hence,

$$\begin{aligned} g_1(z) &= \frac{R^2}{z - \overline{q_1}} + q_1 \\ &= \frac{q_1 z - q_1\overline{q_1} + R^2}{z - \overline{q_1}}. \end{aligned}$$

Because it is a linear fraction, it can be mapped into a two-by-two matrix as follows.

$$\begin{pmatrix} q_1 & -q_1\overline{q_1} + R^2 \\ 1 & -\overline{q_1} \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, we can also have

$$\frac{\partial g_1}{\partial z} = \frac{ad - bc}{(cz + d)^2}.$$

For $g_2(z)$,

$$h_2(z) = \left(\frac{(1 - \sqrt{3}i)}{(1 + \sqrt{3}i)} \right) \overline{z} := H_2\overline{z}.$$

Thus,

$$\begin{aligned} g_2(z) &= \frac{R^2}{\overline{H}_2 z - \overline{q_2}} + q_2 \\ &= \frac{q_2\overline{H}_2 z - q_2\overline{q_2} + R^2}{\overline{H}_2 z - \overline{q_2}} \end{aligned}$$

which is corresponding to the following matrix

$$\begin{pmatrix} \overline{H}_2 q_2 & -q_2\overline{q_2} + R^2 \\ \overline{H}_2 & -\overline{q_2} \end{pmatrix}.$$

Similarly, for g_3,

$$g_3(z) = \frac{q_3\overline{H}_3 z - q_3\overline{q_3} + R^2}{\overline{H}_3 z - \overline{q_3}}$$

which corresponds to the matrix below

$$\begin{pmatrix} \overline{H}_3 q_2 & -q_2\overline{q_3} + R^2 \\ \overline{H}_3 & -\overline{q_3} \end{pmatrix}$$

where

$$\overline{H}_3 := \left(\frac{(-1-\sqrt{3}i)}{(-1+\sqrt{3}i)}\right).$$

We can now utilize the three generators mentioned above to compute the first level approximation using McMullen's technique.

Let A and B be two intercepts of I_{g_1} and $\partial\mathbb{B}^2$, and θ be the angle $\angle[A, O, B]$, where O is the origin.

Example: $\theta = \frac{2\pi}{3}$**.**

We have $x_2 = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$, and $q_1 = \left(\sqrt{1+R^2}, 0\right) = (2, 0)$, since $R = \tan\left(\frac{\theta}{2}\right)$.

Then,

$$\left|\frac{\partial g_1}{\partial z}\right| = \frac{R^2}{|x_2 - q_1|^2} = \frac{3}{7}.$$

We have the transition matrix through symmetry:

$$T_{12} = T_{21} = T_{13} = T_{31} = T_{32} = T_{23} = \left|\frac{\partial g_1}{\partial z}\right| = \frac{3}{7},$$

and for the diagonal entries,

$$T_{11} = T_{22} = T_{33} = 0.$$

Thus, based on McMullen's algorithm, for the first level (or the first iteration),

$$2\left(\frac{3}{7}\right)^{\alpha} = 1.$$

Solve the above equation for α, we derived

$$\alpha = \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{3}{7}\right)} \approx 0.818068.$$

This result can be used to verify whether our implementation makes sense in Chapter 6.

Furthermore, the formula

$$\left|\frac{\partial g_1}{\partial z}\right| = \frac{R^2}{|x_2 - q_1|^2}$$

can be used to check whether our conjecture can produce the same approximation based on this algorithm.

Furthermore, in general, by using cosine law,

$$|x_2 - q_1|^2 = 2 + \left(\tan^2\left(\frac{\theta}{2}\right)\right) + \sec\left(\frac{\theta}{2}\right).$$

Thus, in general, for $\theta \in (0, \frac{2\pi}{3}]$, and for the first level approximation, we can obtain the following

$$\left|\frac{\partial g_1}{\partial z}\right| = \frac{\tan^2\left(\frac{\theta}{2}\right)}{2 + \left(\tan^2\left(\frac{\theta}{2}\right)\right) + \sec\left(\frac{\theta}{2}\right)},$$

where $R = \tan\left(\frac{\theta}{2}\right)$.

Moreover, in general, by solving the equation

$$2t^\alpha = 1,$$

where $t = \left|\frac{\partial g_1}{\partial z}\right|$, then

$$\delta(G) \approx \alpha = \frac{\ln\left(\frac{1}{2}\right)}{\ln(t)} = \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2+\left(\tan^2\left(\frac{\theta}{2}\right)\right)+\sec\left(\frac{\theta}{2}\right)}\right)}.$$

5.1 Comparing with our conjecture on small angles

Recall that at the end of the last Chapter, for a well-distributed Schottky group of rank two we obtained

$$\delta(G) = \frac{\ln(3)}{\ln(\cosh(r))},$$

where

$$\cosh(r) = \frac{e^r + e^{-r}}{2}, r = \ln\left(\frac{1+\cos\left(\frac{\theta}{2}\right)}{1-\cos\left(\frac{\theta}{2}\right)}\right).$$

Then,

$$\cosh(r) = \csc\left(\frac{\theta}{2}\right)^2 + \cot\left(\frac{\theta}{2}\right)^2$$

$$= \frac{1}{\sin\left(\frac{\theta}{2}\right)^2} + \frac{1}{\tan\left(\frac{\theta}{2}\right)^2}.$$

For $\theta \to 0$,

$$\sin\left(\frac{\theta}{2}\right)^2 \approx \left(\frac{\theta}{2}\right)^2 \approx \tan\left(\frac{\theta}{2}\right)^2.$$

Thus,

$$\cosh(r) \simeq \frac{8}{\theta^2}.$$

On the other hand, for $m = 2$,

$$R = \tan\left(\frac{\theta}{2}\right) \approx \frac{\theta}{2},$$

and

$$|x_2 - q_1| = \sqrt{2}.$$

Therefore, from the first level approximation of McMullen's algorithm,

$$\left|\frac{\partial g_1}{\partial z}\right| = \frac{R^2}{|x_2 - q_1|^2} \approx \frac{\theta^2}{8}.$$

Thus, for $\theta \to 0$, both the first level approximation of McMullen's algorithm and our conjecture give an identical estimation

$$\delta(G) \approx \frac{\ln(3)}{\ln\left(\frac{\theta^2}{8}\right)}.$$

5.2 Using the first level approximation to verify our implementation in C

When McMullen's results[23] are compared to our first level approximation result, we can see that they both coincide at the first digit after the decimal point from 1° to 80°.

Please keep in mind that McMullen's results were based on a level 18 computation. Additionally, it was stated in [23] that there were about 600,000 Markov partitions in this level, and $600,000 \approx 600 \cdot 2^{10} < 6 \cdot 2^{17} = 3 \cdot 2^{18}$, but in our implementation, at level 1 we only use three Markov partitions.

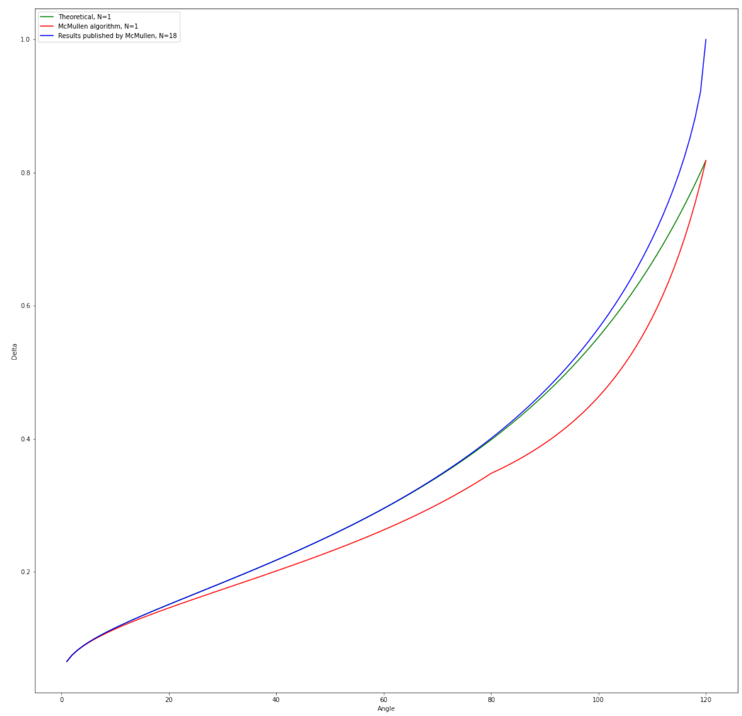

Figure 5.2: Level 1 ($N = 1$) approximations, including McMullen's published data[23][Table 12] (colored in blue), our $N = 1$ theoretical approximation (green), and $N = 1$ numerical approximation based on McMullen's algorithm (red). The angle θ is running from $1°$ to $120°$.

In the above figure, we plotted three results together, including the data published by McMullen [23][Table 12] for $N = 18$ which is colored in blue, the output of our implementation for $N = 1$ which is colored in red, and the theoretical result $\delta(G) \approx \alpha = \frac{\ln\left(\frac{1}{2}\right)}{\ln(t)} = \frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2+\left(\tan^2\left(\frac{\theta}{2}\right)\right)+\sec\left(\frac{\theta}{2}\right)}\right)}$ for $N = 1$ which is colored in green.

We can see that the $N = 1$ theoretical result can rather well resemble McMullen's $N = 18$ finding[23] for $\theta \in [0°, 80°]$. Furthermore, we can see that, owing to McMullen's practical concern, the outcome of our implementation's $N = 1$ instance did not correspond with the theoretical findings or the $N = 18$ results provided by McMullen.

This, however, may be improved angle by angle. It takes longer when N exceeds 8 (after $N > 8$, each level may take more than 2 hours before parallel computing), but it is achievable.

For instance, if we focus on $\theta = 50°$, then we can have the following improvement

level	degree	$\alpha_N(\Gamma)$
1	50	0.23066000000008990
2	50	0.27209000000013134
3	50	0.25797000000011720
4	50	0.24197000000010122
5	50	0.25848000000011770
6	50	0.25839000000011764
7	50	0.25825000000011750
8	50	0.25839000000011764
9	50	0.25839000000011764

Table 5.1: An improvement for the case $\theta = 50°$ with precision$= 0.00001$.

The meaning of the above table is twofold: First and foremost, it demonstrates that if we fix a precision to a specific digit, the sequence of α_N would ultimately converge (in $O(N)$ time complexity). Second, we can observe an improvement when we compare the first two digits after the decimal point for the $N = 1$ result, which is 0.23, to McMullen's $N = 18$ result, which is 0.25, i.e. when $N > 4$ the sequence becomes stable. Furthermore, the figure after 25 may vary as our precision improves.

Furthermore, in the following table, for large θ, i.e. $\theta \to \frac{2\pi}{3}$, we can observe that the sequence tends to converge to 1.

level	degree	$\alpha_N(\Gamma)$
1	120	0.8180799999989117
2	120	0.8739599999986574
3	120	0.9045899999985180
4	120	0.9240099999984296
5	120	0.9372899999983691
6	120	0.9468699999983256
7	120	0.9540399999982929
8	120	0.9595799999982677
9	120	0.9639899999982476

Table 5.2: Improve $\alpha_N(\Gamma)$ such that it converges to $\delta(\Gamma) = 1$ with precision= 0.00001.

Based on the above observation, we can also notice that compared to $N = 1$, the more level we can reach, the larger value $\alpha_N(\Gamma)$ will be, and different angles need different N to converge when the same precision was given. Furthermore, based on these above results, including our $N = 1$ theoretical formula can fit McMullen's result well for $\theta \leq 80°$, and when $\theta > 80°$, as shown in the above two examples, we can have a good approximation compare to McMullen's results by increasing the number of level, N.

Finally, we also use the same understanding to implement McMullen's algorithm for $m = 2$ well-distributed Schottky group and plot the following results with the results generated by our main theorem and conjecture all together.

For $m = 2$, by using cosine law for $\frac{\pi}{2}$, and solving α for $3(t)^{\alpha} = 1$, we have the first level approximation as follows

$$\delta(G) \approx \alpha = \frac{\ln\left(\frac{1}{3}\right)}{\ln(t)} = \frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2+\left(\tan^2\left(\frac{\theta}{2}\right)\right)}\right)}.$$

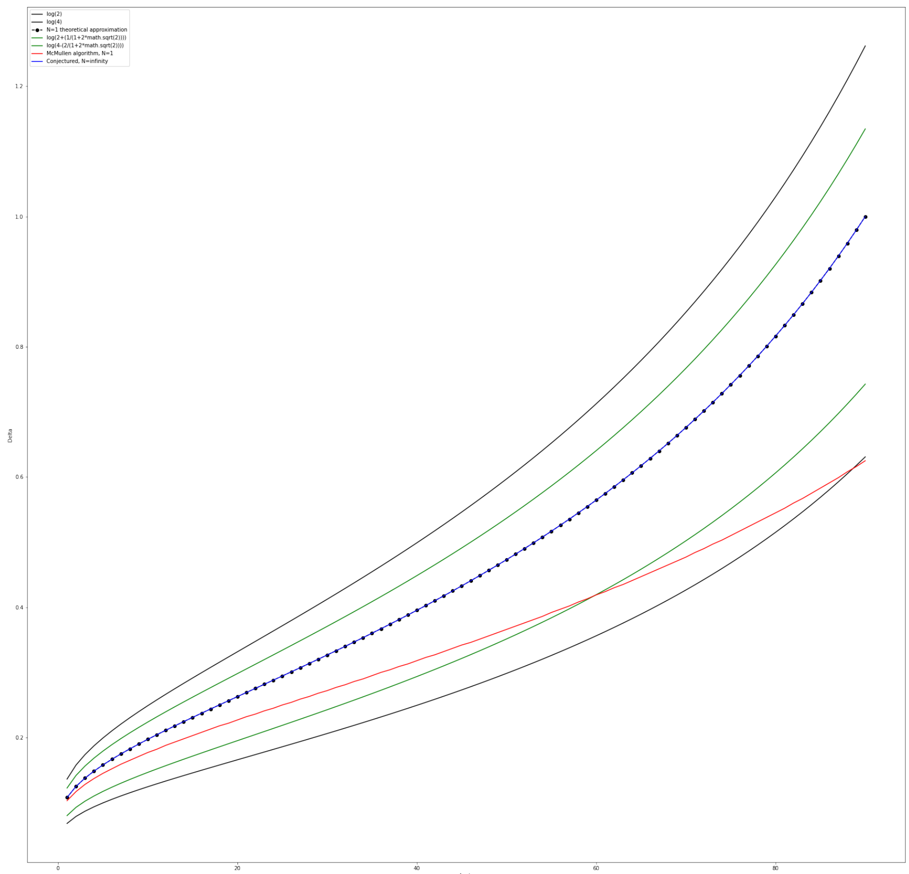

Figure 5.3: Results from our main theorems (colored in green and black), conjecture (blue), theoretical $N = 1$ approximation (dotted black), and numerical $N = 1$ approximation implementation of McMullen algorithm (red).

Although we have not had enough time to improve the results of our implementation of McMullen's algorithm, based on the above examples, we believe that once N increases, the curve will be shifted towards the conjectured curve.

Surprisingly, the findings of the $N = 1$ approximation precisely match our conjecture for $N \to \infty$. Therefore, if our conjecture is true, then this implies that for a well-distributed Schottky group of rank 2, $\delta(\Gamma) = \alpha_1(\Gamma)$. It may also imply that for all N, we have

$$\delta(\Gamma) = \alpha_N(\Gamma) = \alpha_1(\Gamma) = \frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2+\left(\tan^2\left(\frac{\theta}{2}\right)\right)}\right)} = \frac{\ln(3)}{\ln(\cosh(r))},$$

where

$$r = \ln\left(\frac{1+\cos\left(\frac{\theta}{2}\right)}{1-\cos\left(\frac{\theta}{2}\right)}\right),$$

since $T0 = \cos\left(\frac{\theta}{2}\right)$. For the last equation, it can be demonstrated that

$$\cosh(r) = \frac{1+\cos^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)},$$

and

$$\frac{\tan^2\left(\frac{\theta}{2}\right)}{2+\left(\tan^2\left(\frac{\theta}{2}\right)\right)} = \left(\frac{1+\cos^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}\right)^{-1}.$$

Hence, we always have $\frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2+\left(\tan^2\left(\frac{\theta}{2}\right)\right)}\right)} = \frac{\ln(3)}{\ln(\cosh(r))}$ for all $0 < \theta < \frac{\pi}{2}$. This alternate strategy might also shed some light on the meaning of $cosh(r)$ in our conjecture.

A proof of the conjecture, a generalization to $m > 2$, and enhancements to our implementations are on the horizon for our future work. Due to heavy matrix operations involved, an implementation on parallel computing in a contemporary computer might potentially enhance McMullen's results.

Our source code can be downloaded from GitHub:

`https://github.com/williamchuang/well-distributed-schottky-groups/tree/main/code`

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Appendix A: Properties of Schottky Groups and Limit Sets of Schottky Groups

The purpose of this appendix is to provide a brief summary for introducing the properties of Schottky Groups and limit sets of Schottky groups that are not used in the development of our main theorem.

Let $S(g_1, ..., g_p)$ denote a Schottky group generated by $\{g_1, .., g_p\}$.

The following lemma demonstrates that we only need two non-elliptic isometries to form a Schottky group and it is a direct result of Lemma 2.2 and 2.3.

Lemma 5.4 (Dal'Bo)**.** *Let g and g' be two non-elliptic isometries in G which have no common fixed points. Then there exists $N > 0$ such that g^N and g'^N generate a Schottky group $S(g^N, g'^N)$.*

In order to introduce some well-known properties of limit sets in $\partial\mathbb{B}^2$ of discrete subgroup of $\mathrm{PSU}(1,1)$, let us recall two definitions from Topology:

Definition 5.5. [28] A subset A of a metric space X is *nowhere dense in X*, if $\mathrm{int}(\overline{A}) = \emptyset$.

Definition 5.6. [28] A subset A of a metric space X is *perfect*, if it is closed and if every point in A is an accumulation point of A, i.e. there exists a sequence $x_n \to x \in A$, and all x_n are distinct.

Proposition 5.5. *Let $G \subset PSU(1,1)$ be discrete, and $L(G)$ be the limit set of G, then*

1. *$A(L(G)) = L(AGA^{-1})$, where $A \in PSU(1,1)$ and $A^{-1}(\infty) \notin L(G)$,*
2. *$L(G)$ is G-invariant, i.e. $AL(G) = L(G)$, where $A \in G$,*

3. *$L(G)$ is uncountable and perfect ($L(G)$ has no isolated points), and*

4. *Either $L(G)$ is nowhere dense or $L(G) = \partial\mathbb{B}^2$.*

Remark: Please notice the difference between the first and second properties are due to A is chosen from $\mathrm{PSU}(1,1) \setminus G$ or from G. To have the second property, we must have $A \in G$.

Proof of 1, [9]. Let $z_+ = \left(\lim_{n\to\infty} T^n 0\right) \in L(G)$, and $T \in G$, $A \in \mathrm{PSU}(1,1)$, i.e. A can be in $\mathrm{PSU}(1,1) \setminus G$, and $A^{-1}(\infty) \notin L(G)$. With the given condition we have the limit point

$$z_+ = \left(\lim_{n\to\infty} T^n 0\right) \in L(G).$$

By taking the conjugation map on G with A, we have

$$w_+ = \left(\lim_{n\to\infty} (ATA^{-1})^n 0\right) \in L(AGA^{-1}).$$

However,

$$(ATA^{-1})^n = AT^nA^{-1},$$

and A is conformal, so it is continuous, thus

$$w_+ = A\left(\lim_{n\to\infty} T^n A^{-1} 0\right) = Az_+ \in AL(G).$$

Since $w_+ \in L(AGA^{-1})$, we also have $Az_+ \in AL(G)$, and vice versa. Therefore, $AL(G) = L(AGA^{-1})$. □

Proof of 2, [28]. Let $z_+ = \left(\lim_{n\to\infty} T^n 0\right) \in L(G)$. Similar to the previous proof, this time we have $A \in G$. So, let $T \in G$ and $A \in G$. With the given condition we have the limit point

$$z_+ = \left(\lim_{n\to\infty} T^n 0\right) \in L(G).$$

By taking the conjugation map on T with A, we have

$$w_+ = \left(\lim_{n\to\infty} (ATA^{-1})^n 0\right) \in L(G).$$

Furthermore,

$$(ATA^{-1})^n = AT^nA^{-1},$$

and A is conformal, so it is continuous, thus

$$w_+ = A\left(\lim_{n\to\infty} T^nA^{-1}0\right) = Az_+ \in AL(G).$$

Since $w_+ \in L(G)$, we also have $Az_+ \in L(G)$, and vice versa. Therefore, $L(G)$ is G-invariance. $\square$

Proof of 3, [28]. Let $z \in L(G)$. Then by definition, there exists a sequence $\{g_n\}_{n\geq 1}$ where $g_n \in G$ are hyperbolic such that $z = \lim g_n^+$ be an accumulation point, if g_n are all different. However, if g_n are not all distinct, then there exists a non-elliptic $g \in G$ such that $z = g^+$. Take $z' \in L(G)$, and $z' \neq z^\pm$, then there exists a sequence $\{h^n\}$ such that $h^n(z') \to z$ and all points $h^n(z')$ are distinct. Therefore, A is perfect. It follows from lemma 5.7 that A is uncountable. $\square$

Proof of 4, [28]. If $L(G) \neq \partial\mathbb{B}^2$. Let $z \in \partial\mathbb{B}^2 \setminus \Lambda$. Recall $L(G)$ is closed, there is an open set U_z such that $U_z \subset \partial\mathbb{B}^2 \setminus L(G)$. By G-invariance, i.e. by 1, $g(U_z) \cap L(G) = \emptyset$, for all $g \in G$. Let O be an open set with $O \cap L(G) \neq \emptyset$, and want to find a non-empty open set $V \subset O$ with $V \cap A = \emptyset$. Take $y \in O \cap A$. By Theorem 3.7, the accumulation points of the orbit of z is equal to $L(G)$. Hence, take $g_n \in G$ with $g_n(z) \to y$. Then, ultimately, $g_n(z) \in O$ and consequently $g_n(U_z) \cap O \neq \emptyset$. Since $g_n(U_z)$ is open and contains $g_n(z) \to y \in L(G)$. $\square$

Corollary 5.6. *[28] If $L(G) \neq \partial\mathbb{B}^2$, then $L(G)$ is homeomorphic to the middle third Cantor set.*

Proof. Since $L(G)$ is nowhere dense, $L(G)$ is totally disconnected. Take $z, w \in L(G)$, then $\partial\mathbb{B}^2\, \{z, w\}$ have two open intervals U_1 and U_2. Since $L(G)$ is nowhere dense, in each interval U_i, there exists $v_i \in U_i$, where $v_i \notin L(G)$. Hence v_1 and v_2 separate $\partial\mathbb{B}^2$ into two open interval V_1 and V_2 each has one of z, w as interior point. Hence, $V_i \bigcup L(G)$ disconnect $L(G)$. It follows from the theorem that every perfect totally disconnected subset of $\mathbb{R}$ is homeomorphic to the middle third Cantor set. $\square$

Lemma 5.7. *[28] A perfect subset A of a complete metric space X is uncountable.*

Proof. Let $A \subset X$ be a perfect set. Then, A is closed, and it is a complete metric space, hence Baire Category theorem[2] can be applied on A. Claim: For all $a \in A$, a is nowhere dense in A. Since $\{a\}$ is closed, we want to show $\text{int}(\{a\}) = \emptyset$. Since $\text{int}(\{a\}) \subset \{a\}$, thus if $\text{int}(\{a\}) \neq \emptyset$, then $\text{int}(\{a\}) = \{a\}$. Since interior is open, so there exists U open in X such that $\text{int}(\{a\}) = U \cap A$. However, since A is perfect, so if $U \subset X$ is open, and $a \in U$, then there exist some $y \in U \cap A$, but $y \neq a$. Thus, $\text{int}(\{a\}) = \{a\} \neq U \cap A$. $\square$

Proposition 5.8. *Let $G(g_1, g_2)$ be a Schottky group. Then,*

$$L(G(g_1, g_2)) = \bigcap_{n=1}^{\infty} \bigcup_{len(g_I)=1}^{len(g_I)=n} \overline{D(g_I)}.$$

Proof. (i) To show the right hand side is a subset of the left hand side: Let $z \in L(G)$, and $(g_i)_{\geq 1}$ is a sequence in G such that $g_I 0 \to z$ as $\text{len}(g_I) = n \to \infty$. Since the alphabet $\mathcal{A}$ is finite, by switching to a subsequence, we can assume there is a sequence $(h_i)_{\geq 1}$ with $h_{i+1} \neq h_i^{-1}$, and a sequence of positive integers $(m_k)_{k \geq 1}$ which is strictly increasing such that $g_I = h_1 \dots h_{m_n}$. Since $h_{m_n} 0 \in D(h_{m_n})$, $g_I 0 \in D(h_J)$, where $h_J = h_1 \dots h_{m_n}$ for all $n \geq 1$. Since the sets $D(h_1 h_2 \dots h_n)$ are nested and by Lemma 3.11 their radii converge to zero, and recall that $\text{len}(h_J) = n$, we have

$$\{z\} = \bigcap_{n=1}^{\infty} \overline{D(h_J)}.$$

(ii) To show $L(G(g_1, g_2))$ is a subset of the right hand side of the equality: This result follows from Proposition 3.10 and Lemma 3.11. $\square$

[2]A complete metric space is not the countable union of closed subsets with empty interior (i.e. nowhere dense subsets)[6, Theorem 6.76].