DEDEKIND'S TRANSPOSITION PRINCIPLE FOR LATTICES OF EQUIVALENCE RELATIONS

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Abstract. We prove a version of Dedekind's Transposition Principle that holds in lattices of equivalence relations.

In this note we prove a version of Dedekind's Transposition Principle[1] that holds in all (not necessarily modular) lattices of equivalence relations. Let X be a set and let $\operatorname{Eq} X$ denote the lattice of equivalence relations on X. Given $\alpha, \beta \in \operatorname{Eq} X$, we define the *interval sublattice of equivalence relations above α and below β*, denoted $[\![\alpha, \beta]\!]$, as follows:

$$[\![\alpha, \beta]\!] := \{\gamma \in \operatorname{Eq} X \mid \alpha \leqslant \gamma \leqslant \beta\}.$$

Let L be a sublattice of $\operatorname{Eq} X$. Given $\alpha, \beta \in L$, let $[\![\alpha, \beta]\!]_L := [\![\alpha, \beta]\!] \cap L$, which we call an *interval sublattice of L*, or more simply, an *interval of L*. Given $\alpha, \beta, \theta \in L$, let $[\![\alpha, \beta]\!]_L^\theta$ denote the set of equivalence relations in the interval $[\![\alpha, \beta]\!]_L$ that permute with θ. That is,

$$[\![\alpha, \beta]\!]_L^\theta := \{\gamma \in L \mid \alpha \leqslant \gamma \leqslant \beta \text{ and } \gamma \circ \theta = \theta \circ \gamma\}.$$

Lemma 1. *If $\eta, \theta \in L \leqslant \operatorname{Eq} X$, and if $\eta \circ \theta = \theta \circ \eta$, then*

$$[\![\theta, \eta \vee \theta]\!]_L \cong [\![\eta \wedge \theta, \eta]\!]_L^\theta \leqslant [\![\eta \wedge \theta, \eta]\!]_L.$$

The lemma states that the sublattice $[\![\theta, \eta \vee \theta]\!]_L$ is isomorphic to the lattice, $[\![\eta \wedge \theta, \eta]\!]_L^\theta$, of relations in L that are below η, above $\eta \wedge \theta$, and permute with θ; moreover, $[\![\eta \wedge \theta, \eta]\!]_L^\theta$ is a sublattice of $[\![\eta \wedge \theta, \eta]\!]_L$. To prove this, we need the following generalized version of *Dedekind's Rule*:[2]

Lemma 2. *If $\alpha, \beta, \gamma \in L \leqslant \operatorname{Eq} X$, and if $\alpha \leqslant \beta$, then we have the following identities of subsets of X^2:*

$$\alpha \circ (\beta \cap \gamma) = \beta \cap (\alpha \circ \gamma), \tag{1}$$

$$(\beta \cap \gamma) \circ \alpha = \beta \cap (\gamma \circ \alpha). \tag{2}$$

Date: November 14, 2012.

[1] If L is a modular lattice, then for any two elements $a, b \in L$ the intervals $[\![b, a \vee b]\!]$ and $[\![a \wedge b, a]\!]$ are isomorphic. See [1], or [2, page 57].

[2] In the group theory setting, the well known Dedekind's Rule states that if A, B, C are subgroups of a group, and $A \leqslant B$, then we have the following identity of sets: $A(B \cap C) = B \cap AC$.

Proof. We prove (1); the proof of (2) is similar. First we check that $\alpha \circ (\beta \cap \gamma) \subseteq \beta \cap (\alpha \circ \gamma)$. Indeed, since $\alpha \leqslant \beta$, we have

$$\alpha \circ (\beta \cap \gamma) \subseteq \alpha \vee (\beta \cap \gamma) \leqslant \beta \vee (\beta \cap \gamma) = \beta.$$

Also, $\beta \cap \gamma \leqslant \gamma$ implies $\alpha \circ (\beta \cap \gamma) \subseteq \alpha \circ \gamma$. Therefore, $\alpha \circ (\beta \cap \gamma) \subseteq \beta \cap (\alpha \circ \gamma)$.

For the reverse inclusion, fix $(x, y) \in \beta \cap (\alpha \circ \gamma)$. Since $(x, y) \in \alpha \circ \gamma$, there exists $c \in X$ such that $x \,\alpha\, c \,\gamma\, y$. We must produce $d \in X$ such that $x \,\alpha\, d \,(\beta \cap \gamma)\, y$. In fact, $d = c$ works, since $(x, c) \in \alpha \leqslant \beta$ implies $c \,\beta\, x \,\beta\, y$, so $(c, y) \in \beta \cap \gamma$. $\square$

Proof of Lemma 1. Let $\eta, \theta \in L \leqslant \operatorname{Eq} X$ be permuting equivalence relations in L, so $\eta \circ \theta = \theta \circ \eta = \eta \vee \theta$. Consider the mapping $\varphi : [\![\theta, \eta \vee \theta]\!]_L \to [\![\eta \wedge \theta, \eta]\!]$ given by $\alpha \mapsto \alpha \wedge \eta$. Clearly φ maps $[\![\theta, \eta \vee \theta]\!]_L$ into the sublattice $[\![\eta \wedge \theta, \eta]\!]_L \leqslant [\![\eta \wedge \theta, \eta]\!]$. Moreover, it's easy to see that the range of φ consists of elements of L that permute with θ, so that φ maps into $[\![\eta \wedge \theta, \eta]\!]_L^\theta$. Indeed, if $\alpha \in [\![\theta, \eta \vee \theta]\!]_L$, then by Lemma 2 we have $(\alpha \wedge \eta) \circ \theta = \alpha \cap (\eta \circ \theta) = \alpha \cap (\theta \circ \eta) = \theta \circ (\alpha \wedge \eta)$.

Next, consider the mapping $\psi : [\![\eta \wedge \theta, \eta]\!]_L^\theta \to [\![\theta, \eta \vee \theta]\!]$ given by $\psi(\alpha) = \alpha \circ \theta$. Note that $\psi(\alpha) = \alpha \circ \theta = \alpha \vee \theta$, an element of L, since the domain of ψ is a set of relations in L that permute with θ. We show that the two maps

$$\varphi : [\![\theta, \eta \vee \theta]\!]_L \ni \alpha \longmapsto \alpha \wedge \eta \in [\![\eta \wedge \theta, \eta]\!]_L^\theta \tag{3}$$

$$\psi : [\![\eta \wedge \theta, \eta]\!]_L^\theta \ni \alpha \longmapsto \alpha \circ \theta \in [\![\theta, \eta \vee \theta]\!]_L. \tag{4}$$

are inverse lattice isomorphisms. It is clear that these maps are order preserving. Also, for $\alpha \in [\![\theta, \eta \vee \theta]\!]_L$ we have, by Lemma 2, $\psi\,\varphi(\alpha) = (\alpha \wedge \eta) \circ \theta = \alpha \cap (\eta \circ \theta) = \alpha \cap (\eta \vee \theta) = \alpha$. For $\alpha \in [\![\eta \wedge \theta, \eta]\!]_L^\theta$, we have, by Lemma 2, $\varphi\,\psi(\alpha) = \varphi(\alpha \circ \theta) = (\alpha \circ \theta) \wedge \eta = \alpha \circ (\theta \wedge \eta)$.

To complete the proof of Lemma 1, we show that $[\![\eta \wedge \theta, \eta]\!]_L^\theta$ is a sublattice of $[\![\eta \wedge \theta, \eta]\!]_L$. Fix $\alpha, \beta \in [\![\eta \wedge \theta, \eta]\!]_L^\theta$. We show

$$(\alpha \vee \beta) \circ \theta \subseteq \theta \circ (\alpha \vee \beta), \tag{5}$$

and

$$(\alpha \wedge \beta) \circ \theta \subseteq \theta \circ (\alpha \wedge \beta). \tag{6}$$

The reverse inclusions follow by symmetric arguments.

Fix $(x, y) \in (\alpha \vee \beta) \circ \theta$. Then there exist $c \in X$ and $n < \omega$ such that $x \,(\alpha \circ^{(n)} \beta)\, c \,\theta\, y$. Thus, $(x, y) \in \alpha \circ^{(n)} \beta \circ \theta$. Since θ permutes with both α and β, we have $(x, y) \in \theta \circ \alpha \circ^{(n)} \beta \subseteq \theta \circ (\alpha \vee \beta)$, which proves (5). Fix $(x, y) \in (\alpha \wedge \beta) \circ \theta$. Then $(x, y) \in (\alpha \circ \theta) \cap (\beta \circ \theta) = (\theta \circ \alpha) \cap (\theta \circ \beta)$. Therefore, there exist d_1, d_2 such that $x \,\theta\, d_1 \,\alpha\, y$ and $x \,\theta\, d_2 \,\beta\, y$. Note that $(d_1, y) \in \alpha \leqslant \eta$ and $(d_2, y) \in \beta \leqslant \eta$, so $(d_1, d_2) \in \eta$. Also, $d_1 \,\theta\, x \,\theta\, d_2$, so $(d_1, d_2) \in \theta$. Therefore, $(d_1, d_2) \in \eta \wedge \theta \leqslant \alpha \wedge \beta$. In particular, $d_1 \,\beta\, d_2 \,\beta\, y$, so $(d_1, y) \in \alpha \wedge \beta$. Thus, $x \,\theta\, d_1 \,(\alpha \wedge \beta)\, y$, which proves (6).

$\square$

References

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