

INTERVALS ENFORCEABLE PROPERTIES OF FINITE GROUPS

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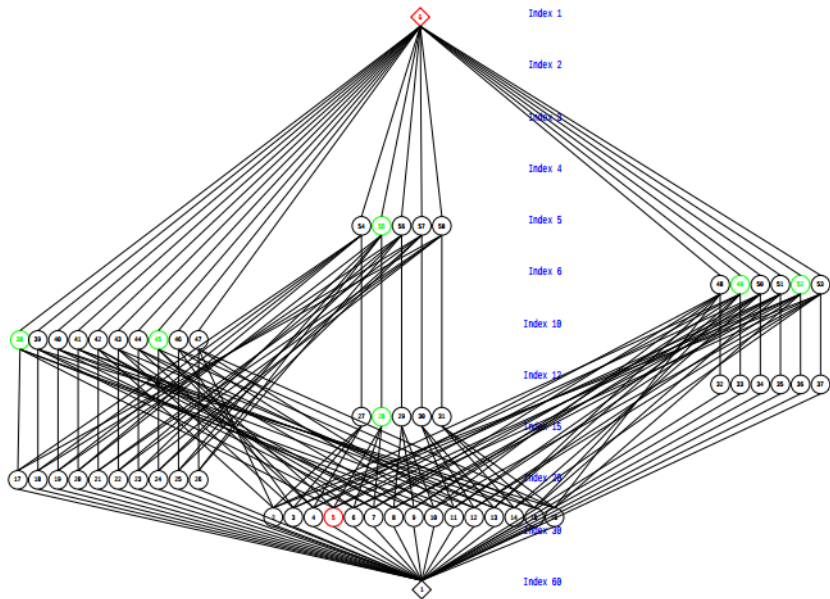


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Historically, much work has focused on:

- inferring properties of a group G from the structure of its lattice of subgroups $\text{Sub}(G)$;
- inferring lattice theoretical properties of $\text{Sub}(G)$ from properties of G .

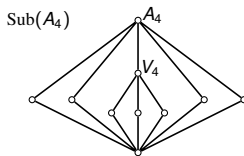
For some groups, $\text{Sub}(G)$ determines G up to isomorphism.

EXAMPLES

The Klein 4-group, V_4 .


The alternating groups, A_n ($n \geq 4$).


Every finite nonabelian simple group.




For other groups, $\text{Sub}(G)$ is isomorphic to the subgroup lattices of all groups in an infinite class of nonisomorphic groups.

EXAMPLES


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
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At the other extreme, there are finite lattices that are not subgroup lattices.

Example?

We are interested in the local structure of subgroup lattices, that is, the possible *intervals*

$$[[H, K]] := \{X \mid H \leq X \leq K\} \leq \text{Sub}(G)$$

where $H \leq K \leq G$.

We restrict our attention to *upper intervals*, where $K = G$, and ask two questions:

- 1 What intervals $[[H, G]]$ are possible?
- 2 What properties of a group G can be inferred from the shape of an upper interval in $\text{Sub}(G)$?

1. WHAT INTERVALS $\llbracket H, G \rrbracket$ ARE POSSIBLE?

There is a remarkable theorem relating this question to the *finite lattice representation problem* (FLRP).

THEOREM (PÁLFY AND PUDLÁK(1980))

The following statements are equivalent:

- (A) *Every finite lattice is isomorphic to the congruence lattice of a finite algebra.*
- (B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

2. WHAT PROPERTIES OF G CAN BE INFERRED FROM $\llbracket H, G \rrbracket$?

A group theoretical property \mathcal{P} (and the associated class $\mathcal{G}_{\mathcal{P}}$) is

- **interval enforceable** (IE) provided there exists a lattice L such that

if $G \in \mathfrak{G}$ and $L \cong \llbracket H, G \rrbracket$, then G has property \mathcal{P} .

- **core-free interval enforceable** (cf-IE) provided $\exists L$ st

if $G \in \mathfrak{G}$, $L \cong \llbracket H, G \rrbracket$, H core-free, then G has property \mathcal{P} .

- **minimal interval enforceable** (min-IE) provided $\exists L$ st

if $G \in \mathfrak{G}$, $L \cong \llbracket H, G \rrbracket$, and if G has minimal order (wrt $L \cong \llbracket H, G \rrbracket$), then G has property \mathcal{P} .

EXAMPLES

Nonsolvability

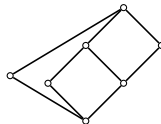
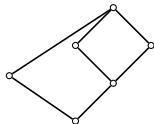
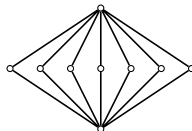
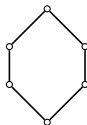
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EXAMPLES

Nonsolvability

It's not hard to find examples of lattices that cannot occur as upper intervals in the subgroup lattices of finite *solvable* groups.

Here are a few



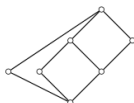
HAS ANYONE SEEN THIS LATTICE?

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Given a lattice L with n elements, are there finite groups $H < G$ such that $L \cong$ the lattice of subgroups between H and G ?

▲
13
▼

If there is no restriction on n , this is a famous [open problem](#). I'm wondering if any recent work has been done for small $n > 6$. I believe the question is answered (positively) for $n = 6$ by Watatani (1996) [MR1409040](#) and Aschbacher (2008) [MR2393428](#). I also believe we can answer it for $n = 7$, with one possible exception. The exceptional case is shown below.



So my two questions are these:

1) Does anyone know of recent work on this special case of the problem (specifically for $n = 7$ or $n = 8$)?

2) Has anyone found a finite group G with a subgroup H such that the interval

$$[H, G] = \{K : H \leq K \leq G\}$$

is the lattice shown above?

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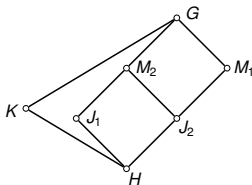
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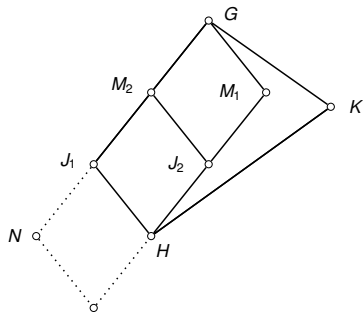


PROPOSITION

Suppose $H < G$, $\text{core}_G(H) = 1$, and $L_7 \cong \llbracket H, G \rrbracket$. Then

- (I) G is a primitive permutation group.
- (II) If $N \triangleleft G$, then $C_G(N) = 1$.
- (III) G contains no non-trivial abelian normal subgroup.
- (IV) G is not solvable.
- (V) G is subdirectly irreducible.
- (VI) With the possible exception of at most one maximal subgroup, M_1 or M_2 , all proper subgroups in the interval $\llbracket H, G \rrbracket$ are core-free.

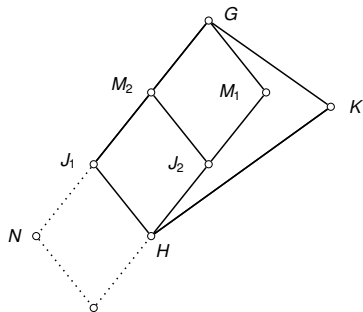
EXAMPLE



Claim: J_1 and J_2 are core-free subgroups of G .

Proof:

EXAMPLE

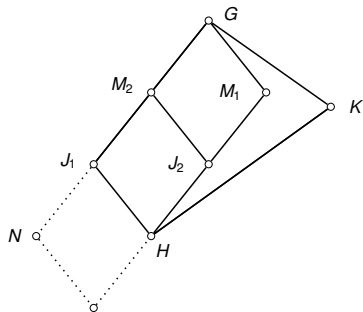


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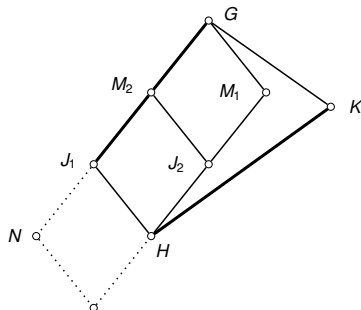


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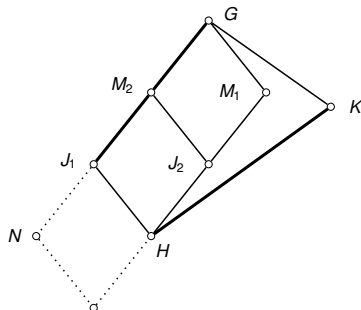
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- If $N \triangleleft G$ then NH permutes with each subgroup containing H .
- If $1 \neq N \leq J_1$, then $NH = J_1$, so J_1 and K permute.
- Since $J_1K = G$ and $J_1 \cap K = H$, our lemma yields

$$[J_1, G] \cong [H, K]^{J_1} = \{X \in [H, K] \mid J_1X = XJ_1\}.$$

EXAMPLE



Claim: J_1 and J_2 are core-free subgroups of G .

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Impossible!

The following are at least core-free interval enforceable:

- $\mathcal{G}_0 = \mathfrak{G}^c =$ the nonsolvable groups
- $\mathcal{G}_1 = \{G \in \mathfrak{G} \mid (\forall n < \omega) (G \neq A_n \text{ and } G \neq S_n)\}$
- $\mathcal{G}_2 =$ the subdirectly irreducible groups
- $\mathcal{G}_3 =$ groups with no nontrivial abelian normal subgroups
- $\mathcal{G}_4 = \{G \in \mathfrak{G} \mid C_G(M) = 1 \text{ for all } 1 \neq M \trianglelefteq G\}.$

If a lattice L is isomorphic to an interval in the subgroup lattice of a finite group, then we call L ***group representable***.

By the Pálffy-Pudlák Theorem, the FLRP has a negative answer if we can find a finite lattice that is not group representable.

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Suppose there exists property \mathcal{P} such that both \mathcal{P} and its negation $\neg\mathcal{P}$ are interval enforceable by the lattices L and L_c , respectively:

$$L \cong \llbracket H, G \rrbracket \implies G \text{ has property } \mathcal{P}$$

$$L_c \cong \llbracket H_c, G_c \rrbracket \implies G_c \text{ does not have property } \mathcal{P}$$

If a lattice L is isomorphic to an interval in the subgroup lattice of a finite group, then we call L **group representable**.

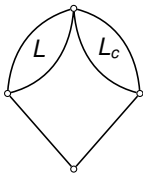
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Then the lattice



wouldn't be group representable.

As the next result shows, however, if a group property and its negation are interval enforceable by L and L_c , then already at least one of these lattices is not group representable.

LEMMA

If \mathcal{P} is a group property that is interval enforceable by a group representable lattice, then $\neg\mathcal{P}$ is not interval enforceable by a group representable lattice.

Nonsolvability is interval enforceable, but solvability is not.

For if $L \cong \llbracket H, G \rrbracket$, then for any nonsolvable group K we have $L \cong \llbracket H \times K, G \times K \rrbracket$, and $G \times K$ is nonsolvable.

Note that the group $H \times K$ at the bottom of the interval is not core-free. So a more interesting question is whether a property and its negation could both be *core-free* IE.

CONJECTURE

If \mathcal{P} is core-free interval enforceable by a group representable lattice, then $\neg\mathcal{P}$ is not core-free interval enforceable by a group representable lattice.

The following lemma shows that any class of groups that omits certain wreath products cannot be core-free interval enforceable by a group representable lattice.

LEMMA

Suppose \mathcal{P} is core-free interval enforceable by a group representable lattice. Then, for any finite nonabelian simple group S , there exists a wreath product group of the form $W = S \wr U$ that has property \mathcal{P} .

COROLLARY

Solvability is not core-free interval enforceable.

Proof Sketch

Let L be a group representable lattice such that if $L \cong \llbracket H, G \rrbracket$ and $\text{core}_G(H) = 1$ then G has property \mathcal{P} .

Since L is group representable, $\exists G \models \mathcal{P}$ with $L \cong \llbracket H, G \rrbracket$.

Proof Sketch

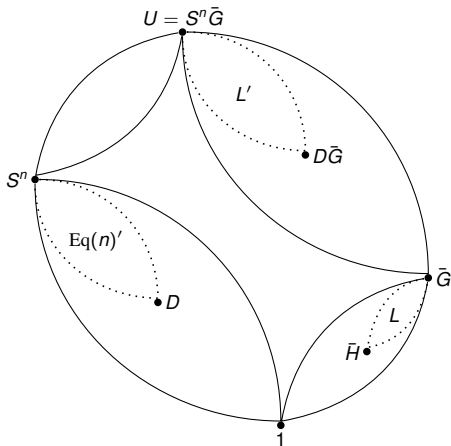
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Since L is group representable, $\exists G \models \mathcal{P}$ with $L \cong \llbracket H, G \rrbracket$.

We apply the idea of Hans Kurzweil twice:



- Fix a finite nonabelian simple group S .
- Suppose the index of H in G is $|G : H| = n$.
- Then the action of G on the cosets of H induces an automorphism of the group S^n by permutation of coordinates.
- Denote this by $\varphi : G \rightarrow \text{Aut}(S^n)$, and let $\varphi(G) = \bar{G} \leq \text{Aut}(S^n)$.



The interval $[D, S^n]$ is isomorphic to $\text{Eq}(n)'$, the dual of the lattice of partitions of an n -element set.

The dual lattice L' is an upper interval of $\text{Sub}(U)$, namely, $L' \cong [D\bar{G}, U]$.

We conclude that a class of groups that does not include wreath products of the form $S \wr G$, where S is an arbitrary finite nonabelian simple group, is not a core-free interval enforceable class. The class of solvable groups is an example.

THEOREM

The following statements are equivalent:

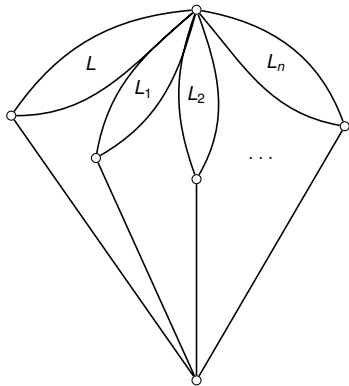
- (B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*
- (C) *For every finite lattice L and every finite collection $\mathcal{G}_1, \dots, \mathcal{G}_n$ of cf-IE classes of groups,*

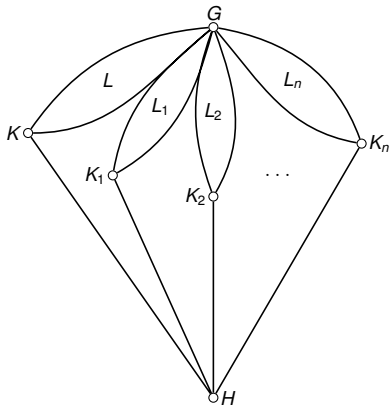
$$\exists G \in \bigcap_{i=1}^n \mathcal{G}_i \text{ such that } L \cong \llbracket H, G \rrbracket \text{ and } \text{core}_G(H) = 1.$$

- (D) *For every finite collection \mathcal{L} of finite lattices, there exists a finite group G such that each $L_i \in \mathcal{L}$ is isomorphic to $\llbracket H_i, G \rrbracket$ for some core-free subgroup $H_i \leq G$.*

By (C), the FLRP would have a negative answer if we could find a collection $\mathcal{G}_1, \dots, \mathcal{G}_n$ of cf-IE classes such that $\bigcap_{i=1}^n \mathcal{G}_i$ is empty.

By (D), it makes sense to consider finite collections of finite lattices and ask what can be proved about a group G if one assumes that all of these lattices are isomorphic to upper intervals of $\text{Sub}(G)$.







ASCHBACHER-O'NAN-SCOTT THEOREM

Let G be a primitive permutation group of degree d , and let $N := \text{Soc}(G) \cong T^m$ with $m \geq 1$. Then one of the following holds.

① N is regular and

- (Affine type) T is cyclic of order p , so $|N| = p^m$. Then $d = p^m$ and G is permutation isomorphic to a subgroup of the affine general linear group $\text{AGL}(m, p)$.
- (Twisted wreath product type) $m \geq 6$, the group T is nonabelian and G is a group of *twisted wreath product type*, with $d = |T|^m$.

② N is non-regular, non-abelian, and

- (Almost simple type) $m = 1$ and $T \leq G \leq \text{Aut}(T)$.
- (Product action type) $m \geq 2$ and G is permutation isomorphic to a subgroup of the product action wreath product $P \wr_{S_m/I}$ of degree $d = nm/I$. The group P is primitive of type 2.(a) or 2.(c), P has degree n and $\text{Soc}(P) \cong T^l$, where $l \geq 1$ divides m .
- (Diagonal type) $m \geq 2$ and $T^m \leq G \leq T^m \cdot (\text{Out}(T) \times S_m)$, with the diagonal action. The degree $d = |T|^{m-1}$.

ASCHBACHER-O'NAN-SCOTT THEOREM

See Peter Cameron's blog at

<http://cameroncounts.wordpress.com/tag/onan-scott-theorem/>

for some history.