1. Background

2. Physical Explanation of KSWCF

3. 2d-4d wall-crossing

1. Background

The "space of BPS states" has been a central concept in SUSY gauge theory and string theory for almost 30 years. Today I'll focus on recent progress in understanding phenomena associated to maginal stability (MS).

What is the BPS states?

Defining the space of BPS states

For definiteness, the following are focus on theories with $d=4, \mathcal{N}=2$ SUSY in asymptotic Minkowski space $\mathbb{M}_4$.

Hilbert space of one-particle states, $\mathcal{H}$, is a rep. of the $d=4, \mathcal{N}=2$ algebra.

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1$$

$$\mathcal{S}_0 = (\underset{\hat{M}_{\mu\nu}}{\mathrm{spin}(1,3)} \otimes \underset{\hat{P}_\mu}{\mathbb{R}^4}) \oplus u(2) \oplus \underset{\hat{Z}}{\mathbb{R}}$$

$$\mathcal{S}_1 = [\,\mathrm{Spinor} \otimes \mathbb{C}^2\,]_{\mathbb{R}}$$

$$Q_{\alpha I},\ \bar{Q}^{\dot{\alpha} I}$$

$$\{Q_{\alpha I}, \bar{Q}_{\dot{\beta}}{}^{J}\} = 2\hat{P}_\mu \sigma^\mu_{\alpha\dot{\beta}} \delta_I{}^J$$

$$\{Q_{\alpha I}, Q_{\beta J}\} = 2\hat{Z}\epsilon_{\alpha\beta}\epsilon_{IJ}$$

$$\Rightarrow \mathcal{H} = \bigoplus_{z\in\mathbb{C}} \mathcal{H}_{\hat{Z}=z}$$

Lemma: $E \geq |Z|$ on $\mathcal{H}_Z$.

Proof: $\mathcal{N}=2$ is a 6d SUSY algebra.

Dimensionally reduced to 4d:

$$\mathrm{Spin}(1,3) \times \mathrm{Spin}\,2 \hookrightarrow \mathrm{Spin}(1,5)$$

$$2_{1/2} \oplus 2^*_{-1/2} = 4$$

$$\{Q_{rA}, Q_{sB}\} = \Gamma^M_{rs} P_M \varepsilon_{AB} \Rightarrow Z = P_4 + iP_5$$

unitary rep: $P^M P_M \geq 0$

Hence, $E^2 - P^2 - |Z|^2 \geq 0$

Def: When $E = |Z|$, $\mathcal{H} = \mathcal{H}^{BPS}$, i.e., the subspace of $\mathcal{H}$ where $E = |Z|$.

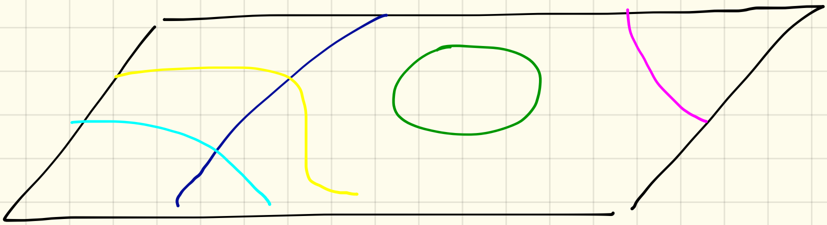
In supersymmetric field/string theories we often interested in BPS states : 1-particle states whose energy is the minimum allowed by the SUSY algebra. Such states are to some extent "protected" from quantum corrections, because of the rigidity of short SUSY representations.

In particular, in many cases one can define an index which counts the # of such states. This index is supossed to be invariant, and not to change when we vary parameters.

How is the walls arise?

(How to construct the walls?)

There are real-codimension-1 loci in parameter space where the mixing will occur. These loci called "wall."



At a wall, $\exists$ marginal bound states; as parameter cross the wall 1-particle states may decay, or convesely appear in the spectrum.

$\Rightarrow$ index counting depend on parameters in a piecewise constant behavior, jumps at the walls.

How does it jump?

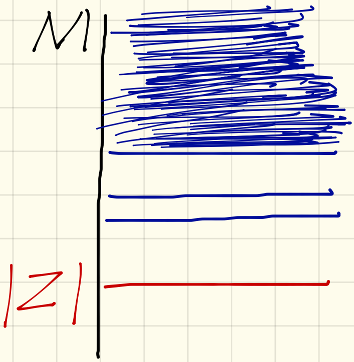
The 1-particle Hilbert space is graded by conserved charges γ. Consider

$$\Omega(\gamma, u) \in \mathbb{Z}$$

counting BPS states with charges γ in theory with parameters u.

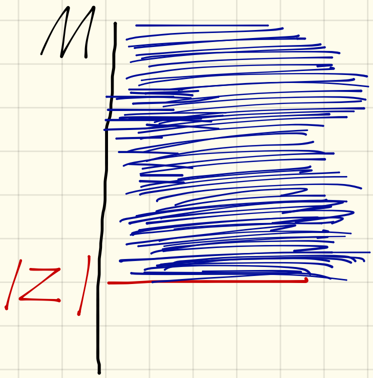
$\Rightarrow$ How does the collection $\{\Omega(\gamma, u)\}_{\gamma \in \Gamma}$ jump when u crosses a wall in the parameter space?

. This index counting method was developed in string approach to black hole entropy (Strominger-Vafa). However, invariant only works for the case that 1-particle BPS Hilber space doesn't mix with the multiparticle continuum.



without mixing

⇒ invariance preserved



with mixing

⇒ jump

⇒ walls arise!

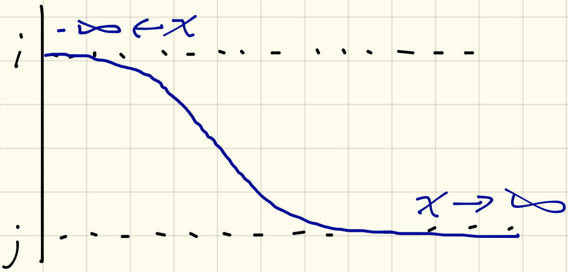
There are two general settings:

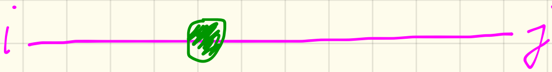
- $\mathcal{N}=(2,2)$ theories in $d=2$ (Cecotti-Vafa, Cecotti-Fendley-Intriligator-Vafa)
- $\mathcal{N}=2$, $d=4$ (SW, Denef-Moore, Kontsevich-Soibelman, Gaiotto-Moore-Neitzke, Cecotti-Vafa, Dimofte-Gukov-Soibelman)

$\mathcal{N}=(2,2)$ $d=2$

Suppose one have a massive $\mathcal{N}=(2,2)$ theory in $d=2$, depending on parameters t. Discrete vacua, labeled by $i=1,\dots,n$. Consider ij-solitons.

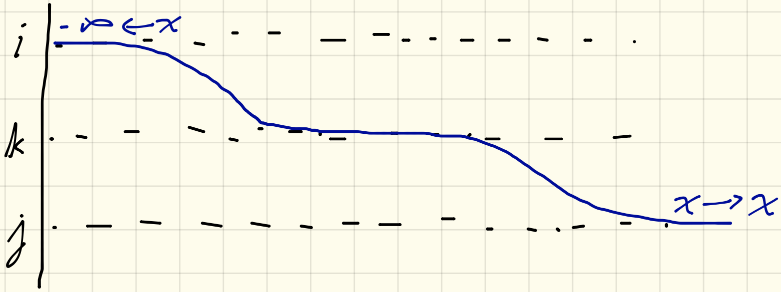
* Index $\mu(ij,t) \in \mathbb{Z}$ counts BPS solitons.





BPS bound is $M \geq |Z_{ij}|$ where the "central charges" $Z_{ij}(t) \in \mathbb{C}$ obey $Z_{ij} + Z_{jk} = Z_{ik}$.

. As we vary t, $\mu(ij,t)$ can jump when t crosses a wall. Walls are loci where some $Z_{ik}/Z_{kj} \in \mathbb{R}^+$. Then a BPS ij-soliton can decay into ik-soliton plus kj-soliton.

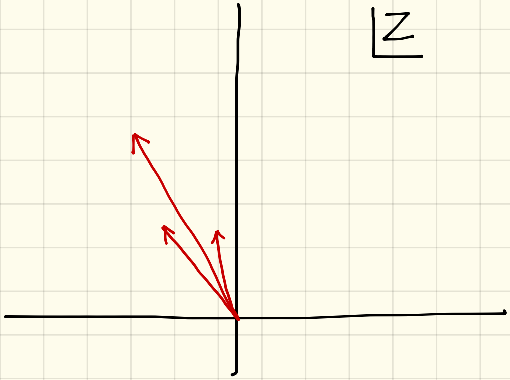


The jump at the wall is

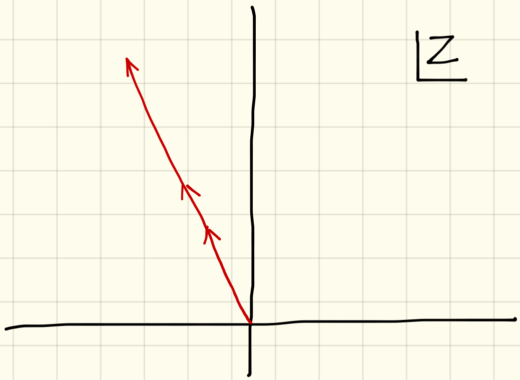
$$\mu(ij,t_+) - \mu(ij,t_-) = \pm \mu(ik)\,\mu(kj)$$ (Cecotti–Vafa)

This is a wall-crossing formula (2d).

At the wall, some collection of solitons become aligned, i.e., their central charges Z are all lying on the same ray in $\mathbb{C}$.



Near the wall.



On the wall.

To each participating ij-soliton, assign an $n \times n$ matrix:

$$S_{ij} = \mathbb{I} + e_{ij}.$$

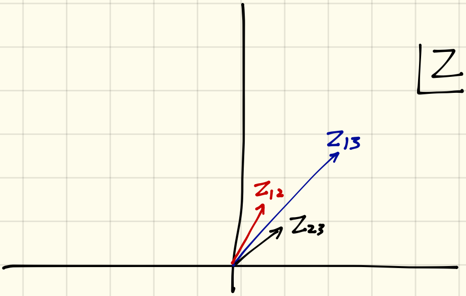
Now consider the object

$$: \prod_{ij} S_{ij}^{\mu(ij)} :$$

where :: means we multiply in order of the phase of Z_{ij}. The WCF is the statement that this object is the same on both sides of the wall.

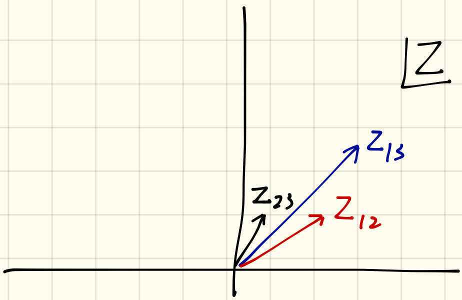
Example:

$$\begin{pmatrix} 1 & \mu(12) & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \mu(13) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu(23) \\ 0 & 0 & 1 \end{pmatrix}$$



equals

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu'(23) \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \mu(13) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \mu'(12) & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



iff

$\mu'(12) = \mu(12)$

$\mu'(23) = \mu(23)$

$\mu'(13) = \mu(13) + \mu(12)\mu(23)$

which is the WCF!

tt^* geometry

Originally, 2d WCF was proved by tt^* geometry.
The idea is compactify the 2d theory on a circle and look at the metric $\langle i|j\rangle$ on the space of ground states, i.e., cylinder path integral.

As we vary parameters t, the space of ground states forms a rank n complex vector bundle over moduli space.

and carries a Hermitian metric obeying an interesting system of integrable PDEs, e.g., Hitchin equations.

Puzzle:

This quantity receives quantum corrections from solitons going around the compactification circle. Solitons can appear and disappear as t varies. But the answer should be continuous as a func. of t ($\because$ the theory has no phase transition).

$\Rightarrow$ Multisoliton contributions become comparable to 1-soliton contributions at the wall, ensure smoothness. But only if the WCF is satisfied! This gives an indirect proof of the WCF.

$\mathcal{N}=2,\ d=4$

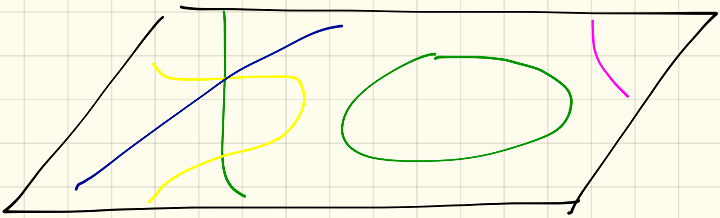
Usually, the theory has a Coulomb branch (moduli space of vacua of vector multiplets): complex manifold of dim $= r$.

IR physics on the Coulomb branch is simple: supersymmetric abelian gauge theory, gauge group $U(1)^r$ couplings determined by prepotential $\mathcal{F}$. Particles carry electromagnetic charge γ. (e.g. for rank $r=1$, $\gamma=(p,q)$ for $p,q\in\mathbb{Z}$.) DSZ pairing $\langle\gamma,\gamma'\rangle:\langle(p,q),(p',q')\rangle = pq'-qp'$. Central charges $Z_\gamma\in\mathbb{C}$ obeying $Z_\gamma+Z_{\gamma'}=Z_{\gamma+\gamma'}$.

. BPS bound: $M \geq |Z_\gamma|$.

Introduce index $\Omega(\gamma, u)$ counting BPS states of charge γ, in Coulomb branch vacuum u.

Walls occur at u for which $Z_\gamma / Z_{\gamma'} \in \mathbb{R}^+$ for some γ, γ' with $\langle \gamma, \gamma' \rangle \neq 0$.



Basically parallel to 2d case.

Structure of WCF is also parallel to 2d case, but we need to replace the finite-dimensional matrices S_{ij} by something fancier:

$\hookrightarrow$ Next page!

Torus algebra with one generator X_γ for each γ, $X_\gamma X_{\gamma'} = X_{\gamma+\gamma'}$.

Automorphism K_γ of this algebra:

$K_\gamma : X_{\gamma'} \mapsto (1+X_\gamma)^{\langle \gamma, \gamma' \rangle} X_{\gamma'}$.

At a wall, some group of BPS particles become aligned. (Maybe ∞)

To each participating particle, assign the automorphism K_γ.

Now consider the object $:\prod_\gamma K_\gamma^{\Omega(\gamma)}:$ where $::$ means we multiply in order of the phase of Z_γ. The WCF is the statement that this object is the same on both sides of the wall.

(Kontsevich-Soibelman).

Example : Knowing one side determine the other by purely algebraic means.

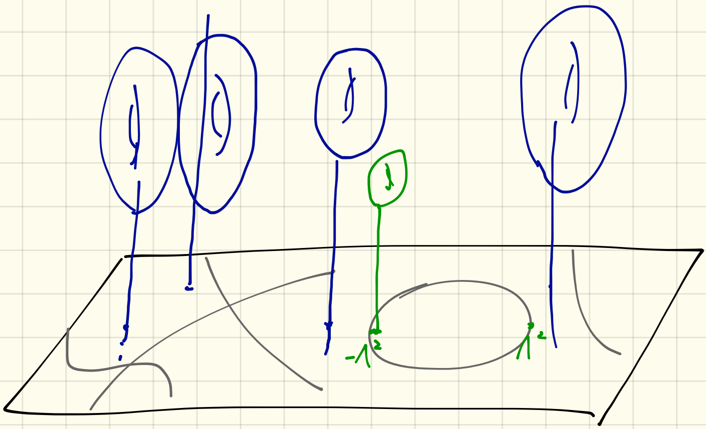
- $$\underbrace{\underset{\text{monople}}{K_{1,0}}\,\underset{\text{electron}}{K_{0,1}} = K_{0,1}}_{\text{form a single dyon bound state, which can appear/decay at the wall}}\underset{\text{dyon bound state}}{K_{1,1}}\,K_{1,0}$$

- Another e.x. is taken from SU(2) SW

$$\underset{\text{monople}}{K_{1,0}}\,\underset{\text{dyon}}{K_{-1,2}} = \underbrace{\left(\prod_{n=1}^{\infty} K_{-1,2n}\right)}_{\text{infty dyons}} \underset{\text{boson}}{K_{0,2}^{-2}} \underbrace{\left(\prod_{n=\infty}^{0} K_{1,2n}\right)}_{\text{infty dyons}}.$$

In order to prove the $\mathcal{N}=4$, $d=2$ WCF, in the following derivation one should compactify the theory on S^1 of radius R. In the IR, the resulting theory looks 3-dimensional.

Dualize all gauge fields into scalars to get a sigma model, whose target $\mathcal{M}$ is a fibration by compact $2r$-tori over the 4d Coulomb branch.



SUSY implies $\mathcal{M}$ is hyperkähler.
If our $d=4$ theory was obtained from the $d=6$ $(2,0)$ SCFT, then $\mathcal{M}$ is the Hitchin moduli space.

How to calculate the metric on M?

Näively, dim-reduction ($\mathbb{R}^4 \to \mathbb{R}^3 \times S^1$) gives an approximation ("semiflat metric"), exact in the limit $R \to \infty$, away from singular fibers. (Like the SYZ picture of a CY 3-fold, exact in the large complex structure limit.) (Cecotti-Ferrara-Girardello)

$\because$ The exact metric receives quantum corrections from BPS instanton: the 4d BPS particles going around S^1. And, these BPS particles appear and disappear as u varies.

$\to$ However, the metric should be continuous (i.e. without phase trans.)

. From the technology of instantons, (incorporating their corrections)

multiparticle contributions become comparable to 1-particle contributions at the wall, ensure smoothness of the metric on $\mathcal{M}$. (only if the WCF be satisfied!)

Hyperkähler Geometry

1. This theory gives an interpretation of the 4d WCF, and the torus algebra: it's just the algebra of functions on a cood. patch on $\mathcal{M}$.

2. This theory provides a new approach to describe hyperkähler metrics on total spaces of integrable systems as well.

→ Data: (i) $\mathcal{N}=2$ theory

(ii) SW sol. (4d IR effective action)

(iii) BPS spectrum.

Hyperkähler geometry and TBA

To write an explicit formula for the metric on $\mathcal{M}$, one has to solve some interesting integral equations:

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{sf} \exp\left[\sum_{\gamma'} \Omega(\gamma') \langle \gamma, \gamma' \rangle \frac{1}{4\pi i} \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \left(\frac{\zeta' + \zeta}{\zeta' - \zeta}\right) \cdot \log(1 - \mathcal{X}_{\gamma'}(\zeta'))\right]$$

Here $\mathcal{X}_\gamma$ are "holomorphic Darboux cood." on $\mathcal{M}$, also functions of twistor parameter $\zeta \in \mathbb{C}$ which keep track of the complex structures on $\mathcal{M}$.

These equations have exactly the form of the thermodynamic Bethe ansatz for a 2d theory w/ factorized scattering.
(Zamolodchikov)

Open questions:

1. Where did this 2d theory come from?

   (Why on earth would the rapidity be related to the twistor parameter?)

2. What does wall-crossing mean in the 2d theory?

1. Background ✓

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2. Physical Explanation of KSWCF

A connetion to $\mathcal{N}=4$

Suppose we take our $\mathcal{N}=2$ theory to be the n-the "n-th Argyres-Douglas-type SCFT," characterized by SW curve $y^2 = x^{n+2} + (\text{lower order})$

* This is one of the examples where $\mathcal{M}$ is a Hitchin system (with irregular singularity).

This Hitchin system is the same one that governs strings in AdS_3 with the "polygon boundary conditions" that appeared in the strong-coupling $\mathcal{N}=4$ SYM computations. (Alday - Guiotto - Madacena - Sever - Vieira)

OpnQ: What's the implication of this connection?

1. Background ✓

2. Physical Explanation of KSWCF ✓

3. 2d-4d wall-crossing

3. 2d-4d wall-crossing

Wall-crossing for combined 2d-4d theories

Combine 2d & 4d.

Consider a 4d $\mathcal{N}=2$ theory with a surface defect preserving $d=2$, $\mathcal{N}=(2,2)$ SUSY.

e.g. 4d $\mathcal{N}=2$ $SU(2)$ gauge theory in $\mathbb{R}^4$.

2d supersymmetric sigma model into $\mathbb{CP}^1$, supported on $\mathbb{R}^2 \subset \mathbb{R}^4$.

Couple the two by using 4d gauge fields to gauge the global $SU(2)$ isometry group of $\mathbb{CP}^1$.

Wall-crossing for combined 2d-4d theories

In the IR:

4d abelian gauge theory, as before.
Assume surface defect is massive in the IR. Factor out the time direction. Surface defect looks look like a string in space.

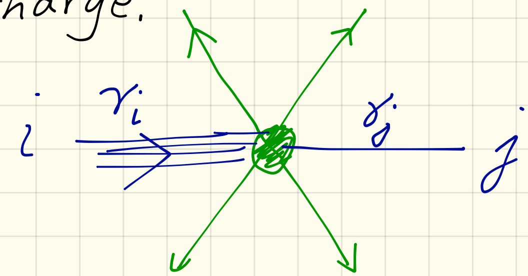
It creates a boundary condition for the gauge fields: fixed holonomy around the string.
So particle transpose around the string pick up a phase;

Wall-crossing for combined 2d-4d theories

like the AB effect created by a solenoid. The flux through the solenoid dep. on the IR data: both the Coulomb branch modulus u and the discrete choice of vacuum i on surface defect. In particular, if we have a soliton on the surface defect, the flux changes across the soliton, in a non-quantized way: some of it must have escaped into the 4d bulk, i.e., 2d solitons

Wall-crossing for combined 2d-4d theories

carry (fractional) 4d gauge charge.



$$\gamma_{ij} = \gamma_i - \gamma_j + \gamma.$$

So they can form bound states with/decay into 4d particles as well as other 2d-4d wall-crossing. 2d-4d WC phenomena are governed by a kind of hybrid of the

Wall-crossing for combined 2d-4d theories

two WCF we had before. Each BPS state corresponds to a certain automorphism:

2d	a vector space
4d	a complex space
2d-4d	a vector bundle over a complex torus

A 2d BPS state of charge γ_{ij} gives an endomophism $S_{\gamma_{ij}}$ of the bundle. A 4d BPS state of charge γ gives an automorphism K_γ of the torus, lifted to act on the bundle.

Wall-crossing for combined 2d-4d theories

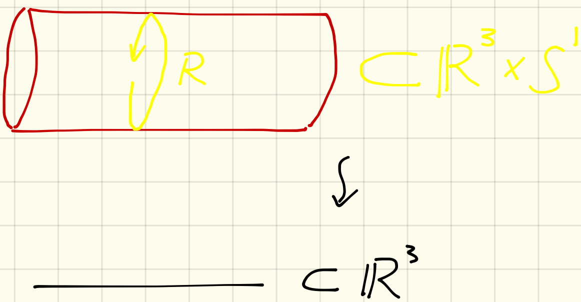
2d-4d WCF

$$:\prod_{i,j,\gamma_{ij}} S_{ij}^{\mu(\gamma_{ij})} \prod_{\gamma} K_{\gamma}^{\Omega(\gamma)}:$$

remains constant as one cross wall.

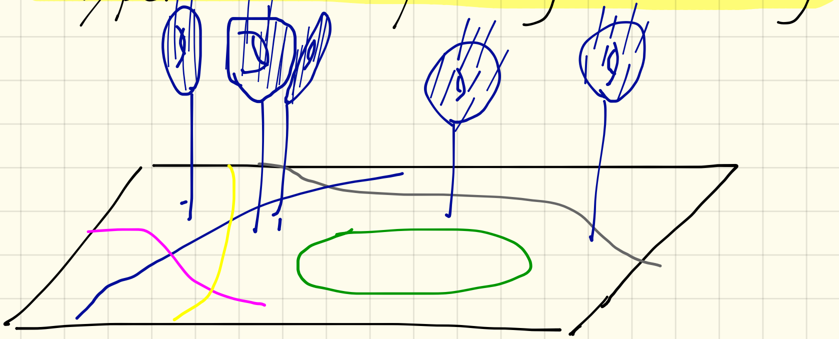
Hyperholomorphic geometry

Upon compactification to 3d, we get a sigma model into hyperkähler manifold M, now with an extra line operator inserted.



The line operator couples to a connection A in a vector bundle V over M. V is just the bundle of vacua of the surface defect on S^1.

Hyperholomorphic geometry

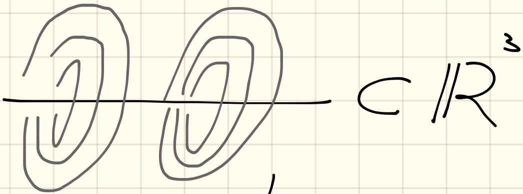


SUSY requires that A is a hyperholomorphic connection. (Curvature of type $(1,1)$ in all complex structures.)

To capture a picture here, compactify from 3d to 2d on a circle surrounding the line operator: get 2d sigma model on a half-space, with boundary condition coming from

Hyperholomorphic geometry

the line operator.



This can be understand of the construction of the brane as mirror symmetry: the IR Lagrangian fixes a certain BAA brane which is mirror to the hyperholomorphic bundle, i.e., BBB brane, which one is constructing.

Hyperholomorphic geometry

1. This theory explains why the 2d-4d WCF is true.

2. This theory provides a new approach to describe hyperkähler spaces with hyperholomorphic vector bundles.

→ Data:

(i) $\mathcal{N}=2$ theory with surface defect.

(ii) IR action (SW sol. in 4d plus effective superpotential in 2d)

(iii) 2d-4d spectrum.

A special case : Constructing sols. to Hitchin equations.

This relates to some classical geometric questions!

(e.g.) Revisiting the application to strong-coupling $\mathcal{N}=4$ computations, it would give not just the minimal area of the string worldsheet, but the actual minimizing configugation. It relates to the classical problem of uniformization as well.

*To write an explicit formula for the hyperholomorphic connection, one again has to solve some interesting integral equations : generalization of the TBA we had before.

1. Background ✓

2. Physical Explanation of KSWCF ✓

3. 2d-4d wall-crossing ✓