PERMUTING EQUIVALENCE RELATIONS AND PERMUTING SUBGROUPS

WILLIAM DEMEO

1. Permuting Equivalence Relations

Let X be a set and let $\mathbf{Eq(X)} = \langle \operatorname{Eq} X, \wedge, \vee \rangle$ denote the lattice of equivalence relations on X. Given $\alpha, \beta \in \operatorname{Eq} X$, the *meet* of α and β is the usual intersection of subsets: $\alpha \wedge \beta = \alpha \cap \beta$. The *join* of α and β, denoted $\alpha \vee \beta$, is the smallest equivalence relation containing both α and β.

Given $\alpha, \beta \in \operatorname{Eq} X$, define the *composition* of α and β to be the binary relation

$$\alpha \circ \beta = \{(x, y) \in X^2 \mid (\exists z \in X)\, x \;\alpha\; z \;\beta\; y\}.$$

Note that $\circ$ is associative: $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$.

Let $\alpha \circ^1 \beta = \alpha$ and, for each $n \in \mathbb{N}$, define the n-fold composition of α and β to be the binary relation

$$\alpha \circ^n \beta = \alpha \circ \beta \circ^{n-1} \alpha.$$

For example, $\alpha \circ^1 \beta = \alpha$; $\alpha \circ^2 \beta = \alpha \circ \beta$; $\alpha \circ^3 \beta = \alpha \circ \beta \circ \alpha$; etc.

For $\alpha, \beta \in \operatorname{Eq} X$, it is clear that both α and β are contained in $\alpha \circ \beta$, which is contained in $\alpha \vee \beta$. In general, $\circ$ is not a binary operation on the set $\operatorname{Eq} X$, since $\alpha \circ \beta$ is not always an equivalence relation; it may be strictly contained in $\alpha \vee \beta$. In fact, we have

$$\alpha \vee \beta = \bigcup_{k \in \mathbb{N}} \alpha \circ^k \beta.$$

Suppose X is a set and $\alpha, \beta \in \operatorname{Eq} X$. It is easy to check that the following facts hold for all positive integers i, j, and n:

Fact 1. If $i < j$, then $\alpha \circ^i \beta \subseteq \beta \circ^j \alpha$.

Fact 2. If $\alpha \circ^n \beta = \beta \circ^n \alpha$, then $\alpha \circ^{n+1} \beta = \alpha \circ^n \beta = \beta \circ^{n+1} \alpha$.

Fact 3. If $\alpha \circ^n \beta = \beta \circ^n \alpha$, then $\alpha \circ^n \beta = \alpha \vee \beta$.

The relations α and β are said to be *permuting*, or *permutable*, provided $\alpha \circ \beta = \beta \circ \alpha$; they are *$n$-permuting*, or *$n$-permutable*, if $\alpha \circ^n \beta = \beta \circ^n \alpha$.

The converse of Fact 3 is not true in general. (For instance, when $n = 3$ it is easy to find examples where $\alpha \circ \beta \circ \alpha = \alpha \vee \beta$ and yet $\beta \circ \alpha \circ \beta$ is strictly

Date: May 20, 2013.

contained in $\alpha \circ \beta \circ \alpha$.) However, for all *even* $n > 1$ (and trivially for $n = 1$) the converse of Fact 3 holds, and we have the following lemma.

Lemma 1. If n is a positive even integer, then the following are equivalent:

(i) $\alpha \circ^n \beta = \alpha \vee \beta$
(ii) $\alpha \circ^n \beta = \beta \circ^n \alpha$
(iii) $\alpha \circ^n \beta \subseteq \beta \circ^n \alpha$

Proof. (i) $\Rightarrow$ (ii): If (i) holds, then $\beta \circ^n \alpha \subseteq \alpha \circ^n \beta$. Suppose $(x, y) \in \alpha \circ^n \beta$. (We show $(x, y) \in \beta \circ^n \alpha$.) Then, there is a sequence $c_1, \dots, c_{n-1}$ such that $x \, \alpha \, c_1 \, \beta \, c_2 \, \alpha \, c_3 \cdots c_{n-1} \, \beta \, y$. Using the reversed sequence, $c_{n-1}, \dots, c_1$, we have $(y, x) \in \beta \circ^n \alpha \subseteq \alpha \vee \beta$. (Note: this depends on the assumption that n is even.) Thus $(y, x) \in \alpha \vee \beta = \alpha \circ^n \beta$, so there is a sequence $d_1, \dots, d_{n-1}$ such that $y \, \alpha \, d_1 \, \beta \, d_2 \, \alpha \, d_3 \cdots d_{n-1} \, \beta \, x$. Again, because n is even, we can use the reversed sequence to arrive at $(x, y) \in \beta \circ^n \alpha$, as desired.

(ii) $\Rightarrow$ (iii): obvious.

(iii) $\Rightarrow$ (i): If (iii) holds, then it is not hard to check that $\beta \circ^n \alpha$ is an equivalence relation. Therefore, $\beta \circ^n \alpha = \alpha \vee \beta$. Now, by symmetry, the argument used above to prove (i) $\Rightarrow$ (ii) can be applied to show that (ii) holds, so we have $\alpha \circ^n \beta = \beta \circ^n \alpha = \alpha \vee \beta$.

□

2. Permuting Subgroups

Let G be a group, and let $H, K \leqslant G$; i.e., H and K are subgroups of G. Define the *composition* of H and K to be the set[1]

$$H \circ K = \{hk \mid h \in H, \, k \in K\}, \tag{1}$$

where juxtaposition, hk, denotes group multiplication. Let $H \circ^1 K = H$ and, for each $n \in \mathbb{N}$, define

$$H \circ^n K = H \circ K \circ^{n-1} H. \tag{2}$$

Note that the sets defined in (1) and (2) are not necessarily groups, but it is easy to very that $H \circ K$ is a group if and only if $H \circ K = K \circ H$.

The subgroups H and K are said to *permute* (to be *permuting*, or to be *permutable*) provided $H \circ K = K \circ H$; they are *n-permuting*, or *n-permutable*, provided $H \circ^n K = K \circ^n H$.

Recall that if G is a group and if H is a (possibly trivial) subgroup of G, then the transitive G-set $\mathbf{A} = \langle G/H, \bar{G} \rangle$ is the algebra with universe $G/H =$ the left cosets of H in G, and basic operations $\bar{G} = \{g^{\mathbf{A}} : g \in G\}$, where for each $xH \in G/H$, we have

$$g^{\mathbf{A}}(xH) = (gx)H.$$

[1]The notation $H \circ K$ is non-standard. This set is usually called HK, but the $\circ$ notation works well in the context of these notes.

In other words, each $g \in G$ acts on the set of left cosets of H by left multiplication. A standard result about the algebra $\mathbf{A}$ is that its congruence lattice, $\operatorname{Con}\mathbf{A}$, is isomorphic to an interval in the subgroup lattice of G (see [1, Lemma 4.12]). More precisely,

$$\operatorname{Con}\mathbf{A} \cong [\![H, G]\!] := \{K \mid H \leqslant K \leqslant G\}.$$

The isomorphism $[\![H, G]\!] \ni K \mapsto \theta_K \in \operatorname{Con}\mathbf{A}$ is given by

$$\theta_K = \{(xH, yH) \in G/H \times G/H \mid y^{-1}x \in K\}$$

and the inverse isomorphism $\operatorname{Con}\mathbf{A} \ni \theta \mapsto K_\theta \in [\![H, G]\!]$ is given by

$$K_\theta = \{g \in G \mid (H, gH) \in \theta\}.$$

It follows that every lattice property that is true of all congruence lattices must also be true of all (intervals of) subgroup lattices. Moreover, the following is easy to prove:

Lemma 2. In $\operatorname{Con}\langle G/H, \bar{G}\rangle$, two congruences θ_{K_1} and θ_{K_2} permute if and only if the corresponding subgroups K_1 and K_2 permute.

Lemma 2 is well known (e.g., it is stated without proof in [2, Lemma 1]), and the following extension is not much harder to prove:

Lemma 3. In $\operatorname{Con}\langle G/H, \bar{G}\rangle$, two congruences θ_{K_1} and θ_{K_2} n-permute if and only if the corresponding subgroups K_1 and K_2 n-permute.

Proof. (coming soon) □

By the correspondence between subgroups and congruences of G-sets described above, and by Lemma 3, Facts 1–3 above imply that for all subgroups H, K of a group G, and for all positive integers i, j, and n, we have

Fact 1'. If $i < j$, then $H \circ^i K \subseteq K \circ^j H$.

Fact 2'. If $H \circ^n K = K \circ^n H$, then $H \circ^{n+1} K = H \circ^n K = K \circ^{n+1} H$.

Fact 3'. If $H \circ^n K = K \circ^n H$, then $H \circ^n K = H \vee K$.

Moreover, we have the following subgroup version of Lemma 1:

Lemma 1'. If n is a positive even integer, then the following are equivalent:

(i) $H \circ^n K = H \vee K$
(ii) $H \circ^n K = K \circ^n H$
(iii) $H \circ^n K \subseteq K \circ^n H$

References

[1] Ralph N. McKenzie, George F. McNulty, and Walter F. Taylor. *Algebras, lattices, varieties. Vol. I.* Wadsworth & Brooks/Cole, Monterey, CA, 1987.

[2] P. P. Pálfy and J. Saxl. Congruence lattices of finite algebras and factorizations of groups. *Comm. Algebra*, 18(9):2783–2790, 1990.

E-mail address: `williamdemeo@gmail.com`
URL: `http://williamdemeo.wordpress.com`

Department of Mathematics, University of South Carolina, Columbia 29208, USA