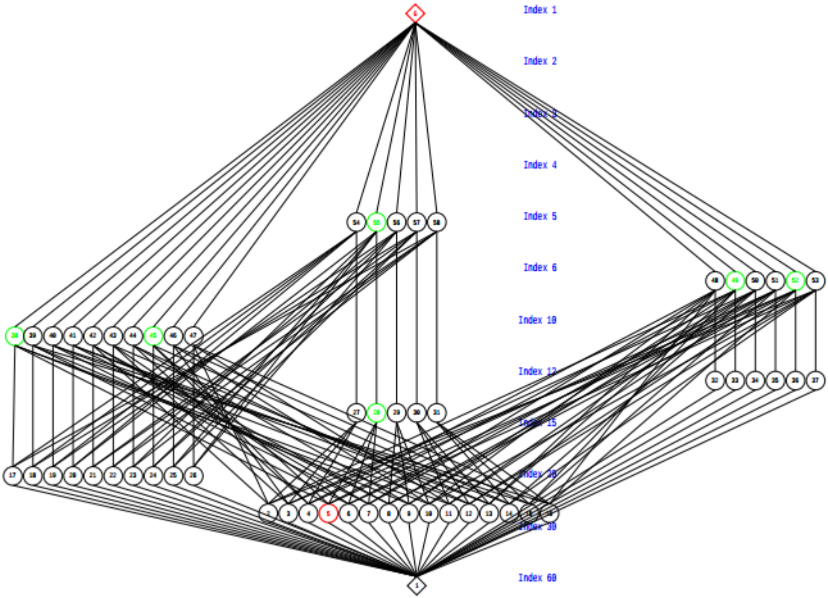
Intervals in Subgroup Lattices of Finite Groups

William DeMeo
williamdemeo@gmail.com
University of South Carolina

Western Carolina University

Mathematics and Computer Science Colloquium

November 1, 2012


Index 1
Index 2
Index 3
Index 4
Index 5
Index 6
Index 10
Index 12
Index 15
Index 20
Index 30
Index 60

Historically, much work has focused on:

- inferring properties of a group G from the structure of its lattice of subgroups $\mathrm{Sub}(G)$;
- inferring lattice theoretical properties of $\mathrm{Sub}(G)$ from properties of G.

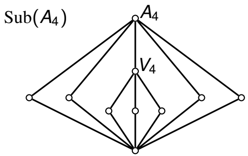
For some groups, $\mathrm{Sub}(G)$ determines G up to isomorphism.

Examples

The Klein 4-group, V_4.

The alternating groups, A_n ($n \geqslant 4$).

Every finite nonabelian simple group.

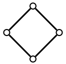


For other groups, $\mathrm{Sub}(G)$ is isomorphic to the subgroup lattices of all groups in an infinite class of nonisomorphic groups.

Examples

$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of prime order.

$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of order p^2.

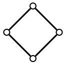
$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of order pq.

For other groups, $\mathrm{Sub}(G)$ is isomorphic to the subgroup lattices of all groups in an infinite class of nonisomorphic groups.

EXAMPLES

$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of prime order.

$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of order p^2.

$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of order pq.

At the other extreme, there are finite lattices that are not subgroup lattices.

We are interested in the local structure of subgroup lattices, that is, the possible ***intervals***

$$[H, K] := \{X \mid H \leqslant X \leqslant K\} \leqslant \mathrm{Sub}(G)$$

where $H \leqslant K \leqslant G$.

We restrict our attention to ***upper intervals***, where $K = G$, and ask two questions:

1. *What intervals $[H, G]$ are possible?*
2. *What properties of a group G can be inferred from the shape of an upper interval in Sub(G)?*

1. What intervals $[H, G]$ are possible?

There is a remarkable theorem relating this question to the *finite lattice representation problem* (FLRP).

Theorem (Pálfy and Pudlák(1980))

The following statements are equivalent:

(A) *Every finite lattice is isomorphic to the congruence lattice of a finite algebra.*

(B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

Subgroup Lattice Basics

Let U and H be subgroups of a finite group.

- By UH we mean the *set* $\{uh \mid u \in U, h \in H\}$.

Subgroup Lattice Basics

Let U and H be subgroups of a finite group.

- By UH we mean the *set* $\{uh \mid u \in U, h \in H\}$.
- $U \vee H = \langle U, H \rangle$ means the group generated by U and H.

Subgroup Lattice Basics

Let U and H be subgroups of a finite group.

- By UH we mean the *set* $\{uh \mid u \in U, h \in H\}$.
- $U \vee H = \langle U, H \rangle$ means the group generated by U and H.
- $UH \subseteq \langle U, H \rangle$ and equality holds iff U and H permute:

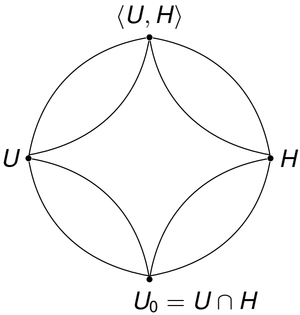
$$UH = \langle U, H \rangle \quad \Leftrightarrow \quad UH = HU.$$

Subgroup Lattice Basics

Let U and H be subgroups of a finite group.

- By UH we mean the *set* $\{uh \mid u \in U, h \in H\}$.
- $U \vee H = \langle U, H \rangle$ means the group generated by U and H.
- $UH \subseteq \langle U, H \rangle$ and equality holds iff U and H permute:

$$UH = \langle U, H \rangle \quad \Leftrightarrow \quad UH = HU.$$

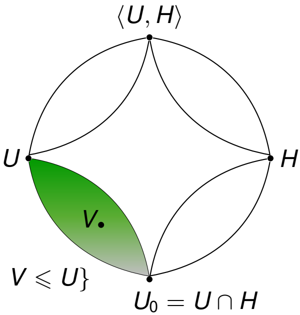


Subgroup Lattice Basics

Let U and H be subgroups of a finite group.

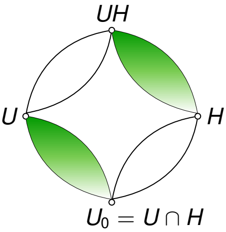
- By UH we mean the *set* $\{uh \mid u \in U, h \in H\}$.
- $U \vee H = \langle U, H \rangle$ means the group generated by U and H.
- $UH \subseteq \langle U, H \rangle$ and equality holds iff U and H permute:

$$UH = \langle U, H \rangle \quad \Leftrightarrow \quad UH = HU.$$



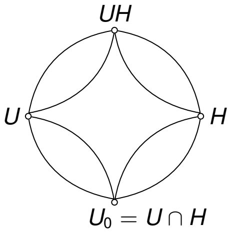
$[U_0, U] := \{V \mid U_0 \leqslant V \leqslant U\}$

Interval Isomorphisms



- If $H \trianglelefteq \langle U, H \rangle$, then $UH = \langle U, H \rangle$ and $[U_0, U] \cong [H, UH]$.

Interval Isomorphisms

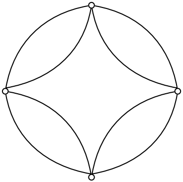


- If $H \trianglelefteq \langle U, H\rangle$, then $UH = \langle U, H\rangle$ and $[U_0, U] \cong [H, UH]$.
- Instead of $H \trianglelefteq \langle U, H\rangle$, assume only $UH = \langle U, H\rangle$ and define

$$[U_0, U]^H := \{V \in [U_0, U] \mid VH = HV\},$$

the *H*-permuting subgroups.

Interval Isomorphisms



- If $H \trianglelefteq \langle U, H \rangle$, then $UH = \langle U, H \rangle$ and $[U_0, U] \cong [H, UH]$.
- Instead of $H \trianglelefteq \langle U, H \rangle$, assume only $UH = \langle U, H \rangle$ and define

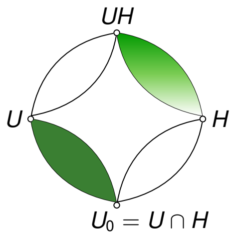
  $$[U_0, U]^H := \{V \in [U_0, U] \mid VH = HV\},$$

  the *H*-permuting subgroups.
- If $U \trianglelefteq UH$, define

  $$[U_0, U]_H := \{V \in [U_0, U] \mid H \leqslant N_{UH}(V)\},$$

  the *H*-invariant subgroups: $V^h = V$ $(\forall h \in H)$.

Interval Isomorphisms



- If $H \trianglelefteq \langle U, H\rangle$, then $UH = \langle U, H\rangle$ and $[U_0, U] \cong [H, UH]$.
- Instead of $H \trianglelefteq \langle U, H\rangle$, assume only $UH = \langle U, H\rangle$ and define

  $$[U_0, U]^H := \{V \in [U_0, U] \mid VH = HV\},$$

  the H-permuting subgroups.
- If $U \trianglelefteq UH$, define

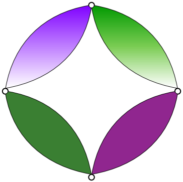
  $$[U_0, U]_H := \{V \in [U_0, U] \mid H \leqslant N_{UH}(V)\},$$

  the H-invariant subgroups: $V^h = V$ $(\forall h \in H)$.

Lemma

1. $[H, UH] \cong [U_0, U]^H \leqslant [U_0, U]$
2. *If* $U \trianglelefteq UH$, *then* $[U_0, U]_H = [U_0, U]^H \leqslant [U_0, U]$.
3. *If* $H \trianglelefteq UH$, *then* $[U_0, U]_H = [U_0, U]^H = [U_0, U]$.

Interval Isomorphisms



- If $H \trianglelefteq \langle U, H\rangle$, then $UH = \langle U, H\rangle$ and $[U_0, U] \cong [H, UH]$.
- Instead of $H \trianglelefteq \langle U, H\rangle$, assume only $UH = \langle U, H\rangle$ and define

  $$[U_0, U]^H := \{V \in [U_0, U] \mid VH = HV\},$$

  the *H*-permuting subgroups.
- If $U \trianglelefteq UH$, define

  $$[U_0, U]_H := \{V \in [U_0, U] \mid H \leqslant N_{UH}(V)\},$$

  the *H*-invariant subgroups: $V^h = V \; (\forall h \in H)$.

Lemma

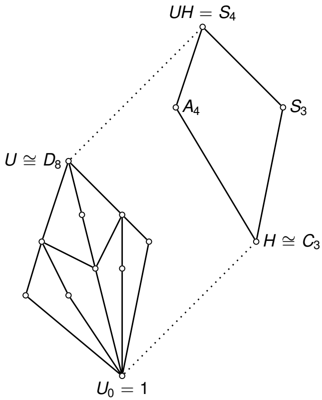
1. $[H, UH] \cong [U_0, U]^H \leqslant [U_0, U]$
2. *If* $U \trianglelefteq UH$, *then* $[U_0, U]_H = [U_0, U]^H \leqslant [U_0, U]$.
3. *If* $H \trianglelefteq UH$, *then* $[U_0, U]_H = [U_0, U]^H = [U_0, U]$.

EXAMPLE

- The group S_4 has subgroups $U \cong D_8$ and $H \cong C_3$ that permute but neither one normalizes the other.

EXAMPLE

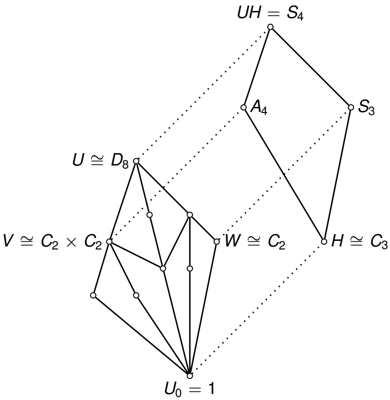
- The group S_4 has subgroups $U \cong D_8$ and $H \cong C_3$ that permute but neither one normalizes the other.



- Only four subgroups of U permute with H

EXAMPLE

- The group S_4 has subgroups $U \cong D_8$ and $H \cong C_3$ that permute but neither one normalizes the other.



- Only four subgroups of U permute with H, including

$$U \cap A_4 \cong C_2 \times C_2, \qquad U \cap S_3 \cong C_2.$$

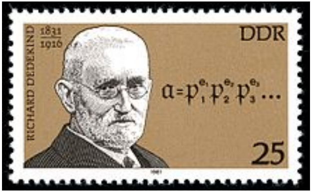
Proof of the Interval Isomorphism Lemma

Theorem (Dedekind's Rule)

Let A, B, C be subgroups of G with $A \leqslant B$. Then,

$$A(B \cap C) = B \cap AC \quad \textit{and} \quad (B \cap C)A = B \cap CA.$$

In other words, no pentagons.



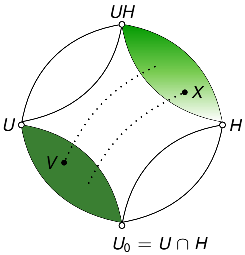
Proof of Interval Isomorphism Lemma

Claim (1)

$[H, UH] \cong [U_0, U]^H$ via

$\varphi : [H, UH] \ni X \mapsto U \cap X \in [U_0, U]^H$

$\psi : [U_0, U]^H \ni V \mapsto VH \in [H, UH].$

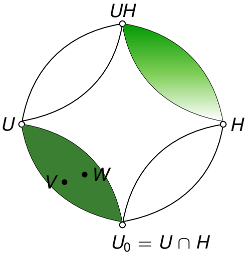


Proof.

1. For $X \in [H, UH]$, check $U \cap X \in [U_0, U]^H$ by Dedekind's rule.
2. For $V \in [U_0, U]^H$, VH is a group in $[H, UH]$.
3. Check $\psi\varphi$ and $\varphi\psi$ are the identity maps.
4. Check φ and ψ are order preserving.

$\square$

Proof of Interval Isomorphism Lemma



Claim (2)

$[U_0, U]^H$ is a sublattice of $[U_0, U]$.

Proof.

Fix $V, W \in [U_0, U]^H$.

1. Check $V \vee W = \langle V, W \rangle$ permutes with H. (easy)
2. Check $V \cap W$ permutes with H.

□

2. What properties of G can be inferred from $[H, G]$?

A group theoretical property $\mathcal{P}$ (and the associated class $\mathscr{G}_{\mathcal{P}}$) is

- ***interval enforceable*** (IE) provided there exists a lattice L such that

  if $G \in \mathfrak{G}$ and $L \cong [H, G]$, then G has property $\mathcal{P}$.

- ***core-free interval enforceable*** (cf-IE) provided $\exists L$ st

  if $G \in \mathfrak{G}$, $L \cong [H, G]$, H core-free, then G has property $\mathcal{P}$.

- ***minimal interval enforceable*** (min-IE) provided $\exists L$ st
  if $G \in \mathfrak{G}$, $L \cong [H, G]$, and if G has minimal order (wrt $L \cong [H, G]$), then G has property $\mathcal{P}$.

Insolubility

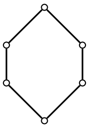
It's not hard to find examples of lattices that cannot occur as upper intervals in the subgroup lattices of finite soluble groups.

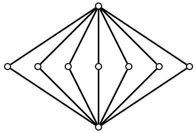
EXAMPLES

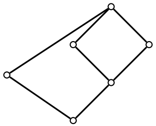
Insolubility

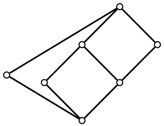
It's not hard to find examples of lattices that cannot occur as upper intervals in the subgroup lattices of finite soluble groups.

Here are a few









Has anyone seen this lattice?

login | faq | how to ask | meta

search

mathoverflow

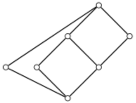
Questions | Tags | Users | Badges | Unanswered | Ask Question

Given a lattice L with n elements, are there finite groups H < G such that L ≅ the lattice of subgroups between H and G?

13

6

If there is no restriction on n, this is a famous open problem. I'm wondering if any recent work has been done for small $n > 6$. I believe the question is answered (positively) for $n = 6$ by Watatani (1996) MR1409040 and Aschbacher (2008) MR2393428. I also believe we can answer it for $n = 7$, with one possible exception. The exceptional case is shown below.



So my two questions are these:

1) Does anyone know of recent work on this special case of the problem (specifically for $n = 7$ or $n = 8$)?

2) Has anyone found a finite group G with a subgroup H such that the interval

$$[H, G] = \{K : H \leq K \leq G\}$$

is the lattice shown above?

Welcome to MathOverflow

A place for mathematicians to ask and answer questions.

tagged

finite-groups × 343

open-problem × 216

lattices × 157

universal-algebra × 55

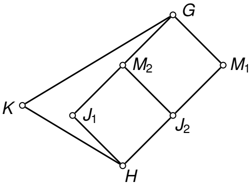
congruences × 7

asked

8 months ago

viewed

468 times

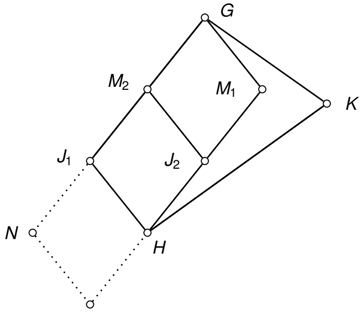


Proposition

Suppose $H < G$, $\mathrm{core}_G(H) = 1$, and $L_7 \cong [H, G]$. Then

(i) *G is a primitive permutation group.*
(ii) *If $N \lhd G$, then $C_G(N) = 1$.*
(iii) *G contains no non-trivial abelian normal subgroup.*
(iv) *G is not solvable.*
(v) *G is subdirectly irreducible.*
(vi) *With the possible exception of at most one maximal subgroup, M_1 or M_2, all proper subgroups in the interval $[H, G]$ are core-free.*

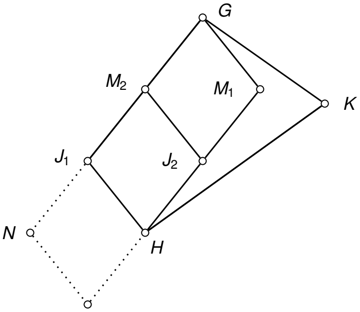
Example



Claim: J_1 and J_2 are core-free subgroups of G.

Proof:

EXAMPLE

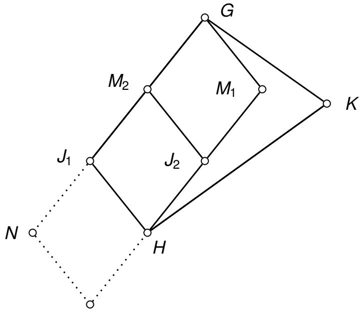


Claim: J_1 and J_2 are core-free subgroups of G.

Proof:

- If $N \lhd G$ then NH permutes with each subgroup containing H.

EXAMPLE

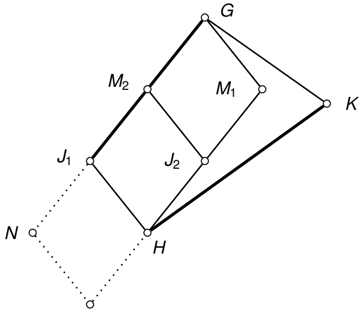


Claim: J_1 and J_2 are core-free subgroups of G.

Proof:

- If $N \lhd G$ then NH permutes with each subgroup containing H.
- If $1 \neq N \leqslant J_1$, then $NH = J_1$, so J_1 and K permute.

EXAMPLE



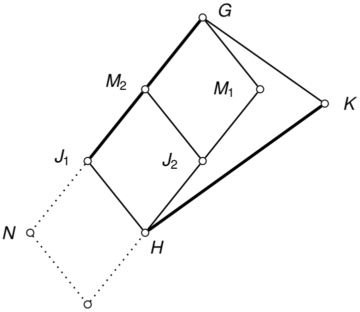
Claim: J_1 and J_2 are core-free subgroups of G.

Proof:

- If $N \lhd G$ then NH permutes with each subgroup containing H.
- If $1 \neq N \leqslant J_1$, then $NH = J_1$, so J_1 and K permute.
- Since $J_1 K = G$ and $J_1 \cap K = H$, our lemma yields

$$[J_1, G] \cong [H, K]^{J_1} = \{X \in [H, K] \mid J_1 X = X J_1\}.$$

Example



Claim: J_1 and J_2 are core-free subgroups of G.

Proof:

- If $N \lhd G$ then NH permutes with each subgroup containing H.
- If $1 \neq N \leqslant J_1$, then $NH = J_1$, so J_1 and K permute.
- Since $J_1 K = G$ and $J_1 \cap K = H$, our lemma yields

$$[J_1, G] \cong [H, K]^{J_1} = \{X \in [H, K] \mid J_1 X = X J_1\}.$$

Impossible!

The following are at least core-free interval enforceable:

- $\mathscr{G}_0 = \mathfrak{S}^c =$ the insoluble groups
- $\mathscr{G}_1 = \{G \in \mathfrak{G} \mid (\forall n < \omega)\ (G \neq A_n \text{ and } G \neq S_n)\}$
- $\mathscr{G}_2 =$ the subdirectly irreducible groups
- $\mathscr{G}_3 =$ groups with no nontrivial abelian normal subgroups
- $\mathscr{G}_4 = \{G \in \mathfrak{G} \mid C_G(M) = 1 \text{ for all } 1 \neq M \trianglelefteq G\}$.

If a lattice L is isomorphic to an interval in the subgroup lattice of a finite group, then we call L ***group representable***.

By the Pálfy-Pudlák Theorem, the FLRP has a negative answer if we can find a finite lattice that is not group representable.

If a lattice L is isomorphic to an interval in the subgroup lattice of a finite group, then we call L ***group representable***.

By the Pálfy-Pudlák Theorem, the FLRP has a negative answer if we can find a finite lattice that is not group representable.

Suppose there exists property $\mathcal{P}$ such that both $\mathcal{P}$ and its negation $\neg\mathcal{P}$ are interval enforceable by the lattices L and L_c, respectively:

$$L \cong [H, G] \implies G \text{ has property } \mathcal{P}$$

$$L_c \cong [H_c, G_c] \implies G_c \text{ does not have property } \mathcal{P}$$

If a lattice L is isomorphic to an interval in the subgroup lattice of a finite group, then we call L ***group representable***.

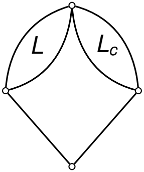
By the Pálfy-Pudlák Theorem, the FLRP has a negative answer if we can find a finite lattice that is not group representable.

Suppose there exists property $\mathcal{P}$ such that both $\mathcal{P}$ and its negation $\neg\mathcal{P}$ are interval enforceable by the lattices L and L_c, respectively:

$$L \cong [H, G] \implies G \text{ has property } \mathcal{P}$$

$$L_c \cong [H_c, G_c] \implies G_c \text{ does not have property } \mathcal{P}$$

Then the lattice



wouldn't be group representable.

As the next result shows, however, if a group property and its negation are interval enforceable by L and L_c, then already at least one of these lattices is not group representable.

LEMMA

If $\mathcal{P}$ is a group property that is interval enforceable by a group representable lattice, then $\neg\mathcal{P}$ is not interval enforceable by a group representable lattice.

Insolubility is interval enforceable, but solubility is not.

For if $L \cong [H, G]$, then for any insoluble group K we have $L \cong [H \times K, G \times K]$, and $G \times K$ is insoluble.

Note that the group $H \times K$ at the bottom of the interval is not core-free. So a more interesting question is whether a property and its negation could both be *core-free* IE.

Conjecture

If $\mathcal{P}$ is core-free interval enforceable by a group representable lattice, then $\neg\mathcal{P}$ is not core-free interval enforceable by a group representable lattice.

The following lemma shows that any class of groups that omits certain wreath products cannot be core-free interval enforceable by a group representable lattice.

Lemma

Suppose $\mathcal{P}$ is core-free interval enforceable by a group representable lattice. Then, for any finite nonabelian simple group S, there exists a wreath product group of the form $W = S \wr U$ that has property $\mathcal{P}$.

Corollary

Solubility is not core-free interval enforceable.

Proof Sketch

Let L be a group representable lattice such that if $L \cong [H, G]$ and $\mathrm{core}_G(H) = 1$ then G has property $\mathcal{P}$.

Since L is group representable, $\exists\, G \vDash \mathcal{P}$ with $L \cong [H, G]$.

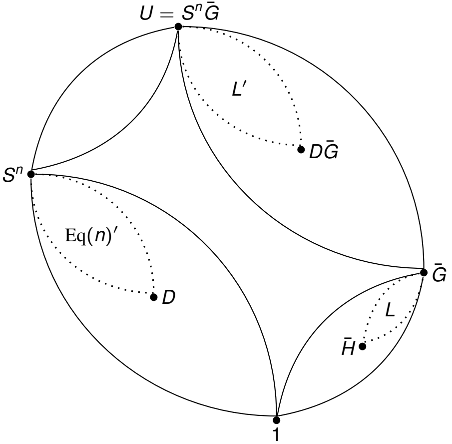
Proof Sketch

Let L be a group representable lattice such that if $L \cong [H, G]$ and $\mathrm{core}_G(H) = 1$ then G has property $\mathcal{P}$.

Since L is group representable, $\exists\, G \models \mathcal{P}$ with $L \cong [H, G]$.

We apply the idea of Hans Kurzweil twice:

- Fix a finite nonabelian simple group S.
- Suppose the index of H in G is $|G : H| = n$.
- Then the action of G on the cosets of H induces an automorphism of the group S^n by permutation of coordinates.
- Denote this by $\varphi : G \to \mathrm{Aut}(S^n)$, and let $\varphi(G) = \bar{G} \leqslant \mathrm{Aut}(S^n)$.



The interval $[D, S^n]$ is isomorphic to $\mathrm{Eq}(n)'$, the dual of the lattice of partitions of an n-element set.

The dual lattice L' is an upper interval of $\mathrm{Sub}(U)$, namely, $L' \cong [D\bar{G}, U]$.

We conclude that a class of groups that does not include wreath products of the form $S \wr G$, where S is an arbitrary finite nonabelian simple group, is not a core-free interval enforceable class. The class of soluble groups is an example.

Theorem

The following statements are equivalent:

(B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

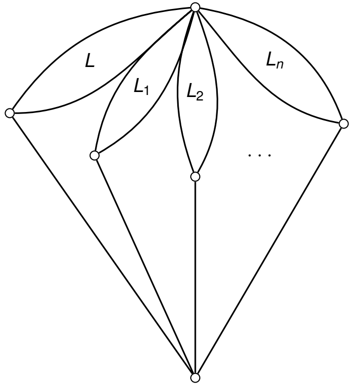
(C) *For every finite lattice L and every finite collection $\mathscr{G}_1, \ldots, \mathscr{G}_n$ of cf-IE classes of groups,*

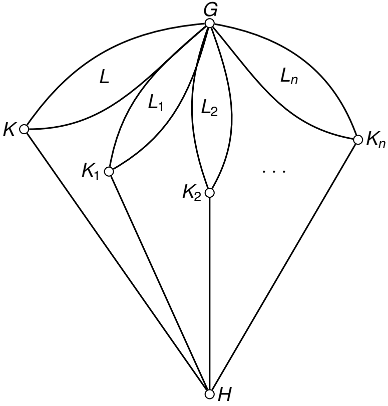
$$\exists\, G \in \bigcap_{i=1}^{n} \mathscr{G}_i \text{ such that } L \cong [H, G] \text{ and } \mathrm{core}_G(H) = 1.$$

(D) *For every finite collection $\mathscr{L}$ of finite lattices, there exists a finite group G such that each $L_i \in \mathscr{L}$ is isomorphic to $[H_i, G]$ for some core-free subgroup $H_i \leqslant G$.*

By (C), the FLRP would have a negative answer if we could find a collection $\mathscr{G}_1, \ldots, \mathscr{G}_n$ of cf-IE classes such that $\bigcap_{i=1}^{n} \mathscr{G}_i$ is empty.

By (D), it makes sense to consider finite collections of finite lattices and ask what can be proved about a group G if one assumes that all of these lattices are isomorphic to upper intervals of $\mathrm{Sub}(G)$.


L
L_1
L_2
L_n
$\cdots$


G
L
L_1
L_2
L_n
K
K_1
K_2
K_n
. . .
H


JUL 9 2003

ASCHBACHER-O'NAN-SCOTT THEOREM

Let G be a primitive permutation group of degree d, and let $N := \mathrm{Soc}(G) \cong T^m$ with $m \geqslant 1$. Then one of the following holds.

1. N is regular and
   - (Affine type) T is cyclic of order p, so $|N| = p^m$. Then $d = p^m$ and G is permutation isomorphic to a subgroup of the affine general linear group $\mathrm{AGL}(m, p)$.
   - (Twisted wreath product type) $m \geqslant 6$, the group T is nonabelian and G is a group of *twisted wreath product type*, with $d = |T|^m$.
2. N is non-regular, non-abelian, and
   - (Almost simple type) $m = 1$ and $T \leqslant G \leqslant \mathrm{Aut}(T)$.
   - (Product action type) $m \geqslant 2$ and G is permutation isomorphic to a subgroup of the product action wreath product $P \wr S_{m/l}$ of degree $d = nm/l$. The group P is primitive of type 2.(a) or 2.(c), P has degree n and $\mathrm{Soc}(P) \cong T^l$, where $l \geqslant 1$ divides m.
   - (Diagonal type) $m \geqslant 2$ and $T^m \leqslant G \leqslant T^m.(\mathrm{Out}(T) \times S_m)$, with the diagonal action. The degree $d = |T|^{m-1}$.

See Peter Cameron's blog at

`http://cameroncounts.wordpress.com/tag/onan-scott-theorem/`

for some history.