

DEDEKIND'S TRANSPOSITION PRINCIPAL FOR LATTICES OF EQUIVALENCE RELATIONS

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ABSTRACT. We prove a version of Dedekind's Transposition Principal that holds in lattices of equivalence relations.

In this note we prove a version of Dedekind's Transposition Principal¹ that holds in all (not necessarily modular) lattices of equivalence relations. Let X be a set and let $\text{Eq } X$ denote the lattice of equivalence relations on X . Given $\alpha, \beta \in \text{Eq } X$ and $n \in \mathbb{N}$, let $\alpha \circ^n \beta$ denote the relation $\alpha \circ \beta \circ \alpha \circ \dots \circ \beta \circ \alpha$ in which the symbol \circ appears $n-1$ times. The relations α and β *n-permute* provided $\alpha \circ^n \beta = \beta \circ^n \alpha$. We define the *interval sublattice of equivalence relations above α and below β* as follows:

$$[\alpha, \beta] := \{\gamma \in \text{Eq } X \mid \alpha \leq \gamma \leq \beta\}.$$

Let L be a sublattice of $\text{Eq } X$. Given $\alpha, \beta \in L$, let $[\alpha, \beta]_L := [\alpha, \beta] \cap L$, which we call an *interval sublattice of L* , or more simply, an *interval of L* . Given $\alpha, \beta, \theta \in L$, let $[\alpha, \beta]_L^{\theta(n)}$ denote the set of equivalence relations in the interval $[\alpha, \beta]_L$ that n -permute with θ . That is,

$$[\alpha, \beta]_L^{\theta(n)} := \{\gamma \in L \mid \alpha \leq \gamma \leq \beta \text{ and } \gamma \circ^n \theta = \theta \circ^n \gamma\}.$$

Lemma 1. *If $\eta, \theta \in L \leq \text{Eq } X$ and $\gamma \circ^n \theta = \theta \circ^n \gamma$ for some $n \in \mathbb{N}$, then*

$$[\theta, \eta \vee \theta]_L \cong [\eta \wedge \theta, \eta]_L^{\theta(n)} \leq [\eta \wedge \theta, \eta]_L.$$

The lemma states that if η and θ n -permute, then the sublattice $[\theta, \eta \vee \theta]_L$ is isomorphic to the lattice, $[\eta \wedge \theta, \eta]_L^{\theta(n)}$ of relations in L that are below η , above $\eta \wedge \theta$, and n -permute with θ ; moreover, $[\eta \wedge \theta, \eta]_L^{\theta(n)}$ is a sublattice of $[\eta \wedge \theta, \eta]_L$.

To prove this, we need the following generalized version of *Dedekind's Rule*:²

Lemma 2. *If $\alpha, \beta, \gamma \in L \leq \text{Eq } X$, and if $\alpha \leq \beta$, then we have the following identities of subsets of X^2 :*

$$\alpha \circ^n (\beta \cap \gamma) = \beta \cap (\alpha \circ^n \gamma), \tag{1}$$

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¹If L is a modular lattice, then for any two elements $a, b \in L$ the intervals $[b, a \vee b]$ and $[a \wedge b, a]$ are isomorphic. See [1], or [2, page 57].

²In the group theory setting, the well known Dedekind's Rule states that if A, B, C are subgroups of a group, and $A \leq B$, then we have the following identity of sets: $A(B \cap C) = B \cap AC$.

$$(\beta \cap \gamma) \circ^n \alpha = \beta \cap (\gamma \circ^n \alpha). \quad (2)$$

Proof. It is obvious that $\alpha \circ^n (\beta \cap \gamma) \subseteq \beta \cap (\alpha \circ^n \gamma)$ and $(\beta \cap \gamma) \circ^n \alpha \subseteq \beta \cap (\gamma \circ^n \alpha)$. We will prove (1) and (2) by establishing the reverse inclusions simultaneously by induction. In case $n = 1$, the reverse inclusions are easy to prove. For example, if $(x, y) \in \beta \cap (\alpha \circ \gamma)$, then since $(x, y) \in \alpha \circ \gamma$, there exists $c \in X$ such that $x \alpha c \gamma y$. To produce $d \in X$ such that $x \alpha d (\beta \cap \gamma) y$, simply note that $d = c$ works, since $(x, c) \in \alpha \leq \beta$ implies $c \beta x \beta y$, so $(c, y) \in \beta \cap \gamma$.

For the inductive step, we assume the following identities hold:

$$\alpha \circ^{n-2} (\beta \cap \gamma) = \beta \cap (\alpha \circ^{n-2} \gamma), \quad (3)$$

$$(\beta \cap \gamma) \circ^{n-2} \alpha = \beta \cap (\gamma \circ^{n-2} \alpha). \quad (4)$$

Fix $(x, y) \in \beta \cap (\alpha \circ^n \gamma)$. Since $(x, y) \in \alpha \circ^n \gamma$, there exist $c_1, \dots, c_{n-1} \in X$ such that $x \alpha c_1 \gamma c_2 \alpha \dots \gamma c_{n-1} \alpha y$. We must produce $d_1, \dots, d_{n-1} \in X$ such that $x \alpha d_1 (\beta \cap \gamma) d_2 \alpha \dots (\beta \cap \gamma) d_{n-1} \alpha y$. In fact, we will see that $d_i = c_i$ works. Note that $c_1 \alpha x \beta y \alpha c_{n-1}$. Thus, $(c_1, c_{n-1}) \in \alpha \circ \beta \circ \alpha = \beta$, since $\alpha \leq \beta$. Therefore, $(c_1, c_{n-1}) \in \beta \cap (\gamma \circ^{n-2} \alpha)$, and by the inductive hypothesis we have $(c_1, c_{n-1}) \in (\beta \cap \gamma) \circ^{n-2} \alpha$.

THIS LEMMA MIGHT ACTUALLY BE FALSE. THE PROOF STILL NEEDS TO BE COMPLETED AND CHECKED. \square

Proof of Lemma 1. Let $\eta, \theta \in L \leq \text{Eq } X$ be permuting equivalence relations in L , so $\eta \circ \theta = \theta \circ \eta = \eta \vee \theta$. Consider the mapping $\varphi : [\![\theta, \eta \vee \theta]\!]_L \rightarrow [\![\eta \wedge \theta, \eta]\!]_L$ given by $\alpha \mapsto \alpha \wedge \eta$. Clearly φ maps $[\![\theta, \eta \vee \theta]\!]_L$ into the sublattice $[\![\eta \wedge \theta, \eta]\!]_L \leq [\![\eta \wedge \theta, \eta]\!]_L$. Moreover, it's easy to see that the range of φ consists of elements of L that permute with θ , so that φ maps into $[\![\eta \wedge \theta, \eta]\!]_L^\theta$. Indeed, if $\alpha \in [\![\theta, \eta \vee \theta]\!]_L$, then by Lemma 2 we have $(\alpha \wedge \eta) \circ \theta = \alpha \cap (\eta \circ \theta) = \alpha \cap (\theta \circ \eta) = \theta \circ (\alpha \wedge \eta)$.

Next, consider the mapping $\psi : [\![\eta \wedge \theta, \eta]\!]_L^\theta \rightarrow [\![\theta, \eta \vee \theta]\!]_L$ given by $\psi(\alpha) = \alpha \circ \theta$. Note that $\psi(\alpha) = \alpha \circ \theta = \alpha \vee \theta$, an element of L , since the domain of ψ is a set of relations in L that permute with θ . We show that the two maps

$$\varphi : [\![\theta, \eta \vee \theta]\!]_L \ni \alpha \mapsto \alpha \wedge \eta \in [\![\eta \wedge \theta, \eta]\!]_L^\theta \quad (5)$$

$$\psi : [\![\eta \wedge \theta, \eta]\!]_L^\theta \ni \alpha \mapsto \alpha \circ \theta \in [\![\theta, \eta \vee \theta]\!]_L. \quad (6)$$

are inverse lattice isomorphisms. It is clear that these maps are order preserving. Also, for $\alpha \in [\![\theta, \eta \vee \theta]\!]_L$ we have, by Lemma 2, $\psi \varphi(\alpha) = (\alpha \wedge \eta) \circ \theta = \alpha \cap (\eta \circ \theta) = \alpha \cap (\eta \vee \theta) = \alpha$. For $\alpha \in [\![\eta \wedge \theta, \eta]\!]_L^\theta$, we have, by Lemma 2, $\varphi \psi(\alpha) = \varphi(\alpha \circ \theta) = (\alpha \circ \theta) \wedge \eta = \alpha \circ (\theta \wedge \eta)$.

To complete the proof of Lemma 1, we show that $[\![\eta \wedge \theta, \eta]\!]_L^\theta$ is a sublattice of $[\![\eta \wedge \theta, \eta]\!]_L$. Fix $\alpha, \beta \in [\![\eta \wedge \theta, \eta]\!]_L^\theta$. We show

$$(\alpha \vee \beta) \circ \theta \subseteq \theta \circ (\alpha \vee \beta), \quad (7)$$

and

$$(\alpha \wedge \beta) \circ \theta \subseteq \theta \circ (\alpha \wedge \beta). \quad (8)$$

The reverse inclusions follow by symmetric arguments.

Fix $(x, y) \in (\alpha \vee \beta) \circ \theta$. Then there exist $c \in X$ and $n < \omega$ such that $x (\alpha \circ^n \beta) c \theta y$. Thus, $(x, y) \in \alpha \circ^n \beta \circ \theta$. Since θ permutes with both α and β , we have $(x, y) \in \theta \circ \alpha \circ^n \beta \subseteq \theta \circ (\alpha \vee \beta)$, which proves (7). Fix $(x, y) \in (\alpha \wedge \beta) \circ \theta$. Then $(x, y) \in (\alpha \circ \theta) \cap (\beta \circ \theta) = (\theta \circ \alpha) \cap (\theta \circ \beta)$. Therefore, there exist d_1, d_2 such that $x \theta d_1 \alpha y$ and $x \theta d_2 \beta y$. Note that $(d_1, y) \in \alpha \leq \eta$ and $(d_2, y) \in \beta \leq \eta$, so $(d_1, d_2) \in \eta$. Also, $d_1 \theta x \theta d_2$, so $(d_1, d_2) \in \theta$. Therefore, $(d_1, d_2) \in \eta \wedge \theta \leq \alpha \wedge \beta$. In particular, $d_1 \beta d_2 \beta y$, so $(d_1, y) \in \alpha \wedge \beta$. Thus, $x \theta d_1 (\alpha \wedge \beta) y$, which proves (8). □

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