Intervals in Subgroup Lattices of Finite Groups

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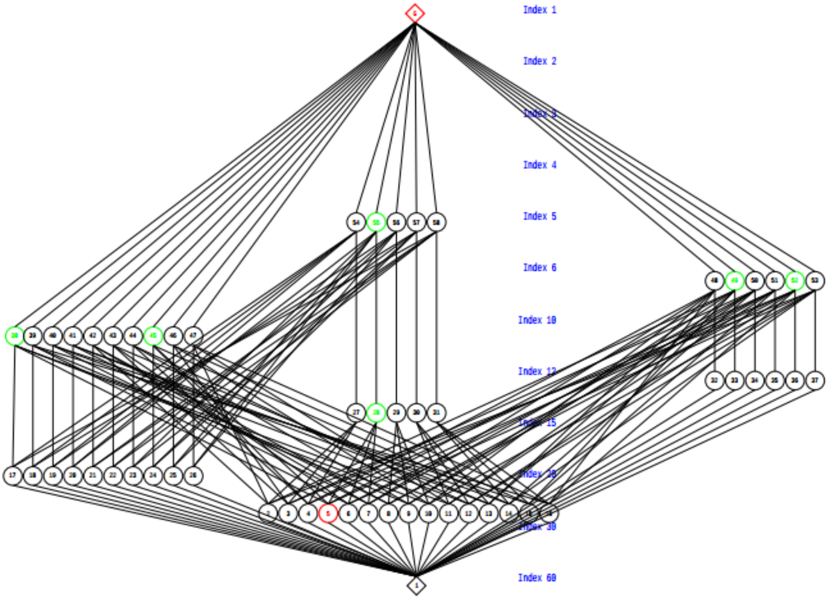

Index 1
Index 2
Index 4
Index 5
Index 6
Index 10
Index 60

Historically, much work has focused on:

- inferring properties of a group G from the structure of its lattice of subgroups $\mathrm{Sub}(G)$;
- inferring lattice theoretical properties of $\mathrm{Sub}(G)$ from properties of G.

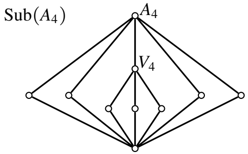
For some groups, $\mathrm{Sub}(G)$ determines G up to isomorphism.

Examples

The Klein 4-group, V_4.

The alternating groups, A_n $(n \geqslant 4)$.

Every finite nonabelian simple group.



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Examples

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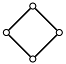
$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of order pq.

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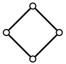
$\mathrm{Sub}(G) \cong$  if and only if G is cyclic of order pq.

At the other extreme, there are finite lattices that are not subgroup lattices.

We are interested in the local structure of subgroup lattices, that is, the possible ***intervals***

$$[\![H, K]\!] := \{X \mid H \leqslant X \leqslant K\} \leqslant \mathrm{Sub}(G)$$

where $H \leqslant K \leqslant G$.

We restrict our attention to ***upper intervals***, where $K = G$, and ask

1. *What intervals $[\![H, G]\!]$ are possible?*
2. *What properties of G can be deduced from the shape of $[\![H, G]\!]$?*

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There is a remarkable theorem relating this question to the *finite lattice representation problem* (FLRP).

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If these equivalent statements turn out to be true, we say, *"the FLRP has a positive answer."* Otherwise, *"the FLRP has a negative answer."*

Subgroup Lattice Basics

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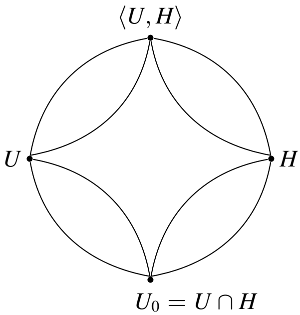

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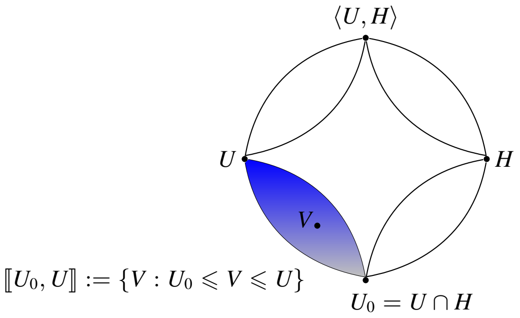

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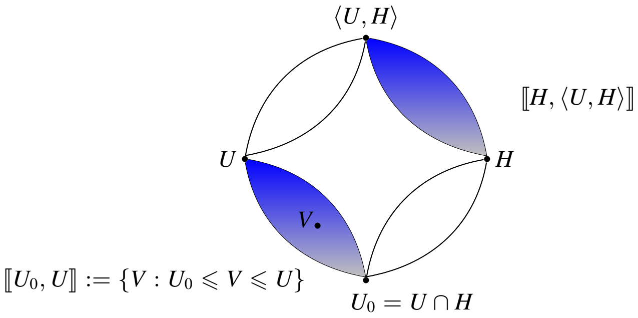
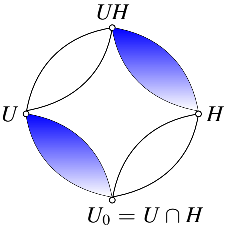


Interval Isomorphisms



- If $H \trianglelefteq \langle U, H\rangle$, then $UH = \langle U, H\rangle$ and $[\![U_0, U]\!] \cong [\![H, UH]\!]$.

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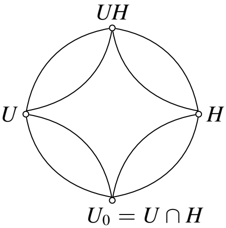

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- Instead of $H \trianglelefteq \langle U, H\rangle$, assume only $UH = \langle U, H\rangle$ and define

$$[\![U_0, U]\!]^H := \{V \in [\![U_0, U]\!] : VH = HV\},$$

the *H-permuting subgroups*.

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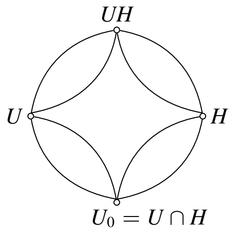

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- If $U \trianglelefteq UH$, define

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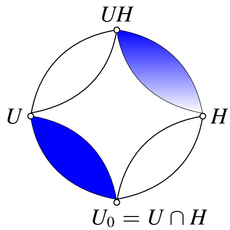

- If $H \trianglelefteqslant \langle U, H\rangle$, then $UH = \langle U, H\rangle$ and $[\![U_0, U]\!] \cong [\![H, UH]\!]$.
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Lemma

1. $[\![H, UH]\!] \cong [\![U_0, U]\!]^H \leqslant [\![U_0, U]\!]$
2. *If* $U \trianglelefteqslant UH$, *then* $[\![U_0, U]\!]_H = [\![U_0, U]\!]^H \leqslant [\![U_0, U]\!]$.
3. *If* $H \trianglelefteqslant UH$, *then* $[\![U_0, U]\!]_H = [\![U_0, U]\!]^H = [\![U_0, U]\!]$.

Example

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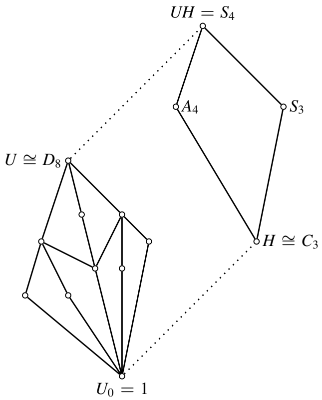

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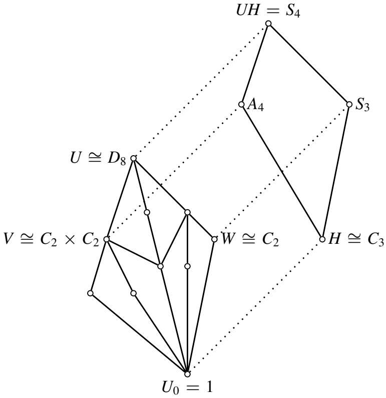


- Only four subgroups of U permute with H, including

$$U \cap A_4 \cong C_2 \times C_2, \qquad U \cap S_3 \cong C_2.$$

2. What properties of G can be inferred from $[\![H, G]\!]$?

Let $\mathfrak{G}$ denote the class of all finite groups.

A group theoretical property $\mathcal{P}$ (and the associated class $\mathscr{G}_{\mathcal{P}}$) is

- ***interval enforceable*** (IE) provided there exists a lattice L such that

  if $G \in \mathfrak{G}$ and $L \cong [\![H, G]\!]$, then G has property $\mathcal{P}$.

- ***core-free interval enforceable*** (cf-IE) provided $\exists L$ st

  if $G \in \mathfrak{G}$ and $L \cong [\![H, G]\!]$ and H core-free, then G has property $\mathcal{P}$.

- ***minimal interval enforceable*** (min-IE) provided $\exists L$ st
  if $G \in \mathfrak{G}$, $L \cong [\![H, G]\!]$, and if G has minimal order (wrt $L \cong [\![H, G]\!]$), then G has property $\mathcal{P}$.

Insolubility

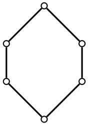
It's not hard to find examples of lattices that cannot occur as upper intervals in the subgroup lattices of finite soluble groups.

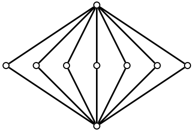
EXAMPLES

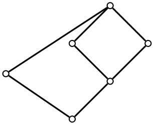
Insolubility

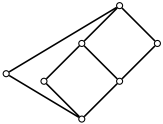
It's not hard to find examples of lattices that cannot occur as upper intervals in the subgroup lattices of finite soluble groups.

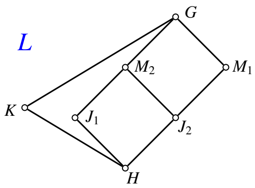
Here are a few











Proposition

Suppose $H < G$, $\text{core}_G(H) = 1$, *and* $L \cong [\![H, G]\!]$. *Then*

(I) G *is a primitive permutation group.*
(II) *If* $N \lhd G$, *then* $C_G(N) = 1$.
(III) G *contains no non-trivial abelian normal subgroup.*
(IV) G *is not solvable.*
(V) G *is subdirectly irreducible.*
(VI) *With the possible exception of at most one maximal subgroup,* M_1 *or* M_2, *all proper subgroups in the interval* $[\![H, G]\!]$ *are core-free.*

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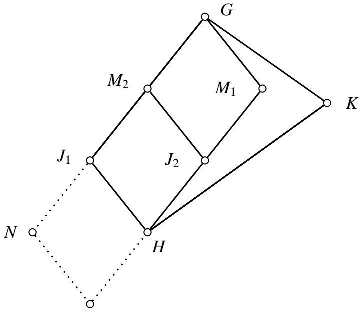

Claim: J_1 and J_2 are core-free subgroups of G.

Proof:

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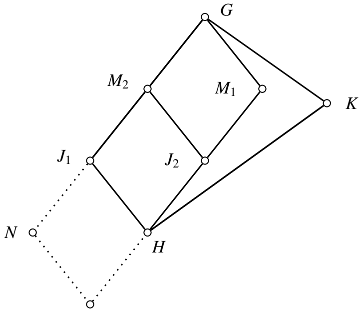

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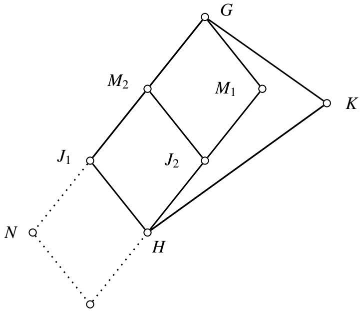

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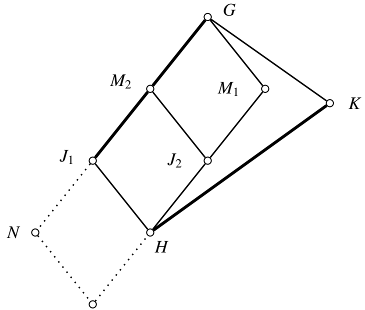

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- Since $J_1K = G$ and $J_1 \cap K = H$, our lemma yields

$$[\![J_1, G]\!] \cong [\![H, K]\!]^{J_1} = \{X \in [\![H, K]\!] \mid J_1X = XJ_1\}.$$

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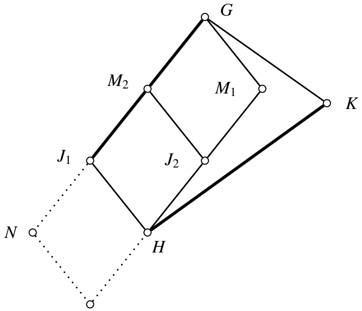

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Impossible!

The following are at least core-free interval enforceable:

- $\mathscr{G}_0 = \mathfrak{S}^c =$ the insoluble groups
- $\mathscr{G}_1 = \{G \in \mathfrak{G} \mid (\forall n < \omega)\ (G \neq A_n \text{ and } G \neq S_n)\}$
- $\mathscr{G}_2 =$ the subdirectly irreducible groups
- $\mathscr{G}_3 =$ groups with no nontrivial abelian normal subgroups
- $\mathscr{G}_4 = \{G \in \mathfrak{G} \mid C_G(M) = 1 \text{ for all } 1 \neq M \trianglelefteq G\}$.

If a lattice L is isomorphic to an interval in the subgroup lattice of a finite group, then we call L ***group representable***.

By the Pálfy-Pudlák Theorem, the FLRP has a negative answer if we can find a finite lattice that is not group representable.

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Suppose there exists property $\mathcal{P}$ such that both $\mathcal{P}$ and its negation $\neg\mathcal{P}$ are interval enforceable by the lattices L and L_c, respectively:

$$L \cong [\![H, G]\!] \implies G \text{ has property } \mathcal{P}$$

$$L_c \cong [\![H_c, G_c]\!] \implies G_c \text{ does not have property } \mathcal{P}$$

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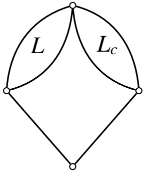

wouldn't be group representable.

But this strategy surely fails!

Lemma

If $\mathcal{P}$ is a group property that is interval enforceable by a group representable lattice, then $\neg\mathcal{P}$ is not interval enforceable by a group representable lattice.

But this strategy surely fails!

LEMMA

If $\mathcal{P}$ is a group property that is interval enforceable by a group representable lattice, then $\neg\mathcal{P}$ is not interval enforceable by a group representable lattice.

PROOF.

Assume both $\mathcal{P}$ and $\neg\mathcal{P}$ are IE by group representable lattices L and L_c.

Let G and G_c be groups for which $L \cong [\![H, G]\!]$ and $L_c \cong [\![H_c, G_c]\!]$.

Then $G \times G_c$ has upper intervals

$$L \cong [\![H \times G_c, G \times G_c]\!] \text{ and } L_c \cong [\![G \times H_c, G \times G_c]\!].$$

Therefore,

$$G \times G_c \text{ both has and has not property } \mathcal{P}.$$

$\square$

Example

Insolubility is interval enforceable, but solubility is not.

For if $L \cong [\![H, G]\!]$, then for any insoluble group K we have $L \cong [\![H \times K, G \times K]\!]$, and $G \times K$ is insoluble.

Conjecture

If $\mathcal{P}$ is core-free interval enforceable by a group representable lattice, then $\neg\mathcal{P}$ is not core-free interval enforceable by a group representable lattice.

Any class of groups that omits certain wreath products cannot be core-free interval enforceable by a group representable lattice.

LEMMA

Suppose $\mathcal{P}$ is core-free interval enforceable by a group representable lattice. Then, for any finite nonabelian simple group S, there exists a wreath product group of the form $W = S \wr U$ that has property $\mathcal{P}$.

COROLLARY

Solubility is not core-free interval enforceable.

Proof Sketch

Let L be a group representable lattice such that if $L \cong [\![H, G]\!]$ and $\mathrm{core}_G(H) = 1$ then G has property $\mathcal{P}$.

Since L is group representable, $\exists\, G \vDash \mathcal{P}$ with $L \cong [\![H, G]\!]$.

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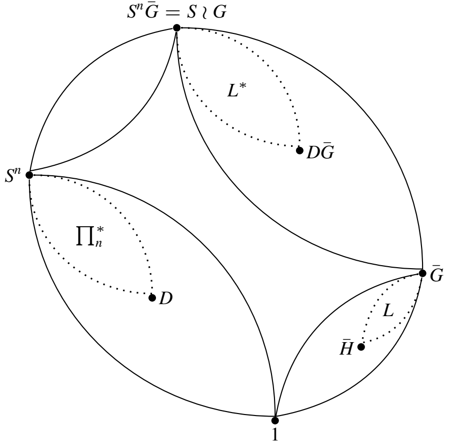
We apply the idea of Hans Kurzweil twice:

- Fix a finite nonabelian simple group S.
- Suppose the index of H in G is $|G : H| = n$.
- Then the action of G on the cosets of H induces an automorphism of the group S^n by permutation of coordinates.
- Denote this by $\varphi : G \to \mathrm{Aut}(S^n)$, and let $\varphi(G) = \bar{G} \leqslant \mathrm{Aut}(S^n)$.



The interval $[\![D, S^n]\!]$ is isomorphic to $\prod_n^*$, the dual of the lattice of partitions of an n-element set.

The dual lattice L^* is an upper interval of $\mathrm{Sub}(S \wr G)$, namely, $L^* \cong [\![D\bar{G}, S \wr G]\!]$.

Repeat to get $L = L^{**}$ as an upper interval, and then check core-free!

We conclude that a class of groups that does not include wreath products of the form $S \wr U$, where S is an arbitrary finite nonabelian simple group, is not a core-free interval enforceable class.

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Examples: soluble groups, simple groups, almost simple groups.

THEOREM

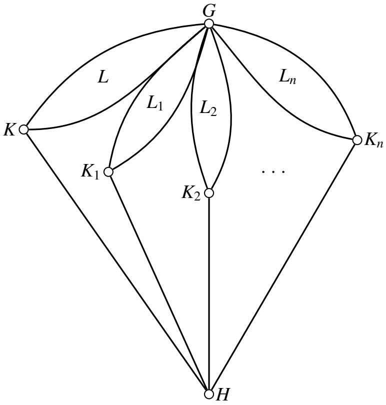
The following statements are equivalent:

(B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

(C) *For every finite lattice* L *and every finite collection* $\mathscr{G}_1, \ldots, \mathscr{G}_n$ *of cf-IE classes of groups,*

$$\exists\ G \in \bigcap_{i=1}^{n} \mathscr{G}_i \text{ such that } L \cong [\![H, G]\!] \text{ and } \operatorname{core}_G(H) = 1.$$

(D) *For any finite collection* $\mathscr{L}$ *of finite lattices, there exists a single finite group* G *such that each* $L_i \in \mathscr{L}$ *is isomorphic to* $[\![H_i, G]\!]$ *for some core-free subgroup* $H_i \leqslant G$*.*


G
L
L_1
L_2
L_n
K
K_1
K_2
K_n
. . .
H

By (C), the FLRP would have a negative answer if we could find a collection $\mathscr{G}_1, \ldots, \mathscr{G}_n$ of cf-IE classes such that $\bigcap_{i=1}^{n} \mathscr{G}_i$ is empty.

By (D), we consider finite collections of finite lattices and ask what can be proved about G if we assume that all of these lattices are isomorphic to upper intervals of $\mathrm{Sub}(G)$.

Aschbacher-O'Nan-Scott Theorem

Let G be a primitive permutation group of degree d, and let $N := \mathrm{Soc}(G) \cong T^m$ with $m \geqslant 1$. Then one of the following holds.

1. N is regular and
   - (Affine type) T is cyclic of order p, so $|N| = p^m$. Then $d = p^m$ and G is permutation isomorphic to a subgroup of the affine general linear group $\mathrm{AGL}(m,p)$.
   - (Twisted wreath product type) $m \geqslant 6$, the group T is nonabelian and G is a group of *twisted wreath product type*, with $d = |T|^m$.
2. N is non-regular, non-abelian, and
   - (Almost simple type) $m = 1$ and $T \leqslant G \leqslant \mathrm{Aut}(T)$.
   - (Product action type) $m \geqslant 2$ and G is permutation isomorphic to a subgroup of the product action wreath product $P \wr S_{m/l}$ of degree $d = nm/l$. The group P is primitive of type 2.(a) or 2.(c), P has degree n and $\mathrm{Soc}(P) \cong T^l$, where $l \geqslant 1$ divides m.
   - (Diagonal type) $m \geqslant 2$ and $T^m \leqslant G \leqslant T^m.(\mathrm{Out}(T) \times S_m)$, with the diagonal action. The degree $d = |T|^{m-1}$.


JUL 9 2003