Applied Maths III Lecture 3

Power Series

Some comments about convergence of series

Some Definitions

Suppose $a_1, a_2 \cdots a_n \cdots$ is a "sequence" and $a \in \mathbb{C}$ such that for all $\varepsilon > 0$, there is an integer N s.t. for all $n \geq N$, we have

$$|a_n - a| < \varepsilon \quad \text{if} \quad n \geq N$$

Then the sequence $a_1, a_2 \cdots a_n$ is convergent with a being its limit if

$$\lim_{n \to \infty} a_n = a$$

When this happens the sequence converges to a. If no such limit exists the sequence diverges.

eg The sequence $a_n = i^n/n$ converges to 0. Given $\epsilon > 0$, can choose $N > 1/\epsilon$, then for any $n \geq N$

$$\left|\frac{i^n}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$$ $\rightsquigarrow$ Can then take $N \to \infty$, $\epsilon \to 0$ to reach the unique limit

eg The sequence $a_n = i^n$ diverges. Given $a \in \mathbb{C}$, choose $\epsilon = 1/2$

if $Re(a) \geq 0$ for any N, choose $n \geq N$ such that $a_n = -1$ i.e. $n = 4k+2$

$$|a - a_n| = |a+1| > 1/2$$ $\sim$ limit does not exist

if $Re(a) < 0$ for any N, choose $n \geq N$ such that $a_n = +1$, i.e. $n = 4k$

$$|a - a_n| = |a-1| > 1/2$$ $\sim$ limit does not exist

An infinite series of complex numbers

$$S_\infty = \sum_{k=1}^{\infty} a_k \quad , \quad a_k \in \mathbb{C} \;, \; k=1,2,\cdots\infty$$

converges to the sum S if the sequence $S_n = \sum_{k=1}^{n} a_k$ of partial sum converges to S, i.e

$$\lim_{n\to\infty} S_n = S$$

if such limit does not exist, the series S_∞ diverges.

<u>Theorem</u> Let $a_k = x_k + i y_k$, and $S = X + iY$, then $\sum_{k=1}^{\infty} a_k = S$ iff

$$\sum_{k=1}^{\infty} x_k = X \;, \quad \sum_{k=1}^{\infty} y_k = Y$$

Proof Trivial, splitting partial sum $S_N = X_N + iY_N$, then show X_N and Y_N converges to X and Y respectively

This theorem allows us to invoke results for real series.

e.g For a real series to converge, $\lim_{n\to\infty} a_n = 0$, $a_n \in \mathbb{R}$, this implies a necessary condition for S_∞ to be convergent is that

$$\lim_{n\to\infty} a_n = 0 \quad , \quad a_n \in \mathbb{C}$$

Absolutely Convergent S_∞ is "absolute convergent" if the real series

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \sqrt{x_k^2 + y_k^2} \qquad \text{converges}$$

As $\sqrt{x_k^2+y_k^2} \geq |x_k|, |y_k|$, $\sum_{k=1}^{\infty} |x_k|$ & $\sum_{k=1}^{\infty} |y_k|$ also converge, this implies that $\sum_{k=1}^{\infty} x_k$ & $\sum_{k=1}^{\infty} y_k$ converge $\rightarrow \sum_{k=1}^{\infty} a_k$ converges

We deduce that absolute convergence of a complex series S_∞ implies convergence. Converse is NOT TRUE! c.f $\sum_{k=1}^{\infty} \frac{1}{k}$ vs $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

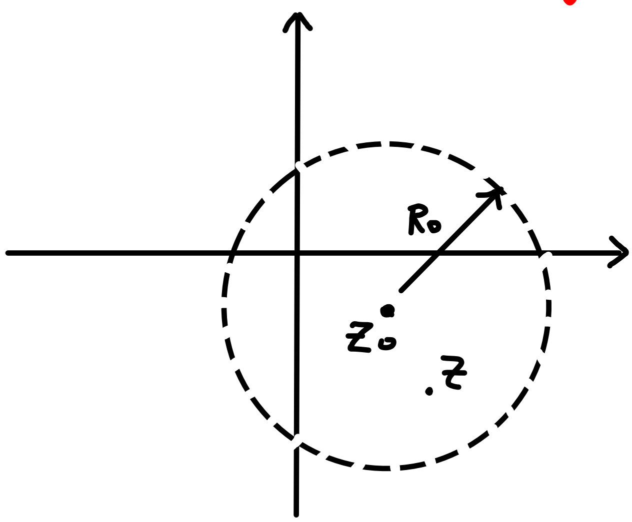
Taylor Series

Taylor's Theorem Suppose a function $f(z)$ is analytic throughout a disk $|z-z_0|<R_0$, then $f(z)$ has the power series representation

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \quad , \qquad |z-z_0| < R_0$$

$$a_k = \frac{f^{(k)}(z_0)}{k!} \quad \rightsquigarrow \ k\text{-th derivative}$$

That is the series converges to $f(z)$ when z lies in $|z-z_0|<R_0$

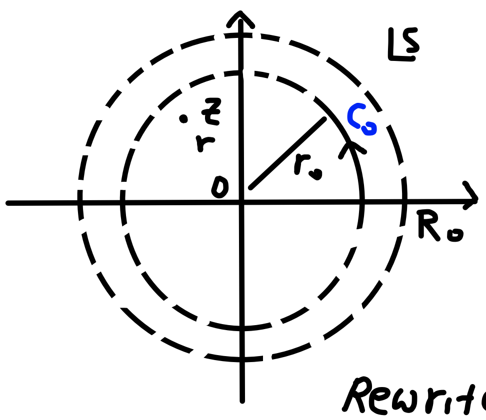


Proof Prove the theorem at $z_0=0$, the proof for other $z_0 \neq 0$ follows trivially

Setting $z_0=0$, $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$

$|z|<R_0$ (Maclaurin's)

To derive this, consider $|z|<r$, and C_0 $|z|=r_0$ contour



$f(z)$ is analytic inside and on C_0 an z in C_0, from Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{ds\, f(s)}{s-z}$$

Rewrite $\frac{1}{s-z} = \sum_{k=0}^{N-1} \frac{z^k}{s^{k+1}} + \frac{z^N/s^N}{(s-z)}$ for some N

(Using $\frac{1-x^N}{(1-x)} = \sum_{k=0}^{N-1} x^k$) $\rightsquigarrow$ Now substituting this back into above

Cont'd

$$\frac{1}{2\pi i}\int_{C_0}\frac{ds\, f(s)}{s-z} = \frac{1}{2\pi i}\sum_{k=0}^{N-1}\int_{C_0}\frac{f(s)\,ds}{s^{k+1}} + z^N\int_{C_0}\frac{f(s)\,ds}{(s-z)s^N}$$

$$= \sum_{k=0}^{N-1}\frac{f^{(k)}(0)}{k!}z^k + \underbrace{\frac{z^N}{2\pi i}\int_{C_0}\frac{ds\, f(s)}{(s-z)s^N}}_{\rho_N(z)}$$

Need to show this vanishes when $N \to \infty$

Cauchy's Differential formula

To show this, notice that $r_0 > r$ so that if s is a point on C_0

$$|s-z| \geq \big||s| - |z|\big| = r_0 - r$$

Max value of $|f(s)|$ on C_0

$$|\rho_N(z)| \leq \frac{r^N}{2\pi}\frac{M}{(r_0-r)r_0^N}\, 2\pi r_0 = \frac{M r_0}{r-r_0}\left(\frac{r}{r_0}\right)^N$$

Maximal value thrm

Clearly $\lim_{N\to\infty}\left(\frac{r}{r_0}\right)^N = 0$ ⇝ Proof established for $z_0 = 0$

Can repeat for abitrary z_0

<u>Corollary</u> If $f(z)$ is analytic in $S \subset \mathbb{C}$, containing z_0 then

$$f(z) = \sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

is valid in the largest open disk of center z_0 contained in S

Few well-known Taylor Series e^z ~ entire function

$$e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!}, \quad \cos z = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}z^{2k}, \quad \sin z = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}z^{2k+1}$$

Functions with branch cuts Choosing appropriate branch to ensure single valued

e.g.

$$(1+z)^\mu = 1 + \mu z + \binom{\mu}{2}z^2 + \cdots + \binom{\mu}{n}z^n + \cdots, \quad \mu \sim \text{fraction}$$

$$\binom{\mu}{n} = \frac{\mu(\mu-1)(\mu-2)\cdots(\mu-n+1)}{1\cdot 2\cdot 3\cdots n}$$

Radius of convergence is at least 1 (R)

⇝ Single valued + Analytic in $|z| < 1$, but if radius of convergence > 1

$\Rightarrow (1+z)^\mu$ is bounded → Contradiction! Resolution → $R = 1$

Laurent Series

If a function $f(z)$ fails to be analytic at some point z_0, Taylor's Theorem does not apply, we can however generalize it to include negative powers of z, ie including series of the form

$b_0 + b_1/z + b_2/z^2 + \cdots \frac{b_n}{z^n} + \cdots$ ~> Convergent when $|z| > R_1$

e.g. $\exp(1/z) = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots + \frac{1}{k!}\frac{1}{z^k} + \cdots$

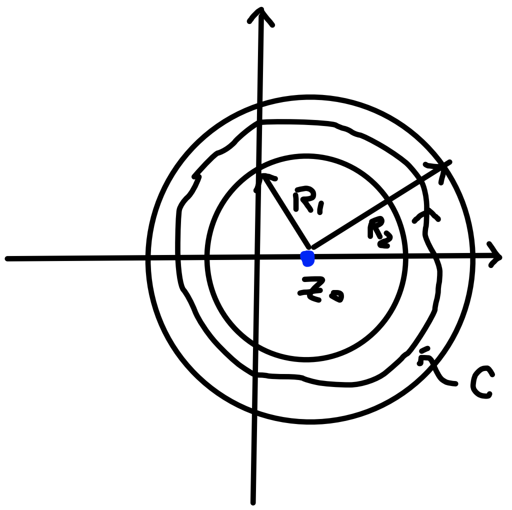
While the positive power series

$a_0 + a_1 z + a_2 z^2 + \cdots$ ~> convergent when $|z| < R_2$

$R_1 < |z| < R_2$ or $R_1 < |z - z_0| < R_2$

Laurent's Theorem

Suppose a function $f(z)$ is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C be any positively oriented contour around z_0 and in that domain, then at each pt in that domain



$$f(z) = \sum_{k=-\infty}^{\infty} C_k (z - z_0)^k$$

$R_1 < |z - z_0| < R_2$

$$C_k = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z - z_0)^{k+1}}, \quad k = 0, \pm 1, \pm 2$$

~ Laurent Series

Proof Basically we need that $f(z)$ can be splitted into (in toroidal region)

$$f(z) = f_+(z) + f_-(z)$$

$f_+(z)$ analytic when $|z - z_0| < R_2$, $f_-(z)$ analytic when $|z - z_0| > R_1$ → Cont'd

Also just need to show that

$f_+(z)$ becomes series containing non-negative power of $(z-z_0)$.
$f_-(z)$ becomes series containing non-positive power of $(z-z_0)$.

For $|z-z_0|<r<R_2$, we have $f_+(z)=\frac{1}{2\pi i}\int_{|s-z_0|=r}\frac{f(s)\,ds}{s-z}$

~ analytic in $|z-z_0|<R_2$

For $R_1<r<|z-z_0|$, taking into account of the orientation, we have

$$f_-(z)=-\frac{1}{2\pi i}\int_{|s-z_0|=r}\frac{f(s)\,ds}{s-z}$$ ~ analytic in $|z-z_0|>R_1$

Focusing on $f_+(z)$, using Taylor's series, we have

$$f_+(z)=\sum_{k=0}^{\infty}A_k\underbrace{(z-z_0)^k,\ A_k}_{\text{Taylor + Cauchy}}=\frac{1}{2\pi i}\int_{|s-z_0|=r}\frac{f(s)\,ds}{(s-z_0)^{k+1}}$$

$|z-z_0|<r<R_2$

Note $\frac{1}{s-z}=\frac{1}{(s-z_0)-(z-z_0)}$

For $f_-(z)$, Set $s=z_0+\frac{1}{s'}$, $z=z_0+\frac{1}{z'}$ $\Rightarrow$ $|s-z_0|=1/s'$

$$\Rightarrow f_-(z)=f_-(z_0+1/s')=\frac{1}{2\pi i}\int_{|s'|=1/r}ds'\,\frac{z'}{s'}\,\frac{f(z_0+1/s')}{s'-z'}=\sum_{k=1}^{\infty}B_k(z')^k$$

$\frac{1}{|z-z_0|}<\frac{1}{r}<\frac{1}{R_1}$

$=z'$, s'

Note that

$$\frac{z'}{s'}\frac{1}{s'-z'}=\frac{z'}{(s')^2}\sum_{k=0}^{\infty}\left(\frac{z'}{s'}\right)^k$$

where $B_k=\frac{1}{2\pi i}\int_{|s'|=1/r}\frac{f(z_0+1/s')}{(s')^{k+1}}ds'=\frac{1}{2\pi i}\int_{|s-z_0|=r}ds\ f(s)(s-z_0)^{k-1}$

$$\Rightarrow f_-(z)=\sum_{k=1}^{\infty}B_k(z-z_0)^{-k}$$

Combining $f_+(z)$ & $f_-(z)$, we complete the proof for Laurent series

Some Examples

1 $f(z)=\frac{1}{(z-i)^2}$ $\rightsquigarrow$ Already of the form of Laurent Series around $z=i$

Suppose $f(z)=\sum_{k=-\infty}^{\infty} C_k (z-i)^k$, $0<|z-i|<\infty$

$$C_k=\frac{1}{2\pi i}\int_C \frac{ds}{(s-i)^{k+3}} \quad , \quad k=0,\pm1,\pm2,\pm3 \;--$$

C ~ any positively oriented circle
$|z-i|=R$, about the pt $z_0=i$

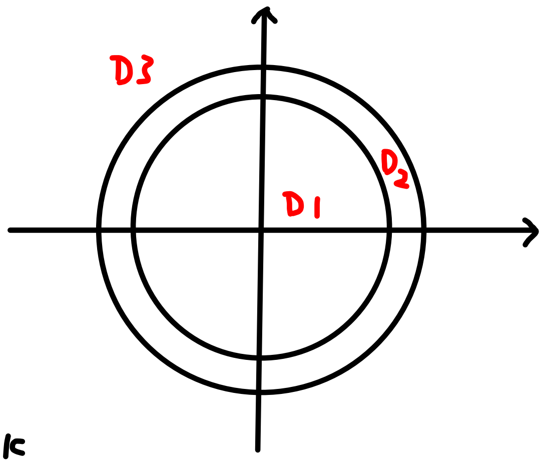
$$\frac{1}{2\pi i}\int_C \frac{dz}{(z-i)^{k+3}}=\begin{cases}0 & k\neq -2\\ 1 & k=-2\end{cases}$$ $\rightsquigarrow$ only C_{-2} non-vanishing, as expected

2 $f(z)=\frac{-1}{(z-1)(z-2)}=\frac{1}{z-1}-\frac{1}{z-2}$ $\rightsquigarrow$ two singularities $z=1$ & $z=2$

$\rightsquigarrow$ analytic in the domain

$|z|<1$, $1<|z|<2$ and $2<|z|<\infty$

D1 D2 D3



D1 $f(z)=\frac{-1}{1-z}+\frac{1}{2}\frac{1}{(1-z/2)}$

$$=-\sum_{k=0}^{\infty} z^k+\sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}}=\sum_{k=0}^{\infty}(2^{-k-1}-1)z^k$$

D2 $f(z)=\frac{1}{z}\frac{1}{1-(\frac{1}{z})}+\frac{1}{2}\frac{1}{1-z/2}=\sum_{k=0}^{\infty}\frac{1}{z^{k+1}}+\sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}}$

$$=\sum_{k=0}^{\infty}\frac{1}{z^{k+1}}+\sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}}$$

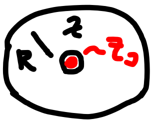
D3 $f(z)=\frac{1}{z}\frac{1}{1-1/z}-\frac{1}{z}\frac{1}{1-2/z}$, $|1/z|<1$, $|2/z|<1$ in D3

$$=\sum_{k=0}^{\infty}\frac{1}{z^{k+1}}-\sum_{k=0}^{\infty}\frac{2^k}{z^{k+1}}=\sum_{k=0}^{\infty}\frac{1-2^k}{z^{k+1}}=\sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n}$$ #

Classification of Singularities

The Laurent Series generalizes the Taylor series in the presence of singularities, that is a point z_0 where $f(z)$ fails to be analytic at z_0, but analytic everywhere else

An isolated Singularity A singularity z_0 is isolated if there is a deleted neighborhood/punctured disk, $0<|z-z_0|<R$ for $R>0$ but not at $z=z_0$.



Punctured Disk

Removable Singularity If a singularity z_0 is removable if there is a function $g(z)$ is holomorphic in $|z-z_0|<R$ such that $f(z)=g(z)$ in $0<|z-z_0|<R$

A pole If $\lim\limits_{z\to z_0} |f(z)| = \infty$

Essential Singularity If z_0 is neither removable nor a pole

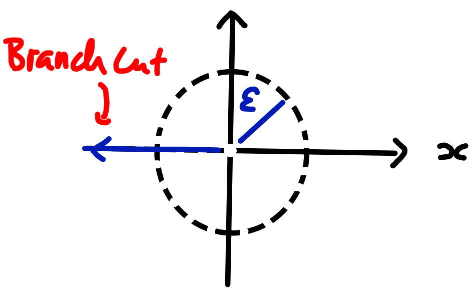
Punchline **Laurent Series** allows us to examine the nature of singularity

Example 1 $f(z) = \dfrac{\sin z}{z}$ around $z=0$, for $z \neq 0$ ~ Replace by $g(z)=1$ in $|z|<\varepsilon$

$$\frac{\sin z}{z} = \frac{1}{z} \sum_{k\geq 0} \frac{(-1)^k}{(2k+1)!} z^{k+1} = \sum_{k\geq 0} \frac{(-1)^k}{(2k+1)!} z^{2k}$$

Example 2 $f(z) = \log z = \log r + i\Theta \quad (r>0,\ -\pi<\Theta<\pi)$ principal branch

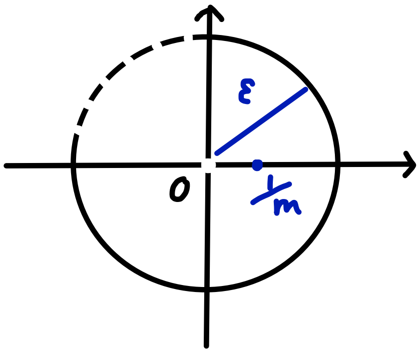
$z=0$ is a singularity, but not an isolated singularity, since every punctured disk $0<|z|<\varepsilon$ contain points on negative real axis, this branch is not defined



Example 3 $f(z)=\frac{1}{\sin(\pi/z)}$, has the singular pts $z=0$ and $z=1/n$

$n=\pm1,\pm2,\cdots$ Each singular pt except $z=0$ is isolated

For $z=0$ is NOT isolated, because every punctured disk of radius ε around $z=0$ contains other singularities, we can always find m st $\varepsilon>\frac{1}{m}$



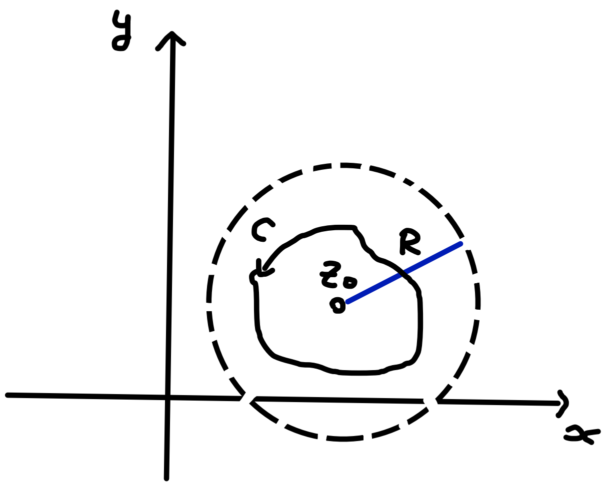
In fact, if z_0 is an isolated singularity of a function $f(z)$, there exists $R>0$ such that $f(z)$ is analytic in $0<|z-z_0|<R$,
we can represent $f(z)$ using Laurent series

$$f(z)=\sum_{k=0}^{\infty}a_k(z-z_0)^k+\frac{b_1}{(z-z_0)}+\frac{b_2}{(z-z_0)^2}+\cdots\frac{b_n}{(z-z_0)^n}+\cdot$$

$$0<|z-z_0|<R$$

where $b_n=\frac{1}{2\pi i}\int_C\frac{f(z)\,dz}{(z-z_0)^{1-n}}$

C is any positively oriented simple closed contour around z_0 and in punctured disk $0<|z-z_0|<R$



If $n=1$,

Only surviving term!

$$b_1=\frac{1}{2\pi i}\int_C f(z)\,dz$$

(or C_{-1})

is called the "Residue" of $f(z)$ at isolated singularity z_0

$$\operatorname*{Res}_{z=z_0} f(z)=\frac{1}{2\pi i}\int_C f(z)\,dz$$ → Common Notation

→ Doing integral from series expansion!

More generally, if z_0 is an isolated singularity with Laurent series

$$f(z) = \sum_{k\in\mathbb{Z}} C_k (z-z_0)^k, \quad C_k = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{k+1}}$$

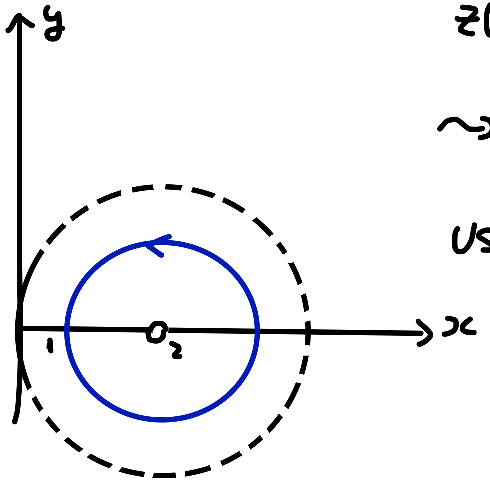
- z_0 = removable singularity, if $C_k = 0$ for $k<0$, ie becoming a Power Series
- z_0 = Pole of order/degree n, $n>0$, if $C_{-n} \neq 0$, $n<\infty$, ie finitely many negative exponents $\sim (z-z_0)^n f(z)$ is not singular
- z_0 = Essential Singularity if $C_{-\infty} \neq 0$, infinitely many negative exponents

<u>More Examples</u> Meromorphic function: A function $f(z)$ is meromorphic if it is analytic everywhere except for Poles

4 $\displaystyle\int_C \frac{dz}{z(z-2)^4}$, C = Positively Oriented Circle $|z-2|=1$

Since the integrand is analytic everywhere except pts $z=0$ (Simple Pole) and $z=2$ (4-th order pole), it has Laurent expansion in punctured disk

$0<|z-2|<2$



$$\frac{1}{z(z-2)^4} = \frac{1}{2(z-2)^4}\,\frac{1}{1-\left(-\frac{z-2}{2}\right)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} (z-2)^{k-4}$$

$\rightsquigarrow$ coefficient for $(z-2)^{-1}$ term $= \frac{-1}{16}$ $(k=3)$

Using $2\pi i \operatorname*{Res}_{z=z_0} f(z) = \int_C dz\, f(z)$, we obtain

$$2\pi i\left(-\frac{1}{16}\right) = \frac{-\pi i}{8} = \int_C \frac{dz}{z(z-2)^4}$$

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5 $\int_C dz \exp(1/z^2)$, C is a unit circle $|z|=1$, since $\exp(1/z^2)$ is analytic everywhere except $z=0$, consider the punctured disk

$0<|z|<\infty \Rightarrow \exp(1/z^2) = 1 + \frac{1}{1!}\frac{1}{z^2} + \frac{1}{2!}\frac{1}{z^4} + \cdots$ Residue $=0$

Laurent Series

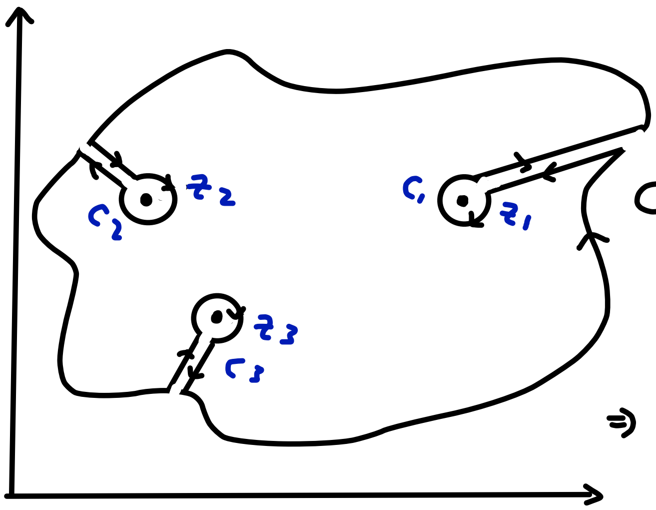
$\Rightarrow$ Integral $=0$

Cauchy's Residue Theorem (Generalizing earlier examples)

Suppose $f(z)$ is analytic inside and on a simple, closed, positively oriented contour C, except for finite number of isolated singularities $z_1, \quad z_n$ inside C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

Proof (By deforming the contour C to go around $z_1, z_2 \quad z_n$



$$0 = \int_C dz\, f(z) - \sum_{k=1}^{n} \underbrace{\int_{C_k} dz\, f(z)}_{2\pi i \operatorname*{Res}_{z=z_0} f(z)}$$

~ definition

$$\Rightarrow \int dz\, f(z) = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

Corollary (beyond simple pole)

Suppose z_0 is an order n pole of $f(z)$, then

$$\operatorname*{Res}_{z=z_0}(f(z)) = \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}\left((z-z_0)^n f(z)\right)$$

Proof By definition of "order n" Pole, we know that the Laurent series is

$$f(z) = \sum_{k\geq -n} c_k (z-z_0)^k$$

$$(z-z_0)^n f(z) = \sum_{k\geq -n} c_k (z-z_0)^{k+n}$$ $\to$ can use Taylor's formula to get c_{-1} and match with the limit of derivatives

Useful little results

a) Let f have a simple pole at z_0 and $g(z)$ is holomorphic at z_0, then

$$\operatorname*{Res}_{z=z_0}\left(f(z)g(z)\right) = g(z_0)\operatorname*{Res}_{z=z_0}\left(f(z)\right)$$

b) If $f(z_0)=0$, $f'(z_0)\neq 0$ Then $1/f$ has a simple pole at z_0 and the residue of $1/f$ at z_0 is $1/f'(z_0)$

Proof for a) Let $f(z) = \frac{a_{-1}}{(z-z_0)}$ + higher terms Say $z_0=0$ for simplicity, we have

$$f(z)g(z) = \left(\frac{a_{-1}}{z} + \quad\right)\left(b_0 + b_1 z + \quad\right) = \frac{a_{-1}b_0}{z} + \text{higher terms}$$

$$\Rightarrow \operatorname{Res}\left(f(z)g(z)\right) = g(0)\operatorname*{Res}_{z=0}\left(f(z)\right)$$ $\rightsquigarrow$ Proof for $z\neq 0$ follows trivially

Proof for b) Let $f(z_0)=0$, $f'(z_0)\neq 0$, then $f(z) = a_1(z-z_0)$ + higher terms, and $a_1 \neq 0$ Let $z_0=0$ for simplicity Then

$f(z) = a_1 z(1+h)$ with order $h \geq 1 \rightarrow \frac{1}{f(z)} = \frac{1}{a_1 z}(1-h+h^2- \quad) = \frac{1}{a_1 z}$ + higher

$\Rightarrow \operatorname*{Res}_{z=0}\left(1/f(z)\right) = 1/a_1 = 1/f'(0)$ as claimed Similar for $z_0 \neq 0$

Elementary Usage of Residue Theorem

From $$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$ $\rightsquigarrow$ Doing contour integrals is the same as finding residue !

Example 1

Find the residue of $f(z) = \frac{z^2}{(z^2-1)}$ at $z=1$

Note $f(z) = \frac{z^2}{(z-1)(z+1)}$, $\frac{z^2}{(z+1)}$ is analytic at $z=1 \Rightarrow$ Residue at $z=1$, is $1/2$

Example 2 Find the residue of $\frac{\sin z}{z^2}$ at $z = 0$

Using Taylor's series $\frac{\sin z}{z^2} = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \qquad\right)$

$$= \frac{1}{z} + \text{higher terms} \rightsquigarrow \text{Residue} = 1$$

Example 3 Find $\displaystyle\int_C \frac{dz\, z^2}{\underbrace{(z+1)(z-1)^2}_{f(z)}}$, $C = |z-1| = 1$, unit circle centered at 1

The function $f(z)$ has only two singularities, at $z = \pm 1$, C contains $z = 1$, we need to evaluate the $\operatorname*{Res}_{z=1} f(z)$

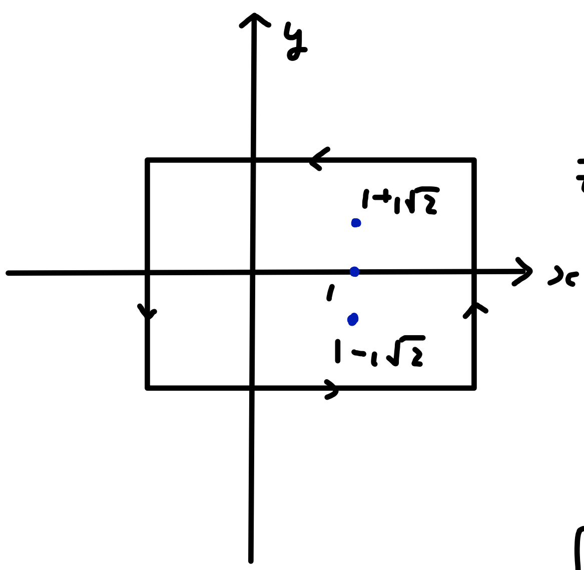
We can Taylor expand $\frac{z^2}{z+1}$ about $z=1$, and isolate $\frac{1}{(z-1)^2}$ piece

$$\frac{z^2}{z+1} = \frac{1}{2}\left(1 + \frac{3}{2}(z-1) + \cdot \quad\right) \Rightarrow \frac{z^2}{z+1}\frac{1}{(z-1)^2} = \frac{1}{2(z-1)^2} + \frac{3}{4}\frac{1}{(z-1)} +$$

Residue at $z=1$

$$\Rightarrow \int_C dz\, f(z) = 2\pi i \quad \frac{3}{4} = \frac{3\pi i}{2}$$

Example 4 Let $f(z) = z^2 - 2z + 3$, and C be rectangle, find $\int_C f(z)$



$$\int_C dz \frac{1}{f(z)} = \int_C dz \frac{1}{(z-z_+)(z-z_-)}$$

$z_\pm = 1 \pm i\sqrt{2}$ ⇝ Simple Poles

$$\operatorname*{Res}_{z=z_+} \frac{1}{f(z)} = \frac{1}{z_+ - z_-}$$

$$\operatorname*{Res}_{z=z_-} \frac{1}{f(z)} = \frac{1}{z_- - z_+}$$

$$\int_C dz \frac{1}{f(z)} = 2\pi i\left(\operatorname*{Res}_{z=z_+} \frac{1}{f(z)} + \operatorname*{Res}_{z=z_-} \frac{1}{f(z)}\right) = 0$$

What about $f(z) = z^2 + az + b$?

Argument Principle

Meromorphic function A function $f(z)$ that is holomorphic everywhere except poles of finite degrees

Suppose a meromorphic function $f(x)$ in S, which has poles at $P_1, P_2, \cdots, P_K \subset S$, of degrees $m_1, m_2, \cdots, m_K$ respectively,

The Logarithmic derivative

$$\frac{f'(z)}{f(z)} = -\frac{m_1}{z-P_1} - \frac{m_2}{z-P_2} - \frac{m_3}{z-P_3} - \cdots - \frac{m_K}{z-P_K} + \frac{g'(z)}{g(z)}$$

← $g(z)$ has no poles in S

Furthermore, $f(z)$ can also have zeros at $z_1, z_2 \cdots Z_L$ of order $n_1, \cdots n_L$ respectively, and we can express the Logarithmic derivative

$$\frac{f'(z)}{f(z)} = \frac{n_1}{z-z_1} + \frac{n_2}{z-z_2} + \cdots \frac{n_L}{z-z_L} + \frac{\tilde{g}'(z)}{\tilde{g}(z)}$$

Combining these poles and zeros, we have in S

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{L} \frac{n_i}{(z-z_i)} - \sum_{j=1}^{K} \frac{m_j}{(z-P_j)} + \frac{h'(z)}{h(z)}$$

← $h(z)$ is a holomorphic function in S and has no zeros

Now if we consider a closed, smooth, positively oriented contour C which encloses these poles and zeros, we can consider the contour integral

$$\int_C dz \frac{f'(z)}{f(z)} = \sum_{i=1}^{L} n_i \int_C \frac{dz}{(z-z_i)} - \sum_{j=1}^{K} m_j \int_C \frac{dz}{(z-P_j)} + \int_C dz \frac{h'(z)}{h(z)}$$

$$= 2\pi i \Big(\sum_{i=1}^{L} n_i - \sum_{j=1}^{K} m_j \Big) + \int_C dz \frac{h'(z)}{h(z)}$$

← vanishes by Cauchy's Principle

From "deforming" the contour trick

Number of zeros

Number of Poles

we have derived the so-called "Argument Principle"

$$\frac{1}{2\pi i}\int_C dz \frac{f'(z)}{f(z)} = Z(f,c) - P(f,c)$$

$Z(f,c)$ = Number of zeros of $f(z)$ inside C, counted with multiplicity

$P(f,c)$ = Number of Poles of $f(z)$ inside C, counted with multiplicity

Corollary (Rouche's Theorem)

Suppose $f(z)$ & $g(z)$ are holomorphic in region S, and C is a smooth, positively oriented, simple, closed, S-contractible curve, such that for all $z \in C$, $|f(z)| > |g(z)|$, we have

$$Z(f+g,\gamma) = Z(f,\gamma)$$ ~ Number of zeros

Proof From previous result, we have

$$Z(f+g,C) = \frac{1}{2\pi i}\int_C \frac{(f+g)'}{(f+g)}dz = \frac{1}{2\pi i}\int_C dz\left(\frac{f'}{f} + \frac{(1+g/f)'}{(1+g/f)}\right)$$

$$= Z(f,c) + \frac{1}{2\pi i}\int_C dz \frac{d}{dz}\log(1+g/f)$$

Now $\frac{|g(z)|}{|f(z)|} < 1 \Rightarrow (1+g/f)$ is away from non-negative real axis (branch cut), this implies that $\log(1+g/f)$ is holomorphic, and

$$\int_C dz \frac{d}{dz}\log(1+g/f) = 0$$ ⇒ Completes our proof

Example Let $P(z) = z^8 - 5z^3 + z - 2$, we want to know the number of roots $P(z)=0$ inside unit circle $|z|=1$

Let $P(z) = f(z) + g(z)$, $f(z) = -5z^3$, $g(z) = -z^8 - z + 2$

$\Rightarrow |g(z)| < |f(z)|$ inside $|z|=1 \Rightarrow -5z^3 = 0$ has 1 zero but multiplicity 3 in $|z|=1$

$\Rightarrow P(z)$ has 3 zeros in C