INTEGRAL CHOW RINGS OF $\overline{\mathcal{M}}_{1,n}$ VIA $\mathrm{CH}^*(\mathcal{M}_{1,n},1)$

ABSTRACT. For $n \leq 4$, we compute the integral indecomposible higher Chow groups $\mathrm{CH}^*(\mathcal{M}_{1,n},1)_{\mathrm{ind}}$ and use these to compute the integral Chow rings $\mathrm{CH}*(\mathcal{M}_{1,n})$.

1. INTRODUCTION

Introduced by Bloch in [], the higher Chow group $\mathrm{CH}_i(X,j)$ are interesting invariants of a space X that extend the usual Chow groups. We are particularly interested in the groups $\mathrm{CH}^*(\mathcal{M}_{g,n},1)$, as they can be used to compute $\mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$: there is an exact sequence

$$\mathrm{CH}^*(\mathcal{M}_{g,n},1) \xrightarrow{\partial_1} \mathrm{CH}^{*-1}(\partial\overline{\mathcal{M}}_{g,n}) \to \mathrm{CH}^*(\overline{\mathcal{M}}_{g,n}) \to \mathrm{CH}^*(\mathcal{M}_{g,n}) \to 0.$$

Thus, $\mathrm{CH}^*(\mathcal{M}_{g,n},1)$, together with the map ∂_1, determine which classes on $\partial\overline{\mathcal{M}}_{g,n}$ map to 0 on $\overline{\mathcal{M}}_{g,n}$. Using this, we compute $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,n})$ for $n \leq 4$. This technique was inspired by [Bae-Schmidt II] and [Larson].

Higher Chow groups are usually very large. For example, for any variety X over the field k, $\mathrm{CH}^1(X,1)$ contains $k^\times$. For this reason, we instead study a variant $\mathrm{CH}^*(X,1)_{\mathrm{ind}}$, the indecomposible higher Chow groups (though see Remarks 4.4 and 4.7). These groups are finitely generated in the cases we are interested in. In [Larosn,Bishop,Bishop], the notions of ℓ-adic higher Chow groups were used for similar reasons. We show how indecomposible Chow groups relate to ℓ-adic Chow groups in Proposition 2.11.

Computation of ∂_1....

We summarize the main results of this paper with the following 2 Theorems

Theorem 1.1. *For $n \leq 4$, one has $\mathrm{CH}^1(\mathcal{M}_{1,n},1)_{ind} = 0$, and for $i \geq 2$, we have*

$$\mathrm{CH}^i(\mathcal{M}_{1,1},1)_{ind} = 0$$

$$\mathrm{CH}^i(\mathcal{M}_{1,2},1)_{ind} = \frac{\mathbb{Z}}{2\mathbb{Z}}$$

$$\mathrm{CH}^*(\mathcal{M}_{1,3},1)_{ind} = \begin{cases} P \oplus \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 & i = 2 \\ \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 & i \geq 3 \end{cases}$$

$$\mathrm{CH}^*(\mathcal{M}_{1,n},1)_{ind} = \begin{cases} \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^6 & i = 2 \\ \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 & i \geq 3 \end{cases}$$

where P is a cyclic group of order $2^a 3^b$ for some $a, b \geq 0$.

Theorem 1.2. *The Chow rings of $\overline{\mathcal{M}}_{1,n}$ are given by*

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) = \frac{\mathbb{Z}[\lambda]}{(24\lambda^2)}$$

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}) = \frac{\mathbb{Z}[\lambda,\delta]}{(24\lambda^2, \lambda^2+\delta\lambda)}$$

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3}) = \frac{\mathbb{Z}[\lambda,\delta_1,\delta_2,\delta_3,\delta_\emptyset]}{I_3}$$

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,4}) = \frac{\mathbb{Z}[\lambda,\{\delta_{ij}\}_{i\neq j\in[4]},\{\delta_i\}_{i\in[4]},\delta_\emptyset]}{I_4}$$

where I_3 and I_4 are as described in Theorem 9.13 and Theorem 9.18, respectively.

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Notation. k is a field whose characteristic is not 2 or 3. All stacks X are separated finite type quotient stacks over k.

2. Indecomposible Higher Chow Groups

2.1. **Higher Chow Groups.** Bloch defined higher Chow groups $\mathrm{CH}_i(X,j)$ for a scheme X in [Bloch]. The relevant properties are as follows:

(1) $\mathrm{CH}_i(-,j)$ has proper pushforwards, flat pull-backs, and pull-backs for arbitrary maps between smooth schemes.
(2) $\mathrm{CH}_i(X,0) = \mathrm{CH}_i(X)$, and the functorialities agree.
(3) If X contains an open subscheme $j : U \hookrightarrow X$, with complement $\iota : Z \hookrightarrow X$, there is an exact sequence

$$\cdots \to \mathrm{CH}_i(Z,j) \xrightarrow{\iota_*} \mathrm{CH}_i(X,j) \xrightarrow{j^*} \mathrm{CH}_i(U,j) \xrightarrow{\partial} \mathrm{CH}_i(Z,j-1) \to \dots$$

(4) There is a product

$$\times : \mathrm{CH}_i(X,j)\times \mathrm{CH}_k(Y,\ell) \to \mathrm{CH}_{i+k}(X\times Y, j+\ell)$$

For smooth X, setting $X=Y$ and pulling back along the diagonal gives the structure of a bi-graded ring on $\mathrm{CH}^*(X,*) = \bigoplus_{i,j}\mathrm{CH}^i(X,j)$.
(5) (dimension vanishing)

For equidimensional X, we set $\mathrm{CH}^i(X,j) := \mathrm{CH}_{\dim X - i}(X,j)$. Note this is not the same as opperational Chow.

In [EG], Edidin and Graham extended Chow groups and higher Chow groups [well not technically, but...] to quotient stacks, $[X/G]$. This is done by setting

$$\mathrm{CH}_i([X/G],j) := CH_{i+\dim(V)}((X\times U)/G,j),$$

where U is an open subset of a G-representation, V, whose complement has codimension $\geq$???. All of the above properties of higher Chow groups extend to stacks (modify dimension statement, unclear on pushes and pulls). (Subtly involving X needing to be quasiprojective.)

We can use higher Chow groups to prove things about ordinary Chow groups, like the following

Lemma 2.1. *Suppose we have a proper map of quotient stacks $g: X \to Y$, with a closed substack $j: Z \subseteq Y$ and open complement $U \subseteq Y$, such that $g^{-1}(U) \xrightarrow{g} U$ is an isomorphism. Then the map*

$$(g_*, j_*) : \mathrm{CH}^*(X) \coprod_{\mathrm{CH}^*(g^{-1}Z)} \mathrm{CH}^*(Z) \to \mathrm{CH}^*(Y)$$

is an isomorphism.

Proof. We get the diagram

$$\begin{tikzcd}
\mathrm{CH}^*(f^{-1}(U),1) \arrow[r] \arrow[d, "\sim"] & \mathrm{CH}^*(g^{-1}(Z)) \arrow[r] \arrow[d] & \mathrm{CH}^*(X) \arrow[r] \arrow[d] & \mathrm{CH}^*(f^{-1}(U)) \arrow[r] \arrow[d, "\sim"] & 0 \\
\mathrm{CH}^*(U,1) \arrow[r] & \mathrm{CH}^*(Z) \arrow[r] & \mathrm{CH}^*(Y) \arrow[r] & \mathrm{CH}^*(U) \arrow[r] & 0
\end{tikzcd}$$

This diagram commutes by a property of higher Chow. Now the lemma follows from a diagram chase. $\square$

2.2. Indecomposible Higher Chow.

Note that $\mathrm{CH}_i(k,1)$ is only nonzero for $i=-1$: for dimension reasons, it can only be nonzero for $i=-1,0$, and $\mathrm{CH}^0(X,1)=0$ for all X (reference).

Definition 2.2. The decomposable first higher Chow groups of a quotient stack X, $\mathrm{CH}_i(X,1)_{\mathrm{dec}}$ is defined to be the image of

$$\times : \mathrm{CH}_{i+1}(X)\otimes \mathrm{CH}_{-1}(k,1) \to \mathrm{CH}_i(X\times_k k,1) \xrightarrow{\sim} \mathrm{CH}_i(X,1).$$

We then define the indecomposible first higher Chow group of a quotient stack X to be

$$\mathrm{CH}_i(X,1)_{\mathrm{ind}} := \mathrm{CH}_i(X,1)/\mathrm{CH}_i(X,1)_{\mathrm{dec}}.$$

If X is smooth with structure morphism $f: X\to \mathrm{Spec}(k)$, note that for $\alpha\in \mathrm{CH}_{i+1}(X)$ and $\beta \in \mathrm{CH}_{-1}(k,1)$, under the identification of $X\times_k k$ with X,we have $\alpha\times\beta = \pi_1^*\alpha\cup\pi_2^*\beta = \alpha\cup f^*\beta$.

Warning 2.3. *Warn*

The idea behind using indecomposible higher Chow groups is....

For a proper map $f : X \to Y$, compatibility between the product $\times$ and proper pushforward f_* implies that $f_*(\mathrm{CH}_i(X,1)_{\mathrm{dec}}) \subseteq \mathrm{CH}_i(Y,1)_{\mathrm{dec}}$. Hence, f_* induces a map $f_* : \mathrm{CH}_i(X,1)_{\mathrm{ind}} \to \mathrm{CH}_i(Y,1)_{\mathrm{ind}}$. Similarly, for a flat map of relative dimension d, compatibility between the product and f^* implies that $f^*(\mathrm{CH}_i(Y,1))_{\mathrm{dec}} \subseteq \mathrm{CH}_{i+d}(X,1)_{\mathrm{dec}}$. Hence, f^* induces a map $f^* : \mathrm{CH}_i(Y,1)_{\mathrm{ind}} \to \mathrm{CH}_{i+d}(X,1)_{\mathrm{ind}}$.

Moreover, we still have the last part of the localization exact sequence:

Lemma 2.4. *If U is an open substack of X with complement Z, $\partial : \mathrm{CH}_i(U,1) \to \mathrm{CH}_i(Z)$ is zero on $\mathrm{CH}_i(U,1)_{dec}$, and the induced sequence*

$$\mathrm{CH}_i(Z,1)_{ind} \to \mathrm{CH}_i(X,1)_{ind} \to \mathrm{CH}_i(U,1)_{ind} \to \mathrm{CH}_i(Z) \to \mathrm{CH}_i(X) \to \mathrm{CH}_i(U) \to 0$$

is exact.

Proof. We need only check exactness at the first four terms in the sequence, as the rest of the sequence is unchanged from the standard localization exact sequence. Consider the diagram

$$\begin{tikzcd}
\mathrm{CH}_i(Z,1) \arrow[r] & \mathrm{CH}_i(X,1) \arrow[r, "j^*"] & \mathrm{CH}_i(U,1) \arrow[r, "\partial"] & \mathrm{CH}_i(Z) \\
\mathrm{CH}_{-1}(k,1) \otimes \mathrm{CH}_{i+1}(Z) \arrow[r] \arrow[u] & \mathrm{CH}_{-1}(k,1) \otimes \mathrm{CH}_{i+1}(X) \arrow[r] \arrow[u] & \mathrm{CH}_{-1}(k,1) \otimes \mathrm{CH}_{i+1}(U) \arrow[r] \arrow[u] & 0 \arrow[u] \arrow[r] & {}
\end{tikzcd}$$

Note that both rows are exact sequences, because the localization exact sequence is exact and because tensor is right-exact. As noted above, compatibility between $\times$ and pullbacks/pushforwards gives that the first two squares commute. To see that the third square commutes, we need that $\partial(a \times \alpha) = 0 \in \mathrm{CH}_i(Z)$ for $a \in \mathrm{CH}_{-1}(k,1)$ and $\alpha \in \mathrm{CH}_i(U)$. This is true because we can lift α to $\widetilde{\alpha} \in \mathrm{CH}_i(X)$, and then

$$\partial(a \times \alpha) = \partial(a \times j^*(\widehat{\alpha})) = \partial(j^*(a \times \widetilde{\alpha})) = 0.$$

Thus, this diagram of exact sequences commutes. The sequence we want to be exact is precisely the cokernel complex

$$\mathrm{CH}_i(Z,1)_{\mathrm{ind}} \to \mathrm{CH}_i(X,1)_{\mathrm{ind}} \to \mathrm{CH}_i(U,1)_{\mathrm{ind}} \xrightarrow{\partial} \mathrm{CH}_i(Z) \to \mathrm{CH}_i(X).$$

This is a routine diagram chase. $\square$

Remark 2.5. One could define indecomposible groups $\mathrm{CH}_i(X,j)_{\mathrm{ind}}$ for any i and j in an analogous way by taking them to be the (i,j) graded piece of the cokernel of

$$\mathrm{CH}_{\geq 1}(k,*) \otimes \mathrm{CH}_*(X) \to \mathrm{CH}_*(X,*).$$

Some good properties that this definition have:

(1) These groups have the same functorialities enjoyed by higher Chow groups

(2) For any cohomology functor H and any natural transformation $\mathrm{CH}^*(-,*) \implies H^*(-)$ (a "higher cycle class map") that sends $\mathrm{CH}^i(X,j)$ to $H^{2i-j}(X)$ factors through this definition of indecomposible higher Chow groups
(3) The proof of Proposition 2.7 goes through to give $\mathrm{CH}^*(X,j)_{\mathrm{ind}} = 0$ for $j > 0$ for smooth proper varieties X with a strong Chow-Kunneth decomposition.

One would also want the localization sequence to be exact. While the maps exist, being induced by the maps on the usual exact sequence, the author was unable to show exactness.

2.3. **The Motivic Kunneth Property.** In [], Totaro defined the Motivic Kunneth Property (MKP). One says that a stack X is has the Motivic Kunneth Property if for all schemes Y (equivalently all stacks) a certain spectral sequence converges to $\mathrm{CH}^*(X \times Y, *)$. He also defines the compactly supported motive $M^c(X)$ of a quotient stack X, and shows that X has the MKP if and only if $M^c(X)$ is pure Tate.

We just need the following properties of the Motivic Kunneth Property:

Proposition 2.6. (1) *Suppose X has the MKP. Then for any stack Y*

$$\mathrm{CH}(X \times Y) = \mathrm{CH}(X) \otimes_{\mathbb{Z}} \mathrm{CH}(Y)$$

(that is, the Motivic Kunneth Property implies the Chow Kunneth Property). [Section 6]

(2) *For a stack X and closed substack Z with open complement U, if two out of three of X, Z, U have the MKP, then so does the third. [Lemma 9.1]*
(3) *If X and Y has the MKP, so does $X \times Y$.*
(4) *If $X \times \mathbb{A}^1$ have the MKP if (and only if) X has the MKP.*
(5) *$\mathbb{P}^n$, $\mathbb{G}_m$, and μ_n for n invertible in k have the MKP. [Corollary 9.10]*

If X has the MKP, then the class of the diagonal $\Delta \in \mathrm{CH}^*(X \times X)$ is in the image of $\mathrm{CH}^*(X) \otimes \mathrm{CH}^*(X) \to \mathrm{CH}^*(X \times X)$ by (1). Stacks X with this property are said to have a strong Chow Kunneth decomposition.

(C.f. Theorem 5.2)

Proposition 2.7. *Suppose X is a smooth proper Deligne-Mumford stack with a strong Chow Kunneth decomposition. Then $\mathrm{CH}^*(X,1)_{ind} \otimes \mathbb{Z}[\frac{1}{N}] = 0$, where N is the lcm of the orders of the stabilizers of points of X.*

Proof. Let $n = \dim(X)$. Because X has the CKgP, we know

$$\mathrm{CH}^*(X) \otimes_{\mathbb{Z}} \mathrm{CH}^*(X) \to \mathrm{CH}^*(X \times X)$$

is surjective. Then, we can write

$$[\Delta] = \sum_i \alpha_i \times \beta_{n-i},$$

for $\alpha_i, \beta_i \in \mathrm{CH}^i(X)$, where $\Delta \overset{\delta}{\hookrightarrow} X \times X$ is the diagonal. Because X is smooth, $\alpha_i \times \beta_{n-i} = \pi_1^*(\alpha_i) \cup \pi_2^*(\beta_{n-i})$.

Because X is proper, we can consider pushforwards along π_i after we invert N. Note that for $\gamma \in \mathrm{CH}^i(X,1)$, we have

$$\pi_{1*}([\Delta]\cup\pi_2^*(\gamma)) = \pi_{1*}(\delta_*(1)\cup\pi_2^*(\gamma)) = \pi_{1*}(\delta_*(1\cup\delta^*\pi_2^*(\gamma))) = (\pi_1\circ\delta)_*(\pi_2\circ\delta)^*(\gamma) = \gamma,$$

using the projection formula. But we also have

$$\pi_{1*}([\Delta]\cup\pi_2^*(\gamma)) = \sum_i \pi_{1*}(\pi_1^*(\alpha_i)\cup\pi_2^*(\beta_{n-i})\cup\pi_2^*(\gamma)) = \sum_i \alpha_i\cup\pi_{1*}\pi_2^*(\beta_{n-i}\cup\gamma).$$

Now, considering

$$\begin{tikzcd} X \times X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & \mathrm{Spec}(k), \end{tikzcd}$$

we see that the classes $\pi_{1*}\pi_2^*(\beta_{n-i}\cup\gamma) = f^*f_*(\beta_{n-i}\cup\gamma)$ are pulled back from $\mathrm{Spec}(k)$ (flat statement). Thus,

$$\gamma = \sum_i \alpha_i\cup\pi_{1*}\pi_2^*(\beta_{n-i}\cup\gamma) = \sum_i \alpha_i\cup f^*f_*(\beta_{n-1}\cup\gamma) \in \mathrm{CH}^i(X,1)_{\mathrm{dec}}\otimes\mathbb{Z}[\frac{1}{N}].$$

□

Corollary 2.8. $\mathrm{CH}^*(\mathbb{P}^n,1)_{ind} = 0$ *and* $\mathrm{CH}^*(B\mathbb{G}_m,1)_{ind} = 0$.

Proof. As $\mathbb{P}^n$ is a scheme, hence a Deligne-Mumford stack with lcm of stabilizers equal to 1, with the MKP, this follows immediately from the proposition. Additionally, we have

$$\mathrm{CH}^i(B\mathbb{G}_m,1)_{\mathrm{ind}} = \mathrm{CH}^i(\mathbb{P}^n,1)_{\mathrm{ind}} = 0$$

for n sufficiently large compared to i. □

This proposition also gives $\mathrm{CH}^*(B\mu_n,1)_{\mathrm{ind}}[\frac{1}{n}] = 0$, but we need not invert n:

Lemma 2.9. *For n invertible in k, we have* $\mathrm{CH}^*(B\mu_n) = \frac{\mathbb{Z}[t]}{(nt)}$ *and* $\mathrm{CH}^*(B\mu_n,1)_{ind} = 0$.

Proof. Let $\mathbb{G}_m$ act on $\mathbb{A}^1$ by $t\cdot x = t^n x$. In the quotient $[\mathbb{A}^1/\mathbb{G}_m]$, we have the origin gives a closed substack isomorphic to $B\mathbb{G}_m$. The complement is $\mathbb{G}_m/(\mathbb{G}_m)^n \cong B\mu_n$. Thus, the localization exact sequence gives

$$\mathrm{CH}([\mathbb{A}^1/\mathbb{G}_m],1)_{\mathrm{ind}} \to \mathrm{CH}(B\mu_n,1)_{\mathrm{ind}} \to \mathrm{CH}(B\mathbb{G}_m) \to \mathrm{CH}([\mathbb{A}^1/\mathbb{G}_m]).$$

The class $[B\mathbb{G}_m] \in \mathrm{CH}^*([\mathbb{A}^1/\mathbb{G}_m]) = \mathbb{Z}[t]$ is equal to nt, so by the following lemma, we have that the pushforward is given by

$$\mathbb{Z}[t] = \mathrm{CH}^*(B\mathbb{G}_m) \to \mathrm{CH}^*([\mathbb{A}^1/\mathbb{G}_m]) = \mathbb{Z}[t]$$
$$t \mapsto nt.$$

Thus, the cokernel is $\mathrm{CH}^*(B\mu_n) = \frac{\mathbb{Z}[t]}{(nt)}$ and the kernel is 0. Moreover, by Corollary 2.8, we have $\mathrm{CH}^*([\mathbb{A}^1/\mathbb{G}_m],1)_{\mathrm{ind}} = 0$, so exactness of the sequence gives $\mathrm{CH}^*(B\mu_n,1)_{\mathrm{ind}} = 0$. □

Aside from the proof of the previous, the next lemma will be used quite a bit in section [???].

Lemma 2.10. *Given actions of G on $\mathbb{A}^n, \mathbb{A}^m$, and a G-equivariant map $f : \mathbb{A}^n \to \mathbb{A}^m$, the pullback $f^* : \mathrm{CH}^*_G(\mathbb{A}^m, j) \to \mathrm{CH}^*_G(\mathbb{A}^n, j)$ is an isomorphism. Moreover, if f is proper, we have*

$$f_*(\beta) = \beta \cdot f_*(1)$$

under the identification of $\mathrm{CH}^*_G(\mathbb{A}^m, j)$ and $\mathrm{CH}^*_G(\mathbb{A}^n, j)$ by f^*.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^*_G(\mathbb{A}^m) & & \mathrm{CH}^*_G(\mathbb{A}^n) \arrow[ll, "f^*"'] \\
& \mathrm{CH}^*_G(\mathrm{Spec}(k)) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

By homotopy invariance, the maps out of $\mathrm{CH}^*(\mathrm{Spec}(k))$ are isomorphisms, hence f^* must also be an isomorphism.

If f is proper, and we use f^* to identify the elements of $\mathrm{CH}^*_G(\mathbb{A}^m)$ with $\mathrm{CH}^*_G(\mathbb{A}^n)$, the projection formula gives

$$f_*(\beta) = f_*(f^*(\beta)) = \beta \cdot f_*(1).$$

□

2.4. **Comparison With ℓ-adic Higher Chow Groups.** In [Larson], Larson introduced the notion of ℓ-adic higher Chow groups. If the groups $\mathrm{CH}_i(X_{\overline{k}}, 1, \mathbb{Z}/\ell^m\mathbb{Z}) := H_1(z_i(X_{\overline{k}}, *) \otimes \mathbb{Z}/\ell \mathbb{Z})$ are finitely generated, then their definition is equivalent to

$$\mathrm{CH}_i(X, 1, \mathbb{Z}_\ell) = \varprojlim \mathrm{CH}_i(X_{\overline{k}}, 1, \mathbb{Z}/\ell^m\mathbb{Z}).$$

These groups share many of the same properties as indecomposible higher Chow groups [Bishop23]. For example, $\mathrm{CH}(\mathrm{Spec}(k), 1, \mathbb{Z}_\ell) = 0$, meaning that these groups are also able to "get rid of" the $k^\times$ in $\mathrm{CH}^1(\mathrm{Spec}(k), 1)$. This allows these ℓ-adic higher Chow groups to be finitely generated in

We have the following result relating ℓ-adic higher Chow groups to indecomposible higher Chow groups. This result explains why there must be similiarities between these groups.

Proposition 2.11. *Let X be a stack with the $\mathrm{CH}_i(X), \mathrm{CH}_i(X, 1)_{ind}$ finitely generated for all i. Then*

$$\mathrm{CH}_i(X, 1, \mathbb{Z}_\ell) = \mathrm{CH}_i(X_{\overline{k}}, 1)_{ind} \otimes \mathbb{Z}_\ell.$$

Furthermore, if X has the MKP then

$$\mathrm{CH}_i(X, 1, \mathbb{Z}_\ell) = \mathrm{CH}_i(X, 1)_{ind} \otimes \mathbb{Z}_\ell.$$

Remark 2.12. If a stack X has the MKP, the groups $\mathrm{CH}_i(X)$ and $\mathrm{CH}_i(X, 1)_{\mathrm{ind}}$ are automatically finitely generated. In all instances where ℓ-adic higher Chow groups have been used, the spaces involvded have the MKP.

Proof. We have an exact sequence

$$\mathrm{CH}_{i+1}(X_{\overline{k}}) \otimes k^\times \to \mathrm{CH}_i(X_{\overline{k}}, 1) \to \mathrm{CH}_i(X_{\overline{k}}, 1)_{\mathrm{ind}} \to 0.$$

Tensoring this with $\mathbb{Z}/\ell^m\mathbb{Z}$, we have an exact sequence

$$0 \to \mathrm{CH}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z} \to \mathrm{CH}_i(X_{\overline{k}}, 1)_{\mathrm{ind}} \otimes \mathbb{Z}/\ell^m\mathbb{Z} \to 0,$$

because $k^\times$ is divisible and tensor is right exact. Thus, we have $\mathrm{CH}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z} \cong \mathrm{CH}_i(X_{\overline{k}}, 1)_{\mathrm{ind}} \otimes \mathbb{Z}/\ell^m\mathbb{Z}$.

Next, by the universal coefficient theorem, we have a split exact sequence

$$0 \to \mathrm{CH}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z} \to \mathrm{CH}_i(X_{\overline{k}}, 1, \mathbb{Z}/\ell^m\mathbb{Z}) \to \mathrm{Tor}^1(\mathrm{CH}_i(X_{\overline{k}}), \mathbb{Z}/\ell^m\mathbb{Z}) \to 0.$$

Because $\mathrm{CH}_i(X_{\overline{k}}, 1)_{\mathrm{ind}}$ is finitely generated, so is $\mathrm{CH}_i(X_{\overline{k}}, 1)_{\mathrm{ind}} \otimes \mathbb{Z}/\ell^m\mathbb{Z} = \mathrm{CH}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z}$. And because $\mathrm{CH}_i(X_{\overline{k}})$ is finitely generated, so is $\mathrm{Tor}^1(\mathrm{CH}_i(X_{\overline{k}}), \mathbb{Z}/\ell^m\mathbb{Z})$. Thus, $\mathrm{CH}_i(X_{\overline{k}}, 1, \mathbb{Z}/\ell^m\mathbb{Z})$ is finitely generated for all m, so we have

$$\mathrm{CH}_i(X, 1, \mathbb{Z}_\ell) = \varprojlim \mathrm{CH}_i(X_{\overline{k}}, 1, \mathbb{Z}/\ell^m\mathbb{Z}).$$

In taking the limit, the torsion groups $\mathrm{Tor}^1(\mathrm{CH}_i(X_{\overline{k}}), \mathbb{Z}/\ell^m\mathbb{Z})$ go to 0, so we have

$$\begin{aligned}\mathrm{CH}_i(X, 1, \mathbb{Z}_\ell) &= \varprojlim \mathrm{CH}_i(X_{\overline{k}}, 1, \mathbb{Z}/\ell^m\mathbb{Z}) \\ &= \varprojlim \mathrm{CH}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z} \\ &= \varprojlim \mathrm{CH}_i(X_{\overline{k}}, 1)_{\mathrm{ind}} \otimes \mathbb{Z}/\ell^m\mathbb{Z} \\ &= \mathrm{CH}_i(X_{\overline{k}}, 1)_{\mathrm{ind}} \otimes \mathbb{Z}_\ell,\end{aligned}$$

where the last equality holds because $\mathrm{CH}_i(X_{\overline{k}}, 1)_{\mathrm{ind}}$ is finitely generated.

Now, if X has the MKP, we have Tor. □

2.5. Universal Separable Homeomorphisms.

Definition 2.13. A map of stacks $f : X \to Y$ is a universal separable homeomorphism if it is a integral surjective map such that, for all $p \in Y$, there is a unique preimage $q \in X$ and the map $\kappa(p) \to \kappa(q)$ is an isomorphism.

Example 2.14. Consider the normalization of the node:

$$f : \mathrm{Spec}(k[t]) \to \mathrm{Spec}(k[x, y]/(y^2 - x^3)$$
$$t \mapsto (t^2, t^3).$$

This is a finite, hence integral, surjection, being a normalization. Away from the cusp, $(0, 0)$, this map is an isomorphism, hence all other points in $\mathrm{Spec}(k[x, y]/(y^2 - x^3)$ have unique preimages with induced isomorphisms on the residue fields. The point $(0, 0)$ also has a unique preimage, 0, and the map on residue fields is an isomorphism. Hence, f is a universal separable homeomorphism.

Lemma 2.15. *Universal separable homeomorphisms are closed under base change.*

Proof. Suppose $f : X \to Y$ is a universal separable homeomorphism, and consider a fiber diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

Being integral and surjective are closed under base change, so we just need to show preimages are unique and the maps on residue fields are isomorphisms.

Take $p' \in Y$ and $p = g(p')$. Because f is integral, it is affine, so we can write $f^{-1}(p) = \mathrm{Spec}(A)$. The ring A must be a Artinian ring over $K := \kappa(p)$. Because f is a universal separable homeomorphism, we know that the residue field of A is K.

Note $f'^{-1}(p') = \mathrm{Spec}(A \otimes_K L)$, where $L = \kappa(p')$. It suffices to show that the residue field of $A \otimes_K L$ is equal to L. Let $\mathfrak{m}$ be the maximal ideal of A. We have an exact sequence

$$0 \to \mathfrak{m} \otimes_K L \to A \otimes_K L \to K \otimes_K L = L \to 0.$$

The exact sequence implies that $\mathfrak{m} \otimes_K L$ must be a maximal ideal, because its quotient is a field. Moreover, any prime ideal of $A \otimes_K L$ restricts to $\mathfrak{m}$, and hence contains the ideal generated by $\mathfrak{m} \otimes 1$, which is $\mathfrak{m} \otimes_K L$. Thus, $A \otimes_K L$ has a unique prime ideal, and the residue field of $A \otimes_K L$ is indeed L. □

Remark 2.16. As the name suggests, universal separable homeomorphisms are precisely the universal homeomorphisms which induce separable field extensions on residue fields. One direction is clear: our universal separable homeomorphisms are homeomorphisms, being closed continuous bijections, and are universal homeomorphisms because this lemma shows the property is preserved by base change. The other direction is true because universal homeomorphisms are exactly maps which are integral, surjective, every point in the target has a unique preimage, and the induced map on residue fields are purely inseparable (see [**?**, Tag 04DF] and [**?**, Tag 01S2]).

Proposition 2.17. *Let $f : X \to Y$ be a representable universal separable homeomorphism. Then the induced maps $f_* : \mathrm{CH}_i(X, j) \to \mathrm{CH}_i(Y, j)$ are isomorphisms.*

Proof. Note that unviersal separable homeomorphisms are proper, since they are closed and preserved by base change, so we really can pushforward along them. It suffices to assume that X and Y are schemes.

The pushforward $f_* : \mathrm{CH}_i(X, j) \to \mathrm{CH}_i(Y, j)$ is induced by the map $f_* : z_i(X, *) \to z_i(Y, *)$ on complexes, given by the pushforward of cycles. Because $f : X \to Y$ is a universal separable homeomorphism, so is f :

$X \times \Delta^n \to Y \times \Delta^n$. Therefore, for any closed, irreducible $V \subseteq X \times \Delta^n$, we have $f_*(V) = \deg(V/f_*(V)) \cdot f(V) = f(V)$, as the map on residue fields is an isomorphism. And now, we see that $f_* : z_i(X, j) \to z_i(Y, j)$ has an inverse, sending W to $f^{-1}(W)$. Thus, the map of complexes is an isomorphism, so the induced maps $f_* : \mathrm{CH}_i(X, j) \to \mathrm{CH}_i(Y, j)$ are isomorphisms. □

3. Presentations of Stacks

Definition 3.1.

$$\mathcal{M}^0_{1,2} = \{(C, p_1, p_2) \in \mathcal{M}_{1,2} | 2p_1 \not\sim 2p_2\}$$

$$\mathcal{M}^0_{1,3} = \{(C, p_1, p_2, p_3) \in \mathcal{M}_{1,3} | 2p_1 \not\sim p_2 + p_3\}$$

Lemma 3.2. *Let D_1, D_2 be degree 2 divisors on a genus 1 curve. Then $|\mathcal{O}(D_1)| \times |\mathcal{O}(D_2)|$ gives a morphism to $\mathbb{P}^1 \times \mathbb{P}^1$, which is an embedding if and only if $D_1 \nsim D_2$.*

Let

$$P_1 := ([1:0], [0:1])$$
$$P_2 := ([0:1], [1:0])$$
$$P_3 := ([1:1], [1:0])$$
$$P_4 := ([1:0], [1:1])$$

and $V_4 \subseteq V$ be the 5-dimensional subspace consisting of curves going through P_1, P_2, P_3, P_4. We can write V_4 as the locus where the following equations hold

$$0 = a_2 = a_6 = a_0 + a_1 = a_0 + a_3,$$

so that

$$V_4 = \{a_0x^2y^2 - a_0x^2yz - a_0xwy^2 + a_4xwyz + a_5xwz^2 + a_7w^2yz + a_8w^2z^2\}.$$

Let $\Delta_4 \subseteq \mathbb{P}(V_4)$ be the locus of singular curves.

Define

$$Z_4 := \{(C, p_1, p_2) \in \mathcal{M}_{1,2} | p_1 + p_3 \sim p_2 + p_4\}$$

and let U_4 be the complement of Z_4 in $\mathcal{M}_{1,4}$.

Lemma 3.3. $U_4 \cong \mathbb{P}(V_4) \setminus \Delta_4$.

Proof. We get a morphism $\mathbb{P}(V_4) \setminus \Delta_4 \to \mathcal{M}_{1,4}$ coming from the tautological family, together with the sections corresponding to P_1, P_2, P_3, and P_4. The inverse is obtained by mapping a curve $(C, p_1, p_2, p_3, p_4) \in U_4$ to $\mathbb{P}^1 \times \mathbb{P}^1$ via the linear series $|\mathcal{O}(p_1 + p_3)| \times |\mathcal{O}(p_2 + p_4)|$ (which is an embedding because $p_1 + p_3 \nsim p_3 + p_4$), and preforming the unique automorphism so that

$$p_1 \mapsto P_1$$
$$p_2 \mapsto P_2$$
$$p_3 \mapsto P_3$$
$$p_4 \mapsto P_4.$$

The resulting curve is in $\mathbb{P}(V_4) \setminus \Delta_4$ because it goes through the required points. $\square$

4. Equivariant Calculations

4.1. $\mathcal{M}_{1,1}$**.** Recall our presentation of $\mathcal{M}_{1,1}$ from the previous section, $\mathcal{M}_{1,1} = [U_1/\mathbb{G}_m]$, where $U_1 = \{(a,b) \in \mathbb{A}^2 | x^3 + ax + b \text{ has distinct roots}\}$, and $\mathbb{G}_m$ acts with weights $(-4,-6)$. The discriminant of x^3+ax+b is $-4a^3-27b^2$. So the complement of U_1 in $\mathbb{A}^2$ is given by $\iota_1 : \Delta_1 := V(-4a^3-27b^2) \hookrightarrow \mathbb{A}^2$.

We study the map $\iota_{1*} : \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_1) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)$. Let

$$f_1 : \mathbb{A}^1 \to \Delta_1$$

$$c \mapsto (-3c^2, 2c^3)$$

be the normalization. This map is equivariant if we act by $\mathbb{G}_m$ on $\mathbb{A}^1$ with weight -2.

Proposition 4.1. *$\mathcal{M}_{1,1}$ has some sort of CKP.*

Proof. $\square$

Lemma 4.2. *$\iota_{1*} : \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_1) \xrightarrow{\iota_*} \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) = \mathbb{Z}[t]$ is injective and has image generated by $12t$.*

Proof. Let $\widetilde{i_1} := \iota_1 \circ f_1$, so that we have

$$\begin{tikzcd} \mathbb{A}^1 \arrow[rr, "\widetilde{\iota}_1"] \arrow[dr, "f"'] & & \mathbb{A}^2 \\ & \Delta_1 \arrow[ur, "\iota_1"'] & \end{tikzcd}$$

By Lemma 2.10, we have $\widetilde{\iota}_1^*$ is an isomorphism and $\widetilde{\iota}_{1*}$ is multiplication by

$$\widetilde{\iota}_{1*}([\mathbb{A}^1]) = \iota_{1*} f_{1*}([\mathbb{A}^1]) = \iota_{1*}([\Delta_1]) = [\Delta_1].$$

We can compute........, so $[\Delta_1] = -12t$. And so $\widetilde{\iota}_{1*}$ has image generated by $12t$ and is injective.

Now, f_1 is a universal separable homeomorphism by Example 2.14, so by Corollary **??**, f_{1*} is an isomorphism. This says that ι_* has the same image, $(12t)$, and is injective. $\square$

Theorem 4.3.

$$\mathrm{CH}^*(\mathcal{M}_{1,1}) = \mathbb{Z}[t]/(12t),$$

and

$$\mathrm{CH}^*(\mathcal{M}_{1,1},1)_{ind} = 0.$$

Proof. Note

$$\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2,1)_{\mathrm{ind}} \cong \mathrm{CH}^*_{\mathbb{G}_m}(\mathrm{pt},1)_{\mathrm{ind}} = \mathrm{CH}^*(B\mathbb{G}_m,1)_{\mathrm{ind}} = 0$$

where the first isomorphism is by homotopy invariance, and the second follows from Proposition 2.7 and Lemma ??. The localization exact sequence then gives

$$0 \to \mathrm{CH}^*(\mathcal{M}_{1,1},1)_{\mathrm{ind}} \to \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_1) \xrightarrow{\iota_{1*}} \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \to \mathrm{CH}^*(\mathcal{M}_{1,1}) \to 0.$$

By Lemma 4.2, ι_* is injective and has image $(12t) \subseteq \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) = \mathbb{Z}[t]$. Thus, exactness gives $\mathrm{CH}^*(\mathcal{M}_{1,1},1)_{\mathrm{ind}} = 0$ and $\mathrm{CH}^*(\mathcal{M}_{1,1}) = \mathbb{Z}[t]/(12t)$. □

Remark 4.4. One can adapt the above to calculate the full higher Chow groups of $\mathcal{M}_{1,1}$. For $i \geq 1$ we have

$$\mathrm{CH}^i_{\mathbb{G}_m}(\mathbb{A}^2,1) = \mathrm{CH}^i_{\mathbb{G}_m}(\mathrm{Spec}(k),1) = \mathrm{CH}^i(\mathbb{P}^n,1) = k^\times$$

and

$$\mathrm{CH}^i_{\mathbb{G}_m}(\Delta_1) = \mathrm{CH}^i_{\mathbb{G}_m}(\mathbb{A}^1,1) = k^\times.$$

Then the localization exact sequence gives

$$k^\times \to k^\times \to \mathrm{CH}^i(\mathcal{M}_{1,1},1)_{\mathrm{ind}} \to 0.$$

An argument analogous to Lemma 2.10 gives that the map $k^\times \to k^\times$ is $t \mapsto t^12$. Thus, one has

$$\mathrm{CH}^*(\mathcal{M}_{1,1},1) = \bigoplus_{i=1}^{\infty} \frac{k^\times}{(k^\times)^{12}}.$$

4.2. $\mathcal{M}_{1,2}$**.** Recall our presentation of $\mathcal{M}_{1,2}$ from the previous section, $\mathcal{M}_{1,2} = [U_2/\mathbb{G}_m]$ for

$$U_2 := \{(a,x_2,y_2) \in \mathbb{A}^3 | -4a^3 - 27B(a,x_2,y_2)^2 \neq 0\}$$

where

$$B(a,x_2,y_2) := y_2^2 - x_2^3 - ax_2$$

and $\mathbb{G}_m$ acts with weights $(-4,-2,-3)$. The complement of U_2 in $\mathbb{A}^3$ is given by $\iota_2 : \Delta_2 := V(-4a^3 - 27b^2) \hookrightarrow \mathbb{A}^2$.

We study the map $\iota_{2*} : \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3)$. We have the $\mathbb{G}_m$-equivariant map

$$\mathbb{A}^3 \to \mathbb{A}^2$$

$$(a,x_2,y_2) \mapsto (a, B(x_2,y_2,a)).$$

This restricts to a map $\Delta_2 \to \Delta_1$, as Δ_2 is, essentially by definition, the inverse image of Δ_1 under this map. Define $\widehat{\Delta}_2 := \mathbb{A}_1 \times_{\Delta_1} \Delta_2$ so that

$$\begin{tikzcd} \widehat{\Delta}_2 \arrow[r, "f_2"] \arrow[d] & \Delta_2 \arrow[d] \\ \mathbb{A}^1 \arrow[r, "f_1"] & \Delta_1 \end{tikzcd}$$

is Cartesian. Everything involved is an affine scheme, so a tensor product computation gives

$$\widehat{\Delta}_2 = \operatorname{Spec} k[x_2, y_2, c]/(y_2^2 - x_2^3 + 3c^2 x_2 - 2c^3).$$

Next, define

$$g_2 : \mathbb{A}^2 \to \widehat{\Delta}_2$$
$$(c, d) \mapsto (d^2 - 2c, d^3 - 3cd, c).$$

To make this map equivariant, we act with weight -2 on c and -1 on d. This map is also finite, hence proper, as both c and d are integral over $k[x_2, y_2, c]/(y_2^2 - x_2^3 + 3c^2 x_2 + 2c^3)$.

Over the locus $W_2 := D(x_2 - c)$, we have that g_2 is an isomorphism, as one can check that

$$(x_2, y_2, c) \mapsto \left(c, \frac{y_2}{x_2 - c}\right)$$

is an inverse on this open subset. The reduced complement, $j_2 : C_2 \hookrightarrow \Delta_2$, of W_2 is given by $C_2 = V(x_2 - c, y_2)$. Then $C_2 = \operatorname{Spec} k[x_2, y_2, c]/(x_2 - c, y_2) \cong \operatorname{Spec} k[c]$. Moreover, $g_2^{-1}(C_2) = \operatorname{Spec} k[c, d]/(d^2 - 3c) \cong \operatorname{Spec} k[d]$.

Lemma 4.5. *The image of $\iota_{2*} : \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3) = \mathbb{Z}[t]$ is generated by $12t$, and the kernel is isomorphic to*

$$\bigoplus_{i=1}^{\infty} \frac{\mathbb{Z}}{2\mathbb{Z}},$$

generated by $f_{2*}(g_{2*}(t^i) - j_{2*}(t^{i-1}))$ in degree i.

Proof. Lemma 2.1 gives us that

$$\mathrm{CH}^*_{\mathbb{G}_m}(\widehat{\Delta}_2) = \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1} C_2)} \mathrm{CH}^*_{\mathbb{G}_m}(C_2).$$

Because $f_2 : \widehat{\Delta}_2 \to \Delta_2$ is a universal separable homeomorphism, we have $f_{2*} : \mathrm{CH}^*_{\mathbb{G}_m}(\widehat{\Delta}_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2)$ is an isomorphism, and so we have

$$\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2) = \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1} C_2)} \mathrm{CH}^*_{\mathbb{G}_m}(C_2).$$

Thus, to compute the image of ι_{2*}, it suffices to compute the images of the pushforwards of $\mathbb{A}^2 \to \mathbb{A}^3$ and $C_2 \to \mathbb{A}^3$.

The map $\mathbb{A}^2 \to \mathbb{A}^3$ is the composition

$$\mathbb{A}^2 \xrightarrow{g_2} \widehat{\Delta}_2 \xrightarrow{f_2} \Delta_2 \xrightarrow{\iota_2} \mathbb{A}^3.$$

The first two maps in this composition are birational, and the second is a closed embedding, so $1 \in \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)$ pushes forward to $[\Delta_2]$. We compute....... $[\Delta_2] = -12t$. By Lemma 2.10, the map $\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3)$ is multiplication by $-12t$, under the identification of the pullback.

The map $C_2 \cong \operatorname{Spec} k[c] \to \mathbb{A}^3$ is the composition

$$C_2 \xrightarrow{j} \widehat{\Delta}_2 \xrightarrow{f_2} \Delta_2 \xrightarrow{\iota_2} \mathbb{A}^3.$$

Using the definitions of these maps, we can compute it explicitly as $c \mapsto (c, 0, -3c^2)$. The image of this is the closed subscheme $V(y_2, a + 3x_2^2)$. We compute the class of this image to be.........$-12t^2$. By Lemma 2.10, the map $\mathrm{CH}^*_{\mathbb{G}_m}(C_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3)$ is multiplication by $-12t^2$, under the identification of the pullback. Combining this with the conclusion of the previous paragraph, we see that the image of ι_{2*} is generated by $12t$.

We next compute the kernel of ι_{2*}. From the above descriptions, we see that the kernel of $\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \oplus \mathrm{CH}^*_{\mathbb{G}_m}(C_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3)$ is $\{(tp(t), -p(t)) | p(t) \in \mathbb{Z}[t]\}$. To get the kernel out of the map out of $\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \amalg_{\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}C_2)} \mathrm{CH}^*_{\mathbb{G}_m}(C_2)$, we need to quotient this out by the image of $\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}(C_2)) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \oplus \mathrm{CH}^*_{\mathbb{G}_m}(C_2)$.

The map $g_2^{-1}(C_2) \cong \operatorname{Spec} k[d] \to \mathbb{A}^2$ is a closed embedding cut out by $d^2 - 3c$. So 1 pushes forward to the class of $V(d^2 - 3c)$..... i.e. $-2t$. By Lemma 2.10, we have $\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}(C_2)) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)$ is given by $p(t) \mapsto -2tp(t)$.

The map $g_2^{-1}(C_2) \to C_2$ is given by $d \mapsto \frac{d^2}{3}$ under the isomorphisms $C_2 \cong \operatorname{Spec} k[c]$ and $g_2^{-1}(C_2) \cong \operatorname{Spec} k[d]$. This is a degree 2 map, so $1 \in \mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}(C_2))$ maps to 2. By Lemma 2.10, we have $\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}(C_2)) \to \mathrm{CH}^*_{\mathbb{G}_m}(C_2)$ is given by $p(t) \mapsto 2p(t)$. Thus, the map $\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}(C_2)) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \oplus \mathrm{CH}^*_{\mathbb{G}_m}(C_2)$ has image $\{(2tp(t), -2p(t)) | p(t) \in \mathbb{Z}[t]\}$, and so

$$\begin{aligned}
\ker(\iota_2) &\cong \ker(\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}C_2)} \mathrm{CH}^*_{\mathbb{G}_m}(C_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3)) \\
&\cong \frac{\ker(\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \oplus \mathrm{CH}^*_{\mathbb{G}_m}(C_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3))}{\mathrm{im}(\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}(C_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \oplus \mathrm{CH}^*_{\mathbb{G}_m}(C_2))} \\
&= \frac{\{(tp(t), -p(t)) | p(t) \in \mathbb{Z}[t]\}}{\{(2tp(t), -2p(t)) | p(t) \in \mathbb{Z}[t]\}} \\
&\cong \bigoplus_{i=1}^{\infty} \frac{\mathbb{Z}}{2\mathbb{Z}}
\end{aligned}$$

□

Theorem 4.6.

$$\mathrm{CH}^*(\mathcal{M}_{1,2}) = \mathbb{Z}[t]/(12t),$$

and

$$\mathrm{CH}^*(\mathcal{M}_{1,2}, 1)_{ind} = \bigoplus_{i=2}^{\infty} \frac{\mathbb{Z}}{2\mathbb{Z}},$$

Proof. We have $\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3,1)_{\mathrm{ind}} = 0$ by Proposition 2.7 and Lemma **??**, so the localization exact sequence gives

$$0 \to \mathrm{CH}^*(\mathcal{M}_{1,2},1)_{\mathrm{ind}} \to \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_1) \xrightarrow{\iota_{2*}} \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \to \mathrm{CH}^*(\mathcal{M}_{1,2}) \to 0.$$

By Lemma 4.5, ι_{2*} has kernel isomorphic to $\bigoplus_{i=1}^{\infty} \frac{\mathbb{Z}}{2\mathbb{Z}}$. Due to the shift of indexing, we conclude $\mathrm{CH}^*(\mathcal{M}_{1,2},1)_{\mathrm{ind}} = \bigoplus_{i=2}^{\infty} \frac{\mathbb{Z}}{2\mathbb{Z}}$. Moreover, the lemma also says that the image of ι_{2*} is $(12t)$, so $\mathrm{CH}^*(\mathcal{M}_{1,2}) = \mathbb{Z}[t]/(12t)$. □

Remark 4.7. One could try and use the above to calculate the full higher Chow groups $\mathrm{CH}(\mathcal{M}_{1,2},1)$. With similar reasoning to Remark 4.4, one gets an exact sequence

$$0 \to \frac{k^\times}{(k^\times)^{12}} \to \mathrm{CH}^i(\mathcal{M}_{1,2},1) \to \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0.$$

Thus, one needs to solve an extension problem to figure out the group $\mathrm{CH}^i(\mathcal{M}_{1,2},1)$. The author was able to show the extension must be trivial if the field k contains a square-root of -1, but does not know what happens otherwise. These group extension problems persist (and increase in number) when trying to compute $\mathrm{CH}^i(\mathcal{M}_{1,n},1)$ for $n = 3,4$.

Proposition 4.8. *$\mathcal{M}_{1,2}$ has some sort of CKP.*

Proof. □

4.3. $\mathcal{M}^0_{1,2}$.

Lemma 4.9. *$\mathcal{M}^0_{1,2}$ has some sort of CKP.*

Proof. Finally, we want $\mathcal{M}^0_{1,2}$ to have the CKP and the hCKgP. Note $[D(y_2)/\mathbb{G}_m]$ has these properties, because it is the complement of an $[\mathbb{A}^1/\mathbb{G}_m]$ in $[\mathbb{A}^2/\mathbb{G}_m]$. And so it suffices to show that $[\Delta^0_2/\mathbb{G}_m]$ has CKP and hCKgP. As noted above, this is isomorphic to $[D(d^3-3cd)/\mathbb{G}_m] \subseteq [\mathbb{A}^2/\mathbb{G}_m]$. To show this has CKP and hCKgP, it suffices to show that $[V(d^3-3cd)/\mathbb{G}_m]$ does. Note $V(d^3-3cd) = V(d) \cup V(d^2-3c)$, both of which are isomorphic to $\mathbb{A}^1$. These intersect at a point, so this space can be cut up into 2 pieces isomorphic to $\mathbb{G}_m$ and one point, which implies $[V(d^3-3cd)/\mathbb{G}_m]$ has CKP and hCKgP by Proposition **??**. □

Recall that $\mathcal{M}^0_{1,2}$ has presentation $[U_2 \backslash V(y_2)/\mathbb{G}_m]$. To compute $\mathrm{CH}^*(\mathcal{M}^0_{1,2}), \mathrm{CH}^*(\mathcal{M}^0_{1,2},1)_{\mathrm{ind}}$, we first remove $V(y_2)$ from $\mathbb{A}^2$, and then remove $V(-4a^3 - 27B(x_2,y_2)^2 \setminus D(y_2)$ from that. One may expect it to be easier to remove $V(y_2) \cap U_2$ from U_2, as we have already done calculations to obtain $\mathrm{CH}^*_{\mathbb{G}_m}(U_2), \mathrm{CH}^*_{\mathbb{G}_m}(U_2,1)_{\mathrm{ind}}$, but this is not the case.

Lemma 4.10.

$$\mathrm{CH}^*_{\mathbb{G}_m}(D(y_2)) = \mathbb{Z}[t]/(3t)$$

and

$$\mathrm{CH}^*_{\mathbb{G}_m}(D(y_2),1)_{ind} = 0$$

Proof. Because $\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2,1)_{\mathrm{ind}} = 0$, the localization exact sequence for $D(y_2) \subseteq \mathbb{A}^2$ gives

$$0 \to \mathrm{CH}^*_{\mathbb{G}_m}(D(y_2),1) \to \mathrm{CH}^*_{\mathbb{G}_m}(V(y_2)) \xrightarrow{q_*} \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3) \to \mathrm{CH}^*_{\mathbb{G}_m}(D(y_2)) \to 0.$$

Note $V(y_2) \cong \mathbb{A}^1$, and $[V(y_2)] = -3t \in \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3)$ because y_2 has weight -3, so by Lemma 2.10, we know that the maps q_* are multiplication by $-3t$, under the identification by the pullback. By exactness, we get

$$\mathrm{CH}^*_{\mathbb{G}_m}(D(y_2)) = \mathbb{Z}[t]/(3t)$$
$$\mathrm{CH}^*_{\mathbb{G}_m}(D(y_2),1)_{\mathrm{ind}} = 0$$

□

Set $\Delta_2^0 := \Delta_2 \cap D(y_2)$, with closed embedding $\iota_2^0 : \Delta_2^0 \hookrightarrow D(y_2)$, so then $\mathcal{M}_{1,2}^0 \cong [(D(y_2) \setminus \Delta_2^0)/\mathbb{G}_m]$.

Lemma 4.11. *$\iota^0_{2*} : \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2^0) \to \mathrm{CH}^*_{\mathbb{G}_m}(D(y_2))$ is equal to zero and $\mathrm{CH}^*(\Delta_2^0) = \mathbb{Z}$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2) \arrow[r, "\iota_{2*}"] \arrow[d] & \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3) \arrow[d] \\
\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2^0) \arrow[r, "\iota^0_{2*}"] & \mathrm{CH}^*_{\mathbb{G}_m}(D(y_2))
\end{tikzcd}$$

As noted above, we can identify the right map with $\mathbb{Z}[t] \to \mathbb{Z}[t]/(3t)$. Moreover, by Lemma 4.5, the image of ι_{2*} is $(12t)$, so composition $\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3) \to \mathrm{CH}^*_{\mathbb{G}_m}(D(y_2))$ is 0. Finally, because Δ_2^0 is open in Δ_2, the left map is surjective, and so $\iota^0_{2*} = 0$.

We pullback $D(y_2)$ through f', a universal separable homeomorphism, and g to obtain

$$\begin{tikzcd}
\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \arrow[r] \arrow[d] & \mathrm{CH}^*_{\mathbb{G}_m}(g^{-1}f'^{-1}(\Delta_2^0)) \arrow[r] \arrow[d] & 0 \\
\mathrm{CH}^*_{\mathbb{G}_m}(\widehat{\Delta}_2) \arrow[r] \arrow[d, "\sim"] & \mathrm{CH}^*_{\mathbb{G}_m}(f'^{-1}(\Delta_2^0)) \arrow[r] \arrow[d, "\sim"] & 0 \\
\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2) \arrow[r] & \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2^0) \arrow[r] & 0
\end{tikzcd}$$

As $\Delta_2^0 = D_{\Delta_2}(y_2)$, we have $f'^{-1}(\Delta_2^0) = f'^{-1}(D_{\Delta_2}(y_2)) = D_{\widehat{\Delta}_2}(y_2)$ and $g^{-1}(f'^{-1}(D_{\Delta_2}(y_2))) = g^{-1}(D_{\widehat{\Delta}_2}(y_2)) = D_{\mathbb{A}^2}(d^3 - 3cd)$. Now, we have

$$f'^{-1}(\Delta_2^0) = D_{\widehat{\Delta}_2}(y_2) \subseteq D_{\widehat{\Delta}_2}(x_2 - c) = W,$$

because, in $\widehat{\Delta}_2$, we have

$$y_2^2 = x_2^3 - 3c^2x_2 + 2c^3 = (x_2 - c)^2(x_2 + 2c).$$

Moreover, we have g is an isomorphism over W, and so our pushforward

$$\mathrm{CH}^*(D(d^3-3cd)) \xrightarrow{(g\circ f')_*} \mathrm{CH}^*(\Delta_2^0)$$

is an isomorphism.

Because $D(d^3-3cd)$ is an open in $D(d)$, we have $\mathrm{CH}^*_{\mathbb{G}_m}(D(d^3-3cd))$ is a quotient of $\mathrm{CH}^*_{\mathbb{G}_m}(D(d))$. Because d is acted on by weight -1, $[V(d)] = -t \in \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)$, so Lemma 2.10 and the exact sequence

$$\mathrm{CH}^*_{\mathbb{G}_m}(V(d)) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \to \mathrm{CH}^*_{\mathbb{G}_m}(D(d)) \to 0$$

imply $\mathrm{CH}^*_{\mathbb{G}_m}(D(d)) = \mathbb{Z}$. Thus, we also have $\mathrm{CH}^*_{\mathbb{G}_m}(D(d^3-3cd)) = \mathbb{Z}$, and so $\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2^0) = \mathbb{Z}$. $\square$

Theorem 4.12.

$$\mathrm{CH}^*(\mathcal{M}^0_{1,2}) = \mathbb{Z}[t]/(3t)$$

and

$$\mathrm{CH}^*(\mathcal{M}^0_{1,2},1)_{ind} = \mathbb{Z}.$$

Proof. We saw $\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^2}(y_2)) = \mathbb{Z}[t]/(3t)$ and $\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^2}(y_2),1)_{\mathrm{ind}} = 0$. So the localization sequence for $U_2 \setminus V(y_2) \subseteq D_{\mathbb{A}^2}(y_2)$ gives

$$0 \to \mathrm{CH}^*(\mathcal{M}^0_{1,2},1)_{\mathrm{ind}} \to \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2^0) \xrightarrow{\iota_2^0} \mathbb{Z}[t]/(3t) \to \mathrm{CH}^*(\mathcal{M}^0_{1,2}) \to 0$$

By Lemma 4.11, we know $\iota_2^0 = 0$ and $\mathrm{CH}^*_{\mathbb{G}_m}(Z_2^0) = \mathbb{Z}$, from which the theorem follows. $\square$

4.4. $\mathcal{M}^0_{1,3}$. Recall our presentation of $\mathcal{M}^0_{1,3}$, $[(U_3/\mathbb{G}_m]$ for

$$U_3 := \{(x_2,y_2,x_3,y_3) \in D_{\mathbb{A}^4}(x_3-x_2) | -4A^3-27B^2 \neq 0\},$$

where

$$A = A(x_2,y_2,x_3,y_3) := \frac{y_3^2-y_2^2-x_3^3+x_2^3}{x_3-x_2}$$
$$B = B(x_2,y_2,x_3,y_3) := y_2^2-x_2^3-Ax_2$$

and $\mathbb{G}_m$ acts with weights $(-2,-3,-2,-3)$.

We first remove $V(x_3-x_2)$ from $\mathbb{A}^4$. Similar to the proof of Lemma 4.10, we have

$$\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3-x_2)) = \mathbb{Z}[t]/(2t)$$
$$\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3-x_2),1)_{\mathrm{ind}} = 0,$$

because the weight of the action on x_3-x_2 is -2. Let $\iota_3 : \Delta_3^0 \hookrightarrow D_{\mathbb{A}^4}(x_3-x_2)$ be the complement of U_3 in $D_{\mathbb{A}^4}(x_3-x_2)$, so $\Delta_3^0 := V(-4A^3-27B^2)$.

We study the map $\iota_{3*} : \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_3^0) \to \mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3-x_2))$. We have the equivariant map

$$D_{\mathbb{A}^4}(x_3-x_2) \to \mathbb{A}^2$$
$$(x_2,y_2,x_3,y_3) \mapsto (A(x_2,y_2,x_3,y_3), B(x_2,y_2,x_3,y_3)).$$

This restricts to a map $\Delta_3^0 \to \Delta_1$, as Δ_3^0 is, essentially by definition, the inverse image of Δ_1 under this map. Define $\widehat{\Delta}_3^0 := \mathbb{A}_1 \times_{\Delta_1} \Delta_3^0$, so

$$\begin{tikzcd} \widehat{\Delta}_3^0 \arrow[r, "f_3"] \arrow[d] & \Delta_3^0 \arrow[d] \\ \mathbb{A}^1 \arrow[r, "f_1"] & \Delta_1 \end{tikzcd}$$

is Cartesian. Everything involved is an affine scheme, so a tensor product computation gives

$$\widehat{\Delta}_3^0 = \operatorname{Spec} k[x_2, y_2, x_3, y_3, c]_{x_3 - x_2}/(A + 3c^2, B - 2c^3).$$

Now, we parameterize $\widehat{\Delta}_3^0$ by

$$(c, d_2, d_3) \mapsto (d_2^2 - 2c, d_2^3 - 3cd_2, d_3^2 - 2c, d_3^3 - 3cd_3, c)$$

Note the similarities with the parameterization $g_2 : \mathbb{A}^2 \to \widehat{\Delta}_2$. This this tuple satisfies the equations $A + 3c^2 = B - 2c^3 = 0$, and so they determine a morphism to $\widehat{\Delta}_3^0$ so long as $x_2 \neq x_3$, i.e. $d_2^2 \neq d_3^2$. And so the above gives a morphism $g_3 : \operatorname{Spec} k[c, d_2, d_3]_{d_3^2 - d_2^2} \to \widehat{\Delta}_3^0$. To make this map equivariant, we act with weights $(-2, -1, -1)$. Note g_3 is finite, hence proper, as c, d_2, d_3 are integral over $k[x_2, y_2, x_3, y_3, c]_{x_3 - x_2}/(A + 3c^2, B - 2c^3)$.

Over the locus $W_3 := D_{\widehat{\Delta}_3^0}((x_2 - c)(x_3 - c))$, we have that g_3 is an isomorphism, as

$$(x_2, y_2, x_3, y_3, c) \mapsto \left(c, \frac{y_2}{x_2 - c}, \frac{y_3}{x_3 - c}\right)$$

gives an inverse on this open subset. The reduced complement of W_3 is given by $C_3 := C_3^2 \cup C_3^3 \subseteq \widehat{\Delta}_3^0$, where

$$C_3^\ell := V(x_\ell - c, y_\ell) \xhookrightarrow{j_3^\ell} \widehat{\Delta}_3^0.$$

Calculations on the ideals give that $C_3^2 = \operatorname{Spec}(R)$ where

$$R = \frac{k[x_2, y_2, x_3, y_3, c]_{x_3 - c}}{(y_2, x_2 - c, y_3^2 - x_3^3 + 3c^2 x_3 - 2c^3)} \cong \frac{k[x_3, y_3, c]_{x_3 - c}}{(y_3^2 - x_3^3 + 3c^2 x_3 - 2c^3)}.$$

Note that this latter expression gives exactly the open $W \subseteq \widehat{\Delta}_2$, up to changing variable names, and recall $g_2^{-1}(W_2) \to W_2$ was an isomorphism. Using this, we have an isomorphism

$$\operatorname{Spec} k[c, d]_{d^2 - 3c} \xrightarrow{\sim} C_3^2$$

$$(c, d) \mapsto (c, 0, d^2 - 2c, d^3 - 3cd, c)$$

from which we can compute the Chow groups of C_3^2. Furthermore, we can compute,

$$g_3^{-1}(C_3^2) = \operatorname{Spec} k[c, d_2, d_3]_{d_3^2 - d_2^2}/(d_2^2 - 3c) \cong \operatorname{Spec} k[d_2, d_3]_{d_3^2 - d_2^2}.$$

Analogously, we get an isomorphism

$$\operatorname{Spec} k[c, d]_{d^2 - 3c} \xrightarrow{\sim} C_3^3$$

$$(c, d) \mapsto (d^2 - 2c, d^3 - 3cd, c, 0, c)$$

and

$$g_3^{-1}(C_3^3) = \operatorname{Spec} k[c, d_2, d_3]_{d_3^2-d_2^2}/(d_3^2 - 3c) \cong \operatorname{Spec} k[d_2, d_3]_{d_3^2-d_2^2}.$$

Lemma 4.13. *$\iota_{3*} : \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_3^0) \to \mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3-x_2))$ is equal to zero and*

$$\mathrm{CH}^*(\Delta_3^0) = \mathbb{Z} \oplus \bigoplus_{i=1}^{\infty} \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$$

generated by $[\Delta_3^0]$ in degree 0 and $f_{3*}j^2_{3*}(t^{i-1}), f_{3*}j^3_{3*}(t^{i-1})$ in degrees $i > 0$.

Proof. First, note that C_3^2 and C_3^3 are disjoint: they are cut out of of $\widehat{\Delta}_3^0$ by (x_2-c, y_2) and (x_3-c, y_3) respectively, and so any point in there intersection would be a zero of x_3-x_2, which cannot happen. Furthermore, this implies $g_3^{-1}(C_3^2)$ and $g_3^{-1}(C_3^3)$ are also disjoint.

Using Lemma 2.1, we have

$$\begin{aligned}\mathrm{CH}^*_{\mathbb{G}_m}(\widehat{\Delta}_3^0) &= \mathrm{CH}^*_{\mathbb{G}_m}(W_3) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g_3^{-1}C_3)} \mathrm{CH}^*_{\mathbb{G}_m}(C_3)\\ &= CH_{\mathbb{G}_m}(W_3) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g_3^{-1}C_3)} (\mathrm{CH}^*_{\mathbb{G}_m}(C_3^2) \oplus \mathrm{CH}^*_{\mathbb{G}_m}(C_3^3)).\end{aligned}$$

And we also know the pushforward $f_{3*} : \mathrm{CH}^*_{\mathbb{G}_m}(\widehat{\Delta}_3^0) \to \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_3^0)$ is an isomorphism, since f_3 is a universal separable homeomorphism. Thus, to show ι_{3*} is zero, it suffices to show that pushforwards of $D(d^2 - e^2) \to D_{\mathbb{A}^4}(x_3 - x_2)$ and $C'_\ell \to D_{\mathbb{A}^4}(x_3 - x_2)$ are zero.

The map $D(d^2 - e^2) \to D_{\mathbb{A}^4}(x_3 - x_2)$ is the composition

$$D(d^2 - e^2) \xrightarrow{g'} \widehat{\Delta}_3^0 \xrightarrow{f''} \Delta_3^0 \xhookrightarrow{\iota_3} D_{\mathbb{A}^4}(x_3 - x_2).$$

Using the definitions of these maps, we can compute it explicitly as $(c, d_2, d_3) \mapsto (d_2^2 - 2c, d_2^3 - 3cd_2, d_3^2 - 2c, d_3^3 - 3cd_3)$. This extends to a morphism $p : \mathbb{A}^3 \to \mathbb{A}^4$. This extension is finite, hence proper, because d_2 is a zero of the monic polynomial $t^3 - 3t(d_2^2 - 2c) + 2(d_2^3 - 3cd_2) = 0$ and d_3 is a zero of the analogous polynomial. By Lemma 2.10, the pushforward on Chow is given by multiplication by $p_*(1)$, under the identification of the pullback p^*. Because $p|_{D(d^2-e^2)} : D(d^2 - e^2) \to \Delta_3^0$ is birational, the pushforward of 1 is $[\Delta_3]$, where Δ_3 is defined to be the closure of Δ_3^0 in $\mathbb{A}^4$. Δ_3^0 is defined by $-4A^3 - 27B^2 = 0$, but this equation has denominators, so we cannot say the same equation defines Δ_3. Instead, we have $\Delta_3 = V((x_3-x_2)^3(-4A^3-27B^2))$, as this is how many factors of x_3-x_2 are needed to clear the denominators. The weight of $(x_3 - x_2)^3(-4A^3 - 27B^2))$ is -18, so the image of p_* is $(18t)$. Using the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3) \arrow[r, "p_*"] \arrow[d] & \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^4) = \mathbb{Z}[t] \arrow[d] \\
\mathrm{CH}^*_{\mathbb{G}_m}(D(d^2-e^2)) \arrow[r] & \mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3-x_2)) = \frac{\mathbb{Z}[t]}{(2t)}
\end{tikzcd}$$

we have that the bottom map is 0, because multiples of $18t$ go to 0 under the right map and the left map is surjective.

The map $C_3^3 \to D_{\mathbb{A}^4}(x_3 - x_2)$ is the composition $C_3^3 \hookrightarrow \widehat{\Delta}_3^0 \xrightarrow{f''} \Delta_3^0 \overset{\iota_3}{\hookrightarrow} D_{\mathbb{A}^4}(x_3 - x_2)$. Using the definitions of these maps, we can compute it explicitly as $(x_2, y_2, c) \mapsto (x_2, y_2, c, 0)$, using our identification $C_3^2 \cong \mathrm{Spec}(k[x_2, y_2, c]_{x_2-c}/(y_2^2 - x_2^3 + 3c^2x_2 - 2c^3))$. Note that this is just the space W. Composing $g^{-1}(W) \to W$ with the above $C_3^3 \to D_{\mathbb{A}^4}(x_3 - x_2)$, we get

$$(c, d) \mapsto (d^2 - 2c, d^3 - 3cd, c) \mapsto (d^2 - 2c, d^3 - 3cd, c, 0).$$

This extends to a finite map $q : \mathbb{A}^2 \to \mathbb{A}^4$. The image of this map is $V(y_3, y_2^2 - x_2^3 + 3x_3^2x_2 - 2x_3^3)$, which has fundamental class $-18t^2$. A similar argument to the last paragraph using this extension q, Lemma 2.10, and a commutative diagram shows that the pushforward of $C_3^3 \to D_{\mathbb{A}^4}(x_3 - x_2)$ is 0, since $-18t^2$ gets killed in $\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3 - x_2)) = \mathbb{Z}[t]/(2t)$. Analogously, the pushforward of $C_3^2 \to D_{\mathbb{A}^4}(x_3 - x_2)$ is 0.

Now, we compute $\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_3^0) \cong CH_{\mathbb{G}_m}(D(d^2-e^2)) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g'^{-1}C')} \mathrm{CH}^*_{\mathbb{G}_m}(C')$. Note $D(e^2 - f^2) \subseteq D(e - f)$. Because $e - f$ has weight -1, arguing as in Lemma 4.10 gives $\mathrm{CH}^*(D(e - f)) = \mathbb{Z}$, and then localization gives a surjection $\mathbb{Z} = \mathrm{CH}^*_{\mathbb{G}_m}(D(e-f)) \twoheadrightarrow \mathrm{CH}^*_{\mathbb{G}_m}(D(d^2-e^2))$, and so $\mathrm{CH}^*_{\mathbb{G}_m}(D(d^2-e^2)) = \mathbb{Z}$. Furthermore, $C'_\ell \cong W \cong g^{-1}(W) = D_{\mathbb{A}^2}(d^2 - 3c)$, and $d^2 - 3c$ has weight -2, so arguing as in Lemma 4.10, we have $\mathrm{CH}^*_{\mathbb{G}_m}(C'_\ell) = \mathbb{Z}[t]/(2t)$.

Now, $g'^{-1}(C')$ is a closed subscheme of $D(d^2 - e^2)$, so the pushforward on $\mathrm{CH}^*_{\mathbb{G}_m}$ must land in degrees ≥ 1. But, as we just saw, $\mathrm{CH}^*_{\mathbb{G}_m}(D(d^2-e^2)) = \mathbb{Z}$, and so the pushforward is the zero map. To compute the pushforward of the other map $g'^{-1}(C') \to C'$, recall $C' = C_3^2 \cup C_3^3$ and $g'^{-1}(C') = g'^{-1}(C_3^2) \cup g'^{-1}(C_3^3)$. It suffices to just compute the pushforward $g'^{-1}(C'_\ell) \to C'_\ell$. Note that

$$\mathrm{Spec}(k[d,e]_{d^2-e^2}) \cong g'^{-1}(C_3^3) \xrightarrow{g'|_{g'^{-1}(C_3^3)}} C_3^3 \cong \mathrm{Spec}(k[x_2, y_2, c]_{x_2-c}/(y_2^2 - x_2^3 + 3c^2x_2 - 2c^3))$$
$$(d, e) \mapsto (d^2 - \frac{2}{3}e^2, d^3 - e^2 d, \frac{e^2}{3})$$

factors as

$$g'^{-1}(C_3^3) \xrightarrow{h} \mathrm{Spec}(k[c, d]_{e^2-3c}) \to C_3^3$$
$$(d, e) \mapsto (\frac{e^2}{3}, d)$$
$$(c, d) \mapsto (d^2 - 2c, d^3 - 3cd, c).$$

Moreover, this second map is precisely the map $g|_{g^{-1}(W)}$, which we know to be an isomorphism. Similar to our computation that $\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^3}(d^2 - e^2)) =$

$\mathbb{Z}$ in the previous paragraph, we have $\mathrm{CH}^*_{\mathbb{G}_m}(g'^{-1}(C_3^3)) = \mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^2}(d^2-e^2)) = \mathbb{Z}$, and we have $\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^2}(e^2-3c)) = \mathbb{Z}[t]/(2t)$, using Lemma 4.10. We can compute $h_*(1)=2$, since h is a degree 2 map between varieties of the same dimension. Because $\mathrm{CH}^*_{\mathbb{G}_m}(g'^{-1}(C_3^3)) = \mathbb{Z}$, this determines h_*.

Now, we have

$$\begin{aligned}
\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_3^0) &\cong \mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^3}(d^2-e^2)) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g'^{-1}C')} \mathrm{CH}^*_{\mathbb{G}_m}(C') \\
&= \mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^3}(d^2-e^2)) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g'^{-1}C_3^2)\oplus \mathrm{CH}^*_{\mathbb{G}_m}(g'^{-1}C_3^3)} (\mathrm{CH}^*_{\mathbb{G}_m}(C_3^2)\oplus \mathrm{CH}^*_{\mathbb{G}_m}(C_3^3)) \\
&= \mathbb{Z} \coprod_{\mathbb{Z}\oplus\mathbb{Z}} \mathbb{Z}[t]/(2t) \oplus \mathbb{Z}[t]/(2t) \\
&\cong \mathbb{Z} \oplus \bigoplus_{i=1}^{\infty} (\mathbb{Z}/(2\mathbb{Z}))^2
\end{aligned}$$

□

Theorem 4.14. *The Chow ring of $\mathcal{M}^0_{1,3}$ is given by*

$$\mathrm{CH}^*(\mathcal{M}^0_{1,3}) = \mathbb{Z}[t]/(2t),$$

and the first higher Chow group of $\mathcal{M}^0_{1,3}$ is given by

$$\mathrm{CH}^*(\mathcal{M}^0_{1,3},1)_{ind} = \mathbb{Z} \oplus \bigoplus_{i>2} (\frac{\mathbb{Z}}{2\mathbb{Z}})^2.$$

Proof. We saw $\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3-x_2)) = \frac{\mathbb{Z}[t]}{(3t)}$ and $\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3-x_2),1)_{\mathrm{ind}} = 0$. So the localization exact sequence for $\mathcal{M}^0_{1,3} \subseteq [D_{\mathbb{A}^4}(x_3-x_2)/\mathbb{G}_m]$ is

$$0 \to \mathrm{CH}^*(\mathcal{M}^0_{1,3},1)_{\mathrm{ind}} \to \mathrm{CH}^{*-1}(\Delta_3^0) \xrightarrow{\iota_{3*}} \frac{\mathbb{Z}[t]}{(3t)} \to \mathrm{CH}^*(\mathcal{M}^0_{1,3}) \to 0.$$

Lemma 4.13 says that $\iota_{3*}=0$, and

$$\mathbb{Z} \oplus \bigoplus_{i=1}^{\infty} \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$$

from which the Theorem follows. □

Next, we establish a relationship between the 2-torsion in $\mathrm{CH}^*(\mathcal{M}^0_{1,3},1)_{\mathrm{ind}}$ and $\mathrm{CH}^*(\mathcal{M}_{1,2},1)_{\mathrm{ind}}$, which will allow us to lift calculations of ∂_1 on $\mathrm{CH}^*(\mathcal{M}_{1,2},1)_{\mathrm{ind}}$ to $\mathcal{M}^0_{1,3}$. We have two morphisms $\pi_2,\pi_3 : \mathcal{M}^0_{1,3} \to \mathcal{M}_{1,2}$, where π_ℓ forgets the i-th marked point.

Proposition 4.15.

$$\pi_2^*\oplus\pi_3^* : \mathrm{CH}^i(\mathcal{M}_{1,2},1)_{ind} \oplus \mathrm{CH}^i(\mathcal{M}_{1,2},1)_{ind} \to \mathrm{CH}^i(\mathcal{M}^0_{1,3},1)_{ind}$$

is an isomorphism for $i \geq 2$.

Proof. Define

$$\widetilde{\pi}_2 : D_{\mathbb{A}^4}(x_3 - x_2) \to \mathbb{A}^3$$

$$(x_2, y_2, x_3, y_3) \mapsto (x_3, y_3, A(x_2, y_2, x_3, y_3))$$

$$\widetilde{\pi}_3 : D_{\mathbb{A}^4}(x_3 - x_2) \to \mathbb{A}^3$$

$$(x_2, y_2, x_3, y_3) \mapsto (x_2, y_2, A(x_2, y_2, x_3, y_3)).$$

Then π_ℓ is obtained by restricting $\widetilde{\pi}_\ell$ to the complement of the singular locus, and quotienting out by $\mathbb{G}_m$. So we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}^i(\mathcal{M}^0_{1,3},1)_{\mathrm{ind}} \arrow[r] & \mathrm{CH}^{i-1}_{\mathbb{G}_m}(\Delta^0_3) \arrow[r, "\iota_{3*}"] & \mathrm{CH}^i_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3 - x_2)) \\
0 \arrow[r] & \mathrm{CH}^i(\mathcal{M}_{1,2},1)_{\mathrm{ind}} \arrow[r] \arrow[u, "\pi_\ell^*"] & \mathrm{CH}^{i-1}_{\mathbb{G}_m}(\Delta_2) \arrow[r, "\iota_{2*}"] \arrow[u, "\widetilde{\pi}_\ell^*"] & \mathrm{CH}^i_{\mathbb{G}_m}(\mathbb{A}^3) \arrow[u, "\widetilde{\pi}_\ell^*"]
\end{tikzcd}$$

Now, $\mathrm{CH}^i(\mathcal{M}_{1,2},1)_{\mathrm{ind}} = \ker(\iota_{2*})$ is generated by $f_{2*}g_*(t^i) - f_{2*}j_*(t^{i-1})$. The diagram

$$\begin{tikzcd}
C_3^\ell \arrow[r, "f_3 \circ j_3^\ell"] \arrow[d, "\rho"] & Z_3^0 \arrow[d, "\widetilde{\pi}_\ell"] \\
C_2 \arrow[r, "f_2 \circ j"] & \Delta_2
\end{tikzcd}$$

is Cartesian, where $\rho(c,d) = c$ under the isomorphisms $C_2 \cong \operatorname{Spec} k[c]$, $C_3^\ell \cong \operatorname{Spec} k[c,d]_{d^2-3c}$. Thus, push-pull gives

$$\widetilde{\pi}_\ell^*(f_{2*}j_*(t^{i-1})) = f_{3*}j^\ell_{3*}(\rho^*(t^{i-1})) = f_{3*}j^\ell_{3*}(t^{i-1}).$$

Additionally, the diagram

$$\begin{tikzcd}
D_{\mathbb{A}^3}(d_2^2 - d_3^2) \arrow[r, "f_3 \circ g_3"] \arrow[d, "\rho'_\ell"] & Z_3^0 \arrow[d, "\widetilde{\pi}_\ell"] \\
\mathbb{A}^2 \arrow[r, "f_2 \circ g"] & \Delta_2
\end{tikzcd}$$

is Cartesian, where $\rho'_\ell(c, d_2, d_3) = (c, d_\ell)$. Thus, push-pull tells us

$$\widetilde{\pi}_\ell^*(f_{2*}g_*(t^i)) = f_{3*}j^\ell_{3*}(\rho'^*(t^i)) = 0,$$

using that

Putting these together, we have $\widetilde{\pi}_\ell^*(f_{2*}g_*(t^i) - f_{2*}j_*(t^{i-1})) = f_{3*}j^\ell_{3*}(t^{i-1})$, which are the generators of $\mathrm{CH}^i(\mathcal{M}^0_{1,3},1)_{\mathrm{ind}}$. □

5. $\mathcal{M}_{1,4}^0$

Recall our presentation of $\mathcal{M}_{1,4}^0$ as $U_4 = \mathbb{P}(V) \setminus \Delta_4$, where

$$V_4 = \{A_0x^2y^2 - A_0x^2yz - A_0xwy^2 + A_4xwyz + A_5xwz^2 + A_7w^2yz + A_8w^2z^2\},$$

and $\Delta_4 = V(A_0F)$, where

$$\begin{aligned}
F := {} & 27A_0^5A_8^4 + 36A_0^4A_4A_5A_8^3 + 36A_0^4A_4A_7A_8^3 - 36A_0^4A_4A_8^4 + 16A_0^4A_5^3A_8^2 - \\
& 24A_0^4A_5^2A_7A_8^2 + 24A_0^4A_5^2A_8^3 - 24A_0^4A_5A_7^2A_8^2 + 12A_0^4A_5A_7A_8^3 - 24A_0^4A_5A_8^4 + \\
& 16A_0^4A_7^3A_8^2 + 24A_0^4A_7^2A_8^3 - 24A_0^4A_7A_8^4 - 16A_0^4A_8^5 + A_0^3A_4^3A_8^3 + \\
& 8A_0^3A_4^2A_5^2A_8^2 + 46A_0^3A_4^2A_5A_7A_8^2 - 46A_0^3A_4^2A_5A_8^3 + 8A_0^3A_4^2A_7^2A_8^2 - 46A_0^3A_4^2A_7A_8^3 + \\
& 8A_0^3A_4^2A_8^4 + 16A_0^3A_4A_5^3A_7A_8 - 16A_0^3A_4A_5^3A_8^2 - 64A_0^3A_4A_5^2A_7^2A_8 + 76A_0^3A_4A_5^2A_7A_8^2 - \\
& 64A_0^3A_4A_5^2A_8^3 + 16A_0^3A_4A_5A_7^3A_8 + 76A_0^3A_4A_5A_7^2A_8^2 - 76A_0^3A_4A_5A_7A_8^3 - 16A_0^3A_4A_5A_8^4 - \\
& 16A_0^3A_4A_7^3A_8^2 - 64A_0^3A_4A_7^2A_8^3 - 16A_0^3A_4A_7A_8^4 - 16A_0^3A_5^4A_7^2 - 16A_0^3A_5^4A_8^2 + \\
& 32A_0^3A_5^3A_7^3 - 16A_0^3A_5^3A_7^2A_8 + 16A_0^3A_5^3A_7A_8^2 - 32A_0^3A_5^3A_8^3 - 16A_0^3A_5^2A_7^4 - \\
& 16A_0^3A_5^2A_7^3A_8 + 66A_0^3A_5^2A_7^2A_8^2 - 16A_0^3A_5^2A_7A_8^3 - 16A_0^3A_5^2A_8^4 + 16A_0^3A_5A_7^3A_8^2 - \\
& 16A_0^3A_5A_7^2A_8^3 - 16A_0^3A_7^4A_8^2 - 32A_0^3A_7^3A_8^3 - 16A_0^3A_7^2A_8^4 + A_0^2A_4^4A_5A_8^2 + \\
& A_0^2A_4^4A_7A_8^2 - A_0^2A_4^4A_8^3 + 8A_0^2A_4^3A_5^2A_7A_8 - 8A_0^2A_4^3A_5^2A_8^2 + 8A_0^2A_4^3A_5A_7^2A_8 - \\
& 57A_0^2A_4^3A_5A_7A_8^2 + 8A_0^2A_4^3A_5A_8^3 - 8A_0^2A_4^3A_7^2A_8^2 + 8A_0^2A_4^3A_7A_8^3 - 8A_0^2A_4^2A_5^3A_7^2 - \\
& 16A_0^2A_4^2A_5^3A_7A_8 - 8A_0^2A_4^2A_5^3A_8^2 - 8A_0^2A_4^2A_5^2A_7^3 + 108A_0^2A_4^2A_5^2A_7^2A_8 - 108A_0^2A_4^2A_5^2A_7A_8^2 + \\
& 8A_0^2A_4^2A_5^2A_8^3 - 16A_0^2A_4^2A_5A_7^3A_8 - 108A_0^2A_4^2A_5A_7^2A_8^2 - 16A_0^2A_4^2A_5A_7A_8^3 - 8A_0^2A_4^2A_7^3A_8^2 + \\
& 8A_0^2A_4^2A_7^2A_8^3 + 16A_0^2A_4A_5^4A_7^2 - 16A_0^2A_4A_5^4A_7A_8 - 64A_0^2A_4A_5^3A_7^3 + 76A_0^2A_4A_5^3A_7^2A_8 - \\
& 64A_0^2A_4A_5^3A_7A_8^2 + 16A_0^2A_4A_5^2A_7^4 + 76A_0^2A_4A_5^2A_7^3A_8 - 76A_0^2A_4A_5^2A_7^2A_8^2 - 16A_0^2A_4A_5^2A_7A_8^3 - \\
& 16A_0^2A_4A_5A_7^4A_8 - 64A_0^2A_4A_5A_7^3A_8^2 - 16A_0^2A_4A_5A_7^2A_8^3 + 16A_0^2A_5^5A_7^2 - 24A_0^2A_5^4A_7^3 + \\
& 24A_0^2A_5^4A_7^2A_8 - 24A_0^2A_5^3A_7^4 + 12A_0^2A_5^3A_7^3A_8 - 24A_0^2A_5^3A_7^2A_8^2 + 16A_0^2A_5^2A_7^5 + \\
& 24A_0^2A_5^2A_7^4A_8 - 24A_0^2A_5^2A_7^3A_8^2 - 16A_0^2A_5^2A_7^2A_8^3 + A_0A_4^5A_5A_7A_8 - A_0A_4^5A_5A_8^2 - \\
& A_0A_4^5A_7A_8^2 - A_0A_4^4A_5^2A_7^2 - 8A_0A_4^4A_5^2A_7A_8 - A_0A_4^4A_5^2A_8^2 - 8A_0A_4^4A_5A_7^2A_8 + \\
& 8A_0A_4^4A_5A_7A_8^2 - A_0A_4^4A_7^2A_8^2 + 8A_0A_4^3A_5^3A_7^2 - 8A_0A_4^3A_5^3A_7A_8 + 8A_0A_4^3A_5^2A_7^3 - \\
& 57A_0A_4^3A_5^2A_7^2A_8 + 8A_0A_4^3A_5^2A_7A_8^2 - 8A_0A_4^3A_5A_7^3A_8 + 8A_0A_4^3A_5A_7^2A_8^2 + 8A_0A_4^2A_5^4A_7^2 + \\
& 46A_0A_4^2A_5^3A_7^3 - 46A_0A_4^2A_5^3A_7^2A_8 + 8A_0A_4^2A_5^2A_7^4 - 46A_0A_4^2A_5^2A_7^3A_8 + 8A_0A_4^2A_5^2A_7^2A_8^2 + \\
& 36A_0A_4A_5^4A_7^3 + 36A_0A_4A_5^3A_7^4 - 36A_0A_4A_5^3A_7^3A_8 + 27A_0A_5^4A_7^4 - A_4^6A_5A_7A_8 + \\
& A_4^5A_5^2A_7^2 - A_4^5A_5^2A_7A_8 - A_4^5A_5A_7^2A_8 + A_4^4A_5^3A_7^2 + A_4^4A_5^2A_7^3 - \\
& A_4^4A_5^2A_7^2A_8 + A_4^3A_5^3A_7^3
\end{aligned}$$

Note that $V(A_0) \subseteq \Delta_4$. It is quick to compute how the (higher) Chow groups of $\mathbb{P}(V_4)$ change when removing $V(a_0)$, as one then obtains affine space, but removing the rest of Δ_4 is more involved. Define $\Delta_4^0 := \Delta_4 \backslash V(a_0)$. Homogenizing with respect to A_0, we have Δ_4^0 is a closed subvariety of $\mathbb{A}^4 = \operatorname{Spec} k[a_4, a_5, a_7, a_8]$, cut out by

$$\begin{aligned}
f := {} & 27a_8^4 + 36a_4a_5a_8^3 + 36a_4a_7a_8^3 - 36a_4a_8^4 + 16a_5^3a_8^2 - \\
& 24a_5^2a_7a_8^2 + 24a_5^2a_8^3 - 24a_5a_7^2a_8^2 + 12a_5a_7a_8^3 - 24a_5a_8^4 + \\
& 16a_7^3a_8^2 + 24a_7^2a_8^3 - 24a_7a_8^4 - 16a_8^5 + a_4^3a_8^3 + \\
& 8a_4^2a_5^2a_8^2 + 46a_4^2a_5a_7a_8^2 - 46a_4^2a_5a_8^3 + 8a_4^2a_7^2a_8^2 - 46a_4^2a_7a_8^3 + \\
& 8a_4^2a_8^4 + 16a_4a_5^3a_7a_8 - 16a_4a_5^3a_8^2 - 64a_4a_5^2a_7^2a_8 + 76a_4a_5^2a_7a_8^2 - \\
& 64a_4a_5^2a_8^3 + 16a_4a_5a_7^3a_8 + 76a_4a_5a_7^2a_8^2 - 76a_4a_5a_7a_8^3 - 16a_4a_5a_8^4 - \\
& 16a_4a_7^3a_8^2 - 64a_4a_7^2a_8^3 - 16a_4a_7a_8^4 - 16a_5^4a_7^2 - 16a_5^4a_8^2 + \\
& 32a_5^3a_7^3 - 16a_5^3a_7^2a_8 + 16a_5^3a_7a_8^2 - 32a_5^3a_8^3 - 16a_5^2a_7^4 - \\
& 16a_5^2a_7^3a_8 + 66a_5^2a_7^2a_8^2 - 16a_5^2a_7a_8^3 - 16a_5^2a_8^4 + 16a_5a_7^3a_8^2 - \\
& 16a_5a_7^2a_8^3 - 16a_7^4a_8^2 - 32a_7^3a_8^3 - 16a_7^2a_8^4 + a_4^4a_5a_8^2 + \\
& a_4^4a_7a_8^2 - a_4^4a_8^3 + 8a_4^3a_5^2a_7a_8 - 8a_4^3a_5^2a_8^2 + 8a_4^3a_5a_7^2a_8 - \\
& 57a_4^3a_5a_7a_8^2 + 8a_4^3a_5a_8^3 - 8a_4^3a_7^2a_8^2 + 8a_4^3a_7a_8^3 - 8a_4^2a_5^3a_7^2 - \\
& 16a_4^2a_5^3a_7a_8 - 8a_4^2a_5^3a_8^2 - 8a_4^2a_5^2a_7^3 + 108a_4^2a_5^2a_7^2a_8 - 108a_4^2a_5^2a_7a_8^2 + \\
& 8a_4^2a_5^2a_8^3 - 16a_4^2a_5a_7^3a_8 - 108a_4^2a_5a_7^2a_8^2 - 16a_4^2a_5a_7a_8^3 - 8a_4^2a_7^3a_8^2 + \\
& 8a_4^2a_7^2a_8^3 + 16a_4a_5^4a_7^2 - 16a_4a_5^4a_7a_8 - 64a_4a_5^3a_7^3 + 76a_4a_5^3a_7^2a_8 - \\
& 64a_4a_5^3a_7a_8^2 + 16a_4a_5^2a_7^4 + 76a_4a_5^2a_7^3a_8 - 76a_4a_5^2a_7^2a_8^2 - 16a_4a_5^2a_7a_8^3 - \\
& 16a_4a_5a_7^4a_8 - 64a_4a_5a_7^3a_8^2 - 16a_4a_5a_7^2a_8^3 + 16a_5^5a_7^2 - 24a_5^4a_7^3 + \\
& 24a_5^4a_7^2a_8 - 24a_5^3a_7^4 + 12a_5^3a_7^3a_8 - 24a_5^3a_7^2a_8^2 + 16a_5^2a_7^5 + \\
& 24a_5^2a_7^4a_8 - 24a_5^2a_7^3a_8^2 - 16a_5^2a_7^2a_8^3 + a_4^5a_5a_7a_8 - a_4^5a_5a_8^2 - \\
& a_4^5a_7a_8^2 - a_4^4a_5^2a_7^2 - 8a_4^4a_5^2a_7a_8 - a_4^4a_5^2a_8^2 - 8a_4^4a_5a_7^2a_8 + \\
& 8a_4^4a_5a_7a_8^2 - a_4^4a_7^2a_8^2 + 8a_4^3a_5^3a_7^2 - 8a_4^3a_5^3a_7a_8 + 8a_4^3a_5^2a_7^3 - \\
& 57a_4^3a_5^2a_7^2a_8 + 8a_4^3a_5^2a_7a_8^2 - 8a_4^3a_5a_7^3a_8 + 8a_4^3a_5a_7^2a_8^2 + 8a_4^2a_5^4a_7^2 + \\
& 46a_4^2a_5^3a_7^3 - 46a_4^2a_5^3a_7^2a_8 + 8a_4^2a_5^2a_7^4 - 46a_4^2a_5^2a_7^3a_8 + 8a_4^2a_5^2a_7^2a_8^2 + \\
& 36a_4a_5^4a_7^3 + 36a_4a_5^3a_7^4 - 36a_4a_5^3a_7^3a_8 + 27a_5^4a_7^4 - a_4^6a_5a_7a_8 + \\
& a_4^5a_5^2a_7^2 - a_4^5a_5^2a_7a_8 - a_4^5a_5a_7^2a_8 + a_4^4a_5^3a_7^2 + a_4^4a_5^2a_7^3 - \\
& a_4^4a_5^2a_7^2a_8 + a_4^3a_5^3a_7^3
\end{aligned}$$

Lemma 5.1. $\mathrm{CH}^i(\Delta_4^0) = \begin{cases} \mathbb{Z} & i=0 \\ \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^6 & i=1 \\ 0 & i>1 \end{cases}$

and Δ_4^0 has the integral CKP and integral CKgP.

Proof. Define

$$u : \mathbb{A}^3 \to \Delta_4^0$$

$(x,y,t) \mapsto (-4xy+2x+2y+2t, 2xy^2-y^2-2yt, 2x^2y-x^2-2xt, -x^2y^2+2xyt)$, and note that switching x and y preserves a_4, a_8 and flips a_5 and a_7.

We claim this map is finite. One can check that x satisfies

$$2x^3 + (-3a_4 - 6a_5)x^2 + (a_4^2 + 4a_5 - 2a_7 - 4a_8)x + (a_4a_7 + 2a_8) = 0$$

and then y must also be integral over Δ_4^0 by symmetry. Additionally, because $a_4 = -4xy + 2x + 2y + 2t$, t must be integral as well, and so u is integral, hence finite.

Additionally, we have $x = \frac{b_1}{b_2}$, where

$$\begin{aligned} b_1 := {} & a_4^4a_5a_7 + a_4^3a_5^2a_7 - a_4^3a_5a_7 + 2a_4^3a_5a_8 + a_4^3a_7a_8 + 4a_4^2a_5^2a_7 + \\ & 2a_4^2a_5^2a_8 + 6a_4^2a_5a_7^2 - 4a_4^2a_5a_7a_8 - 2a_4^2a_5a_8 - 2a_4^2a_7a_8 + 2a_4^2a_8^2 \\ & + 4a_4a_5^3a_7 + 25a_4a_5^2a_7^2 - 4a_4a_5^2a_7a_8 - 4a_4a_5^2a_7 + 8a_4a_5^2a_8 - 4a_4a_5a_7^2 - \\ & 2a_4a_5a_7a_8 - 8a_4a_5a_8^2 + 4a_4a_7^2a_8 - 4a_4a_7a_8^2 - 3a_4a_8^2 + 18a_5^3a_7^2 + 8a_5^3a_8 - \\ & 16a_5^2a_7^2 - 4a_5^2a_7a_8 - 8a_5^2a_8^2 - 8a_5^2a_8 + 8a_5a_7^3 + 16a_5a_7^2a_8 + 8a_5a_7a_8 \\ & + 10a_5a_8^2 - 8a_7^2a_8 - 8a_7a_8^2 - 8a_8^3, \end{aligned}$$

and

$$\begin{aligned} b_2 := {} & -a_4^5a_5 - a_4^4a_5^2 + a_4^4a_5 - a_4^4a_8 - 8a_4^3a_5^2 - 7a_4^3a_5a_7 + 8a_4^3a_5a_8 + a_4^3a_8 - \\ & 8a_4^2a_5^3 - 32a_4^2a_5^2a_7 + 8a_4^2a_5^2a_8 + 8a_4^2a_5^2 + 6a_4^2a_5a_7 - 38a_4^2a_5a_8 - 6a_4^2a_7a_8 + \\ & 8a_4^2a_8^2 - 24a_4a_5^3a_7 - 16a_4a_5^3 + 44a_4a_5^2a_7 - 40a_4a_5^2a_8 - 12a_4a_5a_7^2 - \\ & 44a_4a_5a_7a_8 - 16a_4a_5a_8^2 + 28a_4a_5a_8 + 4a_4a_7a_8 - 28a_4a_8^2 - 16a_5^4 + \\ & 16a_5^3a_7 - 16a_5^3a_8 + 16a_5^3 + 2a_5^2a_7^2 - 16a_5^2a_7a_8 - 24a_5^2a_7 - 16a_5^2a_8^2 + \\ & 8a_5^2a_8 + 8a_5a_7^2 + 20a_5a_7a_8 - 8a_5a_8^2 - 8a_7^2a_8 - 24a_7a_8^2 - 16a_8^3 + 18a_8^2 \end{aligned}$$

and, by symmetry, we can write $y = \frac{c_1}{c_2}$, where $c_i := b_i(a_4, a_7, a_5, a_8)$. And because $a_4 = -4xy + 2x + 2y + 2t$, we have a rational expression for t as well. Thus, the map u has a birational inverse, and so u is an isomorphism onto Δ_4^0 away from the vanishing of the denominators. Let C be the reduced subscheme of $V(b_2c_2) \subseteq \Delta_4^0$. Then Lemma 2.1 applied to $u : \mathbb{A}^3 \to \Delta_4^0$ says

$$\mathrm{CH}^*(\Delta_4^0) = \mathrm{CH}^*(\mathbb{A}^3) \coprod_{\mathrm{CH}^*(p^{-1}(C))} \mathrm{CH}^*(C) = \mathbb{Z} \oplus \operatorname{coker}(\mathrm{CH}^*(p^{-1}(C)) \to \mathrm{CH}^*(C)),$$

using the fact that the pushforward $p^{-1}(C) \to \mathbb{A}^3$ is 0.

Pulling back b_2c_2 along u, we have

$$\widetilde{C} := p^{-1}(C) = \widetilde{C}_0 \cup \widetilde{C}_1 \cup \widetilde{C}_2 \cup \widetilde{C}_3 \cup \widetilde{C}_4 \cup \widetilde{C}_5 \cup \widetilde{C}_6$$

where

$$\begin{aligned} \widetilde{C}_0 &:= V(t^2 - x(x-1)y(y-1)) \\ \widetilde{C}_1 &:= V(x) \cong \mathbb{A}^2 \\ \widetilde{C}_2 &:= V(y) \cong \mathbb{A}^2 \\ \widetilde{C}_3 &:= V(x-1) \cong \mathbb{A}^2 \\ \widetilde{C}_4 &:= V(y-1) \cong \mathbb{A}^2 \\ \widetilde{C}_5 &:= V(2xy - x - y - 2t) \cong \mathbb{A}^2 \\ \widetilde{C}_6 &:= V(2xy - x - y - 2t + 1) \cong \mathbb{A}^2. \end{aligned}$$

Defining $C_i := p(\widetilde{C}_i)$, one can routinely verify that

$$\begin{aligned} C_1 &= V(a_7, a_8) \cong \mathbb{A}^2 \\ C_2 &= V(a_5, a_8) \cong \mathbb{A}^2 \\ C_3 &= V(a_5 + a_8, a_4 + a_7 - 1)) \cong \mathbb{A}^2 \\ C_4 &= V(a_7 + a_8, a_4 + a_5 - 1)) \cong \mathbb{A}^2 \\ C_5 &= V(a_5 - a_7, a_8 + a_7^2 + a_4a_7) \cong \mathbb{A}^2 \\ C_6 &= V(a_5a_7 - a_8, a_4 + a_5 + a_7 - 1) \cong \mathbb{A}^2, \end{aligned}$$

and that the degree of $\widetilde{C}_\ell \to C_\ell$ is 2 for $\ell \geq 1$.

We will show that $\widetilde{C}_0 \to C_0$ induces surjections on Chow groups. Assuming that for now, we complete the proof of the Lemma. We have the commutative diagram

$$\begin{tikzcd} \bigoplus_{\ell=0}^6 \mathrm{CH}_i(\widetilde{C}_\ell) \arrow[r, two heads] \arrow[d, "\bigoplus_\ell (p|_{\widetilde{C}_\ell})_*"] & \mathrm{CH}_i(\widetilde{C}) \arrow[d, "p_*"] \\ \bigoplus_{\ell=0}^6 \mathrm{CH}_i(C_\ell) \arrow[r, two heads] & \mathrm{CH}_i(C). \end{tikzcd}$$

Because the horizontal maps are isomorphisms for $i = 2$ and the degree of $\widetilde{C}_\ell \to C_\ell$ is 2 for $\ell \geq 1$, we get the cokernel of p_* is generated by the classes $[C_\ell]$, $\ell \geq 1$, with each class being 2-torsion. In degrees $i < 2$, note $\mathrm{CH}_i(\widetilde{C}_\ell) = \mathrm{CH}_i(C_\ell) = 0$ for $\ell \geq 1$, and $\mathrm{CH}_i(\widetilde{C}_0) \to \mathrm{CH}_i(C_0)$ is surjective, so the cokernel of p_* is 0 in all other degrees.

Now, we just need that $C_0 \to p(C_0)$ induces is surjective on Chow groups. The map $C_0 \to p(C_0)$ is birational: we can describe a rational inverse by noting $x = \frac{d_1}{d_2}$, where

$$\begin{aligned} d_1 &:= a_4^3 + 2a_4^2a_5 + 4a_4a_5 + 4a_4a_7 - 4a_4a_8 + 8a_5^2 - 4a_5a_7 - 8a_5a_8 + 12a_8 \\ d_2 &:= 2a_4^2 + 24a_4a_5 + 24a_5^2 - 16a_5 + 8a_7 + 16a_8, \end{aligned}$$

which then gives $y = \frac{e_1}{e_2}$ where $e_i := d_i(a_4, a_7, a_5, a_8)$ and we can get an expression for t using $a_4 = -4xy + 2x + 2y + 2t$. Let D be the reduced subscheme of $V_{p(C_0)}(d_2e_2)$. A COMPUTATION SHOWS that

$$\widetilde{D} := p^{-1}(D) = \widetilde{D}_1 \cup \widetilde{D}_2 \cup \widetilde{D}_3 \cup \widetilde{D}_4 \cup \widetilde{D}_5 \cup \widetilde{D}_6,$$

where

$$\begin{aligned} \widetilde{D}_1 &:= V_{\mathbb{A}^3}(x + y - 1, y^2 - y + t) \cong \mathbb{A}^1 \\ \widetilde{D}_2 &:= V_{\mathbb{A}^3}(x - y, y^2 - y - t) \cong \mathbb{A}^1 \\ \widetilde{D}_3 &:= V_{\mathbb{A}^3}(t, x) \cong \mathbb{A}^1 \\ \widetilde{D}_4 &:= V_{\mathbb{A}^3}(t, x + 1) \cong \mathbb{A}^1 \\ \widetilde{D}_5 &:= V_{\mathbb{A}^3}(t, y) \cong \mathbb{A}^1 \\ \widetilde{D}_6 &:= V_{\mathbb{A}^3}(t, y + 1) \cong \mathbb{A}^1. \end{aligned}$$

It is routine to verify that $\widetilde{D}_\ell \to D_\ell := \widetilde{D}_\ell$ had degree 1 for each ℓ. Because all of these maps have degree 1, we have

$$\mathbb{Z}^6 = \mathrm{CH}_1(\widetilde{D}) \to \mathrm{CH}_1(D) = \mathbb{Z}^6$$

is an isomorphism, hence surjective. Next, because $\widetilde{D}_i$ is a rational curve for each ℓ, we know D_ℓ is also rational. Then $\mathrm{CH}_0(\widetilde{D}_\ell), \mathrm{CH}_0(D_\ell) = 0$ for all ℓ, so $\mathrm{CH}_0(\widetilde{D}) \to \mathrm{CH}_0(D)$ is surjective, as both groups are 0.

Lemma 2.1 says that $\mathrm{CH}^*(\widetilde{C}_0) \oplus \mathrm{CH}^*(D) \to \mathrm{CH}^*(C_0)$ is surjective. But because $\mathrm{CH}^*(\widetilde{D}) \to \mathrm{CH}^*(D)$ is surjective, we actually have $\mathrm{CH}^*(\widetilde{C}_0) \to \mathrm{CH}^*(C_0)$ is surjective.

□

Theorem 5.2. $\mathrm{CH}^*(\mathcal{M}^0_{1,4}) = \mathbb{Z}$ *and*

$$\mathrm{CH}^i(\mathcal{M}^0_{1,4}, 1)_{ind} = \begin{cases} \mathbb{Z} & i = 1 \\ \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^6 & i = 2 \\ 0 & i > 2 \end{cases}$$

Proof. □

Remark 5.3. We know that the elements in $\mathrm{CH}^2(\mathcal{M}^0_{1,4}, 1)_{\mathrm{ind}}$ are not pulled back from $\mathrm{CH}^2(\mathcal{M}_{1,3}, 1)_{\mathrm{ind}}$ because...

6. Patching 1

6.1. $\mathcal{M}_{1,3}$.

Theorem 6.1.

$$\mathrm{CH}^*(\mathcal{M}_{1,3}) = \mathbb{Z}[\lambda]/(12\lambda, 6\lambda^2)$$

and

$$\mathrm{CH}^*(\mathcal{M}_{1,3}, 1)_{ind} = im(j_*) \oplus \bigoplus_{i=2}^{\infty} \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$$

and something about pullbacks from both $\mathcal{M}_{1,2}$ and $\mathcal{M}^0_{1,3}$.

Proof. We know

$$\mathrm{CH}^*(\mathcal{M}^0_{1,3}) = \mathbb{Z}[t]/(2t)$$
$$\mathrm{CH}^*(\mathcal{M}^0_{1,3},1)_{\mathrm{ind}} = \mathbb{Z} \oplus \bigoplus_{i=2}^{\infty} \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$$
$$\mathrm{CH}^*(\mathcal{M}^0_{1,2}) = \mathbb{Z}[t]/(3t)$$

by Theorem 4.14 and Theorem 4.12. So in degree 1, the localization exact sequence for $\mathcal{M}^0_{1,3} \subseteq \mathcal{M}_{1,3}$ is

$$0 \to \mathrm{CH}^1(\mathcal{M}_{1,3},1)_{\mathrm{ind}} \to \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \to \mathrm{CH}^1(\mathcal{M}_{1,3}) \to \frac{Z}{2\mathbb{Z}} \to 0$$

We know $\mathrm{CH}^1(\mathcal{M}^0_{1,3},1)_{\mathrm{ind}} = \mathbb{Z}$ is generated by $\Psi_{\mathbb{G}_m}(\frac{\Delta}{(x_3-x_2)^6})$. We wish to compute $\partial(\Psi_{\mathbb{G}_m}(\frac{\Delta}{(x_3-x_2)^6})) = \mathrm{div}(\frac{\Delta}{(x_3-x_2)^6}) = \mathrm{ord}_{Z_{1,3}}(\frac{\Delta}{(x_3-x_2)^6})[Z_{1,3}]$.

Choose a curve $(C = V(y^2 - x^3 - ax - b), \infty, (x_2, y_2)) \in U_2$, with $y_2 \neq 0$. We get a morphism

$$\varphi : C \setminus \{\infty, (x_2, y_2)\} \to \mathcal{M}_{1,3}$$
$$(x, y) \mapsto (C, \infty, (x_2, y_2), (x, y)).$$

This lands in $\mathcal{M}_{1,3} \setminus \mathcal{M}^0_{1,3}$ if and only if $(x, y) = (x_2, -y_2)$, using that $y_2 \neq 0$. By Theorem **??**

$$\mathrm{ord}_{Z_{1,3}}(\frac{\Delta}{(x_3-x_2)^6}) = \mathrm{ord}_{(x_2,-y_2)}(\varphi^{\#}\frac{\Delta}{(x_3-x_2)^6}) = \mathrm{ord}_{(x_2,-y_2)}(\frac{-4a^3-27b^2}{(x-x_2)^6}) = -6,$$

using that $(x - x_2)$ is a uniformizer for C at $(x_2, -y_2)$, which is true because $y_2 \neq 0$.

Thus, in degree 1, our localization exact sequence looks like

$$0 \to \mathrm{CH}^1(\mathcal{M}_{1,3},1)_{\mathrm{ind}} \to \mathbb{Z} \xrightarrow{\cdot -6} \mathbb{Z} \to \mathrm{CH}^1(\mathcal{M}_{1,3}) \to \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0,$$

from which we see $\mathrm{CH}^1(\mathcal{M}_{1,3},1)_{\mathrm{ind}} = 0$. Moreover, we have a short exact sequence

$$0 \to \mathbb{Z}/6\mathbb{Z} \to \mathrm{CH}^1(\mathcal{M}_{1,3}) \to \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0.$$

The group $\mathrm{CH}^1(\mathcal{M}_{1,3})$ must then be isomorphic to either $\frac{\mathbb{Z}}{12\mathbb{Z}}$ or $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}/6\mathbb{Z}$. Let $\lambda \in \mathrm{CH}^1(\mathcal{M}_{1,3})$ denote the first Chern class of the Hodge bundle. Proposition 9.5 implies that λ has order 12. And so, we must have $\mathrm{CH}^1(\mathcal{M}_{1,3}) \cong \frac{\mathbb{Z}}{12\mathbb{Z}}$, generated by λ.

In degree $i > 1$, the localization exact sequence says

$$\mathrm{CH}^{i-1}(\mathcal{M}^0_{1,2},1)_{\mathrm{ind}} \xrightarrow{j_*} \mathrm{CH}^i(\mathcal{M}_{1,3},1)_{\mathrm{ind}} \to \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 \xrightarrow{\partial_1} \frac{\mathbb{Z}}{3\mathbb{Z}} \to \mathrm{CH}^i(\mathcal{M}_{1,3}) \to \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0,$$

where $\mathrm{CH}^{i-1}(\mathcal{M}^0_{1,2},1)_{\mathrm{ind}} = \mathbb{Z}$ if $i = 2$ and vanishing otherwise. We see that ∂_1 has to be 0, which gives

- $\mathrm{CH}^i(\mathcal{M}_{1,3})$ is an extension of $\frac{\mathbb{Z}}{2\mathbb{Z}}$ by $\frac{\mathbb{Z}}{3\mathbb{Z}}$, hence it is isomorphic to $\frac{\mathbb{Z}}{6\mathbb{Z}}$ and

- $\mathrm{CH}^2(\mathcal{M}_{1,3},1)_{\mathrm{ind}}$ is an extension of $\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$ by the cyclic group $\mathrm{im}(j_*)$ and $\mathrm{CH}^i(\mathcal{M}_{1,3},1)_{\mathrm{ind}} = \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$ for $i > 2$.

We know that $\mathrm{CH}^2(\mathcal{M}_{1,3},1)_{\mathrm{ind}}$ is a split extension, meaning that we can write $\mathrm{CH}^2(\mathcal{M}_{1,3},1)_{\mathrm{ind}} = \mathrm{im}(j_*) \oplus \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$, by considering the diagram

$$\begin{tikzcd} \mathrm{CH}^2(\mathcal{M}_{1,3},1)_{\mathrm{ind}} \arrow[r] & \mathrm{CH}^2(\mathcal{M}^0_{1,3},1)_{\mathrm{ind}} \\ \mathrm{CH}^2(\mathcal{M}_{1,2},1)^2_{\mathrm{ind}} \arrow[u] \arrow[ur, "\sim"] & \end{tikzcd}$$

using Proposition 4.15

Now, I claim that $\mathrm{CH}^*(\mathcal{M}_{1,3})$ is generated as a ring by λ. We know λ additively generates $\mathrm{CH}^1(\mathcal{M}_{1,3})$. Because $\iota : \mathcal{M}^0_{1,2} \to \mathcal{M}_{1,3}$ is a section of the forgetful map $\pi : \mathcal{M}_{1,3} \to \mathcal{M}_{1,2}$ over $\mathcal{M}^0_{1,2}$, we have that $\iota^*(\lambda) = \lambda$. Hence help. Thus, $\tilde{t}^n$ has order 6, and so $\tilde{t}$ generates $\mathrm{CH}^*(\mathcal{M}_{1,3})$. Thus, we have a surjection $\mathbb{Z}[\tilde{t}] \to \mathrm{CH}^*(\mathcal{M}_{1,3})$. The elements $12\tilde{t}$ and $6\tilde{t}^2$ are in the kernel of this map, and the induced map

$$\mathbb{Z}[\tilde{t}]/(12\tilde{t}, 6\tilde{t}^2) \to \mathrm{CH}^*(\mathcal{M}_{1,3})$$

must be an isomorphism, because it is surjective the graded pieces of a fixed positive degree have the same finite cardinality.

$$\mathrm{CH}^{*-1}(\mathcal{M}^0_{1,2}) \to \mathrm{CH}^*(\mathcal{M}_{1,3}) \to \mathrm{CH}^*(\mathcal{M}^*_{1,3}) \xrightarrow{\partial} \mathrm{CH}^{*-1}(\mathcal{M}^0_{1,2}).$$

From our above computation with ∂, we have

$$\mathbb{Z} \to \mathrm{CH}^*(\mathcal{M}_{1,3}) \to \bigoplus_{i=2}^{\infty} \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0.$$

□

6.2. $\mathcal{M}_{1,4}$.

Theorem 6.2. $\mathrm{CH}^*(\mathcal{M}_{1,4}) = \frac{\mathbb{Z}[\lambda]}{(12\lambda, 2\lambda^2)}$ *and*

$$\mathrm{CH}^i(\mathcal{M}_{1,4},1)_{ind} = \begin{cases} 0 & i=1 \\ im(j_*) \oplus \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^6 & i=2 \\ \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 & i \geq 3 \end{cases}$$

Moreover, the pullback map

$$\pi^* : \mathrm{CH}^i(\mathcal{M}_{1,3},1)_{ind} \to \mathrm{CH}^i(\mathcal{M}_{1,4},1)_{ind}$$

is injective for $i = 2$ and an isomorphism for $i \geq 3$. And $\mathcal{M}_{1,4}$ has the integral CKP and integral CKgP.

Proof. We have the exact sequence

$$\mathrm{CH}^{*-1}(\mathcal{M}^0_{1,3})_{\mathrm{ind}} \to \mathrm{CH}^*(\mathcal{M}_{1,4},1)_{\mathrm{ind}} \to \mathrm{CH}^*(\mathcal{M}^0_{1,4},1)_{\mathrm{ind}} \xrightarrow{\partial}$$
$$\mathrm{CH}^{*-1}(\mathcal{M}^0_{1,3}) \to \mathrm{CH}^*(\mathcal{M}_{1,4}) \to \mathrm{CH}^*(\mathcal{M}^0_{1.4}) \to 0.$$

We know

$$\mathrm{CH}^*(\mathcal{M}^0_{1,4}) = \mathbb{Z}$$
$$\mathrm{CH}^i(\mathcal{M}^0_{1,4},1)_{\mathrm{ind}} = \begin{cases} \mathbb{Z} & i=1 \\ (\frac{\mathbb{Z}}{2\mathbb{Z}})^6 & i=2 \\ 0 & i>2 \end{cases}$$
$$\mathrm{CH}^*(\mathcal{M}^0_{1,3}) = \frac{\mathbb{Z}[t]}{(2t)}$$
$$\mathrm{CH}^*(\mathcal{M}^0_{1,3},1)_{\mathrm{ind}} = \mathbb{Z} \oplus \bigoplus_{i>2} (\frac{\mathbb{Z}}{2\mathbb{Z}})^2.$$

by Theorem 5.2 and Theorem 4.14.

In degree 1, the exact sequence is

$$0 \to \mathrm{CH}^1(\mathcal{M}_{1,4},1)_{\mathrm{ind}} \to \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \to \mathrm{CH}^1(\mathcal{M}_{1,4}) \to 0,$$

where $\mathrm{CH}^1(\mathcal{M}^0_{1,4},1)_{\mathrm{ind}}$ is generated by f (help). We compute the image of ∂_1 using help. Pick a generic point $(a_4,a_5,a_7,a_8) \in \mathcal{M}^0_{1,4}$ corresponding to the pointed curve (C,P_1,P_2,P_3,P_4). We have a closed embedding

$$g : C \setminus \{P_1,P_2,P_3\} \to \mathcal{M}_{1,4}$$
$$P \mapsto (C,P_1,P_2,P_3,P).$$

This intersects $j(\mathcal{M}^0_{1,3})$ transversely because it is a fiber of $\pi : \mathcal{M}_{1,4} \to \mathcal{M}_{1,3}$, of which j is a section.

This curve intersects $j(\mathcal{M}_{1,4})$ exactly at $([a_8 : -a_5],[0:1])$. We compute $g|^{\mathcal{M}^0_{1,4}}$ in the affine neighborhood $xz \neq 0$. Set $u := \frac{w}{x}$, $v := \frac{y}{z}$, and consider a point $P = (u_0,v_0) \in C$. Following the proof of help, we embed C in $\mathbb{P}^1 \times \mathbb{P}^1$ using $\mathcal{O}(P_1+P)$ and $\mathcal{O}(P_2+P_3)$. The functions $1, \frac{v_0a_0 - u_0a_5 + (u_0v_0a_7 - v_0^2a_0)u}{v_0u - u_0v}$ form a basis of $\mathcal{O}(P_1+P)$, and the functions $1, v$ form a basis for $\mathcal{O}(P_2+P_3)$. Hence, we are supposed to use the embedding

$$C \to \mathbb{P}^1 \times \mathbb{P}^1$$
$$([x:w],[y:z]) \mapsto ([v_0wz - u_0xy : (v_0a_0 - u_0a_5)xz + (u_0v_0a_7 - v_0^2a_0)wz],[y:z])$$

up to the automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which sends P to P_4 and P_i to P_i for $i \in \{1,2,3\}$. After applying the automorphism, we get the embedding

$$C \to \mathbb{P}^1 \times \mathbb{P}^1$$
$$([x:w],[y:z]) \mapsto ([(v_0a_0 - u_0a_5)xz + u_0v_0a_7wz - a_0v_0u_0xy : a_0v_0^2wz - a_0v_0u_0xy],[y:v_0z]).$$

This has image given by the vanishing of

$$x^2y^2 - x^2yz - xwy^2 + \frac{2a_7u_0v_0 + 2a_8u_0 + v_0a_4}{v_0^2a_0}xwyz + \frac{a_5}{v_0^2a_0}xwz^2 +$$

$$\frac{-a_8u_0+v_0a_7-u_0v_0a_7}{v_0^2a_0}w^2yz+\frac{a_0v_0a_8-a_5u_0a_7v_0-a_5a_8u_0}{v_0^4a_0^2}w^2z^2$$

which one can check by noting that composing this expression with the embedding gives a multiple of the defining equation of C. Thus, g is given by

$$(u_0,v_0)\mapsto\left(\frac{2a_7u_0v_0+2a_8u_0+v_0a_4}{v_0^2a_0},\frac{a_5}{v_0^2a_0},\frac{-a_8u_0+v_0a_7-u_0v_0a_7}{v_0^2a_0},\frac{a_0v_0a_8-a_5u_0a_7v_0-a_5a_8u_0}{v_0^4a_0^2}\right)$$

Composing this map with f, one gets

$$\frac{h(u_0,v_0)}{v_0^{20}},$$

for some polynomial $h(u_0,v_0)$. Expanding $h(u_0,v_0)$ around $(u_0,v_0)=(-\frac{a_5}{a_8},0)$, one gets an expression whose lowest term is degree 4 in $(u_0+\frac{a_5}{a_8}),v_0$. The defining equation of C is equivalent to

$$u_0+\frac{a_5}{a_8}=\frac{1}{a_5a_8^2}(a_8^2a_7(u_0+\frac{a_5}{a_8})^2v_0+a_8^3(u_0+\frac{a_5}{a_8})^2+(a_4a_8^2-2a_5a_7a_8)(u_0+\frac{a_5}{a_8})v_0-$$
$$(u_0+\frac{a_5}{a_8})a_8^2v_0^2+(a_5a_8+a_8a_8)v_0^2+(-a_8^2-a_4a_5a_8+a_5^2a_7)v_0).$$

Substituting this expression for $u_0+\frac{a_5}{a_8}$ into h, one gets an expression whose lowest term is degree 8 in $(u_0+\frac{a_5}{a_8}),v_0$. Preforming the same substitution again, one gets v_0^8 plus terms of larger degree. Since v_0 is a uniformizer at $(-\frac{a_5}{a_8},0)$, we have

$$\mathrm{ord}_{\mathcal{M}_{1,3}^0}(f)=\mathrm{ord}_{(-\frac{a_5}{a_8},0)}(f\circ g))=\mathrm{ord}_{(-\frac{a_5}{a_8},0)}(\frac{v_0^8+\text{higher order terms}}{v_0^{20}})=-12.$$

Thus, $\partial_1(f)=-12[\mathcal{M}_{1,3}^0]$, and so

$$\mathrm{CH}^1(\mathcal{M}_{1,4})=\frac{\mathbb{Z}}{12\mathbb{Z}}$$
$$\mathrm{CH}^1(\mathcal{M}_{1,4},1)_{\mathrm{ind}}=0.$$

Now, we use Proposition 9.5 to conclude that λ has order 12 on $\mathrm{CH}^1(\mathcal{M}_{1,4})$, and thus generates $\mathrm{CH}^1(\mathcal{M}_{1,4})$.

For $i>2$, the localization exact sequence is

$$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2\xrightarrow{j_*}\mathrm{CH}^i(\mathcal{M}_{1,4},1)_{\mathrm{ind}}\to 0\to\frac{\mathbb{Z}}{2\mathbb{Z}}\xrightarrow{j_*}\mathrm{CH}^i(\mathcal{M}_{1,4})\to 0$$

hence $\mathrm{CH}^i(\mathcal{M}_{1,4})\cong\frac{\mathbb{Z}}{2\mathbb{Z}}$, generated by $j_*(\lambda^{i-1})$, and $\mathrm{CH}^i(\mathcal{M}_{1,4},1)_{\mathrm{ind}}=\mathrm{im}(j_*)$ for $i>2$. When $i=2$, we have

$$\mathbb{Z}\xrightarrow{j_*}\mathrm{CH}^2(\mathcal{M}_{1,4},1)_{\mathrm{ind}}\to\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^6\xrightarrow{\partial_1}\frac{\mathbb{Z}}{2\mathbb{Z}}\xrightarrow{j_*}\mathrm{CH}^2(\mathcal{M}_{1,4})\to 0.$$

The group $\mathrm{CH}^1(\mathcal{M}_{1,3}^0)=\frac{\mathbb{Z}}{2\mathbb{Z}}$ is generated by λ. Note

$$j_*(\lambda^{i-1})=j_*(j^*(\lambda^{i-1}))=j_*(1)\lambda^{i-1}.$$

For $i > 2$, this is nonzero, so this is must be nonzero when $i = 2$. Hence, j_* is injective on $\mathrm{CH}^2(\mathcal{M}^0_{1,3})$, and $\mathrm{CH}^2(\mathcal{M}_{1,4}) = \frac{\mathbb{Z}}{2\mathbb{Z}}$, generated by $j_*(\lambda)$.

We claim that λ generates $\mathrm{CH}^*(\mathcal{M}_{1,4})$ as a ring. Note that because $j_*(1)$ and λ both generate $\mathrm{CH}^1(\mathcal{M}_{1,4})$, we can write $j_*(1) = a\lambda$ for some $a \in \mathbb{Z}$ invertible mod 12. Then, we know that $\mathrm{CH}^i(\mathcal{M}_{1,4})$ is generated by $j_*(\lambda^{i-1}) = j_*(1)\lambda^{i-1}) = a\lambda^i$, and so λ^i generates $\mathrm{CH}^i(\mathcal{M}_{1,4})$. Thus, we have a surjection

$$\frac{\mathbb{Z}[\lambda]}{(12\lambda, 2\lambda^2)} \to \mathrm{CH}^*(\mathcal{M}_{1,4}).$$

In degree 0, this is $\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}$, and in larger degrees, this is a surjection of finite sets of the same size. Thus, this map is an isomorphism.

Because j_* is injective on $\mathrm{CH}^2(\mathcal{M}^0_{1,3})$, we have an exact sequence

$$0 \to \mathrm{im}(j_*) \to \mathrm{CH}^2(\mathcal{M}_{1,4}, 1)_{\mathrm{ind}} \to \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^6 \to 0.$$

This splits because....

Finally, we claim that $\pi^* : \mathrm{CH}^i(\mathcal{M}_{1,3}, 1)_{\mathrm{ind}} \to \mathrm{CH}^i(\mathcal{M}_{1,4}, 1)_{\mathrm{ind}}$ is injective for $i = 2$ and an isomorphism for $i \geq 3$. We have a commutative diagram

$$\begin{tikzcd} \mathcal{M}^0_{1,3} \arrow[rr, "j"] \arrow[dr] & & \mathcal{M}_{1,4} \arrow[dl, "\pi"] \\ & \mathcal{M}_{1,3} & \end{tikzcd}.$$

Additionally, the pullback $\mathrm{CH}^i(\mathcal{M}_{1,3}, 1)_{\mathrm{ind}} \to \mathrm{CH}^i(\mathcal{M}^0_{1,3}, 1)_{\mathrm{ind}}$ is injective for $i \geq 2$ by Theorem 6.1. This implies that

$$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 \cong \mathrm{CH}^i(\mathcal{M}_{1,3}) \xrightarrow{\pi^*} \mathrm{CH}^i(\mathcal{M}_{1,4})$$

is injective. Moreover, for $i \geq 3$, the localization exact sequence says

$$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 = \mathrm{CH}^i(\mathcal{M}^0_{1,3}, 1)_{\mathrm{ind}} \xrightarrow{j_*} \mathrm{CH}^i(\mathcal{M}_{1,4}, 1)_{\mathrm{ind}} \to 0.$$

As the group $\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$ has both injections and surjections to $\mathrm{CH}^i(\mathcal{M}_{1,4}, 1)_{\mathrm{ind}}$ for $i \geq 3$, we know they are isomorphic. Hence, π^* is an isomorphism for $i \geq 3$.

□

7. Stratification of $\overline{\mathcal{M}}_{g,n}$

Definition 7.1. Let Γ be a stable graph. We define $\mathcal{M}^\Gamma_{g,n}$ to be the locus of curves in $\overline{\mathcal{M}}_{g,n}$ with stable graph Γ. Its closure is denoted $\overline{\mathcal{M}}^\Gamma_{g,n}$. When there is no confusion as to which $\overline{\mathcal{M}}_{g,n}$ we are working in, we omit the subscripts, writing just $\mathcal{M}^\Gamma$ and $\overline{\mathcal{M}}^\Gamma$.

We also define

$$\overline{\mathcal{M}}_\Gamma := \prod_{v\in\Gamma} \overline{\mathcal{M}}_{g_v,n_v}$$

and

$$\mathcal{M}_\Gamma := \prod_{v\in\Gamma} \mathcal{M}_{g_v,n_v}.$$

There is a surjective map

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}^\Gamma$$

that glues together the various curves. Moreover, $\xi_\Gamma(\mathcal{M}_\Gamma) = \mathcal{M}^\Gamma$.

Cite Bae-Schmidt:

Theorem 7.2. *The map ξ_Γ induces an isomorphism*

$$\mathcal{M}_\Gamma/\operatorname{Aut}(\Gamma) \cong \mathcal{M}^\Gamma.$$

Proposition 7.3. *The map $\xi_\Gamma : [\overline{\mathcal{M}}_\Gamma/\operatorname{Aut}(\Gamma)] \to \overline{\mathcal{M}}^\Gamma$ is representable. This seems to be in Bae and Schmidt.*

Proposition 7.4. *Suppose Γ is a stable graph with $\operatorname{Aut}(\Gamma) = 1$. Then the map $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}^\Gamma$ induces an isomorphism on* CH. *The hypothesis needs to be stronger. Also, it would be convenient to show that this is an isomorphism of stacks.*

Proof. Help. □

Now, we explain our general strategy for computing $\mathrm{CH}(\partial\overline{\mathcal{M}}_{g,n})$.

Definition 7.5. For a fixed g,n, let $\overline{\mathcal{M}}^{\mathrm{sep}}$ denote the locus of curves in the boundary with at least one separating node, and let $\overline{\mathcal{M}}^{\mathrm{sep},\geq p}$ denote the union of $\overline{\mathcal{M}}^{\mathrm{sep}}$ with the locus of curves with at least p nodes. Finally, let $\mathcal{M}^{\mathrm{non}=p} = \overline{\mathcal{M}}^{\mathrm{sep},\geq p} \setminus \overline{\mathcal{M}}^{\mathrm{sep},\geq p+1}$.

The following are immediate from the definition

(1) $\overline{\mathcal{M}}^{\mathrm{sep}}$ is the union of $\overline{\mathcal{M}}^\Gamma$ over stable graphs Γ with one node which is separating
(2) $\overline{\mathcal{M}}^{\mathrm{sep},\geq p} = \overline{\mathcal{M}}^{\mathrm{sep}}$ for $p > \dim(\overline{\mathcal{M}}_{g,n})$
(3) $\overline{\mathcal{M}}^{\mathrm{sep},\geq 1} = \partial\overline{\mathcal{M}}_{g,n}$
(4) $\mathcal{M}^{\mathrm{non}=p}$ is the disjoint union of $\mathcal{M}^\Gamma$ over graphs Γ with exactly p nodes, all of which are nonseparating.

Thus, we have a filtration of the boundary $\partial\overline{\mathcal{M}}_{g,n}$. We compute $\mathrm{CH}(\partial\overline{\mathcal{M}}_{g,n})$ using this stratification and the localization exact sequence. To do this, one must know the Chow ring of the bottom piece of the filtration, $\overline{\mathcal{M}}^{\mathrm{sep}}$, the Chow ring of the open parts, $\mathcal{M}^{\mathrm{non}=p}$, and then one uses the localization exact sequence to fit the Chow rings of these spaces together. We compute the Chow ring of $\mathcal{M}^{\text{non}=p}$, using the description of $\mathcal{M}^\Gamma$ given in Theorem 7.2. To compute the Chow group of $\overline{\mathcal{M}}^{\text{sep}}$, note we have an exact sequence

$$\bigoplus_{\Gamma,\Gamma' \in S} \mathrm{CH}^*(\overline{\mathcal{M}}^\Gamma \cap \overline{\mathcal{M}}^{\Gamma'}) \to \bigoplus_{\Gamma \in S} \mathrm{CH}^*(\overline{\mathcal{M}}^\Gamma) \to \mathrm{CH}(\overline{\mathcal{M}}^{\text{sep}}) \to 0$$

where S is the set of stable graphs with one separating node. In this exact sequence, we can compute $\mathrm{CH}^*(\mathcal{M}^\Gamma)$ for $\Gamma \in S$ as $\mathrm{CH}^*(\overline{\mathcal{M}}_{g_1,n_1}) \otimes_{\mathbb{Z}} \mathrm{CH}^*(\overline{\mathcal{M}}_{g_2,n_2})$ for $(g_i, n_i) < (g, n)$ in lex order using Proposition 7.4 and the Chow-Kunneth property. Finally, we use the commutative diagram with exact rows

(7.6)

$$\begin{tikzcd}
\mathrm{CH}^*(\mathcal{M}^\Gamma, 1)_{\text{ind}} \arrow[r, "\partial_1'"] \arrow[d] & \mathrm{CH}^{*-1}([\partial\overline{\mathcal{M}}_\Gamma / \operatorname{Aut}(\Gamma)) \arrow[r] \arrow[d] & \mathrm{CH}^*([\overline{\mathcal{M}}_\Gamma / \operatorname{Aut}(\Gamma)]) \arrow[r] \arrow[d] & \mathrm{CH}^*(\mathcal{M}^\Gamma) \arrow[r] \arrow[d] & {} \\
\mathrm{CH}^*(\mathcal{M}^{\text{non}=p}, 1)_{\text{ind}} \arrow[r, "\partial_1"] & \mathrm{CH}^{*-1}(\overline{\mathcal{M}}^{\text{sep},\geq p+1}) \arrow[r] & \mathrm{CH}^*(\overline{\mathcal{M}}^{\text{sep},\geq p}) \arrow[r] & \mathrm{CH}^*(\mathcal{M}^{\text{non}=p}) \arrow[r] & {}
\end{tikzcd}$$

for $\mathcal{M}^\Gamma \subseteq \mathcal{M}^{\text{non}=p}$ to compute $\mathrm{CH}^*(\overline{\mathcal{M}}^{\text{sep},\geq p})$ after knowing $\mathrm{CH}^*(\overline{\mathcal{M}}^{\text{sep},\geq p+1})$.

8. $[\overline{\mathcal{M}}_{0,n}/\mu_2]$

In this section, we investigate the stacks $[\mathcal{M}_{0,n}/\mu_2]$ and $[\overline{\mathcal{M}}_{0,n}/\mu_2]$, where μ_2 acts by switching the last two markings.

Here are our $\mathcal{M}_{0,n}$ conventions. They are not the standard conventions in order to make the action by μ_2 on $\mathbb{A}^{n-3}$ linear. Suppose $n \geq 3$. Firstly, we label the markings $1, 2, \ldots, n-2, a, b$. Now, given a family $\pi : \mathcal{C} \to S$ of smooth genus 0 curves, with n-sections $\sigma_i : S \to \mathcal{C}$, we can find a unique S-isomorphism $\mathcal{C} \cong \mathbb{P}^1 \times S$ such that

- σ_1 is the constant ∞ section
- σ_a is the constant 1 section
- σ_b is the constant -1 section.

Thus, we have an isomorphism

$$\mathcal{M}_{0,n} \cong \mathbb{A}^{n-3} \setminus V\left(\prod_{p<q}(x_p - x_q)\prod_p (x_p^2 - 1)\right), \tag{8.1}$$

where $\mathbb{A}^{n-3} = \operatorname{Spec} k[x_2, \ldots, x_{n-2}]$, and x_p records the p-th marking. The induced μ_2 action on the right is given by $(x_2, \ldots, x_{n-2}) \mapsto (-x_2, \ldots, -x_{n-2})$. With this action, we have

$$\mathrm{CH}^*_{\mu_2}(\mathbb{A}^{n-3}) = \mathbb{Z}[s]/(2s),$$

where s is the class of a hyperplane going through the origin.

We now compute $\mathrm{CH}^*([\mathcal{M}_{0,n}/\mu_2]), \mathrm{CH}^*([\mathcal{M}_{0,n}/\mu_2, 1])_{\text{ind}}$. First a lemma on $[\mathbb{A}^m/\mu_2]$.

Lemma 8.2. *Let $U \subseteq \mathbb{A}^m$ be a μ_2-invariant open subset with $0 \in U$. Then $\mathrm{CH}^*_{\mu_2}(\mathbb{A}^n) \to \mathrm{CH}^*_{\mu_2}(U)$ is an isomorphism.*

Proof. Note that the pullback

$$\mathrm{CH}^*_{\mu_2}(\mathbb{A}^n) \to \mathrm{CH}^*_{\mu_2}(\{0\})$$

is an isomorphism because it is inverse to the pullback $\mathrm{CH}^*_{\mu_2}(\mathrm{Spec}(k)) \to \mathrm{CH}^*_{\mu_2}(\mathbb{A}^n)$, which is an isomorphism by homotopy invariance. Thus, for U an open subset of $\mathbb{A}^n$, if $0 \in U$, $\mathrm{CH}^*_{\mu_2}(\mathbb{A}^n) \to \mathrm{CH}^*_{\mu_2}(U)$ is surjective by the localization exact sequence and injective because we can write composing with the pullback $\mathrm{CH}^*_{\mu_2}(U) \to \mathrm{CH}^*_{\mu_2}(\{0\})$ gives an isomorphism. $\square$

Theorem 8.3.

$$\mathrm{CH}^*([\mathcal{M}_{0,n}/\mu_2]) \cong \begin{cases} \mathrm{CH}^*_{\mu_2}(\mathrm{Spec}(k)) & n \leq 4 \\ \mathbb{Z} & n \geq 5 \end{cases}$$

and

$$\mathrm{CH}^*([\mathcal{M}_{0,n}/\mu_2], 1)_{ind} = \mathrm{CH}^1([\mathcal{M}_{0,n}/\mu_2], 1)_{ind} = (\mathcal{O}_{\mathcal{M}_{0,n}}(\mathcal{M}_{0,n})^\times)^{\mu_2}$$

Proof. The localization exact sequence for $[\mathcal{M}_{0,n}/\mu_2] \subseteq [\mathbb{A}^{n-3}/\mu_2]$ reads

$$0 \to \mathrm{CH}^*_{\mu_2}(\mathcal{M}_{0,n}, 1)_{\mathrm{ind}} \to \mathrm{CH}^{*-1}_{\mu_2}(Z) \to \mathrm{CH}^*_{\mu_2}(\mathbb{A}^{n-3}) \to \mathrm{CH}^*_{\mu_2}(\mathcal{M}_{0,n}) \to 0,$$

using Proposition 2.9 and homotopy invariance to say $\mathrm{CH}^*_{\mu_2}(\mathbb{A}^{n-3}, 1)_{\mathrm{ind}} = 0$. Suppose $n \leq 4$. Then $0 \in \mathcal{M}_{0,n}$, so Lemma 8.2 says $\mathrm{CH}^*([\mathbb{A}^{n-3}/\mu_2]) \to \mathrm{CH}^*([\mathcal{M}_{0,n}/\mu_2])$ is an isomorphism. The localization sequence then gives $\mathrm{CH}^*_{\mu_2}(\mathcal{M}_{0,n}, 1)_{\mathrm{ind}} = \mathrm{CH}^{*-1}_{\mu_2}(Z)$. If $n = 3$, $Z = \emptyset$, so $\mathrm{CH}^*(\mathcal{M}_{0,3}, 1)_{\mathrm{ind}} = 0$. If $n = 4$, $Z = V(x_1^2 - 1)$, so $[Z/\mu_2] = \mathrm{Spec}(k)$, giving $\mathrm{CH}^1([\mathcal{M}_{0,4}/\mu_2], 1) = \mathrm{CH}^0(\mathrm{Spec}(k)) = \mathbb{Z}$.

Now suppose $n \geq 5$. Then $V(x_p - x_q) \cong \mathbb{A}^{n-4} \subsetneq Z$ and $\mathrm{CH}^*_{\mu_2}(\mathbb{A}^m) = \mathbb{Z}[s]/(2s)$. Now $[V(x_p - x_q)] = s$ because $V(x_p - x_q)$ is a hyperplane containing the origin, so pushforward $\mathrm{CH}^{*-1}_{\mu_2}(V(x_p - x_q)) \to \mathrm{CH}^*_{\mu_2}(\mathbb{A}^{n-3})$ is multiplication by s, using Lemma 2.10. This is an isomorphism in positive degrees, and so $\mathrm{CH}^*_{\mu_2}(\mathcal{M}_{0,n}) = \mathbb{Z}$.

We next compute $\mathrm{CH}^*_{\mu_2}(Z)$. We have

$$\bigoplus_p \mathrm{CH}^*_{\mu_2}(V(x_p^2 - 1)) \oplus \bigoplus_{p<q} \mathrm{CH}^*_{\mu_2}(V(x_p - x_q)) \twoheadrightarrow \mathrm{CH}^*_{\mu_2}(Z)$$

with the kernel generated by pushforwards of intersections of components. In particular, this is an isomorphism in degree 0. Note

$$\mathrm{CH}^*_{\mu_2}(V(x_p^2 - 1)) = \mathrm{CH}^*(V(x_p^2 - 1)/\mu_2) = \mathrm{CH}^*(\mathbb{A}^{n-4}) = \mathbb{Z}$$

because the action is free . Moreover, for $p \neq q$, we must have

$$\mathrm{CH}^*_{\mu_2}(V(x_p^2 - 1) \cap V(x_q^2 - 1)) \to \mathrm{CH}^*_{\mu_2}(V(x_p^2 - 1)) \oplus \mathrm{CH}^*_{\mu_2}(V(x_q^2 - 1))$$

equals 0, as it maps into positive degrees. For the same reason the map

$$\mathrm{CH}^*_{\mu_2}(V(x_p^2 - 1) \cap V(x_j - x_\ell)) \to \mathrm{CH}^*_{\mu_2}(V(x_p^2 - 1)) \oplus \mathrm{CH}^*_{\mu_2}(V(x_j - x_\ell))$$

is 0 on the first coordinate. It is also 0 on the second coordinate, because $[V(x_p^2-1)] \in \mathrm{CH}^*_{\mu_2}(\mathbb{A}^m)$ is 0. Finally, the map

$$\mathrm{CH}^*_{\mu_2}(V(x_p-x_q)\cap V(x_j-x_\ell)) \to \mathrm{CH}^*_{\mu_2}(V(x_p-x_q)) \oplus \mathrm{CH}^*_{\mu_2}(V(x_j-x_\ell))$$

is an isomorphism onto either coordinate in positive degrees. Thus,

$$\mathrm{CH}^i_{\mu_2}(Z) = \mathbb{Z}/2\mathbb{Z},$$

for $i \geq 1$.

As noted above, $\mathrm{CH}^{*-1}_{\mu_2}(V(x_p-x_q)) \to \mathrm{CH}^*_{\mu_2}(\mathbb{A}^{n-3})$ is an isomorphism in degrees > 2, and we have just seen $\mathrm{CH}^*_{\mu_2}(V(x_p-x_q)) \to \mathrm{CH}^*_{\mu_2}(Z)$ is an isomorphism in positive degrees, so $\mathrm{CH}^{*-1}_{\mu_2}(Z) \xrightarrow{\iota_*} \mathrm{CH}^*_{\mu_2}(\mathbb{A}^{n-3})$ is an isomorphism in degrees > 1. Thus,

$$\mathrm{CH}^i_{\mu_2}(\mathcal{M}_{0,n},1)_{\mathrm{ind}} = 0$$

for $i \geq 2$. Moreover,

$$\mathrm{CH}^1_{\mu_2}(\mathcal{M}_{0,n},1)_{\mathrm{ind}} = (\mathcal{O}_{\mathcal{M}_{0,n}}(\mathcal{M}_{0,n})^\times)^{\mu_2}$$

by Theorem **??**. □

Next, we calculate the image of $\mathrm{CH}^*([\mathcal{M}_{0,n}/\mu_2]) \to \mathrm{CH}^*([\partial\overline{\mathcal{M}}_{0,n}/\mu_2])$, which which helps give the image of $\mathrm{CH}^*([\mathcal{M}_{0,n}/\mu_2]) \to A \oplus B \oplus \mathbb{Z}\cdot[\mathcal{M}^\Lambda]$.

For $A, B \subseteq \{1,\dots,n-2,a,b\}$ disjoint subsets, we have the divisors $D(A|B) = D(B|A) \subseteq \overline{\mathcal{M}}_{0,n}$ which correspond to curves that have a node separating the markings from sets A and B. Such a divisor is a sum of divisors $D(A'|B')$ where B' is the complement of A'. Let $\widehat{D}(A|B)$ be the image of $D(A|B)$ in $[\overline{\mathcal{M}}_{0,n}/\mu_2]$ and text $\widehat{T}$ text.

Theorem 8.4. *The image of*

$$\mathrm{CH}^*([\mathcal{M}_{0,n}/\mu_2],1)_{ind} \xrightarrow{\partial_1^{\mu_2}} \mathrm{CH}^*([\partial\overline{\mathcal{M}}_{0,n}/\mu_2])$$

is generated by

$$\widehat{D}(pa|rb) - 2\widehat{D}(pr|ab)$$

for $p, r \in \{1,\dots,n-2\}$ *and*

$$\alpha_{pqr} + \alpha_{p'q'r'}$$

for $p,q,r,p',q',r' \in \{1,\dots,n-2\}$ *with* $|\{p,q,r\}| = |\{p',q',r'\}| = 3$, *where*

$$\alpha_{pqr} := \widehat{D}(pqb|ra) + \widehat{D}(rab|jk) - \widehat{D}(rpq|ab) - \widehat{D}(qab|rp) - \widehat{D}(pab|rq).$$

Remark 8.5. The proof actually shows that we need only take the above generators when $r, r' = 1$. Moreover, these generators freely generate (I think). But we write the theorem like this for the sake of symmetry.

Proof. Note $\mathcal{O}_{\mathcal{M}_{0,n}}(\mathcal{M}_{0,n})/k^\times$ is freely generated by $\{x_p \pm 1\}, \{x_p - x_q\}$, and so $(\mathcal{O}_{\mathcal{M}_{0,n}}(\mathcal{M}_{0,n})^\times)^{\mu_2}/k^\times = \mathrm{CH}^*([\mathcal{M}_{0,n}/\mu_2],1)_{\mathrm{ind}}$ is generated by

$$\{x_p^2-1\}_p, \{(x_p-x_q)(x_{p'}-x_{q'})\}_{p\neq q, p'\neq q'}.$$

Therefore, $\partial_1^{\mu_2}$ applied to this generating set generates the image of $\partial_1^{\mu_2}$.

On $\mathcal{M}_{0,4}$, because $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, we have

$$\partial_1(x-1) = D(1b|2a) - D(12|ab).$$

We leverage commutativity of

$$\begin{tikzcd} \mathrm{CH}^1(\mathcal{M}_{0,n},1)_{\mathrm{ind}} \arrow[r, "\partial_1"] & \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{0,n}) \\ \mathrm{CH}^1(\mathcal{M}_{0,4},1)_{\mathrm{ind}} \arrow[r, "\partial_1"] \arrow[u, "\varphi^*"] & \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{0,4}) \arrow[u, "\varphi^*"] \end{tikzcd}$$

induced by morphisms $\varphi : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,4}$ with $\varphi(\mathcal{M}_{0,n}) \subseteq \mathcal{M}_{0,4}$ to extend this computation to general n.

For a general n, given a choice of 4 markings i, j, k, ℓ, we can consider the morphism

$$\varphi_{i,j,k,\ell} : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,4}$$
$$(C, p_1, \dots, p_n) \mapsto (C, p_i, p_j, p_k, p_\ell).$$

Over $\mathcal{M}_{0,n} \subseteq \overline{\mathcal{M}}_{0,n}$, the points p_i, p_j, p_k, p_ℓ are distinct smooth points of $\mathbb{P}^1$, so $\varphi_{i,j,k,\ell}$ maps into $\mathcal{M}_{0,4}$. By the commutativity of the above diagram, we thus have

$$\partial_1(\varphi^*(x-1)) = D(i\ell|jk) - D(ij|k\ell).$$

We can compute $\varphi_{i,j,k,\ell}$ restricted to $\mathcal{M}_{0,n}$ under our identifications made in 8.1 by applying transformations to send $p_i \mapsto \infty$, $p_k \mapsto 1$, $p_\ell \mapsto -1$. This gives the cross ratio, modified to fit with our conventions of $\mathcal{M}_{0,n}$.

When $i = 1$, $k = a$, and $\ell = b$, the cross ratio is x_j, and so the function $x - 1$ on $\mathcal{M}_{0,4}$ pulls back to $x_j - 1$. Thus,

$$\partial_1(x_j - 1) = D(1b|ja) - D(1j|ab).$$

Similarly, when $i = 1$, $k = b$, and $\ell = b$, the cross ratio is $-x_j$, so the function $x - 1$ on $\mathcal{M}_{0,4}$ pulls back to $-x_j - 1$ on $\mathcal{M}_{0,4}$, and so

$$\partial_1(x_j + 1) = D(1a|jb) - D(1j|ab).$$

Finally, when $\ell = a$ and $i = b$, the cross ratio is

$$2\frac{(x_j-1)(x_k+1)}{(x_j+1)(x_k-1)} - 1,$$

so $x - 1$ on $\mathcal{M}_{0,4}$ pulls back to

$$4\frac{x_j - x_k}{(x_j+1)(x_k-1)},$$

meaning

$$\partial_1\left(\frac{x_j - x_k}{(x_j+1)(x_k-1)}\right) = D(jk|ab) - D(jb|ka).$$

Combining this with the above expressions for $\partial_1(x_j+1), \partial_1(x_k-1)$, we get

$$\partial_1(x_j - x_k) = D(jk|ab) - D(jb|ka) + D(1a|jb) - D(1j|ab) + D(1b|ka) - D(1k|ab).$$

A more useful way to write this is to expand out do that every term involves all of the markings we are looking at, using, for example, $D(jk|ab) = D(1jk|ab)+D(jk|1ab)$. In doing this, there are many cancellations, and one gets the expression

$$D(1ab|jk) + D(jkb|1a) + D(jka|1b) - D(1jk|ab) - D(kab|1j) - D(jab|1k)$$

for $\partial_1(x_j - x_k)$. Note this expression is invariant under switching either a, b or j, k, as it should be.

Let $\pi : \overline{\mathcal{M}}_{0,n} \to [\overline{\mathcal{M}}_{0,n}/\mu_2]$ be the quotient map. Consider the diagram

$$\begin{array}{ccc} \mathrm{CH}^1(\mathcal{M}_{0,n},1)_{\mathrm{ind}} & \xrightarrow{\partial_1} & \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{0,n}) \\ \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} \\ \mathrm{CH}^1([\mathcal{M}_{0,n}/\mu_2],1)_{\mathrm{ind}} & \xrightarrow{\partial_1^{\mu_2}} & \mathrm{CH}^0([\partial\overline{\mathcal{M}}_{0,n}/\mu_2]) \end{array}$$

The pullback $\pi^* : \mathrm{CH}^1([\mathcal{M}_{0,n}/\mu_2],1)_{\mathrm{ind}} \to \mathrm{CH}^1(\mathcal{M}_{0,n},1)_{\mathrm{ind}}$ is the inclusion $(\mathcal{O}_{\mathcal{M}_{0,n}}(\mathcal{M}_{0,n})^\times)^{\mu_2}/k^\times \hookrightarrow \mathcal{O}_{\mathcal{M}_{0,n}}(\mathcal{M}_{0,n})^\times/k^\times$ under the identifications. We also have that $\mathrm{CH}^0([\partial\overline{\mathcal{M}}_{0,n}/\mu_2]) = \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{0,n})^{\mu_2}$ via the pullback, because equivariant Chow is just the invariant Chow cycles when looking at CH^0.

For an irreducible boundary divisor $D(A|B) \subseteq \overline{\mathcal{M}}_{0,n}$, we have $\pi^*(\widehat{D}(A|B))$ is $D(A|B)$ if a, b are in the same set and is $D(A|B)+D(A'|B')$ if there are in different sets, where A', B' are the same as A, B, but with a and b switched. We can say the same for $D(A|B)$ not necessarily irreducible, as long as $a, b \in A \cup B$, because the what happens on each irreducible boundary divisor making up $D(A|B)$ will be consistent.

Now, I claim

$$\partial_1^{\mu_2}(x_p^2 - 1) = \widehat{D}(1b|pa) - 2\widehat{D}(1p|ab)$$

$$\partial_1^{\mu_2}((x_p - x_q)(x_{p'} - x_{q'})) = \alpha_{p,q,1} + r_{p',q',1},$$

where $\alpha_{p,q,r}$ is as in the statement of the Theorem. This is true because these statements give exactly what we computed before for $\partial_1(x_p - 1), \partial_1(x_p + 1), \partial_1(x_p - x_q)$ when pulled back along π^*, and therefore hold by injectivity of π^*. □

Corollary 8.6. *There is a unique nontrivial* 2*-torsion element in* $\mathrm{CH}^1([\overline{\mathcal{M}}_{0.n}/\mu_2]) = \mathrm{Pic}([\overline{\mathcal{M}}_{0.n}/\mu_2])$ *given by*

$$\widehat{D}(23b|1a) + \widehat{D}(1ab|23) - \widehat{D}(123|ab) - \widehat{D}(3ab|12) - \widehat{D}(2ab|13)$$

Proof. The Theorem gives that this element is 2-torsion. It is nontrivial because help. It remains to show uniqueness.

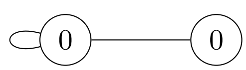
Torsion line bundles of order N are equivalent to μ_N covers. Because $\overline{\mathcal{M}}_{0,n}$ is a rational smooth projective variety, it has trivial étale fundamental group. Hence, we have $\pi_1^{\text{ét}}([\overline{\mathcal{M}}_{0,n}/\mu_2]) = \frac{\mathbb{Z}}{2\mathbb{Z}}$. Thus, there is a unique nontrivial 2-torsion line bundle. □

Proposition 8.7. *Most of this should be scrapped, and replaced with a general statement, belonging to the previous section. Suppose $\hat{D}(A|B) \subseteq [\overline{\mathcal{M}}_{0,n}/\mu_2]$ is an irreducible boundary divisor, and let $\overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{1,n-2}$ be the map that glues markings a and b. Then, the degree of $\hat{D}(A|B) \subseteq [\overline{\mathcal{M}}_{0,n}/\mu_2] \to \overline{\mathcal{M}}_{1,n-2}$ is 2 if markings a and b are on different components and 1 if they are on the same components.*

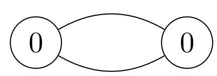
Proof. Note that the map $\overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{1,n-2}$ does descend to $[\overline{\mathcal{M}}_{0,n}]$, since switching the markings being glued does not affect the glued curve. We have the diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_{0,|A|+1} \times \overline{\mathcal{M}}_{0,|B|+1} \arrow[r] \arrow[dd] & \overline{\mathcal{M}}_{0,n} \arrow[d] \\
& {[\overline{\mathcal{M}}_{0,n}/\mu_2]} \arrow[d] \\
{[\overline{\mathcal{M}}_{0,A+1} \times \overline{\mathcal{M}}_{0,B+1} / \operatorname{Aut}(\Gamma)]} \arrow[r] & \overline{\mathcal{M}}_{1,n-2}
\end{tikzcd}$$

where the top map is the gluing map for $D(A|B) \subseteq \overline{\mathcal{M}}_{0,n}$, and the bottom map is the gluing map for the image of $D(A|B)$ under the gluing map $\overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{1,n-2}$, which has graph Γ. The graph Γ looks like one of



or



depending on whether a, b are in the same set or different set, respectively. By Theorem 7.2, the bottom map gives an isomorphism on an open subset. Regardless of the shape of Γ, Γ has two automorphisms, meaning the degree of the map diagonal map in the diagram is always two.

Suppose a, b are in the same set. Then the μ_2 action preserves $D(A|B)$, and is an automorphism of $D(A|B) \to \widehat{D}(A|B)$. When there is at least one more marking with a, b, this action is nontrivial, and so the degree of $D(A|B) \to \widehat{D}(A|B)$ is at least two. When a, b are alone on their component, the action is trivial, but, in the computation of equivariant Chow, we take a product with a space on which the action is free, and so we still have that the degree of $D(A|B) \to \widehat{D}(A|B)$ is at least two. When we compose $D(A|B) \to \widehat{D}(A|B)$ with $\widehat{D}(A|B) \to \overline{\mathcal{M}}_{1,n-2}$, we must get a morphism of degree 2. Hence, the degree of $\widehat{D}(A|B) \to \overline{\mathcal{M}}_{1,n-2}$ must be 1.

If a, b are in different sets, then $\overline{\mathcal{M}}_{0,|A|+1} \times \overline{\mathcal{M}}_{0,|B|+1} \to \widehat{D}(A|B)$ is bijective on closed points, as switching the markings moves you to a different divisor in $\overline{\mathcal{M}}_{0,n}$. This says that the degree of the morphism is 1 in characteristic 0, and some power of $\mathrm{char}(k)$ otherwise. But this degree must divide $2 =$

$\deg(D(A|B) \to \overline{\mathcal{M}}_{1,n-2}$, and $\mathrm{char}(k) \neq 2$, so this degree must be 1. Hence, the degree of $\widehat{D}(A|B) \to \overline{\mathcal{M}}_{1,n-2}$ is 2. □

Proposition 8.8. *Make this look more like the next one? The exact sequence*

$$\mathrm{CH}^*([\mathcal{M}_{0,4}/\mu_2],1)_{ind} \xrightarrow{\partial_1} \mathrm{CH}^{*-1}([\partial\overline{\mathcal{M}}_{0,4}/\mu_2]) \xrightarrow{\iota_*} \mathrm{CH}^*([\overline{\mathcal{M}}_{0,4}/\mu_2]) \to \mathrm{CH}^*([\mathcal{M}_{0,4}/\mu_2]) \to 0$$

can be identified with

$$\mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \oplus \frac{\mathbb{Z}[u]}{(2u)} \xrightarrow{\iota_*} \frac{\mathbb{Z}[v,t]}{(vt, 2(t-v))} \xrightarrow{q} \frac{\mathbb{Z}[t]}{(2t)} \to 0$$

where $v = [\hat{D}(1a|2b)]$, $t = [(\mathbb{P}^1, \infty, 0, 1, -1)]$, and the maps are given by

$$\partial(1) = (2,-1)$$
$$\iota_*(a,b) = (2a+b)v, \quad \iota_*(u^i) = v^{i+1}$$
$$q(v) = 0, \quad q(t) = t.$$

Proof. Help. □

Consider the exact sequence

$$\mathrm{CH}^{*-1}([\partial(\overline{\mathcal{M}}_{0,4}\times\overline{\mathcal{M}}_{0,4})/\mu_2]) \xrightarrow{\iota_*} \mathrm{CH}^*([\overline{\mathcal{M}}_{0,4}\times\overline{\mathcal{M}}_{0,4}/\mu_2]) \to \mathrm{CH}^*([\mathcal{M}_{0,4}\times\mathcal{M}_{0,4}/\mu_2]) \to 0.$$

Because $[\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}/\mu_2]$ is an open in $[\mathbb{A}^2/\mu_2]$ containing the origin, Lemma 8.2 implies its Chow ring is isomorphic to $\frac{\mathbb{Z}[t]}{(2t)}$.

Let $\pi_i : [\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,4}/\mu_2] \to [\overline{\mathcal{M}}_{0,4}/\mu_2]$ be the projections onto the ith factor. As $\partial\overline{\mathcal{M}}_{0,4} = \{D(12|ab), D(1a|2b), D(1b|2a)\}$, we have that $[\partial(\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,4})/\mu_2]$ has the 4 components

$$\pi_1^{-1}\widehat{D}(12|ab), \pi_1^{-1}\widehat{D}(1a|2b), \pi_2^{-1}\widehat{D}(12|ab), \pi_2^{-1}\widehat{D}(1a|2b).$$

Lemma 8.9. (1) *MKP*
(2) *t lifts to the 2-torsion class $\pi_1^*(t-v)$, and t^i lifts to $\pi_1^*(t^i)$ for $i \geq 2$.*
(3) *The kernel of ι_* is generated by* Compare with MKP

$$[\pi_1^{-1}\widehat{D}(1a|2b)] - 2[\pi_1^{-1}\widehat{D}(12|ab)], [\pi_2^{-1}\widehat{D}(1a|2b)] - 2[\pi_2^{-1}\widehat{D}(12|ab)]$$

9. Patching 2

9.1. $\overline{\mathcal{M}}_{1,1}$. The stack $\overline{\mathcal{M}}_{1,1}$ is isomorphic to weighted projective space $\mathbb{P}(4,6)$, and the computation of its integral Chow ring follows readily from this (cite EG). But, in order to illustrate our method, we give a computation based on the localization exact sequence for $\mathcal{M}_{1,1} \subseteq \overline{\mathcal{M}}_{1,1}$. This computation does require some outside input, namely Mumford's relation

$$12\lambda = [\partial\overline{\mathcal{M}}_{1,1}] \tag{9.1}$$

in the rational Chow ring $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1})\otimes\mathbb{Q}$, where λ is the first Chern class of the Hodge bundle (cite something).

Note $\partial\overline{\mathcal{M}}_{1,1} \cong B\mu_2$. Let $\lambda \in \mathrm{CH}^1(\overline{\mathcal{M}}_{1,1})$ be the first Chern class of the Hodge bundle.

Proposition 9.2. *For $\iota : \partial\overline{\mathcal{M}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$, we have $\iota^*(\lambda) = s$.*

Theorem 9.3. $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}[\lambda]/(24\lambda^2)$.

Proof. The complement of $\partial\overline{\mathcal{M}}_{1,1}$ is $\mathcal{M}_{1,1}$, giving a localization exact sequence

$$\mathrm{CH}^*(\mathcal{M}_{1,1},1)_{\mathrm{ind}} \to \mathrm{CH}^{*-1}(\partial\overline{\mathcal{M}}_{1,1}) \to \mathrm{CH}^*(\overline{\mathcal{M}}_1,1) \to \mathrm{CH}^*(\mathcal{M}_{1,1}) \to 0.$$

Using Theorem 4.3 and the fact that $\partial\overline{\mathcal{M}}_{1,1} \cong B\mu_2$, we get

$$0 \to \mathbb{Z}[s]/(2s) \xrightarrow{\iota_*} \mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \to \frac{\mathbb{Z}[\lambda]}{(12\lambda)} \to 0.$$

In degree 1, this is

$$0 \to \mathbb{Z} \xrightarrow{\iota_*} \mathrm{CH}^1(\overline{\mathcal{M}}_{1,1}) \to \frac{\mathbb{Z}}{12\mathbb{Z}} \to 0.$$

Mumford's relation (9.1) implies that $12\lambda = \iota_*(1)$ up to torsion. Becuase $12\lambda = 0$ on $\mathcal{M}_{1,1}$, we know we can write $12\lambda = \iota_*(a)$ for some $a \in \mathbb{Z}$, hence we have $\iota_*(a) - \iota_*(1) = \iota_*(a-1)$ is torsion. Because ι is injective, we need $a = 1$, and so $12\lambda = \iota_*(1)$. Hence, $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}$, and is generated by λ.

Now, I claim that $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1})$ is generated as a ring by λ. We know λ additively generates $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,1})$. The degree $n > 1$ part of the localization exact sequence gives

$$0 \to \frac{\mathbb{Z}}{2\mathbb{Z}} \to \mathrm{CH}^n(\overline{\mathcal{M}}_{1,1}) \to \frac{\mathbb{Z}}{12\mathbb{Z}}.$$

Note $\iota^*(\lambda) = s$ because of a computation. Hence

$$0 \neq \iota_*(s^{n-1}) = \iota_*\iota^*(\lambda^{n-1}) = 12\lambda^n. \tag{9.4}$$

Thus, the subgroup of $\mathrm{CH}^n(\overline{\mathcal{M}}_{1,1})$ generated by λ^n contains the image of ι_* and surjects onto $\mathrm{CH}^n(\mathcal{M}_{1,1})$, as $\lambda^n \mapsto t^n$, so this subgroup must be $\mathrm{CH}^n(\overline{\mathcal{M}}_{1,1})$.

As $2s = 0$, we have $0 = \iota_*(2s) = 24t^2$. Hence, we get a surjective graded ring homomorphism

$$\mathbb{Z}[\lambda]/(24\lambda^2) \to \mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}).$$

This must be an isomorphism, because it is in degrees $0, 1$, and in higher degrees, both groups have the same finite cardinality. ∎

We get some immediate concequences for $\overline{\mathcal{M}}_{1,n}$ using the map $\pi : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,1}$ forgetting all but the first point.

Proposition 9.5. (1) *The order of $\lambda^i \in \mathrm{CH}^*(\overline{\mathcal{M}}_{1,n})$ is 24.*

(2) *For $\overline{\mathcal{M}}_{1,n}^{\Phi} \subseteq \overline{\mathcal{M}}_{1,n}$ the curves with a non-separating node, $[\overline{\mathcal{M}}_{1,n}^{\Phi}] = 12\lambda$*

(3) *If $\mathrm{CH}^1(\mathcal{M}_{1,n},1)_{ind} = 0$, then λ has order 12 in $\mathrm{CH}^1(\mathcal{M}_{1,n})$.*

Proof. (1) Because λ^i has order 24 in $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1})$, the order of $\pi^*(\lambda^i) = \lambda^i \in \mathrm{CH}^*(\overline{\mathcal{M}}_{1,n})$ divides 24. Moreover, the map π has a section, s, given by attaching a fixed Genus 0 curve with $n+1$ point to each $(C,p) \in \overline{\mathcal{M}}_{1,1}$. And so λ^i must have order exactly 24.

(2) We have a commutative diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{1,n}^{\Phi} \arrow[r] \arrow[d, "\pi"] & \overline{\mathcal{M}}_{1,n} \arrow[d, "\pi"] \\ \partial\overline{\mathcal{M}}_{1,1} \arrow[r, "\iota"] & \overline{\mathcal{M}}_{1,1} \end{tikzcd}$$

This is Cartesian: set theoretically, this is clear, and the fiber product is reduced because $\pi : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,1}$ is log-smooth. Note that the vertical maps are flat and the horizontal maps are proper. Then, we have

$$12\lambda = \pi^*(12\lambda) = \pi^*(\iota_*(1)) = \iota_*(\pi^*(1)) = \iota_*(1) = [\overline{\mathcal{M}}_{1,n}^{\Phi}]$$

by push-pull, giving (1).

(3) Note that the above implies λ has order 12 in $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,n} \setminus \overline{\mathcal{M}}_{1,n}^{\Phi})$. If the order λ is not 12 in $\mathrm{CH}^1(\mathcal{M}_{1,n})$, then there must be some relation

$$a\lambda = \sum_{\delta} a_\delta \delta$$

with $a \neq 0$ and some a_δ nonzero, where the sum runs over the classes $\delta = [\Delta]$ of the components of $\partial\overline{\mathcal{M}}_{1,n}$ besides $\overline{\mathcal{M}}_{1,n}^{\Phi}$. Multiplying by 12 and taking a lift to $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,n})$, we get a nontrivial relation between the boundary divisors of $\overline{\mathcal{M}}_{1,n}$. This cannot happen if $\mathrm{CH}^1(\mathcal{M}_{1,n},1)_{\mathrm{ind}} = 0$.

□

9.2. $\overline{\mathcal{M}}_{1,2}$. We define the following stable graphs

$$\Delta :=$$

$$\Theta :=$$

$$\Phi :=$$

In general, for a stable graph Γ, necessarily an uppercase Greek letter, we let the corresponding lowercase Greek letter denote $\gamma := [\overline{\mathcal{M}}^{\Gamma}]$.

Recall $\mathrm{CH}^*(B\mu_2) = \frac{\mathbb{Z}[u]}{(2u)}$. Using Theorem 7.2, we have

$$\mathcal{M}^{\Phi} = [\mathcal{M}_{\Phi}/\operatorname{Aut}(\Phi)] = [\mathcal{M}_{0,4} \times \mathcal{M}_{0,3}/\mu_2] = [\mathcal{M}_{0,4}/\mu_2].$$

Let θ^i denote the pushforward of u^{i-1} to $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2})$ under $\mathcal{M}^{\Theta} \hookrightarrow \partial\overline{\mathcal{M}}_{1,2}$. We also have a point in $P \in \partial\overline{\mathcal{M}}_{1,2}$ isomorphic to $B\mu_2$ corresponding to the irreducible nodal curve whose second marked point is at the 2-torsion point. Let τ^i denote the pushforward of u^{i-1} to $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2})$ along $P \hookrightarrow \partial\overline{\mathcal{M}}_{1,2}$.

Warning: Although the notation suggests that θ^i, τ^i are powers of an element this is not the case. Moreover, when we refer to powers of λ on $\partial\overline{\mathcal{M}}_{1,2}$, these are pushed forward from $\overline{\mathcal{M}}^{\Delta}$. Because $\partial\overline{\mathcal{M}}_{1,2}$ is not smooth, $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2})$ has no product structure.

Proposition 9.6. *The Chow group* $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2})$ *is given by*

- $\mathrm{CH}^0(\partial\overline{\mathcal{M}}_{1,2})$ *is freely generated by* ϕ, δ
- $\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,2})$ *is generated by* λ, θ, τ *subject to the relations*

$$24\lambda - 2\theta$$

$$24\lambda - 2\tau$$

- $\mathrm{CH}^i(\partial\overline{\mathcal{M}}_{1,2})$ *is generated by* $\lambda^i, \theta^i, \tau^i$ *subject to the relations*

$$24\lambda^i$$

$$2\theta^i$$

$$2\tau^i$$

for $i \geq 2$.

Proof. There is only one component in $\overline{\mathcal{M}}^{\mathrm{sep},\geq 3} = \overline{\mathcal{M}}^{\mathrm{sep}}$, corresponding to the graph Δ. Thus, $\overline{\mathcal{M}}^{\mathrm{sep}} \cong \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,3} = \overline{\mathcal{M}}_{1,1}$ and $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) = \frac{\mathbb{Z}[\lambda]}{(24\lambda^2)}$. Moreover, $\overline{\mathcal{M}}^{\Theta} = \mathcal{M}^{\Theta} = [\mathcal{M}_{\Theta}/\operatorname{Aut}(\Theta) = B\mu_2$ as noted before, making $\mathrm{CH}^*(\mathcal{M}^{\Theta}) = \frac{\mathbb{Z}[u]}{(2u)}$. Because $\overline{\mathcal{M}}^{\mathrm{sep},\geq 2} = \mathcal{M}^{\Theta} \amalg \overline{\mathcal{M}}^{\mathrm{sep}}$ as spaces, we have

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) = \frac{\mathbb{Z}[u]}{(2u)} \oplus \frac{\mathbb{Z}[\lambda]}{(24\lambda^2)}.$$

Next, we consider $\partial\overline{\mathcal{M}}_{1,2} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 1} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 2} \cup \mathcal{M}^{\Phi}$. As noted above, $\mathcal{M}^{\Phi} = [\mathcal{M}_{0,4}/\mu_2]$. Moreover, $[\overline{\mathcal{M}}_{\Phi}/\operatorname{Aut}(\Phi)] = [\overline{\mathcal{M}}_{0,4}/\mu_2] \cong [\mathbb{P}^1/\mu_2]$ and $[\partial\overline{\mathcal{M}}_{\Phi}.4/\mu_2] = \operatorname{Spec}(k) \amalg B\mu_2$. The diagram (7.6) in this setting becomes

$$\begin{tikzcd}
\mathrm{CH}^*(\mathcal{M}_{0,4},1)_{\mathrm{ind}} \arrow[r] \arrow[d] & \mathrm{CH}^{*-1}([\partial\overline{\mathcal{M}}_{0,4}/\mu_2) \arrow[r] \arrow[d] & \mathrm{CH}^*([\overline{\mathcal{M}}_{0,4}/\mu_2]) \arrow[r] \arrow[d] & \mathrm{CH}^*([\mathcal{M}_{0,4}/\mu_2]) \arrow[r] \arrow[d] & 0 \\
\mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=1},1)_{\mathrm{ind}} \arrow[r] & \mathrm{CH}^{*-1}(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \arrow[r] & \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2}) \arrow[r] & \mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=1}) \arrow[r] & 0
\end{tikzcd}$$

Filling this out using the above and Theorem 8.8, we get

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\partial_1"] \arrow[d, "\sim"] & \mathbb{Z} \oplus \frac{\mathbb{Z}[u]}{(2u)} \arrow[r] \arrow[d] & \frac{\mathbb{Z}[v,t]}{(vt,2(t-v))} \arrow[r] \arrow[d] & \frac{\mathbb{Z}[t]}{(2t)} \arrow[r] \arrow[d, "\sim"] & 0 \\
\mathbb{Z} \arrow[r, "\partial_1'"] & \frac{\mathbb{Z}[u]}{(2u)} \oplus \frac{\mathbb{Z}[\lambda]}{(24\lambda^2)} \arrow[r] & \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2}) \arrow[r] & \frac{\mathbb{Z}[t]}{(2t)} \arrow[r] & 0
\end{tikzcd}$$

The map $[\overline{\mathcal{M}}_{0,4}/\mu_2] \to \partial\overline{\mathcal{M}}_{1,2}$ sends $\widehat{D}(12|ab)$ to the point in $\overline{\mathcal{M}}^{\Delta} \cap \overline{\mathcal{M}}^{\Phi}$ and $(\mathbb{P}^1, \infty, 0, 1, -1)$ to P. Thus, we get $v^i \mapsto 12\lambda^i$ and $t^i \mapsto \tau^i$.

By Theorem 8.4, we know that the image of ∂_1 is generated by $\widehat{D}(1a|2b) - 2\widehat{D}(12|ab)$, which then pushes forward to $2\theta - 24\lambda$. Also, by the diagram, the exact sequence

$$0 \to \left(\frac{\mathbb{Z}[u]}{(2u)} \oplus \frac{\mathbb{Z}[\lambda]}{(24\lambda^2)}\right)/\langle(2, -24\lambda)\rangle \to \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2}) \to \frac{\mathbb{Z}[t]}{(2t)} \to 0$$

has a splitting by lifting t to $\tau - 12\lambda$ and t^i to τ^i for $i \geq 2$. Putting this together, we get the desired presentation for $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2})$. □

Proposition 9.7. *The image of $\partial_1 : \mathrm{CH}^i(\mathcal{M}_{1,2}, 1)_{ind} \to \mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,2})$ is 0 for $i = 1$ and is generated by $\theta^{i-1} - \tau^{i-1}$ for $i > 1$.*

Proof. Help □

Let λ denote the first Chern class of the Hodge bundle on $\overline{\mathcal{M}}_{1,2}$. Let $\pi : \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{1,1}$ be the morphism forgetting the second point, and let $s : \overline{\mathcal{M}}_{1,1} \xrightarrow{\sim} \overline{\mathcal{M}}^{\Delta} \subseteq \overline{\mathcal{M}}_{1,2}$ be its section.

Theorem 9.8.

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}) = \mathbb{Z}[\lambda, \delta]/(24\lambda^2, \delta(\lambda + \delta))$$

and $\iota_*(\theta^i) = 12\lambda^i(\delta - \lambda)$.

First, we state and prove a lemma that lets us turn an aditive presentation of a ring into a multiplicitive one:

Lemma 9.9. *Suppose $\phi : A \twoheadrightarrow B$ is a surjective homomorphism of abelian groups, and $S \subseteq B$ is a generating subset. For each $s \in S$, choose a lift $\tilde{s} \in A$, and let $\widetilde{S}$ be the set of these lifts. Then ϕ is an isomorphism if both*

(1) *$\widetilde{S}$ generates A and*
(2) *every relations satisfied by elements of S is satisfied by their lifts in $\widetilde{S}$.*

Proof. easy. □

Proof of Theorem 9.8. The localization exact sequence on $\partial\overline{\mathcal{M}}_{1,2} \subseteq \overline{\mathcal{M}}_{1,2}$ is

$$\mathrm{CH}^*(\mathcal{M}_{1,2}, 1)_{\mathrm{ind}} \xrightarrow{\partial_1} \mathrm{CH}^{*-1}(\partial\overline{\mathcal{M}}_{1,2}) \to \mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}) \to \mathrm{CH}^*(\mathcal{M}_{1,2}) \to 0.$$

The terms besides $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2})$ were computed in Theorem 4.6 and Theorem 9.6. In degree 0, it is clear that $\mathrm{CH}^0(\overline{\mathcal{M}}_{1,2}) = \mathbb{Z}$.

In degree 1, the localization exact sequence is is

$$0 \to \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{1,2}) \xrightarrow{\iota_*} \mathrm{CH}^1(\overline{\mathcal{M}}_{1,2}) \to \frac{\mathbb{Z}}{12\mathbb{Z}} \to 0.$$

By Proposition 9.5, we have $\iota_*(\phi) = 12\lambda$. Thus, we have $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,2})$ is freely generated by λ, δ.

In degree $i > 1$, the localization exact sequence gives

$$0 \to \mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,2})/\langle \tau^i - \theta^i\rangle \xrightarrow{\iota_*} \mathrm{CH}^i(\overline{\mathcal{M}}_{1,2}) \to \mathrm{CH}^i(\mathcal{M}_{1,2}) \to 0.$$

Because $12\lambda^i = 0$, we know $12\lambda^i = \iota_*(\alpha)$, for some $\alpha \in \mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,2})$. The order of $\lambda^i \in \mathrm{CH}^i(\overline{\mathcal{M}}_{1,2})$ is 24, by Proposition 9.5, so α must have order 2 in $\mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,2})/\langle\tau^i - \theta^i\rangle$. So $\alpha \in \{12\lambda^{i-1}, \theta^{i-1}, 12\lambda^{i-1} - \theta^{i-1}\}$. Note $\pi_*(\lambda^i) = \pi_*(\pi_*(\lambda^i)) = \lambda^i\pi_*(1) = 0$. If $\alpha = 12\lambda^{i-1}$, then

$$12\lambda^i = \iota_*(\alpha) = \iota_*(12\lambda^{i-1}) = s_*(12\lambda^{i-1}) \implies 12\lambda^{i-1} = 12\pi_*(\lambda^i) = 0,$$

which is not true in $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1})$. If $\alpha = \theta^{i-1}$, then we get $\pi_*(\iota_*(\theta^{i-1})) = \pi_*(12\lambda^i) = 0$. But θ^{i-1} is the pushforward of u^{i-2} from $\overline{\mathcal{M}}^{\Theta} \cong B\mu_2$, and the composition $B\mu_2 \cong \overline{\mathcal{M}}^{\Theta} \to \overline{\mathcal{M}}_{1,2} \xrightarrow{\pi} \overline{\mathcal{M}}_{1,1}$ is the same as the inclusion $\partial\overline{\mathcal{M}}_{1,1} \hookrightarrow \overline{\mathcal{M}}_{1,1}$, and we know the pushforward under this map is nonzero, giving a contradiction. Thus, we must have $\alpha = 12\lambda^{i-1} - \theta^{i-1}$, so

$$12\lambda^i = \iota_*(12\lambda^{i-1} - \theta^{i-1}) = 12s_*(\lambda^{i-1}) - \iota_*(\theta^{i-1}).$$

By the projection formula, we have

$$s_*(\lambda^{i-1}) = s_*(s^*(\lambda^{i-1})) = \lambda^{i-1}s_*(1) = \lambda^{i-1}\delta.$$

And so we can write $\iota_*(\theta^i) = 12\lambda^i(\delta - \lambda)$. By the exact sequence we have $\mathrm{CH}^i(\overline{\mathcal{M}}_{1,2})$ for $i > 1$ is generated by λ^i and $\lambda^{i-1}\delta$ modulo the relations $24\lambda^i, 24\lambda^{i-1}\delta$ for $i > 2$ and $24\lambda^2$ for $i = 2$. Possibly explain more? Write out $0 \to \langle S|R\rangle \to A \to \langle T|U\rangle \to 0 \implies A = \langle S, \tilde{T}|R, \tilde{U}\rangle$?

Next, note that the first Chern class of normal bundle to $\overline{\mathcal{M}}_{1,1}$ in $\overline{\mathcal{M}}_{1,2}$ is given by $-\psi_1 = -\lambda$. Then we have $-\lambda = s^*(\delta)$, which gives

$$-\lambda\delta = s_*(-\lambda) = s_*(s^*(\delta)) = \delta^2.$$

Thus, we have a surjection

$$\mathbb{Z}[\lambda, \delta]/(24\lambda^2, \delta^2 + \delta\lambda) \to \mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}).$$

By Lemma 9.9, it is enough to show that the additive generators we found for $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2})$ above both generate $\mathbb{Z}[\lambda, \delta]/(24\lambda^2, \delta^2 + \delta\lambda)$ and satisfy the same relations they do in $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2})$. We know that monomials of the form $\lambda^i, \lambda^{i-1}\delta$ additively generate $\mathbb{Z}[\lambda, \delta]/(24\lambda^2, \delta^2 + \delta\lambda)$ because we can repeatedly use the relation $\delta^2 + \delta\lambda$ to replace any monomial $\delta^a\lambda^b$ with $\pm\delta\lambda^{a+b-1}$ if $a > 0$. Moreover, the relation $24\lambda^2$ implies all of the additive relations $24\lambda^2$, $24\lambda^i, 24\lambda^{i-1}\delta$ for $i > 3$ in $\mathbb{Z}[\lambda, \delta]/(24\lambda^2, \delta^2 + \delta\lambda)$, and so we are done. $\square$

9.3. $\overline{\mathcal{M}}_{1,3}$**.** Define the graphs $\Delta_1, \Delta_2, \Delta_3, \Delta_\emptyset, \Theta_1, \Theta_2, \Theta_3, \Omega, \Delta_{\emptyset,j}$.

By Theorem 7.4, we have

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_j}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}) = \mathbb{Z}[\lambda_j, \delta_{\emptyset,j}]/(24\lambda_j^2, \delta_{\emptyset,j}^2 + \delta_{\emptyset,j}\mu_j)$$

for $j \in \{1, 2, 3\}$ and

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_\emptyset}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \otimes_{\mathbb{Z}} \mathrm{CH}^*(\overline{\mathcal{M}}_{0,4}) = \mathbb{Z}[\lambda_\emptyset, \delta_{\emptyset,1}]/(24\lambda_\emptyset^2, \delta_{\emptyset,1}^2),$$

using the CKP for $\overline{\mathcal{M}}_{0,4}$.

By Theorem 7.2, we have

$$\mathcal{M}^{\Theta_j} = [\mathcal{M}_{\Theta_j}/\operatorname{Aut}(\Theta_j)] = [\mathcal{M}_{0,4} \times \mathcal{M}_{0,3}/\mu_2] = [\mathcal{M}_{0,4}/\mu_2].$$

There are unique points $P_j \in \mathcal{M}^{\Theta_j} \cong [\mathcal{M}_{0,4}/\mu_2]$ isomorphic to $B\mu_2$. Set τ_j^i to be the pushforward of u^{i-1} to $\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}$ along $P_j \hookrightarrow \overline{\mathcal{M}}^{\mathrm{sep},\geq 2}$.

Theorem 9.10. *The Chow group* $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,3})$ *is given by*

- $\mathrm{CH}^0(\partial\overline{\mathcal{M}}_{1,3})$ *is freely generated by*
$$\phi, \delta_1, \delta_2, \delta_3, \delta_\emptyset$$
- $\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})$ *is generated by*
$$\lambda_1, \lambda_2, \lambda_3, \lambda_\emptyset, \theta_1, \theta_2, \theta_3, \delta_{\emptyset,1}$$
subject to the relations
$$2\theta_1 + 2\theta_2 - 24\lambda_\emptyset - 24\lambda_3$$
$$2\theta_1 + 2\theta_3 - 24\lambda_\emptyset - 24\lambda_2$$
$$4\theta_1 - 24\lambda_\emptyset - 24\lambda_2 - 24\lambda_3 + 24\lambda_1$$
- $\mathrm{CH}^2(\partial\overline{\mathcal{M}}_{1,3})$ *is generated by*
$$\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_\emptyset^2, \lambda_\emptyset\delta_{\emptyset,1}, \tau_1, \tau_2, \tau_3$$
subject to the relations
$$24\lambda_\ell^2 = 0 \textit{ for } \ell \in \{1,2,3,\emptyset\}$$
$$2\tau_\ell = 24\delta_{\emptyset,1}\lambda_\emptyset \textit{ for } \ell \in \{1,2,3\}$$
- *For* $i \geq 3$*,* $\mathrm{CH}^i(\partial\overline{\mathcal{M}}_{1,3})$ *is generated by*
$$\lambda_1^i, \lambda_2^i, \lambda_3^i, t\lambda_\emptyset^{i-1}, \tau_1^{i-1}, \tau_2^{i-1}, \tau_3^{i-1}$$
subject to the relations
$$24\lambda_\ell^i = 0 \textit{ for } \ell \in \{1,2,3,\}$$
$$24\delta_{\emptyset,1}\lambda_\emptyset^{i-1} = 0$$
$$2\tau_\ell^{i-1} = 0 \textit{ for } \ell \in \{1,2,3\}$$

Moreover, $\omega = 2\tau_j$ *for* $j = 1,2,3$*.*

Proof. The components of $\overline{\mathcal{M}}^{\mathrm{sep}.\geq 4} = \overline{\mathcal{M}}^{\mathrm{sep}}$ are given by $\overline{\mathcal{M}}^{\Delta_1}, \overline{\mathcal{M}}^{\Delta_2}, \overline{\mathcal{M}}^{\Delta_3}, \overline{\mathcal{M}}^{\Delta_\emptyset}$. We have

$$\overline{\mathcal{M}}^{\Delta_j} \cap \overline{\mathcal{M}}^{\Delta_k} = \emptyset \text{ for } i \neq j$$
$$\overline{\mathcal{M}}^{\Delta_j} \cap \overline{\mathcal{M}}^{\Delta_\emptyset} = \overline{\mathcal{M}}^{\Delta_{\emptyset,j}}.$$

Thus, we have an exact sequence

$$\bigoplus_{j\in\{1,2,3\}} \mathrm{CH}(\overline{\mathcal{M}}^{\Delta_{\emptyset,j}}) \to \bigoplus_{j\in\{1,2,3,\emptyset\}} \mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_j}) \to \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \to 0.$$

By Theorem 7.4, we have $\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_{\emptyset,j}}) \cong \mathrm{CH}^*(\overline{\mathcal{M}}_{1,1})$. The induced push-forward maps are given by

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \to \mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_j})$$

$$\lambda^i \mapsto \delta_{\emptyset,j}\lambda_j^i$$

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \to \mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_\emptyset})$$

$$\lambda^i \mapsto \delta_{\emptyset,1}\lambda_\emptyset^i.$$

Thus, the exact sequence gives

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) = \bigoplus_{j\in\{1,2,3,\emptyset\}} \mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_j})/\langle\{\delta_{\emptyset,1}\lambda_\emptyset^i - \delta_{\emptyset,j}\lambda_j^i\}_{j=1,2,3}\rangle.$$

Because $\overline{\mathcal{M}}^{\mathrm{sep},\geq 3} = \overline{\mathcal{M}}^{\Omega} \amalg \overline{\mathcal{M}}^{\mathrm{sep}}$ as spaces, we have $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) = \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \oplus \langle\omega\rangle$.

Next, we consider $\overline{\mathcal{M}}^{\mathrm{sep},\geq 2} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 3} \cup \mathcal{M}^{\mathrm{non}=2}$. We have $\mathcal{M}^{\mathrm{non}=2} = \mathcal{M}^{\Theta_1} \amalg \mathcal{M}^{\Theta_2} \amalg \mathcal{M}^{\Theta_3}$ and, as noted above, $\mathcal{M}^{\Theta_j} = [\mathcal{M}_{0,4}/\mu_2]$. The diagram (7.6) for $\Gamma = \Theta_j$ becomes

$$\begin{tikzcd}
\mathrm{CH}^*([\mathcal{M}_{0,4}/\mu_2],1)_{\mathrm{ind}} \arrow[r] \arrow[d] & \mathrm{CH}^{*-1}([\partial\overline{\mathcal{M}}_{0,4}/\mu_2]) \arrow[r] \arrow[d] & \mathrm{CH}^*([\overline{\mathcal{M}}_{0,4}/\mu_2]) \arrow[r] \arrow[d] & \mathrm{CH}^*([\mathcal{M}_{0,4}/\mu_2]) \arrow[r] \arrow[d] & {} \\
\mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=2},1)_{\mathrm{ind}} \arrow[r] & \mathrm{CH}^{*-1}(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) \arrow[r] & \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \arrow[r] & \mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=2}) \arrow[r] & {}
\end{tikzcd}$$

Filling this out using the above and Theorem 8.8, we get

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\partial_1"] \arrow[d, "\sim"] & \mathbb{Z} \oplus \frac{\mathbb{Z}[u]}{(2u)} \arrow[r] \arrow[d] & \frac{\mathbb{Z}[v_j,t_j]}{(v_jt_j,2(t_j-v_j))} \arrow[r] \arrow[d] & \frac{\mathbb{Z}[t_j]}{(2t_j)} \arrow[r] \arrow[d, "\sim"] & 0 \\
\mathbb{Z} \arrow[r, "\partial_1'"] & \mathbb{Z} \oplus \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) \arrow[r] & \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \arrow[r] & \mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=2}) \arrow[r] & 0
\end{tikzcd}$$

Using the formula for $\iota_*(\theta^i)$ from Theorem 9.8, we see $v_j^i \mapsto 12\lambda_j^i(\delta_{\emptyset,j}-\lambda_j)$. Additionally, the map $[\overline{\mathcal{M}}_{0,4}/\mu_2] \to \partial\overline{\mathcal{M}}_{1,3}$ sends $(\mathbb{P}^1,\infty,0,1,-1)$ to P_j, so we have $t_j^i \mapsto \tau_j^i$.

By Theorem 8.4, we know that the image of ∂_1 is generated by $\widehat{D}(1a|2b) - 2\widehat{D}(12|ab)$, which pushes forward to $\omega - 24\lambda_j(\delta_{\emptyset,j}-\lambda_j)$. Also, by the diagram, the exact sequence

$$0 \to \frac{\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3})}{\langle 2(\omega - 12\lambda_j(\mu_j-\lambda_j))\rangle} \to \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \to \bigoplus_{j=1}^{3} \frac{\mathbb{Z}[t_j]}{(2t_j)} \to 0$$

has a splitting by lifting t_j to $\tau_j - 12\lambda_j(\mu_j - \lambda_j)$ and t_j^i to τ_j^i for $i \geq 2$. From this, we can conclude

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) = \frac{\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) \oplus \mathbb{Z}\left\langle \{\theta_j, \tau_j^i\}_{i\geq 1}\right\rangle}{\langle \{2(\tau_j - 12\lambda_j(\mu_j - \lambda_j))\}, \{2\tau_j^i\}_{i\geq 2}, \{\omega - 24\lambda_j(\delta_{\emptyset,j} - \lambda_j)\}\rangle}.$$

But because and $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) = \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \oplus \langle \omega \rangle$, we can write

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) = \frac{\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \oplus \mathbb{Z}\left\langle \{\theta_j, \tau_j^i\}_{i\geq 1}\right\rangle}{\langle \{2(\tau_j - 12\lambda_j(\mu_j - \lambda_j))\}, \{2\tau_j^i\}_{i\geq 2}\rangle},$$

and we remember that $\omega = 2\tau_j$ in $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2})$.

Finally, we consider $\partial\overline{\mathcal{M}}_{1,3} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 1} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 2} \cup \mathcal{M}^{\mathrm{non}=1}$. We have $\mathcal{M}^{\mathrm{non}=1} = M^{\Phi}$ and by Theorem 7.2, we have

$$\mathcal{M}^{\Phi} = [\mathcal{M}_{\Phi}/\operatorname{Aut}(\Phi)] = [\mathcal{M}_{0,5}/\mu_2].$$

The diagram (7.6) becomes

$$\begin{tikzcd}
\mathrm{CH}^*(\mathcal{M}_{0,5},1)_{\mathrm{ind}} \arrow[r, "\partial_1"] \arrow[d] & \mathrm{CH}^{*-1}([\partial\overline{\mathcal{M}}_{0,5}/\mu_2) \arrow[r] \arrow[d] & \mathrm{CH}^*([\overline{\mathcal{M}}_{0,5}/\mu_2]) \arrow[r] \arrow[d] & \mathrm{CH}^*([\mathcal{M}_{0,5}/\mu_2]) \arrow[r] \arrow[d] & 0 \\
\mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=1},1)_{\mathrm{ind}} \arrow[r, "\partial_1"] & \mathrm{CH}^{*-1}(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \arrow[r] & \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,3}) \arrow[r] & \mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=1}) \arrow[r] & 0
\end{tikzcd}$$

By Theorem 8.3, we have $\mathrm{CH}^*([\mathcal{M}_{0,5}/\mu_2]) = \mathbb{Z}$. We can then split the bottom sequence by lifting $[\mathcal{M}^{\Phi}]$ to $[\overline{\mathcal{M}}^{\Phi}] = \phi$. By Theorem 8.4, the image of ∂_1 is generated by

$$\widehat{D}(1a|2b) - 2\widehat{D}(12|ab)$$
$$\widehat{D}(1a|3b) - 2\widehat{D}(13|ab)$$
$$2(\widehat{D}(23b|1a) + \widehat{D}(1ab|23) - \widehat{D}(123|ab) - \widehat{D}(3ab|12) - \widehat{D}(2ab|13)).$$

By Theorem 8.7, these push forward to

$$2\theta_1 + 2\theta_2 - 24\lambda_3 - 24\lambda_\emptyset$$
$$2\theta_1 - 2\theta_3 - 24\lambda_2 - 24\lambda_\emptyset$$
$$2(2\theta_1 + 12\lambda_1 - 12\lambda_\emptyset - 12\lambda_3 - 12\lambda_2))$$

respectively, on $\partial\overline{\mathcal{M}}_{1,3}$. Thus, the quotient of $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \oplus \mathbb{Z}\langle [\phi]\rangle$ by these relations is $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$. Unpacking this expression, we get the description stated in the theorem. □

Proposition 9.11. *The image of* $\partial_1 : \mathrm{CH}^i(\mathcal{M}_{1,3},1)_{ind} \to \mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,3})$ *contains*

$$\theta_1 + 12\lambda_1 - \theta_2 - 12\lambda_2$$
$$\theta_1 + 12\lambda_1 - \theta_3 - 12\lambda_3$$

in degree 2 *and is generated by*

$$\tau_1^i - \tau_2^i$$

$$\tau_1^i - \tau_3^i$$

in degree $i+2$ *for* $i \geq 1$.

We will need the following lemma to prove this proposition:

Lemma 9.12. *Suppose we have stacks* X, Y, Z *with* Z *smooth and morphisms* $f : X \to Y$, $g : Y \to Z$ *so that* f *is proper, birational and* $g, f \circ g$ *are flat. Then* $g^* = f_* \circ (f \circ g)^*$

Proof. We use operational Chow rings (reference Fulton). Because Z is smooth, any element $\alpha \in \mathrm{CH}(Z)$ may be written as $\alpha^\vee \cap [Z]$ for some $\alpha^\vee \in \mathrm{opCH}^*(Z)$. Then, we have

$$(g \circ f)^*(\alpha) = (g \circ f)^*(\alpha^\vee \cap [Z]) = f^* g^*(\alpha^\vee) \cap [X].$$

Using the projection formula and the fact that f is birational, this gives

$$\begin{aligned} f_*((g \circ f)^*(\alpha)) &= f_*(f^* g^*(\alpha^\vee) \cap [X]) \\ &= g^*(\alpha^\vee) \cap f_*([X]) \\ &= g^*(\alpha^\vee) \cap [Y] \\ &= g^*(\alpha^\vee \cap [Z]) \\ &= g^*(\alpha). \end{aligned}$$

□

This should be reworked to use Proposition 4.15. And some notions need to be updated to deal with the changes made to the previous, like the γ should really be τ.

Proof of Proposition 9.11. Consider $\overline{\mathcal{M}}_{1,2}^{\Theta} \subseteq \overline{\mathcal{M}}_{1,2}$. We have $\pi^{-1}(\overline{\mathcal{M}}_{1,2}^{\Theta}) = \overline{\mathcal{M}}_{1,3}^{\Theta_1} \cup \overline{\mathcal{M}}_{1,3}^{\Theta_2}$. As $\overline{\mathcal{M}}_{1,2}^{\Theta} \cong B\mu_2$, we have $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}^{\Theta}) = \mathbb{Z}[s]/(2s)$. We compute the flat pullbacks $\pi^*(s^i) \in \mathrm{CH}^i(\overline{\mathcal{M}}_{1,3}^{\Theta_1} \cup \overline{\mathcal{M}}_{1,3}^{\Theta_2})$. When $i = 0$, we have $\pi^*(1) = [\overline{\mathcal{M}}_{1,3}^{\Theta_1}] + [\overline{\mathcal{M}}_{1,3}^{\Theta_2}]$. As noted above, $\overline{\mathcal{M}}_{1,3}^{\Theta_i} \cong [\mathbb{P}^1/\mu_2]$, and has Chow ring

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3}^{\Theta_\ell}) = \mathbb{Z}[\theta_{\ell,\emptyset}, \tau_\ell]/(2(\theta_{\ell,\emptyset} - \tau_\ell), \tau_\ell \cdot \theta_{\ell,\emptyset}).$$

For $i \geq 1$, the pullback of s^i to $\overline{\mathcal{M}}_{1,3}^{\Theta_\ell}$ is $\theta_{\ell,\emptyset}^i - \tau_\ell^i$. By the next lemma, considering the map $\overline{\mathcal{M}}_{1,3}^{\Theta_1} \amalg \overline{\mathcal{M}}_{1,3}^{\Theta_2} \to \overline{\mathcal{M}}_{1,3}^{\Theta_1} \cup \overline{\mathcal{M}}_{1,3}^{\Theta_2}$, we have that the pullback of s^i is

$$\theta_{1,\emptyset}^i - \tau_1^i + \theta_{2,\emptyset}^i - \tau_2^i = 12\lambda_1^i(\mu_1 - \lambda_1) - \tau_1^i + 12\lambda_\ell^i(\mu_\ell - \lambda_\ell) - \tau_\ell^i$$

Next, consider $U \subseteq \overline{\mathcal{M}}_{1,2}$ which consists only of the point corresponding to a nodal cubic curve, with the marked point being 2-torsion. Then $U \cong B\mu_2$, so $\mathrm{CH}^*(U) \cong \mathbb{Z}[s]/(2s)$. We have the following pull back diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & {[C/\mu_2]} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{1,3} \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & B\mu_2 \arrow[r] & \overline{\mathcal{M}}_{1,2} \end{tikzcd}$$

The two smooth points on C fixed by the μ_2 action are $\sigma_1(U)$ and $\sigma_2(U)$. The inclusion of $\sigma_i(U)$ into $\partial\overline{\mathcal{M}}_{1,3}$ induces a pushforward

$$h_\ell : \mathbb{Z}[s]/(2s) \to \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,3}).$$

By help, for $\ell = 1, 2$, we have $h_\ell(s^i) = \lambda_k^i(\mu_k - \lambda_k)$, where $k = 3 - \ell$. We compute the flat pullback $\pi^*(s^\ell)$. Of course $\pi^*(1) = [C]$. We have the normalization map $\mathbb{P}^1 \to C$, and this map is equivariant with respect to the μ_2 actions. Using the next lemma, we have that the pullback of s^i for $i \geq 1$ is given by

$$h_2(s^{i-1}) - h_1(s^{i-1}) = 12\lambda_1^i(\mu_1 - \lambda_1) - 12\lambda_2^i(\mu_2 - \lambda_2).$$

Now, on $\overline{\mathcal{M}}_{1,2}$, we have that the two maps $B\mu_2 \to \overline{\mathcal{M}}_{1,2}$ corresponding to the points $\overline{\mathcal{M}}_{1,2}^{\Theta}, U$ induce the same pushforward, by help. Thus, on $\overline{\mathcal{M}}_{1,3}$, we should have that these pullbacks of s^i agree, giving

$$[\overline{\mathcal{M}}^{\Theta_1}] + [\overline{\mathcal{M}}^{\Theta_2}] = [C]$$

for $i = 0$ and

$$12\lambda_1^i(\mu_1 - \lambda_1) - \tau_1^i + 12\lambda_2^i(\mu_2 - \lambda_2) - \tau_2^i = 12\lambda_1^i(\mu_1 - \lambda_1) - 12\lambda_2^i(\mu_2 - \lambda_2),$$

or

$$\tau_1^i + \tau_2^i = 24\lambda_2^i\mu_2 = 24t\lambda_\emptyset^i$$

for $i \geq 1$. Using the equality $2\tau_1^i = 24t\lambda_\emptyset^i$, we get $\tau_1^i = \tau_2^i$.

We can compute $[C] \in \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,3})$. Consider the diagram

$$\begin{tikzcd} {[\overline{\mathcal{M}}_{0,5}/\mu_2]} \arrow[r] \arrow[d] & \partial\overline{\mathcal{M}}_{1,3} \arrow[d, "\pi"] \\ {[\overline{\mathcal{M}}_{0,4}/\mu_2]} \arrow[r] & \partial\overline{\mathcal{M}}_{1,2}. \end{tikzcd}$$

We defined C as the fiber π over U. Lift U to a point in $\overline{\mathcal{M}}_{0,4}$, and consider C', the fiber of $\overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,4}$ over this lift. Because $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, all fibers have the same class, and so $D(12|3ab) + D(123|ab)$, the class of the fiber over $D(12|ab) \in \overline{\mathcal{M}}_{0,4}$, is equal to $[C']$. Moreover, if $\widehat{C'}$ is the image of C' in $[\overline{\mathcal{M}}_{0,5}/\mu_2]$, then we have $[\widehat{C'}] = \widehat{D}(12|3ab) + \widehat{D}(123|ab)$, since each of these varities is μ_2 invariant. Pushing forward this to $\partial\overline{\mathcal{M}}_{1,3}$ using Proposition 8.7 we get

$$[C] = 2\theta_1 + 12\lambda_1 - 12\lambda_2$$

Thus,

$$\theta_1 + \theta_2 = [C] = 2\theta_1 + 12\lambda_1 - 12\lambda_2,$$

or

$$\theta_1 + 12\lambda_1 = \theta_2 + 12\lambda_2$$

on $\overline{\mathcal{M}}_{1,3}$.

We obtain more relations on $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$ by pulling back along $\pi' : \overline{\mathcal{M}}_{1,3} \to \overline{\mathcal{M}}_{1,2}$, which forgets the second point, or, equivalently switching the second and third marked point in the relations above. This gives

$$\theta_1 + 12\lambda_1 = \theta_3 + 12\lambda_3$$

and

$$\tau_1^i = \tau_3^i$$

for $i \geq 1$. By exactness of

$$\mathrm{CH}^{+1}(\mathcal{M}_{1,3},1)_{\mathrm{ind}} \to \mathrm{CH}^i(\partial\overline{\mathcal{M}}_{1,3}) \to \mathrm{CH}^{i+1}(\overline{\mathcal{M}}_{1,3}),$$

the fact that these relations hold on $\mathrm{CH}^i(\overline{\mathcal{M}}_{1,3})$ but not on $\mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,3})$ imply that the elements

$$\theta_1 + 12\lambda_1 = \theta_2 + 12\lambda_2$$

$$\theta_1 + 12\lambda_1 = \theta_3 + 12\lambda_3$$

are in $\mathrm{im}(\mathrm{CH}^1(\mathcal{M}_{1,3}) \xrightarrow{\partial_1} \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{1,3}))$ and

$$\tau_1^i - \tau_2^i$$

$$\tau_1^i - \tau_3^i$$

are in $\mathrm{im}(\mathrm{CH}^i(\mathcal{M}_{1,3}) \to \mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,3}))$ for $i \geq 2$. When $i \geq 2$, because $\tau_1^i - \tau_2^i, \tau_1^i - \tau_3^i$ are independent and

$$\mathrm{CH}^i(\mathcal{M}_{1,3},1)_{\mathrm{ind}} = (\frac{\mathbb{Z}}{2\mathbb{Z}})^2$$

we have that $\tau_1^i - \tau_2^i, \tau_1^i - \tau_3^i$ generate the image of ∂_1 in $\mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,3})$. □

Theorem 9.13. $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3}) = \mathbb{Z}[\delta_1, \delta_2, \delta_3, \delta_\emptyset, \lambda]/I$, *where* I *is the ideal generated by the elements*

$$24\lambda^2$$

$$\delta_\ell\delta_k,\ \textit{for } \ell \neq k \in \{1,2,3\}$$

$$\delta_\ell\delta_\emptyset - \delta_k\delta_\emptyset,\ \textit{for } \ell \neq k \in \{1,2,3\}$$

$$\delta_\ell^2 + \lambda\delta_\ell + \delta_1\delta_\emptyset,\ \textit{for } \ell \in \{1,2,3,\emptyset\}$$

$$12\lambda^3 - 12\lambda^2\delta_1 + 12\lambda^2\delta_2 + 12\lambda^2\delta_3 + 12\lambda^2\delta_\emptyset.$$

Moreover, we have

$$\iota_*(\theta_j) = \pm 6\lambda^2 - 6\lambda\delta_j + 6\lambda\delta_k + 6\lambda\delta_\ell + 6\lambda\delta_\emptyset$$

for $\{j,k,\ell\}=[3]$, *and*

$$\iota_*(\tau_j^i) = 12\lambda^{i-2}\delta_1\delta_\emptyset \pm 6\lambda^i + 6\lambda^{i-1}\delta_1 + 6\lambda^{i-1}\delta_2 + 6\lambda^{i-1}\delta_3 + 6\lambda^{i-1}\delta_\emptyset.$$

Proof. We have the exact sequence

$$0 \to \mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,3})/\mathrm{im}(\partial) \xrightarrow{\iota_*} \mathrm{CH}^i(\overline{\mathcal{M}}_{1,3}) \to \mathrm{CH}^i(\mathcal{M}_{1,3}) \to 0.$$

In degree $i=1$, this is

$$0 \to \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{1,3}) \to \mathrm{CH}^1(\overline{\mathcal{M}}_{1,3}) \to \mathrm{CH}^1(\mathcal{M}_{1,3}) \to 0.$$

By Theorem 6.1, we know $\mathrm{CH}^1(\mathcal{M}_{1,3})$ is cyclic of order 12, generated by λ. By Theorem 9.10, $\mathrm{CH}^0(\partial\overline{\mathcal{M}}_{1,3})$ is freely generated by $\phi,\delta_1,\delta_2,\delta_3,\delta_\emptyset$. By Proposition 9.5, $\phi = 12\lambda$. Thus, $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,3})$ is freely generated by $\lambda,\delta_1,\delta_2,\delta_3,\delta_\emptyset$.

Note

$$\lambda^i\delta_\ell = \lambda^i\iota_*(1) = \iota_*(\iota^*(\lambda^i)) = \iota_*(\lambda^i_\ell)$$

for $\ell\in\{1,2,3,\emptyset\}$, where $\iota : \overline{\mathcal{M}}^{\Delta_\ell} \to \overline{\mathcal{M}}_{1,3}$. Moreover, we have that $\iota_*(t\lambda^i_\emptyset) = \lambda^i\delta_\emptyset\delta_\ell$, for any $\ell\in\{1,2,3\}$, because...

In degree $i=2$, the localization exact sequence is

$$0 \to \mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})/\mathrm{im}(\partial) \to \mathrm{CH}^2(\overline{\mathcal{M}}_{1,3}) \to \mathrm{CH}^2(\mathcal{M}_{1,3}) \to 0.$$

By Theorem 6.1, we know $\mathrm{CH}^2(\mathcal{M}_{1,3})$ is generated by λ^2, which has order 6. We have not computed the full image of ∂_1 in $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,3})$. Proposition 9.11 says that the two torsion elements

$$\theta_1 + 12\lambda_1 - \theta_2 - 12\lambda_2$$

$$\theta_1 + 12\lambda_1 - \theta_3 - 12\lambda_3$$

in $\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})$ are contained in the image. Suppose the image of ∂_1 in $\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})$ contained a non-torsion element. We saw that $\overline{\mathcal{M}}_{1,3}$ has the MCKP, and hence the rational CKgP. This gives, by Canning-Larson that Chow is isomorphic to cohomology, rationally. As we just computed, $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,3})$ has rank 5, and $\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})$ also has rank 5. Thus, a nontorsion element in the image of ∂_1 would decreases the rank of $\mathrm{CH}^2(\overline{\mathcal{M}}_{1,3})$ to 4, violating Poincare Duality.

Now, a computation with Smith normal form gives

$$\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})/\langle\{\theta_1 + 12\lambda_1 - \theta_\ell - 12\lambda_\ell\}_{\ell=2,3}\rangle \cong \mathbb{Z}^2 \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}.$$

Because $6\lambda^2 = 0$ on $\mathcal{M}_{1,3}$, we know $6\lambda^2 \in \mathrm{im}(\iota_*)$. Note $6\lambda^2$ has order exactly 4, so it must pushforward from an element in $\mathrm{CH}^1(\partial\mathcal{M}_{1,3})/\mathrm{im}(\partial_1)$ of order exactly 4. Thus, $\mathrm{im}(\partial_1)$ can contain no torsion elements besides what is in $\langle\{\theta_1 + 12\lambda_1 - \theta_\ell - 12\lambda_\ell\}_{\ell=2,3}\rangle$, and so $\mathrm{im}(\partial_1) = \{\theta_1 + 12\lambda_1 - \theta_\ell - 12\lambda_\ell\}_{\ell=2,3}$.

The 4-torsion elements in $\mathrm{CH}^1(\partial\mathcal{M}_{1,3})/\mathrm{im}(\partial_1)$ are

$$\pm(\theta_1 + 6\lambda_1 - 6\lambda_2 - 6\lambda_3 - 6\lambda_\emptyset),$$

and the above says one of these must push forward to $6\lambda^2$. Thus, we can write

$$\iota_*(\theta_1) = \pm 6\lambda^2 - 6\lambda\delta_1 + 6\lambda\delta_2 + 6\lambda\delta_3 + 6\lambda\delta_\emptyset. \tag{9.14}$$

By S_3-symmetry, we have

$$\iota_*(\theta_\ell) = \pm 6\lambda^2 - 6\lambda\delta_\ell + 6\lambda\delta_{\ell'} + 6\lambda\delta_{\ell''} + 6\lambda\delta_\emptyset$$

for $\{\ell, \ell', \ell''\} = \{1,2,3\}$. Moreover, using this expression for $\iota_*(\theta_\ell)$, the pushforward of the relations from $\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})/\mathrm{im}(\partial_1)$ become 0. Thus, we have that $\mathrm{CH}^2(\overline{\mathcal{M}}_{1,3})$ is generated by $\lambda^2, \lambda\delta_1, \lambda\delta_2, \lambda\delta_3, \lambda\delta_\emptyset, \delta_\emptyset\delta_\ell$, subject to $24\lambda^2 = 0$.

Multiplying (9.14) by λ^{i-2} for $i \geq 3$, we get

$$\iota_*(\theta_1)\lambda^{i-2} = \pm 6\lambda^i - 6\lambda^{i-1}\delta_1 + 6\lambda^{i-1}\delta_2 + 6\lambda^{i-1}\delta_3 + 6\lambda^{i-1}\delta_\emptyset.$$

For the morphism $\iota' : \overline{\mathcal{M}}_{1,3}^{\Theta_1} \to \overline{\mathcal{M}}_{1,3}$, we have

$$\iota_*(\theta_1)\lambda^{i-2} = \iota'_*(1)\lambda^{i-2} = \iota'_*(\iota'^*(\lambda^{i-2})).$$

We have the diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_{1,3}^{\Theta_1} \cup \overline{\mathcal{M}}_{1,3}^{\Theta_2} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{1,3}^{\Phi} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{1,3} \arrow[d] \\
\overline{\mathcal{M}}_{1,2}^{\Theta} \arrow[r] \arrow[dr, "\sim"] & \overline{\mathcal{M}}_{1,2}^{\Phi} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{1,2} \arrow[d] \\
& \partial\overline{\mathcal{M}}_{1,1} \arrow[r] & \overline{\mathcal{M}}_{1,1}
\end{tikzcd}$$

from which we see that the pullback of λ^{i-2} to $\overline{\mathcal{M}}_{1,3}^{\Theta_1}$ is the same as the pullback of $s^{i-2} \in \mathrm{CH}^1(\overline{\mathcal{M}}_{1,2}^{\Theta})$, using Proposition 9.2. This was computed in Proposition 9.11 to be $\theta_{1,\emptyset}^{i-2} - \tau_1^{i-2} = 12\lambda_1^{i-2}(\mu_1 - \lambda_1) - \tau_1^i$. Thus,

$$\iota'_*(\iota'^*(\lambda^{i-2})) = \iota_*(\tau_1^i - 12\lambda_1^{i-2}(\mu_1 - \lambda_1)) = \iota_*(\tau_1^i) - 12\lambda^{i-2}\delta_1(\delta_\emptyset - \lambda)$$

and so, we have

$$\iota_*(\tau_1^i) - 12\lambda^{i-2}\delta_1(\delta_\emptyset - \lambda) = \pm 6\lambda^i - 6\lambda^{i-1}\delta_1 + 6\lambda^{i-1}\delta_2 + 6\lambda^{i-1}\delta_3 + 6\lambda^{i-1}\delta_\emptyset$$

or

$$\begin{aligned}
\iota_*(\tau_1^i) &= 12\lambda^{i-2}\delta_1(\delta_\emptyset - \lambda) \pm 6\lambda^i - 6\lambda^{i-1}\delta_1 + 6\lambda^{i-1}\delta_2 + 6\lambda^{i-1}\delta_3 + 6\lambda^{i-1}\delta_\emptyset \\
&= 12\lambda^{i-2}\delta_1\delta_\emptyset \pm 6\lambda^i + 6\lambda^{i-1}\delta_1 + 6\lambda^{i-1}\delta_2 + 6\lambda^{i-1}\delta_3 + 6\lambda^{i-1}\delta_\emptyset
\end{aligned}$$

for $i \geq 3$. Note the pushforward of the relations $\tau_1^i - \tau_2^i, \tau_1^i - \tau_3^i$ are automatically satisfied, because this expression for $\iota_*(\tau_1)$ is S_3 invariant. The relation $2\tau_1^i - 24t\lambda^i = 2(\tau_1^i - 12\lambda^{i-2}(\mu_1 - \lambda_1))$ pushes forward to

$$12\lambda^i - 12\lambda^{i-1}\delta_1 + 12\lambda^{i-1}\delta_2 + 12\lambda^{i-1}\delta_3 + 12\lambda^{i-1}\delta_\emptyset.$$

Thus, using Theorem 9.10 and Proposition 9.11, we have that

- $\mathrm{CH}^3(\overline{\mathcal{M}}_{1,3})$ is generated by $\lambda^3, \lambda^2\delta_1, \lambda^2\delta_2, \lambda^2\delta_3, \lambda^2\delta_\emptyset, \lambda\delta_1\delta_\emptyset$ modulo the relations
$$24\lambda^3$$
$$24\lambda^2\delta_\ell, \text{ for } \ell \in \{1,2,3,\emptyset\}$$
$$12\lambda^3 - 12\lambda^2\delta_1 + 12\lambda^2\delta_2 + 12\lambda^2\delta_3 + 12\lambda^2\delta_\emptyset$$
- $\mathrm{CH}^i(\overline{\mathcal{M}}_{1,3})$ is generated by $\lambda^i, \lambda^{i-1}\delta_1, \lambda^{i-1}\delta_2, \lambda^{i-1}\delta_3, \lambda^{i-1}\delta_\emptyset, \lambda^{i-2}\delta_1\delta_\emptyset$ modulo the relations
$$24\lambda^i$$
$$24\lambda^{i-1}\delta_\ell, \text{ for } \ell \in \{1,2,3,\emptyset\}$$
$$24\lambda^{i-2}\delta_1\delta_\emptyset$$
$$12\lambda^i - 12\lambda^{i-1}\delta_1 + 12\lambda^{i-1}\delta_2 + 12\lambda^{i-1}\delta_3 + 12\lambda^{i-1}\delta_\emptyset$$

This finishes an additive presentation for $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$. From this description, we see that $\delta_1, \delta_2, \delta_3, \delta_\emptyset, \lambda$ generated $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$ as a ring.

We now compute some products of these generators. We can compute
$$\delta_\ell^2 = -\lambda\delta_\ell - \delta_1\delta_\emptyset.$$

Finally, note that $\delta_\ell\delta_k = 0$ for $\ell \neq k \in \{1,2,3\}$, because $\overline{\mathcal{M}}^{\Delta_\ell} \cap \overline{\mathcal{M}}^{\Delta_k} = \emptyset$.

Now, by looking at the relations we have found, the kernel of the surjective map
$$\mathbb{Z}[\delta_1, \delta_2, \delta_3, \delta_\emptyset, \lambda] \twoheadrightarrow \mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$$
contains the following elements
$$24\lambda^2$$
$$\delta_\ell\delta_k, \text{ for } \ell \neq k \in \{1,2,3\}$$
$$\delta_\ell\delta_\emptyset - \delta_k\delta_\emptyset, \text{ for } \ell \neq k \in \{1,2,3\}$$
$$\delta_\ell^2 + \lambda\delta_\ell + \delta_1\delta_\emptyset, \text{ for } \ell \in \{1,2,3,\emptyset\}$$
$$12\lambda^3 - 12\lambda^2\delta_1 + 12\lambda^2\delta_2 + 12\lambda^2\delta_3 + 12\lambda^2\delta_\emptyset,$$
and so it contains I, the ideal they generate. We claim that I is equal to the kernel. By Lemma 9.9, it is enough to show that the additive generators we found for $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$ above both generate $\mathbb{Z}[\delta_1, \delta_2, \delta_3, \delta_\emptyset, \lambda]/I$ and satisfy the same relations that they do in $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$. The latter is clear, as all of the relations we found between these additive generators above are clearly in the ideal I. So it remains to show that monomials of the form
$$\lambda^i, \lambda^i\delta_1, \lambda^i\delta_2, \lambda^i\delta_3, \lambda^i\delta_\emptyset, \lambda^i\delta_1\delta_\emptyset$$
additively generate $\mathbb{Z}[\delta_1, \delta_2, \delta_3, \delta_\emptyset, \lambda]/I$. Note that any monomial in $\mathbb{Z}[\delta_1, \delta_2, \delta_3, \delta_\emptyset, \lambda]/I$ is 0 if it contains distinct δ_ℓ, δ_k, $\ell, k \in \{1,2,3\}$. Moreover, note
$$\delta_\ell^2\delta_\emptyset = \delta_\ell\delta_k\delta_\emptyset = 0,$$
and so the only possibly nonzero monomials look like $\lambda^i\delta_\ell^j$ or $\lambda^i\delta_\emptyset^j\delta_\ell$, for some $\ell \in \{1,2,3\}$. Then, using the self intersection formulas $\delta_\ell^2 = -\delta_\ell\lambda - \delta_\emptyset\delta_1$ for $\ell \in \{1,2,3,\emptyset\}$, we see that these monomials are indeed linear combinations of $\lambda^i, \lambda^i\delta_1, \lambda^i\delta_2, \lambda^i\delta_3, \lambda^i\delta_\emptyset, \lambda^i\delta_1\delta_\emptyset$, and so we are done.

□

9.4. $\overline{\mathcal{M}}_{1,4}$**.** Define the graphs $\Delta_j, \Delta_{jk}, \Delta_\emptyset, \Theta_j, \Theta_{1j}, \Omega_{jk}, \Sigma, \Delta_{j|k}, \Psi_{jk}, \Delta_{\emptyset,j}, \Delta_{\emptyset,jk}$.

By Theorem 7.4 and the CKP for $\overline{\mathcal{M}}_{0,n}$, we have

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_j}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}) \otimes \mathrm{CH}^*(\overline{\mathcal{M}}_{0,4}) = \frac{\mathbb{Z}[\lambda_j, \delta_{\emptyset,j}, \delta_{j|k}]}{(24\lambda_j^2, \delta_{\emptyset,j}^2 + \delta_{\emptyset,j}\lambda_j, \delta_{j|k}^2)}$$

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_{jk}}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,3}) = \frac{\mathbb{Z}[\delta_{j|k}, \delta_{k|j}, \psi_{jk}, \delta_{\emptyset,jk}, \lambda_{jk}]}{I}$$

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_\emptyset}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \otimes_{\mathbb{Z}} \mathrm{CH}^*(\overline{\mathcal{M}}_{0,5}) =$$

$$\frac{\mathbb{Z}[\lambda_\emptyset, \delta_{\emptyset,1}, \delta_{\emptyset,2}, \delta_{\emptyset,3}, \delta_{\emptyset,4}, \delta_{\emptyset,12}]}{(24\lambda_\emptyset^2, \{\delta_{\emptyset,j}\delta_{\emptyset,k}, \delta_{\emptyset,j}^2 - \delta_{\emptyset,k}^2\}_{j\neq k}, \delta_{\emptyset,12}\delta_{\emptyset,3}, \delta_{\emptyset,12}\delta_{\emptyset,4}, \delta_{\emptyset,12}^2 - \delta_{\emptyset,1}^2, \{\delta_{\emptyset,1}^2 + \delta_{\emptyset,12}\delta_{\emptyset,j}\}_{j=1,2}, \delta_{\emptyset,1}^3)}.$$

By Theorem 7.2, we have

$$\mathcal{M}^{\Theta_{jk}} = [\mathcal{M}_{\Theta_{jk}} / \operatorname{Aut}(\Theta_j)] = [\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}/\mu_2].$$

There are unique points $P_{jk} \in \mathcal{M}^{\Theta_{jk}} \cong [\mathcal{M}_{0,4}/\mu_2]$ isomorphic to $B\mu_2$. Set τ_{jk}^i to be the pushforward of u^{i-1} to $\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}$ along $P_j \hookrightarrow \overline{\mathcal{M}}^{\mathrm{sep},\geq 2}$. Moreover, let κ_{jk} be the class of the image of

$$\widehat{T} \times \overline{\mathcal{M}}_{0,4} \hookrightarrow \overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,4} \to [\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,4}/\mu_2] \to \overline{\mathcal{M}}^{\Theta_{jk}}.$$

Proposition 9.15. $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,4}) =$

Proof. The components of $\overline{\mathcal{M}}^{\mathrm{sep}.\geq 5} = \overline{\mathcal{M}}^{\mathrm{sep}}$ are given by $\overline{\mathcal{M}}^{\Delta_j}$ for $j \in \{1,2,3,4\}$, $\overline{\mathcal{M}}^{\Delta_{jk}}$ for $j \neq k \in \{1,2,3,4\}$, and $\overline{\mathcal{M}}^{\Delta_\emptyset}$. We use the exact sequence

$$\bigoplus_{\Gamma\neq\Gamma'} \mathrm{CH}(\overline{\mathcal{M}}^{\Gamma} \cap \overline{\mathcal{M}}^{\Gamma'}) \to \bigoplus_{\Gamma} \mathrm{CH}^*(\overline{\mathcal{M}}^{\Gamma}) \to \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \to 0$$

to calculate $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}})$.

The intersections between these components are

$$\overline{\mathcal{M}}^{\Delta_j} \cap \overline{\mathcal{M}}^{\Delta_k} = \emptyset \text{ for } j \neq k,$$
$$\overline{\mathcal{M}}^{\Delta_j} \cap \overline{\mathcal{M}}^{\Delta_{k\ell}} = \emptyset \text{ for } j \neq k, \ell$$
$$\overline{\mathcal{M}}^{\Delta_{jk}} \cap \overline{\mathcal{M}}^{\Delta_{j\ell}} = \emptyset \text{ for } k \neq \ell$$
$$\overline{\mathcal{M}}^{\Delta_j} \cap \overline{\mathcal{M}}^{\Delta_{jk}} = \overline{\mathcal{M}}^{\Delta_{j|k}}$$
$$\overline{\mathcal{M}}^{\Delta_{jk}} \cap \overline{\mathcal{M}}^{\Delta_{\ell m}} = \overline{\mathcal{M}}^{\Psi_{jk}} \text{ for } \{j,k,\ell,m\} = [4]$$
$$\overline{\mathcal{M}}^{\Delta_j} \cap \overline{\mathcal{M}}^{\Delta_\emptyset} = \overline{\mathcal{M}}^{\Delta_{\emptyset,j}}$$
$$\overline{\mathcal{M}}^{\Delta_{jk}} \cap \overline{\mathcal{M}}^{\Delta_\emptyset} = \overline{\mathcal{M}}^{\Delta_{\emptyset,jk}}.$$

By Theorem 7.4, we have

$$\overline{\mathcal{M}}^{\Delta_{j|k}} \cong \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,3} \cong \overline{\mathcal{M}}_{1,2}$$

$$\overline{\mathcal{M}}^{\Psi_{jk}} \cong \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,3} \cong \overline{\mathcal{M}}_{1,2}$$

$$\overline{\mathcal{M}}^{\Delta_{\emptyset,j}} \cong \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,4} \cong \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,4}$$
$$\overline{\mathcal{M}}^{\Delta_{\emptyset,jk}} \cong \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,3} \cong \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,4}$$

The following table describes the relations in $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}})$ obtained by pushing forward from $\overline{\mathcal{M}}^{\Gamma}$

$\Delta_{j\vert k}$	$\delta_{j\vert k}\lambda^i_j - \delta_{j\vert k}\lambda^i_{jk}$,	$\delta_{j\vert k}\delta_{\emptyset,j}\lambda^i_j - \delta_{j\vert k}\delta_{\emptyset,jk}\lambda^i_{jk}$
Ψ_{jk}	$\psi_{jk}\lambda^i_{jk} - \psi_{\ell m}\lambda^i_{\ell m}$,	$\psi_{jk}\delta_{\emptyset,jk}\lambda^i_{jk} - \psi_{\ell m}\delta_{\emptyset,\ell m}\lambda^i_{\ell m}$
$\Delta_{\emptyset,j}$	$\delta_{\emptyset,j}\lambda^i_j - \delta_{\emptyset,j}\lambda^i_{\emptyset}$,	$\delta_{\emptyset,j}\delta_{jk}\lambda^i_j - \delta_{\emptyset,j}\delta_{\emptyset,jk}\lambda^i_{\emptyset}$
$\Delta_{\emptyset,jk}$	$\delta_{\emptyset,jk}\lambda^i_{jk} - \delta_{\emptyset,jk}\lambda^i_{\emptyset}$,	$\delta_{\emptyset,jk}\delta_{j\vert k}\lambda^i_{jk} - \delta_{\emptyset,jk}\delta_{\emptyset,j}\lambda^i_{\emptyset}$

This determines $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}})$. Because $\overline{\mathcal{M}}^{\mathrm{sep},\geq 4} = \overline{\mathcal{M}}^{\mathrm{sep}} \amalg \overline{\mathcal{M}}^{\Sigma}$ and $\overline{\mathcal{M}}^{\Sigma} = \mathrm{Spec}(k)$ by Theorem 7.2, we have $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 4}) = \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \oplus \langle\sigma\rangle$.

Next, we consider $\overline{\mathcal{M}}^{\mathrm{sep},\geq 3} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 4} \cup \mathcal{M}^{\mathrm{non}=3}$. We have

$$\mathcal{M}^{\mathrm{non}=3} = \coprod_{\{j,k\}\in\binom{[4]}{2}} \mathcal{M}^{\Omega_{jk}}.$$

Theorem 7.2 gives

$$\mathcal{M}^{\Omega_{jk}} = [\mathcal{M}_{\Omega_{jk}}/\operatorname{Aut}(\Omega_{jk})] = \mathcal{M}_{0,3} \times \mathcal{M}_{0,4} \times \mathcal{M}_{0,3} = \mathcal{M}_{0,4}.$$

We have

$$\mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=3},1)_{\mathrm{ind}} \xrightarrow{\partial_1} \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 4}) \to \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) \to \mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=3}) \to 0,$$

which becomes

$$\bigoplus_{\{j,k\}\in\binom{[4]}{2}} \mathrm{CH}^*(\mathcal{M}_{0,4},1)_{\mathrm{ind}} \xrightarrow{\partial_1} \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 4}) \to \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) \to \bigoplus_{\{j,k\}\in\binom{[4]}{2}} \mathrm{CH}^*(\mathcal{M}_{0,4}) \to 0.$$

The diagram (7.6) shows that the image of ∂_1 is generated by the pushforward of the WDVV realtions to $\overline{\mathcal{M}}^{\mathrm{sep},\geq 4}$ from $\partial\overline{\mathcal{M}}_{0,4}$. The relations on $\overline{\mathcal{M}}_{0,4}$ are generated by $D(12|ab) - D(1a|2b), D(12|ab) - D(1a|2b)$. Both $D(1a|2b)$ and $D(1b|2a)$ push forward to σ, and $D(12|ab)$ pushes forward to the pushforward of the class $\omega \in \mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$ inside $\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_{\ell m}})$, where ℓ, m is such that $\{j,k,\ell,m\} = [4]$. From Proposition 9.10, we know $\omega = 2\tau_j$, and Theorem 9.13 gives a computation of τ_j in terms of the ring generators. Thus, we have the relation

$$\sigma = 12(2\delta_{\ell|m}\delta_{\emptyset,\ell m} + \lambda^2_{\ell m} + \lambda_{\ell m}\delta_{\ell|m} + \lambda_{\ell m}\delta_{\ell|m} + \lambda_{\ell m}\psi_{\ell m} + \lambda_{\ell m}\delta_{\emptyset,\ell m}).$$

Moreover, because $\mathrm{CH}^*(\mathcal{M}_{0,4}) = \mathbb{Z}$, we can split the exact sequence by lifting $[\mathcal{M}^{\Omega_{jk}}]$ to $[\overline{\mathcal{M}}^{\Omega_{jk}}]$. So $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3})$ is the quotient of $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 4}) \oplus \langle\omega_{jk}\rangle$ by the above expressions for σ. But because $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 4}) = \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \oplus \langle\sigma\rangle$, the above expression for σ says that we can instead write $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) = \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \oplus \langle\omega_{jk}\rangle$ and remember the above expression for σ inside of $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3})$.

Now, consider $\overline{\mathcal{M}}^{\text{sep},\geq 2} = \overline{\mathcal{M}}^{\text{sep},\geq 3} \cup \mathcal{M}^{\text{non}=2}$. We have

$$\mathcal{M}^{\text{non}=2} = \coprod_{\{j,k\}} \mathcal{M}^{\Theta_{jk}} \amalg \coprod_{j\in[4]} \mathcal{M}^{\Theta_j}.$$

Theorem 7.2 gives

$$\mathcal{M}^{\Theta_{jk}} = [\mathcal{M}_{\Theta_{jk}}/\mu_2] = [\mathcal{M}_{0,4}\times\mathcal{M}_{0,4}/\mu_2]$$

and

$$\mathcal{M}^{\Theta_j} = [\mathcal{M}_{\Theta_j}/\operatorname{Aut}(\Theta_j)] = [\mathcal{M}_{0,5}/\mu_2].$$

By Theorem 8.3, $\mathrm{CH}^*(\mathcal{M}^{\Theta_j}) = \mathrm{CH}^*([\mathcal{M}_{0,5}/\mu_2]) = \mathbb{Z}$, and so we can simply to lift $[\mathcal{M}^{\Theta_j}]$ to $[\overline{\mathcal{M}}^{\Theta_j}]$ when dealing with that piece of $\mathrm{CH}^*(\mathcal{M}^{\text{non}=2})$. Diagram (7.6) for Θ_j and Theorem 8.4 show that the image of ∂_1 coming from $\mathcal{M}^{\Theta_j}$ is generated by the pushforwards of

$$\widehat{D}(ka|\ell b) - 2\widehat{D}(k\ell|ab)$$

$$\widehat{D}(ka|mb) - 2\widehat{D}(km|ab)$$

$$2(\widehat{D}(\ell mb|ka) + \widehat{D}(kab|\ell m) - \widehat{D}(k\ell m|ab) - \widehat{D}(mab|k\ell) - \widehat{D}(\ell ab|km)).$$

We can compute the degrees of these push forwards using Automorphism theorem. Then we get

$$\widehat{D}(ka|\ell mb) \mapsto \Omega_{\ell m}$$

$$\widehat{D}(ab|k\ell m) \mapsto 24\lambda_j(\delta_{\emptyset,j} - \lambda_j)$$

$$\widehat{D}(kab|\ell m) \mapsto 6\lambda_{jk}(\pm\lambda_{jk} - \delta_{j|k} + \delta_{k|j} + \psi_{jk} + \delta_{\emptyset,jk})$$

under the pushforward map, using Theorem 9.8 and Theorem 9.13 for their formulas of $\iota_*(\theta)$ and $\iota_*(\theta_j)$, respectively. Thus, the relations push forward to

$$\omega_{km} + \omega_{\ell m} - 24\lambda_j(\delta_{\emptyset,j} - \lambda_j) - 12\lambda_{jm}(\lambda_{jm} - \delta_{j|m} + \delta_{m|j} + \psi_{jm} + \delta_{\emptyset,jm})$$

$$\omega_{k\ell} + \omega_{\ell m} - 24\lambda_j(\delta_{\emptyset,j} - \lambda_j) - 12\lambda_{j\ell}(\lambda_{j\ell} - \delta_{j|\ell} + \delta_{\ell|j} + \psi_{j\ell}\delta_{\emptyset,j\ell})$$

$$2(\omega_{\ell m} + 6\lambda_{jk}(\pm\lambda_{jk} - \delta_{j|k} + \psi_{jk} + \delta_{k|j} + \delta_{\emptyset,jk}) - 12\lambda_j(\delta_{\emptyset,j} - \lambda_j) -$$

$$6\lambda_{jm}(\pm\lambda_{jm} - \delta_{j|m} + \delta_{m|j} + \psi_{jm} + \delta_{\emptyset,jm}) - 6\lambda_{j\ell}(\pm\lambda_{j\ell} - \delta_{j|\ell} + \delta_{\ell|j} + \psi_{j\ell} + \delta_{\emptyset,j\ell})).$$

For $\mathcal{M}^{\Theta_{jk}}$, we plug $\Gamma = \Theta_{jk}$ into the diagram (7.6) to get

$$\begin{tikzcd}
\mathrm{CH}^*(\mathcal{M}^{\Theta_{jk}},1)_{\text{ind}} \arrow[r] \arrow[d] & \mathrm{CH}^{*-1}([\partial\overline{\mathcal{M}}_{\Theta_{jk}}/\operatorname{Aut}(\Theta_{jk})]) \arrow[r] \arrow[d] & \mathrm{CH}^*([\overline{\mathcal{M}}_{\Theta_{jk}}/\operatorname{Aut}(\Theta_{jk})]) \arrow[r] \arrow[d] & \mathrm{CH}^*(\mathcal{M} \arrow[d] \\
\mathrm{CH}^*(\mathcal{M}^{\text{non}=p},1)_{\text{ind}} \arrow[r, "\partial_1"] & \mathrm{CH}^{*-1}(\overline{\mathcal{M}}^{\text{sep},\geq 3}) \arrow[r] & \mathrm{CH}^*(\overline{\mathcal{M}}^{\text{sep},\geq 2}) \arrow[r] & \mathrm{CH}^*(\mathcal{M}
\end{tikzcd}$$

By Lemma 8.9, we have that $t \in \frac{\mathbb{Z}[t]}{(2t)} = \mathrm{CH}^*([\mathcal{M}_{0,4}\times\mathcal{M}_{0,4}/\mu_2])$ lifts to $\pi_1^*(t-v)\in \mathrm{CH}^*([\overline{\mathcal{M}}_{0,4}\times\overline{\mathcal{M}}_{0,4}/\mu_2])$ and t^i lifts to $\pi_1^*(t^i)$... help

Lemma 8.9 also says that the image of ∂_1 coming from $\mathcal{M}^{\Theta_{jk}}$ is generated by the pushforwards of

$$[\pi_1^{-1}\widehat{D}(ja|kb)] - 2[\pi_1^{-1}\widehat{D}(jk|ab)]$$
$$[\pi_2^{-1}\widehat{D}(\ell a|mb)] - 2[\pi_2^{-1}\widehat{D}(\ell m|ab)].$$

We get

$$\pi_1^{-1}\widehat{D}(ja|kb)\mapsto \omega_{\ell m}$$
$$\pi_1^{-1}\widehat{D}(jk|ab)\mapsto 6\lambda_{\ell m}(\pm\lambda_{\ell m}+\delta_{\ell|m}+\delta_{m|\ell}-\psi_{\ell m}+\delta_{\emptyset,\ell m})$$
$$\pi_2^{-1}\widehat{D}(\ell a|mb)\mapsto \omega_{jk}$$
$$\pi_2^{-1}\widehat{D}(\ell m|ab)\mapsto 6\lambda_{jk}(\pm\lambda_{jk}+\delta_{j|k}+\delta_{k|j}-\psi_{jk}+\delta_{\emptyset,jk})$$

under the pushforward map, using Theorem 9.8 and Theorem 9.13 for its formula of $\iota_*(\theta_j)$. Then the relations pushforward to

$$\omega_{\ell m} - 12\lambda_{\ell m}(\lambda_{\ell m}+\delta_{\ell|m}+\delta_{m\ell}-\psi_{\ell m}+\delta_{\emptyset,\ell m})$$
$$\omega_{jk} - 12\lambda_{jk}(\lambda_{jk}+\delta_{j|k}+\delta_{kj}-\psi_{jk}+\delta_{\emptyset,jk})$$

This shows that we do not need to use ω_{jk} as generators in $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2})$, and we can remember these expressions instead. Combining these relations with the relations obtained from $\mathcal{M}^{\Theta_j}$, we get the relations

after simplifying.

Finally, we consider $\partial\overline{\mathcal{M}}_{1,4} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 1} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 2}\cup\mathcal{M}^{\mathrm{non}=1}$. We have $\mathcal{M}^{\mathrm{non}=1} = M^{\Phi}$ and by Theorem 7.2, we have

$$\mathcal{M}^{\Phi} = [\mathcal{M}_{\Phi}/\operatorname{Aut}(\Phi)] = [\mathcal{M}_{0,6}/\mu_2].$$

The localization exact sequence for $\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}\subseteq \partial\overline{\mathcal{M}}_{1,4}$ is

$$\mathrm{CH}^*([\mathcal{M}_{0,6}/\mu_2],1)_{\mathrm{ind}} \xrightarrow{\partial_1} \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \to \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,4}) \to \mathrm{CH}^*([\mathcal{M}_{0,6}/\mu_2]) \to 0.$$

By Theorem 8.3, we have $\mathrm{CH}^*([\mathcal{M}_{0,6}/\mu_2]) = \mathbb{Z}$, so we can lift $[\mathcal{M}^{\Phi}]$ to $\phi = [\overline{\mathcal{M}}^{\phi}]$ to split the exact sequence. To compute the image of ∂_1, we use the diagram (7.6) to conclude that the image of ∂_1 in the pushforward of the image of ∂_1' along $\mathrm{CH}([\partial\overline{\mathcal{M}}_{0,6}/\mu_2]) \to \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2})$. By Theorem 8.4, the image of ∂_1' is generated by

$$\widehat{D}(ja|kb) - 2\widehat{D}(jk|ab)$$

for $j,k\in\{1,\dots,4\}$ and

$$\alpha_{jk\ell}+\alpha_{j'k'\ell'}$$

for $j,k,\ell,j',k',\ell'\in\{1,\dots,4\}$ with $|\{j,k,\ell\}| = |\{j',k',\ell'\}| = 3$, where

$$\alpha_{j,k,\ell} := \widehat{D}(jkb|\ell a) + \widehat{D}(\ell ab|jk) - \widehat{D}(\ell jk|ab) - \widehat{D}(kab|\ell j) - \widehat{D}(jab|\ell k).$$

Taking the pushforward, we get the relations

$$2(\theta_k+\theta_{j\ell}+\theta_{jm}+\theta_j)-24(\lambda_\emptyset+\lambda_m+\lambda_\ell+\lambda_{\ell m})$$

for $\{j,k,\ell,m\}=[4]$ and

□

Theorem 9.16. *The image of* $\mathrm{CH}^1(\mathcal{M}^0_{1,3},1)_{ind}\to\mathrm{CH}^2(\mathcal{M}_{1,4},1)_{ind}\to\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,4})$ *is generated by*

Proposition 9.17. *The image of* $\partial_1:\mathrm{CH}^*(\mathcal{M}_{1,4},1)_{ind}\to\mathrm{CH}^{*-1}(\partial\overline{\mathcal{M}}_{1,4})$ *is given by*

Proof. Consider the commutative diagram

$$\begin{array}{ccc}\mathrm{CH}^*(\mathcal{M}_{1,4},1)_{\mathrm{ind}} & \xrightarrow{\partial_1} & \mathrm{CH}^{*-1}(\partial\overline{\mathcal{M}}_{1,4})\\ {\scriptstyle\pi^*}\uparrow & & {\scriptstyle\pi^*}\uparrow\\ \mathrm{CH}^*(\mathcal{M}_{1,3},1)_{\mathrm{ind}} & \xrightarrow{\partial_1'} & \mathrm{CH}^{*-1}(\partial\overline{\mathcal{M}}_{1,3}).\end{array}$$

We know that the 2-torsion element

$$\theta_1+12\lambda_1-\theta_2-12\lambda_2$$

is in the image of ∂_1' by Proposition 9.11. Pulling this back along π, we get the 2-torsion element

$$\theta_1+\theta_{14}+12\lambda_1+12\lambda_{14}-\theta_2-\theta_{24}-12\lambda_2-12\lambda_{24}.$$

Applying the S_4 action to the element, we can obtain more elements in the image of ∂_1. There are six linearly independent elements in this orbit, and $\mathrm{CH}^2(\mathcal{M}_{1,4},1)_{\mathrm{ind}}$ help.

Next, we have the 2-torsion elements $\tau_1^i-\tau_2^i$ in the image of ∂_1' by Proposition 9.11.

□

Theorem 9.18. $\mathrm{CH}(\overline{\mathcal{M}}_{1,4})=$

Proof. □

10. Proof of Theorem 9.16

Definition 10.1. Let $\widetilde{\mathcal{M}}_{1,2}$ be the closure of $\mathcal{M}^0_{1,2}\subseteq\overline{\mathcal{M}}_{1.3}$ and let $\widetilde{\mathcal{M}}_{1,3}$ be the closure of $\mathcal{M}^0_{1,3}\subseteq\overline{\mathcal{M}}_{1,4}$.

Lemma 10.2. *The class* $[\widetilde{\mathcal{M}}_{1,2}]\in\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$ *is given by*

$$[\widetilde{\mathcal{M}}_{1,2}]=2\lambda-\delta_1+2\delta_2+2\delta_3+2\delta_\emptyset$$

Proof. By Theorem 9.13, we can write

$$[\widetilde{\mathcal{M}}_{1,2}] = b_0\lambda + b_1\delta_1 + b_2\delta_2 + b_3\delta_3 + b_4\delta_\emptyset \tag{10.3}$$

for some $b_i \in \mathbb{Z}$.

For a curve $(C, p_1, p_2, p_3) \in \mathcal{M}^0_{1,2} \subseteq \mathcal{M}_{1,3}$, elliptic curve inversion swaps the points p_2 and p_3, giving an isomorphism $(C, p_1, p_2, p_3) \xrightarrow{\sim} (C, p_1, p_3, p_2)$. Thus, every point in $\mathcal{M}^0_{1,2} \subseteq \mathcal{M}_{1,3}$ is invariant under (23), hence the same is true for $\widetilde{\mathcal{M}}_{1,3}$.Applying the permutation (23) to both sides of (10.3), we see that $b_2 = b_3$.

Next, the composite $\widetilde{\mathcal{M}}_{1,2} \hookrightarrow \overline{\mathcal{M}}_{1,3} \xrightarrow{\pi} \overline{\mathcal{M}}_{1,2}$ has degree 1, since it has a rational inverse given by $\mathcal{M}^0_{1,2} \hookleftarrow \mathcal{M}_{1,3}$. Thus, after pushing forward along π, (10.3) becomes

$$[\overline{\mathcal{M}}_{1,2}] = (b_1 + b_2)[\overline{\mathcal{M}}_{1,2}],$$

so $b_1 + b_2 = 1$.

Let $\pi' : \overline{\mathcal{M}}_{1,3} \to \overline{\mathcal{M}}_{1,2}$ denote the map forgetting the first point. Given two distinct points p, p' on a smooth genus 1 curve, there are always 4 points q such that $p + p' \sim 2q$. Thus, the composite $\widetilde{\mathcal{M}}_{1,2} \hookrightarrow \overline{\mathcal{M}}_{1,3} \xrightarrow{\pi'} \overline{\mathcal{M}}_{1,2}$ has degree 4. Thus, after pushing forward along π', (10.3) becomes

$$4[\overline{\mathcal{M}}_{1,2}] = (b_2 + b_3)[\overline{\mathcal{M}}_{1,2}],$$

so $b_2+b_3 = 4$. Putting these equations together, we get $b_1 = -1$, $b_2 = b_3 = 2$.

No points in $\overline{\mathcal{M}}^{\Delta_1}_{1,3}$ are invariant under (23), so its intersection with $\widetilde{\mathcal{M}}_{1,2}$ must be empty. Thus, multiplying by δ_2, we have

$$0 = b_0\lambda\delta_2 + b_2\delta_2^2 + b_4\delta_\emptyset\delta_2 = (b_0 - b_2)\lambda\delta_2 + (b_4 - b_2)\delta_\emptyset\delta_2$$

from which we get $b_0 = b_4 = b_2 = 2$. ∎

Proof of Theorem 9.16. Let $\widetilde{\mathcal{M}}_{1,3}$ be the closure of $\mathcal{M}^0_{1,3}$ inside $\overline{\mathcal{M}}_{1,4}$. We utilize the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^1(\mathcal{M}^0_{1,3},1)_{\mathrm{ind}} \arrow[r, "\partial_1"] \arrow[d] & \mathrm{CH}_2(\widetilde{\mathcal{M}}_{1,3} \setminus \mathcal{M}^0_{1,3}) \arrow[d] \\
\mathrm{CH}^2(\mathcal{M}_{1,4},1)_{\mathrm{ind}} \arrow[r, "\partial_1"] & \mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,4})
\end{tikzcd}$$

First, we wish to compute $\mathrm{CH}_2(\widetilde{\mathcal{M}}_{1,3} \setminus \mathcal{M}^0_{1,3})$, i.e. the 2-dimensional components of $\widetilde{\mathcal{M}}_{1,3} \setminus \mathcal{M}^0_{1,3}$. We think about the map $\widetilde{\mathcal{M}}_{1,3} \to \overline{\mathcal{M}}_{1,3}$. The dimension of the image of 2-dimensional component of $\widetilde{\mathcal{M}}_{1.3} \setminus \mathcal{M}^0_{1,3}$ in $\overline{\mathcal{M}}_{1,3} \setminus \mathcal{M}^0_{1,3}$ has dimension either 1, in which case the map is finite, or 2, in which case the fibers are 1 diemsnional . Thus, we only need to consider points in $\widetilde{\mathcal{M}}_{1,3}$ lying over $\mathcal{M}^0_{1.2} \subseteq \overline{\mathcal{M}}_{1,3}$ or $\mathcal{M}^\Gamma_{1,3}$ for Γ a stable graph of codimension 1 or 2.

Note that the morphism

$$\varphi : \mathcal{M}^0_{1,3} \to \mathcal{M}_{1,4}$$

$$(C,p_1,p_2,p_3)\mapsto(C,p_1,p_2,p_3,p_2+p_3)$$

exhibiting $\mathcal{M}^0_{1,3}$ as a closed subset of $\mathcal{M}_{1,4}$ extends to $\mathcal{M}^{\Phi}_{1,3}$ when viewed as a map to $\overline{\mathcal{M}}_{1,4}$. Moreover, this map extends to $\mathcal{M}^0_{1,2}\subseteq\mathcal{M}_{1,3}$ by sending (C,p_1,p_2,p_3) to the curve C with an attached $\mathbb{P}^1$ at p_1 that contains p_1 and p_4. Thus, over $\mathcal{M}^0_{1,2}$ and $\mathcal{M}^{\Phi}_{1,3}$, there are unique 2-dimensional components of $\widetilde{\mathcal{M}}_{1,3}\setminus\mathcal{M}^0_{1,3}$. Let $\widetilde{\mathcal{M}}^{\Phi}_{1,3}:=\overline{\varphi(\mathcal{M}^{\Phi}_{1,3})}$ and $\widetilde{\mathcal{M}}^{\Xi}_{1,3}:=\overline{\varphi(\mathcal{M}^0_{1,2})}$.

Next, note the curves in $\mathcal{M}^0_{1,3}\subseteq\overline{\mathcal{M}}_{1,4}$ are invariant under the permutation $(14)(23)$, as the isomorphism $q\mapsto 2p_4-q$ takes (C,p_1,p_2,p_3,p_4) to (C,p_4,p_3,p_2,p_1) for $(C,p_1,p_2,p_3,p_4)\in\mathcal{M}^0_{1,3}$. Thus, the points in $\widetilde{\mathcal{M}}_{1,3}$ must also be invariant under $(14)(23)$. Moreover, the permutation $(12)(34)$ will not generally fix the curves in $\mathcal{M}^0_{1,3}$, but it does send the locus $\mathcal{M}^0_{1,3}$ to itself.

For $j=2,3)$, over points in $\mathcal{M}^{\Delta_j}_{1,3}$ there is a unique point in $\overline{\mathcal{M}}_{1,4}$ that is invariant under $(14)(23)$, given by attaching a $\mathbb{P}^1$ containing the markings j and 4 to the marking j. Thus, $\overline{\mathcal{M}}^{\Psi_{13}}$ and $\overline{\mathcal{M}}^{\Psi_{12}}$ are the only 2-dimensional components over $\mathcal{M}^{\Delta_2}_{1,3}$ and $\mathcal{M}^{\Delta_3}_{1,3}$. Additionally, over a generic point of $\mathcal{M}^{\Delta_\emptyset}_{1,3}$, there is a unique point invariant under $(14)(23)$. Over the other points of $\mathcal{M}^{\Delta_\emptyset}_{1,3}$, there is exactly one additional curve invariant under $(14)(23)$, meaning there will not be a 2-dimensional component over these points. We can describe the 2-dimensional component of $\widetilde{\mathcal{M}}_{1,3}\setminus\mathcal{M}^0_{1,3}$ over $\overline{\mathcal{M}}^{\Delta_\emptyset}$ as the image of the $(14)(23)$ invariant points in $\overline{\mathcal{M}}_{0,5}\times\overline{\mathcal{M}}_{1,1}$ under the map gluing the 5-th marked point of the genus 0 curve to the only marked point of the genus 1 curve. Call this component $\widetilde{\mathcal{M}}^{\Delta_\emptyset}_{13}$.

There are exactly two $(14)(23)$ invariant curves over every point in $\overline{\mathcal{M}}^{\Delta_1}$. These are given by placing p_4 either on a rational component with p_1 or at the point $2q$, where q is the point on the genus 1 component where the rational curve containing $2,3$ is attached. We have that curves of the latter type are all inside $\widetilde{\mathcal{M}}_{1,3}$ because they are equal to $(12)(34)$ applied to a point of $\varphi(\mathcal{M}^0_{1,2})$. Moreover, the map $\varphi:\overline{\mathcal{M}}_{1,3}\dashrightarrow\overline{\mathcal{M}}_{1,4}$ extends to a morphism over an open subset of $\overline{\mathcal{M}}^{\Delta_1}_{1,3}$, because $\overline{\mathcal{M}}_{1,3}$ is normal and $\overline{\mathcal{M}}_{1,4}$ is proper. Thus, there is only one point in $\widetilde{\mathcal{M}}_{1,3}$ lying over a generic point of $\overline{\mathcal{M}}^{\Delta_1}_{1,3}$. And the remaining points have at most 2 points lying over them, so we can conclude that there is a unique 2-dimensional component lying over $\mathcal{M}^{\Delta_1}_{1,3}$. Call this component $\widetilde{\mathcal{M}}^{\Delta_1}_{1,3}$. This takes care of all of the codimension 1 graphs of $\overline{\mathcal{M}}_{1,3}$.

For all codimension 2 graphs Γ of $\overline{\mathcal{M}}_{1,3}$ besides Θ_1, for any point in $\mathcal{M}^{\Gamma}_{1,3}$, one can see that there are only finitely many $(14)(23)$-invariant preimages over in $\overline{\mathcal{M}}_{1,4}$, meaning there cannot be a 2-dimensional component of $\widetilde{\mathcal{M}}_{1,3}\setminus\mathcal{M}^0_{1,3}$ lying over these $\mathcal{M}^{\Gamma}_{1,3}$. But over a point in $\mathcal{M}^{\Theta_1}_{1,3}$, there are infinitely many preimages invariant under $(14)(23)$, obtained by placing p_4 anywhere on the same component as p_1. These are the points of $\overline{\mathcal{M}}_{1,4}$ in $\mathcal{M}_{1,4}^{\Theta_{1,4}}$, and we do indeed have $\mathcal{M}_{1,4}^{\Theta_{1,4}} \subseteq \widetilde{\mathcal{M}}_{1,3}$, hence $\overline{\mathcal{M}}_{1,4}^{\Theta_{1,4}}$ is a 2-dimensional component of $\widetilde{\mathcal{M}}_{1,3} \setminus \mathcal{M}_{1,3}^0$. To show that $\mathcal{M}_{1,4}^{\Theta_{1,4}} \subseteq \widetilde{\mathcal{M}}_{1,3}$, we provided an explicit deformation: help.

Knowing all of the 2-dimensional components, we can write

$$\partial_1(f_0) = \mathrm{div}(f_0) = a_0[\widetilde{\mathcal{M}}_{1,3}^{\Phi}]+a_1[\widetilde{\mathcal{M}}_{1,3}^{\Delta_1}]+a_2\psi_{13}+a_3\psi_{12}+a_4[\widetilde{\mathcal{M}}_{13}^{\Delta_\emptyset}]+a_5[\widetilde{\mathcal{M}}_{1,3}^{\Xi}]+a_6\theta_{14}$$

for some $a_i \in \mathbb{Z}$. By commutativity of

$$\begin{array}{ccc}
\mathrm{CH}^1(\mathcal{M}_{1,3}^0,1)_{\mathrm{ind}} & \xrightarrow{\partial_1} & \mathrm{CH}_2(\widetilde{\mathcal{M}}_{1,3} \setminus \mathcal{M}_{1,3}^0) \\
\downarrow & & \downarrow \\
\mathrm{CH}^1(\mathcal{M}_{1,3},1)_{\mathrm{ind}} & \xrightarrow{\partial_1'} & \mathrm{CH}_2(\overline{\mathcal{M}}_{1,3} \setminus \mathcal{M}_{1,3}^0)
\end{array}$$

the above equation pushes forward to

$$\partial_1'(f_0) = a_0\phi + a_1\delta_1 + a_2\delta_2 + a_3\delta_3 + a_4\delta_\emptyset + a_5[\widetilde{\mathcal{M}}_{1,2}]$$

on $\partial\overline{\mathcal{M}}_{1,3}$, hence

$$0 = a_0\phi + a_1\delta_1 + a_2\delta_2 + a_3\delta_3 + a_4\delta_\emptyset + a_5[\widetilde{\mathcal{M}}_{1,2}]$$

in $\overline{\mathcal{M}}_{1,3}$.

By Lemma 10.2, we have

$$0 = \phi - 6\lambda_1 + 12\delta_2 + 12\delta_3 + 12\delta_\emptyset - 6[\widetilde{\mathcal{M}}_{1,2}].$$

There is only one relation between these classes up to scaling by an integer. Because this above relation holds on $\overline{\mathcal{M}}_{1,3}$ and it is primitive, we can conclude that either $\mathrm{div}(f_0)$ or $\mathrm{div}(f_0^{-1})$ is given by the right-hand side. But, by definition of f_0, it is clear that the order of vanishing along irreducible nodal curves is negative. Thus, we have $a_0 = -1, a_1 = a_5 = 6$, and $a_2 = a_3 = a_4 = -12$.

Finally, we compute a_6 using the curves approaching $\mathcal{M}^{\Theta_{14}}$ from earlier. We know that these curves are transverse to $\mathcal{M}^{\Theta_{14}}$ because help.

Now, we pushforward $\mathrm{div}(f_0)$ to $\overline{\mathcal{M}}_{1,4}$, giving

$$\partial_1((\mathcal{M}_{1,3}^0, f_0)) = [\widetilde{\mathcal{M}}_{1,3}^{\Phi}]+6[\widetilde{\mathcal{M}}_{1,3}^{\Delta_1}]-12\psi_{13}-12\psi_{12}-12[\widetilde{\mathcal{M}}_{13}^{\Delta_\emptyset}]+6[\widetilde{\mathcal{M}}_{1,3}^{\Xi}]+a_6\theta_{14}$$

We want to express $[\widetilde{\mathcal{M}}_{1,3}^{\Phi}], [\widetilde{\mathcal{M}}_{1,3}^{\Delta_1}], [\widetilde{\mathcal{M}}_{13}^{\Delta_\emptyset}], [\widetilde{\mathcal{M}}_{1,3}^{\Xi}]$ in terms of our generators for the $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,4})$ given in Theorem **??**. Note $\widetilde{\mathcal{M}}_{1,3}^{\Xi}$ is equal to the image of $\widetilde{\mathcal{M}}_{1,2}$ along the section $\overline{\mathcal{M}}_{1,3} \xrightarrow{\sim} \overline{\mathcal{M}}^{\Delta_{2,3}} \hookrightarrow \overline{\mathcal{M}}_{1,4}$. Thus, we have

$$[\widetilde{\mathcal{M}}_{1,3}^{\Xi}] = 2\lambda_{2,3} - \psi_{23} + 2\delta_{2|3} + 2\delta_{3|2} + 2\delta_{\emptyset,23}$$

on $\partial\overline{\mathcal{M}}_{1,4}$. As noted before, the image of $\widetilde{\mathcal{M}}_{1,3}^{\Xi}$ under (12)(34) is $\widetilde{\mathcal{M}}_{1,3}^{\Delta_1}$, so we have

$$[\widetilde{\mathcal{M}}_{1,3}^{\Xi}] = 2\lambda_{1,4} - \psi_{14} + 2\delta_{1|4} + 2\delta_{4|1} + 2\delta_{\emptyset,14}$$

on $\partial\overline{\mathcal{M}}_{1,4}$. $\square$