The Hausdorff Dimension of Limit Sets of Well-distributed Schottky Groups

William Huanshan Chuang

Mathematics Department, San Francisco State University

Abstract

Let a finitely generated Schottky group G be given, and $\mathbb{B}^2$ be a Poincaré disk model of two-dimensional hyperbolic space. An unsolved problem is: How to find an exact value of Hausdorff dimension of the limit set $L(G)$ when $L(G) \neq \partial\mathbb{B}^2$?
To solve one specific case of this problem, a well-distributed Schottky group $\Gamma = \langle T_1, T_2, ..., T_m\rangle$ is defined. Our main theorem was proved based on: properties of Poincaré series, isometric circles, and some nice properties come with the definition of well-distributed Schottky group, especially that we found a way to reconstruct the orbit $\Gamma(0)$ by using only two operators T and R for any $m \in \mathbb{Z}^+$.

Motivation

The following are five top historical reasons that highlight the need of creating a systematic method for computing the critical exponent of the Poincaré sereis. Let G be a finitely generated Schottky group.

- In the prime geodesic theorem[2], $\delta(G)$ plays a role $z = \delta(G) + it$ which is similar to the line $z = 1$ in the prime number theorem where the Riemann zeta function has its only singularity at the point $z = 1$ and no zeros on this line as proved by von Mangoldt in 1895.
- In the spectral gap problem of hyperbolic surfaces[3], a hyperbolic surface has an essential spectral gap if it can meet some conditions in the theorem they proved, meaning that there are finitely many zeros within the region $\Re(s) > \frac{1}{2} - \beta = \delta(G)$.
- In $\mathrm{AdS}_3/\mathrm{CFT}_2$[5], $\delta(G)$ is a criterion for the critical exponent of a scalar field to have stable solutions in 3d quantum gravity.
- Since $\delta(G)$ uniquely determines the first resonance of the corresponding Selberg zeta function of G, and the first resonance is the lowest eigenvalue of eigenfunctions of the Laplacian operated on a Riemann surface (algebraic curve) $\mathbb{H}^2/G$. Since $\delta(G)$ is the convergence exponent of Poincaré series of G, and this series can be written into a Dirichlet $L-$function, plus eigenfunctions of $\mathbb{H}^2/G$ are automorphic forms, hence an exact connection between number theory and harmonic analysis might possibly be established. It maight also be viewed as an illustration of the Langlands program that Dirichlet characters, algebraic curves, and automorphic forms are all interrelated[4, 6, 7, 8].
- Furthermore, since the Patterson-Sullivan measure μ is a constant multiple of the Hausdorff measure $H^{\delta}|_{\Lambda(G)}$, by Sullivan's theorem, we can have the Hausdorff dimension of the limit set of G equals the critical exponent, i.e. $\dim_H \Lambda(G) = \delta$[9].

However, except for the case when $\delta(G) = 1$, so far there is no explicit formula to describe $\delta(G)$.

Poincaré series

The Poincaré series is defined as follows

$$\mathcal{P}(G,t) := \sum_{\forall g_i \in G} \exp\{-t d_{\mathbb{B}^2}(z, g_i(z))\}$$

where $\delta(G) := \inf\{t \in \mathbb{R} : \mathcal{P}(G,t) < \infty\} = \sup\{t \in \mathbb{R} : \mathcal{P}(G,t) = \infty\}$ is called the critical exponent of the Poincaré sereis, the exponent of convergence of the Poincaré sereis, or the Poincaré exponent, and $z, g_i(z) \in \mathbb{B}^2$.

Limit Points and Limit Sets

Let G be a discrete subgroup of sense-preserving isometry group $\mathrm{Isom}^+(\mathbb{B}^2)$. The *limit set* of G is denoted by $L(G)$, is the set of all accumulation points in the intersection of all orbits $G(x)$ for all $x \in \mathbb{B}^2 \cup \mathbb{S}^1_\infty$, where $\mathbb{S}^1_\infty$ is the sphere at infinity and it also represents the boundary of the hyperbolic space $\mathbb{B}^2$[1].

Schottky Groups

Given $D_1, ..., D_{2m}$ disjoint closed disks which intersect the boundary of the unit disk $\mathbb{B}^2$ orthogonally where $m \geq 2$, we let non-elliptic $T_i \in PSU(1,1)$, identified as the linear fractional transformation, be the mapping such that

1. $T_i(\mathbb{B}^2 \setminus D^\circ_{i+m}) = D_i$
2. $T_i^{-1}(\mathbb{B}^2 \setminus D^\circ_i) = D_{i+m}$.

Then, *a Schottky group* Γ *of rank* m is finitely generated by $T_1, ..., T_m$. We will denote it by

$$\Gamma = \langle T_1, ..., T_m\rangle.$$

We denote a cyclic notation for the group generators by $T_{i+m} = T_i^{-1}$.

Well-distributed Schottky groups

We say that Γ is *a well-distributed Schottky group of rank* m with generators $T_1, ..., T_m$, if the following holds.

(1) $d(0, T_i 0)$ are equal to each other for all $i = 1, ..., m$.
(2) Let $r = d(0, T_i 0)$ be the number defined in (1). The argument of the complex numbers $\{T_i(0) : i = 1, ..., 2m\}$, arranged in ascending order, are equal angle apart in the circle $|z| = r$.
(3) Half-spaces $D_0(T_i)$ are closed disjoint disks.

Two methods to plot the orbit $\Gamma(0)$

Method 1: Let Γ be well-distributed. Assume $z \in \mathcal{R}(\Gamma, T_{i_1}) \cap \Gamma(0)$, and $T_{i_1}z = RT_{i_j}z$. Instead of using T_{i_1} to derive $T_{i_1}z$, we want to use T_{i_j}. As a result, first use $R^{-1}z$ to map z to $\mathcal{R}(\Gamma, T_{i_j}) \cap \Gamma(0)$. Then we apply T_{i_j} to it and utilize R to map it back to $\mathcal{R}(\Gamma, T_{i_1}) \cap \Gamma(0)$. Because each point in $\Gamma(0)$ contained by $D_0(T_{i_j})$ may be found by applying elliptic isometry to obtain, i.e. by mapping a related point from another sector $D_0(T_{i_k}), k \neq j$. Thus, instead of considering the generators $T_1, .., T_m, T_1^{-1}, ..., T_m^{-1}$, we can derive:

$$\Gamma = \left\langle T_{i_j}, RT_{i_j}R^{-1}, ...R^{m-1}T_{i_j}R^{-(m-1)}\right\rangle,$$

and for simplicity we can let $T = T_{i_j}$, then we can obtain

$$G' := \left\langle T, RTR^{-1}, ...R^{m-1}TR^{-(m-1)}\right\rangle$$

Then, we can use $G'(0) \cap \Gamma(0)$ to reconstruct $\Gamma(0)$.

Method 2: Assume we already know $\Gamma(0)$. At the highest level, Γ is reconstructed counter-clockwisely and level by level. For $N = 1$, we start with $T_{i_j}0$, and map it to $RT_{i_j}0$, $R^2T_{i_j}0$,..., $R^kT_{i_j}0$, ..., $R^{2m-1}T_{i_j}0$. For $N = 2$, firstly by using $\Gamma(0)$, apply T_{i_j} at each $T_I 0 = R^k T_{i_j} 0$ where $\mathrm{len}(T_I) = 1, T_I \neq T_{i_j}$, then we can determine each node in the second level in $\mathcal{R}(\Gamma, T_{i_j})$. Next, use R^k to map all these nodes to $\mathcal{R}(\Gamma, T_{i_a}), a \neq j$, $a \in \{1, ..., 2m\}$. Assume at $N = K$, all nodes in $\mathcal{R}(\Gamma, T_{i_j})$ and the remaining $\mathcal{R}(\Gamma, T_{i_a})$ are given. Then, for $N = K + 1$ level, we use $\Gamma(0)$ to see where each $T_{i_j}T_I 0$ goes, when $\mathrm{len}(T_I) = K$. Next, we use the R^k to map those $(2m-1)^{K+1}$ nodes represented only by operators R and T_{i_j} in $\mathcal{R}(\Gamma, T_{i_j})$ to $\mathcal{R}(\Gamma, T_{i_a})$.

Our set-up

Let $T \in \mathrm{PSU}(1,1)$, T is hyperbolic, and $T = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}$, where $\alpha \in \mathbb{C}$, $\gamma \in \mathbb{C}$, $\alpha + \overline{\alpha} > 2$, and $|\alpha|^2 - |\gamma|^2 = 1$. The perpendicular bisector of $[0, T0]$ is the isometric circle of T^{-1}. Without loss the generality, assume $\alpha = \left(\sin\left(\frac{\theta}{2}\right)\right)^{-1}$, and $\gamma = \cot\left(\frac{\theta}{2}\right)$ to satisfy the identity. Then we can define

$$T := \begin{pmatrix} \frac{1}{\sin\left(\frac{\theta}{2}\right)} & \cot\left(\frac{\theta}{2}\right) \\ \cot\left(\frac{\theta}{2}\right) & \frac{1}{\sin\left(\frac{\theta}{2}\right)} \end{pmatrix}.$$

Lemma

The Poincareé series can be rewritten using the above definition

$$\mathcal{P}(\Gamma, t) \asymp \sum_{n=1}^{\infty} P_n,$$

where

$$P_n := \sum_{T \in \mathcal{W}_n} \frac{1}{Y_{I_n}^t},$$

and

$$Y_{I_n} := \cosh^n(r) \prod_{j=1}^{n-1} \left(1 - \tanh\left(X_{I_{j+1}}\right) \tanh(r) \cos\left(\frac{i_{j+1}\pi}{m} - \theta_{I_j}\right)\right).$$

Plotting The Orbit $\Gamma(0)$ of Well-distributed Schokky Groups

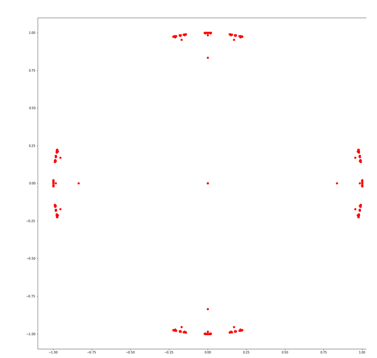

Figure: $m = 2$, $\Lambda = 0.3$, $\theta \approx 33.398473447277695°$, Level 14 ($N = 14$).

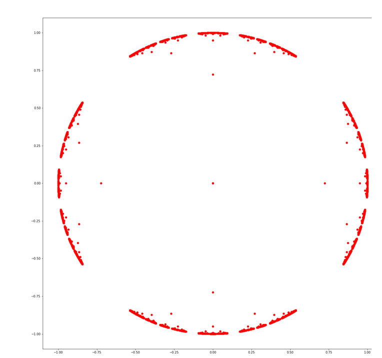

Figure: $m = 2$, $\Lambda = 0.4$, $\theta \approx 43.602794482778144°$, Level 14 ($N = 14$).

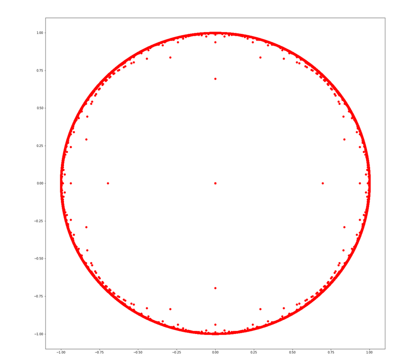

Figure: $m = 2$, $\Lambda = \frac{\sqrt{2}}{2+\sqrt{2}} + 0.01$, $\theta \approx 45.9746105715017°$, Level 14 ($N = 14$).

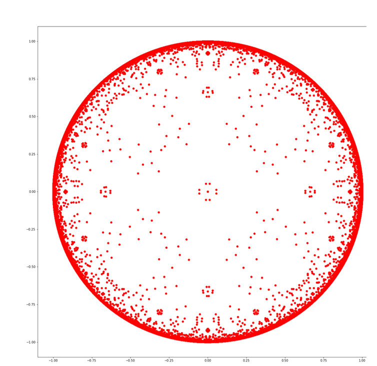

Figure: $m = 2$, $\Lambda = 0.45$, $\frac{\theta}{2} \approx 48.455517824418166°$, Level 14 ($N = 14$).

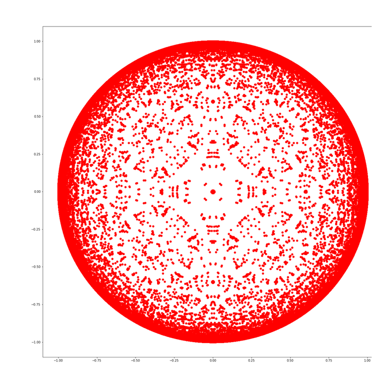

Figure: $m = 2$, $\Lambda = 0.5$, $\theta \approx 53.13010858755201°$, Level 14 ($N = 14$).

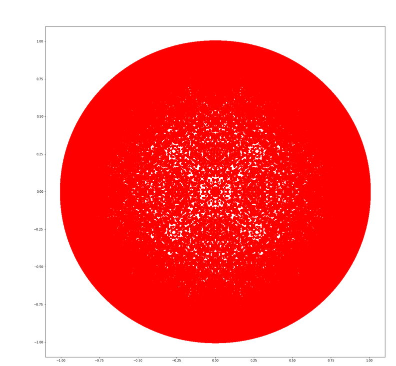

Figure: $m = 2$, $\Lambda = 0.6$, $\theta \approx 61.927531757108724°$, Level 14 ($N = 14$).

Our source code can be downloaded from GitHub:

`https://github.com/williamchuang/well-distributed-schottky-groups/tree/main/code`

Main Theorem

Suppose Γ is a well-distributed Schottky group of order 2. Then,

$$\frac{\ln\left(2 + \frac{1}{1+2\sqrt{2}}\right)}{\ln(\cosh(r))} \leq \delta(\Gamma) \leq \min\left\{\frac{\ln\left(4 - \frac{2}{1+2\sqrt{2}}\right)}{\ln(\cosh(r))}, 1\right\}.$$

Conjecture

Suppose Γ is a well-distributed Schottky group of order 2. Then,

$$\delta(\Gamma) = \frac{\ln(3)}{\ln(\cosh(r))},$$

where $r = \ln\left(\frac{1+|T0|}{1-|T0|}\right) = \ln\left(\frac{1+\cos\left(\frac{\theta}{2}\right)}{1-\cos\left(\frac{\theta}{2}\right)}\right)$, and $T0 = \cos\left(\frac{\theta}{2}\right)$.

Furthermore, the formula

$$\left|\frac{\partial g_1}{\partial z}\right| = \frac{R^2}{|x_2 - q_1|^2}$$

can be used to check whether our conjecture can produce the same approximation based on McMullen's algorithm.
Furthermore, in general, by using cosine law,

$$|x_2 - q_1|^2 = 2 + \left(\tan^2\left(\frac{\theta}{2}\right)\right) + \sec\left(\frac{\theta}{2}\right),$$

where $R = \tan\left(\frac{\theta}{2}\right)$. For $m = 2$, by using cosine law for $\frac{\pi}{2}$, and solving α for $3(t)^\alpha = 1$, we have the first level approximation as follows

$$\delta(G) \approx \alpha = \frac{\ln\left(\frac{1}{3}\right)}{\ln(t)} = \frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2+\left(\tan^2\left(\frac{\theta}{2}\right)\right)}\right)}.$$

Plotting $N = 1$ theoretical, conjecture, and numerical results with bounds given by the main theorem

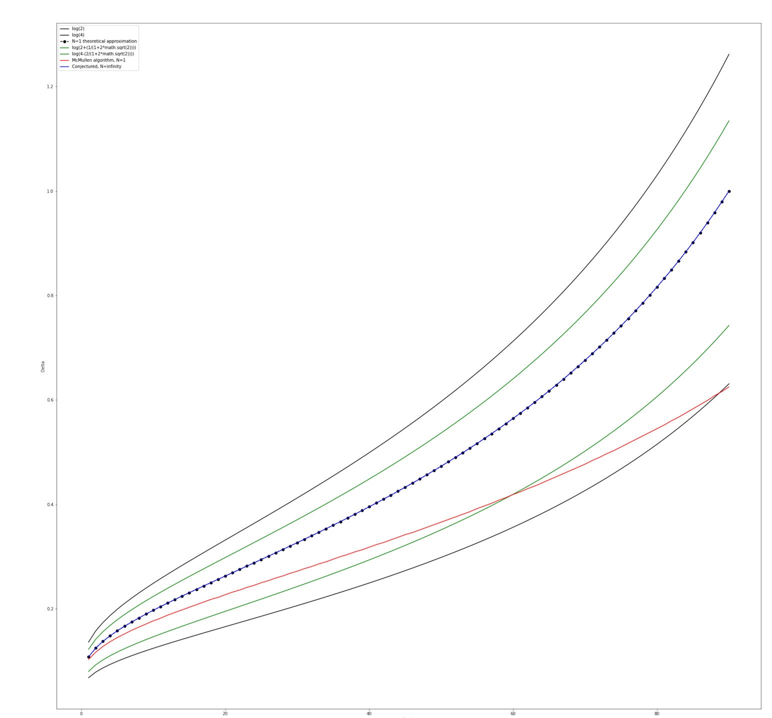

Figure: Results from our main theorems (colored in green and black), conjecture (blue), theoretical $N = 1$ approximation (dotted black), and numerical $N = 1$ approximation implementation of McMullen algorithm (red).

Surprisingly, the findings of the $N = 1$ approximation precisely match our conjecture for $N \to \infty$. Therefore, if our conjecture is true, then this implies that for a well-distributed Schottky group of rank 2, $\delta(\Gamma) = \alpha_1(\Gamma)$. It may also imply that for all N, we have

$$\delta(\Gamma) = \alpha_N(\Gamma) = \alpha_1(\Gamma) = \frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2+\left(\tan^2\left(\frac{\theta}{2}\right)\right)}\right)} = \frac{\ln(3)}{\ln(\cosh(r))}$$

This alternate strategy might also shed some light on the meaning of $cosh(r)$ in our conjecture.

Importance

Our main theorem gives sharp bounds on $\delta(\Gamma)$. Following the proof of the main theorem, for the first time, an exact form expression of Hausdorff dimension for two-generator Schottky group was conjectured and used to generate results against the best approximation derived using McMullen's algorithm.

Future Research

A proof of the conjecture, a generalization to $m > 2$, and enhancements to our implementations are on the horizon for our future work. Due to there are heavily matrix operations involved, an implementation on parallel computing in a contemporary computer might potentially enhance McMullen's results.

References

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