INTEGRAL CHOW RINGS OF $\overline{\mathcal{M}}_{1,n}$ VIA $\mathrm{CH}^*(\mathcal{M}_{1,n}, 1)$

ABSTRACT. For $n \leq 4$, we compute the integral indecomposible higher Chow groups $\overline{\mathrm{CH}}^*(\mathcal{M}_{1,n}, 1)$ and use these to compute the integral Chow rings $\mathrm{CH}*(\mathcal{M}_{1,n})$.

CONTENTS

1. Introduction

Introduced by Bloch in [], the higher Chow group $\mathrm{CH}_i(X,j)$ are interesting invariants of a space X that extend the usual Chow groups. We are particularly interested in the groups $\mathrm{CH}^*(\mathcal{M}_{g,n},1)$, as they can be used to compute $\mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$: there is an exact sequence

$$\mathrm{CH}^*(\mathcal{M}_{g,n},1) \xrightarrow{\partial_1} \mathrm{CH}^{*-1}(\partial\overline{\mathcal{M}}_{g,n}) \to \mathrm{CH}^*(\overline{\mathcal{M}}_{g,n}) \to \mathrm{CH}^*(\mathcal{M}_{g,n}) \to 0.$$

Thus, $\mathrm{CH}^*(\mathcal{M}_{g,n},1)$, together with the map ∂_1, determine which classes on $\partial\overline{\mathcal{M}}_{g,n}$ map to 0 on $\overline{\mathcal{M}}_{g,n}$. Using this, we compute $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,n})$ for $n \leq 4$. This technique was inspired by [Bae-Schmidt II] and [Larson].

Higher Chow groups are usually very large. For example, for any variety X over the field k, $\mathrm{CH}^1(X,1)$ contains $k^\times$. For this reason, we instead study a variant $\overline{\mathrm{CH}}^*(X,1)$, the indecomposible higher Chow groups (though see Remarks 3.3 and 3.6). These groups are finitely generated in the cases we are interested in. In [Larosn,Bishop,Bishop], the notions of ℓ-adic higher Chow groups were used for similar reasons. We show how indecomposible Chow groups relate to ℓ-adic Chow groups in Proposition 2.24.

the following 2 Theorems summarize our main results.

Theorem 1.1. *For $n \leq 4$, one has $\overline{\mathrm{CH}}^1(\mathcal{M}_{1,n},1) = 0$, and for $i \geq 2$, we have*

$$\overline{\mathrm{CH}}^i(\mathcal{M}_{1,1},1) = 0$$
$$\overline{\mathrm{CH}}^i(\mathcal{M}_{1,2},1) = \frac{\mathbb{Z}}{2\mathbb{Z}}$$
$$\overline{\mathrm{CH}}^i(\mathcal{M}_{1,3},1) = \begin{cases} P \oplus \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 & i = 2 \\ \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 & i \geq 3 \end{cases}$$
$$\overline{\mathrm{CH}}^i(\mathcal{M}_{1,n},1) = \begin{cases} \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^6 & i = 2 \\ \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 & i \geq 3 \end{cases}$$

where P is a cyclic group of order 2^a3^b for some $a, b \geq 0$.

Theorem 1.2. *The Chow rings of $\overline{\mathcal{M}}_{1,n}$ are given by*

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) = \frac{\mathbb{Z}[\lambda]}{(24\lambda^2)}$$
$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}) = \frac{\mathbb{Z}[\lambda,\delta]}{(24\lambda^2, \lambda^2+\delta\lambda)}$$
$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3}) = \frac{\mathbb{Z}[\lambda,\delta_1,\delta_2,\delta_3,\delta_\emptyset]}{I_3}$$
$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,4}) = \frac{\mathbb{Z}[\lambda,\{\delta_{ij}\}_{i\neq j\in[4]},\{\delta_i\}_{i\in[4]},\delta_\emptyset]}{I_4}$$

where I_3 and I_4 are as described in Theorem 8.8 and Theorem 8.13, respectively.

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Notation. k is a field whose characteristic is not 2 or 3. All stacks X are separated finite type quotient stacks over k. In general, for a stable graph Γ, necessarily an uppercase Greek letter, we let the corresponding lowercase Greek letter denote the class of $\overline{\mathcal{M}}^{\Gamma}$ in any space containing it, i.e. $\gamma := [\overline{\mathcal{M}}^{\Gamma}]$.

2. Indecomposible Higher Chow Groups

2.1. **Higher Chow Groups.** Bloch defined higher Chow groups $\mathrm{CH}_i(X,j)$ for a scheme X in [Bloch]. The relevant properties are as follows:

(1) $\mathrm{CH}_i(-,j)$ has proper pushforwards, flat pull-backs, and pull-backs for arbitrary maps between smooth schemes.
(2) $\mathrm{CH}_i(X,0) = \mathrm{CH}_i(X)$, and the functorialities agree.
(3) If X contains an open subscheme $j : U \hookrightarrow X$, with complement $\iota : Z \hookrightarrow X$, there is an exact sequence

$$\cdots \to \mathrm{CH}_i(Z,j) \xrightarrow{\iota_*} \mathrm{CH}_i(X,j) \xrightarrow{j^*} \mathrm{CH}_i(U,j) \xrightarrow{\partial} \mathrm{CH}_i(Z,j-1) \to \ldots$$

(4) There is a product

$$\times : \mathrm{CH}_i(X,j) \times \mathrm{CH}_k(Y,\ell) \to \mathrm{CH}_{i+k}(X \times Y, j+\ell)$$

For smooth X, setting $X = Y$ and pulling back along the diagonal gives the structure of a bi-graded ring on $\mathrm{CH}^*(X,*) = \bigoplus_{i,j} \mathrm{CH}^i(X,j)$.
(5) (dimension vanishing)

For equidimensional X, we set $\mathrm{CH}^i(X,j) := \mathrm{CH}_{\dim X - i}(X,j)$. Note this is not the same as opperational Chow.

In [EG], Edidin and Graham extended Chow groups and higher Chow groups [well not technically, but...] to quotient stacks, $[X/G]$. This is done by setting

$$\mathrm{CH}_i([X/G],j) := CH_{i+\dim(V)}((X \times U)/G, j),$$

where U is an open subset of a G-representation, V, whose complement has codimension $\geq$???. All of the above properties of higher Chow groups extend to stacks (modify dimension statement, unclear on pushes and pulls). (Subtly involving X needing to be quasiprojective.)

We can use higher Chow groups to prove things about ordinary Chow groups, like the following

Lemma 2.1. *Suppose we have a proper map of quotient stacks $g : X \to Y$, with a closed substack $j : Z \subseteq Y$ and open complement $U \subseteq Y$, such that $g^{-1}(U) \xrightarrow{g} U$ is an isomorphism. Then the map*

$$(g_*, j_*) : \mathrm{CH}^*(X) \coprod_{\mathrm{CH}^*(g^{-1}Z)} \mathrm{CH}^*(Z) \to \mathrm{CH}^*(Y)$$

is an isomorphism.

Proof. We get the diagram

$$\begin{tikzcd}
\mathrm{CH}^*(f^{-1}(U),1) \arrow[r] \arrow[d, "\sim"] & \mathrm{CH}^*(g^{-1}(Z)) \arrow[r] \arrow[d] & \mathrm{CH}^*(X) \arrow[r] \arrow[d] & \mathrm{CH}^*(f^{-1}(U)) \arrow[r] \arrow[d, "\sim"] & 0 \\
\mathrm{CH}^*(U,1) \arrow[r] & \mathrm{CH}^*(Z) \arrow[r] & \mathrm{CH}^*(Y) \arrow[r] & \mathrm{CH}^*(U) \arrow[r] & 0
\end{tikzcd}$$

This diagram commutes by a property of higher Chow. Now the lemma follows from a diagram chase. $\square$

The only higher Chow groups we will use in this paper are $\mathrm{CH}_i(X,1)$. These groups have the following description:

Proposition 2.2. *Suppose X is a scheme. Consider the group*

$$\bigoplus_{\substack{W \subseteq X \text{ closed, integral} \\ \dim(W)=i+1}} K(W)^\times$$

whose elements we write as $\sum_W (W, f_W)$. Then $\mathrm{CH}_i(X,1)$ is can be identified with a quotient of the subgroup

$$\{\sum_W (W, f_W) | \sum_W div_W(f_W) = 0 \in Z_i(X)\}.$$

Moreover, under this identification, we have the following

(1) *For X irreducible, $\mathrm{CH}^1(X,1) = \{f \in K(X)^\times | div(f) = 0\}$. Hence for X normal and irreducible, $\mathrm{CH}^1(X,1) = \mathcal{O}_X(X)^\times$.*
(2) *For $\iota : Z \to X$ a closed embedding, the pushforward is given by*

$$\iota_* : \mathrm{CH}_i(Z,1) \to \mathrm{CH}_i(X,1).$$

$$\sum_W (Z, f_W) \mapsto \sum_W (W, f_W)$$

(3) *For $j : U \to X$ an open embedding, the pullback is given by*

$$j^* : \mathrm{CH}_i(X,1) \to \mathrm{CH}_i(U,1)$$

$$\sum_W (W, f_W) \mapsto \sum_W (W \cap U, f_W|_{U \cap W}).$$

(4) *For $Z \subseteq X$ closed with complement U, the connecting homomorphism is given by*

$$\partial_1 : \mathrm{CH}_i(U,1) \to \mathrm{CH}_i(Z)$$
$$\sum_W (W, f_W) \mapsto \sum_W div_{\overline{W}}(f_W),$$

where $\overline{W}$ is the closure of W in X and $f_W \in K(W)^\times = K(\overline{W})^\times$.

Remark 2.3. It is only slightly more complicated to describe arbitrary proper pushforwards flat pullbacks using this description.

Now, suppose we have a smooth variety X and closed $Z \subseteq X$ of pure codimension 1 with complement U, and we wish to compute the boundary map

$$\partial_1 : \mathcal{O}_U(U)^\times \to \mathrm{CH}^0(Z).$$

The above proposition says that the map sends f to $\mathrm{div}_X(f)$. To compute this, one must know $\mathrm{ord}_W(f)$ for components $W \subseteq Z$.

Proposition 2.4. *Suppose we have a smooth variety X and closed irreducible subvariety $Z \subseteq X$ of codimension 1 with complement U. Additionally, suppose we have a curve $C \subseteq X$ with C intersecting Z transversely at a point $p \in C$. Then, for $f \in \mathcal{O}_U(U)^\times$, we have $f|_C$ makes sense and*

$$\mathrm{ord}_Z(f) = \mathrm{ord}_p(f|_C).$$

Proof. □

Now, for a quotient stack $[X/G]$, the above implies that we can write elements of $\mathrm{CH}_i([X/G],1)$ as $\sum_W (W, f_W)$ with $\sum_W \mathrm{div}_W(f_W) = 0$, where W are $(i+\dim(V)-\dim(G))$-dimensional G-equivariant integral subvarieties of $X \times V$, where V is a representation of G with free locus $U \subseteq V$ having $\mathrm{codim}(V \setminus U)$ small.

We can give a simple description of $\mathrm{CH}^1([X/G],1)$:

Lemma 2.5. *For a smooth quotient stack $[X/G]$ with X irreducible one has*

$$\mathrm{CH}^1([X/G],1) = (\mathcal{O}_X(X)^\times)^G.$$

Proof. Take a representation of G, V, with G acting freely on the open subset $U \subseteq V$ and $\mathrm{codim}(V \setminus U)$ small. Then, we have

$$\begin{aligned}\mathrm{CH}^1([X/G],1) &= \mathrm{CH}^1(X \times U/G, 1)\\ &= \mathcal{O}_{X\times U/G}(X \times U/G)^\times\\ &= (\mathcal{O}_{X\times U}(X \times U)^\times)^G.\end{aligned}$$

Because $\mathrm{codim}(V \setminus U)$ is small and X is normal, we have that global functions of $X \times U$ are the same as that of X, and so we can write the above as $(\mathcal{O}_{X\times V}(X \times V)^\times)^G$. And this is equal to $(\mathcal{O}_X(X)^\times)^G$, because units on X are the same as units on $X \times \mathbb{A}^1$. □

2.2. **Indecomposible Higher Chow.** Now, we define indecomposible higher Chow groups. These will have essentially all of the same properties as higher Chow groups, with the bonus of being finitely generated and independent of base field in the cases that we care about.

We have $\mathrm{CH}_i(k,1)$ is only nonzero for $i=-1$. This can be seen using Proposition 2.2.

Definition 2.6. The decomposable first higher Chow groups of a quotient stack X, $\mathrm{CH}_i^{\mathrm{dec}}(X,1)$ is defined to be the image of

$$\times : \mathrm{CH}_{i+1}(X) \otimes \mathrm{CH}_{-1}(k,1) \to \mathrm{CH}_i(X \times_k k, 1) \xrightarrow{\sim} \mathrm{CH}_i(X,1).$$

We then define the indecomposible (first) higher Chow group of a quotient stack X to be

$$\overline{\mathrm{CH}}_i(X,1) := \mathrm{CH}_i(X,1)/\,\mathrm{CH}_i^{\mathrm{dec}}(X,1).$$

If X is smooth with structure morphism $f : X \to \mathrm{Spec}(k)$, note that for $\alpha \in \mathrm{CH}_{i+1}(X)$ and $\beta \in \mathrm{CH}_{-1}(k,1)$, under the identification of $X \times_k k$ with X,we have $\alpha \times \beta = \pi_1^*\alpha \cup \pi_2^*\beta = \alpha \cup f^*\beta$.

Warning 2.7. *There are different, nonequivalent notions of indecomposible higher Chow groups. Our definition is the same as those appearing in [Carlino-Fakhrudan] and [Bae-Schmidt II]. Notions of indecomposible cycles were originally defined in order to give some notion of "nontriviality" of a higher cycle, in the sense that it is not decomposing into other cycles in some specified way.*

For a proper map $f : X \to Y$, compatibility between the product $\times$ and proper pushforward f_* implies that $f_*(\mathrm{CH}_i(X,1)_{\mathrm{dec}}) \subseteq \mathrm{CH}_i(Y,1)_{\mathrm{dec}}$. Hence, f_* induces a map $f_* : \overline{\mathrm{CH}}_i(X,1) \to \overline{\mathrm{CH}}_i(Y,1)$. Similarly, for a flat map of relative dimension d, compatibility between the product and f^* implies that $f^*(\mathrm{CH}_i(Y,1))_{\mathrm{dec}} \subseteq \mathrm{CH}_{i+d}(X,1)_{\mathrm{dec}}$. Hence, f^* induces a map $f^* : \overline{\mathrm{CH}}_i(Y,1) \to \overline{\mathrm{CH}}_{i+d}(X,1)$.

Moreover, we still have the last part of the localization exact sequence:

Lemma 2.8. *If U is an open substack of X with complement Z, $\partial : \mathrm{CH}_i(U,1) \to \mathrm{CH}_i(Z)$ is zero on $\mathrm{CH}_i(U,1)_{dec}$, and the induced sequence*

$$\overline{\mathrm{CH}}_i(Z,1) \to \overline{\mathrm{CH}}_i(X,1) \to \overline{\mathrm{CH}}_i(U,1) \to \mathrm{CH}_i(Z) \to \mathrm{CH}_i(X) \to \mathrm{CH}_i(U) \to 0$$

is exact.

Proof. We need only check exactness at the first four terms in the sequence, as the rest of the sequence is unchanged from the standard localization exact sequence. Consider the diagram

$$\begin{tikzcd}
\mathrm{CH}_i(Z,1) \arrow[r] & \mathrm{CH}_i(X,1) \arrow[r, "j^*"] & \mathrm{CH}_i(U,1) \arrow[r, "\partial"] & \mathrm{CH}_i(Z) \\
\mathrm{CH}_{-1}(k,1) \otimes \mathrm{CH}_{i+1}(Z) \arrow[r] \arrow[u] & \mathrm{CH}_{-1}(k,1) \otimes \mathrm{CH}_{i+1}(X) \arrow[r] \arrow[u] & \mathrm{CH}_{-1}(k,1) \otimes \mathrm{CH}_{i+1}(U) \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Note that both rows are exact sequences, because the localization exact sequence is exact and because tensor is right-exact. As noted above, compatibility between $\times$ and pullbacks/pushforwards gives that the first two squares commute. To see that the third square commutes, we need that $\partial(a \times \alpha) = 0 \in \mathrm{CH}_i(Z)$ for $a \in \mathrm{CH}_{-1}(k, 1)$ and $\alpha \in \mathrm{CH}_i(U)$. This is true because we can lift α to $\widetilde{\alpha} \in \mathrm{CH}_i(X)$, and then

$$\partial(a \times \alpha) = \partial(a \times j^*(\widehat{\alpha})) = \partial(j^*(a \times \widetilde{\alpha})) = 0.$$

Thus, this diagram of exact sequences commutes. The sequence we want to be exact is precisely the cokernel complex

$$\overline{\mathrm{CH}}_i(Z, 1) \to \overline{\mathrm{CH}}_i(X, 1) \to \overline{\mathrm{CH}}_i(U, 1) \xrightarrow{\partial} \mathrm{CH}_i(Z) \to \mathrm{CH}_i(X).$$

This is a routine diagram chase. □

Remark 2.9. One could define indecomposible groups $\overline{\mathrm{CH}}_i(X, j)$ for any i and j in an analogous way by taking them to be the (i, j) graded piece of the cokernel of

$$\mathrm{CH}_*(k, \geq 1) \otimes \mathrm{CH}_*(X) \to \mathrm{CH}_*(X, *).$$

Some good properties that this definition have:

(1) These groups have the same functorialities enjoyed by higher Chow groups
(2) For any cohomology functor H and any natural transformation $\mathrm{CH}^*(-, *) \implies H^*(-)$ (a "higher cycle class map") that sends $\mathrm{CH}^i(X, j)$ to $H^{2i-j}(X)$ factors through this definition of indecomposible higher Chow groups
(3) The proof of Proposition 2.19 goes through to give $\overline{\mathrm{CH}}^*(X, j) = 0$ for $j > 0$ for smooth proper varieties X with a strong Chow-Kunneth decomposition.

One would also want the localization sequence to be exact. While the maps exist, being induced by the maps on the usual exact sequence, the author was unable to show exactness.

2.3. **Universal Separable Homeomorphisms.** We now introduce the notion of a universal separable homeomorphism.

Definition 2.10. A map of stacks $f : Z' \to Z$ is a universal separable homeomorphism if it is a representable integral surjective map such that, for all $p \in Z$, there is a unique preimage $q \in Z'$ and the map $\kappa(p) \to \kappa(q)$ is an isomorphism.

Example 2.11. Consider the normalization of the node:

$$f : \mathrm{Spec}(k[t]) \to \mathrm{Spec}(k[x, y]/(y^2 - x^3)$$
$$t \mapsto (t^2, t^3).$$

This is a finite, hence integral, surjection, being a normalization. Away from the cusp, $(0, 0)$, this map is an isomorphism, hence all other points in $\mathrm{Spec}(k[x, y]/(y^2 - x^3)$ have unique preimages with induced isomorphisms on the residue fields. The point $(0,0)$ also has a unique preimage, 0, and the map on residue fields is an isomorphism. Hence, f is a universal separable homeomorphism. We use this example in section ???

We compare our use of this notion to the use of Chow envelopes in the literature. A map of stacks $f : Z' \to Z$ is a Chow envelope if it is a representable, proper, and if for all G-invariant integral substacks $V \subseteq Z'$ there exists some integral substack $W \subseteq Z$ mapping to V birationally. This implies that the pushforward f_* is surjective on integral Chow groups. This has been used in the following way (e.g. [Vistoli]): Suppose you want to compute the Chow groups of $U \subseteq X$ and you know the Chow groups of X and $Z := X \setminus U$. By the localization exact sequence

$$\mathrm{CH}_i(Z) \xrightarrow{\iota_*} \mathrm{CH}_i(X) \to \mathrm{CH}_i(U) \to 0$$

it suffices to compute the image of ι_*. The space Z may be hard to work with (e.g. Z may be singular) so it suffices to find some Chow envelope $f : Z' \to Z$ and compute the image of $(\iota \circ f)_*$, since this is just the image of ι_*.

In this paper, the above is true, but we also would like to compute the kernel of ι_*, because we are also trying to compute the indecomposible higher Chow groups $\overline{\mathrm{CH}}_i(U,1)$, which fit into the sequence in the following way

$$\overline{\mathrm{CH}}_i(X,1) \to \overline{\mathrm{CH}}_i(U,1) \xrightarrow{\partial_1} \mathrm{CH}_i(Z) \xrightarrow{\iota_*} \mathrm{CH}_i(X).$$

Hence, we would like a map $f : Z' \to Z$ which induces an *isomorphism* $f_* : \mathrm{CH}_*(Z') \to \mathrm{CH}_*(Z)$.

Proposition 2.12. *Let $f : Z' \to Z$ be a universal separable homeomorphism. Then the induced maps $f_* : \mathrm{CH}_i(Z', j) \to \mathrm{CH}_i(Z, j)$ are isomorphisms.*

Proof. Note that universal separable homeomorphisms are closed, since they are homeomorphisms. By the next lemma, they are also preserved by base change, so they are proper. Hence, we can indeed consider the pushforward f_*.

Now, to show f_* is an isomorphism, it suffices to assume that Z' and Z are schemes. The pushforward $f_* : \mathrm{CH}_i(Z', j) \to \mathrm{CH}_i(Z, j)$ is induced by the map $f_* : z_i(Z', *) \to z_i(Z, *)$ on complexes, given by the pushforward of cycles. Because $f : Z' \to Z$ is a universal separable homeomorphism, so is $f : Z' \times \Delta^n \to Z \times \Delta^n$. Therefore, for any closed, irreducible $V \subseteq Z' \times \Delta^n$, we have $f_*(V) = \deg(V/f_*(V)) \cdot f(V) = f(V)$, as the map on residue fields is an isomorphism. And now, we see that $f_* : z_i(Z', j) \to z_i(Z, j)$ has an inverse, sending W to $f^{-1}(W)$. Thus, the map of complexes is an isomorphism, so the induced maps $f_* : \mathrm{CH}_i(Z, j) \to \mathrm{CH}_i(Z', j)$ are isomorphisms. □

Now, in the setting mentioned above, there do not always exist a universal separable homeomorphism $Z' \to Z$. For example, if Z is a nodal curve, there is no universal separable homeomorphism from a smooth stack to Z. We will have to deal with such cases in the next section.

Lemma 2.13. *Universal separable homeomorphisms are preserved by base change.*

Proof. Suppose $f : Z' \to Z$ is a universal separable homeomorphism, and consider a fiber diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d, "g"] \\ Z' \arrow[r, "f"] & Z \end{tikzcd}$$

Being integral and surjective are closed under base change, so we just need to show preimages are unique and the maps on residue fields are isomorphisms.

Take $p' \in Y$ and $p = g(p')$. Because f is integral, it is affine, so we can write $f^{-1}(p) = \mathrm{Spec}(A)$. The ring A must be a Artinian ring over $K := \kappa(p)$. Because f is a universal separable homeomorphism, we know that the residue field of A is K.

Note $f'^{-1}(p') = \mathrm{Spec}(A \otimes_K L)$, where $L = \kappa(p')$. It suffices to show that the residue field of $A \otimes_K L$ is equal to L. Let $\mathfrak{m}$ be the maximal ideal of A. We have an exact sequence

$$0 \to \mathfrak{m} \otimes_K L \to A \otimes_K L \to K \otimes_K L = L \to 0.$$

The exact sequence implies that $\mathfrak{m} \otimes_K L$ must be a maximal ideal, because its quotient is a field. Moreover, any prime ideal of $A \otimes_K L$ restricts to $\mathfrak{m}$, and hence contains the ideal generated by $\mathfrak{m} \otimes 1$, which is $\mathfrak{m} \otimes_K L$. Thus, $A \otimes_K L$ has a unique prime ideal, and the residue field of $A \otimes_K L$ is indeed L. □

Remark 2.14. As the name suggests, universal separable homeomorphisms are precisely the universal homeomorphisms which induce separable field extensions on residue fields. One direction is clear: our universal separable homeomorphisms are homeomorphisms, being closed continuous bijections, and are universal homeomorphisms because this lemma shows the property is preserved by base change. The other direction is true because universal homeomorphisms are exactly maps which are integral, surjective, every point in the target has a unique preimage, and the induced map on residue fields are purely inseparable (see [**?**, Tag 04DF] and [**?**, Tag 01S2]).

2.4. **The Chow-Kunneth Generation Property.**

Definition 2.15. A stack X has the Chow-Kunneth Generation Property (CKgP) if for all stacks Y, the induced map

$$\mathrm{CH}^*(X) \otimes_{\mathbb{Z}} \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X \times Y)$$

is surjective.

This property has been used in [Bae-Schmidt II] and [Canning-Larson] to prove results on rational Chow rings of moduli of curves, so the authors are working with Chow rings with rational coefficients. Most properties remain unchanged, with the same proofs:

Proposition 2.16.

(1) *Suppose X is a stack with closed substack Z and set $U := X \setminus Z$. If X and Z have the CKgP then so does U, and if Z and U have the CKgP then so does X.*
(2) *If X and Y has the CKgP, so does $X \times Y$.*
(3) *$X \times \mathbb{A}^1$ has the CKgP if (and only if) X has the CKgP.*
(4) *If $f : X \to Y$ is a universal separable homeomorphism, then X has the CKgP if and only if Y does.*
(5) *$\mathbb{P}^n$ and $B\mathbb{G}_m$ have the CKgP.*

Proof. □

One notable property that holds for the rational CKgP but not the integral CKgP is that, given a proper surjection $X \to Y$, if X has the CKgP then so does Y. Totaro found examples of finite groups G with BG not having the integral CKgP, despite the proper surjection $\mathrm{Spec}(k) \to BG$ [Totaro].

Remark 2.17. More is true in the setting of (2) for the CKgP: one has that if X has the CKgP, then so does U. Nevertheless, we ignore this, and ensure that Z has the CKgP before concluding the same for U. We do this because then all of our proofs show that the stacks in question have the *Motivic Kunneth Property* (MKP) defined in [Totaro], which is true because the MKP satisfies all five properties listed in Proposition 2.16. The MKP is stronger and more robust than the CKgP:

- For any stack X with the MKP, X has the CKP $\mathrm{CH}_*(X) \otimes_{\mathbb{Z}} \mathrm{CH}_*(Y) \to \mathrm{CH}_*(X \times Y)$ is an isomorphism for any stack Y (that is X has the Chow-Kunneth Property).
- In the setting of (2) above, if X and U have the MKP, then so does Z. Hence, if any two of X, Z, U have the MKP, then so does the third.
- If X has the MKP, there is a spectral sequence computing the higher Chow groups $\mathrm{CH}_i(X \times Y, j)$ for any stack Y.

Moreover, X has the MKP if and only if the motive $M^c(X)$ is mixed Tate in Voevodsky's triangulated category of mixed motives. Hence, we have that the compactly supported motives of $\overline{\mathcal{M}}_{1,n}$ are mixed Tate for $n \leq 4$ from the results in this paper (specifically Corollary 7.4). The downside to using the MKP, and the reason we work with the CKgP instead, is that the derived category of mixed motives is only defined in characteristic $p > 0$ with coefficient rings where p is invertible. Thus, we could only compute (higher) Chow groups with $\mathbb{Z}[\frac{1}{p}]$ coefficients when invoking the MKP in characteristic p.

Proposition 2.18. *Suppose X is a smooth proper variety with the CKgP. Then for any stack Y the induced map*

$$\mathrm{CH}_*(X) \otimes_{\mathbb{Z}} \mathrm{CH}_*(Y) \to \mathrm{CH}_*(X \times Y)$$

is an isomorphism.

Proof. □

The following proposition gives a relation between the CKgP and indecomposible higher Chow groups.

Proposition 2.19. *Suppose X is a smooth proper Deligne-Mumford stack with the CKgP. Then $\overline{\mathrm{CH}}^*(X, 1) \otimes \mathbb{Z}[\frac{1}{N}] = 0$, where N is the lcm of the orders of the stabilizers of points of X.*

Proof. Let $n = \dim(X)$. Because X has the CKgP, we know

$$\mathrm{CH}^*(X) \otimes_{\mathbb{Z}} \mathrm{CH}^*(X) \to \mathrm{CH}^*(X \times X)$$

is surjective. Then, we can write

$$[\Delta] = \sum_i \alpha_i \times \beta_{n-i},$$

for $\alpha_i, \beta_i \in \mathrm{CH}^i(X)$, where $\Delta \overset{\delta}{\hookrightarrow} X \times X$ is the diagonal. Because X is smooth, $\alpha_i \times \beta_{n-i} = \pi_1^*(\alpha_i) \cup \pi_2^*(\beta_{n-i})$.

Because X is proper, we can consider pushforwards along π_i after we invert N. Note that for $\gamma \in \mathrm{CH}^i(X, 1)$, we have

$$\pi_{1*}([\Delta]\cup\pi_2^*(\gamma)) = \pi_{1*}(\delta_*(1)\cup\pi_2^*(\gamma)) = \pi_{1*}(\delta_*(1\cup\delta^*\pi_2^*(\gamma))) = (\pi_1\circ\delta)_*(\pi_2\circ\delta)^*(\gamma) = \gamma,$$

using the projection formula. But we also have

$$\pi_{1*}([\Delta]\cup\pi_2^*(\gamma)) = \sum_i \pi_{1*}(\pi_1^*(\alpha_i)\cup\pi_2^*(\beta_{n-i})\cup\pi_2^*(\gamma)) = \sum_i \alpha_i\cup\pi_{1*}\pi_2^*(\beta_{n-i}\cup\gamma).$$

Now, considering

$$\begin{tikzcd} X \times X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & \mathrm{Spec}(k), \end{tikzcd}$$

we see that the classes $\pi_{1*}\pi_2^*(\beta_{n-i} \cup \gamma) = f^*f_*(\beta_{n-i} \cup \gamma)$ are pulled back from $\mathrm{Spec}(k)$ (flat statement). Thus,

$$\gamma = \sum_i \alpha_i\cup\pi_{1*}\pi_2^*(\beta_{n-i}\cup\gamma) = \sum_i \alpha_i\cup f^*f_*(\beta_{n-1}\cup\gamma) \in \mathrm{CH}^i(X, 1)_{\mathrm{dec}}\otimes\mathbb{Z}[\frac{1}{N}].$$

□

If X is an irreducible variety with a compactification $\overline{X}$ that has the CKgP, the proposition says we have an exact sequence

$$0 \to \overline{\mathrm{CH}}^*(X,1) \to \mathrm{CH}^{*-1}(\overline{X} \setminus X) \to \mathrm{CH}^*((\overline{X}),$$

and so the group $\overline{\mathrm{CH}}^*(X,1)$ explains which relations need to hold in $\overline{X}$ between on cycles in $\overline{X} \setminus X$. Thus, we can roughly view $\overline{\mathrm{CH}}^*(X,1)$ as something that measures the failure of compactness, in such situations.

Remark 2.20. When X is a smooth proper Deligne mumford stack with CKgP, it is not necessarily true that $\overline{\mathrm{CH}}^*(X,1)=0$: The stack $\overline{\mathcal{M}}_2$ has the CKgP. In [Larson], it is shown that the pushforward

$$\mathrm{CH}^{*-1}(\Delta_1) \to \mathrm{CH}^*(\overline{\mathcal{M}}_2)$$

is injective despite the fact that $\mathrm{CH}^*(\overline{\mathcal{M}}_2 \backslash \Delta_1, 1, \mathbb{Z}_2) \neq 0$. By the localization exact sequence and our comparison Proposition 2.24, we can conclude that $\overline{\mathrm{CH}}^*(\overline{\mathcal{M}}_2,1)$ has nontrivial 2^m-torsion for some m.

The case of genus one curves seems to be different: The author expects $\overline{\mathrm{CH}}^*(\overline{\mathcal{M}}_{1,n},1) = 0$ for $n \leq 10$. (Corollary 7.4 and the proposition imply $\overline{\mathrm{CH}}^*(\overline{\mathcal{M}}_{1,n},1)_{[\frac{1}{6}]} = 0$ for $n \leq 4$). The quotient stack presentation of $\overline{\mathcal{M}}_{1,1}$ and Lemma 2.23 can be used to show that one does have $\overline{\mathrm{CH}}^*(\overline{\mathcal{M}}_{1,1},1)_{=}0$. Moreover, in [Bishop], it is shown that $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3},1,\mathbb{Z}_\ell) = 0$ for all ℓ, so, by the comparison Proposition 2.24, we have $\overline{\mathrm{CH}}^*(\overline{\mathcal{M}}_{1,3},1)_{=}0$. This suggests that the stack $\overline{\mathcal{M}}_2$ is, in some way, fundamentally different from the stacks $\overline{\mathcal{M}}_{1,n}$ for small n.

Corollary 2.21. *$\overline{\mathrm{CH}}^*(\mathbb{P}^n,1)_{=}0$ and $\overline{\mathrm{CH}}^*(B\mathbb{G}_m,1)_{=}0$.*

Proof. As $\mathbb{P}^n$ is a scheme, hence a Deligne-Mumford stack with lcm of stabilizers equal to 1, with the CKgP, this follows immediately from the proposition. Additionally, we have

$$\overline{\mathrm{CH}}^i(B\mathbb{G}_m,1)_{=}\overline{\mathrm{CH}}^i(\mathbb{P}^n,1)_{=}0$$

for n sufficiently large compared to i. $\square$

From what we know about $B\mathbb{G}_m$, we can prove the following result on $B\mu_n$.

Lemma 2.22. *Suppose n invertible in k. The Chow ring of $B\mu_n$ is give by*

$$\mathrm{CH}^*(B\mu_n) = \frac{\mathbb{Z}[t]}{(nt)}$$

and the higher indecomposible Chow groups are given by

$$\overline{\mathrm{CH}}^*(B\mu_n,1) = 0.$$

Moreover, $B\mu_n$ has the CKgP.

Proof. Let $\mathbb{G}_m$ act on $\mathbb{A}^1$ by $t \cdot x = t^n x$. In the quotient $[\mathbb{A}^1/\mathbb{G}_m]$, we have the origin gives a closed substack isomorphic to $B\mathbb{G}_m$. The complement is $\mathbb{G}_m/(\mathbb{G}_m)^n \cong B\mu_n$. Thus, the localization exact sequence gives

$$\overline{\mathrm{CH}}^*([\mathbb{A}^1/\mathbb{G}_m],1) \to \overline{\mathrm{CH}}^*(B\mu_n,1) \to \mathrm{CH}^{*-1}(B\mathbb{G}_m) \to \mathrm{CH}^*([\mathbb{A}^1/\mathbb{G}_m]).$$

The class $[B\mathbb{G}_m] \in \mathrm{CH}^*([\mathbb{A}^1/\mathbb{G}_m]) = \mathbb{Z}[t]$ is equal to nt, so by the following lemma, we have that the pushforward is given by

$$\mathbb{Z}[t] = \mathrm{CH}^*(B\mathbb{G}_m) \to \mathrm{CH}^*([\mathbb{A}^1/\mathbb{G}_m]) = \mathbb{Z}[t]$$

$$t \mapsto nt.$$

Thus, the cokernel is $\mathrm{CH}^*(B\mu_n) = \frac{\mathbb{Z}[t]}{(nt)}$ and the kernel is 0. Moreover, by Corollary 2.21, we have $\overline{\mathrm{CH}}^*([\mathbb{A}^1/\mathbb{G}_m],1) = 0$, so exactness of the sequence gives $\overline{\mathrm{CH}}^*(B\mu_n,1) = 0$.

Finally, we know $[\mathbb{A}^1/\mathbb{G}_m]$ and $B\mathbb{G}_m$ have the CKgP by Proposition 2.16(3,5). Thus, by Proposition 2.16(1), we have $B\mu_n$ has the CKgP. □

Aside from the proof of the previous, the next lemma will be used quite a bit in section [???].

Lemma 2.23. *Given actions of G on $\mathbb{A}^n, \mathbb{A}^m$, and a G-equivariant map $f : \mathbb{A}^n \to \mathbb{A}^m$, the pullback $f^* : \mathrm{CH}^*_G(\mathbb{A}^m, j) \to \mathrm{CH}^*_G(\mathbb{A}^n, j)$ is an isomorphism. Moreover, if f is proper, we have*

$$f_*(\beta) = \beta \cdot f_*(1)$$

under the identification of $\mathrm{CH}^*_G(\mathbb{A}^m, j)$ and $\mathrm{CH}^*_G(\mathbb{A}^n, j)$ by f^*.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^*_G(\mathbb{A}^m) & & \mathrm{CH}^*_G(\mathbb{A}^n) \arrow[ll, "f^*"'] \\
& \mathrm{CH}^*_G(\mathrm{Spec}(k)) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

By homotopy invariance, the maps out of $\mathrm{CH}^*(\mathrm{Spec}(k))$ are isomorphisms, hence f^* must also be an isomorphism.

If f is proper, and we use f^* to identify the elements of $\mathrm{CH}^*_G(\mathbb{A}^m)$ with $\mathrm{CH}^*_G(\mathbb{A}^n)$, the projection formula gives

$$f_*(\beta) = f_*(f^*(\beta)) = \beta \cdot f_*(1).$$

□

2.5. **Comparison With ℓ-adic Higher Chow Groups.** In [Larson], Larson introduced the notion of ℓ-adic higher Chow groups. If the groups $\mathrm{CH}_i(X_{\overline{k}}, 1, \mathbb{Z}/\ell^m\mathbb{Z}) := H_1(z_i(X_{\overline{k}}, *) \otimes \mathbb{Z}/\ell^{\cdot}\mathbb{Z})$ are finitely generated, then their definition is equivalent to

$$\mathrm{CH}_i(X, 1, \mathbb{Z}_\ell) = \varprojlim \mathrm{CH}_i(X_{\overline{k}}, 1, \mathbb{Z}/\ell^m\mathbb{Z}).$$

These groups share many of the same properties as indecomposible higher Chow groups [Bishop23]. For example, $\mathrm{CH}^*(\mathrm{Spec}(k), 1, \mathbb{Z}_\ell) = 0$, meaning that these groups are also able to "get rid of" the $k^\times$ in $\mathrm{CH}^1(\mathrm{Spec}(k), 1)$. This allows these ℓ-adic higher Chow groups to be finitely generated in

We have the following result relating ℓ-adic higher Chow groups to indecomposible higher Chow groups. This result explains why there must be similarities between these groups.

Proposition 2.24. *Let X be a stack with the $\mathrm{CH}_i(X), \overline{\mathrm{CH}}_i(X,1)$ finitely generated for all i. Then*

$$\mathrm{CH}_i(X, 1, \mathbb{Z}_\ell) = \overline{\mathrm{CH}}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}_\ell.$$

Remark 2.25. Though the assumptions on X are strong, they hold in all cases where ℓ-adic higher Chow groups have been used to compute Chow groups.

Proof. We have an exact sequence

$$\mathrm{CH}_{i+1}(X_{\overline{k}}) \otimes \overline{k}^\times \to \mathrm{CH}_i(X_{\overline{k}}, 1) \to \overline{\mathrm{CH}}_i(X_{\overline{k}}, 1) \to 0.$$

Tensoring this with $\mathbb{Z}/\ell^m\mathbb{Z}$, we have an exact sequence

$$0 \to \mathrm{CH}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z} \to \overline{\mathrm{CH}}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z} \to 0,$$

because $\overline{k}^\times$ is divisible and tensor is right exact. Thus, we have $\mathrm{CH}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z} \cong \overline{\mathrm{CH}}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z}$.

Next, by the universal coefficient theorem, we have a split exact sequence

$$0 \to \mathrm{CH}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z} \to \mathrm{CH}_i(X_{\overline{k}}, 1, \mathbb{Z}/\ell^m\mathbb{Z}) \to \mathrm{Tor}^1(\mathrm{CH}_i(X_{\overline{k}}), \mathbb{Z}/\ell^m\mathbb{Z}) \to 0.$$

Because $\overline{\mathrm{CH}}_i(X_{\overline{k}}, 1)$ is finitely generated, so is $\overline{\mathrm{CH}}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z} = \mathrm{CH}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z}$. And because $\mathrm{CH}_i(X_{\overline{k}})$ is finitely generated, so is $\mathrm{Tor}^1(\mathrm{CH}_i(X_{\overline{k}}), \mathbb{Z}/\ell^m\mathbb{Z})$. Thus, $\mathrm{CH}_i(X_{\overline{k}}, 1, \mathbb{Z}/\ell^m\mathbb{Z})$ is finitely generated for all m, so we have

$$\mathrm{CH}_i(X, 1, \mathbb{Z}_\ell) = \varprojlim \mathrm{CH}_i(X_{\overline{k}}, 1, \mathbb{Z}/\ell^m\mathbb{Z}).$$

In taking the limit, the torsion groups $\mathrm{Tor}^1(\mathrm{CH}_i(X_{\overline{k}}), \mathbb{Z}/\ell^m\mathbb{Z})$ go to 0, because the maps between them are multiplication by ℓ and $\mathrm{CH}_i(X_{\overline{k}})$ is finitely generated, so we have

$$\begin{aligned}
\mathrm{CH}_i(X, 1, \mathbb{Z}_\ell) &= \varprojlim \mathrm{CH}_i(X_{\overline{k}}, 1, \mathbb{Z}/\ell^m\mathbb{Z}) \\
&= \varprojlim \mathrm{CH}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z} \\
&= \varprojlim \overline{\mathrm{CH}}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}/\ell^m\mathbb{Z} \\
&= \overline{\mathrm{CH}}_i(X_{\overline{k}}, 1) \otimes \mathbb{Z}_\ell,
\end{aligned}$$

where the last equality holds because $\overline{\mathrm{CH}}_i(X_{\overline{k}}, 1)$ is finitely generated. □

3. Equivariant Calculations

Recall that if we have an action of $\mathbb{G}_m$ on $\mathbb{A}^n$ and $f \in k[x_1, \ldots, x_n]$ is satisfies $t \cdot f = z^n f$ for $z \in \mathbb{G}_m$, then $[V(f)] = nt \in \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^n) = \mathbb{Z}[t]$.

3.1. $\mathcal{M}_{1,1}$**.** Recall our presentation of $\mathcal{M}_{1,1}$ from the previous section, $\mathcal{M}_{1,1} = [U_1/\mathbb{G}_m]$, where $U_1 = \{(a,b) \in \mathbb{A}^2 | x^3 + ax + b$ has distinct roots$\}$, and $\mathbb{G}_m$ acts with weights $(-4,-6)$. The discriminant of x^3+ax+b is $-4a^3-27b^2$. So the complement of U_1 in $\mathbb{A}^2$ is given by $\iota_1 : \Delta_1 := V(-4a^3-27b^2) \hookrightarrow \mathbb{A}^2$.

We study the map $\iota_{1*} : \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_1) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)$. Let

$$f_1 : \mathbb{A}^1 \to \Delta_1$$
$$c \mapsto (-3c^2, 2c^3)$$

be the normalization. This map is equivariant if we act by $\mathbb{G}_m$ on $\mathbb{A}^1$ with weight -2.

Lemma 3.1. *$\iota_{1*} : \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_1) \xrightarrow{\iota_*} \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) = \mathbb{Z}[t]$ is injective and has image generated by $12t$.*

Proof. Let $\widetilde{i_1} := \iota_1 \circ f_1$, so that we have

$$\begin{tikzcd} \mathbb{A}^1 \arrow[rr, "\widetilde{\iota_1}"] \arrow[dr, "f"'] & & \mathbb{A}^2 \\ & \Delta_1 \arrow[ur, "\iota_1"'] & \end{tikzcd}$$

By Lemma 2.23, we have $\widetilde{\iota_1}^*$ is an isomorphism and $\widetilde{\iota}_{1*}$ is multiplication by

$$\widetilde{\iota}_{1*}([\mathbb{A}^1]) = \iota_{1*} f_{1*}([\mathbb{A}^1]) = \iota_{1*}([\Delta_1]) = [\Delta_1].$$

Because $-4a^3 - 27b^2$ has degree -12, we have $[\Delta_1] = -12t$. And so $\widetilde{\iota}_{1*}$ has image generated by $12t$ and is injective.

Now, f_1 is a universal separable homeomorphism by Example 2.11, so by Proposition 2.12, f_{1*} is an isomorphism. This says that ι_* has the same image, $(12t)$, and is injective. $\square$

Theorem 3.2. *The Chow ring of $\mathcal{M}_{1,1}$ is given by*

$$\mathrm{CH}^*(\mathcal{M}_{1,1}) = \mathbb{Z}[\lambda]/(12\lambda)$$

and the higher indecomposible Chow groups are given by

$$\overline{\mathrm{CH}}^*(\mathcal{M}_{1,1}, 1) = 0.$$

Moreover, $\mathcal{M}_{1,1}$ has the CKgP.

Proof. Note

$$\overline{\mathrm{CH}}^*_{\mathbb{G}_m}(\mathbb{A}^2, 1) \cong \overline{\mathrm{CH}}^*_{\mathbb{G}_m}(\mathrm{pt}, 1) = \overline{\mathrm{CH}}^*(B\mathbb{G}_m, 1) = 0$$

by homotopy invariance and Corollary 2.21. The localization exact sequence then gives

$$0 \to \mathrm{CH}^*(\mathcal{M}_{1,1}, 1)_{\mathrm{ind}} \to \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_1) \xrightarrow{\iota_{1*}} \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \to \mathrm{CH}^*(\mathcal{M}_{1,1}) \to 0.$$

By Lemma 3.1, ι_* is injective and has image $(12t) \subseteq \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) = \mathbb{Z}[t]$. Thus, exactness gives $\overline{\mathrm{CH}}^*(\mathcal{M}_{1,1}, 1) = 0$ and $\mathrm{CH}^*(\mathcal{M}_{1,1}) = \mathbb{Z}[t]/(12t)$.

Finally, we want to say that $\mathcal{M}_{1,1}$ has the CKgP. By Proposition 2.16(1), it suffices to show that $[\mathbb{A}^2/\mathbb{G}_m]$ and $[\Delta_1/\mathbb{G}_m]$ have the CKgP. We know

$[\mathbb{A}^2/\mathbb{G}_m]$ and $[\mathbb{A}^1/\mathbb{G}_m]$ have the CKgP by Proposition 2.16(3,5), and because f_1 is a universal separable homeomorphism, $[\Delta_1/\mathbb{G}_m]$ also has the CKgP by Proposition 2.16(4). □

Remark 3.3. One can adapt the above to calculate the full higher Chow groups of $\mathcal{M}_{1,1}$. For $i \geq 1$ we have

$$\mathrm{CH}^i_{\mathbb{G}_m}(\mathbb{A}^2,1) = \mathrm{CH}^i_{\mathbb{G}_m}(\mathrm{Spec}(k),1) = \mathrm{CH}^i(\mathbb{P}^n,1) = k^\times$$

and

$$\mathrm{CH}^i_{\mathbb{G}_m}(\Delta_1) = \mathrm{CH}^i_{\mathbb{G}_m}(\mathbb{A}^1,1) = k^\times.$$

Then the localization exact sequence gives

$$k^\times \to k^\times \to \overline{\mathrm{CH}}^i(\mathcal{M}_{1,1},1) \to 0.$$

An argument analogous to Lemma 2.23 gives that the map $k^\times \to k^\times$ is $z \mapsto z^{12}$. Thus, one has

$$\mathrm{CH}^*(\mathcal{M}_{1,1},1) = \bigoplus_{i=1}^{\infty} \frac{k^\times}{(k^\times)^{12}}.$$

3.2. $\mathcal{M}_{1,2}$**.** Recall our presentation of $\mathcal{M}_{1,2}$ from the previous section, $\mathcal{M}_{1,2} = [U_2/\mathbb{G}_m]$ for

$$U_2 := \{(a,x_2,y_2) \in \mathbb{A}^3 | -4a^3 - 27B(a,x_2,y_2)^2 \neq 0\}$$

where

$$B(a,x_2,y_2) := y_2^2 - x_2^3 - ax_2$$

and $\mathbb{G}_m$ acts with weights $(-4,-2,-3)$. The complement of U_2 in $\mathbb{A}^3$ is given by $\iota_2 : \Delta_2 := V(-4a^3 - 27b^2) \hookrightarrow \mathbb{A}^2$.

We study the map $\iota_{2*} : \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3)$. We have the $\mathbb{G}_m$-equivariant map

$$\mathbb{A}^3 \to \mathbb{A}^2$$
$$(a,x_2,y_2) \mapsto (a, B(x_2,y_2,a)).$$

This restricts to a map $\Delta_2 \to \Delta_1$, as Δ_2 is, essentially by definition, the inverse image of Δ_1 under this map. Define $\widehat{\Delta}_2 := \mathbb{A}_1 \times_{\Delta_1} \Delta_2$ so that

$$\begin{tikzcd}
\widehat{\Delta}_2 \arrow[r, "f_2"] \arrow[d] & \Delta_2 \arrow[d] \\
\mathbb{A}^1 \arrow[r, "f_1"] & \Delta_1
\end{tikzcd}$$

is Cartesian. Everything involved is an affine scheme, so a tensor product computation gives

$$\widehat{\Delta}_2 = \mathrm{Spec}\, k[x_2,y_2,c]/(y_2^2 - x_2^3 + 3c^2x_2 - 2c^3).$$

Next, define

$$g_2 : \mathbb{A}^2 \to \widehat{\Delta}_2$$
$$(c,d) \mapsto (d^2 - 2c, d^3 - 3cd, c).$$

To make this map equivariant, we act with weight -2 on c and -1 on d. This map is also finite, hence proper, as both c and d are integral over $k[x_2, y_2, c]/(y_2^2 - x_2^3 + 3c^2 x_2 + 2c^3)$.

Over the locus $W_2 := D(x_2 - c)$, we have that g_2 is an isomorphism, as one can check that

$$(x_2, y_2, c) \mapsto \left(c, \frac{y_2}{x_2 - c}\right)$$

is an inverse on this open subset. The reduced complement, $j_2 : C_2 \hookrightarrow \Delta_2$, of W_2 is given by $C_2 = V(x_2 - c, y_2)$. Then $C_2 = \operatorname{Spec} k[x_2, y_2, c]/(x_2 - c, y_2) \cong \operatorname{Spec} k[c]$. Moreover, $g_2^{-1}(C_2) = \operatorname{Spec} k[c, d]/(d^2 - 3c) \cong \operatorname{Spec} k[d]$.

Lemma 3.4. *The image of* $\iota_{2*} : \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3) = \mathbb{Z}[t]$ *is generated by* $12t$*, and the kernel is isomorphic to*

$$\bigoplus_{i=1}^{\infty} \frac{\mathbb{Z}}{2\mathbb{Z}},$$

generated by $f_{2*}(g_{2*}(t^i) - j_{2*}(t^{i-1}))$ *in degree* i*.*

Proof. Lemma 2.1 gives us that

$$\mathrm{CH}^*_{\mathbb{G}_m}(\widehat{\Delta}_2) = \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}C_2)} \mathrm{CH}^*_{\mathbb{G}_m}(C_2).$$

Because $f_2 : \widehat{\Delta}_2 \to \Delta_2$ is a universal separable homeomorphism, we have $f_{2*} : \mathrm{CH}^*_{\mathbb{G}_m}(\widehat{\Delta}_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2)$ is an isomorphism, and so we have

$$\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2) = \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}C_2)} \mathrm{CH}^*_{\mathbb{G}_m}(C_2).$$

Thus, to compute the image of ι_{2*}, it suffices to compute the images of the pushforwards of $\mathbb{A}^2 \to \mathbb{A}^3$ and $C_2 \to \mathbb{A}^3$.

The map $\mathbb{A}^2 \to \mathbb{A}^3$ is the composition

$$\mathbb{A}^2 \xrightarrow{g_2} \widehat{\Delta}_2 \xrightarrow{f_2} \Delta_2 \xrightarrow{\iota_2} \mathbb{A}^3.$$

The first two maps in this composition are birational, and the second is a closed embedding, so $1 \in \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)$ pushes forward to $[\Delta_2]$. Because $-4a^3 - 27B(a, x_2, y_2)$ has degree -12, we have $[\Delta_2] = -12t$. By Lemma 2.23, the map $\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3)$ is multiplication by $-12t$, under the identification of the pullback.

The map $C_2 \cong \operatorname{Spec} k[c] \to \mathbb{A}^3$ is the composition

$$C_2 \xrightarrow{j} \widehat{\Delta}_2 \xrightarrow{f_2} \Delta_2 \xrightarrow{\iota_2} \mathbb{A}^3.$$

Using the definitions of these maps, we can compute it explicitly as $c \mapsto (c, 0, -3c^2)$. The image of this is the closed subscheme $V(y_2, a + 3x_2^2)$. We know $[V(y_2)] = -3t$ and $[V(a + 3x_2^2)] = -4t$, so $[V(y_2, a + 3x_2^2)] = 12t^2$, because the intersection is transverse. By Lemma 2.23, the map $\mathrm{CH}^*_{\mathbb{G}_m}(C_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3)$ is multiplication by $12t^2$, under the identification of the pullback.

Combining this with the conclusion of the previous paragraph, we see that the image of ι_{2*} is generated by $12t$.

We next compute the kernel of ι_{2*}. From the above descriptions, we see that the kernel of $\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)\oplus\mathrm{CH}^*_{\mathbb{G}_m}(C_2)\to\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3)$ is $\{(tp(t),-p(t))|p(t)\in\mathbb{Z}[t]\}$. To get the kernel out of the map out of $\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)\amalg_{\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}C_2)}\mathrm{CH}^*_{\mathbb{G}_m}(C_2)$, we need to quotient this out by the image of $\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}(C_2))\to\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)\oplus\mathrm{CH}^*_{\mathbb{G}_m}(C_2)$.

The map $g_2^{-1}(C_2)\cong\operatorname{Spec}k[d]\to\mathbb{A}^2$ is a closed embedding cut out by d^2-3c. So 1 pushes forward to $[V(d^2-3c)]=-2t$. By Lemma 2.23, we have $\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}(C_2))\to\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)$ is given by $p(t)\mapsto-2tp(t)$.

The map $g_2^{-1}(C_2)\to C_2$ is given by $d\mapsto\frac{d^2}{3}$ under the isomorphisms $C_2\cong\operatorname{Spec}k[c]$ and $g_2^{-1}(C_2)\cong\operatorname{Spec}k[d]$. This is a degree 2 map, so $1\in\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}(C_2))$ maps to 2. By Lemma 2.23, we have $\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}(C_2))\to\mathrm{CH}^*_{\mathbb{G}_m}(C_2)$ is given by $p(t)\mapsto 2p(t)$. Thus, the map $\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}(C_2))\to\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)\oplus\mathrm{CH}^*_{\mathbb{G}_m}(C_2)$ has image $\{(2tp(t),-2p(t))|p(t)\in\mathbb{Z}[t]\}$, and so

$$\begin{aligned}\ker(\iota_2)&\cong\ker(\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)\coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}C_2)}\mathrm{CH}^*_{\mathbb{G}_m}(C_2)\to\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3))\\&\cong\frac{\ker(\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)\oplus\mathrm{CH}^*_{\mathbb{G}_m}(C_2)\to\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3))}{\operatorname{im}(\mathrm{CH}^*_{\mathbb{G}_m}(g_2^{-1}(C_2)\to\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)\oplus\mathrm{CH}^*_{\mathbb{G}_m}(C_2))}\\&=\frac{\{(tp(t),-p(t))|p(t)\in\mathbb{Z}[t]\}}{\{(2tp(t),-2p(t))|p(t)\in\mathbb{Z}[t]\}}\\&\cong\bigoplus_{i=1}^{\infty}\frac{\mathbb{Z}}{2\mathbb{Z}}\end{aligned}$$

□

Theorem 3.5. *The Chow ring of $\mathcal{M}_{1,2}$ is given by*

$$\mathrm{CH}^*(\mathcal{M}_{1,2})=\mathbb{Z}[t]/(12t),$$

and the higher indecomposible Chow groups are given by

$$\overline{\mathrm{CH}}^*(\mathcal{M}_{1,2},1)=\bigoplus_{i=2}^{\infty}\frac{\mathbb{Z}}{2\mathbb{Z}}.$$

Moreover, $\mathcal{M}_{1,2}$ has the CKgP.

Proof. We have $\overline{\mathrm{CH}}^*_{\mathbb{G}_m}(\mathbb{A}^3,1)=0$ by Proposition 2.19 and Lemma **??**, so the localization exact sequence gives

$$0\to\mathrm{CH}^*(\mathcal{M}_{1,2},1)_{\mathrm{ind}}\to\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_1)\xrightarrow{\iota_{2*}}\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)\to\mathrm{CH}^*(\mathcal{M}_{1,2})\to 0.$$

By Lemma 3.4, ι_{2*} has kernel isomorphic to $\bigoplus_{i=1}^{\infty}\frac{\mathbb{Z}}{2\mathbb{Z}}$. Due to the shift of indexing, we conclude $\overline{\mathrm{CH}}^*(\mathcal{M}_{1,2},1)=\bigoplus_{i=2}^{\infty}\frac{\mathbb{Z}}{2\mathbb{Z}}$. Moreover, the lemma also says that the image of ι_{2*} is $(12t)$, so $\mathrm{CH}^*(\mathcal{M}_{1,2})=\mathbb{Z}[t]/(12t)$.

Finally, we want to show that $\mathcal{M}_{1,2}$ has the CKgP. By Proposition 2.16(1), it suffices to show that $[\mathbb{A}^3/\mathbb{G}_m]$ and $[\Delta_2/\mathbb{G}_m]$ have the CKgP. By Proposition 2.16(3,5), $[\mathbb{A}^3/\mathbb{G}_m]$ has the CKgP. By Proposition 2.16(4), it suffices to show that $[\widehat{\Delta}_2/\mathbb{G}_m]$ has the CKgP. The closed substack $[C/\mathbb{G}_m] \subseteq [\widehat{\Delta}_2/\mathbb{G}_m]$ has the CKgP because it is isomorphic to $[\mathbb{A}^1/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts with weight -2, so by Proposition 2.16(1) it suffices to show that the complement $[W/\mathbb{G}_m]$ has the CKgP. For this, we use that $[W/\mathbb{G}_m] \cong [g^{-1}(W)/\mathbb{G}_m]$, and $[g^{-1}(W)/\mathbb{G}_m]$ is open in $[\mathbb{A}^2/\mathbb{G}_m]$ with complement $[g^{-1}(C_2)/\mathbb{G}_m] \cong [\mathbb{A}^1/\mathbb{G}_m]$. These latter two spaces have the CKgP, so by Proposition 2.16(1), so does $[g^{-1}(W)/\mathbb{G}_m]$. $\square$

Remark 3.6. One could try and use the above to calculate the full higher Chow groups $\mathrm{CH}^*(\mathcal{M}_{1,2},1)$. With similar reasoning to Remark 3.3, one gets an exact sequence

$$0 \to \frac{k^\times}{(k^\times)^{12}} \to \mathrm{CH}^i(\mathcal{M}_{1,2},1) \to \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0.$$

Thus, one needs to solve an extension problem to figure out the group $\mathrm{CH}^i(\mathcal{M}_{1,2},1)$. The author was able to show the extension must be trivial if the field k contains a square-root of -1, but does not know what happens otherwise. These group extension problems persist (and increase in number) when trying to compute $\mathrm{CH}^i(\mathcal{M}_{1,n},1)$ for $n=3,4$.

3.3. $\mathcal{M}^0_{1,2}$**.** Recall that $\mathcal{M}^0_{1,2}$ has presentation $[U_2 \setminus V(y_2)/\mathbb{G}_m]$. To compute $\mathrm{CH}^*(\mathcal{M}^0_{1,2}), \overline{\mathrm{CH}}^*(\mathcal{M}^0_{1,2},1)$, we first remove $V(y_2)$ from $\mathbb{A}^2$, and then remove $V(-4a^3-27B(x_2,y_2)^2 \setminus D(y_2)$ from that. One may expect it to be easier to remove $V(y_2) \cap U_2$ from U_2, as we have already done calculations to obtain $\mathrm{CH}^*_{\mathbb{G}_m}(U_2), \overline{\mathrm{CH}}^*_{\mathbb{G}_m}(U_2,1)$, but this is not the case.

Lemma 3.7.

$$\mathrm{CH}^*_{\mathbb{G}_m}(D(y_2)) = \mathbb{Z}[t]/(3t)$$

and

$$\overline{\mathrm{CH}}^*_{\mathbb{G}_m}(D(y_2),1) = 0$$

Proof. Because $\overline{\mathrm{CH}}^*_{\mathbb{G}_m}(\mathbb{A}^2,1)=0$, the localization exact sequence for $D(y_2) \subseteq \mathbb{A}^2$ gives

$$0 \to \mathrm{CH}^*_{\mathbb{G}_m}(D(y_2),1) \to \mathrm{CH}^*_{\mathbb{G}_m}(V(y_2)) \xrightarrow{q_*} \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3) \to \mathrm{CH}^*_{\mathbb{G}_m}(D(y_2)) \to 0.$$

Note $V(y_2) \cong \mathbb{A}^1$, and $[V(y_2)] = -3t \in \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3)$ because y_2 has weight -3, so by Lemma 2.23, we know that the maps q_* are multiplication by $-3t$, under the identification by the pullback. By exactness, we get

$$\mathrm{CH}^*_{\mathbb{G}_m}(D(y_2)) = \mathbb{Z}[t]/(3t)$$

$$\overline{\mathrm{CH}}^*_{\mathbb{G}_m}(D(y_2),1) = 0$$

$\square$

Set $\Delta_2^0 := \Delta_2 \cap D(y_2)$, with closed embedding $\iota_2^0 : \Delta_2^0 \hookrightarrow D(y_2)$, so then $\mathcal{M}_{1,2}^0 \cong [(D(y_2) \setminus \Delta_2^0)/\mathbb{G}_m]$.

Lemma 3.8. *$\iota_{2*}^0 : \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2^0) \to \mathrm{CH}^*_{\mathbb{G}_m}(D(y_2))$ is equal to zero and $\mathrm{CH}^*(\Delta_2^0) = \mathbb{Z}$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2) \arrow[r, "\iota_{2*}"] \arrow[d] & \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3) \arrow[d] \\
\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2^0) \arrow[r, "\iota_{2*}^0"] & \mathrm{CH}^*_{\mathbb{G}_m}(D(y_2))
\end{tikzcd}$$

As noted above, we can identify the right map with $\mathbb{Z}[t] \to \mathbb{Z}[t]/(3t)$. Moreover, by Lemma 3.4, the image of ι_{2*} is $(12t)$, so composition $\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3) \to \mathrm{CH}^*_{\mathbb{G}_m}(D(y_2))$ is 0. Finally, because Δ_2^0 is open in Δ_2, the left map is surjective, and so $\iota_{2*}^0 = 0$.

We pullback $D(y_2)$ through f', a universal separable homeomorphism, and g to obtain

$$\begin{tikzcd}
\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \arrow[r] \arrow[d] & \mathrm{CH}^*_{\mathbb{G}_m}(g^{-1}f'^{-1}(\Delta_2^0)) \arrow[r] \arrow[d] & 0 \\
\mathrm{CH}^*_{\mathbb{G}_m}(\widehat{\Delta}_2) \arrow[r] \arrow[d, "\sim"] & \mathrm{CH}^*_{\mathbb{G}_m}(f'^{-1}(\Delta_2^0)) \arrow[r] \arrow[d, "\sim"] & 0 \\
\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2) \arrow[r] & \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2^0) \arrow[r] & 0
\end{tikzcd}$$

As $\Delta_2^0 = D_{\Delta_2}(y_2)$, we have $f'^{-1}(\Delta_2^0) = f'^{-1}(D_{\Delta_2}(y_2)) = D_{\widehat{\Delta}_2}(y_2)$ and $g^{-1}(f'^{-1}(D_{\Delta_2}(y_2))) = g^{-1}(D_{\widehat{\Delta}_2}(y_2)) = D_{\mathbb{A}^2}(d^3 - 3cd)$. Now, we have

$$f'^{-1}(\Delta_2^0) = D_{\widehat{\Delta}_2}(y_2) \subseteq D_{\widehat{\Delta}_2}(x_2 - c) = W,$$

because, in $\widehat{\Delta}_2$, we have

$$y_2^2 = x_2^3 - 3c^2x_2 + 2c^3 = (x_2 - c)^2(x_2 + 2c).$$

Moreover, we have g is an isomorphism over W, and so our pushforward

$$\mathrm{CH}^*(D(d^3 - 3cd)) \xrightarrow{(g\circ f')_*} \mathrm{CH}^*(\Delta_2^0)$$

is an isomorphism.

Because $D(d^3 - 3cd)$ is an open in $D(d)$, we have $\mathrm{CH}^*_{\mathbb{G}_m}(D(d^3 - 3cd))$ is a quotient of $\mathrm{CH}^*_{\mathbb{G}_m}(D(d))$. Because d is acted on by weight -1, $[V(d)] = -t \in \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2)$, so Lemma 2.23 and the exact sequence

$$\mathrm{CH}^*_{\mathbb{G}_m}(V(d)) \to \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^2) \to \mathrm{CH}^*_{\mathbb{G}_m}(D(d)) \to 0$$

imply $\mathrm{CH}^*_{\mathbb{G}_m}(D(d)) = \mathbb{Z}$. Thus, we also have $\mathrm{CH}^*_{\mathbb{G}_m}(D(d^3 - 3cd)) = \mathbb{Z}$, and so $\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_2^0) = \mathbb{Z}$. $\square$

Theorem 3.9.

$$\mathrm{CH}^*(\mathcal{M}^0_{1,2}) = \mathbb{Z}[t]/(3t)$$

and

$$\overline{\mathrm{CH}}^*(\mathcal{M}^0_{1,2},1) = \mathbb{Z}.$$

Proof. We saw $\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^2}(y_2)) = \mathbb{Z}[t]/(3t)$ and $\overline{\mathrm{CH}}^*_{\mathbb{G}_m}(D_{\mathbb{A}^2}(y_2),1) = 0$. So the localization sequence for $U_2 \setminus V(y_2) \subseteq D_{\mathbb{A}^2}(y_2)$ gives

$$0 \to \overline{\mathrm{CH}}^*(\mathcal{M}^0_{1,2},1) \to \mathrm{CH}^*_{\mathbb{G}_m}(\Delta^0_2) \xrightarrow{\iota^0_2} \mathbb{Z}[t]/(3t) \to \mathrm{CH}^*(\mathcal{M}^0_{1,2}) \to 0$$

By Lemma 3.8, we know $\iota^0_2 = 0$ and $\mathrm{CH}^*_{\mathbb{G}_m}(Z^0_2) = \mathbb{Z}$, from which the theorem follows.

Finally, we want $\mathcal{M}^0_{1,2}$ to have the CKgP. WE can see that $[D(y_2)/\mathbb{G}_m]$ has the CKgP by Proposition 2.16(1,3,5), because it is the complement of an $[\mathbb{A}^1/\mathbb{G}_m]$ in $[\mathbb{A}^2/\mathbb{G}_m]$. And so it suffices to show that $[\Delta^0_2/\mathbb{G}_m]$ has the CKgP. As noted above, this is isomorphic to $[D(d^3-3cd)/\mathbb{G}_m] \subsetneq [\mathbb{A}^2/\mathbb{G}_m]$, so it suffices to show that $[V(d^3-3cd)/\mathbb{G}_m]$ does. Note $V(d^3-3cd) = V(d) \cup V(d^2-3c)$, both of which are isomorphic to $\mathbb{A}^1$. These intersect at a point, so $[V(d^3-3cd)/\mathbb{G}_m]$ can be cut up into an open isomorphic to $[\mathbb{G}_m/\mathbb{G}_m] \amalg [\mathbb{G}_m/\mathbb{G}_m]$ and a closed isomorphic to $B\mathbb{G}_m$. Thus, $[V(d^3-3cd)/\mathbb{G}_m]$ has the CKgP by Proposition 2.16(1). □

3.4. $\mathcal{M}^0_{1,3}$. Recall our presentation of $\mathcal{M}^0_{1,3}$, $[(U_3/\mathbb{G}_m]$ for

$$U_3 := \{(x_2,y_2,x_3,y_3) \in D_{\mathbb{A}^4}(x_3-x_2) | -4A^3 - 27B^2 \neq 0\},$$

where

$$A = A(x_2,y_2,x_3,y_3) := \frac{y_3^2 - y_2^2 - x_3^3 + x_2^3}{x_3 - x_2}$$

$$B = B(x_2,y_2,x_3,y_3) := y_2^2 - x_2^3 - Ax_2$$

and $\mathbb{G}_m$ acts with weights $(-2,-3,-2,-3)$.

We first remove $V(x_3-x_2)$ from $\mathbb{A}^4$. Similar to the proof of Lemma 3.7, we have

$$\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3-x_2)) = \mathbb{Z}[t]/(2t)$$

$$\overline{\mathrm{CH}}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3-x_2),1) = 0,$$

because the weight of the action on $x_3 - x_2$ is -2. Let $\iota_3 : \Delta^0_3 \hookrightarrow D_{\mathbb{A}^4}(x_3-x_2)$ be the complement of U_3 in $D_{\mathbb{A}^4}(x_3-x_2)$, so $\Delta^0_3 := V(-4A^3-27B^2)$.

We study the map $\iota_{3*} : \mathrm{CH}^*_{\mathbb{G}_m}(\Delta^0_3) \to \mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3-x_2))$. We have the equivariant map

$$D_{\mathbb{A}^4}(x_3-x_2) \to \mathbb{A}^2$$

$$(x_2,y_2,x_3,y_3) \mapsto (A(x_2,y_2,x_3,y_3), B(x_2,y_2,x_3,y_3)).$$

This restricts to a map $\Delta^0_3 \to \Delta_1$, as Δ^0_3 is, essentially by definition, the inverse image of Δ_1 under this map. Define $\widehat{\Delta}^0_3 := \mathbb{A}_1 \times_{\Delta_1} \Delta^0_3$, so

$$\begin{array}{ccc} \widehat{\Delta}_3^0 & \xrightarrow{f_3} & \Delta_3^0 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \xrightarrow{f_1} & \Delta_1 \end{array}$$

is Cartesian. Everything involved is an affine scheme, so a tensor product computation gives

$$\widehat{\Delta}_3^0 = \operatorname{Spec} k[x_2, y_2, x_3, y_3, c]_{x_3 - x_2}/(A + 3c^2, B - 2c^3).$$

Now, we parameterize $\widehat{\Delta}_3^0$ by

$$(c, d_2, d_3) \mapsto (d_2^2 - 2c, d_2^3 - 3cd_2, d_3^2 - 2c, d_3^3 - 3cd_3, c)$$

Note the similarities with the parameterization $g_2 : \mathbb{A}^2 \to \widehat{\Delta}_2$. This this tuple satisfies the equations $A + 3c^2 = B - 2c^3 = 0$, and so they determine a morphism to $\widehat{\Delta}_3^0$ so long as $x_2 \neq x_3$, i.e. $d_2^2 \neq d_3^2$. And so the above gives a morphism $g_3 : \operatorname{Spec} k[c, d_2, d_3]_{d_3^2 - d_2^2} \to \widehat{\Delta}_3^0$. To make this map equivariant, we act with weights $(-2, -1, -1)$. Note g_3 is finite, hence proper, as c, d_2, d_3 are integral over $k[x_2, y_2, x_3, y_3, c]_{x_3 - x_2}/(A + 3c^2, B - 2c^3)$.

Over the locus $W_3 := D_{\widehat{\Delta}_3^0}((x_2 - c)(x_3 - c))$, we have that g_3 is an isomorphism, as

$$(x_2, y_2, x_3, y_3, c) \mapsto \left(c, \frac{y_2}{x_2 - c}, \frac{y_3}{x_3 - c}\right)$$

gives an inverse on this open subset. The reduced complement of W_3 is given by $C_3 := C_3^2 \cup C_3^3 \subseteq \widehat{\Delta}_3^0$, where

$$C_3^\ell := V(x_\ell - c, y_\ell) \overset{j_3^\ell}{\hookrightarrow} \widehat{\Delta}_3^0.$$

Calculations on the ideals give that $C_3^2 = \operatorname{Spec}(R)$ where

$$R = \frac{k[x_2, y_2, x_3, y_3, c]_{x_3 - c}}{(y_2, x_2 - c, y_3^2 - x_3^3 + 3c^2 x_3 - 2c^3)} \cong \frac{k[x_3, y_3, c]_{x_3 - c}}{(y_3^2 - x_3^3 + 3c^2 x_3 - 2c^3)}.$$

Note that this latter expression gives exactly the open $W \subseteq \widehat{\Delta}_2$, up to changing variable names, and recall $g_2^{-1}(W_2) \to W_2$ was an isomorphism. Using this, we have an isomorphism

$$\operatorname{Spec} k[c, d]_{d^2 - 3c} \xrightarrow{\sim} C_3^2$$
$$(c, d) \mapsto (c, 0, d^2 - 2c, d^3 - 3cd, c)$$

from which we can compute the Chow groups of C_3^2. Furthermore, we can compute,

$$g_3^{-1}(C_3^2) = \operatorname{Spec} k[c, d_2, d_3]_{d_3^2 - d_2^2}/(d_2^2 - 3c) \cong \operatorname{Spec} k[d_2, d_3]_{d_3^2 - d_2^2}.$$

Analogously, we get an isomorphism

$$\operatorname{Spec} k[c, d]_{d^2 - 3c} \xrightarrow{\sim} C_3^3$$
$$(c, d) \mapsto (d^2 - 2c, d^3 - 3cd, c, 0, c)$$

and

$$g_3^{-1}(C_3^3) = \operatorname{Spec} k[c,d_2,d_3]_{d_3^2-d_2^2}/(d_3^2-3c) \cong \operatorname{Spec} k[d_2,d_3]_{d_3^2-d_2^2}.$$

Lemma 3.10. *$\iota_{3*}: \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_3^0) \to \mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3-x_2))$ is equal to zero and*

$$\mathrm{CH}^*(\Delta_3^0) = \mathbb{Z} \oplus \bigoplus_{i=1}^{\infty} \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$$

generated by $[\Delta_3^0]$ in degree 0 and $f_{3*}j_{3*}^2(t^{i-1}), f_{3*}j_{3*}^3(t^{i-1})$ in degrees $i > 0$.

Proof. First, note that C_3^2 and C_3^3 are disjoint: they are cut out of of $\widehat{\Delta}_3^0$ by (x_2-c,y_2) and (x_3-c,y_3) respectively, and so any point in there intersection would be a zero of x_3-x_2, which cannot happen. Furthermore, this implies $g_3^{-1}(C_3^2)$ and $g_3^{-1}(C_3^3)$ are also disjoint.

Using Lemma 2.1, we have

$$\begin{aligned}\mathrm{CH}^*_{\mathbb{G}_m}(\widehat{\Delta}_3^0) &= \mathrm{CH}^*_{\mathbb{G}_m}(W_3) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g_3^{-1}C_3)} \mathrm{CH}^*_{\mathbb{G}_m}(C_3) \\ &= CH_{\mathbb{G}_m}(W_3) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g_3^{-1}C_3)} (\mathrm{CH}^*_{\mathbb{G}_m}(C_3^2) \oplus \mathrm{CH}^*_{\mathbb{G}_m}(C_3^3)).\end{aligned}$$

And we also know the pushforward $f_{3*}: \mathrm{CH}^*_{\mathbb{G}_m}(\widehat{\Delta}_3^0) \to \mathrm{CH}^*_{\mathbb{G}_m}(\Delta_3^0)$ is an isomorphism, since f_3 is a universal separable homeomorphism. Thus, to show ι_{3*} is zero, it suffices to show that pushforwards of $D(d^2-e^2) \to D_{\mathbb{A}^4}(x_3-x_2)$ and $C'_\ell \to D_{\mathbb{A}^4}(x_3-x_2)$ are zero.

The map $D(d^2-e^2) \to D_{\mathbb{A}^4}(x_3-x_2)$ is the composition

$$D(d^2-e^2) \xrightarrow{g'} \widehat{\Delta}_3^0 \xrightarrow{f''} \Delta_3^0 \xhookrightarrow{\iota_3} D_{\mathbb{A}^4}(x_3-x_2).$$

Using the definitions of these maps, we can compute it explicitly as $(c,d_2,d_3) \mapsto (d_2^2-2c, d_2^3-3cd_2, d_3^2-2c, d_3^3-3cd_3)$. This extends to a morphism $p: \mathbb{A}^3 \to \mathbb{A}^4$. This extension is finite, hence proper, because d_2 is a zero of the monic polynomial $t^3-3t(d_2^2-2c)+2(d_2^3-3cd_2) = 0$ and d_3 is a zero of the analogous polynomial. By Lemma 2.23, the pushforward on Chow is given by multiplication by $p_*(1)$, under the identification of the pullback p^*. Because $p|_{D(d^2-e^2)}: D(d^2-e^2) \to \Delta_3^0$ is birational, the pushforward of 1 is $[\Delta_3]$, where Δ_3 is defined to be the closure of Δ_3^0 in $\mathbb{A}^4$. Δ_3^0 is defined by $-4A^3-27B^2 = 0$, but this equation has denominators, so we cannot say the same equation defines Δ_3. Instead, we have $\Delta_3 = V((x_3-x_2)^3(-4A^3-27B^2))$, as this is how many factors of x_3-x_2 are needed to clear the denominators. The weight of $(x_3-x_2)^3(-4A^3-27B^2))$ is -18, so the image of p_* is $(18t)$. Using the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^3) \arrow[r, "p_*"] \arrow[d] & \mathrm{CH}^*_{\mathbb{G}_m}(\mathbb{A}^4) = \mathbb{Z}[t] \arrow[d] \\
\mathrm{CH}^*_{\mathbb{G}_m}(D(d^2-e^2)) \arrow[r] & \mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3-x_2)) = \frac{\mathbb{Z}[t]}{(2t)}
\end{tikzcd}$$

we have that the bottom map is 0, because multiples of $18t$ go to 0 under the right map and the left map is surjective.

The map $C_3^3 \to D_{\mathbb{A}^4}(x_3-x_2)$ is the composition $C_3^3 \hookrightarrow \widehat{\Delta}_3^0 \xrightarrow{f''} \Delta_3^0 \overset{\iota_3}{\hookrightarrow} D_{\mathbb{A}^4}(x_3-x_2)$. Using the definitions of these maps, we can compute it explicitly as $(x_2,y_2,c) \mapsto (x_2,y_2,c,0)$, using our identification $C_3^2 \cong \mathrm{Spec}(k[x_2,y_2,c]_{x_2-c}/(y_2^2-x_2^3+3c^2x_2-2c^3))$. Note that this is just the space W. Composing $g^{-1}(W) \to W$ with the above $C_3^3 \to D_{\mathbb{A}^4}(x_3-x_2)$, we get

$$(c,d) \mapsto (d^2-2c, d^3-3cd, c) \mapsto (d^2-2c, d^3-3cd, c, 0).$$

This extends to a finite map $q : \mathbb{A}^2 \to \mathbb{A}^4$. The image of this map is $V(y_3, y_2^2-x_2^3+3x_3^2x_2-2x_3^3)$, which has fundamental class $-18t^2$. A similar argument to the last paragraph using this extension q, Lemma 2.23, and a commutative diagram shows that the pushforward of $C_3^3 \to D_{\mathbb{A}^4}(x_3-x_2)$ is 0, since $-18t^2$ gets killed in $\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3-x_2)) = \mathbb{Z}[t]/(2t)$. Analogously, the pushforward of $C_3^2 \to D_{\mathbb{A}^4}(x_3-x_2)$ is 0.

Now, we compute $\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_3^0) \cong CH_{\mathbb{G}_m}(D(d^2-e^2)) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g'^{-1}C')} \mathrm{CH}^*_{\mathbb{G}_m}(C')$. Note $D(e^2-f^2) \subseteq D(e-f)$. Because $e-f$ has weight -1, arguing as in Lemma 3.7 gives $\mathrm{CH}^*(D(e-f)) = \mathbb{Z}$, and then localization gives a surjection $\mathbb{Z} = \mathrm{CH}^*_{\mathbb{G}_m}(D(e-f)) \twoheadrightarrow \mathrm{CH}^*_{\mathbb{G}_m}(D(d^2-e^2))$, and so $\mathrm{CH}^*_{\mathbb{G}_m}(D(d^2-e^2)) = \mathbb{Z}$. Furthermore, $C'_\ell \cong W \cong g^{-1}(W) = D_{\mathbb{A}^2}(d^2-3c)$, and d^2-3c has weight -2, so arguing as in Lemma 3.7, we have $\mathrm{CH}^*_{\mathbb{G}_m}(C'_\ell) = \mathbb{Z}[t]/(2t)$.

Now, $g'^{-1}(C')$ is a closed subscheme of $D(d^2-e^2)$, so the pushforward on $\mathrm{CH}^*_{\mathbb{G}_m}$ must land in degrees ≥ 1. But, as we just saw, $\mathrm{CH}^*_{\mathbb{G}_m}(D(d^2-e^2)) = \mathbb{Z}$, and so the pushforward is the zero map. To compute the pushforward of the other map $g'^{-1}(C') \to C'$, recall $C' = C_3^2 \cup C_3^3$ and $g'^{-1}(C') = g'^{-1}(C_3^2) \cup g'^{-1}(C_3^3)$. It suffices to just compute the pushforward $g'^{-1}(C'_\ell) \to C'_\ell$. Note that

$$\mathrm{Spec}(k[d,e]_{d^2-e^2}) \cong g'^{-1}(C_3^3) \xrightarrow{g'|_{g'^{-1}(C_3^3)}} C_3^3 \cong \mathrm{Spec}(k[x_2,y_2,c]_{x_2-c}/(y_2^2-x_2^3+3c^2x_2-2c^3))$$

$$(d,e) \mapsto (d^2-\frac{2}{3}e^2, d^3-e^2d, \frac{e^2}{3})$$

factors as

$$g'^{-1}(C_3^3) \xrightarrow{h} \mathrm{Spec}(k[c,d]_{e^2-3c}) \to C_3^3$$

$$(d,e) \mapsto (\frac{e^2}{3}, d)$$

$$(c,d) \mapsto (d^2-2c, d^3-3cd, c).$$

Moreover, this second map is precisely the map $g|_{g^{-1}(W)}$, which we know to be an isomorphism. Similar to our computation that $\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^3}(d^2-e^2)) =$

$\mathbb{Z}$ in the previous paragraph, we have $\mathrm{CH}^*_{\mathbb{G}_m}(g'^{-1}(C_3^3)) = \mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^2}(d^2 - e^2)) = \mathbb{Z}$, and we have $\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^2}(e^2 - 3c)) = \mathbb{Z}[t]/(2t)$, using Lemma 3.7. We can compute $h_*(1) = 2$, since h is a degree 2 map between varieties of the same dimension. Because $\mathrm{CH}^*_{\mathbb{G}_m}(g'^{-1}(C_3^3)) = \mathbb{Z}$, this determines h_*.

Now, we have

$$\begin{aligned}
\mathrm{CH}^*_{\mathbb{G}_m}(\Delta_3^0) &\cong \mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^3}(d^2 - e^2)) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g'^{-1}C')} \mathrm{CH}^*_{\mathbb{G}_m}(C') \\
&= \mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^3}(d^2 - e^2)) \coprod_{\mathrm{CH}^*_{\mathbb{G}_m}(g'^{-1}C_3^2) \oplus \mathrm{CH}^*_{\mathbb{G}_m}(g'^{-1}C_3^3)} (\mathrm{CH}^*_{\mathbb{G}_m}(C_3^2) \oplus \mathrm{CH}^*_{\mathbb{G}_m}(C_3^3)) \\
&= \mathbb{Z} \coprod_{\mathbb{Z}\oplus\mathbb{Z}} \mathbb{Z}[t]/(2t) \oplus \mathbb{Z}[t]/(2t) \\
&\cong \mathbb{Z} \oplus \bigoplus_{i=1}^{\infty} (\mathbb{Z}/(2\mathbb{Z}))^2
\end{aligned}$$

$\square$

Theorem 3.11. *The Chow ring of $\mathcal{M}^0_{1,3}$ is given by*

$$\mathrm{CH}^*(\mathcal{M}^0_{1,3}) = \mathbb{Z}[t]/(2t),$$

and the first higher Chow group of $\mathcal{M}^0_{1,3}$ is given by

$$\overline{\mathrm{CH}}^*(\mathcal{M}^0_{1,3}, 1) = \mathbb{Z} \oplus \bigoplus_{i>2} (\frac{\mathbb{Z}}{2\mathbb{Z}})^2.$$

Proof. We saw $\mathrm{CH}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3 - x_2)) = \frac{\mathbb{Z}[t]}{(3t)}$ and $\overline{\mathrm{CH}}^*_{\mathbb{G}_m}(D_{\mathbb{A}^4}(x_3 - x_2), 1) = 0$. So the localization exact sequence for $\mathcal{M}^0_{1,3} \subseteq [D_{\mathbb{A}^4}(x_3 - x_2)/\mathbb{G}_m]$ is

$$0 \to \overline{\mathrm{CH}}^*(\mathcal{M}^0_{1,3}, 1) \to \mathrm{CH}^{*-1}(\Delta_3^0) \xrightarrow{\iota_{3*}} \frac{\mathbb{Z}[t]}{(3t)} \to \mathrm{CH}^*(\mathcal{M}^0_{1,3}) \to 0.$$

Lemma 3.10 says that $\iota_{3*} = 0$, and

$$\mathbb{Z} \oplus \bigoplus_{i=1}^{\infty} \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$$

from which the Theorem follows. $\square$

Remark 3.12. We have the higher Chow cycles on $\mathcal{M}^0_{1,3}$ come from $\mathcal{M}_{1,2}$ in the following sense:

Let $\pi : \mathcal{M}^0_{1,3} \to \mathcal{M}_{1,2}$ be the map forgetting the third point and $\pi' : \mathcal{M}^0_{1,3} \to \mathcal{M}_{1,2}$ be the map forgetting the second point. Then one can verify that

$$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 = \overline{\mathrm{CH}}^i(\mathcal{M}_{1,2}, 1) \xrightarrow{\pi^* \oplus \pi'^*} \overline{\mathrm{CH}}^i(\mathcal{M}^0_{1,3}, 1) = \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$$

is an isomorphism by explicitly computing the pullbacks. Alternatively, this will follow from computations on the boundary $\overline{\mathcal{M}}_{1,3}$ in Proposition 8.6.

4. $\mathcal{M}^0_{1,4}$

Recall our presentation of $\mathcal{M}^0_{1,4}$ as $U_4 = \mathbb{P}(V) \setminus \Delta_4$, where

$$V_4 = \{A_0x^2y^2 - A_0x^2yz - A_0xwy^2 + A_4xwyz + A_5xwz^2 + A_7w^2yz + A_8w^2z^2\},$$

and $\Delta_4 = V(A_0F)$, where

$$
\begin{aligned}
F := {} & 27A_0^5A_8^4 + 36A_0^4A_4A_5A_8^3 + 36A_0^4A_4A_7A_8^3 - 36A_0^4A_4A_8^4 + 16A_0^4A_5^3A_8^2 - \\
& 24A_0^4A_5^2A_7A_8^2 + 24A_0^4A_5^2A_8^3 - 24A_0^4A_5A_7^2A_8^2 + 12A_0^4A_5A_7A_8^3 - 24A_0^4A_5A_8^4 + \\
& 16A_0^4A_7^3A_8^2 + 24A_0^4A_7^2A_8^3 - 24A_0^4A_7A_8^4 - 16A_0^4A_8^5 + A_0^3A_4^3A_8^3 + \\
& 8A_0^3A_4^2A_5^2A_8^2 + 46A_0^3A_4^2A_5A_7A_8^2 - 46A_0^3A_4^2A_5A_8^3 + 8A_0^3A_4^2A_7^2A_8^2 - 46A_0^3A_4^2A_7A_8^3 + \\
& 8A_0^3A_4^2A_8^4 + 16A_0^3A_4A_5^3A_7A_8 - 16A_0^3A_4A_5^3A_8^2 - 64A_0^3A_4A_5^2A_7^2A_8 + 76A_0^3A_4A_5^2A_7A_8^2 - \\
& 64A_0^3A_4A_5^2A_8^3 + 16A_0^3A_4A_5A_7^3A_8 + 76A_0^3A_4A_5A_7^2A_8^2 - 76A_0^3A_4A_5A_7A_8^3 - 16A_0^3A_4A_5A_8^4 - \\
& 16A_0^3A_4A_7^3A_8^2 - 64A_0^3A_4A_7^2A_8^3 - 16A_0^3A_4A_7A_8^4 - 16A_0^3A_5^4A_7^2 - 16A_0^3A_5^4A_8^2 + \\
& 32A_0^3A_5^3A_7^3 - 16A_0^3A_5^3A_7^2A_8 + 16A_0^3A_5^3A_7A_8^2 - 32A_0^3A_5^3A_8^3 - 16A_0^3A_5^2A_7^4 - \\
& 16A_0^3A_5^2A_7^3A_8 + 66A_0^3A_5^2A_7^2A_8^2 - 16A_0^3A_5^2A_7A_8^3 - 16A_0^3A_5^2A_8^4 + 16A_0^3A_5A_7^3A_8^2 - \\
& 16A_0^3A_5A_7^2A_8^3 - 16A_0^3A_7^4A_8^2 - 32A_0^3A_7^3A_8^3 - 16A_0^3A_7^2A_8^4 + A_0^2A_4^4A_5A_8^2 + \\
& A_0^2A_4^4A_7A_8^2 - A_0^2A_4^4A_8^3 + 8A_0^2A_4^3A_5^2A_7A_8 - 8A_0^2A_4^3A_5^2A_8^2 + 8A_0^2A_4^3A_5A_7^2A_8 - \\
& 57A_0^2A_4^3A_5A_7A_8^2 + 8A_0^2A_4^3A_5A_8^3 - 8A_0^2A_4^3A_7^2A_8^2 + 8A_0^2A_4^3A_7A_8^3 - 8A_0^2A_4^2A_5^3A_7^2 - \\
& 16A_0^2A_4^2A_5^3A_7A_8 - 8A_0^2A_4^2A_5^3A_8^2 - 8A_0^2A_4^2A_5^2A_7^3 + 108A_0^2A_4^2A_5^2A_7^2A_8 - 108A_0^2A_4^2A_5^2A_7A_8^2 + \\
& 8A_0^2A_4^2A_5^2A_8^3 - 16A_0^2A_4^2A_5A_7^3A_8 - 108A_0^2A_4^2A_5A_7^2A_8^2 - 16A_0^2A_4^2A_5A_7A_8^3 - 8A_0^2A_4^2A_7^3A_8^2 + \\
& 8A_0^2A_4^2A_7^2A_8^3 + 16A_0^2A_4A_5^4A_7^2 - 16A_0^2A_4A_5^4A_7A_8 - 64A_0^2A_4A_5^3A_7^3 + 76A_0^2A_4A_5^3A_7^2A_8 - \\
& 64A_0^2A_4A_5^3A_7A_8^2 + 16A_0^2A_4A_5^2A_7^4 + 76A_0^2A_4A_5^2A_7^3A_8 - 76A_0^2A_4A_5^2A_7^2A_8^2 - 16A_0^2A_4A_5^2A_7A_8^3 - \\
& 16A_0^2A_4A_5A_7^4A_8 - 64A_0^2A_4A_5A_7^3A_8^2 - 16A_0^2A_4A_5A_7^2A_8^3 + 16A_0^2A_5^5A_7^2 - 24A_0^2A_5^4A_7^3 + \\
& 24A_0^2A_5^4A_7^2A_8 - 24A_0^2A_5^3A_7^4 + 12A_0^2A_5^3A_7^3A_8 - 24A_0^2A_5^3A_7^2A_8^2 + 16A_0^2A_5^2A_7^5 + \\
& 24A_0^2A_5^2A_7^4A_8 - 24A_0^2A_5^2A_7^3A_8^2 - 16A_0^2A_5^2A_7^2A_8^3 + A_0A_4^5A_5A_7A_8 - A_0A_4^5A_5A_8^2 - \\
& A_0A_4^5A_7A_8^2 - A_0A_4^4A_5^2A_7^2 - 8A_0A_4^4A_5^2A_7A_8 - A_0A_4^4A_5^2A_8^2 - 8A_0A_4^4A_5A_7^2A_8 + \\
& 8A_0A_4^4A_5A_7A_8^2 - A_0A_4^4A_7^2A_8^2 + 8A_0A_4^3A_5^3A_7^2 - 8A_0A_4^3A_5^3A_7A_8 + 8A_0A_4^3A_5^2A_7^3 - \\
& 57A_0A_4^3A_5^2A_7^2A_8 + 8A_0A_4^3A_5^2A_7A_8^2 - 8A_0A_4^3A_5A_7^3A_8 + 8A_0A_4^3A_5A_7^2A_8^2 + 8A_0A_4^2A_5^4A_7^2 + \\
& 46A_0A_4^2A_5^3A_7^3 - 46A_0A_4^2A_5^3A_7^2A_8 + 8A_0A_4^2A_5^2A_7^4 - 46A_0A_4^2A_5^2A_7^3A_8 + 8A_0A_4^2A_5^2A_7^2A_8^2 + \\
& 36A_0A_4A_5^4A_7^3 + 36A_0A_4A_5^3A_7^4 - 36A_0A_4A_5^3A_7^3A_8 + 27A_0A_5^4A_7^4 - A_4^6A_5A_7A_8 + \\
& A_4^5A_5^2A_7^2 - A_4^5A_5^2A_7A_8 - A_4^5A_5A_7^2A_8 + A_4^4A_5^3A_7^2 + A_4^4A_5^2A_7^3 - \\
& A_4^4A_5^2A_7^2A_8 + A_4^3A_5^3A_7^3
\end{aligned}
$$

Note that $V(A_0) \subseteq \Delta_4$. It is quick to compute how the (higher) Chow groups of $\mathbb{P}(V_4)$ change when removing $V(a_0)$, as one then obtains affine space, but removing the rest of Δ_4 is more involved. Define $\Delta_4^0 := \Delta_4 \backslash V(a_0)$. Homogenizing with respect to A_0, we have Δ_4^0 is a closed subvariety of $\mathbb{A}^4 = \operatorname{Spec} k[a_4, a_5, a_7, a_8]$, cut out by

$$
\begin{aligned}
f := {} & 27a_8^4 + 36a_4a_5a_8^3 + 36a_4a_7a_8^3 - 36a_4a_8^4 + 16a_5^3a_8^2 - \\
& 24a_5^2a_7a_8^2 + 24a_5^2a_8^3 - 24a_5a_7^2a_8^2 + 12a_5a_7a_8^3 - 24a_5a_8^4 + \\
& 16a_7^3a_8^2 + 24a_7^2a_8^3 - 24a_7a_8^4 - 16a_8^5 + a_4^3a_8^3 + \\
& 8a_4^2a_5^2a_8^2 + 46a_4^2a_5a_7a_8^2 - 46a_4^2a_5a_8^3 + 8a_4^2a_7^2a_8^2 - 46a_4^2a_7a_8^3 + \\
& 8a_4^2a_8^4 + 16a_4a_5^3a_7a_8 - 16a_4a_5^3a_8^2 - 64a_4a_5^2a_7^2a_8 + 76a_4a_5^2a_7a_8^2 - \\
& 64a_4a_5^2a_8^3 + 16a_4a_5a_7^3a_8 + 76a_4a_5a_7^2a_8^2 - 76a_4a_5a_7a_8^3 - 16a_4a_5a_8^4 - \\
& 16a_4a_7^3a_8^2 - 64a_4a_7^2a_8^3 - 16a_4a_7a_8^4 - 16a_5^4a_7^2 - 16a_5^4a_8^2 + \\
& 32a_5^3a_7^3 - 16a_5^3a_7^2a_8 + 16a_5^3a_7a_8^2 - 32a_5^3a_8^3 - 16a_5^2a_7^4 - \\
& 16a_5^2a_7^3a_8 + 66a_5^2a_7^2a_8^2 - 16a_5^2a_7a_8^3 - 16a_5^2a_8^4 + 16a_5a_7^3a_8^2 - \\
& 16a_5a_7^2a_8^3 - 16a_7^4a_8^2 - 32a_7^3a_8^3 - 16a_7^2a_8^4 + a_4^4a_5a_8^2 + \\
& a_4^4a_7a_8^2 - a_4^4a_8^3 + 8a_4^3a_5^2a_7a_8 - 8a_4^3a_5^2a_8^2 + 8a_4^3a_5a_7^2a_8 - \\
& 57a_4^3a_5a_7a_8^2 + 8a_4^3a_5a_8^3 - 8a_4^3a_7^2a_8^2 + 8a_4^3a_7a_8^3 - 8a_4^2a_5^3a_7^2 - \\
& 16a_4^2a_5^3a_7a_8 - 8a_4^2a_5^3a_8^2 - 8a_4^2a_5^2a_7^3 + 108a_4^2a_5^2a_7^2a_8 - 108a_4^2a_5^2a_7a_8^2 + \\
& 8a_4^2a_5^2a_8^3 - 16a_4^2a_5a_7^3a_8 - 108a_4^2a_5a_7^2a_8^2 - 16a_4^2a_5a_7a_8^3 - 8a_4^2a_7^3a_8^2 + \\
& 8a_4^2a_7^2a_8^3 + 16a_4a_5^4a_7^2 - 16a_4a_5^4a_7a_8 - 64a_4a_5^3a_7^3 + 76a_4a_5^3a_7^2a_8 - \\
& 64a_4a_5^3a_7a_8^2 + 16a_4a_5^2a_7^4 + 76a_4a_5^2a_7^3a_8 - 76a_4a_5^2a_7^2a_8^2 - 16a_4a_5^2a_7a_8^3 - \\
& 16a_4a_5a_7^4a_8 - 64a_4a_5a_7^3a_8^2 - 16a_4a_5a_7^2a_8^3 + 16a_5^5a_7^2 - 24a_5^4a_7^3 + \\
& 24a_5^4a_7^2a_8 - 24a_5^3a_7^4 + 12a_5^3a_7^3a_8 - 24a_5^3a_7^2a_8^2 + 16a_5^2a_7^5 + \\
& 24a_5^2a_7^4a_8 - 24a_5^2a_7^3a_8^2 - 16a_5^2a_7^2a_8^3 + a_4^5a_5a_7a_8 - a_4^5a_5a_8^2 - \\
& a_4^5a_7a_8^2 - a_4^4a_5^2a_7^2 - 8a_4^4a_5^2a_7a_8 - a_4^4a_5^2a_8^2 - 8a_4^4a_5a_7^2a_8 + \\
& 8a_4^4a_5a_7a_8^2 - a_4^4a_7^2a_8^2 + 8a_4^3a_5^3a_7^2 - 8a_4^3a_5^3a_7a_8 + 8a_4^3a_5^2a_7^3 - \\
& 57a_4^3a_5^2a_7^2a_8 + 8a_4^3a_5^2a_7a_8^2 - 8a_4^3a_5a_7^3a_8 + 8a_4^3a_5a_7^2a_8^2 + 8a_4^2a_5^4a_7^2 + \\
& 46a_4^2a_5^3a_7^3 - 46a_4^2a_5^3a_7^2a_8 + 8a_4^2a_5^2a_7^4 - 46a_4^2a_5^2a_7^3a_8 + 8a_4^2a_5^2a_7^2a_8^2 + \\
& 36a_4a_5^4a_7^3 + 36a_4a_5^3a_7^4 - 36a_4a_5^3a_7^3a_8 + 27a_5^4a_7^4 - a_4^6a_5a_7a_8 + \\
& a_4^5a_5^2a_7^2 - a_4^5a_5^2a_7a_8 - a_4^5a_5a_7^2a_8 + a_4^4a_5^3a_7^2 + a_4^4a_5^2a_7^3 - \\
& a_4^4a_5^2a_7^2a_8 + a_4^3a_5^3a_7^3
\end{aligned}
$$

Lemma 4.1. $\mathrm{CH}^i(\Delta_4^0) = \begin{cases} \mathbb{Z} & i = 0 \\ \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^6 & i = 1 \\ 0 & i > 1 \end{cases}$

and Δ_4^0 has the integral CKP and integral CKgP.

Proof. Define

$$u : \mathbb{A}^3 \to \Delta_4^0$$

$(x,y,t) \mapsto (-4xy+2x+2y+2t, 2xy^2-y^2-2yt, 2x^2y-x^2-2xt, -x^2y^2+2xyt)$, and note that switching x and y preserves a_4, a_8 and flips a_5 and a_7.

We claim this map is finite. One can check that x satisfies

$$2x^3 + (-3a_4 - 6a_5)x^2 + (a_4^2 + 4a_5 - 2a_7 - 4a_8)x + (a_4a_7 + 2a_8) = 0$$

and then y must also be integral over Δ_4^0 by symmetry. Additionally, because $a_4 = -4xy + 2x + 2y + 2t$, t must be integral as well, and so u is integral, hence finite.

Additionally, we have $x = \frac{b_1}{b_2}$, where

$$\begin{aligned}b_1 := {} & a_4^4a_5a_7 + a_4^3a_5^2a_7 - a_4^3a_5a_7 + 2a_4^3a_5a_8 + a_4^3a_7a_8 + 4a_4^2a_5^2a_7 + \\ & 2a_4^2a_5^2a_8 + 6a_4^2a_5a_7^2 - 4a_4^2a_5a_7a_8 - 2a_4^2a_5a_8 - 2a_4^2a_7a_8 + 2a_4^2a_8^2 \\ & + 4a_4a_5^3a_7 + 25a_4a_5^2a_7^2 - 4a_4a_5^2a_7a_8 - 4a_4a_5^2a_7 + 8a_4a_5^2a_8 - 4a_4a_5a_7^2 - \\ & 2a_4a_5a_7a_8 - 8a_4a_5a_8^2 + 4a_4a_7^2a_8 - 4a_4a_7a_8^2 - 3a_4a_8^2 + 18a_5^3a_7^2 + 8a_5^3a_8 - \\ & 16a_5^2a_7^2 - 4a_5^2a_7a_8 - 8a_5^2a_8^2 - 8a_5^2a_8 + 8a_5a_7^3 + 16a_5a_7^2a_8 + 8a_5a_7a_8 \\ & + 10a_5a_8^2 - 8a_7^2a_8 - 8a_7a_8^2 - 8a_8^3,\end{aligned}$$

and

$$\begin{aligned}b_2 := {} & -a_4^5a_5 - a_4^4a_5^2 + a_4^4a_5 - a_4^4a_8 - 8a_4^3a_5^2 - 7a_4^3a_5a_7 + 8a_4^3a_5a_8 + a_4^3a_8 - \\ & 8a_4^2a_5^3 - 32a_4^2a_5^2a_7 + 8a_4^2a_5^2a_8 + 8a_4^2a_5^2 + 6a_4^2a_5a_7 - 38a_4^2a_5a_8 - 6a_4^2a_7a_8 + \\ & 8a_4^2a_8^2 - 24a_4a_5^3a_7 - 16a_4a_5^3 + 44a_4a_5^2a_7 - 40a_4a_5^2a_8 - 12a_4a_5a_7^2 - \\ & 44a_4a_5a_7a_8 - 16a_4a_5a_8^2 + 28a_4a_5a_8 + 4a_4a_7a_8 - 28a_4a_8^2 - 16a_5^4 + \\ & 16a_5^3a_7 - 16a_5^3a_8 + 16a_5^3 + 2a_5^2a_7^2 - 16a_5^2a_7a_8 - 24a_5^2a_7 - 16a_5^2a_8^2 + \\ & 8a_5^2a_8 + 8a_5a_7^2 + 20a_5a_7a_8 - 8a_5a_8^2 - 8a_7^2a_8 - 24a_7a_8^2 - 16a_8^3 + 18a_8^2\end{aligned}$$

and, by symmetry, we can write $y = \frac{c_1}{c_2}$, where $c_i := b_i(a_4, a_7, a_5, a_8)$. And because $a_4 = -4xy + 2x + 2y + 2t$, we have a rational expression for t as well. Thus, the map u has a birational inverse, and so u is an isomorphism onto Δ_4^0 away from the vanishing of the denominators. Let C be the reduced subscheme of $V(b_2c_2) \subseteq \Delta_4^0$. Then Lemma 2.1 applied to $u : \mathbb{A}^3 \to \Delta_4^0$ says

$$\mathrm{CH}^*(\Delta_4^0) = \mathrm{CH}^*(\mathbb{A}^3) \coprod_{\mathrm{CH}^*(p^{-1}(C))} \mathrm{CH}^*(C) = \mathbb{Z} \oplus \mathrm{coker}(\mathrm{CH}^*(p^{-1}(C) \to \mathrm{CH}^*(C)),$$

using the fact that the pushforward $p^{-1}(C) \to \mathbb{A}^3$ is 0.

Pulling back b_2c_2 along u, we have

$$\widetilde{C} := p^{-1}(C) = \widetilde{C}_0 \cup \widetilde{C}_1 \cup \widetilde{C}_2 \cup \widetilde{C}_3 \cup \widetilde{C}_4 \cup \widetilde{C}_5 \cup \widetilde{C}_6$$

where

$$\begin{aligned}\widetilde{C}_0 &:= V(t^2 - x(x-1)y(y-1)) \\ \widetilde{C}_1 &:= V(x) \cong \mathbb{A}^2 \\ \widetilde{C}_2 &:= V(y) \cong \mathbb{A}^2\end{aligned}$$

$$\widetilde{C}_3 := V(x-1) \cong \mathbb{A}^2$$
$$\widetilde{C}_4 := V(y-1) \cong \mathbb{A}^2$$
$$\widetilde{C}_5 := V(2xy-x-y-2t) \cong \mathbb{A}^2$$
$$\widetilde{C}_6 := V(2xy-x-y-2t+1) \cong \mathbb{A}^2.$$

Defining $C_i := p(\widetilde{C}_i)$, one can routinely verify that

$$C_1 = V(a_7, a_8) \cong \mathbb{A}^2$$
$$C_2 = V(a_5, a_8) \cong \mathbb{A}^2$$
$$C_3 = V(a_5 + a_8, a_4 + a_7 - 1)) \cong \mathbb{A}^2$$
$$C_4 = V(a_7 + a_8, a_4 + a_5 - 1)) \cong \mathbb{A}^2$$
$$C_5 = V(a_5 - a_7, a_8 + a_7^2 + a_4a_7) \cong \mathbb{A}^2$$
$$C_6 = V(a_5a_7 - a_8, a_4 + a_5 + a_7 - 1) \cong \mathbb{A}^2,$$

and that the degree of $\widetilde{C}_\ell \to C_\ell$ is 2 for $\ell \geq 1$.

We will show that $\widetilde{C}_0 \to C_0$ induces surjections on Chow groups. Assuming that for now, we complete the proof of the Lemma. We have the commutative diagram

$$\begin{tikzcd}
\bigoplus_{\ell=0}^6 \mathrm{CH}_i(\widetilde{C}_\ell) \arrow[r, two heads] \arrow[d, "\bigoplus_\ell (p|_{\widetilde{C}_\ell})_*"] & \mathrm{CH}_i(\widetilde{C}) \arrow[d, "p_*"] \\
\bigoplus_{\ell=0}^6 \mathrm{CH}_i(C_\ell) \arrow[r, two heads] & \mathrm{CH}_i(C).
\end{tikzcd}$$

Because the horizontal maps are isomorphisms for $i = 2$ and the degree of $\widetilde{C}_\ell \to C_\ell$ is 2 for $\ell \geq 1$, we get the cokernel of p_* is generated by the classes $[C_\ell]$, $\ell \geq 1$, with each class being 2-torsion. In degrees $i < 2$, note $\mathrm{CH}_i(\widetilde{C}_\ell) = \mathrm{CH}_i(C_\ell) = 0$ for $\ell \geq 1$, and $\mathrm{CH}_i(\widetilde{C}_0) \to \mathrm{CH}_i(C_0)$ is surjective, so the cokernel of p_* is 0 in all other degrees.

Now, we just need that $C_0 \to p(C_0)$ induces is surjective on Chow groups. The map $C_0 \to p(C_0)$ is birational: we can describe a rational inverse by noting $x = \frac{d_1}{d_2}$, where

$$d_1 := a_4^3 + 2a_4^2a_5 + 4a_4a_5 + 4a_4a_7 - 4a_4a_8 + 8a_5^2 - 4a_5a_7 - 8a_5a_8 + 12a_8$$
$$d_2 := 2a_4^2 + 24a_4a_5 + 24a_5^2 - 16a_5 + 8a_7 + 16a_8,$$

which then gives $y = \frac{e_1}{e_2}$ where $e_i := d_i(a_4, a_7, a_5, a_8)$ and we can get an expression for t using $a_4 = -4xy + 2x + 2y + 2t$. Let D be the reduced subscheme of $V_{p(C_0)}(d_2e_2)$. A COMPUTATION SHOWS that

$$\widetilde{D} := p^{-1}(D) = \widetilde{D}_1 \cup \widetilde{D}_2 \cup \widetilde{D}_3 \cup \widetilde{D}_4 \cup \widetilde{D}_5 \cup \widetilde{D}_6,$$

where

$$\widetilde{D}_1 := V_{\mathbb{A}^3}(x+y-1, y^2-y+t) \cong \mathbb{A}^1$$
$$\widetilde{D}_2 := V_{\mathbb{A}^3}(x-y, y^2-y-t) \cong \mathbb{A}^1$$

$$\widetilde{D}_3 := V_{\mathbb{A}^3}(t,x) \cong \mathbb{A}^1$$
$$\widetilde{D}_4 := V_{\mathbb{A}^3}(t,x+1) \cong \mathbb{A}^1$$
$$\widetilde{D}_5 := V_{\mathbb{A}^3}(t,y) \cong \mathbb{A}^1$$
$$\widetilde{D}_6 := V_{\mathbb{A}^3}(t,y+1) \cong \mathbb{A}^1.$$

It is routine to verify that $\widetilde{D}_\ell \to D_\ell := \widetilde{D}_\ell$ had degree 1 for each ℓ. Because all of these maps have degree 1, we have

$$\mathbb{Z}^6 = \mathrm{CH}_1(\widetilde{D}) \to \mathrm{CH}_1(D) = \mathbb{Z}^6$$

is an isomorphism, hence surjective. Next, because $\widetilde{D}_i$ is a rational curve for each ℓ, we know D_ℓ is also rational. Then $\mathrm{CH}_0(\widetilde{D}_\ell), \mathrm{CH}_0(D_\ell) = 0$ for all ℓ, so $\mathrm{CH}_0(\widetilde{D}) \to \mathrm{CH}_0(D)$ is surjective, as both groups are 0.

Lemma 2.1 says that $\mathrm{CH}^*(\widetilde{C}_0) \oplus \mathrm{CH}^*(D) \to \mathrm{CH}^*(C_0)$ is surjective. But because $\mathrm{CH}^*(\widetilde{D}) \to \mathrm{CH}^*(D)$ is surjective, we actually have $\mathrm{CH}^*(\widetilde{C}_0) \to \mathrm{CH}^*(C_0)$ is surjective.

□

Theorem 4.2. $\mathrm{CH}^*(\mathcal{M}^0_{1,4}) = \mathbb{Z}$ *and*

$$\overline{\mathrm{CH}}^i(\mathcal{M}^0_{1,4},1) = \begin{cases} \mathbb{Z} & i=1 \\ \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^6 & i=2 \\ 0 & i>2 \end{cases}$$

Proof. □

Remark 4.3. We know that the elements in $\overline{\mathrm{CH}}^2(\mathcal{M}^0_{1,4},1)$ are not pulled back from $\overline{\mathrm{CH}}^2(\mathcal{M}_{1,3},1)$ because...

5. The Chow Rings of $\overline{\mathcal{M}}_{1,1}, \mathcal{M}_{1,3}, \mathcal{M}_{1,4}$

5.1. $\overline{\mathcal{M}}_{1,1}$**.** We use consequences of the computation of $\overline{\mathcal{M}}_{1,1}$ in our computations for $\mathcal{M}_{1,3}$ and $\mathcal{M}_{1,4}$, so we present this now, instead of in section help.

The stack $\overline{\mathcal{M}}_{1,1}$ is isomorphic to weighted projective space $\mathbb{P}(4,6)$, and the computation of its integral Chow ring follows readily from this (cite EG). But, in order to illustrate our method, we give a computation based on the localization exact sequence for $\mathcal{M}_{1,1} \subseteq \overline{\mathcal{M}}_{1,1}$. This computation does require some outside input, namely Mumford's relation

$$12\lambda = [\partial\overline{\mathcal{M}}_{1,1}] \tag{5.1}$$

in the rational Chow ring $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \otimes \mathbb{Q}$, where λ is the first Chern class of the Hodge bundle (cite something).

Note $\partial\overline{\mathcal{M}}_{1,1} \cong B\mu_2$. Let $\lambda \in \mathrm{CH}^1(\overline{\mathcal{M}}_{1,1})$ be the first Chern class of the Hodge bundle.

Proposition 5.2. *For* $\iota : \partial\overline{\mathcal{M}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$*, we have* $\iota^*(\lambda) = s$.

Theorem 5.3. $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}[\lambda]/(24\lambda^2)$.

Proof. The complement of $\partial\overline{\mathcal{M}}_{1,1}$ is $\mathcal{M}_{1,1}$, giving a localization exact sequence

$$\overline{\mathrm{CH}}^*(\mathcal{M}_{1,1},1) \to \mathrm{CH}^{*-1}(\partial\overline{\mathcal{M}}_{1,1}) \to \mathrm{CH}^*(\overline{\mathcal{M}}_1,1) \to \mathrm{CH}^*(\mathcal{M}_{1,1}) \to 0.$$

Using Theorem 3.2 and the fact that $\partial\overline{\mathcal{M}}_{1,1} \cong B\mu_2$, we get

$$0 \to \mathbb{Z}[s]/(2s) \xrightarrow{\iota_*} \mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \to \frac{\mathbb{Z}[\lambda]}{(12\lambda)} \to 0.$$

In degree 1, this is

$$0 \to \mathbb{Z} \xrightarrow{\iota_*} \mathrm{CH}^1(\overline{\mathcal{M}}_{1,1}) \to \frac{\mathbb{Z}}{12\mathbb{Z}} \to 0.$$

Mumford's relation (5.1) implies that $12\lambda = \iota_*(1)$ up to torsion. Because $12\lambda = 0$ on $\mathcal{M}_{1,1}$, we know we can write $12\lambda = \iota_*(a)$ for some $a \in \mathbb{Z}$, hence we have $\iota_*(a) - \iota_*(1) = \iota_*(a-1)$ is torsion. Because ι is injective, we need $a = 1$, and so $12\lambda = \iota_*(1)$. Hence, $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}$, and is generated by λ.

Now, I claim that $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1})$ is generated as a ring by λ. We know λ additively generates $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,1})$. The degree $n > 1$ part of the localization exact sequence gives

$$0 \to \frac{\mathbb{Z}}{2\mathbb{Z}} \to \mathrm{CH}^n(\overline{\mathcal{M}}_{1,1}) \to \frac{\mathbb{Z}}{12\mathbb{Z}}.$$

Note $\iota^*(\lambda) = s$ because of a computation. Hence

$$0 \neq \iota_*(s^{n-1}) = \iota_*\iota^*(\lambda^{n-1}) = 12\lambda^n. \tag{5.4}$$

Thus, the subgroup of $\mathrm{CH}^n(\overline{\mathcal{M}}_{1,1})$ generated by λ^n contains the image of ι_* and surjects onto $\mathrm{CH}^n(\mathcal{M}_{1,1})$, as $\lambda^n \mapsto t^n$, so this subgroup must be $\mathrm{CH}^n(\overline{\mathcal{M}}_{1,1})$.

As $2s = 0$, we have $0 = \iota_*(2s) = 24t^2$. Hence, we get a surjective graded ring homomorphism

$$\mathbb{Z}[\lambda]/(24\lambda^2) \to \mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}).$$

This must be an isomorphism, because it is in degrees $0, 1$, and in higher degrees, both groups have the same finite cardinality. ∎

We get some immediate concequences for $\overline{\mathcal{M}}_{1,n}$ using the map $\pi : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,1}$ forgetting all but the first point.

Proposition 5.5.
(1) *The order of $\lambda^i \in \mathrm{CH}^*(\overline{\mathcal{M}}_{1,n})$ is 24.*
(2) *For $\overline{\mathcal{M}}^{\Phi}_{1,n} \subseteq \overline{\mathcal{M}}_{1,n}$ the curves with a non-separating node, $[\overline{\mathcal{M}}^{\Phi}_{1,n}] = 12\lambda$*
(3) *If $\overline{\mathrm{CH}}^1(\mathcal{M}_{1,n},1) = 0$, then λ has order 12 in $\mathrm{CH}^1(\mathcal{M}_{1,n})$.*

Proof. (1) Because λ^i has order 24 in $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1})$, the order of $\pi^*(\lambda^i) = \lambda^i \in \mathrm{CH}^*(\overline{\mathcal{M}}_{1,n})$ divides 24. Moreover, the map π has a section, s, given by attaching a fixed Genus 0 curve with $n+1$ point to each $(C,p) \in \overline{\mathcal{M}}_{1,1}$. And so λ^i must have order exactly 24.

(2) We have a commutative diagram

$$\begin{tikzcd} \overline{\mathcal{M}}^{\Phi}_{1,n} \arrow[r] \arrow[d, "\pi"] & \overline{\mathcal{M}}_{1,n} \arrow[d, "\pi"] \\ \partial\overline{\mathcal{M}}_{1,1} \arrow[r, "\iota"] & \overline{\mathcal{M}}_{1,1} \end{tikzcd}$$

This is Cartesian: set theoretically, this is clear, and the fiber product is reduced because $\pi : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,1}$ is log-smooth. Note that the vertical maps are flat and the horizontal maps are proper. Then, we have

$$12\lambda = \pi^*(12\lambda) = \pi^*(\iota_*(1)) = \iota_*(\pi^*(1)) = \iota_*(1) = [\overline{\mathcal{M}}^{\Phi}_{1,n}]$$

by push-pull, giving (1).

(3) Note that the above implies λ has order 12 in $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,n} \setminus \overline{\mathcal{M}}^{\Phi}_{1,n})$. If the order λ is not 12 in $\mathrm{CH}^1(\mathcal{M}_{1,n})$, then there must be some relation

$$a\lambda = \sum_{\delta} a_\delta \delta$$

with $a \neq 0$ and some a_δ nonzero, where the sum runs over the classes $\delta = [\Delta]$ of the components of $\partial\overline{\mathcal{M}}_{1,n}$ besides $\overline{\mathcal{M}}^{\Phi}_{1,n}$. Multiplying by 12 and taking a lift to $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,n})$, we get a nontrivial relation between the boundary divisors of $\overline{\mathcal{M}}_{1,n}$. This cannot happen if $\overline{\mathrm{CH}}^1(\mathcal{M}_{1,n}, 1) = 0$.

□

5.2. **Group Lemmas.** Here, we give two facts about abelian groups which will be useful to us in the rest of this section and in section help. The first one concerns how to calculate a presentation of the middle term of a short exact sequence given presentations of the outer terms.

Lemma 5.6. *Suppose we have a short exact sequence*

$$0 \to A \to B \to C \to 0$$

of abelian groups with presentations $A = \langle S|R\rangle$, $C = \langle T|U\rangle$, and a set of lifts $\widetilde{T}$ of each $t \in T$ to $\tilde{t} \in B$. For $p(t) \in U$, let $q_p(s)$ be the expression in the $s \in S$ so that $p(\tilde{t}) = q_p(s)$ in B, which exists by exactness. Then, we have a presentation $B = \langle S \cup \widetilde{T}|R \cup \widetilde{U}\rangle$, where $\widetilde{U} = \{(p(\tilde{t}) - q_p(s)|p \in U\}$.

Do I need to prove this?

The following will allow us to turn an additive presentation of a ring into a multiplicative one.

Lemma 5.7. *Suppose $\phi : A \twoheadrightarrow B$ is a surjective homomorphism of abelian groups, and $S \subseteq B$ is a generating subset. For each $s \in S$, choose a lift $\tilde{s} \in A$, and let $\widetilde{S}$ be the set of these lifts. Then ϕ is an isomorphism if both*

(1) *$\widetilde{S}$ generates A and*

(2) *every relations satisfied by elements of S is satisfied by their lifts in $\widetilde{S}$.*

Proof. Take an element $a \in \ker(\phi)$. Because $\widetilde{S}$ generates A, we can write

$$a = \sum_{s\in S} n_s \tilde{s}$$

for some $n_s \in \mathbb{Z}$. Then

$$0 = \phi(a) = \sum_{s\in S} n_s s.$$

Thus, we have a relation on the elements of S, which then holds for the elements of $\widetilde{S}$ by assumption. Hence $a = 0$. ☐

5.3. $\mathcal{M}_{1,3}$.

Theorem 5.8.

$$\mathrm{CH}^*(\mathcal{M}_{1,3}) = \mathbb{Z}[\lambda]/(12\lambda, 6\lambda^2)$$

and

$$\overline{\mathrm{CH}}^*(\mathcal{M}_{1,3},1) = im(j_*) \oplus \bigoplus_{i=2}^{\infty} \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$$

and something about pullbacks from both $\mathcal{M}_{1,2}$ and $\mathcal{M}_{1,3}^0$.

Proof. We know

$$\mathrm{CH}^*(\mathcal{M}_{1,3}^0) = \mathbb{Z}[t]/(2t)$$

$$\overline{\mathrm{CH}}^*(\mathcal{M}_{1,3}^0,1) = \mathbb{Z} \oplus \bigoplus_{i=2}^{\infty} \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$$

$$\mathrm{CH}^*(\mathcal{M}_{1,2}^0) = \mathbb{Z}[t]/(3t)$$

by Theorem 3.11 and Theorem 3.9. So in degree 1, the localization exact sequence for $\mathcal{M}_{1,3}^0 \subseteq \mathcal{M}_{1,3}$ is

$$0 \to \overline{\mathrm{CH}}^1(\mathcal{M}_{1,3},1) \to \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \to \mathrm{CH}^1(\mathcal{M}_{1,3}) \to \frac{Z}{2\mathbb{Z}} \to 0$$

We know $\overline{\mathrm{CH}}^1(\mathcal{M}_{1,3}^0,1) = \mathbb{Z}$ is generated by $\Psi_{\mathbb{G}_m}(\frac{\Delta}{(x_3-x_2)^6})$. We wish to compute $\partial(\Psi_{\mathbb{G}_m}(\frac{\Delta}{(x_3-x_2)^6})) = \mathrm{div}(\frac{\Delta}{(x_3-x_2)^6}) = \mathrm{ord}_{Z_{1,3}}(\frac{\Delta}{(x_3-x_2)^6})[Z_{1,3}]$.

Choose a curve $(C = V(y^2 - x^3 - ax - b), \infty, (x_2, y_2)) \in U_2$, with $y_2 \neq 0$. We get a morphism

$$\varphi : C \setminus \{\infty, (x_2,y_2)\} \to \mathcal{M}_{1,3}$$

$$(x,y) \mapsto (C, \infty, (x_2,y_2), (x,y)).$$

This lands in $\mathcal{M}_{1,3} \setminus \mathcal{M}_{1,3}^0$ if and only if $(x,y) = (x_2, -y_2)$, using that $y_2 \neq 0$. By Theorem **??**

$$\mathrm{ord}_{Z_{1,3}}(\frac{\Delta}{(x_3-x_2)^6}) = \mathrm{ord}_{(x_2,-y_2)}(\varphi^{\#}\frac{\Delta}{(x_3-x_2)^6}) = \mathrm{ord}_{(x_2,-y_2)}(\frac{-4a^3-27b^2}{(x-x_2)^6}) = -6,$$

using that $(x-x_2)$ is a uniformizer for C at $(x_2,-y_2)$, which is true because $y_2 \neq 0$.

Thus, in degree 1, our localization exact sequence looks like

$$0 \to \overline{\mathrm{CH}}^1(\mathcal{M}_{1,3},1) \to \mathbb{Z} \xrightarrow{\cdot -6} \mathbb{Z} \to \mathrm{CH}^1(\mathcal{M}_{1,3}) \to \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0,$$

from which we see $\overline{\mathrm{CH}}^1(\mathcal{M}_{1,3},1) = 0$. Moreover, we have a short exact sequence

$$0 \to \frac{\mathbb{Z}}{6\mathbb{Z}} \to \mathrm{CH}^1(\mathcal{M}_{1,3}) \to \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0.$$

The group $\mathrm{CH}^1(\mathcal{M}_{1,3})$ must then be isomorphic to either $\frac{\mathbb{Z}}{12\mathbb{Z}}$ or $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}/6\mathbb{Z}$. Let $\lambda \in \mathrm{CH}^1(\mathcal{M}_{1,3})$ denote the first Chern class of the Hodge bundle. Proposition 5.5 implies that λ has order 12. And so, we must have $\mathrm{CH}^1(\mathcal{M}_{1,3}) \cong \frac{\mathbb{Z}}{12\mathbb{Z}}$, generated by λ.

In degree $i > 1$, the localization exact sequence says

$$\overline{\mathrm{CH}}^{i-1}(\mathcal{M}^0_{1,2},1) \xrightarrow{j_*} \overline{\mathrm{CH}}^i(\mathcal{M}_{1,3},1) \to \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 \xrightarrow{\partial_1} \frac{\mathbb{Z}}{3\mathbb{Z}} \to \mathrm{CH}^i(\mathcal{M}_{1,3}) \to \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0,$$

where $\overline{\mathrm{CH}}^{i-1}(\mathcal{M}^0_{1,2},1) = \mathbb{Z}$ if $i = 2$ and vanishing otherwise. We see that ∂_1 has to be 0, which gives

- $\mathrm{CH}^i(\mathcal{M}_{1,3})$ is an extension of $\frac{\mathbb{Z}}{2\mathbb{Z}}$ by $\frac{\mathbb{Z}}{3\mathbb{Z}}$, hence it is isomorphic to $\frac{\mathbb{Z}}{6\mathbb{Z}}$
- $\overline{\mathrm{CH}}^2(\mathcal{M}_{1,3},1)$ is an extension of $\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$ by the cyclic group $\mathrm{im}(j_*)$ and $\overline{\mathrm{CH}}^i(\mathcal{M}_{1,3},1) = \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$ for $i > 2$.

We know that $\overline{\mathrm{CH}}^2(\mathcal{M}_{1,3},1)$ is a split extension, meaning that we can write $\overline{\mathrm{CH}}^2(\mathcal{M}_{1,3},1) = \mathrm{im}(j_*) \oplus \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$, by considering the diagram

$$\begin{tikzcd}
\overline{\mathrm{CH}}^2(\mathcal{M}_{1,3},1) \arrow[r] & \overline{\mathrm{CH}}^2(\mathcal{M}^0_{1,3},1) \\
\overline{\mathrm{CH}}^2(\mathcal{M}_{1,2},1)^2 \arrow[u] \arrow[ur, "\sim"] &
\end{tikzcd}$$

using Proposition **??**

Now, I claim that $\mathrm{CH}^*(\mathcal{M}_{1,3})$ is generated as a ring by λ. We know λ additively generates $\mathrm{CH}^1(\mathcal{M}_{1,3})$. Because $\iota : \mathcal{M}^0_{1,2} \to \mathcal{M}_{1,3}$ is a section of the forgetful map $\pi : \mathcal{M}_{1,3} \to \mathcal{M}_{1,2}$ over $\mathcal{M}^0_{1,2}$, we have that $\iota^*(\lambda) = \lambda$. Hence help. Thus, $\tilde{t}^n$ has order 6, and so $\tilde{t}$ generates $\mathrm{CH}^*(\mathcal{M}_{1,3})$. Thus, we have a surjection $\mathbb{Z}[\tilde{t}] \to \mathrm{CH}^*(\mathcal{M}_{1,3})$. The elements $12\tilde{t}$ and $6\tilde{t}^2$ are in the kernel of this map, and the induced map

$$\mathbb{Z}[\tilde{t}]/(12\tilde{t}, 6\tilde{t}^2) \to \mathrm{CH}^*(\mathcal{M}_{1,3})$$

must be an isomorphism, because it is surjective the graded pieces of a fixed positive degree have the same finite cardinality.

$$\mathrm{CH}^{*-1}(\mathcal{M}^0_{1,2}) \to \mathrm{CH}^*(\mathcal{M}_{1,3}) \to \mathrm{CH}^*(\mathcal{M}^*_{1,3}) \xrightarrow{\partial} \mathrm{CH}^{*-1}(\mathcal{M}^0_{1,2}).$$

From our above computation with ∂, we have

$$\mathbb{Z} \to \mathrm{CH}^*(\mathcal{M}_{1,3}) \to \bigoplus_{i=2}^{\infty} \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0.$$

□

5.4. $\mathcal{M}_{1,4}$.

Theorem 5.9. $\mathrm{CH}^*(\mathcal{M}_{1,4}) = \frac{\mathbb{Z}[\lambda]}{(12\lambda, 2\lambda^2)}$ *and*

$$\overline{\mathrm{CH}}^i(\mathcal{M}_{1,4},1) = \begin{cases} 0 & i=1 \\ im(j_*) \oplus \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^6 & i=2 \\ \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 & i \geq 3 \end{cases}$$

Moreover, the pullback map

$$\pi^* : \overline{\mathrm{CH}}^i(\mathcal{M}_{1,3},1) \to \overline{\mathrm{CH}}^i(\mathcal{M}_{1,4},1)$$

is injective for $i = 2$ and an isomorphism for $i \geq 3$. And $\mathcal{M}_{1,4}$ has the integral CKP and integral CKgP.

Proof. We have the exact sequence

$$\overline{\mathrm{CH}}^{*-1}(\mathcal{M}^0_{1,3}) \to \overline{\mathrm{CH}}^*(\mathcal{M}_{1,4},1) \to \overline{\mathrm{CH}}^*(\mathcal{M}^0_{1,4},1) \xrightarrow{\partial}$$

$$\mathrm{CH}^{*-1}(\mathcal{M}^0_{1,3}) \to \mathrm{CH}^*(\mathcal{M}_{1,4}) \to \mathrm{CH}^*(\mathcal{M}^0_{1.4}) \to 0.$$

We know

$$\mathrm{CH}^*(\mathcal{M}^0_{1,4}) = \mathbb{Z}$$

$$\overline{\mathrm{CH}}^i(\mathcal{M}^0_{1,4},1) = \begin{cases} \mathbb{Z} & i=1 \\ \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^6 & i=2 \\ 0 & i>2 \end{cases}$$

$$\mathrm{CH}^*(\mathcal{M}^0_{1,3}) = \frac{\mathbb{Z}[t]}{(2t)}$$

$$\overline{\mathrm{CH}}^*(\mathcal{M}^0_{1,3},1) = \mathbb{Z} \oplus \bigoplus_{i>2} (\frac{\mathbb{Z}}{2\mathbb{Z}})^2.$$

by Theorem 4.2 and Theorem 3.11.

In degree 1, the exact sequence is

$$0 \to \overline{\mathrm{CH}}^1(\mathcal{M}_{1,4},1) \to \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \to \mathrm{CH}^1(\mathcal{M}_{1,4}) \to 0,$$

where $\overline{\mathrm{CH}}^1(\mathcal{M}^0_{1,4},1)$ is generated by f (help). We compute the image of ∂_1 using help. Pick a general point $(a_4, a_5, a_7, a_8) \in \mathcal{M}^0_{1,4}$ corresponding to the pointed curve (C, P_1, P_2, P_3, P_4). We have a closed embedding

$$g : C \setminus \{P_1, P_2, P_3\} \to \mathcal{M}_{1,4}$$

$$P \mapsto (C, P_1, P_2, P_3, P).$$

This intersects $j(\mathcal{M}^0_{1,3})$ transversely because it is a fiber of $\pi : \mathcal{M}_{1,4} \to \mathcal{M}_{1,3}$, of which j is a section.

This curve intersects $j(\mathcal{M}_{1,4})$ exactly at $([a_8 : -a_5], [0 : 1])$. We compute $g|^{\mathcal{M}^0_{1,4}}$ in the affine neighborhood $xz \neq 0$. Set $u := \frac{w}{x}$, $v := \frac{y}{z}$, and consider a point $P = (u_0, v_0) \in C$. Following the proof of help, we embed C in $\mathbb{P}^1 \times \mathbb{P}^1$ using $\mathcal{O}(P_1 + P)$ and $\mathcal{O}(P_2 + P_3)$. The functions $1, \frac{v_0a_0 - u_0a_5 + (u_0v_0a_7 - v_0^2a_0)u}{v_0u - u_0v}$ form a basis of $\mathcal{O}(P_1+P)$, and the functions $1, v$ form a basis for $\mathcal{O}(P_2+P_3)$. Hence, we are supposed to use the embedding

$$C \to \mathbb{P}^1 \times \mathbb{P}^1$$

$$([x : w], [y : z]) \mapsto ([v_0wz - u_0xy : (v_0a_0 - u_0a_5)xz + (u_0v_0a_7 - v_0^2a_0)wz], [y : z])$$

up to the automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which sends P to P_4 and P_i to P_i for $i \in \{1, 2, 3\}$. After applying the automorphism, we get the embedding

$$C \to \mathbb{P}^1 \times \mathbb{P}^1$$

$$([x : w], [y : z]) \mapsto ([(v_0a_0 - u_0a_5)xz + u_0v_0a_7wz - a_0v_0u_0xy : a_0v_0^2wz - a_0v_0u_0xy], [y : v_0z]).$$

This has image given by the vanishing of

$$x^2y^2 - x^2yz - xwy^2 + \frac{2a_7u_0v_0 + 2a_8u_0 + v_0a_4}{v_0^2a_0}xwyz + \frac{a_5}{v_0^2a_0}xwz^2 +$$
$$\frac{-a_8u_0 + v_0a_7 - u_0v_0a_7}{v_0^2a_0}w^2yz + \frac{a_0v_0a_8 - a_5u_0a_7v_0 - a_5a_8u_0}{v_0^4a_0^2}w^2z^2$$

which one can check by noting that composing this expression with the embedding gives a multiple of the defining equation of C. Thus, g is given by

$$(u_0, v_0) \mapsto \left(\frac{2a_7u_0v_0 + 2a_8u_0 + v_0a_4}{v_0^2a_0}, \frac{a_5}{v_0^2a_0}, \frac{-a_8u_0 + v_0a_7 - u_0v_0a_7}{v_0^2a_0}, \frac{a_0v_0a_8 - a_5u_0a_7v_0 - a_5a_8u_0}{v_0^4a_0^2}\right)$$

Composing this map with f, one gets

$$\frac{h(u_0, v_0)}{v_0^{20}},$$

for some polynomial $h(u_0, v_0)$. Expanding $h(u_0, v_0)$ around $(u_0, v_0) = (-\frac{a_5}{a_8}, 0)$, one gets an expression whose lowest term is degree 4 in $(u_0 + \frac{a_5}{a_8}), v_0$. The defining equation of C is equivalent to

$$u_0+\frac{a_5}{a_8} = \frac{1}{a_5a_8^2}(a_8^2a_7(u_0+\frac{a_5}{a_8})^2v_0+a_8^3(u_0+\frac{a_5}{a_8})^2+(a_4a_8^2-2a_5a_7a_8)(u_0+\frac{a_5}{a_8})v_0-$$
$$(u_0 + \frac{a_5}{a_8})a_8^2v_0^2 + (a_5a_8 + a_8a_8)v_0^2 + (-a_8^2 - a_4a_5a_8 + a_5^2a_7)v_0).$$

Substituting this expression for $u_0 + \frac{a_5}{a_8}$ into h, one gets an expression whose lowest term is degree 8 in $(u_0 + \frac{a_5}{a_8}), v_0$. Preforming the same substitution again, one gets v_0^8 plus terms of larger degree. Since v_0 is a uniformizer at $(-\frac{a_5}{a_8},0)$, we have

$$\mathrm{ord}_{\mathcal{M}^0_{1,3}}(f) = \mathrm{ord}_{(-\frac{a_5}{a_8},0)}(f\circ g)) = \mathrm{ord}_{(-\frac{a_5}{a_8},0)}(\frac{v_0^8 + \text{higher order terms}}{v_0^{20}}) = -12.$$

Thus, $\partial_1(f) = -12[\mathcal{M}^0_{1,3}]$, and so

$$\mathrm{CH}^1(\mathcal{M}_{1,4}) = \frac{\mathbb{Z}}{12\mathbb{Z}}$$

$$\overline{\mathrm{CH}}^1(\mathcal{M}_{1,4},1) = 0.$$

Now, we use Proposition 5.5 to conclude that λ has order 12 on $\mathrm{CH}^1(\mathcal{M}_{1,4})$, and thus generates $\mathrm{CH}^1(\mathcal{M}_{1,4})$.

For $i > 2$, the localization exact sequence is

$$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 \xrightarrow{j_*} \overline{\mathrm{CH}}^i(\mathcal{M}_{1,4},1) \to 0 \to \frac{\mathbb{Z}}{2\mathbb{Z}} \xrightarrow{j_*} \mathrm{CH}^i(\mathcal{M}_{1,4}) \to 0$$

hence $\mathrm{CH}^i(\mathcal{M}_{1,4}) \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$, generated by $j_*(\lambda^{i-1})$, and $\overline{\mathrm{CH}}^i(\mathcal{M}_{1,4},1) = \mathrm{im}(j_*)$ for $i > 2$. When $i = 2$, we have

$$\mathbb{Z} \xrightarrow{j_*} \overline{\mathrm{CH}}^2(\mathcal{M}_{1,4},1) \to \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^6 \xrightarrow{\partial_1} \frac{\mathbb{Z}}{2\mathbb{Z}} \xrightarrow{j_*} \mathrm{CH}^2(\mathcal{M}_{1,4}) \to 0.$$

The group $\mathrm{CH}^1(\mathcal{M}^0_{1,3}) = \frac{\mathbb{Z}}{2\mathbb{Z}}$ is generated by λ. Note

$$j_*(\lambda^{i-1}) = j_*(j^*(\lambda^{i-1})) = j_*(1)\lambda^{i-1}.$$

For $i > 2$, this is nonzero, so this is must be nonzero when $i = 2$. Hence, j_* is injective on $\mathrm{CH}^2(\mathcal{M}^0_{1,3})$, and $\mathrm{CH}^2(\mathcal{M}_{1,4}) = \frac{\mathbb{Z}}{2\mathbb{Z}}$, generated by $j_*(\lambda)$.

We claim that λ generates $\mathrm{CH}^*(\mathcal{M}_{1,4})$ as a ring. Note that because $j_*(1)$ and λ both generate $\mathrm{CH}^1(\mathcal{M}_{1,4})$, we can write $j_*(1) = a\lambda$ for some $a \in \mathbb{Z}$ invertible mod 12. Then, we know that $\mathrm{CH}^i(\mathcal{M}_{1,4})$ is generated by $j_*(\lambda^{i-1}) = j_*(1)\lambda^{i-1}) = a\lambda^i$, and so λ^i generates $\mathrm{CH}^i(\mathcal{M}_{1,4})$. Thus, we have a surjection

$$\frac{\mathbb{Z}[\lambda]}{(12\lambda, 2\lambda^2)} \to \mathrm{CH}^*(\mathcal{M}_{1,4}).$$

In degree 0, this is $\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}$, and in larger degrees, this is a surjection of finite sets of the same size. Thus, this map is an isomorphism.

Because j_* is injective on $\mathrm{CH}^2(\mathcal{M}^0_{1,3})$, we have an exact sequence

$$0 \to \mathrm{im}(j_*) \to \overline{\mathrm{CH}}^2(\mathcal{M}_{1,4},1) \to \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^6 \to 0.$$

This splits because....

Finally, we claim that $\pi^* : \overline{\mathrm{CH}}^i(\mathcal{M}_{1,3},1) \to \overline{\mathrm{CH}}^i(\mathcal{M}_{1,4},1)$ is injective for $i = 2$ and an isomorphism for $i \geq 3$. We have a commutative diagram

$$\begin{tikzcd} \mathcal{M}^0_{1,3} \arrow[rr, "j"] \arrow[dr] & & \mathcal{M}_{1,4} \arrow[dl, "\pi"] \\ & \mathcal{M}_{1,3} & \end{tikzcd}.$$

Additionally, the pullback $\overline{\mathrm{CH}}^i(\mathcal{M}_{1,3},1) \to \overline{\mathrm{CH}}^i(\mathcal{M}^0_{1,3},1)$ is injective for $i \geq 2$ by Theorem 5.8. This implies that

$$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 \cong \mathrm{CH}^i(\mathcal{M}_{1,3}) \xrightarrow{\pi^*} \mathrm{CH}^i(\mathcal{M}_{1,4})$$

is injective. Moreover, for $i \geq 3$, the localization exact sequence says

$$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2 = \overline{\mathrm{CH}}^i(\mathcal{M}^0_{1,3},1) \xrightarrow{j_*} \overline{\mathrm{CH}}^i(\mathcal{M}_{1,4},1) \to 0.$$

As the group $\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$ has both injections and surjections to $\overline{\mathrm{CH}}^i(\mathcal{M}_{1,4},1)$ for $i \geq 3$, we know they are isomorphic. Hence, π^* is an isomorphism for $i \geq 3$. $\square$

6. Stratification of $\overline{\mathcal{M}}_{g,n}$

Definition 6.1. Let Γ be a stable graph. We define $\mathcal{M}^\Gamma_{g,n}$ to be the locus of curves in $\overline{\mathcal{M}}_{g,n}$ with stable graph Γ. Its closure is denoted $\overline{\mathcal{M}}^\Gamma_{g,n}$. When there is no confusion as to which $\overline{\mathcal{M}}_{g,n}$ we are working in, we omit the subscripts, writing just $\mathcal{M}^\Gamma$ and $\overline{\mathcal{M}}^\Gamma$.

We also define

$$\overline{\mathcal{M}}_\Gamma := \prod_{v\in\Gamma} \overline{\mathcal{M}}_{g_v,n_v}$$

and

$$\mathcal{M}_\Gamma := \prod_{v\in\Gamma} \mathcal{M}_{g_v,n_v}.$$

There is a surjective map

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}^\Gamma$$

that glues together the various curves. This map is representable by [Bae-Schmidt]. Moreover, $\xi_\Gamma(\mathcal{M}_\Gamma) = \mathcal{M}^\Gamma$.

Cite Bae-Schmidt:

Theorem 6.2. *The map ξ_Γ induces an isomorphism*

$$\mathcal{M}_\Gamma/\operatorname{Aut}(\Gamma) \cong \mathcal{M}^\Gamma.$$

Proposition 6.3. *Suppose Γ is a genus 1 stable graph with a (necessarily unique) genus 1 vertex, v_0. Then $\xi_\Gamma : [\overline{\mathcal{M}}_\Gamma/\operatorname{Aut}(\Gamma)] \to \overline{\mathcal{M}}^\Gamma$ is a universal separable homeomorphism.*

Proof. Note that in this instance, $\mathrm{Aut}(\Gamma)$ is trivial. Thus, we want to show $\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}^\Gamma$ is an isomorphism. Consider a graph Γ' with $\mathcal{M}^{\Gamma'} \subseteq \overline{\mathcal{M}}^\Gamma$. We have the following pullback diagram

$$\begin{tikzcd} \prod_{v\in\Gamma} \mathcal{M}^{\Gamma'_v} \arrow[r] \arrow[d] & \mathcal{M}^{\Gamma'} \arrow[d] \\ \overline{\mathcal{M}}_\Gamma \arrow[r, "\xi_\Gamma"] & \overline{\mathcal{M}}^\Gamma \end{tikzcd}$$

where Γ'_v is the graph inserted into Γ at v to make Γ'. Using Theorem 6.2, we can write the top as the map

$$\prod_{v\in\Gamma}[\mathcal{M}_{\Gamma'_v}/\operatorname{Aut}(\Gamma'_v)] \to [\mathcal{M}_{\Gamma'}/\operatorname{Aut}(\Gamma')].$$

Further, if we set $\mathrm{Aut}_\Gamma(\Gamma') = \{\varphi \in \mathrm{Aut}(\Gamma') | \varphi(\Gamma'_v) = \Gamma'_v\}$, then the above map can be expressed as

$$[\mathcal{M}_{\Gamma'}/\operatorname{Aut}_\Gamma(\Gamma')] \to [\mathcal{M}_{\Gamma'}/\operatorname{Aut}(\Gamma')].$$

Thus, all we need to argue is $\mathrm{Aut}_\Gamma(\Gamma') = \mathrm{Aut}(\Gamma')$. To see this, take an automorphism $\varphi \in \mathrm{Aut}(\Gamma')$. Define the subgraph $\Gamma'' \subseteq \Gamma'_{v_0}$ to be the "genus 1 part" of Γ'_{v_0}, meaning either the genus 1 vertex in Γ'_{v_0} if it exists and otherwise just the vertices on the unique cycle making Γ'_{v_0} have genus 1. Certainly $\varphi(\Gamma'') \subseteq \Gamma''$. Next consider, the graph Γ'/Γ'' obtained by collapsing Γ'' to a point. Then φ induces an automorphism of Γ'/Γ'', and this graph is a tree. The leaves of this tree must be fixed, since they are either the vertex $\{\Gamma''\}$ or otherwise have marked point attached, and any automorphism of a tree fixing the leaves is the identity. Thus, φ fixes every vertex outside of Γ'' pointwise, hence $\varphi(\Gamma'_v) = \Gamma'_v$ for all v. □

Remark 6.4. It is not always true that ξ_Γ is a universal separable homeomorphism. The graph Θ defined in 8.1 is one example. However, ξ_Γ is a universal separable homeomorphism in many more cases. For example, if every vertex of Γ contains a marked point, then ξ_Γ is a universal separable homeomorphism. The proof is essentially the same, because then $\mathrm{Aut}(\Gamma)$ is trivial and one has $\mathrm{Aut}_\Gamma(\Gamma') = \mathrm{Aut}(\Gamma)$ for these graphs as well. The author tried and failed to come up with a general, easily checkable, criteria on Γ which implies ξ_Γ is an isomorphism and generalizes these two cases. Finally, it is plausible to the author that ξ_Γ is actually an isomorphism when it is a universal separable homeomorphism. This would follow smoothness of $\overline{\mathcal{M}}^\Gamma$, but $\overline{\mathcal{M}}^\Theta$ is not smooth, so we do not have that $\overline{\mathcal{M}}^\Gamma$ is always smooth.

Now, we lay out our strategy for computing $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{g,n})$.

Definition 6.5. For a fixed g, n, let $\overline{\mathcal{M}}^{\mathrm{sep}}$ denote the locus of curves in the boundary with at least one separating node, and let $\overline{\mathcal{M}}^{\mathrm{sep},\geq p}$ denote the union of $\overline{\mathcal{M}}^{\mathrm{sep}}$ with the locus of curves with at least p nodes. Finally, let $\mathcal{M}^{\mathrm{non}=p} = \overline{\mathcal{M}}^{\mathrm{sep},\geq p} \setminus \overline{\mathcal{M}}^{\mathrm{sep},\geq p+1}$.

The following are immediate from the definition

(1) $\overline{\mathcal{M}}^{\mathrm{sep}}$ is the union of $\overline{\mathcal{M}}^{\Gamma}$ over stable graphs Γ with one node which is separating
(2) $\overline{\mathcal{M}}^{\mathrm{sep},\geq p} = \overline{\mathcal{M}}^{\mathrm{sep}}$ for $p > \dim(\overline{\mathcal{M}}_{g,n})$
(3) $\overline{\mathcal{M}}^{\mathrm{sep},\geq 1} = \partial\overline{\mathcal{M}}_{g,n}$
(4) $\mathcal{M}^{\mathrm{non}=p}$ is the disjoint union of $\mathcal{M}^{\Gamma}$ over graphs Γ with exactly p nodes, all of which are non-separating.

Thus, we have a filtration of the boundary $\partial\overline{\mathcal{M}}_{g,n}$. We compute $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{g,n})$ using this stratification and the localization exact sequence. To do this, one must know the Chow ring of the bottom piece of the filtration, $\overline{\mathcal{M}}^{\mathrm{sep}}$, the Chow ring of the open parts, $\mathcal{M}^{\mathrm{non}=p}$, and then one uses the localization exact sequence to fit the Chow rings of these spaces together. We compute the Chow ring of $\mathcal{M}^{\mathrm{non}=p}$, using the description of $\mathcal{M}^{\Gamma}$ given in Theorem 6.2. To compute the Chow group of $\overline{\mathcal{M}}^{\mathrm{sep}}$, note we have an exact sequence

$$\bigoplus_{\Gamma,\Gamma'\in S} \mathrm{CH}^*(\overline{\mathcal{M}}^{\Gamma} \cap \overline{\mathcal{M}}^{\Gamma'}) \to \bigoplus_{\Gamma\in S} \mathrm{CH}^*(\overline{\mathcal{M}}^{\Gamma}) \to \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \to 0$$

where S is the set of stable graphs with one separating node. In this exact sequence, we can compute $\mathrm{CH}^*(\overline{\mathcal{M}}^{\Gamma})$ and $\mathrm{CH}^*(\overline{\mathcal{M}}^{\Gamma} \cap \overline{\mathcal{M}}^{\Gamma'})$ for $\Gamma, \Gamma' \in S$ using Proposition 6.3 and the CKgP for $\overline{\mathcal{M}}_{0,n}$. Finally, we use the commutative diagram with exact rows

(6.6)

$$\begin{tikzcd}
\overline{\mathrm{CH}}^*(\mathcal{M}^{\Gamma},1) \arrow[r, "\partial_1'"] \arrow[d] & \mathrm{CH}^{*-1}([\partial\overline{\mathcal{M}}_{\Gamma}/\operatorname{Aut}(\Gamma)) \arrow[r] \arrow[d] & \mathrm{CH}^*([\overline{\mathcal{M}}_{\Gamma}/\operatorname{Aut}(\Gamma)]) \arrow[r] \arrow[d] & \mathrm{CH}^*(\mathcal{M}^{\Gamma}) \arrow[r] \arrow[d] & {} \\
\overline{\mathrm{CH}}^*(\mathcal{M}^{\mathrm{non}=p},1) \arrow[r, "\partial_1"] & \mathrm{CH}^{*-1}(\overline{\mathcal{M}}^{\mathrm{sep},\geq p+1}) \arrow[r] & \mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq p}) \arrow[r] & \mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=p}) \arrow[r] & {}
\end{tikzcd}$$

for $\mathcal{M}^{\Gamma} \subseteq \mathcal{M}^{\mathrm{non}=p}$ to compute $\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq p})$ after knowing $\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq p+1})$.

7. $\overline{\mathcal{M}}_{0,n}$ AND $[\overline{\mathcal{M}}_{0,n}/\mu_2]$

In this section, we study the stacks

- $\mathcal{M}_{0,n}$
- $\overline{\mathcal{M}}_{0,n}$
- $[\mathcal{M}_{0,n}/\mu_2]$
- $[\overline{\mathcal{M}}_{0,n}/\mu_2]$

, where μ_2 acts by switching the last two markings.

Here are our $\mathcal{M}_{0,n}$ conventions. They are not the standard conventions in order to make the action by μ_2 on $\mathbb{A}^{n-3}$ linear. Suppose $n \geq 3$. Firstly, we label the markings $1, 2, \ldots, n-2, a, b$. Now, given a family $\pi : \mathcal{C} \to S$ of smooth genus 0 curves, with n-sections $\sigma_i : S \to \mathcal{C}$, we can find a unique S-isomorphism $\mathcal{C} \cong \mathbb{P}^1 \times S$ such that

- σ_1 is the constant ∞ section

- σ_a is the constant 1 section
- σ_b is the constant -1 section.

Thus, we have an isomorphism

$$\mathcal{M}_{0,n} \cong \mathbb{A}^{n-3} \setminus V\left(\prod_{p<q}(x_p - x_q)\prod_{p}(x_p^2 - 1)\right), \tag{7.1}$$

where $\mathbb{A}^{n-3} = \operatorname{Spec} k[x_2, \ldots, x_{n-2}]$, and x_p records the p-th marking. The induced μ_2 action on the right is given by $(x_2, \ldots, x_{n-2}) \mapsto (-x_2, \ldots, -x_{n-2})$. With this action, we have

$$\mathrm{CH}^*_{\mu_2}(\mathbb{A}^{n-3}) = \mathbb{Z}[s]/(2s),$$

where s is the class of a hyperplane going through the origin.

7.1. CKgP in Genus 0.

Proposition 7.2. *The spaces $\mathcal{M}_{0,n}, \overline{\mathcal{M}}_{0,n}, [\mathcal{M}_{0,n}/\mu_2].[\mathcal{M}_{0,n} \times \mathcal{M}_{0,m}/\mu_2]$ have the CKgP.*

Proof. First, we have that if $Z \subseteq \mathbb{A}^N$ is a union of hyperplanes then Z has the CKgP. To see this, one repeatedly uses Proposition 2.16(1) and the fact that intersections of hyperplanes always give an affine space, which has the CKgP by Proposition 2.16(3). Because $\mathcal{M}_{0,n}$ is the complement of a union of hyperplanes inside an affine space, both of which have the CKgP,we have that $\mathcal{M}_{0,n}$ has the CKgP. Similarlly, suppose we have $[Z/\mu_2] \subseteq [\mathbb{A}^N/\mu_2]$ with Z is a union of hyperplanes, where μ_2 acts on $\mathbb{A}^N$ by negation. For any collection of components of Z, the intersection is either isomrophic to an affine space, if one of the components does not contain the origin, and is otherwise isomorphic to $[\mathbb{A}^m/\mu_2]$, which has the CKgP by Proposition 2.16(3) and Lemma 2.22. Thus, both $[\mathcal{M}_{0,n}/\mu_2]$ and $[\mathcal{M}_{0,n} \times \mathcal{M}_{0,m}/\mu_2]$ have the CKgP because they are complements of such $[Z/\mu_2]$ inside some $[\mathbb{A}^N/\mu_2]$.

Now, we know $\overline{\mathcal{M}}_{0,n}$ has a stratification into

$$\mathcal{M}^\Gamma = [\mathcal{M}_\Gamma/\operatorname{Aut}(\Gamma)] = [\prod_{v\in\Gamma}\mathcal{M}_{g(v),n(v)}/\operatorname{Aut}(\Gamma)]$$

for genus 0 stable n-pointed graphs Γ. Because the graph has genus 0, all vertices must have genus 0, and the graph must be a tree. Then, any leaf on the graph must have marked points attached, to ensure that there are at least 3 special points on the component. An automorphism of Γ must then fix all leaves, and is therefore the identity. Thus, we have

$$\mathcal{M}^\Gamma = \prod_{v\in\Gamma}\mathcal{M}_{0,n(v)}.$$

By the above and Proposition 2.16(2), these $\mathcal{M}^\Gamma$ have the CKgP. Thus we have $\overline{\mathcal{M}}_{0,n}$ has the CKgP using Proposition 2.16(1) and the fact that these $\mathcal{M}^\Gamma$ stratify $\overline{\mathcal{M}}_{0,n}$. ☐

Corollary 7.3. *If $\mathcal{M}_{1,n}$ has the CKgP, then so does $\overline{\mathcal{M}}_{1,n}$.*

Corollary 7.4. *$\overline{\mathcal{M}}_{1,n}$ has the CKgP for $n \leq 4$.*

7.2. **Chow in Genus 0.** We now compute $\mathrm{CH}^*([\mathcal{M}_{0,n}/\mu_2]), \overline{\mathrm{CH}}^*([\mathcal{M}_{0,n}/\mu_2],1)$. First a lemma on $[\mathbb{A}^m/\mu_2]$.

Lemma 7.5. *Let $U \subseteq \mathbb{A}^m$ be a μ_2-invariant open subset with $0 \in U$. Then $\mathrm{CH}^*_{\mu_2}(\mathbb{A}^n) \to \mathrm{CH}^*_{\mu_2}(U)$ is an isomorphism.*

Proof. Note that the pullback
$$\mathrm{CH}^*_{\mu_2}(\mathbb{A}^n) \to \mathrm{CH}^*_{\mu_2}(\{0\})$$
is an isomorphism because it is inverse to the pullback $\mathrm{CH}^*_{\mu_2}(\mathrm{Spec}(k)) \to \mathrm{CH}^*_{\mu_2}(\mathbb{A}^n)$, which is an isomorphism by homotopy invariance. Thus, for U an open subset of $\mathbb{A}^n$, if $0 \in U$, $\mathrm{CH}^*_{\mu_2}(\mathbb{A}^n) \to \mathrm{CH}^*_{\mu_2}(U)$ is surjective by the localization exact sequence and injective because we can write composing with the pullback $\mathrm{CH}^*_{\mu_2}(U) \to \mathrm{CH}^*_{\mu_2}(\{0\})$ gives an isomorphism. □

Theorem 7.6.
$$\mathrm{CH}^*([\mathcal{M}_{0,n}/\mu_2]) \cong \begin{cases} \mathrm{CH}^*(B\mu_2) & n \leq 4 \\ \mathbb{Z} & n \geq 5 \end{cases}$$
and
$$\overline{\mathrm{CH}}^*([\mathcal{M}_{0,n}/\mu_2],1) = \overline{\mathrm{CH}}^1([\mathcal{M}_{0,n}/\mu_2],1) = (\mathcal{O}_{\mathcal{M}_{0,n}}(\mathcal{M}_{0,n})^\times)^{\mu_2}.$$

Proof. The localization exact sequence for $[\mathcal{M}_{0,n}/\mu_2] \subseteq [\mathbb{A}^{n-3}/\mu_2]$ reads
$$0 \to \overline{\mathrm{CH}}^*_{\mu_2}(\mathcal{M}_{0,n},1) \to \mathrm{CH}^{*-1}_{\mu_2}(Z) \to \mathrm{CH}^*_{\mu_2}(\mathbb{A}^{n-3}) \to \mathrm{CH}^*_{\mu_2}(\mathcal{M}_{0,n}) \to 0,$$
using Proposition 2.22 and homotopy invariance to say $\overline{\mathrm{CH}}^*_{\mu_2}(\mathbb{A}^{n-3},1) = 0$. Suppose $n \leq 4$. Then $0 \in \mathcal{M}_{0,n}$, so Lemma 7.5 says $\mathrm{CH}^*([\mathbb{A}^{n-3}/\mu_2]) \to \mathrm{CH}^*([\mathcal{M}_{0,n}/\mu_2])$ is an isomorphism. The localization sequence then gives $\overline{\mathrm{CH}}^*_{\mu_2}(\mathcal{M}_{0,n},1) = \mathrm{CH}^{*-1}_{\mu_2}(Z)$. If $n=3$, $Z = \emptyset$, so $\overline{\mathrm{CH}}^*(\mathcal{M}_{0,3},1) = 0$. If $n=4$, $Z = V(x_1^2-1)$, so $[Z/\mu_2] = \mathrm{Spec}(k)$, giving $\mathrm{CH}^1([\mathcal{M}_{0,4}/\mu_2],1) = \mathrm{CH}^0(\mathrm{Spec}(k)) = \mathbb{Z}$.

Now suppose $n \geq 5$. Then $V(x_p - x_q) \cong \mathbb{A}^{n-4} \subseteq Z$ and $\mathrm{CH}^*_{\mu_2}(\mathbb{A}^m) = \mathbb{Z}[s]/(2s)$. Now $[V(x_p-x_q)] = s$ because $V(x_p - x_q)$ is a hyperplane containing the origin, so pushforward $\mathrm{CH}^{*-1}_{\mu_2}(V(x_p-x_q)) \to \mathrm{CH}^*_{\mu_2}(\mathbb{A}^{n-3})$ is multiplication by s, using Lemma 2.23. This is an isomorphism in positive degrees, and so $\mathrm{CH}^*_{\mu_2}(\mathcal{M}_{0,n}) = \mathbb{Z}$.

We next compute $\mathrm{CH}^*_{\mu_2}(Z)$. We have
$$\bigoplus_p \mathrm{CH}^*_{\mu_2}(V(x_p^2-1)) \oplus \bigoplus_{p<q} \mathrm{CH}^*_{\mu_2}(V(x_p-x_q)) \twoheadrightarrow \mathrm{CH}^*_{\mu_2}(Z)$$
with the kernel generated by pushforwards of intersections of components. In particular, this is an isomorphism in degree 0. Note $[V(x_p^2-1)/\mu_2] \cong \mathbb{A}^{n-4}$ because the action is free, so
$$\mathrm{CH}^*_{\mu_2}(V(x_p^2-1)) = \mathrm{CH}^*(\mathbb{A}^{n-4}) = \mathbb{Z}.$$

Moreover, for $p \neq q$, we must have

$$\mathrm{CH}^*_{\mu_2}(V(x_p^2-1)\cap V(x_q^2-1)) \to \mathrm{CH}^*_{\mu_2}(V(x_p^2-1)) \oplus \mathrm{CH}^*_{\mu_2}(V(x_q^2-1))$$

equals 0, as it maps into positive degrees. For the same reason the map

$$\mathrm{CH}^*_{\mu_2}(V(x_p^2-1)\cap V(x_j-x_\ell)) \to \mathrm{CH}^*_{\mu_2}(V(x_p^2-1)) \oplus \mathrm{CH}^*_{\mu_2}(V(x_j-x_\ell))$$

is 0 on the first coordinate. It is also 0 on the second coordinate, because $[V(x_p^2-1)] \in \mathrm{CH}^*_{\mu_2}(\mathbb{A}^m)$ is 0. Finally, the map

$$\mathrm{CH}^*_{\mu_2}(V(x_p-x_q)\cap V(x_j-x_\ell)) \to \mathrm{CH}^*_{\mu_2}(V(x_p-x_q)) \oplus \mathrm{CH}^*_{\mu_2}(V(x_j-x_\ell))$$

is an isomorphism onto either coordinate in positive degrees. Thus,

$$\mathrm{CH}^i_{\mu_2}(Z) = \mathbb{Z}/2\mathbb{Z},$$

for $i \geq 1$.

As noted above, $\mathrm{CH}^{*-1}_{\mu_2}(V(x_p-x_q)) \to \mathrm{CH}^*_{\mu_2}(\mathbb{A}^{n-3})$ is an isomorphism in degrees > 2, and we have just seen $\mathrm{CH}^*_{\mu_2}(V(x_p-x_q)) \to \mathrm{CH}^*_{\mu_2}(Z)$ is an isomorphism in positive degrees, so $\mathrm{CH}^{*-1}_{\mu_2}(Z) \xrightarrow{\iota_*} \mathrm{CH}^*_{\mu_2}(\mathbb{A}^{n-3})$ is an isomorphism in degrees > 1. Thus,

$$\overline{\mathrm{CH}}^i_{\mu_2}(\mathcal{M}_{0,n},1) = 0$$

for $i \geq 2$. Moreover,

$$\overline{\mathrm{CH}}^1_{\mu_2}(\mathcal{M}_{0,n},1) = (\mathcal{O}_{\mathcal{M}_{0,n}}(\mathcal{M}_{0,n})^\times)^{\mu_2}$$

by Lemma 2.5. □

Now, we move on to thinking about $[\overline{\mathcal{M}}_{0,n}/\mu_2]$. For $A, B \subseteq \{1,\dots,n-2,a,b\}$ disjoint subsets, we have the divisors $D(A|B) = D(B|A) \subseteq \overline{\mathcal{M}}_{0,n}$ which correspond to curves that have a node separating the markings from sets A and B. Such a divisor is a sum of irreducible divisors $D(A'|B')$ where B' is the complement of A'. Let $\widehat{D}(A|B)$ be the image of $D(A|B)$ in $[\overline{\mathcal{M}}_{0,n}/\mu_2]$. By abuse of notation, we also say $D(A|B), \widehat{D}(A|B)$ are the corresponding classes in the respective Chow rings.

For $n \geq 5$, Theorem 7.6 says $\mathrm{CH}^*([\mathcal{M}_{0,n}/\mu_2]) = \mathbb{Z}$, which means that when $\mathcal{M}^\Gamma = [\mathcal{M}_{0,n}/\mu_2]$, it is easy to split the top row of the exact sequence in the diagram 6.6. But this is not the case when $n = 4$, so we need a result that explains how to split the exact sequence. Let $\widehat{T}$ denote the $B\mu_2$-point in $[\mathcal{M}_{0,4}/\mu_2]$.

Lemma 7.7. *The surjection*

$$\mathrm{CH}^*([\overline{\mathcal{M}}_{0,4}/\mu_2]) \twoheadrightarrow \mathrm{CH}^*([\mathcal{M}_{0,4}/\mu_2]) = \frac{\mathbb{Z}[t]}{(2t)}$$

has a splitting given by

$$1 \mapsto 1$$
$$t^i \mapsto \iota_{1*}(u^{i-1}) - \iota_{2*}(u^{i-1})$$

for $i \geq 1$, where ι_1 and ι_2 are the inclusions of the $B\mu_2$-points $\widehat{T}$ and $\widehat{D}(12|ab)$ into $[\overline{\mathcal{M}}_{0,4}/\mu_2]$, respectively.

Proof. We have $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$. Then one can use the projective bundle formula for $[\mathbb{P}^1/\mu_2] \to B\mu_2$ to compute

$$\mathrm{CH}^*([\overline{\mathcal{M}}_{0,4}/\mu_2]) = \frac{\mathbb{Z}[u,t]}{(2u, t^2+ut)},$$

where $t = [\widehat{T}]$ and $t+u = \widehat{D}(12|ab)$. The map $\mathrm{CH}^*([\overline{\mathcal{M}}_{0,4}/\mu_2]) \twoheadrightarrow \mathrm{CH}^*([\mathcal{M}_{0,4}/\mu_2]) = \frac{\mathbb{Z}[t]}{(2t)}$ sends $t \to t$ and $u \to t$. Thus, we have a splitting given by $1 \mapsto 1$ and $t^i \mapsto u^i$ for $i \geq 1$. Finally, we can see that $u^{i-1}t = \iota_{1*}(u^{i-1})$ and $(t+u)u^i = \iota_{2*}(u^{i-1})$ by the projection formula, using that $\iota_1^*(t) = \iota_2^*(s) = u$. Thus, $u^i = \iota_{1*}(u^{i-1}) - \iota_{2*}(u^{i-1})$. □

Next, we calculate the image of $\overline{\mathrm{CH}}^*([\mathcal{M}_{0,n}/\mu_2],1) \to \mathrm{CH}^*([\partial\overline{\mathcal{M}}_{0,n}/\mu_2])$.

Theorem 7.8. *The image of*

$$\overline{\mathrm{CH}}^*([\mathcal{M}_{0,n}/\mu_2],1) \xrightarrow{\partial_1^{\mu_2}} \mathrm{CH}^*([\partial\overline{\mathcal{M}}_{0,n}/\mu_2])$$

is freely generated by

$$\widehat{D}(pa|1b) - 2\widehat{D}(1p|ab)$$

for $p \in \{2,\ldots,n-2\}$ and

$$\alpha_{23} + \alpha_{pq}$$

for $p,q \in \{2,\ldots,n-2\}$ with $p \neq q$, where

$$\alpha_{pq} := \widehat{D}(pqb|1a) + \widehat{D}(1ab|pq) - \widehat{D}(1pq|ab) - \widehat{D}(qab|1p) - \widehat{D}(pab|1q).$$

Proof. Note $\mathcal{O}_{\mathcal{M}_{0,n}}(\mathcal{M}_{0,n})/k^\times = \overline{\mathrm{CH}}^*(\mathcal{M}_{0,n},1)$ is freely generated by $\{x_p \pm 1\}, \{x_p - x_q\}$, and so $(\mathcal{O}_{\mathcal{M}_{0,n}}(\mathcal{M}_{0,n})^\times)^{\mu_2}/k^\times = \overline{\mathrm{CH}}^*([\mathcal{M}_{0,n}/\mu_2],1)$ is freely generated by

$$\{x_p^2 - 1\}_p, \{(x_2 - x_3)(x_p - x_q)\}_{p\neq q}.$$

Therefore, $\partial_1^{\mu_2}$ applied to this generating set generates the image of $\partial_1^{\mu_2}$.

On $\mathcal{M}_{0,4}$, because $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, we have

$$\partial_1(x-1) = D(1b|2a) - D(12|ab).$$

We leverage commutativity of

$$\begin{array}{ccc} \overline{\mathrm{CH}}^1(\mathcal{M}_{0,n},1) & \xrightarrow{\partial_1} & \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{0,n}) \\ {\scriptstyle\varphi^*}\uparrow & & {\scriptstyle\varphi^*}\uparrow \\ \overline{\mathrm{CH}}^1(\mathcal{M}_{0,4},1) & \xrightarrow{\partial_1} & \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{0,4}) \end{array}$$

induced by morphisms $\varphi : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,4}$ with $\varphi(\mathcal{M}_{0,n}) \subseteq \mathcal{M}_{0,4}$ to extend this computation to general n.

For a general n, given a choice of 4 markings i,j,k,ℓ, we can consider the morphism

$$\begin{aligned} \varphi_{i,j,k,\ell} : \overline{\mathcal{M}}_{0,n} &\to \overline{\mathcal{M}}_{0,4} \\ (C,p_1,\ldots,p_n) &\mapsto (C,p_i,p_j,p_k,p_\ell). \end{aligned}$$

Over $\mathcal{M}_{0,n} \subseteq \overline{\mathcal{M}}_{0,n}$, the points p_i, p_j, p_k, p_ℓ are distinct smooth points of $\mathbb{P}^1$, so $\varphi_{i,j,k,\ell}$ maps into $\mathcal{M}_{0,4}$. By the commutativity of the above diagram, we thus have

$$\partial_1(\varphi^*(x-1)) = D(i\ell|jk) - D(ij|k\ell).$$

We can compute $\varphi_{i,j,k,\ell}$ restricted to $\mathcal{M}_{0,n}$ under our identifications made in 7.1 by applying transformations to send $p_i \mapsto \infty$, $p_k \mapsto 1$, $p_\ell \mapsto -1$. This gives the cross ratio, modified to fit with our conventions of $\mathcal{M}_{0,n}$.

When $i = 1$, $k = a$, and $\ell = b$, the cross ratio is x_j, and so the function $x - 1$ on $\mathcal{M}_{0,4}$ pulls back to $x_j - 1$. Thus,

$$\partial_1(x_j - 1) = D(1b|ja) - D(1j|ab).$$

Similarly, when $i = 1$, $k = b$, and $\ell = b$, the cross ratio is $-x_j$, so the function $x - 1$ on $\mathcal{M}_{0,4}$ pulls back to $-x_j - 1$ on $\mathcal{M}_{0,4}$, and so

$$\partial_1(x_j + 1) = D(1a|jb) - D(1j|ab).$$

Finally, when $\ell = a$ and $i = b$, the cross ratio is

$$2\frac{(x_j-1)(x_k+1)}{(x_j+1)(x_k-1)} - 1,$$

so $x - 1$ on $\mathcal{M}_{0,4}$ pulls back to

$$4\frac{x_j - x_k}{(x_j+1)(x_k-1)},$$

meaning

$$\partial_1\Big(\frac{x_j - x_k}{(x_j+1)(x_k-1)}\Big) = D(jk|ab) - D(jb|ka).$$

Combining this with the above expressions for $\partial_1(x_j+1), \partial_1(x_k-1)$, we get

$$\partial_1(x_j - x_k) = D(jk|ab) - D(jb|ka) + D(1a|jb) - D(1j|ab) + D(1b|ka) - D(1k|ab).$$

A more useful way to write this is to expand out do that every term involves all of the markings we are looking at, using, for example, $D(jk|ab) = D(1jk|ab) + D(jk|1ab)$. In doing this, there are many cancellations, and one gets the expression

$$D(1ab|jk) + D(jkb|1a) + D(jka|1b) - D(1jk|ab) - D(kab|1j) - D(jab|1k)$$

for $\partial_1(x_j - x_k)$. Note this expression is invariant under switching either a, b or j, k, as it should be.

Let $\pi : \overline{\mathcal{M}}_{0,n} \to [\overline{\mathcal{M}}_{0,n}/\mu_2]$ be the quotient map. Consider the diagram

$$\begin{tikzcd}
\overline{\mathrm{CH}}^1(\mathcal{M}_{0,n},1) \arrow[r, "\partial_1"] & \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{0,n}) \\
\overline{\mathrm{CH}}^1([\mathcal{M}_{0,n}/\mu_2],1) \arrow[r, "\partial_1^{\mu_2}"] \arrow[u, "\pi^*"] & \mathrm{CH}^0([\partial\overline{\mathcal{M}}_{0,n}/\mu_2]) \arrow[u, "\pi^*"']
\end{tikzcd}$$

The pullback $\pi^* : \overline{\mathrm{CH}}^1([\mathcal{M}_{0,n}/\mu_2], 1) \to \overline{\mathrm{CH}}^1(\mathcal{M}_{0,n}, 1)$ is the inclusion $(\mathcal{O}_{\mathcal{M}_{0,n}}(\mathcal{M}_{0,n})^\times)^{\mu_2}/k^\times \hookrightarrow \mathcal{O}_{\mathcal{M}_{0,n}}(\mathcal{M}_{0,n})^\times/k^\times$ under the identifications. We also have that $\mathrm{CH}^0([\partial\overline{\mathcal{M}}_{0,n}/\mu_2]) = \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{0,n})^{\mu_2}$ via the pullback, because equivariant Chow is just the invariant Chow cycles when looking at CH^0.

For $D(A|B) \subseteq \overline{\mathcal{M}}_{0,n}$ with $A \cup B = [n]$, we have $\pi^*(\widehat{D}(A|B))$ is $D(A|B)$ if a, b are in the same set and is $D(A|B) + D(A'|B')$ if there are in different sets, where A', B' are the same as A, B, but with a and b switched. Then, have that the same for $D(A|B)$ with $A \cup B$ not necessarily equal to $[n]$, as long as $a, b \in A \cup B$, because the what happens on each irreducible boundary divisor making up $D(A|B)$ will be consistent.

Now, we claim

$$\partial_1^{\mu_2}(x_p^2 - 1) = \widehat{D}(1b|pa) - 2\widehat{D}(1p|ab)$$
$$\partial_1^{\mu_2}((x_2 - x_3)(x_{p'} - x_{q'})) = \alpha_{2,3} + \alpha_{p,q},$$

where $\alpha_{p,q,r}$ is as in the statement of the Theorem. This is true because these statements give exactly what we computed before for $\partial_1(x_p - 1), \partial_1(x_p + 1), \partial_1(x_p - x_q)$ when pulled back along π^*, and therefore hold by injectivity of π^*. Thus, the claimed generators do indeed generate.

We now wish to say that they generate the image freely. Because we took the image of a free generating set under $\partial_1^{\mu_2}$, it suffices to show that $\partial_1^{\mu_2}$ is injective. The localization exact sequence says

$$\overline{\mathrm{CH}}^1([\overline{\mathcal{M}}_{0,n}/\mu_2], 1) \to \overline{\mathrm{CH}}^1([\mathcal{M}_{0,n}/\mu_2], 1) \xrightarrow{\partial_1^{\mu_2}} \mathrm{CH}^0([\partial\overline{\mathcal{M}}_{0,n}/\mu_2]).$$

By Lemma 2.5, we have $\overline{\mathrm{CH}}^1([\overline{\mathcal{M}}_{0,n}/\mu_2], 1) = (k^\times)_2^\mu/k^\times = 0$, so $\partial_1^{\mu_2}$ is injective. $\square$

Corollary 7.9. *There is a unique nontrivial* 2*-torsion element in* $\mathrm{CH}^1([\overline{\mathcal{M}}_{0.n}/\mu_2]) = \mathrm{Pic}([\overline{\mathcal{M}}_{0.n}/\mu_2])$ *given by*

$$\alpha_{23} = \widehat{D}(23b|1a) + \widehat{D}(1ab|23) - \widehat{D}(123|ab) - \widehat{D}(3ab|12) - \widehat{D}(2ab|13),$$

which is also the pullback of u *along* $[\overline{\mathcal{M}}_{0,n}/\mu_2] \to B\mu_2$*.*

This should, in principle, follow directly from analyzing the relations for $\mathrm{Pic}([\overline{\mathcal{M}}_{0,n}/\mu_2])$ given by the theorem, but we take an alternative approach.

Proof. Torsion line bundles of order N are equivalent to μ_N covers. Because $\overline{\mathcal{M}}_{0,n}$ is a rational smooth projective variety, it has trivial étale fundamental group. Hence, we have $\pi_1^{\text{ét}}([\overline{\mathcal{M}}_{0,n}/\mu_2]) = \frac{\mathbb{Z}}{2\mathbb{Z}}$. Thus, there is a unique nontrivial 2-torsion line bundle. Moreover, this bundle is pulled back from $B\mu_2$.

Because $n \geq 5$, Theorem 7.6 gives $\mathrm{CH}^1([\mathcal{M}_{0,n}/\mu_2]) = 0$, hence

$$\mathrm{Pic}([\overline{\mathcal{M}}_{0,n}/\mu_2]) = \mathrm{CH}^0([\partial\overline{\mathcal{M}}_{0,n}/\mu_2])/\mathrm{im}(\partial_1^{\mu_2}).$$

Then, the theorem gives that α_{23} is 2-torsion class in this group. Moreover, it is nontrivial as long as α_{23} is not in the image of $\partial_1^{\mu_2}$, which is true because $2\alpha_{23}$ is part of a free generating set of $\mathrm{im}(\partial_1^{\mu_2})$. $\square$

Now, all but one of the $\mathcal{M}^\Gamma$ appearing in the boundary of $\overline{\mathcal{M}}_{1,n}$ for $n \leq 4$ are isomorphic to either $[\mathcal{M}_{0,m}/\mu_2]$ or $\mathcal{M}_{0,m}$ for some m. There is one exception, where $\mathcal{M}^\Gamma$ is isomorphic to $[\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}/\mu_2]$. We need the analogue of the results we have proven in this section for this space.

Consider the exact sequence

$$\mathrm{CH}^{*-1}([\partial(\overline{\mathcal{M}}_{0,4}\times\overline{\mathcal{M}}_{0,4})/\mu_2]) \xrightarrow{\iota_*} \mathrm{CH}^*([\overline{\mathcal{M}}_{0,4}\times\overline{\mathcal{M}}_{0,4}/\mu_2]) \to \mathrm{CH}^*([\mathcal{M}_{0,4}\times\mathcal{M}_{0,4}/\mu_2]) \to 0.$$

Because $[\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}/\mu_2]$ is an open in $[\mathbb{A}^2/\mu_2]$ containing the origin, Lemma 7.5 implies its Chow ring is isomorphic to $\frac{\mathbb{Z}[t]}{(2t)}$.

Let $\pi_i : [\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,4}/\mu_2] \to [\overline{\mathcal{M}}_{0,4}/\mu_2]$ be the projections onto the ith factor. As $\partial\overline{\mathcal{M}}_{0,4} = \{D(12|ab), D(1a|2b), D(1b|2a)\}$, we have that $[\partial(\overline{\mathcal{M}}_{0,4}\times \overline{\mathcal{M}}_{0,4})/\mu_2]$ has the 4 components

$$\pi_1^{-1}\widehat{D}(12|ab), \pi_1^{-1}\widehat{D}(1a|2b), \pi_2^{-1}\widehat{D}(12|ab), \pi_2^{-1}\widehat{D}(1a|2b).$$

Lemma 7.10.

(1) *The kernel of ι_* is generated by*

$$[\pi_1^{-1}\widehat{D}(1a|2b)] - 2[\pi_1^{-1}\widehat{D}(12|ab)], [\pi_2^{-1}\widehat{D}(1a|2b)] - 2[\pi_2^{-1}\widehat{D}(12|ab)].$$

(2) *The surjection*

$$\mathrm{CH}^*([\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,4}/\mu_2]) \twoheadrightarrow \mathrm{CH}^*([\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}/\mu_2]) = \frac{\mathbb{Z}[t]}{(2t)}$$

has a splitting given by

$$1 \mapsto 1$$
$$t \mapsto [\pi_1^{-1}\widehat{T}] - [\pi_1^{-1}\widehat{D}(12|ab)]$$
$$t^i \mapsto \iota_{1*}(u^{i-1}) - \iota_{2*}(u^{i-1}) + \iota_{3*}(u^{i-1}) - \iota_{4*}(u^{i-1})$$

for $i \geq 1$, where the $\iota_1, \iota_2, \iota_3$ and ι_4 are the inclusions of $B\mu_2$-points $\widehat{T} \times \widehat{T}$, $\widehat{T} \times \widehat{D}(12|ab), \widehat{D}(12|ab) \times \widehat{T}$, and $\widehat{D}(12|ab) \times \widehat{D}(12|ab)$ into $[\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,4}/\mu_2]$, respectively.

Proof.

(1) The higher indecomposible Chow groups $\overline{\mathrm{CH}}^*([\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}/\mu_2], 1)$ can be computed in a similar manner to the proof of Theorem 7.6. One gets

$$\overline{\mathrm{CH}}^i([\mathcal{M}_{0,4}\times\mathcal{M}_{0,4}/\mu_2], 1) = \overline{\mathrm{CH}}^1([\mathcal{M}_{0,4}\times\mathcal{M}_{0,4}/\mu_2], 1) = \langle x_2^1-1, y_2^2-1\rangle \cong \mathbb{Z}^2.$$

By using push-pull on the diagram

$$\begin{array}{ccc} \overline{\mathrm{CH}}^1([\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}/\mu_2], 1) & \xrightarrow{\partial_1'} & \mathrm{CH}^0([\partial(\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,4})/\mu_2]) \\ \uparrow{\scriptstyle \pi_i^*} & & \uparrow{\scriptstyle \pi_i^*} \\ \overline{\mathrm{CH}}^1([\mathcal{M}_{0,4}/\mu_2], 1) & \xrightarrow{\partial_1} & \mathrm{CH}^0([\partial(\overline{\mathcal{M}}_{0,4})/\mu_2]) \end{array}$$

for $i = 1, 2$, we see that $\operatorname{im}(\partial_1') = \ker(\iota_*)$ is generated by

$$[\pi_1^{-1}\widehat{D}(1a|2b)] - 2[\pi_1^{-1}\widehat{D}(12|ab)], [\pi_2^{-1}\widehat{D}(1a|2b)] - 2[\pi_2^{-1}\widehat{D}(12|ab)]$$

(2) Similar to the proof of Lemma 7.7, we have

$$\mathrm{CH}^*([\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,4}/\mu_2]) = \mathbb{Z}[t_1, t_2, u]/(2u, t_1^2 + ut_1, t_2^2 + ut_2)$$

where $t_i = \pi_i^{-1}\widehat{T}$, $t_i + u = \pi_i^{-1}\widehat{D}(12|ab)$ using the projective bundle formula and the pullback diagram

$$\begin{tikzcd}
{[\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,4}/\mu_2]} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & {[\overline{\mathcal{M}}_{0,4}/\mu_2]} \arrow[d] \\
{[\overline{\mathcal{M}}_{0,4}/\mu_2]} \arrow[r] & B\mu_2.
\end{tikzcd}$$

We have that the map

$$\mathrm{CH}^*([\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,4}/\mu_2]) \to \mathrm{CH}^*([\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}/\mu_2]) = \frac{\mathbb{Z}[t]}{(2t)}$$

sends each of the generators t_1, t_2, u to t. Thus, we have a splitting given by $t^i \mapsto u^i$. For $i = 1$, note $u = [\pi_1^{-1}\widehat{T}] - [\pi_1^{-1}\widehat{D}(12|ab)]$. For $i = 2$, we have

$$\begin{aligned}
u^2 &= t_1t_2 - t_1(t_2 + u) + (t_1 + u)t_2 - (t_1 + u)(t_2 + u) \\
&= [\widehat{T} \times \widehat{T}] - [\widehat{T} \times \widehat{D}(12|ab)] + [\widehat{D}(12|ab) \times \widehat{T}] - [\widehat{D}(12|ab) \times \widehat{D}(12|ab)].
\end{aligned}$$

And the, we have

$$u^i = \iota_{1*}(u^{i-1}) - \iota_{2*}(u^{i-1}) + \iota_{3*}(u^{i-1}) - \iota_{4*}(u^{i-1})$$

using the projection formula and the fact that $\iota_i^*(u) = u$.

□

8. The Chow Ring of $\overline{\mathcal{M}}_{1,n}$ for $n = 2, 3, 4$

8.1. $\overline{\mathcal{M}}_{1,2}$. Recall our notation that for a stable graph Γ, we set $\gamma := [\overline{\mathcal{M}}^{\Gamma}]$. Define the following stable graphs

$$\Delta :=$$

$$\Theta :=$$

$$\Phi :=$$

By Theorem 6.3 and Theorem 5.3, we have

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) = \frac{\mathbb{Z}[\lambda_0]}{(24\lambda_0^2)}$$

(where the subscript 0 is to distinguish it from the class $\lambda \in \mathrm{CH}^*(\overline{\mathcal{M}}_{1,2})$). By Theorem 6.2, we have

$$\mathcal{M}^{\Theta} = \overline{\mathcal{M}}^{\Theta} = [\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}/\mu_2] = B\mu_2,$$

and

$$\mathcal{M}^\Phi = [\mathcal{M}_\Phi/\operatorname{Aut}(\Phi)] = [\mathcal{M}_{0,4}\times\mathcal{M}_{0,3}/\mu_2] = [\mathcal{M}_{0,4}/\mu_2].$$

Let θ^i denote the image of u^{i-1} under the pushforward $\mathbb{Z}[u]/(2u) = \mathrm{CH}^*(\overline{\mathcal{M}}^\Theta) \to \mathrm{CH}^*\,\partial\overline{\mathcal{M}}_{1,2})$. We also have a $B\mu_2$-point $P \in \mathcal{M}^\Phi$ corresponding to $\widehat{T} \in [\mathcal{M}_{0,4}/\mu_2]$. Let τ^i denote the image of u^{i-1} under the pushforward $\mathbb{Z}[u]/(2u) = \mathrm{CH}^*\,P) \to \mathrm{CH}^*\,\partial\overline{\mathcal{M}}_{1.2})$.

Warning 8.1. *Although the notation suggests that θ^i, τ^i are powers of an element this is not the case. Moreover, when we refer to powers of λ on $\partial\overline{\mathcal{M}}_{1,2}$, these are pushed forward from $\overline{\mathcal{M}}^\Delta$. Because $\partial\overline{\mathcal{M}}_{1,2}$ is not smooth, $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2})$ has no product structure.*

Proposition 8.2. *The Chow group $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2})$ is given by*

- *$\mathrm{CH}^0(\partial\overline{\mathcal{M}}_{1,2})$ is freely generated by ϕ, δ*
- *$\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,2})$ is generated by λ_0, θ, τ subject to the relations*
$$24\lambda_0 - 2\theta$$
$$24\lambda_0 - 2\tau$$
- *$\mathrm{CH}^i(\partial\overline{\mathcal{M}}_{1,2})$ is generated by $\lambda_0^i, \theta^i, \tau^i$ subject to the relations*
$$24\lambda_0^i$$
$$2\theta^i$$
$$2\tau^i$$

  for $i \geq 2$.

Proof. There is only one component in $\overline{\mathcal{M}}^{\mathrm{sep},\geq 3} = \overline{\mathcal{M}}^{\mathrm{sep}}$, corresponding to the graph Δ. Thus, $\overline{\mathcal{M}}^{\mathrm{sep}} = \overline{\mathcal{M}}^\Delta$ and $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) = \frac{\mathbb{Z}[\lambda_0]}{(24\lambda_0^2)}$. Moreover, $\overline{\mathcal{M}}^\Theta = \mathcal{M}^\Theta = [\mathcal{M}_\Theta/\operatorname{Aut}(\Theta) = B\mu_2$ as noted before, making $\mathrm{CH}^*(\mathcal{M}^\Theta) = \frac{\mathbb{Z}[u]}{(2u)}$. Because $\overline{\mathcal{M}}^{\mathrm{sep},\geq 2} = \mathcal{M}^\Theta \amalg \overline{\mathcal{M}}^{\mathrm{sep}}$ as spaces, we have

$$\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) = \frac{\mathbb{Z}[u]}{(2u)} \oplus \frac{\mathbb{Z}[\lambda_0]}{(24\lambda_0^2)}.$$

Next, we consider $\partial\overline{\mathcal{M}}_{1,2} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 1} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 2} \cup \mathcal{M}^\Phi$. As noted above, $\mathcal{M}^\Phi = [\mathcal{M}_{0,4}/\mu_2]$. Moreover, $[\overline{\mathcal{M}}_\Phi/\operatorname{Aut}(\Phi)] = [\overline{\mathcal{M}}_{0,4}/\mu_2] \cong [\mathbb{P}^1/\mu_2]$ and $[\partial\overline{\mathcal{M}}_\Phi.4/\mu_2] = \mathrm{Spec}(k) \amalg B\mu_2$. The diagram (6.6) in this setting becomes

$$\begin{tikzcd}
\overline{\mathrm{CH}}^*(\mathcal{M}_{0,4},1) \arrow[r] \arrow[d] & \mathrm{CH}^{*-1}([\partial\overline{\mathcal{M}}_{0,4}/\mu_2) \arrow[r] \arrow[d] & \mathrm{CH}^*([\overline{\mathcal{M}}_{0,4}/\mu_2]) \arrow[r] \arrow[d] & \mathrm{CH}^*([\mathcal{M}_{0,4}/\mu_2]) \arrow[r] \arrow[d] & 0 \\
\overline{\mathrm{CH}}^*(\mathcal{M}^{\mathrm{non}=1},1) \arrow[r] & \mathrm{CH}^{*-1}(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \arrow[r] & \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2}) \arrow[r] & \mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=1}) \arrow[r] & 0
\end{tikzcd}$$

Filling this out using the above and Theorem 7.7, we get

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\partial_1"] \arrow[d, "\sim"] & \mathbb{Z} \oplus \frac{\mathbb{Z}[u]}{(2u)} \arrow[r] \arrow[d] & \frac{\mathbb{Z}[v,t]}{(vt,2(t-v))} \arrow[r] \arrow[d] & \frac{\mathbb{Z}[t]}{(2t)} \arrow[r] \arrow[d, "\sim"] & 0 \\
\mathbb{Z} \arrow[r, "\partial_1'"] & \frac{\mathbb{Z}[u]}{(2u)} \oplus \frac{\mathbb{Z}[\lambda_0]}{(24\lambda_0^2)} \arrow[r] & \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2}) \arrow[r] & \frac{\mathbb{Z}[t]}{(2t)} \arrow[r] & 0
\end{tikzcd}$$

The map $[\overline{\mathcal{M}}_{0,4}/\mu_2] \to \partial\overline{\mathcal{M}}_{1,2}$ sends $\widehat{D}(12|ab)$ to the point in $\overline{\mathcal{M}}^{\Delta} \cap \overline{\mathcal{M}}^{\Phi}$ and $(\mathbb{P}^1, \infty, 0, 1, -1)$ to P. Thus, we get $v^i \mapsto 12\lambda_0^i$ and $t^i \mapsto \tau^i$.

By Theorem 7.8, we know that the image of ∂_1 is generated by $\widehat{D}(1a|2b) - 2\widehat{D}(12|ab)$, which then pushes forward to $2\theta - 24\lambda_0$. Also, by the diagram, the exact sequence

$$0 \to \left(\frac{\mathbb{Z}[u]}{(2u)} \oplus \frac{\mathbb{Z}[\lambda_0]}{(24\lambda_0^2)}\right)/\langle(2, -24\lambda_0)\rangle \to \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2}) \to \frac{\mathbb{Z}[t]}{(2t)} \to 0$$

has a splitting by lifting t to $\tau - 12\lambda_0$ and t^i to τ^i for $i \geq 2$. Putting this together, we get the desired presentation for $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,2})$. □

Proposition 8.3. *The image of $\partial_1 : \overline{\mathrm{CH}}^i(\mathcal{M}_{1,2}, 1) \to \mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,2})$ is 0 for $i = 1$ and is generated by $\theta^{i-1} - \tau^{i-1}$ for $i > 1$.*

Proof. Help □

Let λ denote the first Chern class of the Hodge bundle on $\overline{\mathcal{M}}_{1,2}$. Let $\pi : \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{1,1}$ be the morphism forgetting the second point, and let $s : \overline{\mathcal{M}}_{1,1} \xrightarrow{\sim} \overline{\mathcal{M}}^{\Delta} \subseteq \overline{\mathcal{M}}_{1,2}$ be its section.

Theorem 8.4.

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}) = \mathbb{Z}[\lambda, \delta]/(24\lambda^2, \delta(\lambda + \delta))$$

and $\iota_*(\theta^i) = 12\lambda^i(\delta - \lambda)$.

Proof of Theorem 8.4. The localization exact sequence on $\partial\overline{\mathcal{M}}_{1,2} \subseteq \overline{\mathcal{M}}_{1,2}$ is

$$\overline{\mathrm{CH}}^*(\mathcal{M}_{1,2}, 1) \xrightarrow{\partial_1} \mathrm{CH}^{*-1}(\partial\overline{\mathcal{M}}_{1,2}) \to \mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}) \to \mathrm{CH}^*(\mathcal{M}_{1,2}) \to 0.$$

The terms besides $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2})$ were computed in Theorem 3.5 and Theorem 8.2. In degree 0, it is clear that $\mathrm{CH}^0(\overline{\mathcal{M}}_{1,2}) = \mathbb{Z}$.

In degree 1, the localization exact sequence is is

$$0 \to \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{1,2}) \xrightarrow{\iota_*} \mathrm{CH}^1(\overline{\mathcal{M}}_{1,2}) \to \frac{\mathbb{Z}}{12\mathbb{Z}} \to 0.$$

By Proposition 5.5, we have $\iota_*(\phi) = 12\lambda$. Thus, we have $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,2})$ is freely generated by λ, δ.

In degree $i > 1$, the localization exact sequence gives

$$0 \to \mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,2})/\langle\tau^i - \theta^i\rangle \xrightarrow{\iota_*} \mathrm{CH}^i(\overline{\mathcal{M}}_{1,2}) \to \mathrm{CH}^i(\mathcal{M}_{1,2}) \to 0.$$

Because $12\lambda^i = 0$, we know $12\lambda^i = \iota_*(\alpha)$, for some $\alpha \in \mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,2})$. The order of $\lambda^i \in \mathrm{CH}^i(\overline{\mathcal{M}}_{1,2})$ is 24 by Proposition 5.5, so α must have order

2 in $\mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,2})/\langle\tau^i-\theta^i\rangle$. So $\alpha\in\{12\lambda_0^{i-1},\theta^{i-1},12\lambda_0^{i-1}-\theta^{i-1}\}$. Note $\pi_*(\lambda^i)=\pi_*(\pi^*(\lambda^i))=\lambda^i\pi_*(1)=0$.

- If $\alpha=12\lambda_0^{i-1}$, then
$$12\lambda^i=\iota_*(\alpha)=\iota_*(12\lambda_0^{i-1})=s_*(12\lambda_0^{i-1}).$$
Taking π_* of both sides, we get $0=12\pi_*(\lambda^i)=12\lambda^{i-1}$, which is not true in $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1})$.
- If $\alpha=\theta^{i-1}$, then we get $\pi_*(\iota_*(\theta^{i-1}))=\pi_*(12\lambda^i)=0$. But θ^{i-1} is the pushforward of u^{i-2} from $\overline{\mathcal{M}}^{\Theta}\cong B\mu_2$, and the composition $B\mu_2\cong\overline{\mathcal{M}}^{\Theta}\to\overline{\mathcal{M}}_{1,2}\xrightarrow{\pi}\overline{\mathcal{M}}_{1,1}$ is the same as the inclusion $\partial\overline{\mathcal{M}}_{1,1}\hookrightarrow\overline{\mathcal{M}}_{1,1}$, and we know the pushforward under this map is nonzero, giving a contradiction.

Thus, we must have $\alpha=12\lambda_0^{i-1}-\theta^{i-1}$, so
$$12\lambda^i=\iota_*(12\lambda_0^{i-1}-\theta^{i-1})=12s_*(\lambda_0^{i-1})-\iota_*(\theta^{i-1}).$$
By the projection formula, we have
$$s_*(\lambda_0^{i-1})=s_*(s^*(\lambda^{i-1}))=\lambda^{i-1}s_*(1)=\lambda^{i-1}\delta.$$
And so we can write $\iota_*(\theta^i)=12\lambda^i(\delta-\lambda)$. By the exact sequence we have $\mathrm{CH}^i(\overline{\mathcal{M}}_{1,2})$ for $i>1$ is generated by λ^i and $\lambda^{i-1}\delta$ modulo the relations $24\lambda^i,24\lambda^{i-1}\delta$ for $i>2$ and $24\lambda^2$ for $i=2$.

Next, note that the first Chern class of normal bundle to $\overline{\mathcal{M}}_{1,1}$ in $\overline{\mathcal{M}}_{1,2}$ is given by $-\psi_1=-\lambda$. Then we have $-\lambda=s^*(\delta)$, which gives
$$-\lambda\delta=s_*(-\lambda)=s_*(s^*(\delta))=\delta^2.$$
Thus, we have a surjection
$$\mathbb{Z}[\lambda,\delta]/(24\lambda^2,\delta^2+\delta\lambda)\to\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}).$$
By Lemma 5.7, it is enough to show that the additive generators we found for $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2})$ above both generate $\mathbb{Z}[\lambda,\delta]/(24\lambda^2,\delta^2+\delta\lambda)$ and satisfy the same relations they do in $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2})$. We know that monomials of the form $\lambda^i,\lambda^{i-1}\delta$ additively generate $\mathbb{Z}[\lambda,\delta]/(24\lambda^2,\delta^2+\delta\lambda)$ because we can repeatedly use the relation $\delta^2+\delta\lambda$ to replace any monomial $\delta^a\lambda^b$ with $\pm\delta\lambda^{a+b-1}$ if $a>0$. Moreover, the relation $24\lambda^2$ implies all of the additive relations $24\lambda^2$, $24\lambda^i,24\lambda^{i-1}\delta$ for $i>3$ in $\mathbb{Z}[\lambda,\delta]/(24\lambda^2,\delta^2+\delta\lambda)$, and so we are done. □

8.2. $\overline{\mathcal{M}}_{1,3}$. Define the graphs $\Delta_1,\Delta_2,\Delta_3,\Delta_\emptyset,\Theta_1,\Theta_2,\Theta_3,\Omega,\Delta_{\emptyset,j}$.

By Theorem 6.3, we have
$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_j})=\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2})=\mathbb{Z}[\lambda_j,\delta_{\emptyset,j}]/(24\lambda_j^2,\delta_{\emptyset,j}^2+\delta_{\emptyset,j}\mu_j)$$
for $j\in\{1,2,3\}$ and
$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_\emptyset})=\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1})\otimes_{\mathbb{Z}}\mathrm{CH}^*(\overline{\mathcal{M}}_{0,4})=\mathbb{Z}[\lambda_\emptyset,\delta_{\emptyset,1}]/(24\lambda_\emptyset^2,\delta_{\emptyset,1}^2),$$
using the CKP for $\overline{\mathcal{M}}_{0,4}$.

By Theorem 6.2, we have

$$\mathcal{M}^{\Theta_j} = [\mathcal{M}_{\Theta_j}/\operatorname{Aut}(\Theta_j)] = [\mathcal{M}_{0,4} \times \mathcal{M}_{0,3}/\mu_2] = [\mathcal{M}_{0,4}/\mu_2].$$

There are unique points $P_j \in \mathcal{M}^{\Theta_j} \cong [\mathcal{M}_{0,4}/\mu_2]$ isomorphic to $B\mu_2$. Set τ_j^i to be the pushforward of u^{i-1} to $\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}$ along $P_j \hookrightarrow \overline{\mathcal{M}}^{\mathrm{sep},\geq 2}$.

Theorem 8.5. *The Chow group* $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,3})$ *is given by*

- $\mathrm{CH}^0(\partial\overline{\mathcal{M}}_{1,3})$ *is freely generated by*
$$\phi, \delta_1, \delta_2, \delta_3, \delta_\emptyset$$
- $\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})$ *is generated by*
$$\lambda_1, \lambda_2, \lambda_3, \lambda_\emptyset, \theta_1, \theta_2, \theta_3, \delta_{\emptyset,1}$$
subject to the relations
$$2\theta_1 + 2\theta_2 - 24\lambda_\emptyset - 24\lambda_3$$
$$2\theta_1 + 2\theta_3 - 24\lambda_\emptyset - 24\lambda_2$$
$$4\theta_1 - 24\lambda_\emptyset - 24\lambda_2 - 24\lambda_3 + 24\lambda_1$$
- $\mathrm{CH}^2(\partial\overline{\mathcal{M}}_{1,3})$ *is generated by*
$$\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_\emptyset^2, \lambda_\emptyset\delta_{\emptyset,1}, \tau_1, \tau_2, \tau_3$$
subject to the relations
$$24\lambda_\ell^2 = 0 \text{ for } \ell \in \{1,2,3,\emptyset\}$$
$$2\tau_\ell = 24\delta_{\emptyset,1}\lambda_\emptyset \text{ for } \ell \in \{1,2,3\}$$
- *For* $i \geq 3$, $\mathrm{CH}^i(\partial\overline{\mathcal{M}}_{1,3})$ *is generated by*
$$\lambda_1^i, \lambda_2^i, \lambda_3^i, t\lambda_\emptyset^{i-1}, \tau_1^{i-1}, \tau_2^{i-1}, \tau_3^{i-1}$$
subject to the relations
$$24\lambda_\ell^i = 0 \text{ for } \ell \in \{1,2,3,\}$$
$$24\delta_{\emptyset,1}\lambda_\emptyset^{i-1} = 0$$
$$2\tau_\ell^{i-1} = 0 \text{ for } \ell \in \{1,2,3\}$$

Moreover, $\omega = 2\tau_j$ *for* $j = 1,2,3$.

Proof. The components of $\overline{\mathcal{M}}^{\mathrm{sep}.\geq 4} = \overline{\mathcal{M}}^{\mathrm{sep}}$ are given by $\overline{\mathcal{M}}^{\Delta_1}, \overline{\mathcal{M}}^{\Delta_2}, \overline{\mathcal{M}}^{\Delta_3}, \overline{\mathcal{M}}^{\Delta_\emptyset}$. We have

$$\overline{\mathcal{M}}^{\Delta_j} \cap \overline{\mathcal{M}}^{\Delta_k} = \emptyset \text{ for } i \neq j$$
$$\overline{\mathcal{M}}^{\Delta_j} \cap \overline{\mathcal{M}}^{\Delta_\emptyset} = \overline{\mathcal{M}}^{\Delta_{\emptyset,j}}.$$

Thus, we have an exact sequence

$$\bigoplus_{j\in\{1,2,3\}} \mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_{\emptyset,j}}) \to \bigoplus_{j\in\{1,2,3,\emptyset\}} \mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_j}) \to \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \to 0.$$

By Theorem 6.3, we have $\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_{\emptyset,j}}) \cong \mathrm{CH}^*(\overline{\mathcal{M}}_{1,1})$. The induced push-forward maps are given by

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \to \mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_j})$$

$$\lambda^i \mapsto \delta_{\emptyset,j}\lambda_j^i$$

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \to \mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_\emptyset})$$

$$\lambda^i \mapsto \delta_{\emptyset,1}\lambda_\emptyset^i.$$

Thus, the exact sequence gives

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) = \bigoplus_{j\in\{1,2,3,\emptyset\}} \mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_j})/\langle\{\delta_{\emptyset,1}\lambda_\emptyset^i - \delta_{\emptyset,j}\lambda_j^i\}_{j=1,2,3}\rangle.$$

Because $\overline{\mathcal{M}}^{\mathrm{sep},\geq 3} = \overline{\mathcal{M}}^{\Omega} \amalg \overline{\mathcal{M}}^{\mathrm{sep}}$ as spaces, we have $\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) = \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \oplus \langle\omega\rangle$.

Next, we consider $\overline{\mathcal{M}}^{\mathrm{sep},\geq 2} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 3} \cup \mathcal{M}^{\mathrm{non}=2}$. We have $\mathcal{M}^{\mathrm{non}=2} = \mathcal{M}^{\Theta_1} \amalg \mathcal{M}^{\Theta_2} \amalg \mathcal{M}^{\Theta_3}$ and, as noted above, $\mathcal{M}^{\Theta_j} = [\mathcal{M}_{0,4}/\mu_2]$. The diagram (6.6) for $\Gamma = \Theta_j$ becomes

$$\begin{tikzcd}
\overline{\mathrm{CH}}^*([\mathcal{M}_{0,4}/\mu_2],1) \arrow[r] \arrow[d] & \mathrm{CH}^{*-1}([\partial\overline{\mathcal{M}}_{0,4}/\mu_2]) \arrow[r] \arrow[d] & \mathrm{CH}^*([\overline{\mathcal{M}}_{0,4}/\mu_2]) \arrow[r] \arrow[d] & \mathrm{CH}^*([\mathcal{M}_{0,4}/\mu_2]) \arrow[r] \arrow[d] & 0 \\
\overline{\mathrm{CH}}^*(\mathcal{M}^{\mathrm{non}=2},1) \arrow[r] & \mathrm{CH}^{*-1}(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) \arrow[r] & \mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \arrow[r] & \mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=2}) \arrow[r] & 0
\end{tikzcd}$$

Filling this out using the above and Theorem 7.7, we get

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\partial_1"] \arrow[d, "\sim"] & \mathbb{Z} \oplus \frac{\mathbb{Z}[u]}{(2u)} \arrow[r] \arrow[d] & \frac{\mathbb{Z}[v_j,t_j]}{(v_jt_j,2(t_j-v_j))} \arrow[r] \arrow[d] & \frac{\mathbb{Z}[t_j]}{(2t_j)} \arrow[r] \arrow[d, "\sim"] & 0 \\
\mathbb{Z} \arrow[r, "\partial_1'"] & \mathbb{Z} \oplus \mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) \arrow[r] & \mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \arrow[r] & \mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=2}) \arrow[r] & 0
\end{tikzcd}$$

Using the formula for $\iota_*(\theta^i)$ from Theorem 8.4, we see $v_j^i \mapsto 12\lambda_j^i(\delta_{\emptyset,j} - \lambda_j)$. Additionally, the map $[\overline{\mathcal{M}}_{0,4}/\mu_2] \to \partial\overline{\mathcal{M}}_{1,3}$ sends $(\mathbb{P}^1, \infty, 0, 1, -1)$ to P_j, so we have $t_j^i \mapsto \tau_j^i$.

By Theorem 7.8, we know that the image of ∂_1 is generated by $\widehat{D}(1a|2b) - 2\widehat{D}(12|ab)$, which pushes forward to $\omega - 24\lambda_j(\delta_{\emptyset,j} - \lambda_j)$. Also, by the diagram, the exact sequence

$$0 \to \frac{\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3})}{\langle 2(\omega - 12\lambda_j(\mu_j - \lambda_j))\rangle} \to \mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \to \bigoplus_{j=1}^{3} \frac{\mathbb{Z}[t_j]}{(2t_j)} \to 0$$

has a splitting by lifting t_j to $\tau_j - 12\lambda_j(\mu_j - \lambda_j)$ and t_j^i to τ_j^i for $i \geq 2$. From this, we can conclude

$$\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) = \frac{\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) \oplus \mathbb{Z}\left\langle \{\theta_j, \tau_j^i\}_{i\geq 1}\right\rangle}{\langle \{2(\tau_j - 12\lambda_j(\mu_j-\lambda_j))\}, \{2\tau_j^i\}_{i\geq 2}, \{\omega - 24\lambda_j(\delta_{\emptyset,j} - \lambda_j)\}\rangle}.$$

But because and $\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) = \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \oplus \langle\omega\rangle$, we can write

$$\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) = \frac{\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \oplus \mathbb{Z}\left\langle \{\theta_j, \tau_j^i\}_{i\geq 1}\right\rangle}{\langle \{2(\tau_j - 12\lambda_j(\mu_j-\lambda_j))\}, \{2\tau_j^i\}_{i\geq 2}\rangle},$$

and we remember that $\omega = 2\tau_j$ in $\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2})$.

Finally, we consider $\partial\overline{\mathcal{M}}_{1,3} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 1} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 2} \cup \mathcal{M}^{\mathrm{non}=1}$. We have $\mathcal{M}^{\mathrm{non}=1} = M^{\Phi}$ and by Theorem 6.2, we have

$$\mathcal{M}^{\Phi} = [\mathcal{M}_{\Phi}/\operatorname{Aut}(\Phi)] = [\mathcal{M}_{0,5}/\mu_2].$$

The diagram (6.6) becomes

$$\begin{tikzcd}
\overline{\mathrm{CH}}^*(\mathcal{M}_{0,5},1) \arrow[r, "\partial_1"] \arrow[d] & \mathrm{CH}^{*-1}([\partial\overline{\mathcal{M}}_{0,5}/\mu_2) \arrow[r] \arrow[d] & \mathrm{CH}^*([\overline{\mathcal{M}}_{0,5}/\mu_2]) \arrow[r] \arrow[d] & \mathrm{CH}^*([\mathcal{M}_{0,5}/\mu_2]) \arrow[r] \arrow[d] & 0 \\
\overline{\mathrm{CH}}^*(\mathcal{M}^{\mathrm{non}=1},1) \arrow[r, "\partial_1"] & \mathrm{CH}^{*-1}(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \arrow[r] & \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,3}) \arrow[r] & \mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=1}) \arrow[r] & 0
\end{tikzcd}$$

By Theorem 7.6, we have $\mathrm{CH}^*([\mathcal{M}_{0,5}/\mu_2]) = \mathbb{Z}$. We can then split the bottom sequence by lifting $[\mathcal{M}^{\Phi}]$ to $[\overline{\mathcal{M}}^{\Phi}] = \phi$. By Theorem 7.8, the image of ∂_1 is generated by

$$\widehat{D}(1a|2b) - 2\widehat{D}(12|ab)$$
$$\widehat{D}(1a|3b) - 2\widehat{D}(13|ab)$$
$$2(\widehat{D}(23b|1a) + \widehat{D}(1ab|23) - \widehat{D}(123|ab) - \widehat{D}(3ab|12) - \widehat{D}(2ab|13)).$$

By Theorem **??**, these push forward to

$$2\theta_1 + 2\theta_2 - 24\lambda_3 - 24\lambda_\emptyset$$
$$2\theta_1 - 2\theta_3 - 24\lambda_2 - 24\lambda_\emptyset$$
$$2(2\theta_1 + 12\lambda_1 - 12\lambda_\emptyset - 12\lambda_3 - 12\lambda_2))$$

respectively, on $\partial\overline{\mathcal{M}}_{1,3}$. Thus, the quotient of $\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \oplus \mathbb{Z}\langle[\phi]\rangle$ by these relations is $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$. Unpacking this expression, we get the description stated in the theorem. □

Proposition 8.6. *The image of* $\partial_1 : \overline{\mathrm{CH}}^i(\mathcal{M}_{1,3},1) \to \mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,3})$ *contains*

$$\theta_1 + 12\lambda_1 - \theta_2 - 12\lambda_2$$
$$\theta_1 + 12\lambda_1 - \theta_3 - 12\lambda_3$$

in degree 2 *and is generated by*

$$\tau_1^i - \tau_2^i$$

$$\tau_1^i - \tau_3^i$$

in degree $i+2$ *for* $i \geq 1$.

We will need the following lemma to prove this proposition:

Lemma 8.7. *Suppose we have stacks* X, Y, Z *with* Z *smooth and morphisms* $f : X \to Y$, $g : Y \to Z$ *so that* f *is proper, birational and* $g, f \circ g$ *are flat. Then* $g^* = f_* \circ (f \circ g)^*$

Proof. We use operational Chow rings (reference Fulton). It suffices to assume X, Y and Z are schemes, and Z is connected. Because Z is smooth, any element $\alpha \in \mathrm{CH}^*(Z)$ may be written as $\alpha^\vee \cap [Z]$ for some $\alpha^\vee \in \mathrm{opCH}^*(Z)$. Then, we have

$$(g \circ f)^*(\alpha) = (g \circ f)^*(\alpha^\vee \cap [Z]) = f^* g^*(\alpha^\vee) \cap [X].$$

Using the projection formula and the fact that f is birational, this gives

$$\begin{aligned} f_*((g \circ f)^*(\alpha)) &= f_*(f^* g^*(\alpha^\vee) \cap [X]) \\ &= g^*(\alpha^\vee) \cap f_*([X]) \\ &= g^*(\alpha^\vee) \cap [Y] \\ &= g^*(\alpha^\vee \cap [Z]) \\ &= g^*(\alpha). \end{aligned}$$

□

We need to make this include the pullback information we get on higher Chow.

Proof of Proposition 8.6. Consider $\overline{\mathcal{M}}_{1,2}^{\Theta} \subseteq \overline{\mathcal{M}}_{1,2}$. We have $\pi^{-1}(\overline{\mathcal{M}}_{1,2}^{\Theta}) = \overline{\mathcal{M}}_{1,3}^{\Theta_1} \cup \overline{\mathcal{M}}_{1,3}^{\Theta_2}$. As $\overline{\mathcal{M}}_{1,2}^{\Theta} \cong B\mu_2$, we have $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}^{\Theta}) = \mathbb{Z}[s]/(2s)$. We compute the flat pullbacks $\pi^*(s^i) \in \mathrm{CH}^i(\overline{\mathcal{M}}_{1,3}^{\Theta_1} \cup \overline{\mathcal{M}}_{1,3}^{\Theta_2})$. When $i = 0$, we have $\pi^*(1) = [\overline{\mathcal{M}}_{1,3}^{\Theta_1}] + [\overline{\mathcal{M}}_{1,3}^{\Theta_2}]$. As noted above, $\overline{\mathcal{M}}_{1,3}^{\Theta_i} \cong [\mathbb{P}^1/\mu_2]$, and has Chow ring

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3}^{\Theta_\ell}) = \mathbb{Z}[\theta_{\ell,\emptyset}, \tau_\ell]/(2(\theta_{\ell,\emptyset} - \tau_\ell), \tau_\ell \cdot \theta_{\ell,\emptyset}).$$

For $i \geq 1$, the pullback of s^i to $\overline{\mathcal{M}}_{1,3}^{\Theta_\ell}$ is $\theta_{\ell,\emptyset}^i - \tau_\ell^i$. By the next lemma, considering the map $\overline{\mathcal{M}}_{1,3}^{\Theta_1} \amalg \overline{\mathcal{M}}_{1,3}^{\Theta_2} \to \overline{\mathcal{M}}_{1,3}^{\Theta_1} \cup \overline{\mathcal{M}}_{1,3}^{\Theta_2}$, we have that the pullback of s^i is

$$\theta_{1,\emptyset}^i - \tau_1^i + \theta_{2,\emptyset}^i - \tau_2^i = 12\lambda_1^i(\mu_1 - \lambda_1) - \tau_1^i + 12\lambda_\ell^i(\mu_\ell - \lambda_\ell) - \tau_\ell^i$$

Next, consider $U \subseteq \overline{\mathcal{M}}_{1,2}$ which consists only of the point corresponding to a nodal cubic curve, with the marked point being 2-torsion. Then $U \cong B\mu_2$, so $\mathrm{CH}^*(U) \cong \mathbb{Z}[s]/(2s)$. We have the following pull back diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & {[C/\mu_2]} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{1,3} \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & B\mu_2 \arrow[r] & \overline{\mathcal{M}}_{1,2} \end{tikzcd}$$

The two smooth points on C fixed by the μ_2 action are $\sigma_1(U)$ and $\sigma_2(U)$. The inclusion of $\sigma_i(U)$ into $\partial\overline{\mathcal{M}}_{1,3}$ induces a pushforward

$$h_\ell : \mathbb{Z}[s]/(2s) \to \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,3}).$$

By help, for $\ell = 1, 2$, we have $h_\ell(s^i) = \lambda_k^i(\mu_k - \lambda_k)$, where $k = 3 - \ell$. We compute the flat pullback $\pi^*(s^\ell)$. Of course $\pi^*(1) = [C]$. We have the normalization map $\mathbb{P}^1 \to C$, and this map is equivariant with respect to the μ_2 actions. Using the next lemma, we have that the pullback of s^i for $i \geq 1$ is given by

$$h_2(s^{i-1}) - h_1(s^{i-1}) = 12\lambda_1^i(\mu_1 - \lambda_1) - 12\lambda_2^i(\mu_2 - \lambda_2).$$

Now, on $\overline{\mathcal{M}}_{1,2}$, we have that the two maps $B\mu_2 \to \overline{\mathcal{M}}_{1,2}$ corresponding to the points $\overline{\mathcal{M}}_{1,2}^{\Theta}, U$ induce the same pushforward, by help. Thus, on $\overline{\mathcal{M}}_{1,3}$, we should have that these pullbacks of s^i agree, giving

$$[\overline{\mathcal{M}}^{\Theta_1}] + [\overline{\mathcal{M}}^{\Theta_2}] = [C]$$

for $i = 0$ and

$$12\lambda_1^i(\mu_1 - \lambda_1) - \tau_1^i + 12\lambda_2^i(\mu_2 - \lambda_2) - \tau_2^i = 12\lambda_1^i(\mu_1 - \lambda_1) - 12\lambda_2^i(\mu_2 - \lambda_2),$$

or

$$\tau_1^i + \tau_2^i = 24\lambda_2^i\mu_2 = 24t\lambda_\emptyset^i$$

for $i \geq 1$. Using the equality $2\tau_1^i = 24t\lambda_\emptyset^i$, we get $\tau_1^i = \tau_2^i$.

We can compute $[C] \in \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,3})$. Consider the diagram

$$\begin{tikzcd} {[\overline{\mathcal{M}}_{0,5}/\mu_2]} \arrow[r] \arrow[d] & \partial\overline{\mathcal{M}}_{1,3} \arrow[d, "\pi"] \\ {[\overline{\mathcal{M}}_{0,4}/\mu_2]} \arrow[r] & \partial\overline{\mathcal{M}}_{1,2}. \end{tikzcd}$$

We defined C as the fiber π over U. Lift U to a point in $\overline{\mathcal{M}}_{0,4}$, and consider C', the fiber of $\overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,4}$ over this lift. Because $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, all fibers have the same class, and so $D(12|3ab) + D(123|ab)$, the class of the fiber over $D(12|ab) \in \overline{\mathcal{M}}_{0,4}$, is equal to $[C']$. Moreover, if $\widehat{C'}$ is the image of C' in $[\overline{\mathcal{M}}_{0,5}/\mu_2]$, then we have $[\widehat{C'}] = \widehat{D}(12|3ab) + \widehat{D}(123|ab)$, since each of these varities is μ_2 invariant. Pushing forward this to $\partial\overline{\mathcal{M}}_{1,3}$ using Proposition **??** we get

$$[C] = 2\theta_1 + 12\lambda_1 - 12\lambda_2$$

Thus,

$$\theta_1 + \theta_2 = [C] = 2\theta_1 + 12\lambda_1 - 12\lambda_2,$$

or

$$\theta_1 + 12\lambda_1 = \theta_2 + 12\lambda_2$$

on $\overline{\mathcal{M}}_{1,3}$.

We obtain more relations on $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$ by pulling back along $\pi' : \overline{\mathcal{M}}_{1,3} \to \overline{\mathcal{M}}_{1,2}$, which forgets the second point, or, equivalently switching the second and third marked point in the relations above. This gives

$$\theta_1 + 12\lambda_1 = \theta_3 + 12\lambda_3$$

and

$$\tau_1^i = \tau_3^i$$

for $i \geq 1$. By exactness of

$$\overline{\mathrm{CH}}^{+1}(\mathcal{M}_{1,3}, 1) \to \mathrm{CH}^i(\partial\overline{\mathcal{M}}_{1,3}) \to \mathrm{CH}^{i+1}(\overline{\mathcal{M}}_{1,3}),$$

the fact that these relations hold on $\mathrm{CH}^i(\overline{\mathcal{M}}_{1,3})$ but not on $\mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,3})$ imply that the elements

$$\theta_1 + 12\lambda_1 = \theta_2 + 12\lambda_2$$
$$\theta_1 + 12\lambda_1 = \theta_3 + 12\lambda_3$$

are in $\mathrm{im}(\mathrm{CH}^1(\mathcal{M}_{1,3}) \xrightarrow{\partial_1} \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{1,3}))$ and

$$\tau_1^i - \tau_2^i$$
$$\tau_1^i - \tau_3^i$$

are in $\mathrm{im}(\mathrm{CH}^i(\mathcal{M}_{1,3}) \to \mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,3}))$ for $i \geq 2$. When $i \geq 2$, because $\tau_1^i - \tau_2^i, \tau_1^i - \tau_3^i$ are independent and

$$\overline{\mathrm{CH}}^i(\mathcal{M}_{1,3}, 1) = (\frac{\mathbb{Z}}{2\mathbb{Z}})^2$$

we have that $\tau_1^i - \tau_2^i, \tau_1^i - \tau_3^i$ generate the image of ∂_1 in $\mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,3})$. □

Theorem 8.8. $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3}) = \mathbb{Z}[\delta_1, \delta_2, \delta_3, \delta_\emptyset, \lambda]/I$, *where I is the ideal generated by the elements*

$$24\lambda^2$$
$$\delta_\ell\delta_k, \ \textit{for } \ell \neq k \in \{1,2,3\}$$
$$\delta_\ell\delta_\emptyset - \delta_k\delta_\emptyset, \ \textit{for } \ell \neq k \in \{1,2,3\}$$
$$\delta_\ell^2 + \lambda\delta_\ell + \delta_1\delta_\emptyset, \ \textit{for } \ell \in \{1,2,3,\emptyset\}$$
$$12\lambda^3 - 12\lambda^2\delta_1 + 12\lambda^2\delta_2 + 12\lambda^2\delta_3 + 12\lambda^2\delta_\emptyset.$$

Moreover, we have It seems like one can show that $\pm = +$ by pulling back to $\overline{\mathcal{M}}^{\Delta_k}$ with $k \neq j$

$$\iota_*(\theta_j) = 6\lambda^2 - 6\lambda\delta_j + 6\lambda\delta_k + 6\lambda\delta_\ell + 6\lambda\delta_\emptyset$$

for $\{j,k,\ell\}=[3]$*, and*

$$\iota_*(\tau_j^i)=12\lambda^{i-2}\delta_1\delta_\emptyset+6\lambda^i+6\lambda^{i-1}\delta_1+6\lambda^{i-1}\delta_2+6\lambda^{i-1}\delta_3+6\lambda^{i-1}\delta_\emptyset.$$

Proof. We have the exact sequence

$$0\to \mathrm{CH}^{i-1}(\partial\overline{\mathcal{M}}_{1,3})/\mathrm{im}(\partial)\xrightarrow{\iota_*}\mathrm{CH}^i(\overline{\mathcal{M}}_{1,3})\to \mathrm{CH}^i(\mathcal{M}_{1,3})\to 0.$$

In degree $i=1$, this is

$$0\to \mathrm{CH}^0(\partial\overline{\mathcal{M}}_{1,3})\to \mathrm{CH}^1(\overline{\mathcal{M}}_{1,3})\to \mathrm{CH}^1(\mathcal{M}_{1,3})\to 0.$$

By Theorem 5.8, we know $\mathrm{CH}^1(\mathcal{M}_{1,3})$ is cyclic of order 12, generated by λ. By Theorem 8.5, $\mathrm{CH}^0(\partial\overline{\mathcal{M}}_{1,3})$ is freely generated by $\phi,\delta_1,\delta_2,\delta_3,\delta_\emptyset$. By Proposition 5.5, $\phi=12\lambda$. Thus, $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,3})$ is freely generated by $\lambda,\delta_1,\delta_2,\delta_3,\delta_\emptyset$.

Note

$$\lambda^i\delta_\ell=\lambda^i\iota_*(1)=\iota_*(\iota^*(\lambda^i))=\iota_*(\lambda_\ell^i)$$

for $\ell\in\{1,2,3,\emptyset\}$, where $\iota:\overline{\mathcal{M}}^{\Delta_\ell}\to\overline{\mathcal{M}}_{1,3}$. Moreover, we have that $\iota_*(t\lambda_\emptyset^i)=\lambda^i\delta_\emptyset\delta_\ell$, for any $\ell\in\{1,2,3\}$, because...

In degree $i=2$, the localization exact sequence is

$$0\to \mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})/\mathrm{im}(\partial)\to \mathrm{CH}^2(\overline{\mathcal{M}}_{1,3})\to \mathrm{CH}^2(\mathcal{M}_{1,3})\to 0.$$

By Theorem 5.8, we know $\mathrm{CH}^2(\mathcal{M}_{1,3})$ is generated by λ^2, which has order 6. We have not computed the full image of ∂_1 in $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,3})$. Proposition 8.6 says that the two torsion elements

$$\theta_1+12\lambda_1-\theta_2-12\lambda_2$$

$$\theta_1+12\lambda_1-\theta_3-12\lambda_3$$

in $\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})$ are contained in the image. Suppose the image of ∂_1 in $\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})$ contained a non-torsion element. We saw that $\overline{\mathcal{M}}_{1,3}$ has the CKgP, and hence the rational CKgP. This gives, by Canning-Larson that Chow is isomorphic to cohomology, rationally. As we just computed, $\mathrm{CH}^1(\overline{\mathcal{M}}_{1,3})$ has rank 5, and $\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})$ also has rank 5. Thus, a non-torsion element in the image of ∂_1 would decreases the rank of $\mathrm{CH}^2(\overline{\mathcal{M}}_{1,3})$ to 4, violating Poincare Duality.

Now, a computation with Smith normal form gives

$$\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})/\langle\{\theta_1+12\lambda_1-\theta_\ell-12\lambda_\ell\}_{\ell=2,3}\rangle\cong\mathbb{Z}^2\oplus\frac{\mathbb{Z}}{4\mathbb{Z}}.$$

Because $6\lambda^2=0$ on $\mathcal{M}_{1,3}$, we know $6\lambda^2\in\mathrm{im}(\iota_*)$. Note $6\lambda^2$ has order exactly 4, so it must pushforward from an element in $\mathrm{CH}^1(\partial\mathcal{M}_{1,3})/\mathrm{im}(\partial_1)$ of order exactly 4. Thus, $\mathrm{im}(\partial_1)$ can contain no torsion elements besides what is in $\langle\{\theta_1+12\lambda_1-\theta_\ell-12\lambda_\ell\}_{\ell=2,3}\rangle$, and so $\mathrm{im}(\partial_1)=\{\theta_1+12\lambda_1-\theta_\ell-12\lambda_\ell\}_{\ell=2,3}$.

The 4-torsion elements in $\mathrm{CH}^1(\partial\mathcal{M}_{1,3})/\mathrm{im}(\partial_1)$ are

$$\pm(\theta_1+6\lambda_1-6\lambda_2-6\lambda_3-6\lambda_\emptyset),$$

and the above says one of these must push forward to $6\lambda^2$. Thus, we can write

$$\iota_*(\theta_1) = \pm 6\lambda^2 - 6\lambda\delta_1 + 6\lambda\delta_2 + 6\lambda\delta_3 + 6\lambda\delta_\emptyset. \tag{8.9}$$

By S_3-symmetry, we have

$$\iota_*(\theta_\ell) = \pm 6\lambda^2 - 6\lambda\delta_\ell + 6\lambda\delta_{\ell'} + 6\lambda\delta_{\ell''} + 6\lambda\delta_\emptyset$$

for $\{\ell, \ell', \ell''\} = \{1,2,3\}$. Moreover, using this expression for $\iota_*(\theta_\ell)$, the pushforward of the relations from $\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,3})/\mathrm{im}(\partial_1)$ become 0. Thus, we have that $\mathrm{CH}^2(\overline{\mathcal{M}}_{1,3})$ is generated by $\lambda^2, \lambda\delta_1, \lambda\delta_2, \lambda\delta_3, \lambda\delta_\emptyset, \delta_\emptyset\delta_\ell$, subject to $24\lambda^2 = 0$.

Multiplying (8.9) by λ^{i-2} for $i \geq 3$, we get

$$\iota_*(\theta_1)\lambda^{i-2} = \pm 6\lambda^i - 6\lambda^{i-1}\delta_1 + 6\lambda^{i-1}\delta_2 + 6\lambda^{i-1}\delta_3 + 6\lambda^{i-1}\delta_\emptyset.$$

For the morphism $\iota' : \overline{\mathcal{M}}_{1,3}^{\Theta_1} \to \overline{\mathcal{M}}_{1,3}$, we have

$$\iota_*(\theta_1)\lambda^{i-2} = \iota'_*(1)\lambda^{i-2} = \iota'_*(\iota'^*(\lambda^{i-2})).$$

We have the diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_{1,3}^{\Theta_1} \cup \overline{\mathcal{M}}_{1,3}^{\Theta_2} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{1,3}^{\Phi} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{1,3} \arrow[d] \\
\overline{\mathcal{M}}_{1,2}^{\Theta} \arrow[r] \arrow[dr, "\sim"] & \overline{\mathcal{M}}_{1,2}^{\Phi} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{1,2} \arrow[d] \\
 & \partial\overline{\mathcal{M}}_{1,1} \arrow[r] & \overline{\mathcal{M}}_{1,1}
\end{tikzcd}$$

from which we see that the pullback of λ^{i-2} to $\overline{\mathcal{M}}_{1,3}^{\Theta_1}$ is the same as the pullback of $s^{i-2} \in \mathrm{CH}^1(\overline{\mathcal{M}}_{1,2}^{\Theta})$, using Proposition 5.2. This was computed in Proposition 8.6 to be $\theta_{1,\emptyset}^{i-2} - \tau_1^{i-2} = 12\lambda_1^{i-2}(\mu_1 - \lambda_1) - \tau_1^i$. Thus,

$$\iota'_*(\iota'^*(\lambda^{i-2})) = \iota_*(\tau_1^i - 12\lambda_1^{i-2}(\mu_1 - \lambda_1)) = \iota_*(\tau_1^i) - 12\lambda^{i-2}\delta_1(\delta_\emptyset - \lambda)$$

and so, we have

$$\iota_*(\tau_1^i) - 12\lambda^{i-2}\delta_1(\delta_\emptyset - \lambda) = \pm 6\lambda^i - 6\lambda^{i-1}\delta_1 + 6\lambda^{i-1}\delta_2 + 6\lambda^{i-1}\delta_3 + 6\lambda^{i-1}\delta_\emptyset$$

or

$$\begin{aligned}
\iota_*(\tau_1^i) &= 12\lambda^{i-2}\delta_1(\delta_\emptyset - \lambda) \pm 6\lambda^i - 6\lambda^{i-1}\delta_1 + 6\lambda^{i-1}\delta_2 + 6\lambda^{i-1}\delta_3 + 6\lambda^{i-1}\delta_\emptyset \\
&= 12\lambda^{i-2}\delta_1\delta_\emptyset \pm 6\lambda^i + 6\lambda^{i-1}\delta_1 + 6\lambda^{i-1}\delta_2 + 6\lambda^{i-1}\delta_3 + 6\lambda^{i-1}\delta_\emptyset
\end{aligned}$$

for $i \geq 3$. Note the pushforward of the relations $\tau_1^i - \tau_2^i, \tau_1^i - \tau_3^i$ are automatically satisfied, because this expression for $\iota_*(\tau_1)$ is S_3 invariant. The relation $2\tau_1^i - 24t\lambda^i = 2(\tau_1^i - 12\lambda^{i-2}(\mu_1 - \lambda_1))$ pushes forward to

$$12\lambda^i - 12\lambda^{i-1}\delta_1 + 12\lambda^{i-1}\delta_2 + 12\lambda^{i-1}\delta_3 + 12\lambda^{i-1}\delta_\emptyset.$$

Thus, using Theorem 8.5 and Proposition 8.6, we have that

- $\mathrm{CH}^3(\overline{\mathcal{M}}_{1,3})$ is generated by $\lambda^3, \lambda^2\delta_1, \lambda^2\delta_2, \lambda^2\delta_3, \lambda^2\delta_\emptyset, \lambda\delta_1\delta_\emptyset$ modulo the relations
$$24\lambda^3$$
$$24\lambda^2\delta_\ell, \text{ for } \ell \in \{1,2,3,\emptyset\}$$
$$12\lambda^3 - 12\lambda^2\delta_1 + 12\lambda^2\delta_2 + 12\lambda^2\delta_3 + 12\lambda^2\delta_\emptyset$$
- $\mathrm{CH}^i(\overline{\mathcal{M}}_{1,3})$ is generated by $\lambda^i, \lambda^{i-1}\delta_1, \lambda^{i-1}\delta_2, \lambda^{i-1}\delta_3, \lambda^{i-1}\delta_\emptyset, \lambda^{i-2}\delta_1\delta_\emptyset$ modulo the relations
$$24\lambda^i$$
$$24\lambda^{i-1}\delta_\ell, \text{ for } \ell \in \{1,2,3,\emptyset\}$$
$$24\lambda^{i-2}\delta_1\delta_\emptyset$$
$$12\lambda^i - 12\lambda^{i-1}\delta_1 + 12\lambda^{i-1}\delta_2 + 12\lambda^{i-1}\delta_3 + 12\lambda^{i-1}\delta_\emptyset$$

This finishes an additive presentation for $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$. From this description, we see that $\delta_1, \delta_2, \delta_3, \delta_\emptyset, \lambda$ generated $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$ as a ring.

We now compute some products of these generators. We can compute
$$\delta_\ell^2 = -\lambda\delta_\ell - \delta_1\delta_\emptyset.$$
Finally, note that $\delta_\ell\delta_k = 0$ for $\ell \neq k \in \{1,2,3\}$, because $\overline{\mathcal{M}}^{\Delta_\ell} \cap \overline{\mathcal{M}}^{\Delta_k} = \emptyset$.

Now, by looking at the relations we have found, the kernel of the surjective map
$$\mathbb{Z}[\delta_1, \delta_2, \delta_3, \delta_\emptyset, \lambda] \twoheadrightarrow \mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$$
contains the following elements
$$24\lambda^2$$
$$\delta_\ell\delta_k, \text{ for } \ell \neq k \in \{1,2,3\}$$
$$\delta_\ell\delta_\emptyset - \delta_k\delta_\emptyset, \text{ for } \ell \neq k \in \{1,2,3\}$$
$$\delta_\ell^2 + \lambda\delta_\ell + \delta_1\delta_\emptyset, \text{ for } \ell \in \{1,2,3,\emptyset\}$$
$$12\lambda^3 - 12\lambda^2\delta_1 + 12\lambda^2\delta_2 + 12\lambda^2\delta_3 + 12\lambda^2\delta_\emptyset,$$
and so it contains I, the ideal they generate. We claim that I is equal to the kernel. By Lemma 5.7, it is enough to show that the additive generators we found for $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$ above both generate $\mathbb{Z}[\delta_1, \delta_2, \delta_3, \delta_\emptyset, \lambda]/I$ and satisfy the same relations that they do in $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$. The latter is clear, as all of the relations we found between these additive generators above are clearly in the ideal I. So it remains to show that monomials of the form
$$\lambda^i, \lambda^i\delta_1, \lambda^i\delta_2, \lambda^i\delta_3, \lambda^i\delta_\emptyset, \lambda^i\delta_1\delta_\emptyset$$
additively generate $\mathbb{Z}[\delta_1, \delta_2, \delta_3, \delta_\emptyset, \lambda]/I$. Note that any monomial in $\mathbb{Z}[\delta_1, \delta_2, \delta_3, \delta_\emptyset, \lambda]/I$ is 0 if it contains distinct δ_ℓ, δ_k, $\ell, k \in \{1,2,3\}$. Moreover, note
$$\delta_\ell^2\delta_\emptyset = \delta_\ell\delta_k\delta_\emptyset = 0,$$
and so the only possibly nonzero monomials look like $\lambda^i\delta_\ell^j$ or $\lambda^i\delta_\emptyset^j\delta_\ell$, for some $\ell \in \{1,2,3\}$. Then, using the self intersection formulas $\delta_\ell^2 = -\delta_\ell\lambda - \delta_\emptyset\delta_1$ for $\ell \in \{1,2,3,\emptyset\}$, we see that these monomials are indeed linear combinations of $\lambda^i, \lambda^i\delta_1, \lambda^i\delta_2, \lambda^i\delta_3, \lambda^i\delta_\emptyset, \lambda^i\delta_1\delta_\emptyset$, and so we are done.

□

8.3. $\overline{\mathcal{M}}_{1,4}$. Define the graphs $\Delta_j, \Delta_{jk}, \Delta_\emptyset, \Theta_j, \Theta_{1j}, \Omega_{jk}, \Sigma, \Delta_{j|k}, \Psi_{jk}, \Delta_{\emptyset,j}, \Delta_{\emptyset,jk}$.

By Theorem 6.3 and the CKP for $\overline{\mathcal{M}}_{0,n}$, we have

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_j}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}) \otimes \mathrm{CH}^*(\overline{\mathcal{M}}_{0,4}) = \frac{\mathbb{Z}[\lambda_j, \delta_{\emptyset,j}, \delta_{j|k}]}{(24\lambda_j^2, \delta_{\emptyset,j}^2 + \delta_{\emptyset,j}\lambda_j, \delta_{j|k}^2)}$$

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_{jk}}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,3}) = \frac{\mathbb{Z}[\delta_{j|k}, \delta_{k|j}, \psi_{jk}, \delta_{\emptyset,jk}, \lambda_{jk}]}{I}$$

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_\emptyset}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \otimes_{\mathbb{Z}} \mathrm{CH}^*(\overline{\mathcal{M}}_{0,5}) = \frac{\mathbb{Z}[\lambda_\emptyset, \delta_{\emptyset,1}, \delta_{\emptyset,2}, \delta_{\emptyset,3}, \delta_{\emptyset,4}, \delta_{\emptyset,12}]}{(24\lambda_\emptyset^2, \text{relations from } \overline{\mathcal{M}}_{0,5})}.$$

By Theorem 6.2, we have

$$\mathcal{M}^{\Theta_{jk}} = [\mathcal{M}_{\Theta_{jk}}/\operatorname{Aut}(\Theta_j)] = [\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}/\mu_2].$$

There are unique points $P_{jk} \in \mathcal{M}^{\Theta_{jk}} \cong [\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}/\mu_2]$ isomorphic to $B\mu_2$. Set τ_{jk}^i to be the pushforward of u^{i-1} to $\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}$ along $P_j \hookrightarrow \overline{\mathcal{M}}^{\mathrm{sep},\geq 2}$. Moreover, let κ_{jk} be the class of the image of

$$\pi_1^{-1}\widehat{T} \to [\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,4}/\mu_2] \to \overline{\mathcal{M}}^{\Theta_{jk}}.$$

Proposition 8.10. $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,4}) =$

Proof. The components of $\overline{\mathcal{M}}^{\mathrm{sep},\geq 5} = \overline{\mathcal{M}}^{\mathrm{sep}}$ are given by $\overline{\mathcal{M}}^{\Delta_j}$ for $j \in \{1,2,3,4\}$, $\overline{\mathcal{M}}^{\Delta_{jk}}$ for $j \neq k \in \{1,2,3,4\}$, and $\overline{\mathcal{M}}^{\Delta_\emptyset}$. We use the exact sequence

$$\bigoplus_{\Gamma \neq \Gamma'} \mathrm{CH}^*(\overline{\mathcal{M}}^{\Gamma} \cap \overline{\mathcal{M}}^{\Gamma'}) \to \bigoplus_{\Gamma} \mathrm{CH}^*(\overline{\mathcal{M}}^{\Gamma}) \to \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \to 0$$

to calculate $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}})$.

The intersections between these components are

$$\overline{\mathcal{M}}^{\Delta_j} \cap \overline{\mathcal{M}}^{\Delta_k} = \emptyset \text{ for } j \neq k,$$
$$\overline{\mathcal{M}}^{\Delta_j} \cap \overline{\mathcal{M}}^{\Delta_{k\ell}} = \emptyset \text{ for } j \neq k, \ell$$
$$\overline{\mathcal{M}}^{\Delta_{jk}} \cap \overline{\mathcal{M}}^{\Delta_{j\ell}} = \emptyset \text{ for } k \neq \ell$$
$$\overline{\mathcal{M}}^{\Delta_j} \cap \overline{\mathcal{M}}^{\Delta_{jk}} = \overline{\mathcal{M}}^{\Delta_{j|k}}$$
$$\overline{\mathcal{M}}^{\Delta_{jk}} \cap \overline{\mathcal{M}}^{\Delta_{\ell m}} = \overline{\mathcal{M}}^{\Psi_{jk}} \text{ for } \{j,k,\ell,m\} = [4]$$
$$\overline{\mathcal{M}}^{\Delta_j} \cap \overline{\mathcal{M}}^{\Delta_\emptyset} = \overline{\mathcal{M}}^{\Delta_{\emptyset,j}}$$
$$\overline{\mathcal{M}}^{\Delta_{jk}} \cap \overline{\mathcal{M}}^{\Delta_\emptyset} = \overline{\mathcal{M}}^{\Delta_{\emptyset,jk}}.$$

By Theorem 6.3, we have

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_{j|k}}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,3}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,2})$$
$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Psi_{jk}}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,3}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,2})$$
$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_{\emptyset,j}}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,4}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,4})$$

$$\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_{\emptyset,jk}}) = \mathrm{CH}^*(\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,3}) = \mathrm{CH}^* \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,4})$$

The following table describes the relations in $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}})$ obtained by pushing forward from $\overline{\mathcal{M}}^{\Gamma}$

$\Delta_{j\|k}$	$\delta_{j\|k}\lambda^i_j - \delta_{j\|k}\lambda^i_{jk}$,	$\delta_{j\|k}\delta_{\emptyset,j}\lambda^i_j - \delta_{j\|k}\delta_{\emptyset,jk}\lambda^i_{jk}$
Ψ_{jk}	$\psi_{jk}\lambda^i_{jk} - \psi_{\ell m}\lambda^i_{\ell m}$,	$\psi_{jk}\delta_{\emptyset,jk}\lambda^i_{jk} - \psi_{\ell m}\delta_{\emptyset,\ell m}\lambda^i_{\ell m}$
$\Delta_{\emptyset,j}$	$\delta_{\emptyset,j}\lambda^i_j - \delta_{\emptyset,j}\lambda^i_{\emptyset}$,	$\delta_{\emptyset,j}\delta_{jk}\lambda^i_j - \delta_{\emptyset,j}\delta_{\emptyset,jk}\lambda^i_{\emptyset}$
$\Delta_{\emptyset,jk}$	$\delta_{\emptyset,jk}\lambda^i_{jk} - \delta_{\emptyset,jk}\lambda^i_{\emptyset}$,	$\delta_{\emptyset,jk}\delta_{j\|k}\lambda^i_{jk} - \delta_{\emptyset,jk}\delta_{\emptyset,j}\lambda^i_{\emptyset}$

This determines $\mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}})$. Because $\overline{\mathcal{M}}^{\mathrm{sep},\geq 4} = \overline{\mathcal{M}}^{\mathrm{sep}} \amalg \overline{\mathcal{M}}^{\Sigma}$ and $\overline{\mathcal{M}}^{\Sigma} = \mathrm{Spec}(k)$ by Theorem 6.2, we have $\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 4}) = \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \oplus \langle\sigma\rangle$.

Next, we consider $\overline{\mathcal{M}}^{\mathrm{sep},\geq 3} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 4} \cup \mathcal{M}^{\mathrm{non}=3}$. We have

$$\mathcal{M}^{\mathrm{non}=3} = \coprod_{\{j,k\}\in\binom{[4]}{2}} \mathcal{M}^{\Omega_{jk}}.$$

Theorem 6.2 gives

$$\mathcal{M}^{\Omega_{jk}} = [\mathcal{M}_{\Omega_{jk}} / \mathrm{Aut}(\Omega_{jk})] = \mathcal{M}_{0,3} \times \mathcal{M}_{0,4} \times \mathcal{M}_{0,3} = \mathcal{M}_{0,4}.$$

We have

$$\overline{\mathrm{CH}}^*(\mathcal{M}^{\mathrm{non}=3},1) \xrightarrow{\partial_1} \mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 4}) \to \mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) \to \mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=3}) \to 0,$$

which becomes

$$\bigoplus_{\{j,k\}\in\binom{[4]}{2}} \overline{\mathrm{CH}}^*(\mathcal{M}_{0,4},1) \xrightarrow{\partial_1} \mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 4}) \to \mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) \to \bigoplus_{\{j,k\}\in\binom{[4]}{2}} \mathrm{CH}^*(\mathcal{M}_{0,4}) \to 0.$$

The diagram (6.6) shows that the image of ∂_1 is generated by the pushforward of the WDVV realtions to $\overline{\mathcal{M}}^{\mathrm{sep},\geq 4}$ from $\partial\overline{\mathcal{M}}_{0,4}$. The relations on $\overline{\mathcal{M}}_{0,4}$ are generated by $D(12|ab) - D(1a|2b), D(12|ab) - D(1a|2b)$. Both $D(1a|2b)$ and $D(1b|2a)$ push forward to σ, and $D(12|ab)$ pushes forward to the pushforward of the class $\omega \in \mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$ inside $\mathrm{CH}^*(\overline{\mathcal{M}}^{\Delta_{\ell m}})$, where ℓ, m is such that $\{j,k,\ell,m\} = [4]$. From Proposition 8.5, we know $\omega = 2\tau_j$, and Theorem 8.8 gives a computation of τ_j in terms of the ring generators. Thus, we have the relation

$$\sigma = 12(2\delta_{\ell|m}\delta_{\emptyset,\ell m} + \lambda^2_{\ell m} + \lambda_{\ell m}\delta_{\ell|m} + \lambda_{\ell m}\delta_{\ell|m} + \lambda_{\ell m}\psi_{\ell m} + \lambda_{\ell m}\delta_{\emptyset,\ell m}).$$

Moreover, because $\mathrm{CH}^*(\mathcal{M}_{0,4}) = \mathbb{Z}$, we can split the exact sequence by lifting $[\mathcal{M}^{\Omega_{jk}}]$ to $[\overline{\mathcal{M}}^{\Omega_{jk}}]$. So $\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3})$ is the quotient of $\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 4}) \oplus \langle\omega_{jk}\rangle$ by the above expressions for σ. But because $\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 4}) = \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \oplus \langle\sigma\rangle$, the above expression for σ says that we can instead write $\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) = \mathrm{CH}^*(\overline{\mathcal{M}}^{\mathrm{sep}}) \oplus \langle\omega_{jk}\rangle$ and remember the above expression for σ inside of $\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3})$.

Now, consider $\overline{\mathcal{M}}^{\mathrm{sep},\geq 2} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 3} \cup \mathcal{M}^{\mathrm{non}=2}$. We have

$$\mathcal{M}^{\mathrm{non}=2} = \coprod_{\{j,k\}} \mathcal{M}^{\Theta_{jk}} \amalg \coprod_{j\in[4]} \mathcal{M}^{\Theta_j}.$$

Theorem 6.2 gives

$$\mathcal{M}^{\Theta_{jk}} = [\mathcal{M}_{\Theta_{jk}}/\mu_2] = [\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}/\mu_2]$$

and

$$\mathcal{M}^{\Theta_j} = [\mathcal{M}_{\Theta_j}/\operatorname{Aut}(\Theta_j)] = [\mathcal{M}_{0,5}/\mu_2].$$

By Theorem 7.6, $\mathrm{CH}^*(\mathcal{M}^{\Theta_j}) = \mathrm{CH}^*([\mathcal{M}_{0,5}/\mu_2]) = \mathbb{Z}$, and so we can simply to lift $[\mathcal{M}^{\Theta_j}]$ to $[\overline{\mathcal{M}}^{\Theta_j}]$ when dealing with that piece of $\mathrm{CH}^*(\mathcal{M}^{\mathrm{non}=2})$. Diagram (6.6) for Θ_j and Theorem 7.8 show that the image of ∂_1 coming from $\mathcal{M}^{\Theta_j}$ is generated by the pushforwards of

$$\widehat{D}(ka|\ell b) - 2\widehat{D}(k\ell|ab)$$

$$\widehat{D}(ka|mb) - 2\widehat{D}(km|ab)$$

$$2(\widehat{D}(\ell mb|ka) + \widehat{D}(kab|\ell m) - \widehat{D}(k\ell m|ab) - \widehat{D}(mab|k\ell) - \widehat{D}(\ell ab|km)).$$

We can compute the degrees of these push forwards using Automorphism theorem. Then we get

$$\widehat{D}(ka|\ell mb) \mapsto \omega_{\ell m}$$

$$\widehat{D}(ab|k\ell m) \mapsto 24\lambda_j(\delta_{\emptyset,j} - \lambda_j)$$

$$\widehat{D}(kab|\ell m) \mapsto 6\lambda_{jk}(\pm\lambda_{jk} - \delta_{j|k} + \delta_{k|j} + \psi_{jk} + \delta_{\emptyset,jk})$$

under the pushforward map, using Theorem 8.4 and Theorem 8.8 for their formulas of $\iota_*(\theta)$ and $\iota_*(\theta_j)$, respectively. Thus, the relations push forward to

$$\omega_{km} + \omega_{\ell m} - 24\lambda_j(\delta_{\emptyset,j} - \lambda_j) - 12\lambda_{jm}(\lambda_{jm} - \delta_{j|m} + \delta_{m|j} + \psi_{jm} + \delta_{\emptyset,jm})$$

$$\omega_{k\ell} + \omega_{\ell m} - 24\lambda_j(\delta_{\emptyset,j} - \lambda_j) - 12\lambda_{j\ell}(\lambda_{j\ell} - \delta_{j|\ell} + \delta_{\ell|j} + \psi_{j\ell}\delta_{\emptyset,j\ell})$$

$$2(\omega_{\ell m} + 6\lambda_{jk}(\pm\lambda_{jk} - \delta_{j|k} + \psi_{jk} + \delta_{k|j} + \delta_{\emptyset,jk}) - 12\lambda_j(\delta_{\emptyset,j} - \lambda_j) -$$

$$6\lambda_{jm}(\pm\lambda_{jm} - \delta_{j|m} + \delta_{m|j} + \psi_{jm} + \delta_{\emptyset,jm}) - 6\lambda_{j\ell}(\pm\lambda_{j\ell} - \delta_{j|\ell} + \delta_{\ell|j} + \psi_{j\ell} + \delta_{\emptyset,j\ell})).$$

For $\mathcal{M}^{\Theta_{jk}}$, we plug $\Gamma = \Theta_{jk}$ into the diagram (6.6) to get

$$\begin{tikzcd}
\overline{\mathrm{CH}}^*(\mathcal{M}^{\Theta_{jk}},1) \arrow[r] \arrow[d] & \mathrm{CH}^{*-1}([\partial\overline{\mathcal{M}}_{\Theta_{jk}}/\operatorname{Aut}(\Theta_{jk})]) \arrow[r] \arrow[d] & \mathrm{CH}^*([\overline{\mathcal{M}}_{\Theta_{jk}}/\operatorname{Aut}(\Theta_{jk})]) \arrow[r] \arrow[d] & \mathrm{CH}^*(\mathcal{M}^{\Theta \arrow[d] \\
\overline{\mathrm{CH}}^*(\mathcal{M}^{\mathrm{non}=p},1) \arrow[r, "\partial_1"] & \mathrm{CH}^{*-1}(\overline{\mathcal{M}}^{\mathrm{sep},\geq 3}) \arrow[r] & \mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \arrow[r] & \mathrm{CH}^*(\mathcal{M}^{\mathrm{no}
\end{tikzcd}$$

By Lemma 7.10, we have that $t \in \frac{\mathbb{Z}[t]}{(2t)} = \mathrm{CH}^*([\mathcal{M}_{0,4} \times \mathcal{M}_{0,4}/\mu_2])$ lifts to $\pi_1^*(t-v) \in \mathrm{CH}^*([\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,4}/\mu_2])$ and t^i lifts to $\pi_1^*(t^i)$... help

Lemma 7.10 also says that the image of ∂_1 coming from $\mathcal{M}^{\Theta_{jk}}$ is generated by the pushforwards of

$$[\pi_1^{-1}\widehat{D}(ja|kb)] - 2[\pi_1^{-1}\widehat{D}(jk|ab)]$$
$$[\pi_2^{-1}\widehat{D}(\ell a|mb)] - 2[\pi_2^{-1}\widehat{D}(\ell m|ab)].$$

We get

$$\pi_1^{-1}\widehat{D}(ja|kb) \mapsto \omega_{\ell m}$$
$$\pi_1^{-1}\widehat{D}(jk|ab) \mapsto 6\lambda_{\ell m}(\pm\lambda_{\ell m} + \delta_{\ell|m} + \delta_{m|\ell} - \psi_{\ell m} + \delta_{\emptyset,\ell m})$$
$$\pi_2^{-1}\widehat{D}(\ell a|mb) \mapsto \omega_{jk}$$
$$\pi_2^{-1}\widehat{D}(\ell m|ab) \mapsto 6\lambda_{jk}(\pm\lambda_{jk} + \delta_{j|k} + \delta_{k|j} - \psi_{jk} + \delta_{\emptyset,jk})$$

under the pushforward map, using Theorem 8.4 and Theorem 8.8 for its formula of $\iota_*(\theta_j)$. Then the relations pushforward to

$$\omega_{\ell m} - 12\lambda_{\ell m}(\lambda_{\ell m} + \delta_{\ell|m} + \delta_{m\ell} - \psi_{\ell m} + \delta_{\emptyset,\ell m})$$
$$\omega_{jk} - 12\lambda_{jk}(\lambda_{jk} + \delta_{j|k} + \delta_{kj} - \psi_{jk} + \delta_{\emptyset,jk})$$

This shows that we do not need to use ω_{jk} as generators in $\mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2})$, and we can remember these expressions instead. Combining these relations with the relations obtained from $\mathcal{M}^{\Theta_j}$, we get the relations

after simplifying.

Finally, we consider $\partial\overline{\mathcal{M}}_{1,4} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 1} = \overline{\mathcal{M}}^{\mathrm{sep},\geq 2} \cup \mathcal{M}^{\mathrm{non}=1}$. We have $\mathcal{M}^{\mathrm{non}=1} = \mathcal{M}^{\Phi}$ and by Theorem 6.2, we have

$$\mathcal{M}^{\Phi} = [\mathcal{M}_{\Phi}/\operatorname{Aut}(\Phi)] = [\mathcal{M}_{0,6}/\mu_2].$$

The localization exact sequence for $\overline{\mathcal{M}}^{\mathrm{sep},\geq 2} \subseteq \partial\overline{\mathcal{M}}_{1,4}$ is

$$\overline{\mathrm{CH}}^*([\mathcal{M}_{0,6}/\mu_2],1) \xrightarrow{\partial_1} \mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2}) \to \mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,4}) \to \mathrm{CH}^*([\mathcal{M}_{0,6}/\mu_2]) \to 0.$$

By Theorem 7.6, we have $\mathrm{CH}^*([\mathcal{M}_{0,6}/\mu_2]) = \mathbb{Z}$, so we can lift $[\mathcal{M}^{\Phi}]$ to $\phi = [\overline{\mathcal{M}}^{\phi}]$ to split the exact sequence. To compute the image of ∂_1, we use the diagram (6.6) to conclude that the image of ∂_1 in the pushforward of the image of ∂_1' along $\mathrm{CH}_*([\partial\overline{\mathcal{M}}_{0,6}/\mu_2]) \to \mathrm{CH}_*(\overline{\mathcal{M}}^{\mathrm{sep},\geq 2})$. By Theorem 7.8, the image of ∂_1' is generated by

$$\widehat{D}(ja|kb) - 2\widehat{D}(jk|ab)$$

for $j,k \in \{1,\ldots,4\}$ and

$$\alpha_{jk\ell} + \alpha_{j'k'\ell'}$$

for $j,k,\ell,j',k',\ell' \in \{1,\ldots,4\}$ with $|\{j,k,\ell\}| = |\{j',k',\ell'\}| = 3$, where

$$\alpha_{j,k,\ell} := \widehat{D}(jkb|\ell a) + \widehat{D}(\ell ab|jk) - \widehat{D}(\ell jk|ab) - \widehat{D}(kab|\ell j) - \widehat{D}(jab|\ell k).$$

Taking the pushforward, we get the relations

$$2(\theta_k+\theta_{j\ell}+\theta_{jm}+\theta_j)-24(\lambda_\emptyset+\lambda_m+\lambda_\ell+\lambda_{\ell m})$$

for $\{j,k,\ell,m\}=[4]$ and

□

Theorem 8.11. *The image of* $\overline{\mathrm{CH}}^1(\mathcal{M}^0_{1,3},1)\to\overline{\mathrm{CH}}^2(\mathcal{M}_{1,4},1)\to\mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,4})$ *is generated by*

Proposition 8.12. *The image of* $\partial_1:\overline{\mathrm{CH}}^*(\mathcal{M}_{1,4},1)\to\mathrm{CH}^{*-1}(\partial\overline{\mathcal{M}}_{1,4})$ *is given by*

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
\overline{\mathrm{CH}}^*(\mathcal{M}_{1,4},1) & \xrightarrow{\partial_1} & \mathrm{CH}^{*-1}(\partial\overline{\mathcal{M}}_{1,4}) \\
{\scriptstyle\pi^*}\big\uparrow & & {\scriptstyle\pi^*}\big\uparrow \\
\overline{\mathrm{CH}}^*(\mathcal{M}_{1,3},1) & \xrightarrow{\partial_1'} & \mathrm{CH}^{*-1}(\partial\overline{\mathcal{M}}_{1,3}).
\end{array}$$

We know that the 2-torsion element

$$\theta_1+12\lambda_1-\theta_2-12\lambda_2$$

is in the image of ∂_1' by Proposition 8.6. Pulling this back along π, we get the 2-torsion element

$$\theta_1+\theta_{14}+12\lambda_1+12\lambda_{14}-\theta_2-\theta_{24}-12\lambda_2-12\lambda_{24}.$$

Applying the S_4 action to the element, we can obtain more elements in the image of ∂_1. There are six linearly independent elements in this orbit, and $\overline{\mathrm{CH}}^2(\mathcal{M}_{1,4},1)$ help.

Next, we have the 2-torsion elements $\tau_1^i-\tau_2^i$ in the image of ∂_1' by Proposition 8.6.

□

Theorem 8.13. $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,4})=$

Proof. □

9. Proof of Theorem 8.11

Definition 9.1. Let $\widetilde{\mathcal{M}}_{1,2}$ be the closure of $\mathcal{M}^0_{1,2}\subseteq\overline{\mathcal{M}}_{1.3}$ and let $\widetilde{\mathcal{M}}_{1,3}$ be the closure of $\mathcal{M}^0_{1,3}\subseteq\overline{\mathcal{M}}_{1,4}$.

Lemma 9.2. *The class* $[\widetilde{\mathcal{M}}_{1,2}]\in\mathrm{CH}^*(\overline{\mathcal{M}}_{1,3})$ *is given by*

$$[\widetilde{\mathcal{M}}_{1,2}]=2\lambda-\delta_1+2\delta_2+2\delta_3+2\delta_\emptyset$$

Proof. By Theorem 8.8, we can write

$$[\widetilde{\mathcal{M}}_{1,2}] = b_0\lambda + b_1\delta_1 + b_2\delta_2 + b_3\delta_3 + b_4\delta_\emptyset \tag{9.3}$$

for some $b_i \in \mathbb{Z}$.

For a curve $(C, p_1, p_2, p_3) \in \mathcal{M}^0_{1,2} \subseteq \mathcal{M}_{1,3}$, elliptic curve inversion swaps the points p_2 and p_3, giving an isomorphism $(C, p_1, p_2, p_3) \xrightarrow{\sim} (C, p_1, p_3, p_2)$. Thus, every point in $\mathcal{M}^0_{1,2} \subseteq \mathcal{M}_{1,3}$ is invariant under (23), hence the same is true for $\widetilde{\mathcal{M}}_{1,3}$.Applying the permutation (23) to both sides of (9.3), we see that $b_2 = b_3$.

Next, the composite $\widetilde{\mathcal{M}}_{1,2} \hookrightarrow \overline{\mathcal{M}}_{1,3} \xrightarrow{\pi} \overline{\mathcal{M}}_{1,2}$ has degree 1, since it has a rational inverse given by $\mathcal{M}^0_{1,2} \hookleftarrow \mathcal{M}_{1,3}$. Thus, after pushing forward along π, (9.3) becomes

$$[\overline{\mathcal{M}}_{1,2}] = (b_1 + b_2)[\overline{\mathcal{M}}_{1,2}],$$

so $b_1 + b_2 = 1$.

Let $\pi' : \overline{\mathcal{M}}_{1,3} \to \overline{\mathcal{M}}_{1,2}$ denote the map forgetting the first point. Given two distinct points p, p' on a smooth genus 1 curve, there are always 4 points q such that $p + p' \sim 2q$. Thus, the composite $\widetilde{\mathcal{M}}_{1,2} \hookrightarrow \overline{\mathcal{M}}_{1,3} \xrightarrow{\pi'} \overline{\mathcal{M}}_{1,2}$ has degree 4. Thus, after pushing forward along π', (9.3) becomes

$$4[\overline{\mathcal{M}}_{1,2}] = (b_2 + b_3)[\overline{\mathcal{M}}_{1,2}],$$

so $b_2+b_3 = 4$. Putting these equations together, we get $b_1 = -1$, $b_2 = b_3 = 2$.

No points in $\overline{\mathcal{M}}^{\Delta_1}_{1,3}$ are invariant under (23), so its intersection with $\widetilde{\mathcal{M}}_{1,2}$ must be empty. Thus, multiplying by δ_2, we have

$$0 = b_0\lambda\delta_2 + b_2\delta_2^2 + b_4\delta_\emptyset\delta_2 = (b_0 - b_2)\lambda\delta_2 + (b_4 - b_2)\delta_\emptyset\delta_2$$

from which we get $b_0 = b_4 = b_2 = 2$. □

Proof of Theorem 8.11. We utilize the commutative diagram

$$\begin{tikzcd}
\overline{\mathrm{CH}}^1(\mathcal{M}^0_{1,3}, 1) \arrow[r, "\partial_1"] \arrow[d] & \mathrm{CH}_2(\widetilde{\mathcal{M}}_{1,3} \setminus \mathcal{M}^0_{1,3}) \arrow[d] \\
\overline{\mathrm{CH}}^2(\mathcal{M}_{1,4}, 1) \arrow[r, "\partial_1"] & \mathrm{CH}^1(\partial\overline{\mathcal{M}}_{1,4})
\end{tikzcd}$$

First, we wish to compute $\mathrm{CH}_2(\widetilde{\mathcal{M}}_{1,3} \setminus \mathcal{M}^0_{1,3})$, i.e. the 2-dimensional components of $\widetilde{\mathcal{M}}_{1,3} \setminus \mathcal{M}^0_{1,3}$. We think about the map $\widetilde{\mathcal{M}}_{1,3} \to \overline{\mathcal{M}}_{1,3}$. The dimension of the image of 2-dimensional component of $\widetilde{\mathcal{M}}_{1.3} \setminus \mathcal{M}^0_{1,3}$ in $\overline{\mathcal{M}}_{1,3} \setminus \mathcal{M}^0_{1,3}$ has dimension either 1, in which case the map is finite, or 2, in which case the fibers are 1 diemsnional . Thus, we only need to consider points in $\widetilde{\mathcal{M}}_{1,3}$ lying over $\mathcal{M}^0_{1.2} \subseteq \overline{\mathcal{M}}_{1,3}$ or $\mathcal{M}^\Gamma_{1,3}$ for Γ a stable graph of codimension 1 or 2.

Note that the morphism

$$\varphi : \mathcal{M}^0_{1,3} \to \mathcal{M}_{1,4}$$

$$(C, p_1, p_2, p_3) \mapsto (C, p_1, p_2, p_3, p_2 + p_3)$$

exhibiting $\mathcal{M}_{1,3}^0$ as a closed subset of $\mathcal{M}_{1,4}$ extends to $\mathcal{M}_{1,3}^{\Phi}$ when viewed as a map to $\overline{\mathcal{M}}_{1,4}$. Moreover, this map extends to $\mathcal{M}_{1,2}^0 \subseteq \mathcal{M}_{1,3}$ by sending (C, p_1, p_2, p_3) to the curve C with an attached $\mathbb{P}^1$ at p_1 that contains p_1 and p_4. Thus, over $\mathcal{M}_{1,2}^0$ and $\mathcal{M}_{1,3}^{\Phi}$, there are unique 2-dimensional components of $\widetilde{\mathcal{M}}_{1,3} \setminus \mathcal{M}_{1,3}^0$. Let $\widetilde{\mathcal{M}}_{1,3}^{\Phi} := \overline{\varphi(\mathcal{M}_{1,3}^{\Phi})}$ and $\widetilde{\mathcal{M}}_{1,3}^{\Xi} := \overline{\varphi(\mathcal{M}_{1,2}^0)}$.

Next, note the curves in $\mathcal{M}_{1,3}^0 \subseteq \overline{\mathcal{M}}_{1,4}$ are invariant under the permutation $(14)(23)$, as the isomorphism $q \mapsto 2p_4 - q$ takes (C, p_1, p_2, p_3, p_4) to (C, p_4, p_3, p_2, p_1) for $(C, p_1, p_2, p_3, p_4) \in \mathcal{M}_{1,3}^0$. Thus, the points in $\widetilde{\mathcal{M}}_{1,3}$ must also be invariant under $(14)(23)$. Moreover, the permutation $(12)(34)$ will not generally fix the curves in $\mathcal{M}_{1,3}^0$, but it does send the locus $\mathcal{M}_{1,3}^0$ to itself.

For $j = 2, 3$, over points in $\mathcal{M}_{1,3}^{\Delta_j}$ there is a unique point in $\overline{\mathcal{M}}_{1,4}$ that is invariant under $(14)(23)$, given by attaching a $\mathbb{P}^1$ containing the markings j and 4 to the marking j. Thus, $\overline{\mathcal{M}}^{\Psi_{13}}$ and $\overline{\mathcal{M}}^{\Psi_{12}}$ are the only 2-dimensional components over $\mathcal{M}_{1,3}^{\Delta_2}$ and $\mathcal{M}_{1,3}^{\Delta_3}$. Additionally, over a general point of $\mathcal{M}_{1,3}^{\Delta_\emptyset}$, there is a unique point invariant under $(14)(23)$. Over the other points of $\mathcal{M}_{1,3}^{\Delta_\emptyset}$, there is exactly one additional curve invariant under $(14)(23)$, meaning there will not be a 2-dimensional component over these points. We can describe the 2-dimensional component of $\widetilde{\mathcal{M}}_{1,3} \setminus \mathcal{M}_{1,3}^0$ over $\overline{\mathcal{M}}^{\Delta_\emptyset}$ as the image of the $(14)(23)$ invariant points in $\overline{\mathcal{M}}_{0,5} \times \overline{\mathcal{M}}_{1,1}$ under the map gluing the 5-th marked point of the genus 0 curve to the only marked point of the genus 1 curve. Call this component $\widetilde{\mathcal{M}}_{13}^{\Delta_\emptyset}$.

There are exactly two $(14)(23)$ invariant curves over every point in $\overline{\mathcal{M}}_{1,3}^{\Delta_1}$. These are given by placing p_4 either on a rational component with p_1 or at the point $2q$, where q is the point on the genus 1 component where the rational curve containing $2, 3$ is attached. We have that curves of the latter type are all inside $\widetilde{\mathcal{M}}_{1,3}$ because they are equal to $(12)(34)$ applied to a point of $\varphi(\mathcal{M}_{1,2}^0)$. Moreover, the map $\varphi : \overline{\mathcal{M}}_{1,3} \dashrightarrow \overline{\mathcal{M}}_{1,4}$ extends to a morphism over an open subset of $\overline{\mathcal{M}}_{1,3}^{\Delta_1}$, because $\overline{\mathcal{M}}_{1,3}$ is normal and $\overline{\mathcal{M}}_{1,4}$ is proper. Thus, there is only one point in $\widetilde{\mathcal{M}}_{1,3}$ lying over a general point of $\overline{\mathcal{M}}_{1,3}^{\Delta_1}$. And the remaining points have at most 2 points lying over them, so we can conclude that there is a unique 2-dimensional component lying over $\mathcal{M}_{1,3}^{\Delta_1}$. Call this component $\widetilde{\mathcal{M}}_{1,3}^{\Delta_1}$. This takes care of all of the codimension 1 graphs of $\overline{\mathcal{M}}_{1,3}$.

For all codimension 2 graphs Γ of $\overline{\mathcal{M}}_{1,3}$ besides Θ_1, for any point in $\mathcal{M}_{1,3}^{\Gamma}$, one can see that there are only finitely many $(14)(23)$-invariant preimages over in $\overline{\mathcal{M}}_{1,4}$, meaning there cannot be a 2-dimensional component of $\widetilde{\mathcal{M}}_{1,3} \setminus \mathcal{M}_{1,3}^0$ lying over these $\mathcal{M}_{1,3}^{\Gamma}$. But over a point in $\mathcal{M}_{1,3}^{\Theta_1}$, there are infinitely many preimages invariant under $(14)(23)$, obtained by placing p_4 anywhere on the same component as p_1. These are the points of $\overline{\mathcal{M}}_{1,4}$ in $\mathcal{M}_{1,4}^{\Theta_{1,4}}$, and we do indeed have $\mathcal{M}_{1,4}^{\Theta_{1,4}} \subseteq \widetilde{\mathcal{M}}_{1,3}$, hence $\overline{\mathcal{M}}_{1,4}^{\Theta_{1,4}}$ is a 2-dimensional component of $\widetilde{\mathcal{M}}_{1,3} \setminus \mathcal{M}_{1,3}^0$. To show that $\mathcal{M}_{1,4}^{\Theta_{1,4}} \subseteq \widetilde{\mathcal{M}}_{1,3}$, we provided an explicit deformation: help.

Knowing all of the 2-dimensional components, we can write

$$\partial_1(f_0) = \mathrm{div}(f_0) = a_0[\widetilde{\mathcal{M}}_{1,3}^{\Phi}]+a_1[\widetilde{\mathcal{M}}_{1,3}^{\Delta_1}]+a_2\psi_{13}+a_3\psi_{12}+a_4[\widetilde{\mathcal{M}}_{13}^{\Delta_\emptyset}]+a_5[\widetilde{\mathcal{M}}_{1,3}^{\Xi}]+a_6\theta_{14}$$

for some $a_i \in \mathbb{Z}$. By commutativity of

$$\begin{array}{ccc}
\overline{\mathrm{CH}}^1(\mathcal{M}_{1,3}^0,1) & \xrightarrow{\partial_1} & \mathrm{CH}_2(\widetilde{\mathcal{M}}_{1,3}\setminus \mathcal{M}_{1,3}^0) \\
\downarrow & & \downarrow \\
\overline{\mathrm{CH}}^1(\mathcal{M}_{1,3},1) & \xrightarrow{\partial_1'} & \mathrm{CH}_2(\overline{\mathcal{M}}_{1,3}\setminus \mathcal{M}_{1,3}^0)
\end{array}$$

the above equation pushes forward to

$$\partial_1'(f_0) = a_0\phi + a_1\delta_1 + a_2\delta_2 + a_3\delta_3 + a_4\delta_\emptyset + a_5[\widetilde{\mathcal{M}}_{1,2}]$$

on $\partial\overline{\mathcal{M}}_{1,3}$, hence

$$0 = a_0\phi + a_1\delta_1 + a_2\delta_2 + a_3\delta_3 + a_4\delta_\emptyset + a_5[\widetilde{\mathcal{M}}_{1,2}]$$

in $\overline{\mathcal{M}}_{1,3}$.

By Lemma 9.2, we have

$$0 = \phi - 6\lambda_1 + 12\delta_2 + 12\delta_3 + 12\delta_\emptyset - 6[\widetilde{\mathcal{M}}_{1,2}].$$

There is only one relation between these classes up to scaling by an integer, given by $\mathrm{div}(f_0)$. Because this above relation holds on $\overline{\mathcal{M}}_{1,3}$ and it is primitive, we can conclude that either $\mathrm{div}(f_0)$ or $\mathrm{div}(f_0^{-1})$ is given by the right-hand side. But, by definition of f_0, it is clear that the order of vanishing along irreducible nodal curves is negative. Thus, we have $a_0 = -1, a_1 = a_5 = 6$, and $a_2 = a_3 = a_4 = -12$.

Finally, we compute a_6 using the curves approaching $\mathcal{M}^{\Theta_{14}}$ from earlier. We know that these curves are transverse to $\mathcal{M}^{\Theta_{14}}$ because help.

Now, we pushforward $\mathrm{div}(f_0)$ to $\overline{\mathcal{M}}_{1,4}$, giving

$$\partial_1((\mathcal{M}_{1,3}^0, f_0)) = [\widetilde{\mathcal{M}}_{1,3}^{\Phi}]+6[\widetilde{\mathcal{M}}_{1,3}^{\Delta_1}]-12\psi_{13}-12\psi_{12}-12[\widetilde{\mathcal{M}}_{13}^{\Delta_\emptyset}]+6[\widetilde{\mathcal{M}}_{1,3}^{\Xi}]+a_6\theta_{14}$$

We want to express $[\widetilde{\mathcal{M}}_{1,3}^{\Phi}], [\widetilde{\mathcal{M}}_{1,3}^{\Delta_1}], [\widetilde{\mathcal{M}}_{13}^{\Delta_\emptyset}], [\widetilde{\mathcal{M}}_{1,3}^{\Xi}]$ in terms of our generators for the $\mathrm{CH}^*(\partial\overline{\mathcal{M}}_{1,4})$ given in Theorem 8.10.

Note $\widetilde{\mathcal{M}}_{1,3}^{\Xi}$ is equal to the image of $\widetilde{\mathcal{M}}_{1,2}$ along the section $\overline{\mathcal{M}}_{1,3} \xrightarrow{\sim} \overline{\mathcal{M}}^{\Delta_{2,3}} \hookrightarrow \overline{\mathcal{M}}_{1,4}$. Thus, we have

$$[\widetilde{\mathcal{M}}_{1,3}^{\Xi}] = 2\lambda_{2,3} - \psi_{23} + 2\delta_{2|3} + 2\delta_{3|2} + 2\delta_{\emptyset,23}$$

on $\partial\overline{\mathcal{M}}_{1,4}$ by Lemma 9.2. As noted before, the image of $\widetilde{\mathcal{M}}_{1,3}^{\Xi}$ under (12)(34) is $\widetilde{\mathcal{M}}_{1,3}^{\Delta_1}$, so we have

$$[\widetilde{\mathcal{M}}_{1,3}^{\Delta_1}] = 2\lambda_{1,4} - \psi_{14} + 2\delta_{1|4} + 2\delta_{4|1} + 2\delta_{\emptyset,14}$$

on $\partial\overline{\mathcal{M}}_{1,4}$.

Next, we compute $[\widetilde{\mathcal{M}}_{1,3}^{\Delta_\emptyset}]$. As noted above, it is equal to the pushforward of $V\times\overline{\mathcal{M}}_{1,1}$ under $\overline{\mathcal{M}}_{0,5}\times\overline{\mathcal{M}}_{1,1}\to\overline{\mathcal{M}}_{1,4}^{\Delta_\emptyset}\subseteq\partial\overline{\mathcal{M}}_{1,4}$, where V the subvariety of curves fixed by (14)(23) in $\overline{\mathcal{M}}_{0,5}$. For ease of notation, set $D_{ij}:=D(ij|k\ell m)$. The divisors $D_{15}, D_{25}.D_{35}, D_{45}$, and D_{14} form a basis for $\mathrm{CH}^1(\overline{\mathcal{M}}_{0,5})$, so we can write

$$[V]=a_{15}D_{15}+a_{25}D(134)+a_{35}D_{35}+a_{45}D(123)+a_{14}D_{14}.$$

for some coefficients $a_{ij}\in\mathbb{Z}$. Because this locus is fixed point wise by (14)(23), after applying this automorphism to both sides, we get $a_{15}=a_{45}$ and $a_{25}=a_{35}$. Now we pushforward along maps $\overline{\mathcal{M}}_{0,5}\to\overline{\mathcal{M}}_{0,4}$ forgetting a marked point, noting that the degree of $D_{ij}\to\overline{\mathcal{M}}_{0,4}$ is 1 if the point being forgotten is i or j and 0 otherwise. First we pushforward along the map $\pi:\overline{\mathcal{M}}_{0,5}\to\overline{\mathcal{M}}_{0,4}$ forgetting the fifth marked point. To calculate the degree for the left-hand side, we work over the opens $\mathcal{M}_{0,5}\to\mathcal{M}_{0,4}$. Applying the automorphism (14)(23) to a point $(\mathbb{P}^1,0,1,\infty,x,y)\in\mathcal{M}_{0,5}$ lying over $(\mathbb{P}^1,0,1,\infty,x)$, we get $(\mathbb{P}^1,x,\infty,1,0,y)$, which is equal to $(\mathbb{P}^1.0,1,\infty,x,\frac{y-x}{y-1})$ via the automorphism $t\mapsto\frac{t-x}{t-1}$ of $\mathbb{P}^1$. Thus, such a point is stable under (14)(23) if and only if $\frac{y-x}{y-1}=y$ or $y^2-2y+x=0$. Thus the map $V\cap\mathcal{M}_{0,5}\to\mathcal{M}_{0,4}$ has degree 2, so we have $2=a_{15}+a_{23}+a_{35}+a_{45}$. Pushing forward along the map forgetting the fourth marked point, the same description on the opens shows that this map $V\cap\mathcal{M}_{0,4}$ has degree one, so we get $1=a_{45}+a_{14}$. Finally. pushing forward along the map forgetting the second marked point, the degree of $V\to\overline{\mathcal{M}}_{0,4}$ will be the same by symmetry of V, so we get $1=a_{25}$. Putting these together, we obtain $[V]=D_{25}+D_{35}+D_{14}$. And so

$$[\widetilde{\mathcal{M}}_{1,3}^{\Delta_\emptyset}]=\iota_*([V]\times[\overline{\mathcal{M}}_{1,1}])=\delta_{\emptyset,2}+\delta_{\emptyset,3}+\delta_{\emptyset,14}.$$

And now we compute $[\widetilde{\mathcal{M}}_{1,3}^{\Phi}]$. Recall $\widetilde{\mathcal{M}}_{1,3}^{\Phi}$ is the closure of the locus in $\mathcal{M}_{1,4}^{\Phi}$ where the fourth marked point is the sum of the second and third marked points, and it has codimension one in $\overline{\mathcal{M}}_{1,4}^{\Phi}$. Let $W\subseteq\overline{\mathcal{M}}_{0,6}$ be the pullback of this locus. Because $\widetilde{\mathcal{M}}_{1,3}^{\Phi}\subseteq\widehat{\mathcal{M}}_{1,3}$, it is invariant under (14)(23). Then, we have that elements of W are invariant under either (14)(23) or (14)(23)(56). An element of $\mathcal{M}_{0,5}$ may be written as $(\mathbb{P}^1,\infty,x,y,z,1,-1)$. One can check that the locus of $\mathcal{M}_{0,5}$ invariant under (14)(23) is given by the vanishing of $yz+xz-xy-1, yz+xy-xz-1$ (which can be reduced to the vanishing of $x, yz-1$). One can also check that the locus of $\mathcal{M}_{0,5}$ invariant under (14)(230(56) is given by the vanishing of $xy+xz-yz-1$. The former locus has codimension 2 and the latter has codimension 1, hence the latter must be contained in W. Since W is equal to the closure of its intersection with $\overline{\mathcal{M}}_{0,6}$ by continuity, we have $W\subseteq\overline{\mathcal{M}}_{0,5}$ is the closure of the vanishing of $xy+xz-yz-1$ inside $\mathcal{M}_{0,5}$. Let $\widehat{W}$ be its image inside $[\overline{\mathcal{M}}_{0,6}/\mu_2]$.

For ease of notation, define $\widehat{D}_{ij} := \widehat{D}(ij|k\ell mn)$ and $\widehat{D}_{ijk} := \widehat{D}(ijk|\ell mn)$. One can use the relations in Theorem 7.8 to see that the elements

$$\widehat{D}_{ab}, \widehat{D}_{14a}, \{\widehat{D}_{ij} | i \neq j \in [4]\}, \{\widehat{D}_{iab} | i \in [4]\}$$

generate $\mathrm{Pic}([\overline{\mathcal{M}}_{0,5}/\mu_2])/\langle\alpha\rangle$. Moreover, using Theorem 7.8, this group has a presentation with 18 generators and 6 relations, so these 12 generators must generate freely. So we can write

$$[\widehat{W}] = c_{ab}\widehat{D}_{ab} + c_{14a}\widehat{D}_{14a} + \sum_{1 \leq i < j \leq 4} c_{ij}\widehat{D}_{ij} + \sum_{1 \leq i \leq 4} c_{iab}\widehat{D}_{iab},$$

for some coefficients in $\mathbb{Z}$. The permutations $(14)(23), (12)(34)$, and (14) map $\widehat{W}$ to itself. Applying these symmetries to the above sum, we get all c_{iab} are equal, $c_{14} = c_{23}$, and $c_{12} = c_{13} = c_{24} = c_{34}$. Next, we pull our expression for $\widehat{W}$ back to $\overline{\mathcal{M}}_{0,6}$ to get an expression for $[W]$ and consider push forwards to $\overline{\mathcal{M}}_{0,5}$ forgetting certain marked points. For D an irreducible boundary divisor, the degree of the map $D \subseteq \overline{\mathcal{M}}_{0,6} \to \overline{\mathcal{M}}_{0,5}$ forgetting the i-th marked point is 0 unless $D = D_{ij}$ for some $j \in [6]$. We have that the degree of $W \subseteq \overline{\mathcal{M}}_{0,6} \to \overline{\mathcal{M}}_{0,5}$ forgetting the fourth marked point is 1, by the fact that W is the closure of $V(yz + xz - xy - 1)$. By symmetry, this is also the degree forgetting the fifth marked point. Pushing forward $[W]$ by forgetting the fourth marked point, we get $1 = 2c_{12} + c_{14}$; pushing forward by forgetting the fifth marked point, we get $c_{ab} = 1$. Finally, multiplying our expression for W by D_{123}, we get □