Hausdorff Dimensions and Limit Sets of Schottky Groups

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Conformal Equivalence

The most important example is the conformal equivalence between the unit disk $\mathbb{B}^2$ and the upper half-plane $\mathbb{H}^2$:

Definition 1.1.

The set of all complex numbers with modulus less than one is called **the unit disk** and is denoted by $\mathbb{B}^2$; that is,

$$\mathbb{B}^2 = \{z \in \mathbb{C} : |z| < 1\},$$

and define the unit circle as its boundary, denoted by $\partial\mathbb{B}^2$.

Definition 2.2.

The set of all complex numbers with positive imaginary part is called **the upper half-plane** and is denoted by $\mathbb{H}^2$; that is,

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \Im(z) > 0\}.$$

One of the conformal mappings between the unbounded set $\mathbb{H}^2$ and the unit disk $\mathbb{B}^2$ is denoted by $\Psi : \mathbb{H}^2 \to \mathbb{B}^2$:

$$\Psi(z) = i\frac{z-i}{z+i}$$

which maps 0 to $-i$, 1 to 1, and -1 to $-i$. The point i on the circle corresponds to the point at infinity of the upper half-plane.

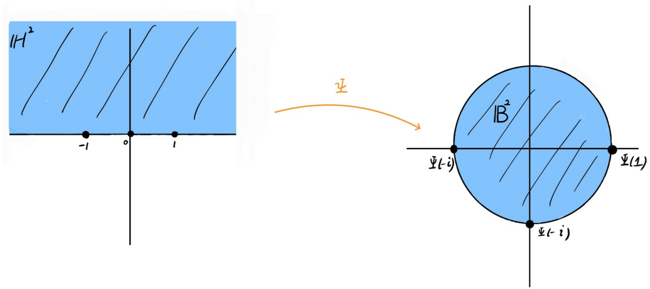

Figure: The conformal mapping $\Psi : \mathbb{H}^2 \to \mathbb{B}^2$.

Isometry Group on $(\mathbb{R}^2, d_{\mathsf{E}})$

Recall that in Euclidean geometry on $\mathbb{R}^2$, by collecting all possible rigid motions: identity map, translation, rotation about a pint, reflection in a line, and glide reflection, we have the Euclidean group $\mathsf{E}(2)$.Furthermore, the length of the line segment is an invariant under **these types of mappings** and the **Euclidean metric** that can be checked by using **the line integral** defined on the 2-dimensional Euclidean space $(\mathbb{R}^2, d_{\mathsf{E}})$. Recall a notion called isometry for each element in $\mathsf{E}(2)$:

Definition 2.9.

Let (X, d) be a metric space. If f is on X that preserves distance, i.e. for a pair of points $z_1, z_2 \in X$:

$$d(z_1, z_2) = d(f(z_1), f(z_2)),$$

then f is called **an isometry of the metric space** (X, d).

Hence, $\mathsf{E}(2)$ is called the isometry group of $(\mathbb{R}^2, d_{\mathsf{E}})$.

Isometry Groups

The idea is that we want to define a notion of length, i.e. a metric on $\mathbb{H}^2$, that is invariant under $PSL(2,\mathbb{R})$.

Isometry Groups

In $\mathbb{H}^2$ and $\mathbb{B}^2$, we would like to have a similar group to $\mathsf{E}(2)$. So we need to find **a set of mappings** and **a metric** to measure length of a line segment while keeping the length intact.
First of all, for $\mathbb{H}^2$ and $\mathbb{B}^2$ we have to find sets of functions that only map $\mathbb{H}^2$ to $\mathbb{H}^2$ and $\mathbb{B}^2$ to $\mathbb{B}^2$.
Hence, for $\mathbb{H}^2$, we use the projective special linear group:

$$\mathsf{PSL}(2,\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ b & d \end{pmatrix} : a,b,c,d \in \mathbb{R}, M \sim -M, \det(M) = 1 \right\}.$$

For $\mathbb{B}^2$, we use the projective special unitary group:

$$\mathsf{PSU}(1,1) = \left\{ M = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} : \alpha,\gamma \in \mathbb{C}, M \sim -M, \det(M) = 1 \right\}.$$

Furthermore, by using the conformal map Ψ, for each g in $\mathsf{PSL}(2,\mathbb{R})$, we can find a T in $\mathsf{PSU}(1,1)$ such that $T = \Psi g \Psi^{-1}$. Conversely, for each T in $\mathsf{PSU}(1,1)$, we can find a g in $\mathsf{PSL}(2,\mathbb{R})$ such that $g = \Psi^{-1} T \Psi$. Based on this conjugation map we can show that $\mathsf{PSU}(1,1) \cong \mathsf{PSL}(2,\mathbb{R})$.

Isometry Groups: The corresponding fractional linear maps

Definition 2.4.

Let $z, a, b, c,$ and d be complex numbers. Then, a mapping with of the following fractional form

$$z \longmapsto \frac{az+b}{cz+d},$$

is called *fractional linear transformations* or *linear fractional transformations*.

Also recall that in Theory of Functions of a Complex Variable (Math 730), we have seen that by using Schwarz lemma[6], for each $g \in \mathrm{PSL}(2,\mathbb{R})$, we assign a linear fractional map denoted as follows:

$$gz = \frac{az+b}{cz+d} \in \mathbb{H}^2, z \in \mathbb{H}^2.$$

Similarly, for each $T \in \mathrm{PSU}(1,1)$, we assign a linear fractional map denoted as follows:

$$Tz = \frac{\alpha z+\overline{\gamma}}{\gamma z+\overline{\alpha}} \in \mathbb{B}^2, z \in \mathbb{B}^2.$$

Isometry Groups: Automorphisms

Definition

Let $\Omega \subseteq \mathbb{C}$ be a region. A conformal mapping from Ω to Ω is *a conformal automorphism of* Ω. The group of conformal automorphisms of Ω is denoted by Aut(Ω).

Hence, we have

$$\mathrm{Aut}(\mathbb{H}^2) = \mathrm{PSL}(2, \mathbb{R})$$

and

$$\mathrm{Aut}(\mathbb{B}^2) = \mathrm{PSU}(1, 1).$$

An observation leads to the isometric invariant

For every $T \in \mathrm{PSL}(2,\mathbb{R})$, $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have a corresponding fractional linear map

$$T(z) = \frac{az+b}{cz+d},$$

and for this fractional linear map, we have:

$$\frac{1}{\Im\left(T(z)\right)}\left|\frac{dT(z)}{dz}\right| = \frac{1}{\Im(z)},$$

that is

$$\frac{|dT(z)|}{\Im\left(T(z)\right)} = \frac{|dz|}{\Im(z)}.$$

Since our goal is to define a way to estimate the arc length of a line segment such that it is invariant under $\mathrm{PSL}(2,\mathbb{R})$, the above identity implies that if **we define length with the following line element**:

$$ds = \frac{|dz|}{\Im(z)},$$

then we have **the following invariant arc length for all possible paths**:

$$\text{Arc length}_{\mathbb{H}^2}(f) = \int_f \frac{1}{\Im(z)}|dz|,$$

where $f : [t_i, t_f] \to \mathbb{H}^2$ is a piecewise-smooth path.

Geodesics

For any two points z_1, z_2, let $K(z_1, z_2)$ be the set of all piecewise-smooth paths connecting z_1 and z_2. Then **consider the function**:

$$d_{\mathbb{H}^2} : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}$$

defined by:

$$d_{\mathbb{H}^2} = \inf \{\text{Arc length}_{\mathbb{H}^2}(f) : f \in K(z_1, z_2)\}.$$

Definition 2.7.

Let $z_1, z_2 \in \mathbb{H}^2$ be distinct. The shortest smooth path on $\mathbb{H}^2$ connects z_1 and z_2 is called *the geodesic* denoted by $[z_1, z_2]$.

The method to derive the geodesic $[z_1, z_2]$ is called the calculus of variations[4].
We immediately derive the invariance of $d_{\mathbb{H}^2}$ for geodesic $[z_1, z_2]$, namely

$$d_{\mathbb{H}^2}(Tz_1, Tz_2) = d_{\mathbb{H}^2}(z_1, z_2)$$

and this proves that each such $T \in \text{PSL}(2, \mathbb{R})$ is an isometry of $(\mathbb{H}^2, d_{\mathbb{H}^2})$. By using the map Ψ, we can also find the geodesics on $\mathbb{B}^2$.

Isometry Groups: A comparison table

Point sets	$\mathbb{R}^2$	$\mathbb{H}^2$	$\mathbb{B}^2$
Line elements ds	$\lvert dz\rvert$	$\frac{\lvert dz\rvert}{\Im z}$	$\frac{2\lvert dz\rvert}{1-\lvert z\rvert^2}$
Isometry subgroups	$\mathsf{E}(2)$	$\mathsf{PSL}(2,\mathbb{R})$	$\mathsf{PSU}(1,1)$
Geodesics	Lines	Vertical lines, Semicircles	Partial circles, Diameters

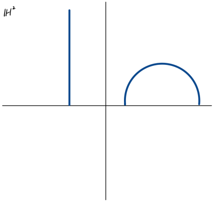

Figure: Geodesics in $\mathbb{H}^2$ are vertical lines or semicircles that are orthogonal to the boundary $\partial\mathbb{B}^2$.

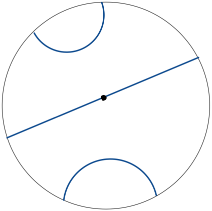

Figure: Geodesics in $\mathbb{B}^2$ are diameters (radial lines) or partial circles that are orthogonal to the boundary $\partial\mathbb{B}^2$.

Examples: `https://graemewilkin.github.io/Geometry/Construct_Hyperbolic_Geodesic.html`

Definition 2.9.

A perpendicular bisector of a geodesic $[v, w]$ in $\mathbb{B}^2$ is a geodesic

$$\left\{z \in \mathbb{B}^2 : \rho_{\mathbb{B}^2}(z, v) = \rho_{\mathbb{B}^2}(z, w)\right\}.$$

Let $g \in \mathsf{PSU}(1,1)$ and $z_0 \in \mathbb{B}^2$. Define the notation:

(i) The perpendicular bisector of the geodesic between z_0 and gz_0, (z_0, gz_0), is denoted be $M_{z_0}(g)$,

(ii) $M_{z_0}(g^{-1})$ is the perpendicular bisector of the geodesic between z_0 and $g^{-1}0$, $(z_0, g^{-1}z_0)$,

(iii) the closed half-space in $\mathbb{B}^2$ bounded by $M_{z_0}(g)$ is denoted by $D_{z_0}(g)$, and

(iv) $D_{z_0}(g^{-1})$ is the closed half-space in $\mathbb{H}^2$ bounded by $M_{z_0}(g^{-1})$.

Remarks:

(a) Both $M_{z_0}(g)$ and $M_{z_0}(g^{-1})$ are also geodesics in $\mathbb{B}^2$.

(b) The reason to call $D_{z_0}(g)$ and $D_{z_0}(g^{-1})$ as half-spaces is that both of them are conformal images of the space $\mathbb{B}^2$.

Lemma 2.2.

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If $g \in G \setminus \{\mathrm{id}\}$, where G is a subgroup of $\mathrm{PSU}(1,1)$, and $z_0 \in \mathbb{B}^2$ which is not fixed by g, then $g(M_{z_0}(g^{-1})) = M_{z_0}(g)$, and $g(\mathrm{int}(D_{z_0}(g^{-1}))) = \mathbb{B}^2 \setminus D_{z_0}(g)$.

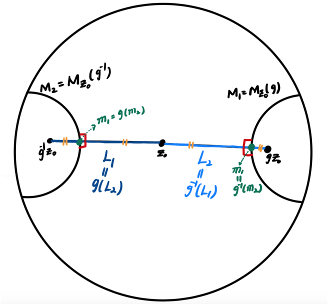

Figure: By applying the property of isometry of $\mathrm{PSU}(1,1)$, we have $g(\rho_{\mathbb{B}^2}(g^{-1}z_0, m_2) = \rho_{\mathbb{B}^2}(z_0, gm_2)$ and $g\left(\rho_{\mathbb{B}^2}(m_2, z_0)\right) = \rho_{\mathbb{B}^2}(gm_2, gz_0)$. Since $g(L_2) = L_1$, this implies $gm_2 = m_1$

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Let a hyperbolic isometry in $T = I_1 \circ I_2 \in \mathrm{PSU}(1,1)$ where I_1 is denoted as $\partial D(T^{-1}) = M_0(T^{-1})$ and I_2 is denoted as $\partial D(T) = M_0(T)$ in the following figure.

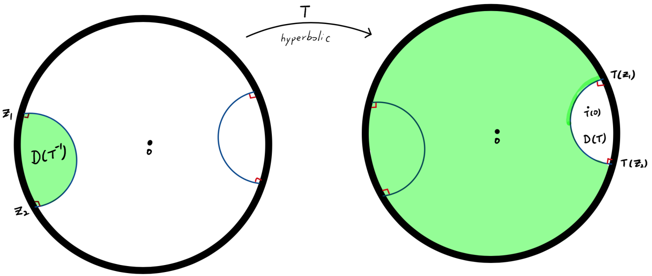

Figure: Let T be a hyperbolic isometry in $\mathrm{PSU}(1,1)$. Then $T(D(T^{-1})) = \mathbb{B}^2 \setminus D(T)$ and $D(T) \cap D(T^{-}1) = \emptyset$.

Lemma 2.2.

If $g \in G \setminus \{\mathrm{id}\}$, where G is a subgroup of $\mathrm{PSU}(1,1)$, and $z_0 \in \mathbb{B}^2$ which is not fixed by g, then $g(M_{z_0}(g^{-1})) = M_{z_0}(g)$, and $g(\mathrm{int}(D_{z_0}(g^{-1}))) = \mathbb{B}^2 \setminus D_{z_0}(g)$.

Let a parabolic isometry in $T = I_1 \circ I_2 \in \mathrm{PSU}(1,1)$ where I_1 is denoted as $\partial D(T^{-1}) = M_0(T^{-1})$ and I_2 is denoted as $\partial D(T) = M_0(T)$ in the following figure.

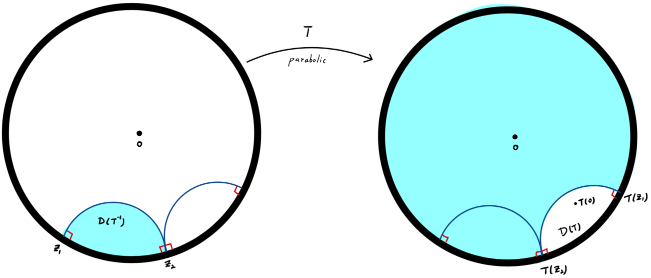

Figure: Let T be a parabolic isometry in $\mathrm{PSU}(1,1)$. Then $T(D(T^{-1})) = \mathbb{B}^2 \setminus D(T)$ and $D(T) \cap D(T^{-}1) = z_2$.

Lemma 2.2.

If $g \in G \setminus \{\mathrm{id}\}$, where G is a subgroup of $\mathrm{PSU}(1,1)$, and $z_0 \in \mathbb{B}^2$ which is not fixed by g, then $g(M_{z_0}(g^{-1})) = M_{z_0}(g)$, and $g(\mathrm{int}(D_{z_0}(g^{-1}))) = \mathbb{B}^2 \setminus D_{z_0}(g)$.

Let a elliptic isometry in $T = I_1 \circ I_2 \in \mathrm{PSU}(1,1)$ where I_1 is denoted as $\partial D(T^{-1}) = M_0(T^{-1})$ and I_2 is denoted as $\partial D(T) = M_0(T)$ in the following figure.

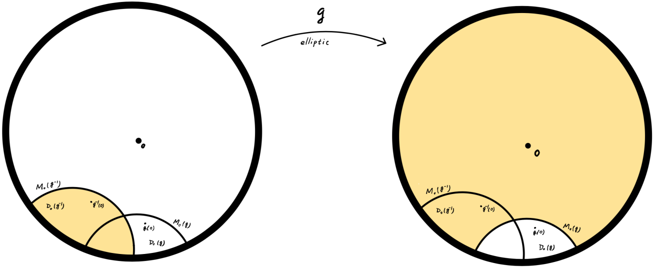

Figure: Let T be a parabolic isometry in $\mathrm{PSU}(1,1)$. Then $T(D(T^{-1})) = \mathbb{B}^2 \setminus D(T)$ and the cardinality of $D(T) \cap D(T^-1)$ is greater than one.

Definition 2.10.

The isometric circle I_T *of* $T = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} \in$ *PSU*$(1,1)$ is defined by

$$I_g := \left\{ z \in \mathbb{B}^2 : |\gamma z + \overline{\alpha}| = 1 \right\};$$

that is with radius $\frac{1}{|\gamma|}$ and centered at $-\frac{\overline{\alpha}}{\gamma}$.

Why isometric circles and why $\mathbb{B}^2$ with PSU$(1,1)$? Because perpendicular bisector is formidable to find, but in $\mathbb{B}^2$, by choosing $z = 0$, we have

$$I_T = M_0(T^{-1}), I_T^{-1} = M_0(T),$$

and isometric circles are easy to find whenever T is given and $\gamma \neq 0$ (γ must be nonzero to make T be hyperbolic).

Remark: The radius of I_T is $\frac{1}{|\gamma|}$ and it is measuring by taking the modulus of complex numbers, hence I_T is an Euclidean circle, not a hyperbolic circle.
Furthermore, by considering $T(z) = \frac{\alpha z + \overline{\gamma}}{\gamma z + \overline{\alpha}}$, the meaning of this definition $|\gamma z + \overline{\alpha}| = |T'| = 1$ is that **isometric circles locally fixes Euclidean length**.
Equivalently, I_T can also be defined by isometry invariant that $|\gamma z + \overline{\alpha}| = 1$ if and only if $|g'(T)| = 1$.

What is a finitely generated Schottky group?

Consider a set of functions $\{T_1, \ldots, T_m, m \geq 2, m \in \mathbb{Z}\}$, where each $T_i : \mathbb{B}^2 \to \mathbb{B}^2$, and consider that there are $2m$ disjoint closed disks associated to the set:

$$K(T_1), K(T_1^{-1}), K(T_2), K(T_2^{-1}), \ldots, K(T_m), K(T_m^{-1})$$

which boundaries are all orthogonal to the boundary of the unit disk $\partial\mathbb{B}^2$. Furthermore, we are taking intersections to make partial closed disk $D(T_i)$ and $D(T_i^{-1})$; that is,

$$D(T_i) = K(T_i) \cap \mathbb{B}^2,$$

and

$$D(T_i^{-1}) = K(T_i^{-1}) \cap \mathbb{B}^2.$$

If we have each T_i taking $\mathbb{B}^2 \setminus D(T_i^{-1})$ onto int$(D(T_i))$, and T_i^{-1} taking $\mathbb{B}^2 \setminus D(T_i)$ onto int$(D(T_i^{-1}))$, then the group generated by these conformal mappings is a finitely generated Schottky group denoted by $G = \langle T_1, T_2, \ldots, T_m \rangle$.

disjoint $\Leftrightarrow$ non-elliptic

Schottky Groups

Let $\{T_i : 1 \leq i \leq m, m \geq 2\}$ be a set of **non-elliptic** isometries of $\mathrm{PSU}(1,1)$ and each T_i and T_j, $i \neq j$, do not have a common fixed point. Take $z_0 \in \mathbb{B}^2$ be a point that is not fixed by each T_i. Then, if for each T_i we denote the half-spaces $D_0(T_i^{\pm}) := D(T_i^{\pm})$, for $i = 1, \ldots, m$ satisfy $\left(\overline{D(T_i) \cup D(T_i^{-1})}\right) \cap \left(\overline{D(T_j) \cup D(T_j^{-1})}\right) = \emptyset, \forall i \neq j \in \{1, \ldots, m\}$, then $\langle T_i, 1 \leq i \leq m \rangle$ is a **Schottky group**.

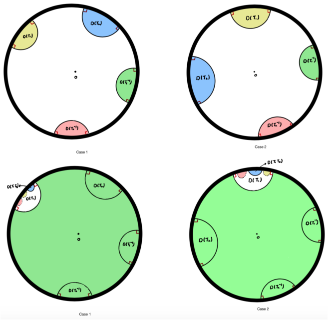

Figure: Let $G = \langle T_1, T_2 \rangle$ be a Schottky group with T_1 and T_2 are hyperbolic. The figure demonstrates the half-space $D(T_1T_2) = D_0(T_1T_2)$ in two possible cases.

Definition 3.8.

[1] Let G be a discrete subgroup of the $\mathrm{PSU}(1,1)$. The *limit set* of G is the set of all accumulation points in the intersection of all orbits $G(x)$ for all $x \in \mathbb{B}^2 \cup \partial\mathbb{B}^2$, and the *limit set* of G is denoted by $L(G)$.

Proposition 3.10.

If G is a Schottky group, then for each point z of $\mathbb{B}^2$, we have $L(G) = \overline{G(z)} \cap \partial\mathbb{B}^2$.

Plotting The Orbit $\Gamma(0)$ of Well-distributed Schokky Groups:

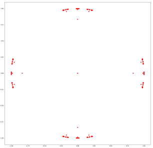

Figure: $m = 2$, $\frac{\theta}{2} \approx 33.3984734472776950^\circ$, Level 14 ($N = 14$).

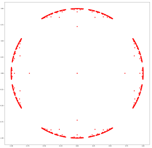

Figure: $m = 2$, $\frac{\theta}{2} \approx 43.6027944827781440^\circ$, Level 14 ($N = 14$).

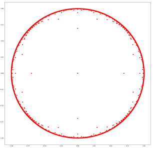

Figure: $m = 2$, $\frac{\theta}{2} \approx 45.9746105715017^\circ$, Level 14 ($N = 14$).

Our set-up

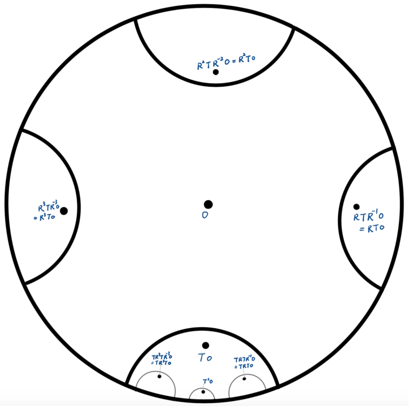

Figure: Let $G' = \left\langle T, RTR^{-1} \right\rangle$ be a well-distributed Schottky group. The figure demonstrates the first level of all four half-spaces: $D(T0)$, $D(RT0) = D(RTR^{-1}0)$, $D(R^2T0) = D(R^2TR^{-2}0)$, $D(R^3T0) = D(R^3TR^{-3}0)$; and the second level half-space within $D(T0)$: $D(T^20)$, $D(TRT0) = D(TRTR^{-1}0)$, and $D(TR^3T0) = D(TR^3TR^{-3}0)$.

Our set-up

In essence, since Γ is a well-distributed Schottky group, assume $\Gamma = \langle T_1, ..., T_m \rangle$. Denote $T_1^{-1} = T_{m+1}, ..., T_m^{-1} = T_{2m}$. Assume for $\text{len}(T_I) = 1$, assume the half-space $D_0(T_i)$ is surrounding by $D_0(T_j)$ and $D_0(T_k)$, and the half-space $D_0(T_a)$ is surrounding by $D_0(T_b)$ and $D_0(T_c)$, where $a \neq i$. Then since Γ is well-distributed, we must have not only

$$\angle[0,\ T_i0,\ T_j0] = \angle[0,\ T_i0,\ T_k0] = \angle[0,\ T_a0,\ T_b0] = \angle[0,\ T_a0,\ T_c0],$$

but also

$$\rho_{\mathbb{B}^2}(0, T_i0) = \rho_{\mathbb{B}^2}(0, T_i^{-1}0) = \rho_{\mathbb{B}^2}(0, T_a0) = \rho_{\mathbb{B}^2}(0, T_a^{-1}0),$$

for all $i \neq a$.
Denote the angle $\angle[0,\ T_i0,\ T_j0] := \phi$.

Then, we can construct the rotation operator R. We would like it to be the function $R(z) = e^{i\phi}z$ which is a counter-clockwise rotating operation, where $\phi := \frac{\pi}{m} \geq \theta$.

Our set-up

Let $T \in \mathrm{PSU}(1,1)$, T is hyperbolic, and $T = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}$, where $\alpha \in \mathbb{C}$, $\gamma \in \mathbb{C}$, $\alpha + \overline{\alpha} > 2$, and $|\alpha|^2 - |\gamma|^2 = 1$. The perpendicular bisector of $[0, T0]$ is the isometric circle of T^{-1}.

Without loss the generality, assume $\alpha = \left(\sin\left(\frac{\theta}{2}\right)\right)^{-1}$, and $\gamma = \cot\left(\frac{\theta}{2}\right)$ to satisfy the identity. Recall **the definition of isometric circles**, with the above information, once we know the center and radius of an isometric circle, we can derive γ and α immediately and derive T as follows

$$T = \begin{pmatrix} \frac{1}{\sin\left(\frac{\theta}{2}\right)} & \cot\left(\frac{\theta}{2}\right) \\ \cot\left(\frac{\theta}{2}\right) & \frac{1}{\sin\left(\frac{\theta}{2}\right)} \end{pmatrix} \in \mathrm{PSU}(1,1).$$

We also define the rotation matrix as follows:

$$R = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \in \mathrm{PSU}(1,1).$$

Poincaré series

The Poincaré series is defined as follows

$$\mathcal{P}(G,t) := \sum_{g_i \in G} \exp\left\{-t\rho_{\mathbb{B}^2}(z, g_i(z))\right\}$$

where $\delta(G) := \inf\{t \in \mathbb{R} : \mathcal{P}(G,t) < \infty\} = \sup\{t \in \mathbb{R} : \mathcal{P}(G,t) = \infty\}$ is called **the critical exponent of the Poincaré sereis**, the exponent of convergence of the Poincaré sereis, or the Poincaré exponent, and $z, g_i(z) \in \mathbb{B}^2$. The Poincareé series can be rewritten as follows

$$\mathcal{P}(\Gamma, t) \asymp \sum_{n=1}^{\infty} P_n,$$

where

$$P_n = \sum_{T \in \mathcal{W}_n} \frac{1}{Y_{I_n}^t}, Y_{I_n} = \cosh^n(r) \prod_{j=1}^{n-1} \left(1 - \tanh\left(X_{I_{j+1}}\right) \tanh(r) \cos\left(\frac{i_{j+1}\pi}{m} - \theta_{I_j}\right)\right).$$

Theorem 3.18.

[7]Let G be a finitely generated Schottky group. The Patterson-Sullivan measure μ is a constant multiple of the Hausdorff measure $H^{\delta(G)}|_{L(G)}$. Furthermore, $\dim_H L(G) = \delta(G)$.

Dennis Parnell Sullivan was awarded the 2022 Abel Prize for Mathematics. Photos by John Griffin.

Figure: Proved by Dennis Sullivan, published in 1979 on "The density at infinity of a discrete group of hyperbolic motions." Awards: Abel Prize (2022), Wolf Prize in Mathematics (2010), National Medal of Science for Mathematics and Computer Science (2006). Notable students: Curtis T. McMullen, Hal Abelson, Elmar Winkelnkemper.

Theorem 4.8.

Suppose Γ is a well-distributed Schottky group of order 2. Then,

$$\frac{\ln\left(2+\frac{1}{1+2\sqrt{2}}\right)}{\ln\left(\cosh\left(r\right)\right)} \leq \delta\left(\Gamma\right) \leq \min\left\{\frac{\ln\left(4-\frac{2}{1+2\sqrt{2}}\right)}{\ln\left(\cosh\left(r\right)\right)}, 1\right\}.$$

Then, **we conjectured**

$$\delta(\Gamma) = \frac{\ln\left(3\right)}{\ln\left(\cosh\left(r\right)\right)},$$

where $r = \ln\left(\frac{1+|T0|}{1-|T0|}\right) = \ln\left(\frac{1+\cos\left(\frac{\theta}{2}\right)}{1-\cos\left(\frac{\theta}{2}\right)}\right)$, and $T0 = \cos\left(\frac{\theta}{2}\right)$.

Furthermore, the formula

$$\left|\frac{\partial g_1}{\partial z}\right| = \frac{R^2}{|x_2 - q_1|^2}$$

can be used to check whether our conjecture can produce the same approximation based on McMullen's algorithm.

Conjecture

Furthermore, in general, by using cosine law,

$$|x_2 - q_1|^2 = 2 + \left(\tan^2\left(\frac{\theta}{2}\right)\right) + \sec\left(\frac{\theta}{2}\right),$$

where $R = \tan\left(\frac{\theta}{2}\right)$. For $m = 2$, by using cosine law for $\frac{\pi}{2}$, and solving α for $3(t)^{\alpha} = 1$, we have the first level approximation as follows

$$\delta(G) \approx \alpha = \frac{\ln\left(\frac{1}{3}\right)}{\ln\left(t\right)} = \frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2+\left(\tan^2\left(\frac{\theta}{2}\right)\right)}\right)}.$$

Surprisingly, the findings of the $N = 1$ approximation precisely match our conjecture for $N \to \infty$. This alternate strategy might also shed some light on the meaning of $cosh(r)$ in our conjecture.

Plotting $N = 1$ theoretical, conjecture, and numerical results with bounds given by the main theorem

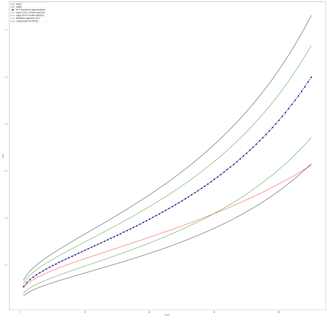

Figure: Results from our main theorems (colored in green and black), conjecture (blue), theoretical $N = 1$ approximation (dotted black), and numerical $N = 1$ approximation implementation of McMullen algorithm (red).

Conclusion

Importance

Our main theorem gives sharp bounds on $\delta(\Gamma)$. Following the proof of the main theorem, for the first time, an exact form of Hausdorff dimension for two-generator well-distributed Schottky groups was conjectured and used to generate results against the best approximation derived using McMullen's algorithm.

Future Research

A proof of the conjecture, a generalization to $m > 2$, and enhancements to our implementations are on the horizon for our future work. Due to there are heavily matrix operations involved, an implementation on parallel computing in a contemporary computer might potentially enhance McMullen's results.

References

Our source code can be downloaded from GitHub:

`https://github.com/williamchuang/well-distributed-schottky-groups/tree/main/code`

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Thank you!!

Q & A

Backup slide 1

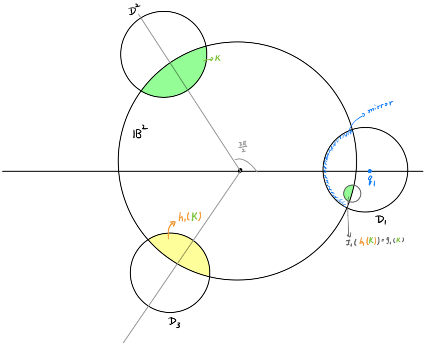

Figure: Please keep in mind that $I_i(z)$ and $h_i(z)$ are non-analytic if we only have one of them alone, but any composition of even numbers of them is analytic.

Backup slide 2: Abstract

Let a finitely generated Schottky group G be given, and $\mathbb{B}^2$ be a Poincaré disk model of two-dimensional hyperbolic space. An unsolved problem is: How to find an exact value of Hausdorff dimension of the limit set $L(G)$ when $L(G) \neq \partial\mathbb{B}^2$?

To solve one specific case of this problem, a well-distributed Schottky group $\Gamma = \langle T_1, T_2, ..., T_m \rangle$ is defined. Our main theorem gives sharp bounds on the critical exponent of Poincaré series, and the theorem was proved based on: properties of Poincaré series, isometric circles, and some nice properties come with the definition of well-distributed Schottky group, especially that we found a way to reconstruct the orbit $\Gamma(0)$ by using only two operators T and R for any $m \in \mathbb{Z}^+$. Following the proof of the main theorem, for the first time, an exact form of Hausdorff dimension for all possible well-distributed Schottky groups of rank two was conjectured and used to generate results against the best approximation derived using McMullen's algorithm.

Backup slide 3: More Definitions, Lemmas, and Theorems

Lemma 4.5.

Let $\Gamma = \langle T_1, \dots, T_m \rangle$ be well-distributed Schottky group, and let

$$X_{I_n} = \rho_{\mathbb{B}^2}\left(0, T_I 0\right).$$

Consider the infinite index of a reduced word $T_{I_n} = T_{i_1} T_{i_2} \dots T_{i_n} \dots \in \mathcal{A} \times \mathcal{A} \cdots \times \mathcal{A}$, where each $\mathcal{A}$ is an alphabet, and $\mathcal{W}_n = \left\{ T_{I_n} : T_{i_j} \in \mathcal{A} \right\}$. Then from hyperbolic cosine law, we can have

$$\cosh\left(X_{I_n}\right) = \cosh^n(r) \prod_{j=1}^{n-1} \left(1 - \tanh\left(X_{I_{j+1}}\right) \tanh(r) \cos\left(\frac{i_{j+1}\pi}{m} - \theta_{I_j} \right) \right)$$

Backup slide 4: More Definitions, Lemmas, and Theorems

Corollary 3.13.

[5] If $L(G) \neq \partial\mathbb{B}^2$, then $L(G)$ is homeomorphic to the middle third Cantor set.

Backup slide 5: More Definitions, Lemmas, and Theorems

Theorem 3.9.

[1, P.80] Let G be non-elementary. $L(G) = L_{z_i}$ for all $z_i \in \overline{\mathbb{B}^n}$.

Theorem 4.7.

Suppose Γ is a well-distributed Schottky group of order 2. Then,

$$\frac{\ln\left(2+\frac{1}{1+2\sqrt{2}}\right)}{\ln\left(\cosh\left(r\right)\right)} \leq \delta\left(\Gamma\right) \leq \min\left\{\frac{\ln\left(4-\frac{2}{1+2\sqrt{2}}\right)}{\ln\left(\cosh\left(r\right)\right)}, 1\right\}.$$

Backup slide 6: More Definitions, Lemmas, and Theorems

Proposition 3.12.

Let $G \subset \mathrm{PSU}(1,1)$ be discrete, and $L(G)$ be the limit set of G, then

1. $A(L(G)) = L(AGA^{-1})$, where $A \in \mathrm{PSU}(1,1)$ and $A^{-1}(\infty) \notin L(G)$,
2. $L(G)$ is G-invariant, i.e. $AL(G) = L(G)$, where $A \in G$,
3. $L(G)$ is uncountable and perfect ($L(G)$ has no isolated points), and
4. Either $L(G)$ is nowhere dense or $L(G) = \partial\mathbb{B}^2$.

Proposition 3.15.

Let $G(g_1, g_2)$ be a Schotkky group, generated by two hyperbolic isometries g_1, g_2 from $\mathrm{PSU}(1,1)$. Recall the notation that $g_I = g_{i_1} g_{i_2} \cdots g_{i_n}$ is a reduced word. Then we have the following properties[3]:

(i) $g_I(\mathbb{B}^2 \setminus \mathrm{int}(g_{i_n}^{-1})) \subset D(g_{i_1})$.

(ii) If $n \geq 2$, $D(g_I) \subset D(g_I g_n^{-1})$.

(iii) If g_I and g_I' are two distinct reduced words, then $D(g_I)$ and $D(g_I')$ are disjoint.

Backup slide 7: What if $\frac{\theta}{2} > 45$ degree?

Our focus in on the limit set $L(\Gamma) \subset \partial\mathbb{B}^2$.

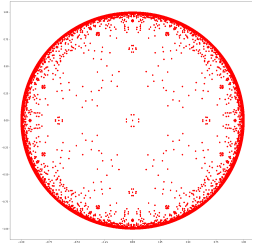

Figure: $m = 2$, $\frac{\theta}{2} \approx 48.4555178244418166°$, Level 14 ($N = 14$).

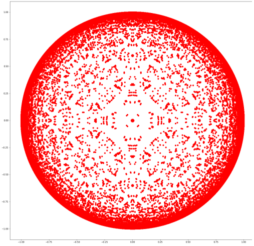

Figure: $m = 2$, $\theta \approx 53.13010858755201°$, Level 14 ($N = 14$).

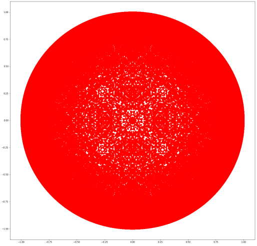

Figure: $m = 2$, $\theta \approx 61.92753175710872 4°$, Level 14 ($N = 14$).

Backup slide 8: Why the length is invariant?

This length is invariant under $\mathrm{PSL}(2,\mathbb{R})$, since if we take $T \in \mathrm{PSL}(2,\mathbb{R})$, then for $T(\gamma)$, by using the above identity, we have

$$\text{Arc length}_{\mathbb{H}^2}(\gamma) = \int_\gamma \frac{1}{\Im(z)}|dz| = \int_{t_i}^{t_f} \frac{1}{\Im(\gamma(t))}|\gamma'(t)|dt$$

$$= \int_{T(t_i)}^{T(t_f)} \frac{1}{\Im(T(\gamma))}|T'(\gamma)|d\gamma = \text{Arc length}_{\mathbb{H}^2}(T \circ \gamma)$$

Backup slide 9: How to derive geodesics on $\mathbb{H}^2$

Given two points to find the equation that describes the path that gives the infimum with respect to the metric $d_{\mathbb{H}^2}$, **all it takes is only the calculus level math**. However, because **this shortest path is an important building block in this paper**, it is worthwhile to provide the derivation in order to construct a strong bridge for audience with a background earned from undergraduate and graduate level courses to the development of our main theorem.

Backup slide 10: How to derive geodesics on $\mathbb{H}^2$

The goal is to determine the path function $y(x)$ such that the integral of the arc length

$$J := \int_{x_1}^{x_2} f\left\{y(x), y'(x); x\right\} dx$$

is an extremum, where $y' := \frac{dy}{dx}$, and the semicolon in f separates the independent variable x from the dependent variable $y(x)$ and its derivative $y'(x)$.

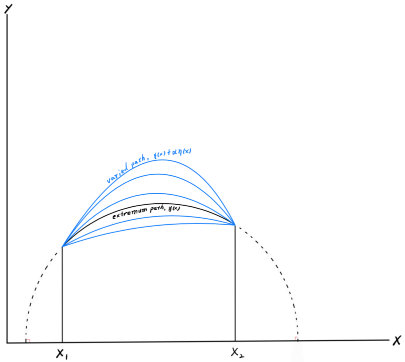

Figure: The path that makes the functional $J(\alpha)$ an extremum is the function $y(x)$. The adjacent function $y(x) + \alpha\eta(x)$ vanishes and can be near to $y(x)$, but they are not the extremum.

Backup slide 11: How to derive geodesics on $\mathbb{H}^2$

The integral J is determined by the path function $y(x)$, and the path endpoints x_1 and x_2 are fixed. After then, the function $y(x)$ is to be modified until an extreme value of J is obtained. A parametric representation to all possible functions y is defined in the form $y = y(\alpha, x)$, such that for $\alpha = 0$, so we now have the function $y = y(0, x) = y(x)$ produces **an extremum for** J:

$$y(\alpha, x) = y(0, x) + \alpha\eta(x)$$

where η has a continuous first derivative and disappears at x_1 and x_2, i.e. $\eta(x_1) = \eta(x_2) = 0$.

Backup slide 12: How to derive geodesics on $\mathbb{H}^2$

The integral J is redefined as a functional of the parametric α:

$$J(\alpha) = \int_{x_1}^{x_2} f\left\{y(\alpha, x), y'(\alpha, x); x\right\} dx.$$

The necessary condition that the integral have an extremum occurs when $\frac{\partial J}{\partial \alpha} = 0$ as $\alpha = 0$. Hence, to solve it, we apply **the Leibniz integral rule** as follows

$$\frac{\partial J(\alpha)}{\partial \alpha} = f\frac{dx_2}{d\alpha} - f\frac{dx_1}{d\alpha} + \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha} f\left\{y(\alpha, x), y'(\alpha, x); x\right\} dx.$$

Since x_1 and x_2 are constant,

$$\begin{aligned}\frac{\partial J(\alpha)}{\partial \alpha} &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right) dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta}{dx}\right) dx.\end{aligned}$$

The second term can be integrated by parts, and using $\eta(x_1) = \eta(x_2) = 0$, then we have

$$\frac{\partial J(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) \eta(x) dx.$$

Since $\eta(x)$ is arbitrary chosen, to have $\frac{\partial J(\alpha)}{\partial \alpha} = 0$ as $\alpha = 0$, we have **the following Euler-Lagrange equation**:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.$$

Backup slide 13: How to derive geodesics on $\mathbb{H}^2$

Since

$$ds = \frac{|dz|}{\Im(z)} = \frac{\sqrt{(dx)^2 + (dy)^2}}{y} = \frac{\sqrt{1 + y'^2}}{y}dx,$$

we have

$$f(x) = \frac{\sqrt{1 + y'^2}}{y}.$$

Then,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = -y^{-2}\left(1 + y'^2\right)^{\frac{1}{2}} - \frac{yy'' - (y')^2(1 + y'^2)}{y^2(1 + y'^2)^{\frac{3}{2}}} = 0.$$

This Euler-Largrange equation can be simplified to

$$yy'' + y'^2 + 1 = 0.$$

To solve the differential equation, firstly, let $u(y) = \frac{dy}{dx}$. Then we have

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = u\frac{du}{dy}.$$

Hence, we solve the following differential equation first:

$$u^2 + yu\frac{du}{dy} + 1 = 0.$$

Backup slide 14: How to derive geodesics on $\mathbb{H}^2$

The differential equation is separable,

$$-\int \frac{u}{1+u^2}\,du = \int \frac{dy}{y}$$

$$u = \frac{\pm\sqrt{c_1 - y^2}}{y}.$$

Since $u(y) = \frac{dy}{dx}$, we solve the following separable differential equation:

$$\frac{dy}{dx} = \frac{\pm\sqrt{c_1 - y^2}}{y},$$

and this gives

$$y = \pm\sqrt{c_1^2 - (x - c_2)^2}.$$

Since in $\mathbb{H}^2$, $y > 0$, we discard the negative. Then, **it is a semicircle centered at the real axis** at $(c_2, 0)$ and radius c_1; that is,

$$(x - c_2)^2 + y^2 = c_1^2,$$

where $y > 0$. Furthermore, since $(x - c_2)^2 + y^2 = c_1^2 \Rightarrow \frac{x^2 - 2c_1 x + y^2}{c_1} = 0$, if we take the radius c_1 to infinity

$$\lim_{c_1 \to \infty} \left(\frac{x^2 + y^2}{c_1} - 2x \right) = 0 \Rightarrow -2x = 0,$$

then after dividing both sides by -2 we have

$$x = 0$$

which is **a vertical line**. Since it can be translated by linear fractional maps in $\mathrm{PSL}(2, \mathbb{R})$, in $\mathbb{H}^2$, **geodesics are vertical lines and semi-circles that are centered on real axis.**

Backup slide 15: The Poincaré Disk

Recall the definition of $\mathbb{B}^2$:

$$\mathbb{B}^2 = \{z \in \mathbb{C} : |z| > 1\}.$$

The map $\Psi(z) = i\frac{z-i}{z+i}$ is a 1-1 map of $\mathbb{H}^2$ onto $\mathbb{B}^2$, hence $\rho_{\mathbb{B}^2}$ given by

$$\rho_{\mathbb{B}^2}(z,w) = d_{\mathbb{H}^2}(\Psi^{-1}z, \Psi^{-1}w),$$

where $z, w \in \mathbb{B}^2$. Then, $\rho_{\mathbb{B}^2}(z,w)$ is a metric on $\mathbb{B}^2$, and the arc length between z and w measured by the metric $\rho_{\mathbb{B}^2}(z,w)$ is also called **the hyperbolic length between z and w**. However, since

$$\frac{1}{\Im(z)} = \left|\frac{d\Psi^{-1}}{dz}\right| \frac{1}{\Im\left(\Psi^{-1}(z)\right)} = \frac{|-2|}{|z-i|^2}\frac{|z-i|^2}{1-|z|^2} = \frac{2}{1-|z|^2},$$

and with the condition $\rho_{\mathbb{B}^2}(z,w) = d_{\mathbb{H}^2}(\Psi^{-1}z, \Psi^{-1}w)$, **define the line element as follows[2, 6]:**

$$ds = \frac{2|dz|}{1-|z|^2}.$$

Backup slide 16: How to define an invariant metric?

Then, **to construct a metric space**, we use the line element to define a metric in the following.
Firstly, for any two points in $\mathbb{B}^2$, every path function γ has an invariant length

$$\int_\gamma \frac{2|dz|}{1-|z|^2}.$$

The shortest arc, i.e. the geodesic, can be attained by firstly mapping the two distinct endpoints $z_1, z_2 \in \mathbb{B}^2$ to $\mathbb{H}^2$ using Ψ^{-1}. Then we can find the unique geodesic connects $\Psi^{-1}(z_1)$ and $\Psi^{-1}(z_2)$ by using the solutions we derived previously; that is, either a vertical line, or a semicircle $y = \sqrt{c_1^2 - (x - c_2)^2}$ which orthogonal to the real axis.

Backup slide 17: Why fractional linear transformations preserve circles and lines?

Next, recall that in complex analysis[2, 6] we have an important result for the conformal inversion map $z \longmapsto \frac{1}{z}$; that is, **it maps a circle or a line into either a circle or a line**. Since for any fractional linear transformation $T(z) = \frac{az+b}{cz+d}$, we can rewrite it as

$$T(z) = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d},$$

if $c \neq 0$. Then, $u = Tz$ is a sequence of compositions of the following maps:

$$z \mapsto u_1 = cz + d \mapsto u_2 = \frac{1}{u_1} \mapsto u = \frac{a}{c} + \left(b - \frac{ad}{c}\right) u_2.$$

Hence, except $u_2 = \frac{1}{u_2}$, **all other maps are affine maps that also map circles and lines into circles and lines.**

Since the conformal mapping we used is Ψ, and **it can be decomposed into the above maps**, it also carries vertical lines and semi-circles on $\mathbb{H}^2$, i.e. geodesics on $\mathbb{H}^2$, to diameters (radial lines) or partial circles that are orthogonal to the boundary $\partial\mathbb{B}^2$.

Backup slide 18: How to derive the invariant metric on $\mathbb{B}^2$?

Since **the metric in $\mathbb{B}^2$ is defined based on the metric in $\mathbb{H}^2$** such that

$$d_{\mathbb{H}^2}(z_1, z_2) = \rho_{\mathbb{B}^2}(\Psi(z_1), \Psi(z_2)),$$

if γ is the geodesic on $\mathbb{H}^2$ that connects z_1 and z_2, then $\Psi(\gamma)$ is the geodesic on $\mathbb{B}^2$ that connects $\Psi(z_1)$, and $\Psi(z_2)$.
That is, geodesics are circles or diameters orthogonal to $|z| = 1$. The hyperbolic distance measured by using the metric $\rho_{\mathbb{B}^2}(0, r)$ from 0 to $r > 0$ along a diameter γ of the unit disk is

$$\int_0^r \frac{2dr}{1 - r^2} = \ln \frac{1 + r}{1 - r}. \tag{1}$$

Now, to define a metric such that $\mathbb{B}^2$ becomes a metric space:

$$\rho_{\mathbb{B}^2}(a, b) := \int_\gamma \frac{2|dz|}{1 - |z|^2}$$

where $a = \gamma(0)$, $b = \gamma(1)$, and γ is a geodesic on $\mathbb{B}^2$ parameterized by $[0, 1]$.

Backup slide 19: How to prove McMullen's Algorithm?

Proof idea: to prove the algorithm converges we "discretize" the integral $\int_E |f'(x)|^\delta d\mu$:
A collection of conformal maps $f : U(f) \to S^n$, where $S^n = \mathbb{R}^n \cup \{\infty\}$ and $U(f)$ is an open set in S^n, is a a conformal dynamical system $\mathcal{F}$. Then, an $\mathcal{F}$-invariant density of dimension δ is a finite positive measure μ on S^n such that

$$\mu(f(E)) = \int_E |f'(x)|^\delta d\mu$$

whenever $f|E$ is injective, $E \subset U(f)$ is a Borel set and $f \in \mathcal{F}$. In McMullen's algorithm, it is expected to have $\alpha(\mathcal{P}) \approx \delta$. The above transition law implies $\mu(P_i)$ is an approximate eigenvector for T_{ij}^δ with the maximized eigenvalue λ to be unit one. In other words,

$$\mu(P_i) = \sum_{i \mapsto j} \mu(f_i^{-1}(P_j)) = \sum \int_{P_j} |(f_i^{-1})'(x)|^\delta d\mu$$

$$\approx \sum |f_i'(y_{ij})|^{-\delta} \mu(P_j) = \sum_j T_{ij}^\delta m_j.$$

Backup slide 20

Because the focus of this study is on Schottky groups, **those groups will be defined by selecting disks with isometric circles of matching group generators as the boundaries of those disks**. Secondly, since for each circle with radius R and center at $q \in \mathbb{C}$ on the complex plane $\mathbb{C}$, **we can find a reflection (or inversion) of that circle which is a Möbius transformation**:

$$I(z) = \frac{R^2}{\overline{z} - \overline{q}} + q.$$

Then, by choosing two circles, depending on the locations of circles is in $\mathbb{H}^2$ or in $\mathbb{B}^2$, we can compose an isometry either in PSL$(2, \mathbb{R})$ or PSU$(1, 1)$. Conversely, any isometry in either PSL$(2, \mathbb{R})$ or PSU$(1, 1)$ can be decomposed into two reflections in the form of Möbius mappings. Since two reflections can compose an isometry $g \in$ PSU$(1, 1)$, we can choose one circle to be the isometric circle of g to obtain a reflection I_1, and find the other circle to be the other reflection to compose $g = I_1 \circ \sigma$.

Backup slide 21: Proof of Lemma 2.2

Let $L_1 := [z_0, gz_0]$ denote the geodesic between z_0 and gz_0, and $L_2 := [z_0, g^{-1}z_0]$ be the geodesic between z_0 and $g^{-1}z_0$, and we also denote $M_1 := M_{z_0}(g)$ and $M_2 := M_{z_0}(g^{-1})$. Additionally, denote the intersection between L_i and M_i by m_i.
(i) Firstly, it can be checked that

$$g(L_2) = g([z_0, g^{-1}z_0]) = [gz_0, z_0] = L_1,$$

and

$$g^{-1}(L_1) = g^{-1}([z_0, gz_0]) = [g^{-1}z_0, z_0] = L_2.$$

Furthermore, based on the above result and the property of isometry, since $m_i \in L_i$, and

$$\rho^2_{\mathbb{B}}(g^{-1}z_0, m_2) = \rho^2_{\mathbb{B}}(z_0, gm_2) = \rho^2_{\mathbb{B}}(z_0, gm_2),$$

and

$$\rho^2_{\mathbb{B}}(m_2, z_0) = \rho^2_{\mathbb{B}}(gm_2, gz_0)$$

, these two conditions uniquely determine

$$gm_2 = m_1.$$

Likewise,

$$g^{-1}m_1 = m_2.$$

Backup slide 22: Proof of Lemma 2.2

(ii) Since g is conformal, so the angle between L_i and M_i will be preserved. Since $M_{z_0}(g^{-1})$ is the perpendicular bisector of points z_0 and $g^{-1}(z_0)$ and $M_{z_0}(g)$ is the perpendicular bisector of points z_0 and $g(z_0)$, $M_{z_0}(g^{-1})$ orthogonally intersects L_2 at the midpoint m_2 of L_2 with angle ϑ_2, and $M_{z_0}(g)$ orthogonally intersects L_1 at the midpoint m_1 of L_1 with angle ϑ_1.
Recall that:

(1) a geodesic will be mapped into another geodesic by a non-identity isometry,
(2) angles are preserved: $\vartheta_1 = \vartheta_2 = \frac{\pi}{2}$,
(3) intersections are mapped to each other: $g(m_1) = m_2$ and $g^{-1}(m_2) = m_1$,
(4) geodesic is unique at each point $m_i \in \mathbb{B}^2$ if a tangent vector at that point m_i is given, and
(5) it can be checked that each M_i is a geodesic of $\mathbb{B}^2$.

These conditions uniquely determine the image of M_1 mapped by g is M_2, and the image of M_2 mapped by g^{-1} is M_1.
That is,

$$g^{-1}(M_1) = M_2,$$

and

$$g(M_2) = M_1.$$

(iv) To check the interior, we can verify it by using the point z_0 in the region $\mathbb{B}^2 \setminus D(g^{-1})$ and check that it will be mapped into the interior of $M_{z_0}(g)$ by definition of the perpendicular bisector $M_{z_0}(g)$. Since g is a conformally (i.e. a bijection) continuous open map, by the open map theorem[6], we have

$$g(\text{int}(D_{z_0}(g^{-1}))) = \mathbb{B}^2 \setminus D_{z_0}(g).$$

Backup slide 23: Conformal Mappings

Question: does there exist a holomorphic bijection between two open sets U and V in $\mathbb{C}$?
This question leads to the definition of conformal mappings:

Definition 2.2.

Let U and V be two open sets in $\mathbb{C}$ and f a complex-valued function on U. *A confromal map or biholomorphism* is a bijective holomorphic function $f : U \to V$.

If there exists such a mapping f, then U and V are called *conformally equivalent*.