Applied Maths III   Lecture 13

Plancherel's and Parseval's Identities

Without entering the full details, let us briefly introduce the concept of Hilbert Space of Complex Squared integrable functions, which is sometimes denoted as $\mathcal{L}^2 = \mathcal{L}^2(\mathbb{R})$

It is an infinite dimensional vector space consisting of all complex-valued functions $f(x)$, $x \in \mathbb{R}$ such that

$$\| f \|^2 = \int_{-\infty}^{\infty} dx \, |f(x)|^2 < \infty \sim \text{finite } \mathcal{L}^2\text{-norm}$$

eg $|f(x)| \sim \frac{1}{|x|^{\frac{1}{2}+\delta}}$, if $|x| \gg 0$, $\delta > 0$

$\Rightarrow f(x) \in \mathcal{L}^2$, however $\mathcal{L}^2$ can contain other functions

We can in general define the inner product on $\mathcal{L}^2$ as

$$\langle f, g \rangle = \int_{-\infty}^{\infty} dx \, f(x) \bar{g}(x)$$

$$|\langle f, g \rangle|^2 \leq \| f \|^2 \| g \|^2 \sim \text{Cauchy-Schwarz inequality}$$

$\Rightarrow$ If $f(x), g(x) \in \mathcal{L}^2$, $\langle f, g \rangle \sim$ finite

In particular, if $f(x) \in \mathcal{L}^2$, then its Fourier transform $\hat{f}(k) \in \mathcal{L}^2$

Parseval's Formula

If $f(x), g(x) \in \mathcal{L}^2$ then

$$\int_{-\infty}^{\infty} dx \, f(x) \bar{g}(x) = \int_{-\infty}^{\infty} dk \, \hat{f}(k) \overline{\hat{g}(k)}, \quad \hat{f}(k) = F[f(x)], \quad \hat{g}(k) = F[g(x)]$$

Proof

$$\int_{-\infty}^{\infty} dk \, \hat{f}(k) \overline{\hat{g}(k)} = \int_{-\infty}^{\infty} dk \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, f(x) e^{-ikx} \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy \, \overline{g(y)} e^{iky} \right)$$

$$= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \, f(x) \overline{g(y)} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{-ik(x-y)} = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \, f(x) \overline{g(y)} \, \delta(x-y)$$

$\gg$

$$\Rightarrow \int_{-\infty}^{\infty} dk\, \hat{f}(k)\, \overline{\hat{g}(k)} = \int_{-\infty}^{\infty} dx\, f(x)\, \overline{g(x)}$$

In particular if $f(x)$ & $g(x)$ are "orthogonal" w r t $\langle\ ,\ \rangle$, i.e $\langle f, g\rangle = 0 \Rightarrow \langle \hat{f}, \hat{g}\rangle = 0$

Also if $f = g$

$$\Rightarrow \int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} dk\, |\hat{f}(k)|^2$$ ~ Plancherel Identity

The "Norm" of a function $f(x)$ is invariant under Fourier transform

Example

① $f(x) = 1, \quad |x| < a$
$\quad\;\; = 0, \quad |x| > a$ $\Rightarrow \hat{f}(k) = \sqrt{\frac{2}{\pi}}\, \frac{\sin ak}{k}$

We can use the identities deduced above

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} dk\, |\hat{f}(k)|^2 = 2a = \frac{2}{\pi} \int_{-\infty}^{\infty} dk \left(\frac{\sin ak}{k}\right)^2$$

$\Rightarrow$ We obtain non-trivial integral identity

$$\int_{-\infty}^{\infty} dk \left(\frac{\sin ak}{k}\right)^2 = \pi a$$

② $f(x) = e^{-a|x|}$, $\Rightarrow \hat{f}(k) = \sqrt{\frac{2}{\pi}}\, \frac{a}{a^2+k^2}$

$$\Rightarrow \int_{-\infty}^{\infty} dx \left(e^{-a|x|}\right)^2 = \int_{-\infty}^{\infty} dk \left(\sqrt{\frac{2}{\pi}}\, \frac{a}{a^2+k^2}\right)^2 = \frac{2}{\pi} \int_{-\infty}^{\infty} dk\, \frac{a^2}{(a^2+k^2)^2}$$

$$\Rightarrow \frac{1}{a} = \frac{2a^2}{\pi} \int_{-\infty}^{\infty} dk\, \frac{1}{(a^2+k^2)^2} \Rightarrow \frac{\pi}{2a^3} = \int_{-\infty}^{\infty} \frac{dk}{(a^2+k^2)^2}$$

Uncertainty Principle / Minimal Wave Packet

From the identities deduced above to show the "uncertainty Principle" in Quantum Mechanics

Without introducing too much details, we state here that the "uncertainty of position"

$$(\Delta x)^2 = \frac{\int_{-\infty}^{\infty} dx\, (x-x_0)^2 \, |\psi(x)|^2}{\int_{-\infty}^{\infty} dx\, |\psi(x)|^2}$$

$\psi(x) \sim$ "Wave Function"

This measures the "Dispersion" or "Spread" of the position x of the "Particle/wave"

We can also define the "Uncertainty of momentum" as

$$(\Delta p)^2 = \frac{\int_{-\infty}^{\infty} dp\, (p-p_0)^2 \, |\hat{\psi}(p)|^2}{\int_{-\infty}^{\infty} dp\, |\hat{\psi}(p)|^2}$$

$\rightsquigarrow \hat{\psi}(p) = F_p[\psi(x)]$ "Wave Function in p space"

$\rightsquigarrow$ can motivate this expression by considering in QM,

"Momentum Operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$" $\Rightarrow \hat{p}\,\psi(x) = -i\hbar \frac{\partial}{\partial x}\psi(x)$

"Fourier Transform" it into p-space, $F_p\left[-i\hbar \frac{\partial}{\partial x}\psi(x)\right] = \hbar p \hat{\psi}(p)$

Uncertainty Principle

$$(\Delta x)^2 (\Delta p)^2 \geq \frac{\hbar^2}{4}$$

Proof (Prove for $x_0 = p_0 = 0$, non-zero x_0, p_0 can be proved by translational properties)

Consider $\int_{-\infty}^{\infty} dx\, x^2 |\psi(x)|^2 \int_{-\infty}^{\infty} dp\, p^2 |\hat{\psi}(p)|^2$

$= \int_{-\infty}^{\infty} dx\, |x\psi(x)|^2 \int_{-\infty}^{\infty} dx\, \left|-i\frac{\partial}{\partial x}\psi(x)\right|^2 \hbar^2$ Parseval Id

$= \int_{-\infty}^{\infty} dx\, |x\psi(x)|^2 \int_{-\infty}^{\infty} dx\, \left|\frac{\partial \psi(x)}{\partial x}\right|^2 \hbar^2$

Cont'd $\rightsquigarrow$

Now consider

$$\left[\frac{1}{2}\int_{-\infty}^{\infty} dx\, x \left[\psi(x)\frac{\partial}{\partial x}\bar{\psi}(x) + \overline{\psi(x)}\frac{\partial}{\partial x}\psi(x)\right]\right]^2$$

$$= \left[\mathrm{Re}\left(\int_{-\infty}^{\infty} dx\, x\,\psi(x)\frac{\partial\bar{\psi}(x)}{\partial x}\right)\right]^2 \leqslant \left|\int_{-\infty}^{\infty} dx\, x\,\psi(x)\frac{\partial\bar{\psi}(x)}{\partial x}\right|^2$$

$$\leqslant \int_{-\infty}^{\infty} dx\, |x\,\psi(x)|^2 \int_{-\infty}^{\infty} dx\, \left|\frac{\partial\psi(x)}{\partial x}\right|^2 = \underbrace{\int_{-\infty}^{\infty} dx\, x^2|\psi(x)|^2}_{(\Delta x)^2} \underbrace{\int_{-\infty}^{\infty} dx\left|\frac{\partial\psi}{\partial x}\right|^2}_{\frac{(\Delta p)^2}{\hbar^2}}$$

Now we would like to show that

$$-\int_{-\infty}^{\infty} dx\, x\left(\psi(x)\frac{d}{dx}\overline{\psi(x)} + \bar{\psi}(x)\frac{d}{dx}\psi(x)\right) = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2$$

$$* \int_{-\infty}^{\infty} dx\, x\,\psi(x)\left(-\frac{d}{dx}\bar{\psi}(x)\right) = -\int_{-\infty}^{\infty} dx\, \frac{d}{dx}\left(x|\psi|^2\right) + \int_{-\infty}^{\infty} dx\, x\frac{d}{dx}\psi(x)\,\overline{\psi(x)}$$
$$+ \int_{-\infty}^{\infty} \frac{dx}{dx}\, |\psi|^2\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = -\int_{-\infty}^{\infty} dx\, x\left(\psi(x)\frac{d\bar{\psi}(x)}{dx} + \bar{\psi}(x)\frac{d\psi(x)}{dx}\right)$$ as required

$$= \int_{-\infty}^{\infty} dp\, |\hat{\psi}(p)|^2$$

Collecting the pieces, we have

$$\frac{1}{4}\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 \int_{-\infty}^{\infty} dp\, |\hat{\psi}(p)|^2 \leqslant \int_{-\infty}^{\infty} dx\, x^2|\psi(x)|^2 \int_{-\infty}^{\infty} dx\, \left|\frac{\partial\psi(x)}{\partial x}\right|^2$$
$$= \int_{-\infty}^{\infty} dx\, x^2|\psi(x)|^2 \int_{-\infty}^{\infty} dp\, \frac{p^2}{\hbar^2}|\hat{\psi}(p)|^2$$

$$\Rightarrow \frac{\int_{-\infty}^{\infty} dx\, x^2|\psi(x)|^2}{\int_{-\infty}^{\infty} dx\, |\psi(x)|^2}\;\frac{\int_{-\infty}^{\infty} dp\, p^2|\hat{\psi}(p)|^2}{\int_{-\infty}^{\infty} dp\, |\hat{\psi}(p)|^2} \geqslant \frac{1}{4}\hbar^2$$

or $(\Delta x)(\Delta p) \geq \hbar/2$ ~ Heisenberg Uncertainty Principle

Minimal Wave Packet

We can also ask the question,

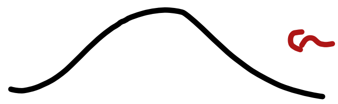
"What kind of $\psi(x)$ (or $\hat{\psi}(k)$) saturates the inequality?"

First we notice that the inequality is saturated, if

$$\psi(x)\,\overline{\frac{\partial\psi(x)}{\partial x}} = \bar{\psi}(x)\,\frac{\partial\psi}{\partial x} \rightsquigarrow \text{First inequality}$$

$$\frac{d}{dx}\psi(x) = K x\,\psi(x) \rightsquigarrow \text{Cauchy-Schwarz} \quad K \in \mathbb{R}$$

$$\Rightarrow \psi(x) = c\,e^{K\frac{x^2}{2}}, \quad \text{if} \quad K \in \mathbb{R}^-$$



← Gaussian wave

$c \rightsquigarrow$ Arbitrary Real const

$$\text{or} \quad \psi(x) = c\,e^{-K\frac{x^2}{2}}, \quad K \in \mathbb{R}^+$$ ← "Minimal Uncertainty"

(Notice the Fourier Transform of Gaussian is also Gaussian)

⇒ Can plug $\psi(x)$ back into the equation to check!

⇒ Try proving this for general x_0, p_0, and other observables

Shannon Sampling Theorem

If the Fourier Transform $\hat{f}(\omega)$ of $f(t)$ has a cutoff frequency ω_c such that

$$\hat{f}(\omega) = 0, \quad \text{for} \quad |\omega| > \omega_c > 0$$ ← e.g. acoustic signal

Cutoff in Human Hearing

A first step in processing such signal is to do "Sampling", that is we replace continuous $f(x)$ by its "Samples" at regular interval

$$\{ f(nT),\ n \in \mathbb{Z} \}$$

Shannon's Sampling theorem states that we can "reconstruct" $f(x)$ from its samples $\{f(nT), n\in\mathbb{Z}\}$, provided the sampling frequency $\sim \frac{1}{T}$ is sufficiently large, $\left(\frac{2\pi}{T} \geq \omega_c\right)$

We can demonstrate this by considering $f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \hat{f}(\omega)\, e^{i\omega t}$ ↘

$$\Rightarrow f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\omega_c}^{\omega_c} d\omega\, \hat{f}(\omega)\, e^{i\omega t}$$ $\Rightarrow$ "Sample at $t = nT$

$$f(nT) = \frac{1}{\sqrt{2\pi}} \int_{-\omega_c}^{\omega_c} d\omega\, \hat{f}(\omega)\, e^{i\omega nT}$$

Now if let $T = \frac{\pi}{\omega_c}$ or $\frac{2\pi}{T} = 2\omega_c$

Fourier coefficient

$$\Rightarrow f\left(\frac{n\pi}{\omega_c}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\omega_c}^{\omega_c} d\omega\, \hat{f}(\omega)\, e^{i\omega\frac{n\pi}{\omega_c}} \rightsquigarrow = \frac{2\omega_c}{\sqrt{2\pi}} \times C_n$$

$$\Rightarrow \hat{f}_p(\omega) = \sqrt{\frac{\pi}{2}}\, \frac{1}{\omega_c} \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\omega_c}\right) \exp\left(-\frac{in\pi\omega}{\omega_c}\right)$$

This is actually a "Periodic Extension of $\hat{f}(\omega)$ outside $|\omega| \geq \omega_c$

To cut this off, we can multiply $\hat{f}_p(\omega)$ with

$$\hat{h}(\omega) = 1, \quad |\omega| \leq \omega_c$$
$$= 0, \quad |\omega| > \omega_c$$

$$\Rightarrow \hat{f}(\omega) = \sqrt{\frac{\pi}{2}}\, \frac{1}{\omega_c} \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\omega_c}\right) \exp\left(-\frac{in\pi\omega}{\omega_c}\right) \times \hat{h}(\omega)$$

Finally, we can Fourier transform $\hat{f}(\omega)$ back into $f(t)$

$$f(t) = \frac{1}{2\omega_c} \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\omega_c}\right) \times \int_{-\infty}^{\infty} d\omega\, \exp\left(i\omega\left(t - \frac{n\pi}{\omega_c}\right)\right) \hat{h}(\omega)$$

$$= \frac{1}{2\omega_c} \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\omega_c}\right) \times \int_{-\omega_c}^{\omega_c} d\omega\, \exp\left(i\omega\left(t - \frac{n\pi}{\omega_c}\right)\right)$$

$$= \frac{1}{2\omega_c} \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\omega_c}\right) \left[\frac{\exp\left(i\omega\left(t - \frac{n\pi}{\omega_c}\right)\right)}{i\left(t - n\pi/\omega_c\right)} \right]_{-\omega_c}^{\omega_c}$$

$$= \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\omega_c}\right) \frac{\sin\left(\omega_c\left(t - \frac{n\pi}{\omega_c}\right)\right)}{\omega_c\left(t - n\pi/\omega_c\right)} = \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\omega_c}\right) \frac{\sin(\omega_c t - n\pi)}{(\omega_c t - n\pi)}$$

$\rightsquigarrow$ Shannon Sampling Theorem

The frequency limited signals can be represented for continuous time by its discretely sampled values, provided the sampling frequency $\frac{2\pi}{T} \geq 2\omega_c$ $\rightsquigarrow$ Fundamental importance in digital musics

Laplace Transform - An Introduction

In electrical/electronic, another integral transform plays prominent role, known as Laplacian Transform. We can regard it as the "Real Counterpart" of Fourier Transform. That is

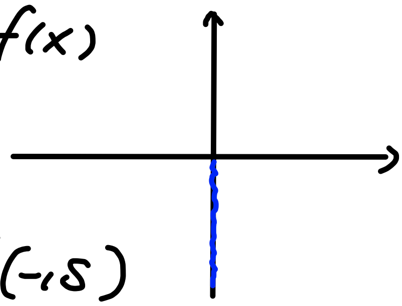
$$\underset{k\in\mathbb{R}}{e^{-ikx}} \xrightarrow{k=-is} \underset{s\in\mathbb{R}}{e^{-sx}}$$

⇝ Evavaluate on imaginary axis instead of real axis

As a result, Laplace transform enjoys many similar properties to Fourier transform, such as Linearity, or transforming derivatives into algebra. However one distinct feature of Laplace transform is that it is more suited for "initial valued problem" (usually with decaying pulse, $t>0$) rather than periodic pulses

Given Fourier Transform

$$F[f(x)] = \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)$$



The Laplace Transform is

$$\mathcal{L}[f(x)] = F(s) = \int_0^{\infty} dx\, f(x)\, e^{-sx} = \sqrt{2\pi}\,\hat{f}(-is)$$

$s\in\mathbb{R}$

"Real Integral" ⇝ Taking real function to real function

⇝ Also important, as e^{-sx} now decays at $t\to\infty$, so we do not need to restrict to $f(x)\to 0$, $x\to\infty$

Examples

① $f(x) = e^{\alpha x}$, $\alpha\in\mathbb{C}$

If $\mathrm{Re}(s-\alpha)>0$

$$\mathcal{L}[f(x)] = F(s) = \int_0^{\infty} dx\, e^{(\alpha-s)x} = \frac{e^{(\alpha-s)\infty}}{(\alpha-s)} - \frac{1}{(\alpha-s)} = \frac{1}{(s-\alpha)}$$

⇝ Laplace Transform $F(s)$ is only defined if $\mathrm{Re}(s-\alpha)>0$

⇝ From this we also obtain that if $\mathrm{Re}(\alpha)=0$ e.g. $\alpha = i\omega$ & $s>0$

convergence

$$\mathcal{L}[e^{i\omega x}] = \frac{1}{s-i\omega} = \frac{s+i\omega}{s^2+\omega^2} = \mathcal{L}[\cos\omega x + i\sin\omega x]$$

By linearity $\mathcal{L}[\cos\omega x] = \frac{s}{\omega^2+s^2}$, $\mathcal{L}[\sin\omega x] = \frac{\omega}{\omega^2+s^2}$

Also if $\omega = 0$ $\Rightarrow$ $\mathcal{L}[1] = 1/s$

Analogously, if α is purely real, i.e $\alpha = \pm k$ & $s \mp k > 0$

$\Rightarrow$ $\mathcal{L}[e^{\pm k}] = \frac{1}{s \mp k}$ $\Rightarrow$ $\mathcal{L}[\cosh kx] = \frac{s}{s^2-k^2}$

$\mathcal{L}[\sinh kx] = \frac{+k}{s^2-k^2}$

Furthermore if we consider differentiation under integration

$$\mathcal{L}\left[\frac{\partial}{\partial\alpha} e^{\alpha x}\right] = \frac{\partial}{\partial\alpha}\int_0^\infty dx\, e^{\alpha x} e^{-sx} = \int_0^\infty dx\, x\, e^{\alpha x} e^{-sx} = \frac{\partial}{\partial\alpha}\frac{1}{s-\alpha}$$

$$= \frac{1}{(s-\alpha)^2}$$

General expression

$$\mathcal{L}\left[\left(\frac{\partial}{\partial\alpha}\right)^n e^{\alpha x}\right] = \mathcal{L}[x^n e^{\alpha x}] = \frac{\partial^n}{\partial\alpha^n}\frac{1}{(s-\alpha)} = \frac{n!}{(s-\alpha)^{n+1}}$$

Setting $\alpha = 0$, we have $\mathcal{L}[x^n] = \int_{-\infty}^{\infty} dx\, x^n e^{-sx} = \frac{1}{s^{n+1}}\int_{-\infty}^{\infty} dy\, e^{-y} y^n$

Gamma Function $= \frac{1}{s^{n+1}} \times \Gamma(n+1) = \frac{n!}{s^{n+1}}$

To give a precise definition for the class of functions $\{f(x)\}$ whose Laplace Transform $\mathcal{L}[f(x)]$ is well-defined, we define

Def $f(x)$ is said to have Exponential Growth of Order α, if

$$|f(x)| < M e^{\alpha x} \quad \text{for all } x > x_0, \quad M > 0, \quad x_0 > 0$$

Statement If $f(x)$ is piecewise continuous and has exponential Growth of order α, then $F(s) = \mathcal{L}[f(x)]$ is defined for all $s > \alpha$

i.e Since $|f(x)e^{-sx}| < M e^{(\alpha-s)x}$, $x \to \infty$

$\rightsquigarrow$ Integrand vanishes as $x \to \infty$ $\rightsquigarrow$ Ensures Convergence

Properties of Laplace Transform

As mentioned earlier, Laplace Transform $\mathcal{L}[f(x)]$ shares many similar properties with Fourier Transform $\mathcal{F}[f(x)]$, such as

Shifting $\mathcal{L}[e^{cx} f(x)] = F(s-c)$

Dilatation $\mathcal{L}[f(cx)] = \frac{1}{c} F(s/|c|)$

Inversion $\mathcal{L}[f(-x)] = -F(s)$

Proof left as Exercise

Sudden Pulse
$$\mathcal{L}[\sigma(x-c) f(x-c)] = \int_c^\infty dx\, e^{-sx} f(x-c) = e^{-sc} F(s)$$

Product (Differentiation)
$$\frac{d}{ds}\int_0^\infty dx\, e^{-sx} f(x) = -\int_{-\infty}^\infty dx\, e^{-sx}\, x f(x) = \frac{d}{ds} F[s]$$

$$\Rightarrow \mathcal{L}[x f(x)] = -\frac{d}{ds} F[s] \Rightarrow \mathcal{L}[x^n f(x)] = (-1)^n \frac{d^n}{ds^n} F[s]$$

Derivatives, Convolution and Integration

To apply Laplace Transform to help us solving differential equations, we also need to consider the action of $\mathcal{L}[\ ]$ on derivatives

Consider
$$\mathcal{L}\left[\frac{d}{dx} f(x)\right] = \int_0^\infty dx\, e^{-sx} \frac{d}{dx} f(x) = \int_0^\infty dx \frac{d}{dx}\left(f(x) e^{-sx}\right) - \int_0^\infty dx\, (-s)\, e^{-sx} f(x)$$

$$\Rightarrow \mathcal{L}\left[\frac{d}{dx} f(x)\right] = sF(s) - f(0)$$

We can continue iteratively for n-times differentiable $f(x)$

$$\Rightarrow \mathcal{L}\left[\frac{d}{dx}\left(\frac{d}{dx} f\right)\right] = s\left(\mathcal{L}\left[\frac{d}{dx} f\right]\right) - \frac{d}{dx} f\Big|_{x=0}$$
$$= s(sF(s) - f(0)) - f'(0)$$
$$= s^2 F(s) - s f(0) - f'(0)$$

~ Notice this is different from Fourier transform

General Result

$$\mathcal{L}[f^{(n)}(x)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

Trivially Proved by induction

We can also define analogue of "Convolution" between $f(x)$ & $g(x)$ + $(\mathcal{L}(f(x)) = F(s),\ \mathcal{L}(g(x)) = G(s))$

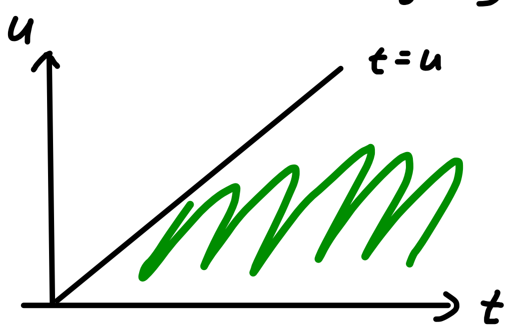
$$(f * g)(x) = \int_0^x dx'\, f(x-x')\, g(x') = \int_0^x dx'\, f(x')\, g(x-x')$$

$$\Rightarrow F(s)\,G(s) = \int_0^\infty du\, e^{-su} f(u) \int_0^\infty dv\, e^{-sv} g(v)$$

$$= \int_0^\infty du \int_0^\infty dv\, f(u)\, g(v)\, e^{-s(u+v)}$$

$(u+v = x)$

$$= \int_0^\infty du \int_u^\infty dt\, f(u)\, g(x-u)\, e^{-sx}$$



$$= \int_0^\infty dt \int_0^x du\, f(u)\, g(x-u)\, e^{-sx}$$ ~ Exchanging the order of t & u integration

$$= \mathcal{L}[f * g(x)]$$

$$\Rightarrow \mathcal{L}[f * g(x)] = F(s)\,G(s)$$ ~ Laplace Transform of Convolution becomes Product of the Transforms

In particular if we set $f(x) = 1$, $F(s) = 1/s$

$$\Rightarrow \mathcal{L}\left[\int_0^x dx'\, g(x')\right] = \frac{1}{s} G(s)$$ ~ Integral Formula for Laplace transform

~ Simpler than the Fourier transform

Usual Properties of Convolution apply

e.g. $f * g = g * f$, $f * (g * h) = (f * g) * h$, $f * \delta(x) = f(x)$ etc

<u>EX</u> Square wave

$$f(x) = \begin{cases} 1 & x_1 \le x \le x_2 \\ 0 & \text{otherwise} \end{cases} = \sigma(x-x_1) - \sigma(x-x_2) \;, \quad x_2 > x_1 > 0$$

$$\mathcal{L}[f(x)] = \int_{x_1}^{\infty} dx\, e^{-sx} - \int_{x_2}^{\infty} dx\, e^{-sx} = \frac{+1}{s}\left[e^{-sx_1} - e^{-sx_2}\right]$$

$\Rightarrow$ Truncating Pulse

$$f * g(x) = \int_0^x dy\, f(y)\, g(x-y) = \int_{x_1}^{x_2} dy\, g(x-y) \qquad \text{if } x \ge x_2$$

$$\Rightarrow \mathcal{L}[f*g(x)] = G(s)\left[\frac{e^{-sx_1} - e^{-sx_2}}{s}\right]$$

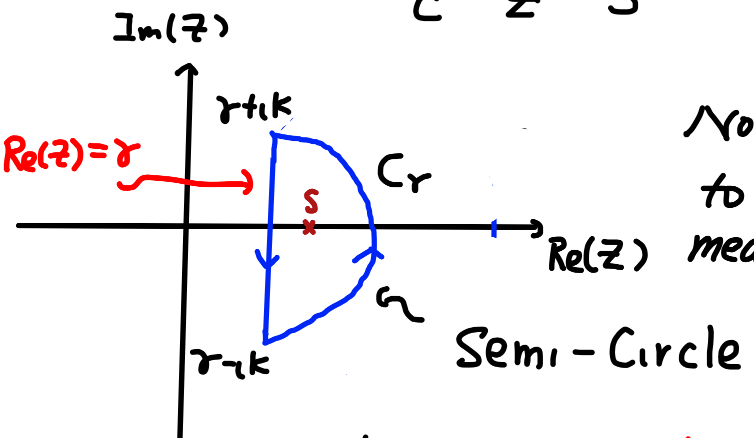
Inverse Laplace Transform

So far we have postponed the discussion on the inversion of Laplace Transform $\mathcal{L}^{-1}[\,\cdot\,]$, as it takes rather different form from $\mathcal{L}[\;]$. Unlike in Fourier Transform, $\{e^{-sx}\}$ does not form orthogonal basis, however, we can still derive a (contour) integral representation of $\mathcal{L}^{-1}[\,\cdot\,]$

Let us consider a contour integral representation of $F(s) = \mathcal{F}[f(x)]$

› We assume that $F(s)$ is analytic on and to the right of $Re(z) = \gamma$

$$\Rightarrow F(s) = \frac{1}{2\pi i}\oint_C \frac{dz\, F(z)}{z-s} = \frac{1}{2\pi i}\oint_{C_r} \frac{dz\, F(z)}{z-s} - \frac{1}{2\pi i}\int_{\gamma-iK}^{\gamma+iK} \frac{dz\, F(z)}{z-s}$$



Now $F(z)$ is analytic and continuous to the right of $Re(z) = \gamma$, this means $\underbrace{|F(z)| < M}_{\text{Bounded}}$ on C_r } Semi-Circle

$s \in \mathbb{R}^+$ i.e no singularity of $F(z)$ has real part $\ge \gamma$

We can now use the maximum length theorem in complex analysis

$$\left|\frac{1}{2\pi i}\int_{C_r} dz\, \frac{F(z)}{z-s}\right| \le \frac{1}{2\pi}\,\frac{M \quad \pi K}{\mathrm{Min}\,|z-s|}$$

$$|z-s| = |z-\gamma-(s-\gamma)| \ge |z-\gamma| - |s-\gamma| \ge K - |s-\gamma|$$

$$\Rightarrow \left|\frac{1}{2\pi i}\int_{C_r} dz\, \frac{F(z)}{z-s}\right| \le \frac{1}{2}\,\frac{M K}{K-|s-\gamma|} = \frac{1}{2}\,\frac{M(K)}{1-|s-\gamma|/K}$$

But if $f(x) \sim e^{+\alpha x} \rightsquigarrow \mathcal{F}[f(x)] \sim \int_0^\infty dx\, e^{(\alpha-s)x} = \frac{1}{s-\alpha} \to 0$

as $s \simeq b \to \infty$

$\Rightarrow$ This implies $M \to 0$ ((cf $f(k) \to 0$, $|k| \to \infty$ for Fourier)

$\Rightarrow$ This gives $\left|\frac{1}{2\pi i}\int_{C_r} dz\, \frac{F(z)}{z-s}\right| \to 0$ as $K \to \infty$

Therefore we have

$$\mathcal{F}(s) = \frac{1}{2\pi i}\int_{\gamma+i\infty}^{\gamma-i\infty} dz\, \frac{\mathcal{F}(z)}{z-s} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dz\, \frac{\mathcal{F}(z)}{s-z}$$

Now to recover $f(x) = \mathcal{L}^{-1}[\mathcal{F}(s)]$

$$= \mathcal{L}^{-1}\left[\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dz\, \frac{\mathcal{F}(z)}{s-z}\right]$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dz\, \mathcal{F}(z)\, \mathcal{L}^{-1}\left[\frac{1}{s-z}\right]$$

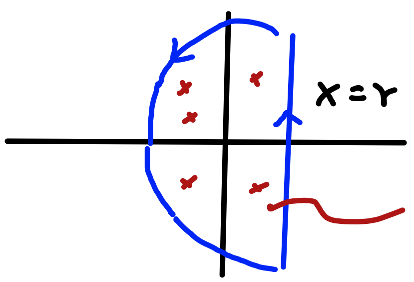
Only inverting s-dependent part

$$\mathcal{L}^{-1}\left[\frac{1}{s-z}\right] = e^{xz} \rightsquigarrow f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dz\, \mathcal{F}(z)\, e^{xz}$$

$\Rightarrow$ Inverse Laplace Transform "Bromwich Integral"

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dz\, \mathcal{F}(z)\, e^{xz}$$

Evaluate by contour integral



$$= \sum_{\{\mathrm{Re}(z_0) < \gamma\}} \mathrm{Res}\,(F(z))_{z=z_0}$$

Poles of $F(z)e^{xz}$, i.e. Poles of $F(z)$

Ex $F(s) = \frac{1}{s-i\omega} + \frac{1}{s+i\omega}$ ⇝ Poles at $s = \pm i\omega$

$$\Rightarrow \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dz \left[\frac{1}{z-i\omega} + \frac{1}{z+i\omega}\right] e^{zx}$$

$$= e^{i\omega x} + e^{-i\omega x} = 2\cos\omega x$$ ⇝ Match with Laplace Transform earlier

But a lot of time we don't even need to perform Inverse Laplace Transform, if we apply the properties of the Laplace Transform cleverly

More Examples next lecture + Application to solve differential eqn