The Hausdorff Dimension of Limit Sets of Well-distributed Schottky Groups

A thesis presented to the faculty of
San Francisco State University
In partial fulfillment of
The Requirements for
The Degree

Master of Arts
In
Mathematics

by

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May 2022

CERTIFICATION OF APPROVAL

I certify that I have read *The Hausdorff Dimension of Limit Sets of Well-distributed Schottky Groups* by William Huanshan Chuangand that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Artsin Mathematicsat San Francisco State University.

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The Hausdorff Dimension of Limit Sets of Well-distributed Schottky Groups

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2022

The problem we are trying to solve: Let a finitely generated Schottky group G be given. How to find bounds and an exact value of Hausdorff dimension of the limit set $L(G)$?

To solve one specific case of the problem, a well-distributed Schottky group $\Gamma = \langle T_1, T_2, ..., T_m \rangle$ is defined. For a well-distributed Schottky group in two-dimensional hyperbolic space $\mathbb{B}^2$, our main theorem was proved based on: properties of Poincaré series, isometric circles, and some nice properties come with the definition of well-distributed Schottky group, and for the first time, our main theorem gives sharp bounds on $\delta(\Gamma)$.

Following the proof of the main theorem, an exact form expression of Hausdorff dimension for two-generator Schottky group was conjectured and used to generate results against the best approximation derived using McMullen's algorithm.

I certify that the Abstract is a correct representation of the content of this thesis.

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ACKNOWLEDGMENTS

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Chapter 1

Introduction

Considering $G = \langle T_1, T_2, ..., T_m \rangle$ is a finitely generated Schottky group(see 3.3), then its associated Poincaré series (see Chapter 4) is:

$$\mathcal{P}(G, t) := \sum_{\forall g_i \in G} \exp\left\{-t d_{\mathbb{B}^2}(z, g_i(z))\right\}$$

where $\delta(G) := \inf\left\{t \in \mathbb{R} : \mathcal{P}(G, t) < \infty\right\} = \sup\left\{t \in \mathbb{R} : \mathcal{P}(G, t) = \infty\right\}$ is called the critical exponent of the Poincaré sereis or the Poincaré exponent, and $z, g_i(z) \in \mathbb{B}^2$.

The following are four top historical reasons that highlight the need of creating a systematic method for computing the critical exponent of the Poincaré sereis, albeit the technical specifics will not be discussed in the thesis. They are: the exponent of Poincaré series has some important meaning in the prime geodesic theorem[10], it plays an important role in the spectral gap problem of hyperbolic surfaces[12], it is a criterion of instability in AdS_3/CFT[16], and it is an example of the connections between Dirichlet characters, algebraic curves, and automorphic

forms[13, 20, 21, 22, 24].

- In prime geodesic theorem, $\delta(G)$ plays a role $z = \delta(G) + it$ which is similar to the line $z = 1$ in the prime number theorem where the Riemann zeta function has its only singularity at the point $z = 1$ and no zeros on this line as proved by von Mangoldt in 1895.

- For the spectral gap problem of hyperbolic surfaces, in Bourgain and Dyatlov's paper, a hyperbolic surface has an essential spectral gap if it can meet some conditions in the theorem they prove, meaning that there are finitely many zeros within the region $\Re(s) > \frac{1}{2} - \beta = \delta(G)$.

- In $\mathrm{AdS}_3/\mathrm{CFT}$[16], $\delta(G)$ is a criterion for the critical exponent of a scalar field to have stable solutions in 3d quantum gravity. Furthermore, $\delta(G)$ uniquely determines the first resonance of the corresponding Selberg zeta function of G, i.e. $\lambda_1 = \delta(G)(1 - \delta(G))$, where λ_1 is the lowest eigenvalue of eigenfunctions of the Laplacian operated on a Riemann surface (algebraic curve) $\mathbb{H}^2/G$.

- Since $\delta(G)$ is the convergence exponent of Poincaré series of G, and this series can be written into a Dirichlet $L-$function, plus eigenfunctions of $\mathbb{H}^2/G$ are automorphic forms, hence an exact connection between number theory and harmonic analysis might possibly be established. It maight also be viewed as an illustration of the theory of the Langlands program that Dirichlet characters, algebraic curves, and automorphic forms are all interrelated.

Thus, the development of a technique to precisely compute the critical exponent $\delta(G)$ of a Poincaré series of a Schottky group G has a long history–however, so far there is no explicit formula to describe $\delta(G)$, all we have are approximation. The link between a specific given finitely generated Schottky group G to the exact value of its Poincaré exponent $\delta(G)$ has been missing since the notions were defined.

The earliest modern investigations of the behavior of the exponent of convergent Poincaré series were done by Akaza and Beardon independently[3, 4, 6, 7, 8]. Their studies focused on determining the range of $\delta(G)$ of some specific type of discrete group G. In [23], McMullen constructed an algorithm to estimate $\delta(G)$ numerically. However, its accuracy is upto $O(N)$, i.e. given N digits, the accuracy can be approximately accurate after upto $C \cdot N$ iterations of applying McMullen's algorithm, where C is a constant. Furthermore, in implementing the algorithm, one extra assumption was added in [23] which is it assumed that at each iteration all perpendicular bisectors are assumed to be having an identical diameter. This assumption is another source of errors in addition to the error generated due to the not enough large N, and the errors accumulated in each floating point operation on a computer system.

In this thesis, a geometric method was developed to conjecture $\delta(G)$ exactly by computing the lower bound and upper bound of $\delta(G)$, where G are some Schottky groups that satisfy a predefined set of conditions. Then, this might be the first time to derive an exact value of λ_1 of a specific Schottky group.

Chapters 2, 3 and 4 provide the background for much of the paper.

In Chapter 2, we show how to build hyperbolic spaces in the Poincar'e disk model, as well as how to build automorphisms specified on this model. The methods of isometric circles and perpendicular bisectors, which correspond to functions defined on the Poincaré disk, are also described in Chapter 2 as our tools for building concepts in the next chapters.

Following the introduction of functions defined on the Poincaré disk, Chapter 3 describes a specific collection of a certain type of functions called Schottky groups, which is a discrete subgroup of $\mathrm{PSU}(1,1)$[1] with derivations of Schottky group features.

Chapter 4 investigates the critical exponent of Poincaré series and Sullivan's theorem that proved that the critical exponent of Poincaré series of every Schottky group equals the Hausdorff dimension of the limit set of the Schottky group. In addition, in Chapter 4, we looked at certain aspects of Poincaré exponents, such as how it is an invariant under a conformal map and how it is independent of the point

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[1]Let $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then we can define the unitary group $\mathrm{U}(1,1) := \{M : M^\dagger J M = J\}$. Thus, the special unitary group is

$$\mathrm{SU}(1,1) := \{M \in \mathrm{U}(1,1) : \det(M) = 1\} = \left\{ \begin{pmatrix} \alpha & \gamma \\ \overline{\gamma} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 - |\gamma|^2 = 1 \right\}$$

$$= \left\{ \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} : |\alpha|^2 - |\gamma|^2 = 1 \right\}$$

where $\alpha \in \mathbb{C}$ and $\gamma \in \mathbb{C}$. The center of $\mathrm{U}(1,1)$ is λJ where $\lambda = e^{i\phi} \in \mathrm{U}(1), \phi \in \mathbb{R}$. Then, by solving λ for $SU(1,1)$, we have the center of $\mathrm{SU}(1,1)$ which is $\pm iJ := Z(\mathrm{SU}(1,1))$. Then the projective special unitary group is $\mathrm{PSU}(1,1) := \mathrm{SU}(1,1)/Z(\mathrm{SU}(1,1))$.

chosen to be the input of the Poincaré series.

Chapter 5 studies a special case of Schottky groups that can meet our well-distributed configuration requirement. In this chapter, our main theorem establishes tight bounds on rank two well-distributed Schottky groups. Based on our understanding, this might be the first time in history that we have such constraints for Schottky groups other than the case when the Hausdorff dimension approaches the unit one. Based on our derivations of the main theorem, in this chapter, we also provides a conjecture for the exact form of Hausdorff dimension of the limit set of a rank two well-distributed Schottky group.

Chapter 6 provides an introduction of McMullen's algorithm which is by far the most accurate algorithm to approximate the the Hausdorff dimension of limit sets of Schottky groups numerically. However, because the original source code used to get the results was never disclosed, we re-implemented the method in C to reproduce the results to ensure that our understanding of Patterson-Sullivan theory and McMullen's technique were correct. Furthermore, based on McMullen's derivation, for small angles, we derived a result that coincides to the result generated by our conjecture is also presented in this chapter.

Chapter 2

Two-dimensional hyperbolic geometry

The purpose of this chapter is to demonstrate how to construct hyperbolic spaces in the Poincaré disk disk model, as well as automorphisms given on this model. The methods of isometric circles and perpendicular bisectors, which relate to functions defined on the Poincaré disk disk, are also explained in Chapter 2 as tools for developing notions in the following chapters. We presume that readers are familiar with complex analysis and differential geometry.

2.1 The Poincaré Disk Model

Everybody is familiar with $\mathbb{C}$. Starting from two-dimension complex plane, now we collect all points in a unit disk

$$\mathbb{B}^2 := \{z \in \mathbb{C} : |z| < 1\},$$

and define the unit circle as its boundary, denoted by $\partial\mathbb{B}^2$.

Define linear fractional transformations that operate on this disk as follows:

$$g(z) := \frac{\alpha z + \overline{\gamma}}{\gamma z + \overline{\alpha}}$$

with a constraint $|\alpha|^2 - |\gamma|^2 = 1$. This transformation conformally maps $\mathbb{B}^2$ onto itself.

Let the following differential arc length be the hyperbolic (Riemannian) metric of the space:

$$ds = 2\frac{|dz|}{1 - |z|^2}$$

which is conformal invariant under conformal self mappings of the disk[2, 31]. Then, in the context of metric space, use it to define a metric. Firstly, for any two points in $\mathbb{B}^2$, every rectifiable arc γ has an invariant length

$$\int_{\gamma} \frac{2|dz|}{1 - |z|^2}.$$

The shortest arc can be attained (by variational calculus in general) firstly from 0 to any other point along a radius. That is, geodesics are circles orthogonal to $|z| = 1$. The non-euclidean distance from 0 to $r > 0$ is

$$\int_0^r \frac{2dr}{1 - r^2} = \ln\frac{1 + r}{1 - r}. \tag{2.1}$$

Now, to define a metric such that $\mathbb{B}^2$ becomes a metric space:

$$\rho_{\mathbb{B}^2}(a, b) := \int_\gamma \frac{2|dz|}{1 - |z|^2}$$

where $a = \gamma(0)$, $b = \gamma(1)$, γ is parameterized by $[0, 1]$.

An isometry group that acts on $\mathbb{B}^2$ is a collection that each of its elements is a transformation denoted by g such that $\rho_{\mathbb{B}^2}(a, b) = \rho_{\mathbb{B}^2}(g(a), g(b))$. Since hyperbolic (Riemannian) metric ds is derived by using conformal map, i.e. linear fractional transformation $g(z) := \frac{\alpha z + \overline{\gamma}}{\gamma z + \overline{\alpha}}$ with a constraint $|\alpha|^2 - |\gamma|^2 = 1$ with an assumption to make g be an isometry, the following set is an isometry group of the Poincaré disk space $\mathrm{Isom}^+(\mathbb{B}^2) := \left\{ \frac{\alpha z + \overline{\gamma}}{cz + \overline{\alpha}} : |\alpha|^2 - |\gamma|^2 = 1 \right\}$. The group $\mathrm{Isom}^+(\mathbb{B}^2)$ can be proved that it is isomorphic to $\mathrm{PSU}(1, 1)$ (defined in the footnote in Chapter 1), that is

$$\mathrm{Isom}^+(\mathbb{B}^2) \simeq \mathrm{PSU}(1, 1),$$

and for sure all other isometry groups of the hyperbolic space $\mathbb{H}^2_{\mathbb{R}}$, but may be implemented in some other models. In general, including $\mathrm{Isom}^+(\mathbb{B}^2)$ they are all simply denoted by $\mathrm{Isom}^+(\mathbb{H}^2_{\mathbb{R}})$.

By using isometry group, i.e. let g be arbitrarily chosen from $\mathrm{Isom}^+(\mathbb{B}^2)$, then the following identity can be derived:

$$\sinh^2\left(\frac{1}{2}\rho_{\mathbb{B}^2}(z, w)\right) = \frac{|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}, \tag{2.2}$$

or alternatively, by using $\cosh^2\left(\frac{1}{2}\rho(z,w)\right) = 1 + \sinh^2\left(\frac{1}{2}\rho(z,w)\right)$

$$\cosh^2\left(\frac{1}{2}\rho_{\mathbb{B}^2}(z,w)\right) = \frac{|1 - z\overline{w}|^2}{(1 - |z|^2)(1 - |w|^2)}. \tag{2.3}$$

Let $g \in \mathrm{Isom}^+(\mathbb{B}^2)$ and $g = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}$, where $|\alpha|^2 - |\gamma|^2 = 1$, then $gz = z$ is a quadratic equation, and by using it discriminant $\mathrm{Tr}(g)^2 - 4$, g can be classified into three types:

- The operator g is *elliptic*, if $0 \le \mathrm{Tr}^2(g) < 4$;

- The operator g is *parabolic*, if $\mathrm{Tr}^2(g) = 4$;

- The operator g is *hyperbolic*, if $\mathrm{Tr}^2(g) > 4$.

Furthermore, fixed points of g are solutions of the equation $gz = z$, and their existence is guaranteed by Brouwer fixed point theorem[25, 18, 30].

2.2   Isometric Circles

Definition 2.1. *The isometric circle I_g of $g = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} \in Isom^+(\mathbb{B}^2)$ is defined by the formula $|\gamma z + \overline{\alpha}| = 1$.*

Equivalently, I_g can also be defined by isometry invariant that $|\gamma z + \overline{\alpha}| = 1$ if and only if $|g'(z)| = 1$.

There are a lot of advantages to use Poincaré disk model $\mathbb{B}^2$ instead of using the upper-half plane model in the development of the main theorems of this thesis. Results in not only the symmetry of the form of entries of isometries on this model, but also each isometry $g = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}$ with $|\alpha|^2 - |\gamma|^2 = 1$ has its own isometric circle as one of reflection geodesic lines to form the Möbius map g on $\mathbb{B}^2$.

Theorem 2.1. *If take $z = 0$, then the perpendicular bisector of $[0, g^{-1}0]$, i.e. $\{z \in \mathbb{B}^2 : d(z, 0) = d(z, g^{-1}0)\}$ is the isometric circle of g denoted as I_g. Furthermore, the isometric circles I_g and $I_{g^{-1}}$ have an identical radius $\frac{1}{|c|}$.*

Proof. Let $g = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}$ with $|\alpha|^2 - |\gamma|^2 = 1$. Firstly, by the hyperbolic length on $\mathbb{B}^2$, $\rho(z, 0) = \rho(z, g0)$ implies $\frac{z\overline{z}}{1 - z\overline{z}} = \frac{(z - g0)\overline{z - g0}}{(1 - z\overline{z})(1 - g0\overline{g0})} \Rightarrow |z|^2 \left(1 - \left|\frac{\overline{\gamma}}{\alpha}\right|^2\right) = \left|z - \frac{\overline{\gamma}}{\alpha}\right|^2$ $\Rightarrow |\alpha|^2|z|^2 - |\gamma|^2|z|^2 = |\overline{\alpha}z - \overline{\gamma}|^2$. Since $|\alpha|^2 - |\gamma|^2 = 1$, thus $|z|^2 = |\overline{\alpha}z - \overline{\gamma}|^2$. Secondly, compute $|\overline{\alpha}z - \overline{\gamma}|^2 - |\gamma z - \alpha|^2 = (|\alpha|^2|z|^2 + |\gamma|^2) - (|\gamma|^2|z|^2 + |a|^2) = (|\alpha|^2 - |\gamma|^2)|z|^2 - |\alpha|^2 + |\gamma|^2 = |z|^2 - 1$. Finally, multiplying minus one to the first result on both sides of the equation, and adding it to the second result gives $|\gamma z - \alpha| = 1$ which is $\left|z - \frac{\alpha}{\gamma}\right| = \frac{1}{|\gamma|}$, and that is the isometric circle of g^{-1}. $\square$

The following lemma is proved in [19, Theorem 3.3.7] and in [14] for n-dimension.

Lemma 2.2. *Given any infinite sequence of distinct isometric spheres $I_1, I_2...$ of Möbius transformations of the group G with radii $r_1, r_2, ...$, then $\lim_{n \to \infty} r_n = 0$ in the sense of Euclidean metric.*

A similar result can also be proved for perpendicular bisectors $M_{z_0}(g_i)$ from an iteration of applying lemma 2.4.

Fixed points of g are solutions of a quadratic equation $gz = z$, and each of them can be reached by a convergent sequence $g^{\pm n}z_0$ for some $z_0 \in \mathbb{B}^2$ (they converge in a sense of Euclidean metric):

Lemma 2.3. *Let g^+ be the fixed point of $g \in Isom^+(\mathbb{B}^2)$ (the positive solution of the equation $gz = z$), where g is non-elliptic. Then, $g(g^+) = g^+$ if and only if $\lim\limits_{n\to\infty} g^n z = g^+$ for all $z \in \mathbb{H}^2$. Similarly, the negative solution g^- of the equation $gz = z$ also has a similar result.*

Proof. For the reason that $g^+ = \lim\limits_{n\to\infty} g^n z$ is given, hence $g \lim\limits_{n\to\infty} g^n z = gg^+$. Thus, $gg^+ = g \lim\limits_{n\to\infty} g^n z = \lim\limits_{n\to\infty} g^{n+1} z = \lim\limits_{n\to\infty} g^n z = g^+$.

In the opposite direction, if $gg^+ = g^+$ is given, then on claim: for all $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that for all $i > N$ we have $|g^i z - g^+| < \epsilon$. To show this direction, let $\epsilon > 0$ be given, recall lemma 2.4, result from the region enclosed by perpendicular bisector (which can be checked that its equation is an Euclidean circle) and the unit circle is nested, thus there exists $N = i + k, k \in \mathbb{Z}^+$ such that $2r_{i+k} < \epsilon$ where r_{i+k} is the radius of the perpendicular bisector of $i + k$-level. Then $|g^{i+k} z - g^{i+k} g^+| = |g^{i+k} z - g^+| < \epsilon$. This completes the proof. $\square$

Hence,

1. the attracting fixed point $g^+ = \lim\limits_{n\to\infty} g^n(z)$ which is also the positive solution of $gz = z$, and

2. the repelling fixed point $g^- = \lim\limits_{n\to\infty} g^{-n}(z)$ which is also the negative solution of $gz = z$.

2.3 Perpendicular Bisectors

In dimension 2, the perpendicular bisector $M_{z_0=0}(g)$ and the isometric circle $I_{g^{-1}}$ is identical if we are using Poincaré disk model, i.e. in $\mathbb{B}^2$. In general, for instance, if $z_0 \neq 0$, perpendicular bisectors need not to be isometric spheres, but have a more general usage which can also give an alternative proof of main theorems of this thesis.

The following results are proved in two-dimensional hyperbolic plane, but can be generalized to higher dimensions.

Recall the notation $M_{z_0}(g)$ stands for the perpendicular bisector of the geodesic $(0, g0)$, $M_{z_0}(g^{-1})$ is the perpendicular bisector of the geodesic $(0, g^{-1}0)$, $D_{z_0}(g)$ is the closed half-plane in $\mathbb{H}^2$ bounded by $M_{z_0}(g)$, and $D_{z_0}(g^{-1})$ is the closed half-plane in $\mathbb{H}^2$ bounded by $M_{z_0}(g^{-1})$.

Lemma 2.4. *If $g \in G\backslash\{\mathrm{id}\}$, where G is a subgroup of $Isom^+(\mathbb{B}^2)$, and $z_0 \in \mathbb{B}^2$ which is not fixed by g, then $g(M_{z_0}(g^{-1})) = M_{z_0}(g)$, and $g(int(D_{z_0}(g^{-1}))) = \mathbb{H}^2 \setminus D_{z_0}(T)$.*

Proof. Claim: The operator g and the operation of constructing a perpendicular bisector $M_{z_0}(g)$ from a given point m_i and an angle ϑ commute.

(i) Let $L_1 := [z_0, gz_0]$ denote the geodesic between z_0 and gz_0, and $L_2 :=$

$[z_0, g^{-1}z_0]$ be the geodesic between z_0 and $g^{-1}z_0$. Now, firstly it can be checked that $g(z_0) = gz_0$ and $gg^{-1}z_0 = z_0$, and $g^{-1}(z_0) = g^{-1}z_0$ and $g^{-1}gz_0 = z_0$. Since g is a continuous, conformal, and isometric map, (or recall that a geodesic will be mapped into another geodesic by a non-identity isometry), hence $g(L_2) = L_1$ and $g^{-1}(L_1) = L_2$.

(ii) Since $M_{z_0}(g^{-1})$ is the perpendicular bisector of points z_0 and $g^{-1}(z_0)$ and $M_{z_0}(g)$ is the perpendicular bisector of points z_0 and $g(z_0)$, $M_{z_0}(g^{-1})$ orthogonally intersects L_2 at the midpoint m_2 of L_2 with angle ϑ_2, and $M_{z_0}(g)$ orthogonally intersects L_1 at the midpoint m_1 of L_1 with angle ϑ_1. A map can be defined $\Upsilon(L_i) = (m_i, \vartheta_i)$.

(iii) Because each geodesic can be uniquely defined by an angle and a point on two-dimensional hyperbolic space, as a result, after denoting $M_1 := M_{z_0}(g)$ and $M_2 := M_{z_0}(g^{-1})$, a bijection can be defined as $\Phi : M_i = (m_i, \vartheta_i)$, and $\Phi^{-1}(m_i, \vartheta_i) = M_i$. It can be checked the equation of M_i is a Euclidean circle intersects real axis perpendicularly, hence each M_i is a geodesic of $\mathbb{H}^2$. Recall again that a geodesic will be mapped into another geodesic by a non-identity isometry, and result from $\vartheta_1 = \vartheta_2 = \frac{\pi}{2}$, $g(m_1) = m_2$ and $g^{-1}(m_2) = m_1$, hence $g(M_1) = M_2$ and $g(M_2) = M_1$.

Therefore the two operations g and Φ are commutative:

$$
\begin{array}{ccc}
L_1 & \xleftrightarrow[g^{-1}]{g} & L_2 \\
{\scriptstyle\Upsilon}\downarrow & & \downarrow{\scriptstyle\Upsilon} \\
m_1, \vartheta_1 & & m_2, \vartheta_2 \\
\downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} \\
M_1 & \xleftrightarrow[g^{-1}]{g} & M_2
\end{array}
$$

Define Ω as a composition of Υ and Φ: $\Omega := \Upsilon\Phi$. Therefore,

$$\Omega g = g\Omega.$$

The use of the above result is that then the operator g now can be directly applied on M_i. Whence, $g(M_{z_0}(g^{-1})) = M_{z_0}(g)$ and $g^{-1}(M_{z_0}(g)) = M_{z_0}(g^{-1})$.

(iv) To check the interior, we can verify it by taking a point in the region say outside of $M_{z_0}(g)$ and check that it will be mapped into the interior of $M_{z_0}(g)$; however, this is clearly true, result from g is a continuous open map, and it is already known that $g(z_0) \in M_{z_0}(g)$. Hence, this result automatically follows once the commutativity $\Omega g = g\Omega$ is proved.

(v) Finally, since g is an invertible continuous open map, and the input is a connected region, the exterior of $M_{z_0}(g)$ will be mapped into the interior of $M_{z_0}(g)$ except the interior of $M_{z_0}(g^{-1})$, as $g(z_0) \in M_{z_0}(g)$. To show it is a bijection, recall g a combination of reflection, and in this case, it can be decomposed to a reflection

in $M_{z_0}(g)$ to be the last operation, so $M_{z_0}(g)$ is like a mirror. In $\mathbb{R}^2$, a reflection is a bijection. $\qquad\square$

Remark: Isometric circles also have this property; actually, in $\mathbb{B}^2$ isometric circle of g is identical to the perpendicular bisector $M_0(g^{-1})$.

Theorem 2.5. *Let $z_0 \in \mathbb{B}^2$. A Möbius map $g \in Isom^+(\mathbb{B}^2)$ is hyperbolic if and only if the closures of the geodesics $M_{z_0}(g)$ and $M_{z_0}(g^{-1})$ are disjoint in $\overline{\mathbb{B}^2}$.*

Similarly, elliptic and parabolic isometries exhibit a comparable feature using the same technique and proof ideas:

Corollary 2.6. *Let $z_0 \in \mathbb{B}^2$. A Möbius map $g \in Isom^+(\mathbb{B}^2)$ is parabolic if and only if the geodesics $M_{z_0}(g)$ and $M_{z_0}(g^{-1})$ have one endpoint (which is the fixed point of g) in $\partial\mathbb{B}^2$ in common. Similarly, g is elliptic if and only if $M_{z_0}(g)$ and $M_{z_0}(g^{-1})$ intersect at the fixed point of g within $\mathbb{B}^2$.*

Chapter 3

Schottky Groups

The goal of this chapter is to describe a specific collection of a certain type of functions known as Schottky groups, which is a discrete subgroup of $\mathrm{PSU}(1,1)$ with derivations of Schottky group features, after the introduction of functions was defined on the Poincaré disk.

3.1 Properly Discontinuous Actions

Historically, in Poincaré's study, the concept of discrete group arose from the concept of discontinuous group. One of Poincaré's aims was to create a non-constant meromorphic function that is automorphic in respect to a group G to which it belongs. In other words, a collection of such non-constant meromorphic functions with a group structure such that those functions become automorphic to the group, where the type of group is discontinuous. The second reason for making discontinuity on the actions of a group of Poincaré's functions an essential condition is for tessellation of

the hyperbolic space using fundamental domain[27]. If the tiling group is improperly discontinuous, an overlap is unavoidable. There will, however, be no overlaps if the group is properly discontinuous.

The group Poincaré used to build these discontinuous subgroup is $\text{Isom}^+(\mathbb{H}^2_\mathbb{R})$ represented by $\text{PSU}(1, 1)$. In brief, to be a discontinuous group, it must be a topological group first.

Topological group is defined by continuity. Recall the definition of continuous functions: If for every open set $U \subseteq G$, $g^{-1}(U)$ is open in the domain of g, then g is called *continuous. A topological group* is a group G that is also a topological space such that multiplication $(g, h) \mapsto gh$ and inversion $g \mapsto g^{-1}$ in G are both continuous functions.

Now let us go back to the beginning of this section, to understand why $\text{Isom}^+(\mathbb{H}^2_\mathbb{R})$ is topological. Consider it is isomorphic to a projective group, that is $\text{PSU}(1, 1)$. $\text{PSU}(1, 1)$ is a quotient group inherits subspace topologies from $\text{SU}(2, \mathbb{C})$.

Further, in general, for all $n \geq 2$ dimensions, define a norm on $GL(n, \mathbb{C})$:

$$|A| := \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}.$$

Thus, a metric can be defined for this space:

$$d(A, B) := |A - B|.$$

Then, for subspace topology, or quotient topology, we have

Theorem 3.1 (Theorem 5.1.2 in [28])**.** *If N is a normal subgroup of a topological group G, then G/N is a topological group with the quotient topology.*

Therefore, the following quotient groups are also topological group:

$$PGL(n, \mathbb{C}) = GL(n, \mathbb{C})/N$$

, where $N = \{kI : k \in \mathbb{C}^*\}$,

$$PSL(n, \mathbb{C}) = SL(n, \mathbb{C})/\{wI : w \text{ is an } n\text{th root of unity}\},$$

$$PSU(n) = SU/\{wI : w \text{ is an } n\text{th root of unity}\}.$$

We can see that for dimension 2 and 3, Möbius transformations can be represented by a topological matrix group $PSL(2, \mathbb{C}) := SL(2, \mathbb{C})/\{\pm I\}$. Since they are topological groups, so they are also topological spaces.

With the necessary condition of being a discontinuous group in mind, we should have a formal definition for discontinuous group[19, 28, 14]. First we need the definition of locally finite

Definition 3.1. Let $\mathcal{F} = \{M_i : i \in I\}$ be a family of subsets of a metric space X, I is an index set. Then, $\mathcal{F}$ is *locally finite* if for any compact subset $K \subset X$, $M_i \cap K \neq \emptyset$ for only finitely many $i \in I$.

By the definition of locally finite,

Definition 3.2. Let G be a topological group. The group actions of G are discontinuous on a topological space X if and only if for all orbits of G, i.e. considering $G(x)$, for all $x \in X$, the family of $G(x)$ is locally finite.

Equivalently, G acts properly discontinuous on X if G acts on X and for every compact set K of X, there exist only finitely many $g \in G$ such that

$$g(K) \cap K \neq \emptyset.$$

To begin the topic of discrete groups, examine a direct conclusion of the previous definition: if the action of a group G on X is discontinuous, then for all $x \in X$, $G(x) \subset X$ is a close discrete subset.

It is analogous to Montel's theorem if the family $\mathcal{F}$ (recall Definition 3.1) is regarded as a normal family using the following definition:

Definition 3.3. The region of discontinuity of G is denoted by $\Omega(G)$ which is of all the points in $\mathbb{B}^2 \cup \mathbb{S}^1$ where G acts discontinuously.

Remark: for each point $x \in \Omega(G)$, there is a neighborhood U_x that intersects only finitely many points in its orbit $G(x)$. Later on, once the limit set of G and ordinary set $O(G)$ are defined, $\Omega(G) = \mathbb{B}^2 \cup \mathbb{S}^1 \setminus L(G)$, and $O(G) = \Omega(G) \setminus \mathbb{B}^2$.

Now, define discrete group as follows: a subgroup G of a topological group Γ is *discrete* if G has no accumulation points in Γ. Furthermore, since the focus of

this thesis is on sense-preserving isometry groups of hyperbolic n-space, so we can further focus on discrete groups which are also sense-preserving isometry group such as $M(\mathbb{B}^2)$, or $M(\mathbb{S}^{n-1})$ (by a stereographic projection). By considering a subgroup G of $M(\mathbb{B}^2)$ (or $M(\mathbb{S}^1)$) is a topological group, i.e. a topological space equipped with the metric induced by the norm $|A| := \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}$, we define the following

Definition 3.4. A subgroup G of a topological group Γ is discrete in which the relative topology of G is discrete.

Remark: This is equivalent to say, there exists an open neighborhood U_{id} of id $\in G$ such that $G \bigcap U_{\text{id}} = \{\text{id}\}$. Or, there exists an open neighborhood U_{id} of id such that for each $A \in \Gamma$, the cardinality of the set $G \bigcap g(U_{\text{id}})$ is less than 2.

The following results can be used to relate the concepts of discontinuous groups and discrete groups: If G is a discrete subgroup of Γ, then G acts discontinuously on $\Omega(G)$. That is, if the relative subspace topology of G is given as discrete, then the action of G to Γ (i.e. $G \times \Gamma \to \Gamma$, and $g \mapsto gA$, where $g \in G$) is *properly discontinuous*.

In the following, we start to let the larger topological

$$\Gamma = \text{Isom}^+(\mathbb{H}_{\mathbb{R}}^2) \simeq \text{Isom}^+(\mathbb{B}^2)$$

Then, by using the above result, we have: If a group $G < \text{Isom}^+(\mathbb{H}_{\mathbb{R}}^2)$ is discrete if and only if G is discontinuous.

The following conditions are equivalent: (i) $G < \mathrm{Isom}^+(\mathbb{H}^2_{\mathbb{R}})$ is discrete; (ii) $\Omega(G) \neq \emptyset$; (iii) $\Omega(G) = \mathrm{int}(\mathrm{Isom}^+(\mathbb{H}^2_{\mathbb{R}}))$. A proof of this result may be found in [14, Theorem 1.2.16] which is based on results derived from an application of Arzelá-Ascoli theorem. Furthermore, Theorem 3.4 can also be proved by using this convergence property.

3.2 Limit Sets

Recalling and continuing from the end of the section3.1, there is no accumulation point inside hyperbolic space due to the necessary condition of being a discontinuous group, and since properly discontinuous group G can act improperly on the boundary, thus on the boundary of the hyperbolic space, there will be accumulations in the group orbits, and the collection set of all these accumulations is called the limit set of the group denoted by $L(G)$.

Definition 3.5 (Limit Point and Limit Set). [1] Let G be a discrete subgroup of sense-preserving isometry group $\mathrm{Isom}^+(\mathbb{B}^2)$. The *limit set* of G is denoted by $L(G)$, is the set of all accumulation points in the intersection of all orbits $G(x)$ for all $x \in \mathbb{B}^2 \cup \mathbb{S}^1_\infty$, where $\mathbb{S}^1_\infty$ is the sphere at infinity and it also represents the boundary of the hyperbolic space $\mathbb{B}^2$.

Definition 3.6. A subgroup G of $\mathrm{Isom}^+(\mathbb{B}^2)$ is elementary if and only if G has a finite orbit in the closed disk $\overline{\mathbb{B}^2}$.

We maintained the n-dimension version of this theorem since it is also valid in higher dimensions.

Theorem 3.2. *[1, P.80] Let G be non-elementary. $L(G) = L_{z_i}$ for all $z_i \in \overline{\mathbb{B}^n}$.*

Proof of Theorem 3.2.[1, 28, 15]. Case (i). This first case follows that fact the if two points z_1, z_2 are in $\mathbb{B}^2$, then $L_{z_1} = L_{z_2}$. In other words, there exists a sequence $\{g_n(z_1)\}_{n \geq 1}$, $z_1 \in \mathbb{H}^n$, converges to a point z_2 in $\partial \mathbb{B}^2 \{\infty\}$, and this point does not depend on z_1. A contradiction can be generated by assuming the opposite, and $z_2 = -i$, and let $g \in G$. Since g is an isometry, $d(gz_1, gz_2) = d(z_1, z_2)$ is fixed and for all $z_2 \in \mathbb{B}^2$. Thus $g_n(z_2)$ is a hyperbolic circle centered at $g_n(z_1)$ with ray $d(z_1, z_2)$. The Euclidean length of the diameter of this circle tends to be zero as $g_n(z_1) \to 0$, therefore, $g_n(z_2) \to 0$ as well.

Case (ii). If $z_i \in \partial \mathbb{B}^n$, and assume $G(z_i) \neq \{z_i\}$, z_i is not a fixed point of all elements in G. There exists $z_2 = g_0 z_1 \neq a_1$, $g_0 \in G$. To show $L_O \subset L_{z_1}$. Let $c \in (z_1, z_2)_h$. By case (i), since $L_c = L_O$, for every $e \in L_O$ there exists a sequence $\{g_n\}_{n \geq 1} \subset G$ such that $g_n z \to e \in L_O$.

Consider two subsequences $g_n z_1 \to z_1'$ and $g_n z_2 \to z_2'$. (a) If $z_1' = z_2'$, then by lemma 2.4, $e = z_1' = z_2'$. If $z_1' \neq z_2'$, then again by lemma 2.4, $g_n c$ converges either to z_1' or z_2'. Since $z_2 = g_0 z_1$, thus $z_1' \in \lambda_{z_1} \subset G(z_1)$ and $z_2' \in \Lambda_{z_1} \subset G(z_1)$. Therefore, $g_n c$ converges to a point e in the accumulation set $\Lambda_{z_1} \subset G(z_1)$.

For the opposite direction, take $z_3 \in \Lambda_{z_1}$ and a sequence such that $g_n z_1 \to z_3$, $g_n O \to e$, $g_n^{-1} O \to e'$. Consider the region divided by isometric circle I_g that if

$|g(z)|' = 1$ then $g(z) \in I_g$; if $|g(z)|' < 1$ then $g(z)$ is in the exterior of I_g; and if $|g(z)|' > 1$ then $g(z)$ is in the interior of I_g, then since O is in the exterior of I_{g_n}, its image $g_n O$ lies in the interior of $I_{g_n^{-1}}$. Similarly, $g_n^{-1} O$ lies in the interior of I_{g_n}. Therefore, the interior of $I_{g_n^{-1}}$ shrinks to e and the interior of I_{g_n} shrinks to e'. If $z_1 \neq e'$, then z_1 is in the exterior of I_{g_n} for large n, and since $g_n z_1$ is in the exterior of $I_{g_n}^{-1}$ which shrinks to e, hence $g_n z_1 \to e$, i.e. $z_3 = e \in L_O$. On the other hand, if $z_1 = e' \in \Lambda_O$, then $g_n z_1 \in \Lambda_O$ which is due to $g_n(\Lambda_O) = \Lambda_O$. Since Λ_O is a limit set which is closed, hence $z_3 \in \Lambda_O$. Because z_3 was arbitrarily chosen from Λ_{z_1}, thus $\Lambda_{z_1} \subset \Lambda_O$.

$\square$

Corollary 3.3. *If G is a subgroup of $PSU(1,1)$, then for each point z of $\mathbb{B}^2$, we have $L(G) = \overline{G(z)} \cap \partial \mathbb{B}^2$.*

Finally, we can have a fundamental result

Theorem 3.4. *If G is a discrete group preserving either $\mathbb{B}^2$, and $a \in \mathbb{B}^2$, then the orbit $G(a)$ can only accumulate at the sphere at infinity $\mathbb{S}^1$.*

A straightforward proof[1, p.79] follows Theorem 3.2 which is an application of isometric circles, i.e. this theorem can also be proved by using lemma 2.2.

The set of all accumulation points of $G(a_i), a_i \in \widehat{\mathbb{B}^2}$ is denoted by L_{a_i}. If we take union of all points $a_i \in \overline{\mathbb{B}^2}$, then $L(G) = \bigcup_{a_i} L_{a_i}$. The following theorem shows that it only takes one point to obtain the whole limit set $L(G)$, and it does not depend

on the choice of the point (hence $L(G)$ can be the group invariant). $L(G)$ and all accumulation sets Λ_{z_i} (derived set of $G(z_i)$) are closed, since each set has all of their limit points by definition.

Definition 3.7. The complement $\partial \mathbb{H}^n_{\mathbb{R}} \cup \{\infty\} \setminus L(G)$, or $\mathbb{S}^1_\infty \setminus L(G)$ is called *the ordinary set* denoted by $O(G)$.

We may define the following using the notion of ordinary set:

Definition 3.8. If G is a discrete subgroup of $\text{Isom}^+(\mathbb{B}^2)$, and $O(G) = \emptyset$, then G is of the first kind. Otherwise, G is of the second kind. Furthermore, if $\text{Card}(L(G)) \leq 2$, then G is elementary; otherwise, G is non-elementary.

If G is non-elementary, then there are infinite points in $L(G)$. If G is of second kind then $O(G)$ is an open dense subset of $\mathbb{S}^1_\infty$, and $L(G)$ is a nowhere dense closed subset of $\mathbb{S}^1_\infty$[28, Theorem 12.2.7].

The following are some well-known features of limit sets in $\mathbb{B}^2$, and they hold true in higher dimensions as well.

Theorem 3.5. *Let $z \in \mathbb{B}^2$. If $G < M(\mathbb{B}^2)$, then $L(G)$ have the following properties:*

1. *$L(G)$ is G-invariant,*

2. *$L(G)$ is uncountable and perfect ($L(G)$ has no isolated points), and*

3. *Either $L(G)$ is nowhere dense or $L(G) = \partial \mathbb{B}^2$.*

Proof of 1, [29]. Let $t = \lim\limits_{n \to \infty} g_n^+$. Then, $g g_n^+ \to g(t)$, and $g(g_n^+) = (g g_n g^{-1})^+$. Thus, $(g g_n g^{-1})^+ \to g(t)$. Hence, $g(t)$ is also a limit of non-elliptic fixed points, and thus $L(G)$ is G-invariance. □

Proof of 2, [29]. Let $z \in L(G)$. Then by definition, there exists a sequence $\{g_n\}_{n \geq 1}$ where $g_n \in G$ are hyperbolic such that $z = \lim g_n^+$ be an accumulation point, if g_n are all different. However, if g_n are not all distinct, then there exists a non-elliptic $g \in G$ such that $z = g^+$. Take $z' \in L(G)$, and $z' \neq z^{\pm}$, then there exists a sequence $\{h^n\}$ such that $h^n(z') \to z$ and all points $h^n(z')$ are distinct. Therefore, A is perfect. It follows from lemma 3.10 that A is uncountable. □

Proof of 3, [29]. Case (i) If $L(G) \neq \partial \mathbb{B}^2$. Let $z \in \partial \mathbb{B}^2 \setminus \Lambda$. Recall $L(G)$ is closed, there is an open set U_z such that $U_z \subset \partial \mathbb{B}^2 \setminus L(G)$. By G-invariance, i.e. by 1, $g(U_z) \cap L(G) = \emptyset$, for all $g \in G$. Let O be an open set with $O \cap L(G) \neq \emptyset$, and want to find a non-empty open set $V \subset O$ with $V \cap A = \emptyset$. Take $y \in O \cap A$. By Theorem 3.2, the accumulation points of the orbit of z is equal to $L(G)$. Hence, take $g_n \in G$ with $g_n(z) \to y$. Then, ultimately, $g_n(z) \in O$ and consequently $g_n(U_z) \cap O \neq \emptyset$. Since $g_n(U_z)$ is open and contains $g_n(z) \to y \in L(G)$. □

Corollary 3.6. *[29] If $L(G) \neq \partial \mathbb{B}^2$, then $L(G)$ is homeomorphic to the middle third Cantor set.*

Proof. Since $L(G)$ is nowhere dense, $L(G)$ is totally disconnected. Take $z, w \in L(G)$, then $\partial \mathbb{B}^2 \{z, w\}$ have two open intervals U_1 and U_2. Since $L(G)$ is nowhere dense,

in each interval U_i, there exists $v_i \in U_i$, where $v_i \notin L(G)$. Hence v_1 and v_2 separate $\partial \mathbb{B}^2$ into two open interval V_1 and V_2 each has one of z, w as interior point. Hence, $V_i \bigcup L(G)$ disconnect $L(G)$. It follows from the theorem that every perfect totally disconnected subset of $\mathbb{R}$ is homeomorphic to the middle third Cantor set. $\qquad \square$

The following definition and theorem can be used to demonstrate that a given finitely generated group is discrete. We should define a norm[1] on this set on $GL(n, \mathbb{C})$: $|A| := \left(\sum\limits_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}$. Then, we used a useful theorem[28, See Theorem 5.3.3] to check whether a given Schottky group is discrete.

Theorem 3.7. *Let $SL(n, \mathbb{C})$ be the group of complex $n \times n$ matrices whose determinant is ± 1. A subgroup G of $SL(n, \mathbb{C})$ is discrete if and only if for each $r > 0$, the set $\{A \in G : |A| \le r\}$ is finite.*

The following is a result proved by H. Weyl in 1916. Weyl introduced a function: Let $f : [0, 1] \longrightarrow [0, 1]$ be a periodic function of period 1, i.e. it is defined by mapping $x \mapsto x + \alpha \bmod 1$,

Lemma 3.8. *[29, Proposition 4.12] If $\alpha \notin \mathbb{Q}$, then every orbit is dense.*

Therefore, from the above lemma we also have one more useful result for checking if a given group is discrete:

---

[1]Further, a metric can be define on this space: $d(A, B) := |A - B|$.

Lemma 3.9. *if there exist g_i and g_j, $i \neq j$, in a given generator set of a group G, and there exists an isomorphism such that the subgroup $\langle g_i, g_j \rangle < G$ is isomorphic to $\langle a^m, b^n \rangle$ where $a, b \in \mathbb{R}$ and $\frac{m}{n} \notin \mathbb{Q}$, then G is not discrete.*

Recall a definition in Topology:

Definition 3.9. [29] A subset A of a metric space X is *nowhere dense in X*, if for all non-empty open sets $U \subset X$, there exists an open non-empty set V such that $V \cap A = \emptyset$.

Furthermore, this is equivalent to $\text{int}(\overline{A}) = \emptyset$.

Definition 3.10. [29] A subset A of a metric space X is *perfect*, if it is closed and if every point in A is an limit (accumulation) point of A, i.e. there exists a sequence $x_n \to x \in A$, and all x_n are distinct.

Lemma 3.10. *[29] A perfect subset A of a complex metric space X is uncountable.*

Proof. Let $A \subset X$ be a perfect set. Then, A is closed, and it is a complete metric space, hence Baire Category theorem[2] can be applied on A. Claim: For all $a \in A$, a is nowhere dense in A. Since $\{a\}$ is closed, we want to show $\text{int}(\{a\}) = \emptyset$. Since $\text{int}(\{a\}) \subset \{a\}$, thus if $\text{int}(\{a\}) \neq \emptyset$, then $\text{int}(\{a\}) = \{a\}$. Since interior is open, so there exists U open in X such that $\text{int}(\{a\}) = U \cap A$. However, since A is perfect, so if $U \subset X$ is open, and $a \in U$, then there exist some $y \in U \cap A$, but $y \neq a$. Thus, $\text{int}(\{a\}) = \{a\} \neq U \cap A$. $\qquad \square$

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[2]A complete metric space is not the countable union of closed subsets with empty interior (i.e. nowhere dense subsets)[5, Theorem 6.76].

3.3 Dynamics of Schottky Groups

First of all, let us define Schottky group:

Definition 3.11. Given $D_1, ..., D_{2m}$ disjoint closed disks which intersect the boundary of the unit disk $\mathbb{B}^2$ orthogonally where $m \geq 2$, we let non-elliptic $T_i \in PSU(1,1)$, identified as the linear fractional transformation, be the mapping such that

1. $T_i(\mathbb{B}^2 \setminus D_{i+m}^{\circ}) = D_i$

2. $T_i^{-1}(\mathbb{B}^2 \setminus D_i^{\circ}) = D_{i+m}$.

Then, a *Schottky group* Γ *of rank* m is finitely generated by $T_1, ..., T_m$. We will denote it by

$$\Gamma = \langle T_1, ..., T_m \rangle.$$

We denote a cyclic notation for the group generators by $T_{i+m} = T_i^{-1}$.

To further study some properties of the above definition, let us define the notion of pencil:

Definition 3.12. Let L and L' be two disjoint geodesics, and let L_0 be the common orthogonal geodesic of L and L'. We define Φ to be the family of all geodesics which are orthogonal to L_0.

Since D_i are disjoint, each Schottky group is a convex co-compact discrete subgroup of $PSU(1,1)$.

Then we can have the following properties[9, section 7.32, 7.34]:

- An isometry T_i is parabolic if and only if it can be represented as $T_i = T_{L_1} \circ T_{L_2}$ where T_{L_j} is a reflection in the geodesic L_j, and L_1 and L_2 determine a parabolic pencil, and so are parallel geodesics.

- An isometry T_i is hyperbolic if and only if it can be represented as $T_i = T_{L_1} \circ T_{L_2}$ where T_{L_j} is a reflection in the geodesic L_j, and L_1 and L_2 determine a hyperbolic pencil, and so are parallel geodesics.

Now, since for each group generator of a Schottky group, we have two conditions $T_i(\mathbb{B}^2 \setminus D_{i+m}^\circ) = D_i$ and $T_i^{-1}(\mathbb{B}^2 \setminus D_i^\circ) = D_{i+m}$, we know ∂D_i and ∂D_{i+m} are two geodesics in the pencil determined by ∂D_i and ∂D_{i+m} for T_i. Furthermore, we can see that the way to determine the same Schottky group is not unique. Because, for the same set of group generators, for each T_i, there are infinite many choices to pick two geodesics L_k and L_j in the pencil of T_i determined by L_1 and L_2 such that there are two Möbius reflections T_{L_k} and T_{L_j} compose $T_i = T_{L_k} \circ T_{L_j}$.

This can inspire us to think that what if the isometric circle of T_i is in the pencil Φ now that we have knew several useful properties of isometric circles. So, we have the following theorem for Schottky group (originally it is a definition of a restricted type of Schottky group that has several useful properties due to the way Γ is generated in [15]):

Theorem 3.11. *Let $\{T_i : 1 \leq i \leq m, m \geq 2\}$ be a set of non-elliptic isometries of $PSU(1,1)$ and each T_i and T_j, $i \neq j$, do not have a common fixed point. Take $z_0 \in \mathbb{B}^2$ be a point that does not fixed by each T_i. Then, if for each T_i we denote the half-spaces*

$D_0(T_i^\pm) := D(T_i^\pm)$, *for* $i = 1, ..., m$ *satisfy* $\left(\overline{D(T_i) \cup D(T_i^{-1})}\right) \cap \left(\overline{D(T_j) \cup D(T_j^{-1})}\right) =$

$\emptyset, \forall i \neq j \in \{1, ..., m\}$, *then* $\langle T_i, 1 \leq i \leq m \rangle$ *is a Schottky group.*

Proof. By lemma 2.4, we have $T_i(\mathbb{B}^2 \backslash D_{i+m}^\circ) = D_i$, and because $\left(\overline{D(T_i) \cup D(T_i^{-1})}\right) \cap$

$\left(\overline{D(T_j) \cup D(T_j^{-1})}\right) = \emptyset$, we have $T_i^{-1}\left(\mathbb{B}^2 \backslash D_i^\circ\right) = D_{i+m}$. $\qquad\square$

Remark: The converse is not true. Considering we are in $\mathbb{B}^2$, the perpendicular bisector that bounds $D_0(T_i^{-1})$ is actually the isometric circle of T_i[9, section 7.22]. Furthermore, if we let $T_i = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}$, then the radius of the perpendicular bisector (isometric circle) of T_i is $\frac{1}{|\gamma|}$ which is the same as the radius of the perpendicular bisector (isometric circle) of T_i^{-1}, i.e. the radii of ∂D_i and ∂D_{i+m} are identical. However, in the definition of Schottky group, ∂D_i and ∂D_{i+m} can have distinct radii which results in a configuration that does not have the symmetry that Theorem 3.11 has.

Additionally, compared to definition 3.11, in [10], Schottky group is defined on $\mathbb{H}^2$, with the center of each D_i on real line. Now, let us prove that when a Schottky group is generated by conditions stated in Theorem 3.11,

Lemma 3.12. *it is guaranteed that the center of each isometric circle of T_i must not on the unit circle $\partial \mathbb{B}^2$.*

Proof. Since for T_i we have $|\alpha|^2 - |\gamma|^2 = 1$, hence $|\alpha| > |\gamma|$, and this implies

$$\left|\frac{\alpha}{\gamma}\right| > 1.$$

Recall $\left|\frac{\alpha}{\gamma}\right|$ is the modulus of the center of the isometric circle of T_i and T_i^{-1}. $\qquad\square$

Let $S(g_1, ..., g_p)$ denote a Schottky group generated by $\{g_1, .., g_p\}$.

The following lemma demonstrates that we only need two non-elliptic isometries to form a Schottky group.

Lemma 3.13 (Dal'Bo). *Let g and g' be two non-elliptic isometries in G which have no common fixed points. Then there exists $N > 0$ such that g^N and g'^N generate a Schottky group $S(g^N, g'^N)$.*

Proof. This follows directly from lemma 2.4, 2.4 and 2.5. $\qquad\square$

Proposition 3.14. *Let $S(g_1, g_2)$ be a Schottky group. Then,*

$$L(S(g_1, g_2)) = \bigcap_{n=1}^{\infty} \bigcup \overline{D(s_1, s_2, ..., s_n)}.$$

This proposition can be proved by using Theorem 2.5 and lemma 2.4.

Chapter 4

Poincaré Series

The goal of this chapter is to study the critical exponent of Poincaré series and Sullivan's theorem, which proves that the critical exponent of Poincaré series of every Schottky group equals the Hausdorff dimension of limit sets of Schottky groups. In addition, this chapter investigates at certain properties of Poincaré exponents, such as how it is an invariant under a conformal transformation and how it is independent of the point z selected to be the input of the Poincaré series.

Let $g_1, ..., g_m$ be isometries of $\mathrm{PSU}(1,1)$ on $\mathbb{B}^2$ such that $G = \langle g_1, ..., g_m \rangle$ generates a classical Schottky group[1] of rank m.

Our goal is to compute the exponent of convergence of the Poincare series for the Schottky group.

$$\mathcal{P}(G, t) = \sum_{g \in G} e^{-t\rho_{\mathbb{B}^2}^2(z, gw)}.$$

---

[1]It is a classical Schottky group acts on two dimensional hyperbolic space $\mathbb{B}^2$, and it is a discrete subgroup of isometries of the space.

where z and w are in $\mathbb{B}^2$, and d is the hyperbolic metric on $\mathbb{B}^2$. Here, the exponent of convergence is defined to be

$$\delta(G) = \inf\{t > 0 : P(G,t) < \infty\} = \sup\{t > 0 : P(G,t) = \infty\}.$$

Remark: Recall the definition of Schottky group in Definition 3.11. If $G = \Gamma$ is a Schottky group, then by Patterson and Sullivan theory, $\delta(\Gamma)$ is equal to the Hausdorff dimension of its limit set.

To show $\delta(G)$ does not depend on the choice of z and w, let us recall triangular inequalities, reverse triangular inequalities, and temporarily write Poincaré series as a series that has four arguments G, t, z and w: $\mathcal{P}(G,t,z,w)$. Hence,

$$\rho_{\mathbb{B}^2}(z, Tw) \leq \rho_{\mathbb{B}^2}(z,w) + \rho_{\mathbb{B}^2}(w, Tw)$$

and

$$\rho_{\mathbb{B}^2}(z, Tw) \geq \rho_{\mathbb{B}^2}(w, Tw) - \rho_{\mathbb{B}^2}(z,w)$$

gives

$$e^{-t\rho_{\mathbb{B}^2}(z,w)}\mathcal{P}(G,t,z,z) \leq \mathcal{P}(G,t,z,w) \leq e^{t\rho_{\mathbb{B}^2}(z,w)}\mathcal{P}(G,t,z,z).$$

It follows that $\delta(G)$ does not depend on z or w, because the factors $e^{t\rho_{\mathbb{B}^2}(z,w)}$ and $e^{-t\rho_{\mathbb{B}^2}(z,w)}$ do not affect the convergent behavior of $\mathcal{P}(G,t)$, and it is bounded by $\mathcal{P}(G,t,z,z)$. Thus, from now on, we let $z = w$ and $z = 0$. The problem turns

into estimating $\mathcal{P}(G,t) := \sum_{g_i \in G} e^{-t\rho_{\mathbb{B}^2}(0,g_i 0)}$, where $0 \neq g_i 0 \in \mathbb{B}^2$. The corner case is when there is a generator $g_i \in G$ which is elliptic and fixed 0. However, this can be modified to $0 + a$, $a \in \mathbb{B}^2$ and $a \neq 0$ by applying a conjugation on G. Thus, we can assume the group does not fix 0, since if it does we can use conjugation to modify it.

The following lemma was modified from a result in [17, Section 47, P.107]:

Lemma 4.1. *If $G < PSU(1,1)$, and $T_n \in G$, $T_n = \begin{pmatrix} a_n & \overline{b_n} \\ b_n & \overline{a_n} \end{pmatrix}$, and take $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSU(1,1)$ where $\gamma \neq 0$, i.e. $\frac{\beta}{\delta-\alpha}$ is an ordinary point of A such that $A\frac{\beta}{\delta-\alpha} \neq \frac{\beta}{\delta-\alpha}$, such that there exists $T' = AT_n A^{-1} := \begin{pmatrix} a'_n & b'_n \\ c'_n & d'_n \end{pmatrix} \in PSU(1,1)$, then the alternative series of Poincaré series $\sum |b_n|^{-2\delta}$ and $\sum |c'_n|^{-2\delta}$ both converge or both diverge, where $\delta(G) = \delta(AGA^{-1})$.*

Proof. It is a direct application of isometric circles. $\qquad\square$

Since the proof of the following theorem does not depend on dimension of the hyperbolic space, it should be true in general n-dimensional hyperbolic space which will be introduced in later chapters.

Theorem 4.2. *Let G be a Schottky group. Take $A \in Isom^+(\mathbb{B}^n)$. Denote $\delta(G)$ be the exponent of convergence of $\mathcal{P}(G,t)$, and $\delta(\Gamma)$ be the Poincaré exponent of $\mathcal{P}(\Gamma,t)$, where $\Gamma = AGA^{-1}$. Then, $\delta(G) = \delta(\Gamma)$.*

Proof. Let $T_i = Ag_iA^{-1}$ By lemma 4.1, and since $\sum_{g_i \in G < \text{Isom}^+(\mathbb{B}^2)} (1 - g_i z)^s$, $\sum_{g_i \in G < \text{Isom}^+(\mathbb{B}^2)} e^{-s\rho_{\mathbb{B}^2}(z,g_i z)}$ and $\sum_{g_i \in G < \text{Isom}^+(\mathbb{B}^2)} |g_i'(z)|^s$ all have the same convergent behavior, in n-dimension with the same type of functions, let $\text{arccosh}(x_{g_i}) = \rho_{\mathbb{B}^2}(z, g_i z)$, and $\text{arccosh}(x_{T_i}) = \rho_{\mathbb{B}^2}(z, T_i z)$, then since they are both inverse hyperbolic cosine functions, the two summands of the two series in n-dimension are $\frac{e^{-s \cdot \cosh^{-1}(x_{g_i})}}{e^{-s \cdot \cosh^{-1}(x_{T_i})}} = \frac{f_s(x_{g_i})}{f_s(x_{T_i})} \longrightarrow 1 \in (0, \infty)$ as $i \to \infty$. Thus, by applying Limit Comparison Test, two series both converge or diverge. Hence, we complete the proof. $\square$

Furthermore, not only the images of two limit sets after the transformation are known (by using Theorem 3.5 $L(AGA^{-1}) = A(L(G))$), but the Hausdorff dimensions of the two limit sets are also known (since $\dim_H (L(G)) = \delta(G) = \delta(AGA^{-1}) = \dim_H (L(AGA^{-1}))$ by using the above lemma and Theorem 4.3).

Our next goal is to prove that let G be a Schottky group, then the Hausdorff dimension of $L(G)$ is the Ponincaré exponent $\delta(G)$ of the Poincaré series.

Firstly, we have to define Hausdorff measure:

Definition 4.1. For $A \subset \partial\mathbb{B}^2$, the $s-$dimensional Hausdorff measure is

$$H^s(A) := \lim_{\epsilon \to 0} \inf \left\{ \sum_j |I_j|^s : A \subset \cup_j I_j, |I_j| < \epsilon \right\}.$$

where $|\cdot|$ denotes the Euclidean arc length in $\partial\mathbb{B}^2$.

For some threshold α,

$$H^s(A) = \begin{cases} 0, & \text{if } s < \alpha, \\ \infty, & \text{if } s > \alpha. \end{cases}$$

and the Hausdorff dimension is

$$\dim_H A := \alpha.$$

Secondly, we define a probability measure:

$$\mu^{(s)} := \frac{\sum\limits_{T \in G} e^{-s\rho_{\mathbb{B}^2}(0,T0)} \nu_{T0}}{\sum\limits_{T \in G} e^{-s\rho_{\mathbb{B}^2}(0,T0)}}$$

where ν_z is the point measure at $z \in \mathbb{B}^2$ with total mass one.

Recall the Banach-Alaoglu theorem, i.e. the closed unit ball of the dual space (the space of Borel measures on $\overline{\mathbb{B}^2}$ with norm given by the total mass) of a normed vector space is compact in the weak$-^*$topology, we can have the limiting measure as follows:

Definition 4.2. There exists a sequence $s_j \to \delta$ such that $\mu^{(s_j)}$ converge weakly to a limiting measure, and this measure is called the Patterson-Sullivan measure associated to G:

$$\mu := \lim_{s_j \to \delta} \mu^{(s_j)}.$$

Patterson[26] proves that

$$\mathcal{P}(G,t) > \frac{C}{t - \delta}$$

where C is a constant, meaning that Poincaré series diverges at $t = \delta$.

By Theorem 1 in [32]:

Theorem 4.3. *Let G be a finitely generated Schottky group. The Patterson-Sullivan measure μ is a constant multiple of the Hausdorff measure $H^\delta|_{\Lambda(G)}$. Furthermore, $\dim_H \Lambda(G) = \delta$.*

Proof idea:

For $q \in L(G)$, let I_q denote an interval in $\partial\mathbb{B}^2$ centered at q. Based on Sullivan's shadow lemma[10][Lemma 14.12], it can be showed that if there exists $\epsilon > 0$ such that for any $|I_q| < \epsilon$, we have

$$\mu\left(I_q\right) \asymp |I_q|^\delta,$$

uniformly in q. If $T \in G$, the the pullback measure $T^*\mu$ is defined by $T^*\mu(E) := \mu(TE)$. Then, on one hand, since for a Möbius transformation $T \in G$ the local distortion of Euclidean length is given by $|T'|$, thus $T^*H^s = |T'|^s H^s$. On the other hand, for Patterson-Sullivan measure, for $T \in G$, we have $T^*\mu = |T'|^\delta \mu$ [10][Lemma 14.2]. This hints that Patterson-Sullivan measure μ transforms under the action of G like the Hausdorff measure of dimension δ. Since μ is absolutely continuous with respect to H^δ, we have $d\mu = kdH^\delta$ for some function k on $L(G)$. By ergodicity

[10][Corollary 14.11], k is constant. Recall the definition of Hausdorff measure,

$$H^s(A) := \lim_{\epsilon \to 0} \inf \left\{ \sum_j |I_j|^s : A \subset_j I_j, |I_j| < \epsilon \right\}$$

where $A \subset L(G)$.

Furthermore, we know that when $s \to \delta$, we have $\mu(L(G)) = 1$. Hence, we have

$$\int_{L(G)} d\mu = \lim_{s \to \delta} \mu^{(s)}(L(G)) = \mu(L(G)) = 1 = k \int_{L(G)} dH^\delta.$$

If $s > \delta$, then we have $\sum |I_q|^s \to \infty$, so $H^s(L(G)) \to \infty$; and if $s < \delta$, we have $\sum |I_q|^s \to 0$, thus $H^s(L(G)) \to 0$. Therefore, δ is the Hausdorff dimension of $L(G)$.

Chapter 5

Bounds of the Hausdorff dimension of Limit Sets of Well-Distributed Schottky Groups

The purpose of this chapter is to investigate a specific example of Schottky groups that can satisfy our well-distributed configuration definition. Our main theorem in this chapter sets tight constraints on rank two well-distributed Schottky groups. According to our knowledge, this is maybe the first time in history that we have such bounds for Schottky groups other than the case when the Hausdorff dimension approaches the unit one. We also present a conjecture for the exact form of a general formula for the Hausdorff dimension of the limit set of a rank two well-distributed Schottky group in this chapter, hinted by our derivations of the main theorem.

5.1 Lower Bound for Poincaré series

Using the following proposition, we aim to prove that there exists a lower bound for any Schottky group Γ.

Proposition 5.1. *Let*

$$d_{\max} = \max_{i=1,\dots,m} \rho_{\mathbb{B}^2}(0, T_i(0)).$$

Then $\delta(\Gamma) \geq \frac{\ln(2m-1)}{d_{\max}}$.

Proof. We know that Schottky group is a free group. Therefore, the group Γ are all the reduced words generated by T_i, and Γ_n denotes the set of all reduced words of length n. We can find that $\#\Gamma_n \asymp (2m-1)^n$ by writing out the tree of the orbit $\Gamma(0)$, and see the pattern[1] that there must be $(2m-1)^n$ for each n. We also notice that if $T \in \Gamma_n$, then we can write $T = T_{i_1}....T_{i_n}$ where $T_{i_j} \neq T_{i_{j+1}}^{-1}$ and $T_{i_n} \in \{T_1, ..., T_m, T_1^{-1}, ..., T_m^{-1}\}$. We have

$$\rho_{\mathbb{B}^2}(0, T(0)) \leq \rho_{\mathbb{B}^2}(0, T_{i_1}(0)) + \rho_{\mathbb{B}^2}(T_{i_1}(0), T_{i_1}T_{i_2}(0)) + ... \tag{5.1}$$
$$+ \rho_{\mathbb{B}^2}(T_{i_1}...T_{i_{n-1}}(0), T_{i_1}...T_{i_n}(0)) \leq n d_{\max}$$

by the fact that T_i are isometries in the hyperbolic space. Hence, combining the above information,

$$\mathcal{P}(\Gamma, t) = \sum_{n=1}^{\infty} \sum_{T \in \Gamma_n} e^{-t\rho_{\mathbb{B}^2}(0,T0)} \tag{5.2}$$
$$\geq c \cdot \sum_{n=1}^{\infty} (2m-1)^n e^{-tnd_{\max}}.$$

---

[1]The $(2m-1)$ factor is resulting from the inverse of each parent node.

The above sum diverges if $(2m-1)e^{-td_{\max}} \geq 1$, which means that $t \leq \frac{\ln(2m-1)}{d_{\max}}$. As $\delta(\Gamma) \geq t$, our desired result follows by taking $t \to \frac{\ln(2m-1)}{d_{\max}}$. The proof is complete.

$\square$

Now we have a lower bound

$$\delta'(\Gamma) = \frac{\ln(2m-1)}{d_{\max}} \leq \delta(\Gamma) = \dim_H(L(\Gamma)).$$

Since every Schottky group G contains an unlimited number of Schottky subgroups, we may construct the biggest Schottky subgroup Γ of a given Schottky group G and utilize the lower bound $\delta'(\Gamma)$ to represent the lower bound of $\delta(G)$.

5.2 The Least Upper Bound for Poincaré series

We need a more rigorous setup for $T_i(0)$ to obtain an upper bound for the $\delta(\Gamma)$. The following is our definition.

Definition 5.1. We say that Γ is *well-distributed* Schottky group of rank m with generators $T_1, ..., T_m$, if the following holds.

(1) $d(0, T_i 0)$ are equal to each other for all $i = 1, ..., m$.

(2) Let $r = d(0, T_i 0)$ be the number defined in (1). The argument of the complex numbers $\{T_i(0) : i = 1, ..., 2m\}$, arranged in ascending order, are equal angle apart in the circle $|z| = r$.

(3) Half-spaces $D_0(T_i)$ are closed disjoint disks.

Secondly, it will be useful if there is at least one pair of perpendicular bisectors of T_i and its inverse T_i^{-1} are located next to each other.

Definition 5.2. Let Γ be a well-distributed Schottky group of rank m. If there exists at least one pair of geodesics $[0, T_i 0]$ and $[0, T_i^{-1} 0]$ that are located next to each other, where T_i is a group generator of a Schottky group Γ, then Γ is *specially well-distributed.*

Let us now define some hyperbolic geometry trigonometric notations.

1. $[A, B]$ denotes the geodesic joining the point A, B.

2. $\triangle[A, B, C]$ denotes the hyperbolic triangle with vertices A, B, C

3. $\angle[A,\ B,\ C]$ denotes the angle between the geodesic $[A, B]$ and $[B, C]$.

To continue, we define some useful notations as follows.

Definition 5.3. Fix a Schottky group $\Gamma(T_1, T_2, ..., T_m)$ of rank m. The *alphabet* of this group is the set $\mathcal{A} = \left\{ T_1^{\pm}, T_2^{\pm}, ..., T_m^{\pm} \right\}$.

Then, we define three notions: letter, word, and word length:

Definition 5.4. For each j such that $T_{i_j} \in \mathcal{A}$, i is called a *letter.*

If $n > 1$, a product of n letters $T_{i_1} T_{i_2} ... T_{i_n}$, for each $T_{i_j} \in \mathcal{A}$, $i_1 i_2 \cdots i_n$ is called a *word* of $\Gamma(T_1, T_2, ..., T_m)$.

The integer n is named the *length* of the reduced word

$$T_{i_1} T_{i_2} ... T_{i_n} \in \Gamma(T_1, T_2, ..., T_m)$$

denoted by $\text{len}(T_{i_1} ... T_{i_n}) = n$.

Now, let us define the notion of reduced word:

Definition 5.5. A product of n letters $T_{i_1} T_{i_2} ... T_{i_n}$ in $\mathcal{A}$ is named a *reduced word* of $\Gamma(T_1, T_2, ..., T_m)$, if $n > 1$ and for each *letter* in the *reduced word* we have $T_{i_j} \neq T_{i_{j+1}}^{-1}$ for all $1 \leq j \leq n - 1$.

For instance, fix a Schottky group $\Gamma(T_1, T_2, T_3)$. Then, $T_1 T_2 T_1^{-1} T_1^{-1} T_3$ is a reduced word of length 5. Here is a non-example: Consider that $T_1 T_1^{-1} T_2$ is not a reduced word, since $T_1 T_1^{-1} = \text{id}$, i.e. in this non-example, we have $T_{i_1} = T_1$, $T_{i_2} = T_1^{-1}$, and $T_{i_3} = T_2$. In this way, since $T_{i_1} = T_{i_2}^{-1}$, $T_1 T_1^{-1} T_2$ is not a reduced word, but T_2 is a reduced word of the word $T_1 T_1^{-1} T_2$.

Remark: By Definition 3.11, if $T_k \in \{T_1, T_2, ..., T_m\}$, i.e. T_k is a generator of Γ and k is in the index set $\{1, 2, ..., m\}$, then $T_k^{-1} = T_{k+m}$. Therefore, in a reduced word $T = T_{i_1} T_{i_2} ... T_{i_n}$, if a letter $T_{i_j} = T_k$, then the condition $T_{i_j} \neq T_{i_{j+1}}^{-1}$ in Definition 5.5 is equivalent to $T_{i_j} \neq T_{k+m}$.

Each T_{i_j} is taken from the set $\mathcal{A} = \{T_j^k : 1 \leq j \leq m, k = \pm 1\} = \mathcal{A}$, i.e. the set of all generators of Γ and their inverses which equals the alphabet of Γ.

For the inverses, by Definition 3.11, each element T_j in the set $\mathcal{A}$ is labeled by an integer i_j, and the inverse of T_j is labeled by i_{j+m}, i.e. we have the label $T_k^{-1} = T_{k+m}$.

Now, we can define our dictionary and a cross product abbreviation for each reduced word $T_{i_1}T_{i_2}...T_{i_n}$:

Definition 5.6. Fix a Schottky group $\Gamma(T_1, T_2, ..., T_m)$ of rank m. Let $\mathcal{W}_n$ be the set of all possible reduced words $T = T_{i_1}T_{i_2}...T_{i_n}$, where each $i_j \in \{1, 2, ..., 2m\}$ i.e. $\operatorname{len}(T) = n$. Then, the dictionary of Γ is

$$\bigcup_n W_n,$$

and the cross product abbreviation of $T_{i_1}T_{i_2}...T_{i_n}$ is by defining the index I as a cross product of n alphabets: $I \in \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3... \times \mathcal{A}_n$, and each $i_j \in \mathcal{A}_j$. Therefore, we can denote

$$T_I := T_{i_1}T_{i_2}...T_{i_n},$$

where

$$I = (i_1, i_2, ..., i_n).$$

We will write $T_I = T = T_{i_1}T_{i_2}...T_{i_n}$ and $T_I 0 = T_I(0)$. Furthermore, we have $\#\mathcal{W}_n = m(2m-1)^{n-1}$.

Let $T_I \in \mathcal{W}_n$ and define T_I^- be the *word* for which the last *letter* of T_I is deleted. In other words, $T_I = T_{i_1}T_{i_2}...T_{i_n} \in \mathcal{W}_n$ and define $T_I^- = T_{i_1}T_{i_2}...T_{i_{n-1}}$ be the word for which the last letter of T_I is deleted.

Lemma 5.2. *Let T_I and T_I^- be given. Then, for any T_{i_j} in the generator set $\{T_1, ..., T_m\}$ such that $T_{i_{j+1}} \neq i_j^{-1}$, and $T_{i_{n+1}} \neq T_{i_n}^{-1}$, we have*

$$\angle[T_I\text{-}0,\ T_I0,\ T_I T_{i_{n+1}}0] = \angle[T_{i_n}^{-1}0,\ 0,\ T_{i_{n+1}}0].$$

In particular, this angle is equal to $\frac{j\pi}{m}$ for some $j = 1, 2, ..., m-1$ if Γ is a well-distributed Schottky group.

Proof. Apply T_I^{-1} to $\triangle[T_I\text{-}0,\ T_I0,\ T_I T_{i_{n+1}}0]$. Its image will be $\triangle[T_{i_n}^{-1}0,\ 0,\ T_{i_{n+1}}0]$. However, given T_I^{-1} is a conformal map, the angle must be preserved. The second statement follows immediately from the second condition of well-distributed Schottky group. $\qquad\square$

Lemma 5.3. *Let $T_I \in \mathcal{W}_n$ and Γ is a well-distributed Schottky group. Then*

$$0 < \angle[0,\ T_I0,\ T_I\text{-}0] < \frac{\pi}{m}.$$

Proof. Apply T_I^{-1} to $\triangle[0,\ T_I0,\ T_I\text{-}0]$, its image will be $\triangle[T_{i_n}^{-1}T_{I-}^{-1}0,\ 0,\ T_{i_n}^{-1}0]$. But T_I^{-1} is a conformal map, the angle

$$\angle[0, T_I0, T_I\text{-}0] = \angle[T_{i_n}^{-1}T_{I-}^{-1}0,\ 0,\ T_{i_n}^{-1}0].$$

As $T_{i_n}^{-1}T_{I^-}^{-1}0$ and $T_{i_n}^{-1}0$ are contained in the closed disk $D_{T_{i_n}^{-1}}$ and the geodesic

$$[0, T_{i_n}^{-1}T_{I^-}^{-1}0], \text{ and } [0, T_{i_n}^{-1}0]$$

are straight line, the angle must be less than all $\angle[T_{i_j}0, \ 0, \ T_{i_k}0]$ for all $i_j \neq i_k = 1, ..., 2m$, i.e. T_{i_j} and T_{i_k} are group generators or their inverses. Therefore, by the fact that Γ is well-distributed, $\angle[0, \ T_I0, \ T_{I^-}0] < \pi/m$. It is positive because $T_{i_n}^{-1}T_{I^-}^{-1}0 \neq T_{i_n}^{-1}0$. $\qquad \square$

Lemma 5.4. *Let $T_I \in \mathcal{W}_n$ and Γ is a well-distributed Schottky group. Let $x_I = \rho_{\mathbb{B}^2}(0, T_I0)$,*

$$\ell_I = \min\{\rho_{\mathbb{B}^2}(0, T_IT_{i_{n+1}}0) : T_IT_{i_{n+1}} \in \mathcal{W}_{n+1}\}$$

and $\theta_I = \angle[0, \ T_I0, \ T_{I^-}0]$. Then

$$\cosh \ell_I = \cosh x_I \cosh r - \sinh x_I \sinh r \cos\left(\frac{\pi}{m} - \theta_I\right).$$

Proof. Consider the $\angle[0, \ T_IT_{i_{n+1}}0, \ T_I0]$. Applying the hyperbolic cosine law to this triangle, we obtain

$$\cosh x_{I,T_{i_{n+1}}} = \cosh x_I \cosh r - \sinh x_I \sinh r \cos\alpha_{Ii_{n+1}}.$$

where

$$\alpha_{Ii_{n+1}} = \angle[0,\ T_I 0,\ T_{Ii_{n+1}} 0]$$

$$= \angle[T_{I^-} 0,\ T_I 0,\ T_{Ii_{n+1}} 0] - \angle[0,\ T_I 0,\ T_{I^-} 0]$$

$$= \frac{j\pi}{m} - \theta_I$$

for some $j = 1, 2, ..., 2m - 1$ using Lemma 5.2 and Lemma 5.3. The lemma now follows by noticing that $\cos \alpha_{Ii_{n+1}}$ achieves its maximum when $j = 1$. $\square$

Definition 5.7. A path in $\mathbb{B}^2$ is a continuous map h from the closed interval $[a, b]$ to $\mathbb{B}^2$.

Definition 5.8. A path of a well-distributed Schottky group, $\mathcal{L}(\Gamma)$, is a continuous function with an image that is connected by infinite many geodesics in a hyperbolic space $\mathbb{B}^2$, starting from the origin, and endpoints of each such geodesic are in the set of the orbit of the origin, i.e. $\Gamma(0)$. Furthermore, a path of Γ constructed by $n + 1$ nodes is denoted as follows

$$\mathcal{L}(\Gamma) := [0, T_{i_1} 0, T_{i_1} T_{i_2} 0, ..., T_{i_1}...T_{i_n} 0].$$

Consider the following particular instance of a well-distributed Schottky group:

Definition 5.9. Let Γ be a well-distributed Schottky group. If when $\text{len}(T_{i_j}) = 1$ for all i_j in the alphabet $\mathcal{A}(\Gamma)$, then we have $\angle[T_{i_j} 0, 0, T_{i_j}^{-1} 0] = \pi$, and Γ is called specially well-distributed.

The next step is to illustrate several examples of this sort of Schottky group by charting their $\Gamma(0)$ orbit.

Let Γ be a subgroup of $\mathrm{PSU}(1,1)$ and be a well-distributed Schottky group with m generators and its orbit $\Gamma(0)$ can be constructed only using two operators T, and R.

To construct the operator T, we start from the upper-half plane model by defining $T' \in \mathrm{PSL}(2,\mathbb{R})$ to be a hyperbolic isometry as follows

$$T' := \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix},$$

where T' fixes $\{0,\infty\}$, and without loss the generality, we assume $\Lambda < 1$.

Recall the Cayley transformation $\Psi(z) : \mathbb{H}^2 \to \mathbb{B}^2$,

$$\Psi(z) := \frac{iz + 1}{z + i},$$

then after normalizing its matrix representation, we have the matrix operator

$$\Psi = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}.$$

Thus, the operator T can be derived by using Ψ:

$$T := \Psi T' \Psi^{-1} = \frac{-1}{2} \begin{pmatrix} -\Lambda - \Lambda^{-1} & -i\Lambda + i\Lambda^{-1} \\ i\Lambda - i\Lambda^{-1} & -\Lambda - \Lambda^{-1} \end{pmatrix}.$$

Then, it can be verified that $T(\pm i) = \pm i$, and that these are the attracting and repelling fixed points of T. By exploring only one scaling (dilation) parameter Λ, we may explore dynamical motion (i.e. a point in this space is moved by operating T on it for $n-$many times).

To construct the rotation operator R, we would like it to be the function $\mathbb{R}(z) = e^{i\theta} z$ which is a counter-clockwise rotating operation, where $\theta := \frac{\pi}{m}$.

Then, we can define R as follows

$$R := \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{-i\theta}{2}} \end{pmatrix} \in \mathrm{PSU}(1,1)$$

which fixes 0, and its upper-half plane conjugate $\Psi^{-1} R \Psi$ fixes $+i$ in the upper-half plane.

Then, we can have two methods to plot $\Gamma(0)$. Both of them give the same output.

Method 1: Recall Γ is well-distributed. Assume $z \in \mathcal{R}(\Gamma, T_{i_1}) \cap \Gamma(0)$, and $T_{i_1} z = R T_{i_j} z$. Instead of using T_{i_1} to derive $T_{i_1} z$, we want to use T_{i_j}. Hence, firstly use $R^{-1} z$ to map z to $\mathcal{R}(\Gamma, T_{i_j}) \cap \Gamma(0)$. Then, we apply T_{i_j} on it, and use R to map it back to $\mathcal{R}(\Gamma, T_{i_1}) \cap \Gamma(0)$. Since each point in $\Gamma(0)$ contains by $D_0(T_{i_j})$ can be obtained

by using elliptic isometry to derive by mapping a related point from another sector $D_0(T_{i_k}), k \neq j$.

Thus, instead of considering the generators $T_1, .., T_m, T_1^{-1}, ..., T_m^{-1}$, we have:

$$\Gamma = \left\langle T_{i_j}, RT_{i_j}R^{-1}, ...R^{m-1}T_{i_j}R^{-(m-1)} \right\rangle,$$

and for simplicity we can let $T = T_{i_j}$, then

$$G' := \left\langle T, RTR^{-1}, ...R^{m-1}TR^{-(m-1)} \right\rangle$$

Then, we can use $G'(0) \cap \Gamma(0)$ to reconstruct $\Gamma(0)$.

Method 2: Assume we already know $\Gamma(0)$. At the highest level, Γ is reconstructed counter-clockwisely and level by level. For $N = 1$, we start with $T_{i_j}0$, and map it to $RT_{i_j}0$, $R^2T_{i_j}0$,..., $R^kT_{i_j}0$, ..., $R^{2m-1}T_{i_j}0$. For $N = 2$, firstly by using $\Gamma(0)$, apply T_{i_j} at each $T_I0 = R^kT_{i_j}0$ where $\text{len}(T_I) = 1, T_I \neq T_{i_j}$, then we can determine each node in the second level in $\mathcal{R}(\Gamma, T_{i_j})$. Next, use R^k to map all these nodes to $\mathcal{R}(\Gamma, T_{i_a}), a \neq j, a \in \{1, ..., 2m\}$. Assume at $N = K$, all nodes in $\mathcal{R}(\Gamma, T_{i_j})$ and the remaining $\mathcal{R}(\Gamma, T_{i_a})$ are given. Then, for $N = K+1$ level, we use $\Gamma(0)$ to see where each $T_{i_j}T_I0$ goes, when $\text{len}(T_I) = K$. Then, we use the R^k to map those $(2m-1)^{K+1}$ nodes represented only by operators R and T_{i_j} in $\mathcal{R}(\Gamma, T_{i_j})$ to $\mathcal{R}(\Gamma, T_{i_a})$.

Either way, since T fixes $\pm i$, so we start to plot $\Gamma(0)$ from the sector that is in the lower half-plane for all m. We could like to find the angle between the intercepts

of the half-space $D_0(T)$ and the unit circle $\partial \mathbb{B}^2$.

Consider the picture of this perpendicular bisector of the map $\Psi^{-1}(z)$ on $\mathbb{H}^2$ to answer this problem. Let the intercept of the positive x-axis on $\mathbb{H}^2$ be $(k, 0)$. Then, because $\Psi^{-1}(D_0(T))$ is a circle and orthogonal to the x-axis, it intersects the positive y-axis at $(0, k)$.

Now, $T'i = \Lambda^2 i$, and by the definition of perpendicular bisector, $(0, k)$ must be the midpoint of $(0, 1)$ and $(0, \Lambda^2)$ in the sense of hyperbolic metric.

Then by solving the equation

$$\ln\left(\frac{k}{\Lambda^2}\right) = \ln\left(\frac{1}{k}\right),$$

we can have

$$k = \Lambda.$$

On the other hand, on $\mathbb{B}^2$ since we have

$$\Psi((\Lambda, 0)) = \left(\cos\left(\frac{\theta}{m}\right), \sin\left(\frac{\theta}{m}\right)\right).$$

Hence, we can easily find Λ by specifying θ.

For example, let $m = 2$, if we want to find Λ when $\theta = \frac{\pi}{2}$, we have

$$\Psi^{-1}\left(\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)\right) = \left(\frac{\sqrt{2}}{2 + \sqrt{2}}, 0\right) \in \mathbb{C}.$$

To verify this, in the following, we utilize method 2 to plot the scenario where $m = 2$ to $N = 14$, i.e. each shortened word has a word length of at most 14.
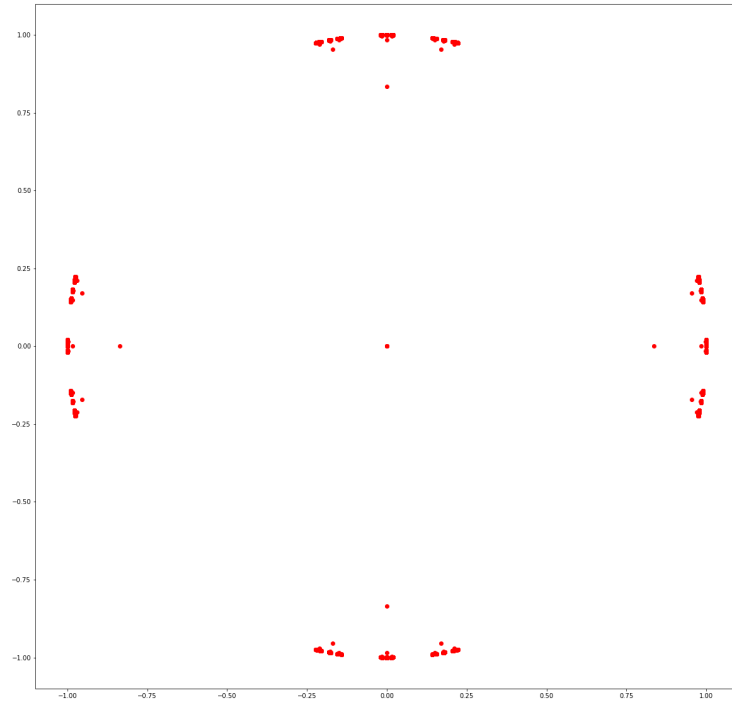
Figure 5.1: $m = 2$, $\Lambda = 0.3$, $\theta \approx 33.398473447277695°$, Level 14 ($N = 14$).
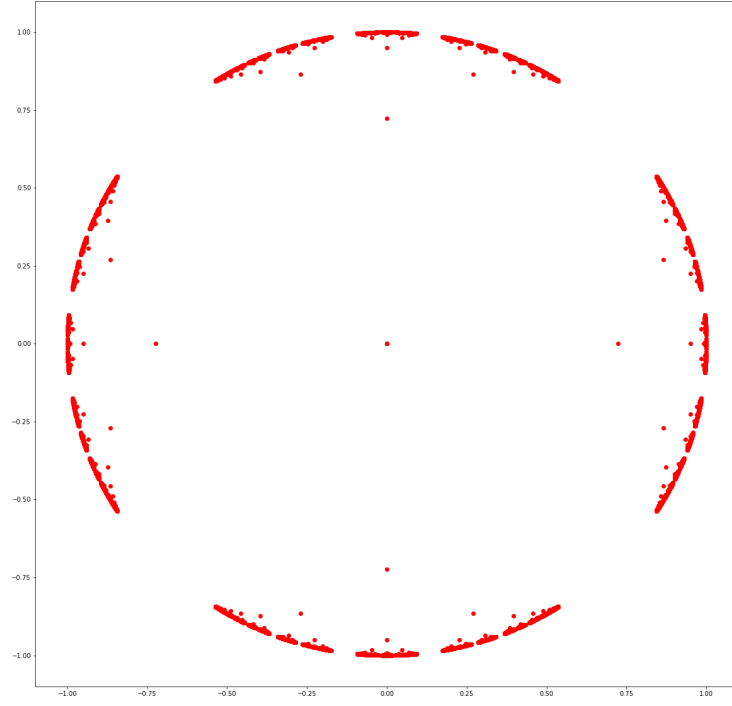
Figure 5.2: $m = 2$, $\Lambda = 0.4$, $\theta \approx 43.602794482778144°$, Level 14 ($N = 14$).
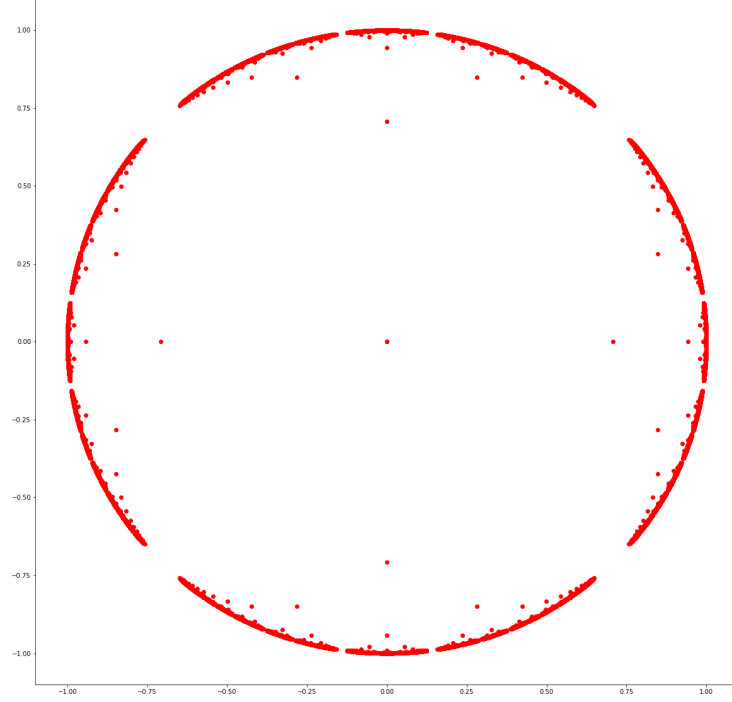
Figure 5.3: $m = 2$, $\Lambda \to \frac{\sqrt{2}}{2+\sqrt{2}}$, $\theta \to 45°$, Level 14 ($N = 14$).

Our computer system has a limit on the number of digits in a floating point number, but an irrational number $\frac{\sqrt{2}}{2+\sqrt{2}}$ has an endless number of digits after the decimal point.

That is, our computer system can only handle rational numbers, i.e., it can only employ rational numbers to approximate irrational numbers. As a result, in this

picture, we have not obtained $\delta(G) = 1$.
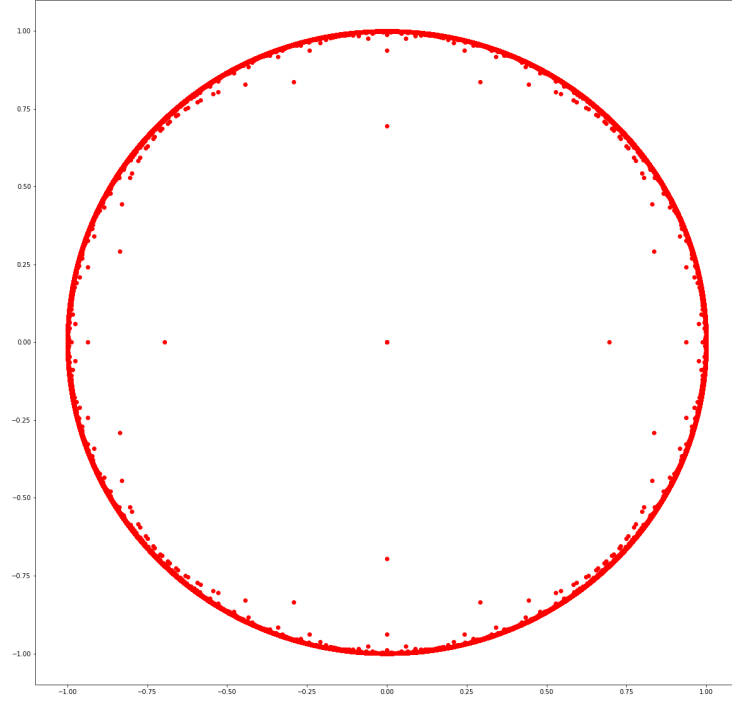
Figure 5.4: $m = 2$, $\Lambda = \frac{\sqrt{2}}{2+\sqrt{2}} + 0.01$, $\theta \approx 45.9746105715017°$, Level 14 ($N = 14$).

To reach $\delta(G) = 1$, we added a small number 0.01 to let Λ pass the number $\frac{\sqrt{2}}{2+\sqrt{2}}$, or equivalently to let θ passes $\frac{\pi}{2}$.
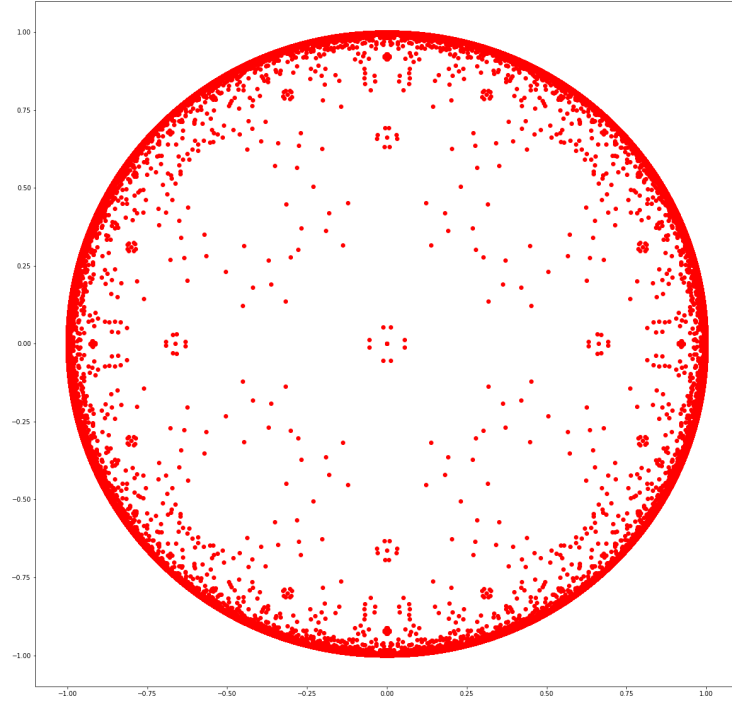
Figure 5.5: $m = 2$, $\Lambda = 0.45$, $\frac{\theta}{2} \approx 48.455517824418166°$, Level 14 ($N = 14$).
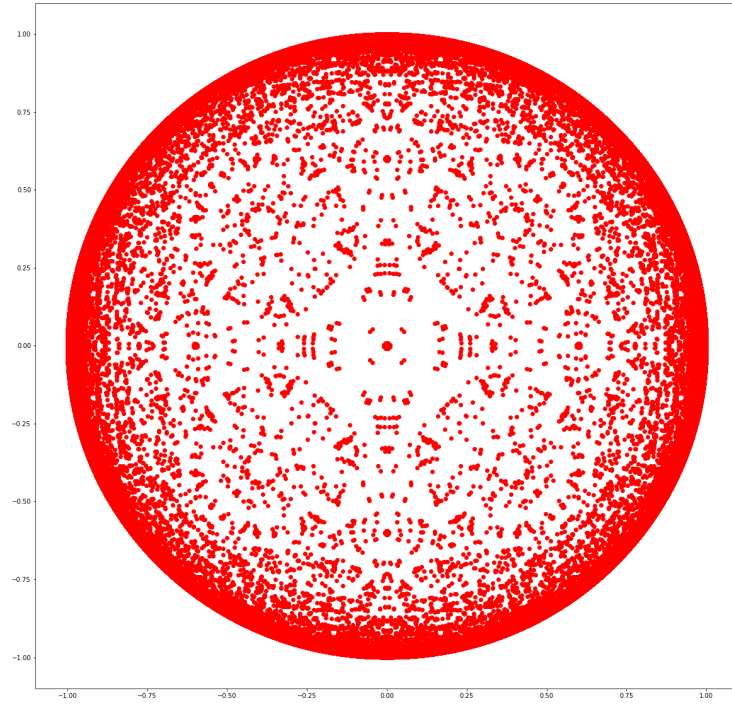
Figure 5.6: $m = 2$, $\Lambda = 0.5$, $\theta \approx 53.13010858755201°$, Level 14 ($N = 14$).
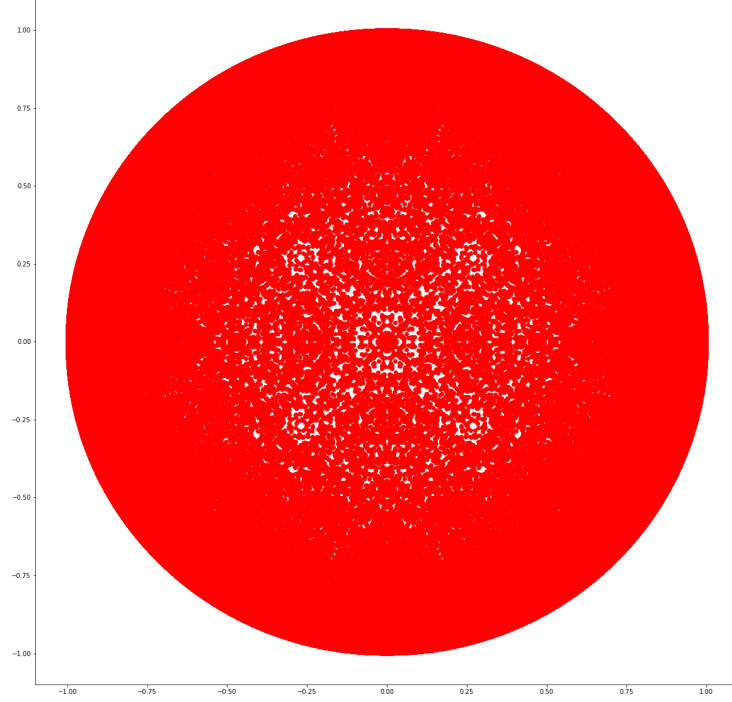
Figure 5.7: $m = 2$, $\Lambda = 0.6$, $\theta \approx 61.927531757108724°$, Level 14 $(N = 14)$.

To begin implementing McMullen's technique, we rotate the orbit by $\frac{\pi}{2}$ counter-clockwise by rotating the above T operator and reassign the result to the symbol T in the following derivation.

In other words, we are going to find an alternative approach that instead of staring from $\mathbb{H}^2$, this time we directly start from $\mathbb{B}^2$ and use $\theta = \frac{\pi}{m}$ to parameterize

T instead of using Λ.

Since on $\mathbb{B}^2$, to be an automorphism, we know T must be in $\text{PSU}(1,1)$. Thus, assume

$$T = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}.$$

Then, for $0 < \theta < \frac{\pi}{4}, \forall m \in \mathbb{Z}^+$, we have

$$T0 = \cos\left(\frac{\theta}{2}\right).$$

Thus

$$|\gamma|^2 = |\alpha|^2 \cos^2\left(\frac{\theta}{2}\right).$$

Since $T \in \text{PSU}(1,1)$, by definition of $\text{PSU}(1,1)$, we have

$$|\alpha|^2 - |\gamma|^2 = 1.$$

Hence,

$$|\alpha|^2 \sin^2\left(\frac{\theta}{2}\right) = 1.$$

That is

$$\left(\sin^2\left(\frac{\theta}{2}\right)\right)^{-1} = |\alpha|^2.$$

Without loss the generality, assume $\alpha = \left(\sin\left(\frac{\theta}{2}\right)\right)^{-1}$, and $\gamma = \cot\left(\frac{\theta}{2}\right)$ to satisfy the

identity. Then we can define

$$
T := \begin{pmatrix} \frac{1}{\sin\left(\frac{\theta}{2}\right)} & \cot\left(\frac{\theta}{2}\right) \\ \cot\left(\frac{\theta}{2}\right) & \frac{1}{\sin\left(\frac{\theta}{2}\right)} \end{pmatrix}.
$$

We can now utilize this revised definition of the operator T to deduce the lemma and main theorem below.

Lemma 5.5. *Let I be the path corresponding to the boundary path, $r = \rho_{\mathbb{B}^2}(0, T0)$, and I_n be the path at the level n. Then,*

$$
\lim_{n \to \infty} \frac{x_{I_n}}{nr} < 1.
$$

Proof. Recall lemma 5.4, we have

$$
\cosh\left(X_{I_n}\right) = \cosh\left(X_{I_{n-1}}\right) \cosh\left(1 - \tanh\left(X_{I_{n-1}} \tanh(r) \cos\left(\frac{\pi}{m} - \theta\right)\right)\right).
$$

Note that $0 < \theta < \frac{\pi}{2m}$, so the factor

$$
1 - \tanh\left(X_{I_{n-1}}\right) \tanh(r) \cos\left(\frac{\pi}{m} - \theta\right) \leq 1 - \cos\left(\frac{\pi}{2m}\right) := C.
$$

Therefore,

$$
\cosh\left(X_{I_n}\right) \leq C \cosh\left(X_{I_{n-1}}\right) \cosh(r)
$$

$$\leq C^2 \cosh\left(X_{I_{n-2}}\right) \cosh^2\left(r\right) \leq \cdots$$

$$\leq C^n \cosh^n\left(r\right).$$

Hence,

$$X_{I_n} \leq \cosh^{-1}\left(C^n \cosh^n\left(r\right)\right).$$

Recall $\cosh^{-1}\left(y\right) = \ln\left(y + \sqrt{y^2 - 1}\right)$.

Therefore,

$$X_{I_n} \leq \cosh^{-1}\left(C^n \cosh^n\left(r\right)\right)$$

$$= \ln\left(C^n \cosh^n\left(r\right) + \sqrt{C^{2n} \cosh^{2n}\left(r\right) - 1}\right)$$

$$\leq \ln\left(2C^n \cosh^n\left(r\right)\right)$$

$$= \ln\left(2\right) + n\ln\left(C\right) + n\ln\left(\cosh\left(r\right)\right).$$

Hence,

$$\frac{X_{I_n}}{nr} \leq \frac{\ln\left(2\right)}{nr} + \frac{\ln\left(C\right)}{r} + \frac{\ln\left(C\right)}{r}.$$

Take n to infinity:

$$\lim_{n\to\infty} \frac{X_{I_n}}{nr} \leq \frac{\ln\left(\cosh\left(r\right)\right)}{r} + \frac{\ln\left(C\right)}{r}.$$

If $C < 1$, then $\ln\left(C\right) < 0$, so we have

$$\lim_{n\to\infty} \frac{X_{I_n}}{nr} < \frac{\ln\left(\cosh\left(r\right)\right)}{r}$$

$$= \frac{\ln\left(\frac{e^r + e^{-r}}{2}\right)}{r} < \frac{\ln\left(e^r\right)}{r} = 1.$$

$\square$

Lemma 5.6. *Let* $\Gamma = \langle T_1, \ldots, T_m \rangle$ *be well-distributed Schottky group, and let*

$$X_{I_n} = \rho_{\mathbb{B}^2}\left(0, T_I 0\right).$$

Consider the infinite path $I_n = (i_1, i_2, \ldots, i_n, \ldots) \in \mathcal{A} \times \mathcal{A} \cdots \times \mathcal{A}$, *where each* $\mathcal{A} = \{0, 1, 2, \ldots, 2m - 1\}$ *is an alphabet, and* $\mathcal{W}_n = \{(i_1, i_2, \ldots, i_n) : i_j \in \mathcal{A}\}$. *Then from hyperbolic cosine law, we have*

$$\cosh\left(X_{I_n}\right) = \cosh^n\left(r\right) \prod_{j=1}^{n-1} \left(1 - \tanh\left(X_{I_{j+1}}\right) \tanh\left(r\right) \cos\left(\frac{i_{j+1}\pi}{m} - \theta_{I_j}\right)\right)$$

Proof. Since after applying $T_{I_{n-1}}^{-1}$ and using the property of isometry, we have the following

$$\angle\left(0, T_{I_{n-1}}0, T_{I_n}0\right) = \angle\left(T_{I_{n-2}}, T_{I_{n-1}}0, T_{I_n}0\right) - \angle\left(T_{I_{n-2}}, T_{I_{n-1}}0, 0\right)$$

$$= \angle\left(T_{i_{n-1}}^{-1}, 0, T_{i_n}0\right) - \angle\left(T_{i_{n-1}}^{-1}0, 0, T_{I_{n-1}}^{-1}0\right)$$

$$= \frac{i_n\pi}{m} - \theta_{I_{n-1}}.$$

Thus, the hyperbolic cosine law can be rewritten as follows

$$\cosh\left(X_{I_n}\right) = \cosh\left(X_{I_{n-1}}\right)\cosh\left(r\right) - \sinh\left(X_{I_{n-1}}\right)\sinh\left(r\right)\cos\left(\angle\left(0, T_{I_{n-1}}0, T_{I_n}0\right)\right)$$

$$= \cosh\left(X_{I_{n-1}}\right)\cosh\left(r\right)\left(1 - \tanh\left(X_{I_{n-1}}\right)\tanh\left(r\right)\cos\left(\frac{i_n\pi}{m} - \theta_{I_{n-1}}\right)\right)$$

$$= \cosh^2\left(r\right)\cosh\left(X_{I_{n-2}}\right)\left(1 - \tanh\left(X_{I_{n-2}}\right)\tanh\left(r\right)\cos\left(\frac{i_{n-1}\pi}{m} - \theta_{I_{n-2}}\right)\right)$$

$$\times\left(1 - \tanh\left(X_{I_{n-1}}\right)\tanh\left(r\right)\cos\left(\frac{i_n\pi}{m} - \theta_{I_{n-1}}\right)\right).$$

By induction, we derived the following result

$$\cosh\left(X_{I_n}\right)$$

$$= \cosh^{n-1}\left(r\right)\cosh\left(X_{I_1}\right)\prod_{j=1}^{n-1}\left(1 - \tanh\left(X_{I_{j+1}}\right)\tanh\left(r\right)\cos\left(\frac{i_{n-1}\pi}{m} - \theta_{I_{n-2}}\right)\right)$$

$$= \cosh^n\left(r\right)\prod_{j=1}^{n-1}\left(1 - \tanh\left(X_{I_{j+1}}\right)\tanh\left(r\right)\cos\left(\frac{i_{j+1}\pi}{m} - \theta_{I_j}\right)\right)$$

$$\square$$

Definition 5.10. Define the symbol for the previous lemma's outcome as follows.

$$Y_{I_n} := \cosh^n\left(r\right)\prod_{j=1}^{n-1}\left(1 - \tanh\left(X_{I_{j+1}}\right)\tanh\left(r\right)\cos\left(\frac{i_{j+1}\pi}{m} - \theta_{I_j}\right)\right)$$

and let

$$P_n := \sum_{T \in \mathcal{W}_n} \frac{1}{Y_{I_n}^t}.$$

Lemma 5.7. *The Poincareé series can be rewritten using the above definition*

$$\mathcal{P}(\Gamma, t) \asymp \sum_{n=1}^{\infty} P_n.$$

Proof. Recall $\rho_{\mathbb{B}^2} = \cosh^{-1}(X_{I_n}) = \ln\left(Y_{I_n} + \sqrt{Y_{I_n}^2 - 1}\right).$

$$\mathcal{P}(\Gamma, t) = \sum_{T \in \Gamma} e^{-t\rho_{\mathbb{B}^2}(0, T0)} \sum_{n=1}^{\infty} \sum_{T \in \mathcal{W}_n} e^{t\rho_{\mathbb{B}^2}(0, T0)} = \sum_{n=1}^{\infty} \sum_{T \in \mathcal{W}_n} \left(\frac{1}{Y_{I_n} + \sqrt{Y_{I_n}^2 - 1}}\right)^t$$

$$\asymp \sum_{n=1}^{\infty} \sum_{T \in \mathcal{W}_n} \frac{1}{Y_{I_n}^t} = \sum_{n=1}^{\infty} P_n.$$

$\square$

Theorem 5.8. *Suppose Γ is a well-distributed Schottky group of order 2. Then,*

$$\frac{\ln(2)}{\ln(\cosh(r))} \leq \delta(\Gamma) \leq \min\left\{\frac{\ln(4)}{\ln(\cosh(r))}, 1\right\}.$$

Proof. Recall definition 5.10,

$$Y_{I_n} = Y_{I_{n-1}i_n} = Y_{I_{n-1}} Y_{i_n}$$

$$= \cosh\left(X_{I_{n-1}}\right)\cosh\left(r\right)\left(1 - \tanh\left(X_{I_{n-1}}\right)\tanh\left(r\right)\cos\left(\frac{i_n\pi}{m} - \theta_{I_{n-1}}\right)\right).$$

Hence,

$$Y_{I_{n+1}} = Y_{I_n}Y_{i_{n+1}} = \cosh\left(X_{I_n}\right)\cosh\left(r\right)\left(1 - \tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n}\right)\right).$$

Thus, for the $n+1$ term in $\mathcal{P}$, we have the following

$$P_{n+1} = \sum_{T \in \mathcal{W}_{n+1}} \frac{1}{Y_{I_{n+1}}^t} = \sum_{T \in \mathcal{W}_n} \sum_{i_{n+1}=1}^{2m-1} \frac{1}{Y_{I_n i_{n+1}}^t}$$

$$= \sum_{T \in \mathcal{W}_n} \frac{1}{\left(\cosh\left(X_{I_n}\right)\right)^t} \frac{1}{\left(\cosh\left(r\right)\right)^t} \sum_{i_{n+1}=1}^{2m-1} \left(1 - \tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n}\right)\right)^{-t}.$$

Recall Newton's generalized binomial theorem: Firstly, for $k \geq 1, k \in \mathbb{Z}$, we have

$$\binom{-t}{k} := \frac{\left(-t-0\right)\left(-t-1\right)\cdots\left(-t-\left(k-1\right)\right)}{k!}$$

$$= \frac{t\left(t+1\right)\cdots\left(t+\left(k-1\right)\right)}{k!}\left(-1\right)^k, \forall k \in \mathbb{Z}^+,$$

and secondly for $k = 0$, we have

$$\binom{-t}{0} := 1.$$

Hence,

$$(x + y)^{-t} = \sum_{k=0}^{\infty} \binom{-t}{k} x^{-t-k} y^k.$$

Now, let $x = 1$, and replace y with $-y$.

$$(1 - y)^{-t} = \sum_{k=0}^{\infty} \binom{-t}{k} (-1)^k y^k$$

$$= \sum_{k=0}^{\infty} \frac{t(t+1)\cdots(t+(k-1))}{k!} (-1)^{2k} y^k$$

$$= \sum_{k=0}^{\infty} \frac{t(t+1)\cdots(t+(k-1))}{k!} y^k$$

$$= 1 + ty + \sum_{k=2}^{\infty} \frac{t(t+1)\cdots(t+(k-1))}{k!} y^k.$$

Thus,

$$\sum_{i_{n+1}=1}^{2m-1} \left(1 - \tanh(X_{I_n}) \tanh(r) \cos\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n} \right) \right)^{-t}$$

$$= \sum_{i_{n+1}=1}^{2m-1} \left(1 + t \tanh(X_{I_n}) \tanh(r) \cos\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n} \right) \right.$$

$$+ \sum_{k=2}^{\infty} \frac{t(t+1)\cdots(t+(k-1))}{k!} \tanh^k(X_{I_n}) \tanh^k(r) \cos^k\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n} \right) \right)$$

$$= 2m - 1 + t \tanh(X_{I_n}) \tanh(r) \left(\sum_{i_{n+1}=1}^{2m-1} \cos\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n} \right) \right)$$

$$+ \sum_{k=2}^{\infty} \frac{t\,(t+1)\cdots(t+(k-1))}{k!} \left(\sum_{i_{n+1}=1}^{2m-1} \tanh^k(X_{I_n})\tanh^k(r)\cos^k\left(\frac{i_{n+1}\pi}{m} - \theta_{I_n}\right) \right)$$

Let $m = 2$. Hence,

$$\sum_{i_{n+1}=1}^{3} \cos^k\left(\frac{i_{n+1}\pi}{2} - \theta_{I_n}\right) = \cos^k\left(\frac{\pi}{2} - \theta_{I_n}\right) + \cos^k(\pi - \theta_{I_n}) + \cos^k\left(\frac{3\pi}{2} - \theta_{I_n}\right)$$

$$= \begin{cases} -\cos^k(\theta_{I_n}) & k \text{ is odd} \\[2mm] 2\sin^k(\theta_{I_n}) + \cos^k(\theta_{I_n}) & k \text{ is even.} \end{cases}$$

The estimation becomes the following

$$\sum_{i_{n+1}=1}^{3} \left(1 - \tanh(X_{I_n})\tanh(r)\cos\left(\frac{i_{n+1}\pi}{2} - \theta_{I_n}\right)\right)^{-t}$$

$$= 3 - t\tanh(X_{I_n})\tanh(r)\cos(\theta_{I_n})$$

$$- \sum_{l=1}^{\infty} \frac{t\,(t+1)\cdots(t+((2l+1)-1))}{(2l+1)!} \left(\tanh^{2l+1}(X_{I_n})\tanh^{2l+1}(r)\cos^{2l+1}(\theta_{I_n})\right)$$

$$+ \sum_{l=1}^{\infty} \frac{t\,(t+1)\cdots(t+((2l)-1))}{(2l)!} \left(\tanh^{2l}(X_{I_n})\tanh^{2l}(r)\left(2\sin^{2l}(\theta_{I_n}) + \cos^{2l}(\theta_{I_n})\right)\right)$$

$$= 3 - t\tanh(X_{I_n})\tanh(r)\cos(\theta_{I_n})$$

$$+ \sum_{l=2}^{\infty} (-1)^l \frac{t\,(t+1)\cdots(t+((l)-1))}{(l)!} \left(\tanh^{2l+1}(X_{I_n})\tanh^{2l+1}(r)\cos^l(\theta_{I_n})\right)$$

$$+ \sum_{l=1}^{\infty} \frac{t(t+1)\cdots(t+((2l)-1))}{(2l)!} \left(\tanh^{2l}(X_{I_n}) \tanh^{2l}(r) 2\sin^{2l}(\theta_{I_n}) \right).$$

Notice $0 < \theta_{I_n} < \frac{\pi}{4}$, $0 < t < 1$,

$$\sum_{l=1}^{\infty} \frac{t(t+1)\cdots(t+((2l)-1))}{(2l)!} \left(\tanh^{2l}(X_{I_n}) \tanh^{2l}(r) 2\sin^{2l}(\theta_{I_n}) \right)$$

$$\leq \sum_{l=1}^{\infty} \frac{1 \cdot 2 \cdot \ldots \cdot 2l}{(2l)!} 2 \cdot \frac{1}{\sqrt{2}^{2k}} = 2 \cdot \sum_{k=2}^{\infty} \frac{1}{2^k} = 1.$$

Hence,

$$\sum_{i_{n+1}=1}^{3} \left(1 - \tanh(X_{I_n}) \tanh(r) \cos\left(\frac{i_{n+1}\pi}{2} - \theta_{I_n} \right) \right)^{-t} \leq 3+$$

$$\left(1 + \sum_{l=2}^{\infty}(-1)^l \frac{t(t+1)\cdots(t+((l)-1))}{(l)!} \left(\tanh^{2l+1}(X_{I_n}) \tanh^{2l+1}(r) \cos^l(\theta_{I_n}) \right) \right)$$

$$= 3 + (1 + \tanh(X_{I_n}) \tanh(r) \cos(\theta_{I_n}))^{-t}.$$

Since

$$(1+y)^{-t} = 1 + \sum_{k=1}^{\infty} \frac{(-t)(-t-1)\cdots(-t-(k-1))}{k!} y^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{t(t+1)\cdots(t+k-1)(-1)^k y^k}{k!},$$

and $\frac{1}{(1+\tanh(X_{I_n})\tanh(r)\cos(\theta_{I_n}))^t} \leq 1$,

$$P_{n+1} \leq \sum_{T \in \mathcal{W}_n} \frac{1}{(\cosh(X_{I_n}))^t (\cosh(r))^t} \left(3 + \frac{1}{(1 + \tanh(X_{I_n}) \tanh(r) \cos(\theta_{I_n}))^t} \right)$$

$$\leq \frac{4}{\cosh^t{(r)}} P_n.$$

Thus,

$$\mathcal{P}\left(\Gamma, t\right) \leq P_1 \sum_{n=1}^{\infty} \left(\frac{4}{(\cosh{(r)})^t}\right)^n.$$

This implies that

$$\delta\left(\Gamma\right) \leq \frac{\ln{(4)}}{\ln{(\cosh{(r)})}}.$$

To obtain a lower bound, recall

$$\sum_{i_{n+1}=1}^{3} \left(1 - \tanh{(X_{I_n})} \tanh{(r)} \cos{\left(\frac{i_{n+1}\pi}{2} - \theta_{I_n}\right)}\right)^{-t}$$

$$= 3 - t \tanh{(X_{I_n})} \tanh{(r)} \cos{(\theta_{I_n})}$$

$$+ \sum_{l=2}^{\infty} (-1)^l \frac{t\,(t+1)\cdots(t+((l)-1))}{(l)!} \left(\tanh^{2l+1}{(X_{I_n})} \tanh^{2l+1}{(r)} \cos^l{(\theta_{I_n})}\right)$$

$$+ \sum_{l=1}^{\infty} \frac{t\,(t+1)\cdots(t+((2l)-1))}{(2l)!} \left(\tanh^{2l}{(X_{I_n})} \tanh^{2l}{(r)}\, 2\sin^{2l}{(\theta_{I_n})}\right)$$

$$\geq 2 + \frac{1}{\left(1 + \tanh{(X_{I_n})} \tanh{(r)} \cos{(\theta_{I_n})}\right)^t}.$$

Implying that

$$P_{n+1} \geq \frac{2}{\cosh^t{(r)}} P_n,$$

and hence

$$\mathcal{P}\left(\Gamma, t\right) \geq P_1 \sum_{n=1}^{\infty} \left(\frac{2}{\cosh^t{(r)}}\right)^n.$$

Therefore,

$$\delta\left(\Gamma\right) \geq \frac{\ln\left(2\right)}{\ln\left(\cosh\left(r\right)\right)}.$$

□

The above bounds can be further sharpen into the following:

Theorem 5.9. *Suppose Γ is a well-distributed Schottky group of order 2. Then,*

$$\frac{\ln\left(2 + \frac{1}{1+2\sqrt{2}}\right)}{\ln\left(\cosh\left(r\right)\right)} \leq \delta\left(\Gamma\right) \leq \min\left\{\frac{\ln\left(4 - \frac{2}{1+2\sqrt{2}}\right)}{\ln\left(\cosh\left(r\right)\right)}, 1\right\}.$$

Proof. Firstly, since $\tanh\left(X_{I_n}\right) \geq \tanh\left(r\right)$, and $0 < \alpha \leq \frac{\pi}{4}$. Thus,

$$\tanh\left(X_{I_n}\right)\tanh\left(r\right)\cos\left(\theta_{I_n}\right) > \tanh^2\left(r\right)\cos\left(\theta_{I_n}\right) > \frac{1}{\sqrt{2}}\tanh^2\left(r\right).$$

Secondly, the choice of the operator T is as follows:

$$T = \begin{pmatrix} \frac{1}{\sin\left(\frac{\alpha}{2}\right)} & \cot\left(\frac{\alpha}{2}\right) \\ \cot\left(\frac{\alpha}{2}\right) & \frac{1}{\sin\left(\frac{\alpha}{2}\right)} \end{pmatrix}.$$

Hence,

$$|T0| = \cos\left(\frac{\alpha}{2}\right).$$

Then,

$$\tanh\left(r\right) = \tanh\left(\ln\left(\frac{1 + |T0|}{1 - |T0|}\right)\right) = \frac{2}{\cos\left(\frac{\alpha}{2}\right)}.$$

Therefore,

$$\frac{1}{(1 + \tanh(X_{I_n})\tanh(r)\cos(\theta_{I_n}))^t} \geq \frac{1}{(1 + 2\sqrt{2})^t} \geq \frac{1}{1 + 2\sqrt{2}}.$$

On the other hand, since

$$t\tanh(X_{I_n})\tanh(r)\cos(\theta_{I_n}) \leq \frac{2}{\cos\left(\frac{\alpha}{2}\right)},$$

and recall that we have

$$\sum_{i_{n+1}=1}^{3}\left(1 - \tanh(X_{I_n})\tanh(r)\cos\left(\frac{i_{n+1}\pi}{2} - \theta_{I_n}\right)\right)^{-t}$$

$$= 3 - t\tanh(X_{I_n})\tanh(r)\cos(\theta_{I_n})$$

$$- \sum_{l=1}^{\infty}\frac{t(t+1)\cdots(t+((2l+1)-1))}{(2l+1)!}\left(\tanh^{2l+1}(X_{I_n})\tanh^{2l+1}(r)\cos^{2l+1}(\theta_{I_n})\right)$$

$$+ \sum_{l=1}^{\infty}\frac{t(t+1)\cdots(t+((2l)-1))}{(2l)!}\left(\tanh^{2l}(X_{I_n})\tanh^{2l}(r)\left(2\sin^{2l}(\theta_{I_n}) + \cos^{2l}(\theta_{I_n})\right)\right).$$

Instead of taking the minimum of the term $-t\tanh(X_{I_n})\tanh(r)\cos(\theta_{I_n})$ by zero, now we use a sharper bound $\frac{2}{\cos\left(\frac{\alpha}{2}\right)}$, and this implies

$$\sum_{i_{n+1}=1}^{3}\left(1 - \tanh(X_{I_n})\tanh(r)\cos\left(\frac{i_{n+1}\pi}{2} - \theta_{I_n}\right)\right)^{-t}$$

$$\leq 3 - \left(\frac{2}{\cos\left(\frac{\alpha}{2}\right)} \right) + \frac{1}{\left(1 + \frac{2}{\cos\left(\frac{\alpha}{2}\right)} \right)^t}.$$

The proof is complete after inserting the aforementioned revised bounds into the prior theorem's proof. □

Based on the above theorems, we conjectured

$$\delta\left(\Gamma\right) = \frac{\ln\left(3\right)}{\ln\left(\cosh\left(r\right)\right)}.$$

To see why this conjecture may be true, we consider $\alpha = \frac{\pi}{4}$. Then,

$$T0 = \cos\left(\frac{\pi}{4}\right),$$

and

$$r = \ln\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right).$$

Then,

$$\cosh\left(r\right) = \frac{e^r + e^{-r}}{2} = \frac{\frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{\sqrt{2}-1}{\sqrt{2}+1}}{2} = 3.$$

Therefore,

$$\frac{\ln\left(3\right)}{\ln\left(\cosh\left(r\right)\right)} = 1.$$

Furthermore, towards the conclusion of the next chapter, this conjecture was analytically tested using an approximation based on McMullen's technique.

Chapter 6

McMullen's Algorithm

The purpose of this chapter is to introduce McMullen's algorithm, which is by far the most accurate approach for numerically approximating the Hausdorff dimension of limit sets of Schottky groups. However, the original source code used to get the findings was never published[23], therefore we re-implemented the algorithm in C and reproduced the results to check that our knowledge and implementation of Patterson-Sullivan theory and McMullen's algorithm were valid. Furthermore, based on McMullen's derivation, we obtained an approximation for small angles that corresponds to the result produced by our conjecture, which is also included in this chapter.

In 1998, McMullen[23] proposed an algorithm for estimating the Hausdorff dimension of a set associated with a conformal dynamical system (as Julia sets or limit set of a geometrically finite Kleinian groups). The dimension of the related Patterson-Sullivan measures coincides with the Hausdorff dimension of its limit set,

and McMullen's approach works to estimate the Hausdorff dimension of the Schottky group limit set.

In [23], some definitions were used for proving a theorem that supports McMullen's algorithm that were used in our set-up of well-distributed group:

Definition 6.1. *A* $G-invariant$ *density of dimension* δ is a finite positive measure μ on $\mathbb{S}^1$ such that

$$\mu\left(T\left(E\right)\right) = \int_E |T'(z)|^\delta d\mu$$

whenever $T|_E$ is injective, $E \subset \partial \mathbb{B}^2 \cup \mathbb{B}^2$ is a Borel set and $T \in G$.

To describe the theorem and McMullen's algorithm, we need to define Markov partitions and their refinements.

Definition 6.2. Let G be a Schottky group. *A Markov partition* for (G, μ) is a nonempty collection $\mathcal{I} \langle (I_i, T_i) \rangle$ of connected compact intervals $I_i \subset \mathbb{S}^1$, $T_i \in G$ meet the following requirements:

- $T_i(I_i) \supset \cup_{i \mapsto j} I_j$, where the relation $i \mapsto j$ means $\mu\left(T_i(I_i) \cap I_j\right) > 0$.

- T_i is a homeomorphism on a neighborhood of $I_i \cap T_i^{-1}(I_j)$, when $i \mapsto j$;

- $\mu(I_i) > 0$;

- $\mu(I_i \cap I_j) = 0$ if $i \neq j$; and

- $\mu(T_i(I_i))\mu(\cup_{i \mapsto j} I_j) = \sum_{i \mapsto j} \mu(I_j)$.

Definition 6.3. *A refinement* of a Markov partition $\mathcal{I}$ is a new Markov partition:

$$\mathcal{R}(\mathcal{I}) = \langle (R_{ij}, T_i) : i \mapsto j \rangle$$

where $R_{ij} := T_i^{-1}(I_j) \cap I_i$.

Definition 6.4. If there exists a smooth conformal metric ρ on $\mathbb{S}^1$ and a constant ν such that

$$|T_i'(x)|_\rho > \nu > 1,$$

whenever $x \in I_i$ and $T_i(x) \in I_j$ for some j, then the Markov partition is *expanding*.

To outline the algorithm's five primary phases for computing the dimension δ of the density μ, firstly, let us consider a Schottky group G, a Markov partition $\mathcal{I} = \langle (I_i, T_i) \rangle$, and sample points $x_i \in I_i$ (in our implementation, the midpoint of each interval I_i was chosen to be x_i). The algorithm computes a sequence of estimates from $\alpha\left(\mathcal{R}^n(I)\right)$ to δ and then proceeds as follows:

- Step 1. For each $i \mapsto j$, solve for $y_{ij} \in I_i$ such that $T_i(y_{ij}) = x_j$.

- Step 2. Compute the transition matrix

$$T_{ij} = \begin{cases} |T_i'(y_{ij})|^{-1} & \text{if } i \mapsto j, \\ 0 & \text{otherwise.} \end{cases}$$

- Step 3. Find $\alpha(\mathcal{I}) \geq 0$ such that the spectral radius (the greatest eigenvalue of the matrix T_{ij}^{δ}, i.e. the transition matrix raised to the power α) equals unity.

- Step 4. Return $\alpha(\mathcal{I})$ as an approximation to δ of this iteration.

- Step 5. Refine the Markov partition, define new sample points $x_{ij} = y_{ij} \in R_{ij}$, and start afresh from Step 1.

The algorithm is supported by the following theorem[23]:

Theorem 6.1 (McMullen). *Let $\mathcal{I}$ be an expanding Markov partition for a Schottky group G with invariant density μ of dimension δ. Then*

$$\alpha(\mathcal{R}^n(\mathcal{I})) \to \delta$$

as $n \to \infty$.

Hence, the dimension of the conformal measure, δ, is approximated by the power α.

The worst case scenario of this theorem could be the case when angle $\theta \to \frac{2\pi}{3}$ in the symmetric pairs of pants example in [23]. Since in this case, it must take at least more than one iteration to obtain an accurate digit for the first figure which is unit one.

Please keep in mind that in McMullen's article [23], this technique was developed in such a way that it was changed to adaptively refine only those partition blocks that exceeded a certain dimension r.

According to [23], the best possible results are produced when all of the blocks are about the same size, because if this extra assumption that all of the blocks are about the same size is not added, then a precise computation that only follows the theorem will be ruined by the inaccuracies introduced by larger blocks of that level.

Additionally, supercomputers in 1990s were about 5 to 10 GFLOPS[1] and 2010s laptops could at least have ranging from 100 to 400 GFLOPS, and laptops of the 2010s could be 5 times as much memory, and access at least 4 times as fast.

In our implementation so far, we have not add this extra assumption into our implementation in C:

`https://github.com/williamchuang/well-distributed-schottky-groups/tree/main/code`

Furthermore, for solving eigenvalues, it usually has $O(n^3)$ time complexity, and $O(n \ln(n))$ complexity for memory allocation. This method of the computation of Hausdorff dimension using eigenvalue algorithm is algebraic and numerical. In this thesis we tried to develop a geometric and analytic method (resulted in our main theorems).

Our goals of studying McMullen's algorithm including to reproduce some results of this paper to show our understanding and implementation of this algorithm were correct, to use an approximation (an example is demonstrated in [23][Theorem 3.5]) based on the first-level approximation of this algorithm to check our conjecture in the next chapter.

---

[1]A computer system with 1 GFLOPS can do one billion (10^9) floating-point calculations per second.

Since in [23], the explicit forms of three generators of the Schottky group $\Gamma = \langle g_1, g_2, g_3 \rangle$ were omitted, to reproduce some results in this paper, we must reconstruct the group.

Hinted by each g_i is an inversion of a circle that intersect the unit circle orthogonally, and based on [9][7.21 The Perpendicular Bisector of a Segment], we can reconstruct Γ by using the idea of isometric circle.

Let $T \in \mathrm{PSU}(1,1)$, T is hyperbolic, and $T = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}$, where $\alpha \in \mathbb{C}$, $\gamma \in \mathbb{C}$, $\alpha + \overline{\alpha} > 2$, and $|\alpha|^2 - |\gamma|^2 = 1$. The perpendicular bisector of $[0, T0]$ is the isometric circle of T^{-1}.

Then the isometric circle of T^{-1} is $|\gamma z + \overline{\alpha}| = 1$, denoted by $I_{T^{-1}}$. Hence, the center of $I_{T^{-1}}$ is $\frac{-\overline{\alpha}}{\gamma}$ denoted by $z_{I_{T^{-1}}}$, and the radius of $I_{T^{-1}}$ is $\frac{1}{|r|}$.

Since $|T0| < 1$, if $|\alpha| \neq 0$, then we have

$$\frac{\overline{\gamma}}{\overline{\alpha}} < 1 \Rightarrow |\alpha| > |\gamma|.$$

Then the modulus of the center of $I_{T^{-1}}$ is

$$|z_{I_{T^{-1}}}| = \left| \frac{-\overline{\alpha}}{\gamma} \right| = \left| \frac{\alpha}{\gamma} \right| > 1,$$

if $|\gamma| \neq 0$.

Thus, we can have the following result:

Lemma 6.2. *If $I_{T^{-1}}$ intercepts $\partial\mathbb{B}^2$ orthogonally, then the Euclidean center of $I_{T^{-1}}$, i.e. $z_{I_{T^{-1}}}$, is always located outside of $\overline{\mathbb{B}^2}$.*

Next, since we know each conformal isometry in $\mathrm{PSU}(1,1)$ can be decomposed into a product of either two reflections, two inversions, or a composition of one reflection and one inversion, we assume each g_i in McMullen's symmetric pairs of pants is composed by a reflection in a line and an inversion in a circle.

Recall that a reflection in a circle centered at q with radius R is a map:

$$I(z) = \frac{R^2}{\bar{z} - \bar{q}} + q,$$

and a reflection in a line $(ax + by = c)$ is a map with the following form:

$$h(z) = \frac{2ic + (b - ai)\bar{z}}{b + ai}.$$

Notice that $I(z)$ and $h(z)$ are non-analytic if we only have one of them alone, but any composition of even number of them is analytic. Hence, we assumed each $g_i(z)$ is decomposed into the following:

$$g_i(z) = I_i(h_i(z)).$$

Therefore, we have the following lemma of half-spaces

Lemma 6.3. *For each g_i, we have*

$$D_0(g_i) = D_0(g_i^{-1}).$$

Proof. Since for each g_i, we have

$$g_i = g_i^{-1}.$$

$\square$

This setting is different compared to group generators of Schottky groups in other places such as [10, 15, 11, 12]. The reason is that according to the set-up of Schottky groups in these papers and books, we have $D_0(T_i) \cap D_0(T_i^{-1}) = \emptyset$.

The next step is to find each g_i explicitly. According to the figure of the example of symmetric pairs of pants in [23], assume the three reflection lines are

$$L_1 : y = 0,$$

$$L_2 : \sqrt{3}x + y = 0, \text{ and}$$

$$L_3 : \sqrt{3}x - y = 0$$

corresponding to g_1, g_2, and g_3 that generate Γ. Therefore, for g_1, we have $h_1(z) = \bar{z}$,

$I_1(z) = \frac{R^2}{\bar{z} - \bar{q_1}} + q_1$. Hence,

$$g_1(z) = \frac{R^2}{z - \overline{q_1}} + q_1$$

$$= \frac{q_1 z - q_1 \overline{q_1} + R^2}{z - \overline{q_1}}.$$

Because it is a linear fraction, it can be mapped into a two-by-two matrix as follows.

$$\begin{pmatrix} q_1 & -q_1\overline{q_1} + R^2 \\ 1 & -\overline{q_1} \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, we can also have

$$\frac{\partial g_1}{\partial z} = \frac{ad - bc}{(cz + d)^2}.$$

For $g_2(z)$, we have

$$h_2(z) = \left(\frac{(1 - \sqrt{3}i)}{(1 + \sqrt{3}i)} \right) \overline{z} := H_2 \overline{z}.$$

Thus,

$$g_2(z) = \frac{R^2}{\overline{H_2}z - \overline{q_2}} + q_2$$

$$= \frac{q_2\overline{H_2}z - q_2\overline{q_2} + R^2}{\overline{H_2}z - \overline{q_2}}$$

which is corresponding to the following matrix

$$
\begin{pmatrix}
\overline{H}_2 q_2 & -q_2 \overline{q_2} + R^2 \\
\overline{H}_2 & -\overline{q_2}
\end{pmatrix}.
$$

Similarly, for g_3, we have

$$
g_3(z) = \frac{q_3 \overline{H}_3 z - q_3 \overline{q_3} + R^2}{\overline{H}_3 z - \overline{q_3}}
$$

which corresponds to the matrix below

$$
\begin{pmatrix}
\overline{H}_3 q_2 & -q_2 \overline{q_3} + R^2 \\
\overline{H}_3 & -\overline{q_3}
\end{pmatrix}
$$

where

$$
\overline{H}_3 := \begin{pmatrix} (-1 - \sqrt{3}i) \\ (-1 + \sqrt{3}i) \end{pmatrix}.
$$

We can now utilize the three generators mentioned above to compute the first level approximation using McMullen's technique.

Let A and B be two intercepts of I_{g_1} and $\partial \mathbb{B}^2$, and θ be the angle $\angle[A, O, B]$, where O is the origin.

Example: $\theta = \frac{2\pi}{3}$.

We have $x_2 = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2} \right)$, and $q_1 = \left(\sqrt{1 + R^2}, 0 \right) = (2, 0)$, since $R = \tan \left(\frac{\theta}{2} \right)$.

Then,

$$\left|\frac{\partial g_1}{\partial z}\right| = \frac{R^2}{|x_2 - q_1|^2} = \frac{3}{7}.$$

We have the transition matrix through symmetry:

$$T_{12} = T_{21} = T_{13} = T_{31} = T_{32} = T_{23} = \left|\frac{\partial g_1}{\partial z}\right| = \frac{3}{7},$$

and for the diagonal entries,

$$T_{11} = T_{22} = T_{33} = 0.$$

Thus, based on McMullen's algorithm, for the first level (or the first iteration), we have

$$2\left(\frac{3}{7}\right)^\alpha = 1.$$

Solve the above equation for α, we derived

$$\alpha = \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{3}{7}\right)} \approx 0.818068.$$

This result can be used to verify whether our implementation make sense in Chapter 6.

Furthermore, the formula

$$\left|\frac{\partial g_1}{\partial z}\right| = \frac{R^2}{|x_2 - q_1|^2}$$

can be used to check whether our conjecture can produce the same approximation based on this algorithm.

Furthermore, in general, by using cosine law,

$$|x_2 - q_1|^2 = 2 + \left(\tan^2\left(\frac{\theta}{2}\right)\right) + \sec\left(\frac{\theta}{2}\right).$$

Thus, in general, for $\theta \in (0, \frac{2\pi}{3}]$, and for the first level approximation, we have

$$\left|\frac{\partial g_1}{\partial z}\right| = \frac{\tan^2\left(\frac{\theta}{2}\right)}{2 + \left(\tan^2\left(\frac{\theta}{2}\right)\right) + \sec\left(\frac{\theta}{2}\right)}.$$

Moreover, in general, by solving the equation

$$2t^\alpha = 1,$$

where $t = \left|\frac{\partial g_1}{\partial z}\right|$, then we have

$$\delta(G) \approx \alpha = \frac{\ln\left(\frac{1}{2}\right)}{\ln(t)} = \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2 + \left(\tan^2\left(\frac{\theta}{2}\right)\right) + \sec\left(\frac{\theta}{2}\right)}\right)}.$$

6.1   Comparing with our conjecture on small angles

Recall that at the end of the last Chapter, for a well-distributed Schottky group of rank two we obtained

$$\delta(G) = \frac{\ln(3)}{\ln(\cosh(r))},$$

where

$$\cosh(r) = \frac{e^r + e^{-r}}{2}, r = \ln\left(\frac{1 + \cos\left(\frac{\theta}{2}\right)}{1 - \cos\left(\frac{\theta}{2}\right)}\right).$$

Then,

$$\cosh(r) = \csc\left(\frac{\theta}{2}\right)^2 + \cot\left(\frac{\theta}{2}\right)^2$$

$$= \frac{1}{\sin\left(\frac{\theta}{2}\right)^2} + \frac{1}{\tan\left(\frac{\theta}{2}\right)^2}.$$

For $\theta \to 0$, we have

$$\sin\left(\frac{\theta}{2}\right)^2 \approx \left(\frac{\theta}{2}\right)^2 \approx \tan\left(\frac{\theta}{2}\right)^2.$$

Thus,

$$\cosh(r) \simeq \frac{8}{\theta^2}.$$

On the other hand, for $m = 2$, we have

$$R = \tan\left(\frac{\theta}{2}\right) \approx \frac{\theta}{2},$$

and

$$|x_2 - q_1| = \sqrt{2}.$$

Therefore, from the first level approximation of McMullen's algorithm, we have

$$\left| \frac{\partial g_1}{\partial z} \right| = \frac{R^2}{|x_2 - q_1|^2} \approx \frac{\theta^2}{8}.$$

Thus, for $\theta \to 0$, both the first level approximation of McMullen's algorithm and our conjecture give an identical estimation

$$\delta(G) \approx \frac{\ln(3)}{\ln\left(\frac{\theta^2}{8}\right)}.$$

6.2 Using the first level approximation to verify our implementation in C

When McMullen's results[23] are compared to our first level approximation result, we can see that they both coincide at the first digit after the decimal point from $1°$ to $80°$.

Please keep in mind that McMullen's results were based on at a level 18 computation. Additionally, it was stated in [23] that there were about 600,000 Markov partitions in this level, and $600,000 \approx 600 \cdot 2^{10} < 6 \cdot 2^{17} = 3 \cdot 2^{18}$, but in our implementation, at level 1 we only use three Markov partitions.
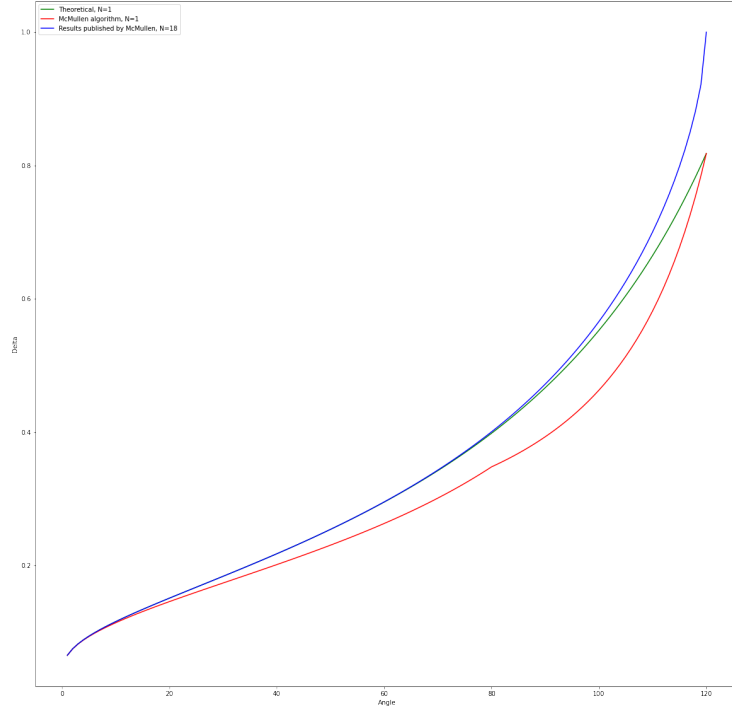
Figure 6.1: Level 1 ($N = 1$) approximations, including McMullen's published data[23][Table 12] (colored in blue), our $N = 1$ theoretical approximation (green), and $N = 1$ numerical approximation based on McMullen's algorithm (red). The angle θ is running from $1°$ to $120°$.

In the above figure, we plotted three results together, including the data published by McMullen [23][Table 12] for $N = 18$ which is colored in blue, the output

of our implementation for $N = 1$ which is colored in red, and the theoretical result

$$\delta(G) \approx \alpha = \frac{\ln\left(\frac{1}{2}\right)}{\ln(t)} = \frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2+\left(\tan^2\left(\frac{\theta}{2}\right)\right)+\sec\left(\frac{\theta}{2}\right)}\right)} \text{ for } N = 1 \text{ which is colored in green.}$$

We can see that the $N = 1$ theoretical result can rather well resemble McMullen's $N = 18$ finding[23] for $\theta \in [0°, 80°]$. Furthermore, we can see that, owing to McMullen's piratical concern, the outcome of our implementation's $N = 1$ instance did not correspond with the theoretical findings or the $N = 18$ results provided by McMullen.

This, however, may be improved angle by angle. It takes longer when N exceeds 8 (after $N > 8$, each level may take more than 2 hours before parallel computing), but it is achievable.

For instance, if we focus on $\theta = 50°$, then we can have the following improvement

level	degree	$\alpha_N(\Gamma)$
1	50	0.23066000000008990
2	50	0.27209000000013134
3	50	0.25797000000011720
4	50	0.24197000000010122
5	50	0.25848000000011770
6	50	0.25839000000011764
7	50	0.25825000000011750
8	50	0.25839000000011764
9	50	0.25839000000011764

Table 6.1: An improvement for the case $\theta = 50°$ with precision= 0.00001.

The meaning of the above table is twofold: First and foremost, it demonstrates

that if we fix a precision to a specific digit, the sequence of α_N would ultimately converge (in $O(N)$ time complexity). Second, we can observe an improvement when we compare the first two digits after the decimal point for the $N = 1$ result, which is 0.23, to McMullen's $N = 18$ result, which is 0.25, i.e. when $N > 4$ the sequence becomes stable. Furthermore, the figure after 25 may vary as our precision improves.

Furthermore, in the following table, for large θ, i.e. $\theta \to \frac{2\pi}{3}$, we can observe that the sequence tend to converge to 1.

level	degree	$\alpha_N(\Gamma)$
1	120	0.8180799999989117
2	120	0.8739599999986574
3	120	0.9045899999985180
4	120	0.9240099999984296
5	120	0.9372899999983691
6	120	0.9468699999983256
7	120	0.9540399999982929
8	120	0.9595799999982677
9	120	0.9639899999982476

Table 6.2: Improve $\alpha_N(\Gamma)$ such that it converges to $\delta(\Gamma) = 1$ with precision= 0.00001.

Based on the above observation, we can also notice that compared to $N = 1$, the more level we can reach, the larger value $\alpha_N(\Gamma)$ will be, and different angles need different N to converge when the same precision was given. Furthermore, based on these above results, including our $N = 1$ theoretical formula can fit McMullen's

result well for $\theta \leq 80°$, and when $\theta > 80°$, as showed in the above two examples, we can have a good approximation compare to McMullen's results by increasing the number of level, N.

Finally, we also use the same understanding to implement McMullen's algorithm for $m = 2$ well-distributed Schottky group and plot the following results with the results generated by our main theorem and conjecture all together.

For $m = 2$, by using cosine law for $\frac{\pi}{2}$, and solving α for $3(t)^\alpha = 1$, we have the first level approximation as follows

$$\delta(G) \approx \alpha = \frac{\ln\left(\frac{1}{3}\right)}{\ln(t)} = \frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2+\left(\tan^2\left(\frac{\theta}{2}\right)\right)}\right)}.$$
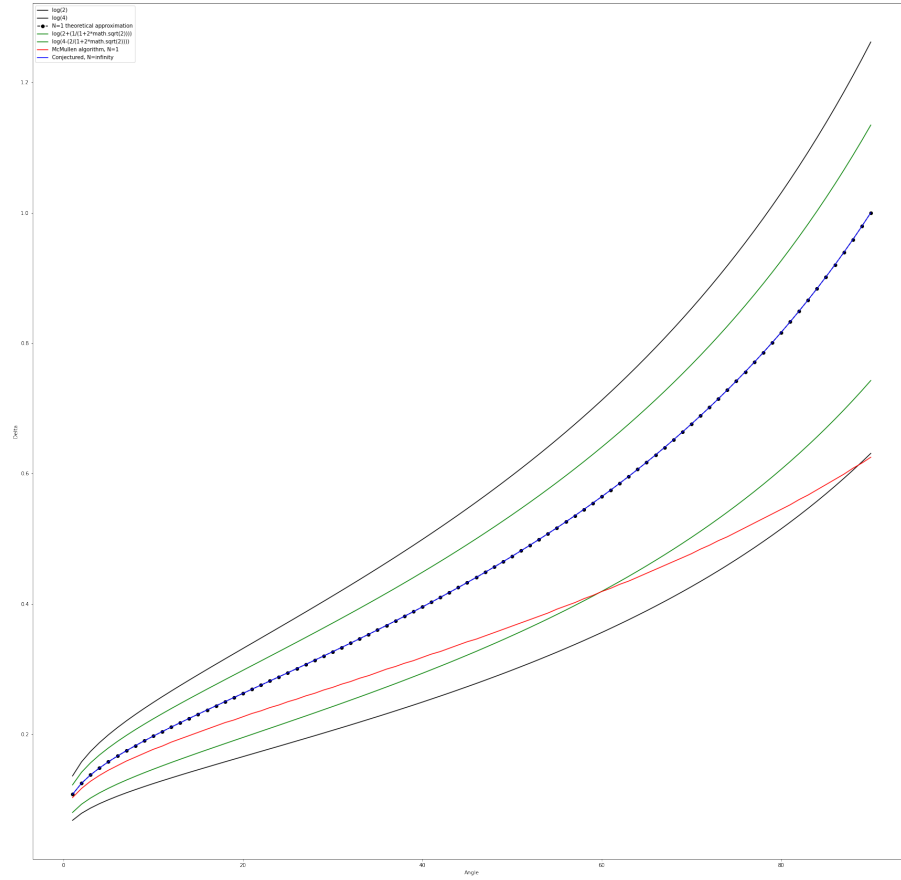
Figure 6.2: Results from our main theorems (colored in green and black), conjecture (blue), theoretical $N = 1$ approximation (dotted black), and numerical $N = 1$ approximation implementation of McMullen algorithm (red).

Since we have not had enough time to improve the results of our implementation of McMullen's algorithm. However, based on the above examples, we believe that once N increases, the curve will be shifted towards the conjectured curve.

Surprisingly, the findings of the $N = 1$ approximation precisely match our conjecture for $N \to \infty$. Therefore, if our conjecture is true, then this implies that for a well-distributed Schottky group of rank 2, $\delta(\Gamma) = \alpha_1(\Gamma)$. It may also imply that for all N, we have

$$\delta(\Gamma) = \alpha_N(\Gamma) = \alpha_1(\Gamma) = \frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2+\left(\tan^2\left(\frac{\theta}{2}\right)\right)}\right)} = \frac{\ln\left(3\right)}{\ln\left(\cosh\left(r\right)\right)},$$

where

$$r = \ln\left(\frac{1 + \cos\left(\frac{\theta}{2}\right)}{1 - \cos\left(\frac{\theta}{2}\right)}\right),$$

since $T0 = \cos\left(\frac{\theta}{2}\right)$. For the last equation, it can be demonstrated that

$$\cosh(r) = \frac{1 + \cos^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)},$$

and

$$\frac{\tan^2\left(\frac{\theta}{2}\right)}{2 + \left(\tan^2\left(\frac{\theta}{2}\right)\right)} = \left(\frac{1 + \cos^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}\right)^{-1}.$$

Hence, we always have $\frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2+\left(\tan^2\left(\frac{\theta}{2}\right)\right)}\right)} = \frac{\ln(3)}{\ln(\cosh(r))}$ for all $0 < \theta < \frac{\pi}{2}$. This alternate strategy might also shed some light on the meaning of $cosh(r)$ in our conjecture. A

proof of the conjecture, a generalization to $m > 2$, and enhancements to our implementations are on the horizon for our future work. Due to there are heavily matrix operations involved, an implementation on parallel computing in a contemporary computer might potentially enhance McMullen's results.

Our source code can be downloaded from GitHub:

`https://github.com/williamchuang/well-distributed-schottky-groups/tree/main/code`

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