Idea of using **isometric circle**.

Source: A.F. Beardon - The Geometry of Discrete Groups (1995, Springer-Verlag)

§ 7.21 The Perpendicular Bisector of a Segment

$\Rightarrow$ The perpendicular bisector of $[0, T_0]$ is the isometric circle of T^{-1}.

Let $T = \begin{pmatrix} \alpha & \bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}$, $\alpha, \gamma \in \mathbb{C}$.

$|\alpha|^2 - |\gamma|^2 = 1$.

$\alpha + \bar{\alpha} \geq 2$. (i.e. T is hyperbolic or parabolic.)

Then the isometric circle of T^{-1} is $|\gamma z + \bar{\alpha}| = 1$, denoted by $I_{T^{-1}}(0)$

$\Rightarrow |z + \frac{\bar{\alpha}}{\gamma}| = \frac{1}{|\gamma|}$.

$\Rightarrow$ Center: $\frac{-\bar{\alpha}}{\gamma}$, radius: $\frac{1}{|\gamma|}$.

Since $|T\circ| < 1$, if $|\alpha| \neq 0$, then

we have $\left|\frac{\bar{\gamma}}{\alpha}\right| < 1$

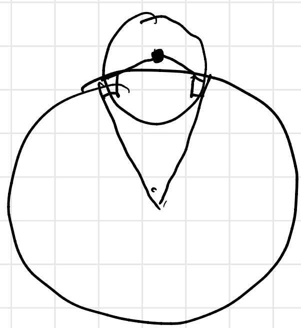
$\Rightarrow |\gamma| < |\alpha|$.

Then the center of $I_T^{-1}(\partial)$

is $|z_c| = \left|\frac{-\bar{\alpha}}{\gamma}\right| = \left|\frac{\alpha}{\gamma}\right| > 1$, if $|\gamma| \neq 0$.

$R = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$, i.e., $Rz = e^{i\theta}z$, $\forall z \in \mathbb{B}$.

Or, geometrically, since ∂D_i intercepts $\partial\mathbb{B}^2$ orthogonally $\Rightarrow$ it's not possible to have Euclidean center of ∂D_i on $\partial\mathbb{B}^2$.

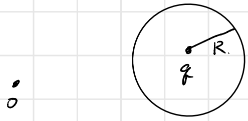


In the definition of Mumford, Series, and Wright (and the majority of others), McMullen's example is a Schottky group.

including William Thurston, Sullivan, Klein, Poincaré, Schottky, Ford, Dal' Bo, etc.

Each map in McMullen's example is composed by one reflection in a circle and a reflection in a line

$$I_k(z) = \frac{R^2}{\bar{z} - \bar{q}} + q$$



where $I_k(z)$ is an inversion in the circle centered at q with radius R.

Recall McMullen's example is based on a group which is isomorphic to a discrete subgroup of $PSL(2,\mathbb{R})$ (i.e Fuchsian) and it is generated by reflections.

Furthermore, it must be differentiable.

Recall: A reflection of a circle

$$\Rightarrow I(z) = \frac{R^2}{\bar{z} - \bar{q}} + q.$$

A reflection of a line $(ax+by=c)$

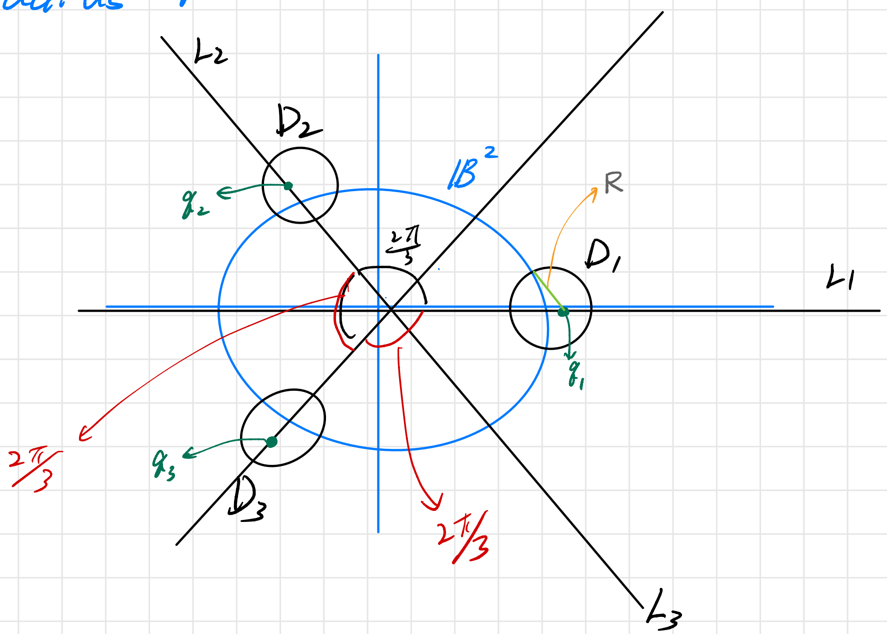
$$h(z) = \frac{2ic + (b-ai)\bar{z}}{b+ai}$$

Both $I(z)$ and $h(z)$ **are non-differentiable.**

However, their composition is differentiable.

The next goal is to reconstruct McMullen's result.

Assume each disk has the same radius $R = \tan\frac{\theta}{2}$



$L_1: \; y = 0$

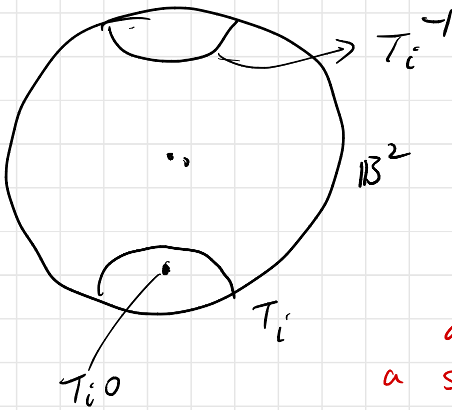
$L_2: \; \sqrt{3}x + y = 0$

$L_3: \; \sqrt{3}x - y = 0$

Assume the Fuchsian group (Schottky) is $\Gamma = \langle g_1, g_2, g_3 \rangle$, genus $= 3$.

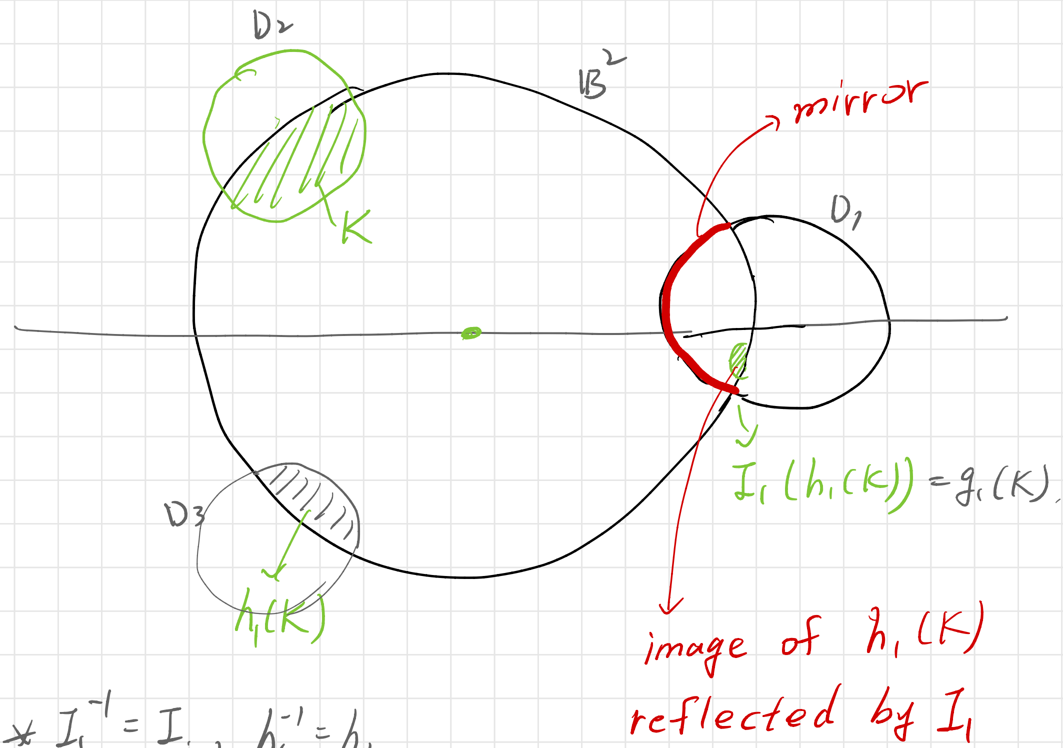
g_1, g_2, and g_3 are corresponding to D_1, D_2, and D_3.

Notice that g_1, g_2, g_3 are very different to T_1, T_2 in Borthwick's $m=2$ example or any examples in Bourgain, Dyatlov, and Borthwick's papers.



$D_0(T_i)$ and $D_0(T_i^{-1})$ are included in $\mathbb{B}^2$,

and for Γ to be a schottky group $D_0(T_i) \cap D_0(T_i^{-1}) = \phi$.



$* \; I_1^{-1} = I_1, \quad h_1^{-1} = h_1$

$$g_1 = I_1 \circ h_1$$

$$\begin{aligned} g_1^{-1} &= h_1^{-1} \circ I_1^{-1} \\ &= h_1^{-1} \circ I_1 \\ &= h_1 \circ I_1 \quad \Rightarrow \quad g_1 z = g_1^{-1} z \quad \forall z \in \mathbb{B}^2 \setminus D_1 \end{aligned}$$

$$\Rightarrow \boxed{D_0(g_1) = D_0(g_1^{-1})}$$

Goal: Find g_1

① reflect about L_1: $y = 0$

$\Rightarrow a = 0,\ b = 1,\ c = 0$

$$h_1(z) = \overline{z}$$

② reflect about ∂D_1: center at $z_1 = (1, 0)$
radius $= R$.

$$g_1(z) = (I_1 \circ h_1)(z)$$

$$= \frac{R^2}{z - \overline{z_1}} + z_1$$

$$= \frac{z_1 z - z_1 \overline{z_1} + R^2}{z - \overline{z_1}}$$

$$\Rightarrow \begin{pmatrix} z_1 & - z_1 \overline{z_1} + R^2 \\ 1 & - \overline{z_1} \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Notice that g_1 hasn't been normalized, so

$$g_1'(z) = \frac{ad - bc}{(cz + d)^2} \neq \frac{1}{(cz+d)^2}$$

To find g_1^{-1}:

$$(g_1 - g_1)(z - \overline{g_1}) = R^2$$

$$z - \overline{g_1} = \frac{R^2}{g_1 - g_1}$$

$$\Rightarrow z = \frac{R^2}{g_1 - g_1} + \overline{g_1}$$

$$\Rightarrow \boxed{g_1^{-1}(z) = \frac{R^2}{z - g_1} + \overline{g_1}}$$

To find g_2.

① reflect about L_2: $\sqrt{3}x + y = 0$

$$a = \sqrt{3},\quad b = 1,\quad C = 0$$

$$h_2(z) = \left[\frac{(1-\sqrt{3}i)}{(1+\sqrt{3}i)}\right]\bar{z} := H_2\bar{z}$$

② reflect about ∂D_2: center at $q_2 = e^{\frac{i2\pi}{3}}$ with radius R.

$$g_2(z) = \frac{R^2}{\overline{H_2 z} - \overline{q_2}} + q_2.$$

$\shortparallel$

$I_2(h_2(z))$

$$= \frac{q_2\overline{H_2}z - q_2\overline{q_2} + R^2}{\overline{H_2}z - \overline{q_2}}$$

To find g_2^{-1}:

$$(g_2 - q_2)(\overline{H_2}z - \overline{q_2}) = R^2$$

$$\overline{H_2}z - \overline{q_2} = \frac{R^2}{g_2 - q_2}$$

$$\overline{H_2}z = \frac{R^2}{g_2 - q_2} + \overline{q_2}$$

$$\Rightarrow z = \frac{1}{\overline{H_2}}\left(\frac{R^2}{g_2 - q_2} + \overline{q_2}\right)$$

$$g_2^{-1}(z) = \frac{1}{\overline{H_2}}\left(\frac{R^2}{z - q_2} + \overline{q_2}\right)$$

Similarly, to find $g_3(z)$:

$$h_3(z) = \left(\frac{-1-\sqrt{3}i}{-1+\sqrt{3}i}\right)\overline{z} := H_3\overline{z}.$$

$$g_3(z) = I_3(h_3(z))$$

$$= \frac{R^2}{\overline{H_3}z - \overline{q_3}} + q_3$$

$$= \frac{q_3\overline{H_3}z - q_3\overline{q_3} + R^2}{\overline{H_3}z - \overline{q_3}}$$

Then, $g_3^{-1}(z) = \dfrac{1}{\overline{H_3}}\left(\dfrac{R^2}{z - q_3} + \overline{q_3}\right)$

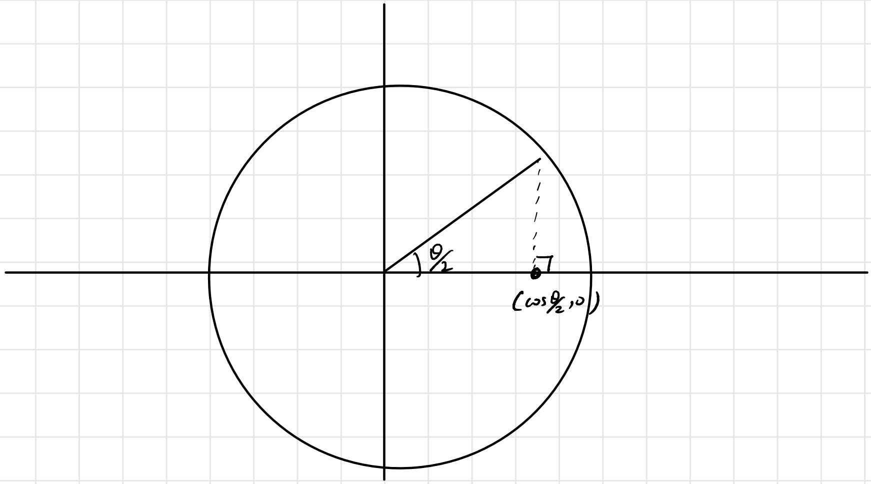
$$T_{ij} = \begin{bmatrix} 0 & t & t \\ t & 0 & t \\ t & t & 0 \end{bmatrix}$$

$\max \lambda = 2t.$

When $\delta = 1 \Rightarrow 2t = 1$

$\Rightarrow t = \frac{1}{2}$ (theoretically)

Example of genus 2. $(I = \langle T_1, T_2 \rangle)$.



Assume $T_1 = \begin{pmatrix} \alpha & \bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}$, $T_2 = R T_1 R^{-1}$

$R = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix}$

$Rz = e^{i\theta} z$.

Know: $T_1 0 = \cos\frac{\theta}{2}$.

$$\Rightarrow \overline{\left(\frac{\gamma}{\alpha}\right)} = \cos\frac{\theta}{2}$$

$$\Rightarrow |\gamma|^2 = |\alpha|^2 \cos^2\frac{\theta}{2}.$$

Recall: $|\alpha|^2 - |\gamma|^2 = 1$

$$\Rightarrow |\alpha|^2\left(1 - \cos^2\frac{\theta}{2}\right) = |\alpha|^2 \sin^2\frac{\theta}{2} = 1$$

$$\Rightarrow \left(\sin^2\frac{\theta}{2}\right)^{-1} = |\alpha|^2$$

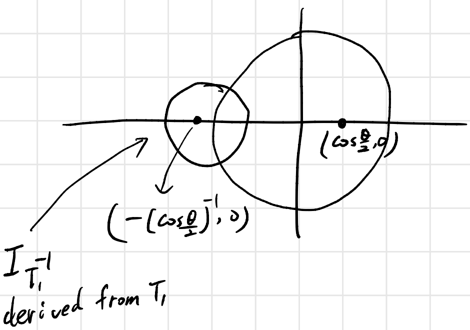
Assume $\alpha = \left(\sin\frac{\theta}{2}\right)^{-1}$, $\gamma = \cot\frac{\theta}{2}$.

Then $T_1 = \begin{pmatrix} \frac{1}{\sin\theta/2} & \cot\frac{\theta}{2} \\ \cot\frac{\theta}{2} & \frac{1}{\sin\frac{\theta}{2}} \end{pmatrix}$

$\Rightarrow$ Isometric circle of T_1^{-1} is

$$\left|\left(\cot\tfrac{\theta}{2}\right)z + \left(\sin\tfrac{\theta}{2}\right)^{-1}\right| = 1$$

$\Rightarrow$ $z_c = -\left(\cos\frac{\theta}{2}\right)^{-1}$, radius $= \frac{1}{|\cot\frac{\theta}{2}|}$



$$T_1^{-1} = \begin{pmatrix} \frac{1}{\sin\theta/2} & -\cot\theta/2 \\ -\cot\theta/2 & \frac{1}{\sin\theta/2} \end{pmatrix}$$

Similarly, by using T_1^{-1}, we can derive

the isometric circle of T_1:

$$\left|\left(\cot\frac{\theta}{2}\right) z + \frac{1}{\sin\frac{\theta}{2}}\right| = 1$$

Goal: To compute the first level using McMullen's

algorithm, firstly, let

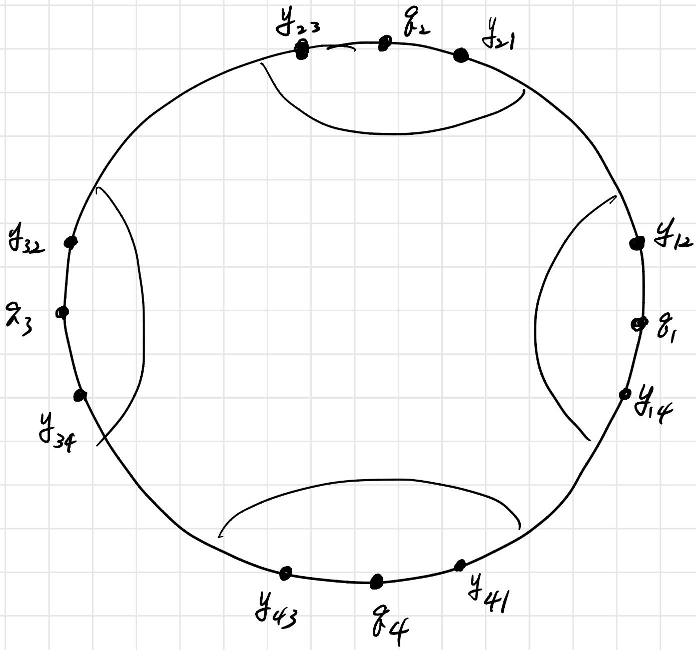
$q_1 = (1,0)$,

$q_2 = (0,1)$,

$q_3 = (-1,0)$, and

$q_4 = (0,-1)$.

$$\Rightarrow \begin{array}{llll} y_{12} = T_1(q_2) & y_{21} = T_2(q_1) & y_{32} = T_3(q_2) & y_{43} = T_4(q_3) \\ y_{14} = T_1(q_4) & y_{23} = T_2(q_3) & y_{34} = T_3(q_4) & y_{41} = T_4(q_1) \\ y_{11} = T_1(q_1) & y_{22} = T_2(q_2) & y_{33} = T_3(q_3) & y_{44} = T_4(q_4). \end{array}$$



$$T_{ij} = \begin{bmatrix} t & t & 0 & t \\ t & t & t & 0 \\ 0 & t & t & t \\ t & 0 & t & t \end{bmatrix}$$

$\max \lambda = 3t.$

when $\delta = 1 \Rightarrow t = \frac{1}{3}$ (theorctically).