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INTERVAL ENFORCEABLE PROPERTIES OF FINITE GROUPS

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ABSTRACT. We propose a classification of group properties according to whether they can be deduced from the assumption that a group's subgroup lattice contains an interval isomorphic to some lattice. Suppose $\mathcal{P}$ is a group property and suppose there exists a lattice L such that if G is a group with L isomorphic to an interval $[\![H, G]\!]$ in $\mathrm{Sub}(G)$, with H core-free, then G has property $\mathcal{P}$. We call such $\mathcal{P}$ *core-free interval enforceable.* Among other things we show that if both a property and its negation could be proved core-free interval enforceable, this would solve an important problem in universal algebra.

1. INTRODUCTION

The study of subgroup lattices has a long history, starting with Richard Dedekind [8] and Ada Rottlaender [26], and later a number of important contributions by Reinhold Baer, Øystein Ore, Michio Suzuki, Roland Schmidt, and many others (see Schmidt [27]). Much of this work focuses on the problem of inferring properties of a group G based on the structure of its lattice of subgroups, $\mathrm{Sub}(G)$, or, conversely, inferring lattice theoretical properties of $\mathrm{Sub}(G)$ from properties of G.

Historically, less attention was paid to the local structure of the subgroup lattice of a finite group, perhaps because it seemed that very little about G could be inferred from knowledge of, say, an *upper interval* $[\![H, G]\!] = \{K \mid H \leqslant K \leqslant G\}$ in the subgroup lattice of G. Recently, however, this topic has attracted more attention (e.g., [2, 3, 5, 7, 16, 18, 20, 21]), mostly owing to its connection to the most important open problem in universal algebra, the *Finite Lattice Representation Problem* (FLRP). This is the problem of characterizing the lattices that are (isomorphic to) congruence lattices of finite algebras (see [6, 10, 21, 22]). There is a remarkable theorem relating this problem to intervals in subgroup lattices of finite groups.

Theorem 1.1 (Pálfy and Pudlák [23])**.** *The following statements are equivalent:*

(A) Every finite lattice is isomorphic to the congruence lattice of a finite algebra.
(B) Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.

If these statements are true (resp., false), then we say the FLRP has a positive (resp., negative) answer. Thus, if we can find a finite lattice L for which it can be

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proved that there is no finite group G with $L \cong [\![H, G]\!]$ for some $H < G$, then the FLRP has a negative answer.

In this paper we propose a new classification of group properties according to whether or not they can be deduced from the assumption that $\mathrm{Sub}(G)$ has an upper interval isomorphic to some finite lattice. We believe that discovering which group properties can (or cannot) be connected in this way to the local structure of a subgroup lattice is itself a worthwhile endeavor, but we will also describe how such a classification could be used to solve the FLRP.

Suppose $\mathcal{P}$ is a *group theoretical property*[1] and suppose there exists a finite lattice L such that if G is a finite group with $L \cong [\![H, G]\!]$ for some $H \leqslant G$, then G has property $\mathcal{P}$. We call such a property $\mathcal{P}$ *interval enforceable* (IE) *by* L. An *interval enforceable class of groups* is a class of all groups having property $\mathcal{P}$, for some interval enforceable property $\mathcal{P}$.

Although it depends on the lattice L, generally speaking it is difficult to deduce very much about a group G from the assumption that an upper interval in $\mathrm{Sub}(G)$ is isomorphic to L. It gets easier easier if, in addition to the hypothesis $L \cong [\![H, G]\!]$, we assume that the subgroup H is *core-free* in G; that is, H contains no nontrivial normal subgroup of G. Properties of G that can be deduced from these assumptions are what we call *core-free interval enforceable* (cf-IE).

Extending this idea, we consider finite collections $\mathscr{L}$ of finite lattices and ask what can be proved about a group G if one assumes that each $L_i \in \mathscr{L}$ is isomorphic to an upper interval $[\![H_i, G]\!] \leqslant \mathrm{Sub}(G)$, with each H_i core-free in G. Clearly, if $\mathrm{Sub}(G)$ has such upper intervals, and if corresponding to each $L_i \in \mathscr{L}$ there is a property $\mathcal{P}_i$ that is cf-IE by L_i, then G must have all of the properties $\mathcal{P}_i$. A related question is the following: Given a set $\mathscr{P}$ of cf-IE properties, is the conjunction $\bigwedge \mathscr{P}$ cf-IE? Corollary 3.11 answers this question affirmatively.

In this paper, we will identify some group properties that are cf-IE, and others that are not. We will see that the cf-IE properties that we have found thus far are negations of common group properties (for example, "not solvable," "not almost simple," "not alternating," "not symmetric"). Moreover, we prove that in these special cases the corresponding group properties ("solvable," "almost simple," "alternating," "symmetric") that are not cf-IE. This and other considerations suggest that a group property and its negation cannot both be cf-IE. As yet, we are unable to prove this. A related question is whether we should expect that, for every group property $\mathcal{P}$, either $\mathcal{P}$ is cf-IE or $\neg\mathcal{P}$ is cf-IE.

One of our main results (Theorem 3.9) connects the foregoing ideas with the FLRP, as follows:

Statement (B) of Theorem 1.1 is equivalent to each of the following statements:

- *(C) For every finite lattice L, for every finite collection $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ of cf-IE classes of groups, there exists a finite group $G \in \bigcap_{i=1}^{n} \mathfrak{X}_i$ such that $L \cong [\![H, G]\!]$ for some subgroup H that is core-free in G.*
- *(D) For every finite collection $\mathscr{L}$ of finite lattices, there exists a finite group G such that for each $L \in \mathscr{L}$ we have $L \cong [\![H, G]\!]$ for some subgroup H that is core-free in G.*

[1]This and other italicized terms in the introduction will be defined more formally in Section 2.

In fact, the arguments proving the equivalence of these statements are easily combined to show that the following is also equivalent to statement (B):

(E) For every finite collection $\mathscr{L}$ of finite lattices, for every finite collection $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ of cf-IE classes of groups, there exists a finite group $G \in \bigcap_{i=1}^{n} \mathfrak{X}_i$ such that each $L_i \in \mathscr{L}$ is isomorphic to an upper interval $[\![H_i, G]\!]$ with H core-free in G.

Core-free interval enforceable properties are intimately related to permutation representations of groups. If H is a core-free subgroup of G, then G has a faithful permutation representation $\varphi : G \hookrightarrow \mathrm{Sym}(G/H)$. Let $\langle G/H, \varphi(G)\rangle$ denote the algebra comprised of the right cosets G/H acted upon by right multiplication by elements of G; that is, $\varphi(g) : Hx \mapsto Hxg$. It is well known that the congruence lattice of this algebra (i.e., the lattice of systems of imprimitivity) is isomorphic to the interval $[\![H, G]\!]$ in the subgroup lattice of G.[2] This puts statement (E) into perspective. If the FLRP has a positive answer, then no matter what we take as our finite collection $\mathscr{L}$—for example, we might take $\mathscr{L}$ to be *all* finite lattices with at most N elements for some large $N < \omega$—we can always find a *single* finite group G such that every lattice in $\mathscr{L}$ is isomorphic to the interval in $\mathrm{Sub}(G)$ above a core-free subgroup. As a result, this group G must have so many faithful representations $G \hookrightarrow \mathrm{Sym}(G/H_i)$ with systems of imprimitivity isomorphic to L_i, one such representation for each distinct $L_i \in \mathscr{L}$. Moreover, the group G having this property can be chosen from the class $\bigcap_{i=1}^{n} \mathfrak{X}_i$, where $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ is an arbitrary collection of cf-IE classes of groups.

2. Notation and definitions

In this paper, *all groups and lattices are finite.* We use $\mathfrak{G}$ to denote the class of all finite groups. Given a group G, we denote the set of subgroups of G by $\mathrm{Sub}(G)$. The algebra $\langle \mathrm{Sub}(G), \wedge, \vee\rangle$ is a lattice where the $\wedge$ ("meet") and $\vee$ ("join") operations are defined for all H and K in $\mathrm{Sub}(G)$ by $H \wedge K = H \cap K$ and $H \vee K = \langle H, K\rangle =$ the smallest subgroup of G containing both H and K. We will refer to the set $\mathrm{Sub}(G)$ as a lattice, without explicitly mentioning the $\wedge$ and $\vee$ operations.

By $H \leqslant G$ (resp., $H < G$) we mean H is a subgroup (resp., proper subgroup) of G. For $H \leqslant G$, the *core of H in G*, denoted by $\mathrm{core}_G(H)$, is the largest normal subgroup of G contained in H. If $\mathrm{core}_G(H) = 1$, then we say that H is *core-free in G*. For $H \leqslant G$, by the *interval* $[\![H, G]\!]$ we mean the set $\{K \mid H \leqslant K \leqslant G\}$, which is a sublattice of $\mathrm{Sub}(G)$. That is, $[\![H, G]\!]$ is the lattice of those subgroups of G that contain H. With this notation, $\mathrm{Sub}(G) = [\![1, G]\!]$. When viewing $[\![H, G]\!]$ as a sublattice of $\mathrm{Sub}(G)$, we sometimes refer to it as an *upper interval.* Given an abstract lattice L, if there is no mention of specific groups H and G, then the expression $L \cong [\![H, G]\!]$ means "there exist (finite) groups $H \leqslant G$ such that L is isomorphic to the interval $\{K \mid H \leqslant K \leqslant G\}$ in the subgroup lattice of G."

By a *group theoretical class*, or *class of groups*, we mean a collection $\mathfrak{X}$ of groups that is closed under isomorphism: if $G_0 \in \mathfrak{X}$ and $G_1 \cong G_0$, then $G_1 \in \mathfrak{X}$. A *group theoretical property*, or simply *property of groups*, is a property $\mathcal{P}$ such that if a

[2] See [19, Lemma 4.20] or [13, Theorem 1.5A].

group G_0 has property $\mathcal{P}$ and $G_1 \cong G_0$, then G_1 has property $\mathcal{P}$.[3] Thus if $\mathfrak{X}_{\mathcal{P}}$ denotes the collection of all groups having the group property $\mathcal{P}$, then $\mathfrak{X}_{\mathcal{P}}$ is a class of groups, and belonging to a particular class of groups is a group theoretical property. Therefore, we need not distinguish between a property of groups and the class of groups that possess this property. A group in the class $\mathfrak{X}$ is called a $\mathfrak{X}$-*group*, and we sometimes write $G \vDash \mathcal{P}$ to indicate that G has property $\mathcal{P}$.

If $\mathscr{K}$ is a class of algebras (e.g., a class of groups), then we say that $\mathscr{K}$ is *closed under homomorphic images* and we write $\mathbf{H}(\mathscr{K}) = \mathscr{K}$ provided $\varphi(G) \in \mathscr{K}$ whenever $G \in \mathscr{K}$ and φ is a homomorphism of G. By the first isomorphism theorem for groups, this is equivalent to: $G/N \in \mathscr{K}$ whenever $G \in \mathscr{K}$ and $N \trianglelefteq G$. For algebras, $\mathbf{H}(\mathscr{K}) = \mathscr{K}$ holds if and only if $\mathbf{A}/\theta \in \mathscr{K}$ for all $\mathbf{A} \in \mathscr{K}$ and all $\theta \in \operatorname{Con}\mathbf{A}$, where $\operatorname{Con}\mathbf{A}$ denotes the lattice of congruence relations of $\mathbf{A}$. Apart from possible notational differences, the foregoing terminology is standard.

We now introduce some new terminology that we find useful.[4] Let $\mathfrak{L}$ denote the class of all finite lattices, and $\mathfrak{G}$ the class of all finite groups. Let $\mathcal{P}$ be a given group theoretical property and $\mathfrak{X}_{\mathcal{P}}$ the associated class of all groups with property $\mathcal{P}$. We call $\mathcal{P}$ (and $\mathfrak{X}_{\mathcal{P}}$)

- *interval enforceable* (IE) provided
$$(\exists L \in \mathfrak{L})\ (\forall G \in \mathfrak{G})\ \big(L \cong [\![H, G]\!] \longrightarrow G \in \mathfrak{X}_{\mathcal{P}}\big)$$
- *core-free interval enforceable* (cf-IE) provided
$$(\exists L \in \mathfrak{L})\ (\forall G \in \mathfrak{G})\ \Big(\big(L \cong [\![H, G]\!] \bigwedge \operatorname{core}_G(H) = 1\big) \longrightarrow G \in \mathfrak{X}_{\mathcal{P}}\Big)$$
- *minimal interval enforceable* (min-IE) provided there exists $L \in \mathfrak{L}$ such that if $L \cong [\![H, G]\!]$ for some group $G \in \mathfrak{G}$ of minimal order (with respect to $L \cong [\![H, G]\!]$), then $G \in \mathfrak{X}_{\mathcal{P}}$.

In this paper we will have little to say about min-IE properties. Nonetheless, we include this class in our list of new definitions because properties of this type arise often (see, e.g., [18]), and a primary aim of this paper is to formalize various notions of interval enforceability that we believe are useful in applications.

3. Results

Clearly, if $\mathcal{P}$ is an interval enforceable property, then it is also core-free interval enforceable. There is an easy sufficient condition under which the converse holds. Suppose $\mathcal{P}$ is a group property, let $\mathfrak{X}_{\mathcal{P}}$ denote the class of all groups with property $\mathcal{P}$, and let $\mathfrak{X}^{c}_{\mathcal{P}}$ denote the class of all groups that do not have property $\mathcal{P}$.

Lemma 3.1. *Suppose $\mathcal{P}$ is a core-free interval enforceable property. If $\mathbf{H}(\mathfrak{X}^{c}_{\mathcal{P}}) = \mathfrak{X}^{c}_{\mathcal{P}}$, then $\mathcal{P}$ is an interval enforceable property.*

Proof. Since $\mathcal{P}$ is cf-IE there is a lattice L such that

$$\big(L \cong [\![H, G]\!] \bigwedge \operatorname{core}_G(H) = 1\big) \longrightarrow G \in \mathfrak{X}_{\mathcal{P}}. \tag{3.1}$$

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[3]It seems there is no single standard definition of *group theoretical class*. While some authors (e.g., [14], [4]) use the same definition we use here, others (e.g. [24], [25]) require that every group theoretical class contains the one element group. In the sequel we consider negations of group properties, and we would like these to qualify as group properties. Therefore, we don't require that every group theoretical class contains the one element group.

[4]The author thanks Bjørn Kjos-Hanssen and David Ross for suggesting improvements to the wording of these definitions.

Under the assumption $\mathbf{H}(\mathfrak{X}^c_{\mathcal{P}}) = \mathfrak{X}^c_{\mathcal{P}}$ we prove

$$L \cong [\![H, G]\!] \;\longrightarrow\; G \in \mathfrak{X}_{\mathcal{P}}. \tag{3.2}$$

If (3.2) fails, then there is a group $G \in \mathfrak{X}^c_{\mathcal{P}}$ with $L \cong [\![H, G]\!]$. Let $N = \mathrm{core}_G(H)$. Then $L \cong [\![H/N, G/N]\!]$ and H/N is core-free in G/N so, by hypothesis (3.1), $G/N \in \mathfrak{X}_{\mathcal{P}}$. But $G/N \in \mathfrak{X}^c_{\mathcal{P}}$, since $\mathfrak{X}^c_{\mathcal{P}}$ is closed under homomorphic images. □

In [21], Péter Pálfy gives an example of a lattice that cannot occur as an upper interval in the subgroup lattice finite solvable group. (We give other examples in §3.3 and §4.) In his Ph.D. thesis [5], Alberto Basile proves that if G is an alternating or symmetric group, then there are certain lattices that cannot occur as upper intervals in $\mathrm{Sub}(G)$. Another class of lattices with this property is described by Aschbacher and Shareshian in [2]. Thus, two classes of groups that are known to be at least cf-IE are the following:

- $\mathfrak{X}_0 = \mathfrak{S}^c =$ nonsolvable finite groups;
- $\mathfrak{X}_1 = \big\{G \in \mathfrak{G} \mid (\forall n < \omega)\ \big(G \neq A_n \bigwedge G \neq S_n\big)\big\}$,

where A_n and S_n denote, respectively, the alternating and symmetric groups on n letters. Note that both classes $\mathfrak{X}_0$ and $\mathfrak{X}_1$ satisfy the hypothesis of 3.1. Explicitly, $\mathfrak{X}_0^c = \mathfrak{S}$, the class of solvable groups, is closed under homomorphic images, as is the class $\mathfrak{X}_1^c$ of alternating and symmetric groups. Therefore, by Lemma 3.1, $\mathfrak{X}_0$ and $\mathfrak{X}_1$ are IE classes. By contrast, suppose there exists a finite lattice L such that

$$\big(L \cong [\![H, G]\!] \bigwedge \mathrm{core}_G(H) = 1\big) \;\longrightarrow\; G \text{ is subdirectly irreducible.}$$

Lemma 3.1 does not apply in this case since the class of subdirectly reducible groups is obviously not closed under homomorphic images.[5] In Sections 3.3 and 4 below we describe lattices with which we can prove that the following classes are at least cf-IE:

- $\mathfrak{X}_2 =$ the subdirectly irreducible groups;
- $\mathfrak{X}_3 =$ the groups having no nontrivial abelian normal subgroups;
- $\mathfrak{X}_4 = \{G \in \mathfrak{G} \mid C_G(M) = 1 \text{ for all } 1 \neq M \trianglelefteq G\}$.

We noted above that $\mathfrak{X}_2$ fails to satisfy the hypothesis of 3.1. The same can be said of $\mathfrak{X}_3$ and $\mathfrak{X}_4$. That is, $\mathbf{H}(\mathfrak{X}^c_i) \neq \mathfrak{X}^c_i$ for $i = 2, 3, 4$. To verify this take $H \in \mathfrak{X}_i$, $K \in \mathfrak{X}^c_i$, and consider $H \times K$. In each case ($i = 2, 3, 4$) we see that $H \times K$ belongs to $\mathfrak{X}^c_i$, but the homomorphic image $(H \times K)/(1 \times K) \cong H$ does not.

3.1. **Negations of interval enforceable properties.** The following definition is useful: if a lattice L is isomorphic to an interval in the subgroup lattice of a finite group, then we call L *group representable*. Recall, Theorem 1.1 says that the FLRP has a negative answer if we can find a finite lattice that is not group representable.

Suppose there exists a property $\mathcal{P}$ such that both $\mathcal{P}$ and its negation $\neg\mathcal{P}$ are interval enforceable by the lattices L and L_c, respectively. That is $L \cong [\![H, G]\!]$ implies $G \in \mathfrak{X}_{\mathcal{P}}$ and $L_c \cong [\![H_c, G_c]\!]$ implies $G_c \in \mathfrak{X}^c_{\mathcal{P}}$. Then clearly the lattice in Figure 1 could not be group representable. As the next result shows, however, if a group property and its negation are interval enforceable by the lattices L and L_c, then already at least one of these lattices is not group representable.

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[5]Recall, for groups *subdirectly irreducible* is equivalent to having a unique minimal normal subgroup. Every algebra, in particular every group G, has a subdirect decomposition into subdirectly irreducibles, say, $G \hookrightarrow G/N_1 \times \cdots \times G/N_n$, so there are always subdirectly irreducible homomorphic images.

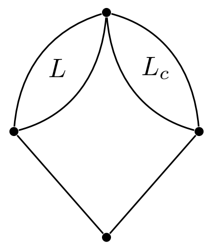


FIGURE 1.

Lemma 3.2. *If $\mathcal{P}$ is a group property that is interval enforceable by a group representable lattice, then $\neg\mathcal{P}$ is not interval enforceable by a group representable lattice.*

Proof. Assume the contrary. Then both $\mathcal{P}$ and its negation $\neg\mathcal{P}$ are interval enforceable by group representable lattices L and L_c, respectively. Let G and G_c be groups for which $L \cong [\![H, G]\!]$ and $L_c \cong [\![H_c, G_c]\!]$ for some $H \leqslant G$ and $H_c \leqslant G_c$. Then the group $G \times G_c$ has upper intervals $L \cong [\![H \times G_c, G \times G_c]\!]$ and $L_c \cong [\![G \times H_c, G \times G_c]\!]$. Thus, by the interval enforceability assumptions, the group $G \times G_c$ both is and is not a $\mathfrak{X}_{\mathcal{P}}$-group. $\square$

To take a concrete example, insolubility is IE. However, solubility is obviously not IE. For, if $L \cong [\![H, G]\!]$ then for any nonsolvable group K we have $L \cong [\![H \times K, G \times K]\!]$, and of course $G \times K$ is nonsolvable. Note that here (and in the proof of Lemma 3.2) the group $H \times K$ at the bottom of the interval is not core-free. So a more interesting question is whether a property and its negation can both be cf-IE. Again, if such a property were found, a lattice of the form in Figure 1 would give a negative answer to the FLRP, though this requires additional justification to address the core-free aspect (see Section 3.3). We suspect the answer is no, as suggested by

Conjecture 3.1. If $\mathcal{P}$ is core-free interval enforceable by a group representable lattice, then $\neg\mathcal{P}$ is not core-free interval enforceable by a group representable lattice.

We will confirm a few special cases of the foregoing conjecture—in particular, the cases when $\mathcal{P}$ means "not solvable" or "not almost simple." Indeed, Lemma 3.4 below implies that the class of solvable groups, and more generally any class of groups that omits certain wreath products, cannot be core-free interval enforceable by a group representable lattice. Before stating and proving this lemma, however, we pause to state a result that gives further support of for the conjecture above.

Proposition 3.3. *The following statments are equivalent...*

Lemma 3.4. *Suppose $\mathcal{P}$ is core-free interval enforceable by a group representable lattice. Then, for any finite nonabelian simple group S, there exists a wreath product group of the form $W = S \wr \bar{U}$ that has property $\mathcal{P}$.*

Proof. Let L be a group representable lattice such that if $L \cong [\![H, G]\!]$ and $\operatorname{core}_G(H) = 1$ then $G \models \mathcal{P}$. Since L is group representable, there exists a $\mathcal{P}$-group G with $L \cong [\![H, G]\!]$. We apply an idea of Hans Kurzweil (see [17]) twice. Fix a finite nonabelian simple group S. Suppose the index of H in G is $|G : H| = n$. Then the action of G on the cosets of H induces an automorphism of the group S^n by permutation of coordinates. Denote this representation by $\varphi : G \to \operatorname{Aut}(S^n)$, and let the image of G be $\varphi(G) = \bar{G} \leqslant \mathrm{Aut}(S^n)$. The wreath product under this action is the group

$$U := S \wr_\varphi G = S^n \rtimes_\varphi G = S^n \rtimes \bar{G},$$

with multiplication given by

$$(s_1, \dots, s_n, x)(t_1, \dots, t_n, y) = (s_1 t_{x(1)}, \dots, s_n t_{x(n)}, xy),$$

for $s_i, t_i \in S$ and $x, y \in \bar{G}$. (For the remainder of the proof, we suppress the semidirect product symbol and write, for example, $S^n\bar{G}$ instead of $S^n \rtimes \bar{G}$.)

An illustration of the subgroup lattice of such a wreath product appears in Figure 2. Note that the interval $[\![D, S^n]\!]$, where D denotes the diagonal subgroup of S^n, is isomorphic to $\mathrm{Eq}(n)'$, the dual of the lattice of partitions of an n-element set. The dual lattice L' is an upper interval of $\mathrm{Sub}(U)$, namely, $L' \cong [\![D\bar{G}, U]\!]$.[6]

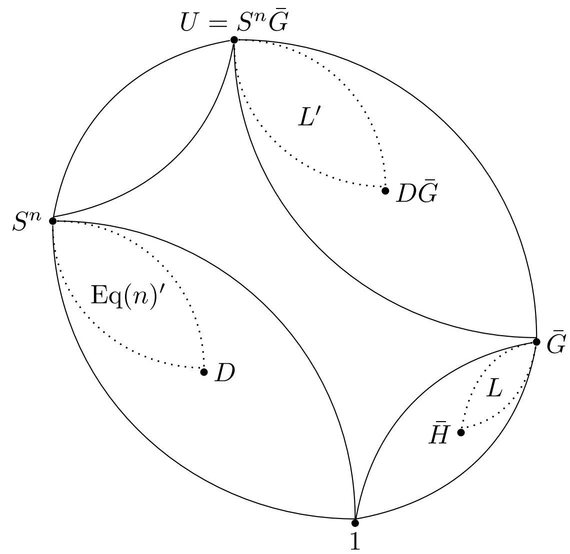


FIGURE 2. Hasse diagram illustrating some features of the subgroup lattice of the wreath product U.

It is important to note (and we prove below) that if H is core-free in G – equivalently, if $\ker \varphi = 1$ – then the foregoing construction results in the subgroup $D\bar{G}$ being core-free in U. Therefore, by repeating the foregoing procedure, with $H_1 = D\bar{G}$ denoting the (core-free) subgroup of U such that $L' \cong [\![H_1, U]\!]$, we find that $L = L'' \cong [\![D_1\bar{U}, S^m\bar{U}]\!]$, where $m = |U : H_1|$, and D_1 denotes the diagonal subgroup of S^m. Since $D_1\bar{U}$ will be core-free in $S^m\bar{U}$ then, it follows by the original hypothesis that $S^m\bar{U} = S \wr \bar{U}$ must have property $\mathcal{P}$.

To complete the proof, we check that starting with a core-free subgroup $H \leqslant G$ in the Kurzweil construction just described results in a core-free subgroup $D\bar{G} \leqslant U$. Let $N = \mathrm{core}_U(D\bar{G})$. Then, for all $w = (d, \dots, d, x) \in N$ and for all $u = (t_1, \dots, t_n, g) \in U$, we have $uwu^{-1} \in N$. Fix $w = (d, \dots, d, x) \in N$. We will choose

[6] These facts, which were proved by Kurzweil in [17], are discussed in greater detail in [10, Section 2.2].

$u \in U$ so that the condition $uwu^{-1} \in N$ implies x acts trivially on $\{1, \dots, n\}$. First note that if $u = (t_1, \dots, t_n, 1)$, then

$$\begin{aligned} uwu^{-1} &= (t_1, \dots, t_n, 1)(d, \dots, d, x)(t_1^{-1}, \dots, t_n^{-1}, 1) \\ &= (t_1 d\, t_{x(1)}^{-1}, \dots, t_n d\, t_{x(n)}^{-1}, 1) \in N, \end{aligned}$$

and this implies that $t_1 d\, t_{x(1)}^{-1} = t_2 d\, t_{x(2)}^{-1} = \cdots = t_n d\, t_{x(n)}^{-1}$. Suppose by way of contradiction that $x(1) = j \neq 1$. Then, since x is a permutation (hence, one-to-one), $x(k) \neq j$ for each $k \in \{2, 3, \dots, n\}$. Pick one such k other than j. (This is possible since $n = |G : H| > 2$; for otherwise $H \trianglelefteq G$ contradicting $\operatorname{core}_G(H) = 1$.) Since $u \in U$ is arbitrary, we may assume $t_1 = t_k$ and $t_{x(1)} = t_j \neq t_{x(k)}$. But this contradicts $t_1 d\, t_{x(1)}^{-1} = t_k d\, t_{x(k)}^{-1}$. Therefore, $x(1) = 1$. The same argument shows that $x(i) = i$ for each $1 \leqslant i \leqslant n$, and we see that $w = (d, \dots, d, x) \in N$ implies $x \in \ker \varphi = 1$. This puts N below D, and the only normal subgroup of U that lies below D is the trivial group. □

By the foregoing result we conclude that a class of groups that does not include wreath products of the form $S \wr G$, where S is an arbitrary finite nonabelian simple group, is not a core-free interval enforceable class. The class of solvable groups is an example.

3.2. **Dedekind's rule and its consequences.** When A and B are subgroups of a group G, by AB we mean the set $\{ab \mid a \in A, b \in B\}$, and we write $A \vee B$ or $\langle A, B\rangle$ to denote the subgroup of G generated by A and B. Clearly $AB \subseteq \langle A, B\rangle$; equality holds if and only if A and B *permute*, by which we mean $AB = BA$.

We will need the following well known result:[7]

Theorem 3.5 (Dedekind's rule)**.** *Let G be a group and let A, B and C be subgroups of G with $A \leqslant B$. Then,*

$$A(C \cap B) = AC \cap B, \qquad \text{and} \tag{3.3}$$

$$(C \cap B)A = CA \cap B. \tag{3.4}$$

For $A \in [\![H, G]\!]$ we let $A^{\perp(H,G)}$ denote the set of complements of A in the interval $[\![H, G]\!]$. That is, $A^{\perp(H,G)} = \{B \in [\![H, G]\!] \mid A \cap B = H, \langle A, B\rangle = G\}$. Clearly $H^{\perp(H,G)} = \{G\}$ and $G^{\perp(H,G)} = \{H\}$. Recall that an *antichain* of a partially ordered set is a subset of pairwise incomparabe elements.

Corollary 3.6. *If $A \in [\![H, G]\!]$ and if A permutes with each subgroup in $A^{\perp(H,G)}$, then $A^{\perp(H,G)}$ is an antichain or empty.*

Proof. If $A^{\perp(H,G)}$ contains fewer that two elements, the result holds trivially. Let B_1 and B_2 denote two distinct elements of $A^{\perp(H,G)}$. We will prove $\neg(B_1 \leqslant B_2)$. Indeed, if $B_1 \leqslant B_2$, then Theorem 3.5 implies

$$B_1 = B_1 H = B_1(A \cap B_2) = B_1 A \cap B_2 = G \cap B_2 = B_2,$$

which is a contradiction. The penultimate equality holds by the hypothesis that A permutes with B_1. □

For the present work, the most important consequence of Dedekind's rule is the following:

[7] See [25, p. 122], for example.

Corollary 3.7. *If $H \leqslant G$ and $N \trianglelefteq G$, then $(HN)^{\perp(H,G)}$ is an antichain.*

Proof. Note that HN permutes with each subgroup in $[\![H, G]\!]$. Indeed, if $H \leqslant A \leqslant G$, then $HNA = HAN = HN = AHN$. Therefore, by Corollary 3.6, the result holds. □

Our next lemma (Lemma 3.8) generalizes a standard result. We find this generalization useful for proving that certain properties are core-free interval enforceable. To state Lemma 3.8, we need some new notation. Let U and H be subgroups of a group, let $U_0 = U \cap H$, and consider the interval $[\![U_0, U]\!] = \{V \mid U_0 \leqslant V \leqslant U\}$. It will be helpful to visualize part of the subgroup lattice of $\langle U, H\rangle$, as shown in Figure 3.

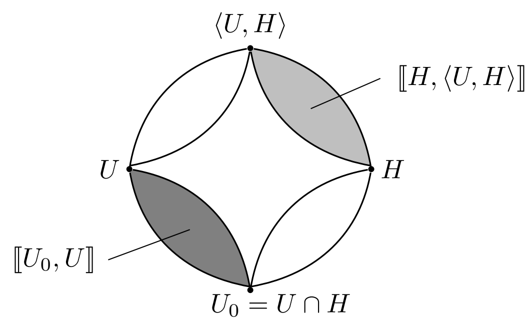


FIGURE 3. Some intervals in a subgroup lattice.

Recall that the usual isomorphism theorem for groups implies that if H is a normal subgroup of $\langle U, H\rangle$, then the interval $[\![H, \langle U, H\rangle]\!]$ is isomorphic to the interval $[\![U \cap H, U]\!]$. The purpose of the next lemma is to relate these two intervals in cases where we drop the assumption $H \trianglelefteq \langle U, H\rangle$ and add the assumption $UH = \langle U, H\rangle$.

If the two subgroups U and H permute, then we define

$$[\![U_0, U]\!]^H = \{V \in [\![U_0, U]\!] \mid VH = HV\}, \tag{3.5}$$

which consists of those subgroups in $[\![U_0, U]\!]$ that permute with H.

If H normalizes U (which implies $UH = HU$), then we define

$$[\![U_0, U]\!]_H = \{V \in [\![U_0, U]\!] \mid H \leqslant N_{UH}(V)\}. \tag{3.6}$$

This is the set consisting of those subgroups in $[\![U_0, U]\!]$ that are normalized by H (sometimes called *H-invariant subgroups* in $[\![U_0, U]\!]$). Notice that to even define $[\![U_0, U]\!]_H$ we must have $H \leqslant N_{UH}(U)$, and in this case, as we will see below, the sublattices $[\![U_0, U]\!]_H$ and $[\![U_0, U]\!]^H$ coincide.

We are now ready to state the main result relating the sets defined in (3.5) and (3.6) to the interval $[\![H, UH]\!]$.

Lemma 3.8. *Suppose U and H are permuting subgroups of a group. Let $U_0 = U \cap H$. Then*

(i) $[\![H, UH]\!] \cong [\![U_0, U]\!]^H \leqslant [\![U_0, U]\!]$.
(ii) If $U \trianglelefteq UH$, then $[\![U_0, U]\!]_H = [\![U_0, U]\!]^H \leqslant [\![U_0, U]\!]$.
(iii) If $H \trianglelefteq UH$, then $[\![U_0, U]\!]_H = [\![U_0, U]\!]^H = [\![U_0, U]\!]$.

Remarks. The expression $[\![H, UH]\!] \cong [\![U_0, U]\!]^H$ means that $[\![H, UH]\!]$ and $[\![U_0, U]\!]^H$ are isomorphic as lattices. Thus, when two subgroups permute, we can identify the interval above either one of them with the set of subgroups below the other that permute with the first. The expression $[\![U_0, U]\!]^H \leqslant [\![U_0, U]\!]$ means that the lattice $[\![U_0, U]\!]^H$ is, in fact, a sublattice of $[\![U_0, U]\!]$. That is, meets and joins in $[\![U_0, U]\!]^H$ coincide with those in $[\![U_0, U]\!]$. Part (ii) of Lemma 3.8 is a standard result (see, e.g., [7] or [15]). Since $G = UH$ is a group, the hypothesis of (ii) is equivalent to $H \leqslant N_G(U)$, and the hypothesis of (iii) is equivalent to $U \leqslant N_G(H)$. Once we have proved (i), the proof of (iii) follows trivially from the standard isomorphism theorem for groups, so we omit the details.

Proof. To prove (i), we first show that the following maps are inverse order isomorphisms:

$$\begin{aligned} \varphi &: [\![H, UH]\!] \ni X \mapsto U \cap X \in [\![U_0, U]\!]^H \\ \psi &: [\![U_0, U]\!]^H \ni V \mapsto VH \in [\![H, UH]\!]. \end{aligned} \tag{3.7}$$

Then we show that $[\![U_0, U]\!]^H$ is a sublattice of $[\![U_0, U]\!]$, that is, $[\![U_0, U]\!]^H \leqslant [\![U_0, U]\!]$.

Fix $X \in [\![H, UH]\!]$. We claim that $U \cap X \in [\![U_0, U]\!]^H$. Indeed,

$$(U \cap X)H = UH \cap X = HU \cap X = H(U \cap X).$$

The first equality holds by (3.4) since $H \leqslant X$, the second holds by assumption, and the third by (3.3). This proves $U \cap X \in [\![U_0, U]\!]^H$. Moreover, by the first equality, $\psi \circ \varphi(X) = (U \cap X)H = UH \cap X = X$, so $\psi \circ \varphi$ is the identity on $[\![H, UH]\!]$.

If $V \in [\![U_0, U]\!]^H$, then $VH = HV$ implies $VH \in [\![H, UH]\!]$. Also, $\varphi \circ \psi$ is the identity on $[\![U_0, U]\!]^H$, since $\varphi \circ \psi(V) = VH \cap U = V(H \cap U) = VU_0 = V$, by (3.3). This proves that φ and ψ are inverses of each other on the sets indicated, and it's easy to see that they are order preserving: $X \leqslant Y$ implies $U \cap X \leqslant U \cap Y$, and $V \leqslant W$ implies $VH \leqslant WH$. Therefore, φ and ψ are inverse order isomorphisms.

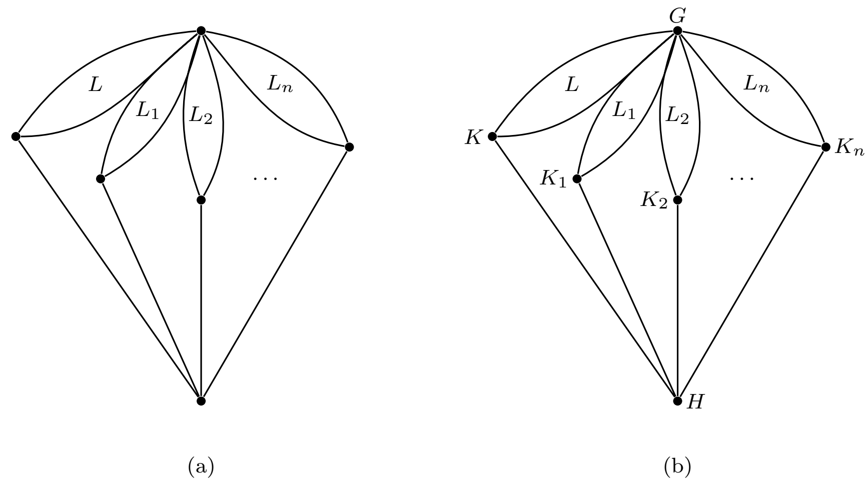
To complete the proof of (i), we show that $[\![U_0, U]\!]^H$ is a sublattice of $[\![U_0, U]\!]$. Suppose V_1 and V_2 are subgroups in $[\![U_0, U]\!]$ that permute with H. It is easy to see that their join $V_1 \vee V_2 = \langle V_1, V_2 \rangle$ also permutes with H, so we just check that their intersection permutes with H. Fix $x \in V_1 \cap V_2$ and $h \in H$. We show $xh = h'x'$ for some $h' \in H$, $x' \in V_1 \cap V_2$. Since V_1 and V_2 permute with H, we have $xh = h_1 v_1$ and $xh = h_2 v_2$ for some $h_1, h_2 \in H$, $v_1 \in V_1$, $v_2 \in V_2$. Therefore, $h_1 v_1 = h_2 v_2$, which implies $v_1 = h_1^{-1} h_2 v_2 \in HV_2$, so v_1 belongs to $V_1 \cap HV_2$. Note that $V_1 \cap HV_2$ is below both V_1 and $U \cap HV_2 = \varphi\psi(V_2) = V_2$. Therefore, $v_1 \in V_1 \cap HV_2 \leqslant V_1 \cap V_2$, and we have proved that $xh = h_1 v_1$ for $h_1 \in H$ and $v_1 \in V_1 \cap V_2$, as desired.

To prove (ii), assuming $U \trianglelefteqslant G := UH$, we show that if $U_0 \leqslant V \leqslant U$, then $VH = HV$ if and only if $H \leqslant N_G(V)$. If $H \leqslant N_G(V)$, then $VH = HV$ (even when $U \ntrianglelefteq G$). Suppose $VH = HV$. We must show $(\forall v \in V)\,(\forall h \in H)\ hvh^{-1} \in V$. Fix $v \in V$, $h \in H$. Then, $hv = v'h'$ for some $v' \in V$, $h' \in H$, since $VH = HV$. Therefore, $v'h'h^{-1} = hvh^{-1} = u$ for some $u \in U$, since $H \leqslant N_G(U)$. This proves that $hvh^{-1} \in VH \cap U = V(H \cap U) = VU_0 = V$, as desired. $\square$

3.3. **Parachute lattices.** We now prove the equivalence of statements (B), (C), and (D) mentioned in Section 1. (That statement (E) of Section 1 is also equivalent to (B) follows easily from the arguments given below, so we omit the details.)

Theorem 3.9. *The following statements are equivalent:*

FIGURE 4. The parachute construction.



(B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

(C) *For every finite lattice L, for every finite collection $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ of cf-IE classes of groups, there exists a finite group $G \in \bigcap_{i=1}^{n} \mathfrak{X}_i$ such that $L \cong [\![H, G]\!]$ for some core-free subgroup $H \leqslant G$.*

(D) *For every finite collection $\mathscr{L}$ of finite lattices, there exists a finite group G such that each $L_i \in \mathscr{L}$ is isomorphic to $[\![H_i, G]\!]$ for some core-free subgroup $H_i \leqslant G$.*

(E) *For every finite collection $\mathscr{L}$ of finite lattices, for every finite collection $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ of cf-IE classes of groups, there exists a finite group $G \in \bigcap_{i=1}^{n} \mathfrak{X}_i$ such that for each $L_i \in \mathscr{L}$ we have $L_i \cong [\![H_i, G]\!]$ for some subgroup H_i that is core-free in G.*

Remark. By (C), the FLRP would have a negative answer if we could find a collection $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ of cf-IE classes such that $\bigcap_{i=1}^{n} \mathfrak{X}_i$ is empty.

Proof. We prove the equivalence of (B) and (C). Obviously (C) implies (B). Assume (B) holds and let L be any finite lattice. Suppose $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ is a collection of cf-IE enforceable classes of groups. Then there exist finite lattices $L_1, \dots, L_n$ such that $L_i \cong [\![H_i, G_i]\!]$ with $\operatorname{core}_G(H_i) = 1$ implies $G_i \in \mathfrak{X}_i$. Construct a new lattice, denoted $\mathscr{P} = \mathscr{P}(L, L_1, \dots, L_n)$, as shown in the Hasse diagram of Figure 4 (a). Note that the bottoms of the L_i sublattices are atoms in $\mathscr{P}$. By (B), there exist groups $H < G$ with $\mathscr{P} \cong [\![H, G]\!]$. We can assume H is a core-free subgroup of G. (If not, replace G and H with G/N and H/N, where $N = \operatorname{core}_G(H)$, and note that $\mathscr{P} \cong [\![H, G]\!] \cong [\![H/N, G/N]\!]$.) Let $K, K_1, \dots, K_n$ be the subgroups of G in which H is maximal and for which $L \cong [\![K, G]\!]$ and $L_i \cong [\![K_i, G]\!]$, $1 \leqslant i \leqslant n$ (Figure 4 (b)). If we can prove for each $1 \leqslant i \leqslant n$ that $\operatorname{core}_G(K_i) = 1$, then $G \in \mathfrak{X}_i$ by hypothesis, and it will follow that $G \in \bigcap_{i=1}^{n} \mathfrak{X}_i$, proving that (B) implies (C).

If $L \cong \mathbf{2}$, the two element lattice, or if $L_j \cong \mathbf{2}$ for all $1 \leqslant j \leqslant n$, then the consequent of the implication in (C) holds trivially. So we can assume without loss of generality that $L \ncong \mathbf{2}$ and that there is at least one $1 \leqslant j \leqslant n$ for which $L_j \ncong \mathbf{2}$. Let N be a minimal normal subgroup of G and suppose $N \leqslant K_i$. Since H is core-free, $K_i = NH$. Suppose $K_i \neq K$. Note that the groups K and K_i permute:

$$KK_i = KNH = NKH = NHK = K_iK.$$

Therefore, by Lemma 3.8 (i) we see that the interval $\llbracket K, G \rrbracket \cong L$ must be isomorphic to a sublattice of the interval $\llbracket H, K_i \rrbracket \cong \mathbf{2}$, but this contradicts $L \ncong \mathbf{2}$. Suppose instead that $K = K_i$. Note that $K \neq K_j$, and since $K = NH$, we see that K and K_j permute. Therefore, by Lemma 3.8 (i) again, the lattice $L_j \ncong \mathbf{2}$ is isomorphic to a sublattice of $\llbracket H, K \rrbracket \cong \mathbf{2}$, which is impossible. This proves that $NH = G$ for all $N \trianglelefteqslant G$, so each K_i is core-free.

The proof that statements (B) and (D) are equivalent follows by a similar construction. Roughly, if $\mathscr{L} = \{L_1, \dots, L_n\}$, we form the lattice $\mathscr{P} = \mathscr{P}(L_1, \dots, L_n)$. If (B) holds, then there exists a group G with $\mathscr{P} \cong \llbracket H, G \rrbracket$ and $\operatorname{core}_G(H) = 1$. The proof that each K_i is core-free, where $L_i \cong \llbracket K_i, G \rrbracket$, is similar to the argument above. □

By a *parachute lattice*, denoted $\mathscr{P}(L_1, \dots, L_m)$, we mean a lattice just like the one illustrated in Figure 4 (a), but with the lattices $L_1, \dots, L_m$ appearing as the upper intervals.

Next we prove that any group that has a nontrivial parachute lattice as an upper interval in its subgroup lattice must have some rather special properties.

Lemma 3.10. *Let $\mathscr{P} = \mathscr{P}(L_1, \dots, L_n)$ with $n \geqslant 2$ and $|L_i| > 2$ for all i, and suppose $\mathscr{P} \cong \llbracket H, G \rrbracket$, with H core-free in G.*

(i) If $1 \neq N \trianglelefteqslant G$, then $NH = G$ and $C_G(N) = 1$.
(ii) G is subdirectly irreducible and nonsolvable.

Remark. If a subgroup $N \leqslant G$ is abelian, then $N \leqslant C_G(N)$, so (i) implies that every nontrivial normal subgroup of G is nonabelian.

Proof. (i) Let $1 \neq N \trianglelefteqslant G$. Then $N \nleqslant H$, since H is core-free in G. Therefore, $H < NH$. As in Section 3.3, we let K_i denote the subgroups of G corresponding to the atoms of $\mathscr{P}$. Then H is covered by each K_i, so $K_j \leqslant NH$ for some $1 \leqslant j \leqslant n$. Suppose, by way of contradiction, that $NH < G$. By assumption, $n \geqslant 2$ and $|L_i| > 2$. Thus for any $i \neq j$ we have $K_i \leqslant Y < Z < G$ for some subgroups Y and Z which satisfy $(NH) \cap Z = H$ and $(NH) \vee Y = G$. Also, $(NH)Y = NY$ is a group, so $(NH)Y = NH \vee Y = G$. But then, by Dedekind's rule, we have

$$Y = HY = ((NH) \cap Z)Y = (NH)Y \cap Z = G \cap Z = Z,$$

contrary to $Y < Z$. This contradiction proves that $NH = G$.

To prove that $C_G(N) = 1$, let M be a minimal normal subgroup of G contained in N. It suffices to prove $C_G(M) = 1$. Assume the contrary. Then, since $C_G(M) \trianglelefteqslant N_G(M) = G$, it follows by what we just proved that $C_G(M)H = G$. Consider any $H < K < G$. Then $1 < M \cap K < M$ (strictly, by Lemma 3.8). Now $M \cap K$

is normalized by H and centralized (hence normalized) by $C_G(M)$. Therefore, $M \cap K \trianglelefteqslant C_G(M)H = G$, contradicting the minimality of M.

To prove (ii) we first show that G has a unique minimal normal subgroup. Let M be a minimal normal subgroup of G and let $N \trianglelefteqslant G$ be any normal subgroup not containing M. We show that $N = 1$. Since both subgroups are normal, the commutator subgroup $[M, N]$ lies in the intersection $M \cap N$, which is trivial by the minimality of M. Thus, M and N centralize each other. In particular, $N \leqslant C_G(M) = 1$, by (i).

Finally, since G has a unique minimal normal subgroup that is nonabelian (see the remark preceding the proof), G is nonsolvable. □

To summarize what we have thus far, the lemmas above imply that (B) holds if and only if every finite lattice is an interval $[\![H, G]\!]$, with H core-free in G, where

(i) G is nonsolvable, not alternating, and not symmetric;
(ii) G has a unique minimal normal subgroup M which satisfies $MH = G$ and $C_G(M) = 1$; in particular, M is nonabelian and $\operatorname{core}_G(X) = 1$ for all $H \leqslant X < G$.

We conclude this section by formalizing the remarks of the previous sentence. Given two group theoretical properties $\mathcal{P}_1, \mathcal{P}_2$, we write $\mathcal{P}_1 \to \mathcal{P}_2$ to denote that property $\mathcal{P}_1$ implies property $\mathcal{P}_2$. In other words, $G \vDash \mathcal{P}_1$ only if $G \vDash \mathcal{P}_2$. Thus $\to$ provides a natural partial order on any given set of properties, as follows:

$$\mathcal{P}_1 \leqslant \mathcal{P}_2 \quad \Longleftrightarrow \quad \mathcal{P}_1 \to \mathcal{P}_2 \quad \Longleftrightarrow \quad \mathfrak{X}_{\mathcal{P}_1} \subseteq \mathfrak{X}_{\mathcal{P}_2},$$

where $\mathfrak{X}_{\mathcal{P}_i} = \{G \in \mathfrak{G} \mid G \vDash \mathcal{P}_i\}$. The following is an immediate corollary of the parachute construction described above.

Corollary 3.11. *If $\mathscr{P} = \{\mathcal{P}_i \mid i \in \mathscr{I}\}$ is a collection of (cf-)IE properties, then $\bigwedge \mathscr{P}$ is (cf-)IE.*

The conjunction $\bigwedge \mathscr{P}$ corresponds to the class $\bigcap_{i \in \mathscr{I}} \mathfrak{X}_{\mathcal{P}_i} = \{G \in \mathfrak{G} \mid (\forall i \in \mathscr{I})\ G \vDash \mathcal{P}_i\}$.

4. An application

We consider an application that demonstrates the utility of Lemma 3.8 for identifying certain core-free interval enforceable properties. The lattice that we study in this section has special relevance to the finite lattice representation problem (FLRP).

In prior work,[8] we considered whether, for every lattice L with at most n elements, there exists a finite algebra with a congruence lattice that is isomorphic to L. For $n = 6$, the problem had already been solved. In fact, Aschbacher [1] and Watatani [28] prove that every lattice with at most 6 elements is group representable (as defined in Section 3.1 above). For $n = 7$, although we have not found them all as intervals in subgroup lattices, we have found congruence lattice representations for all lattice with at most 7 elements with one exception (see [11], [10]). The exceptional lattice, which we call L_7, appears in Figure 5. Thus L_7 is the smallest lattice for which we have not found a representation of the form $L_7 \cong \operatorname{Con} \mathbf{A}$ for some finite algebra $\mathbf{A}$.

[8]Universal Algebra and Lattice Theory Seminar, University of Hawaii, 2010-11; participants: Ralph Freese, Tristan Holmes, Peter Jipsen, Bill Lampe, J. B. Nation and the author.

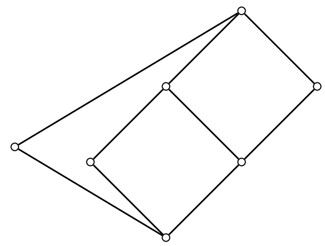


FIGURE 5. The exceptional seven element lattice, L_7.

Suppose $\mathbf{A}$ is a finite algebra with $\operatorname{Con}\mathbf{A} \cong L_7$, and suppose $\mathbf{A}$ is of minimal cardinality among those algebras having a congruence lattice isomorphic to L_7. Then $\mathbf{A}$ must be isomorphic to a transitive G-set. (This fact is proved in the forthcoming article [12].) Therefore, if L_7 is representable, we can assume there is a finite group G with a core-free subgroup $H < G$ such that L_7 is isomorphic to the interval sublattice $[\![H, G]\!] \leqslant \operatorname{Sub}(G)$. In this section we present some restrictions on the possible groups for which this can occur.

The first restriction, which is the easiest to observe, is that G must act primitively on the cosets of one of its maximal subgroups. This suggests the possibility of describing G in terms of the Aschbacher-O'Nan-Scott Theorem which characterizes primitive permutation groups.[9] Ultimately, the goal would be to find enough restrictions on G so as to rule out all finite groups. As yet, we have not achieved this goal. However, the new results in this section reduce the possibilities to special subclasses of the Aschbacher-O'Nan-Scott Theorem. This paves the way for future studies to focus on these subclasses when searching for a group representation of L_7, or proving that none exists.

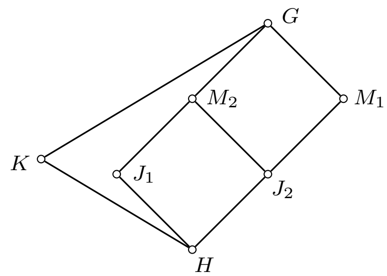
Proposition 4.1. *Suppose G is a finite group with $[\![H, G]\!] \cong L_7$ for some core-free subgroup $H < G$. Then the following hold.*

(i) G is a primitive permutation group.
(ii) If $N \lhd G$, then $C_G(N) = 1$.
(iii) G contains no non-trivial abelian normal subgroup.
(iv) G is nonsolvable.
(v) G is subdirectly irreducible.
(vi) With the possible exception of at most one maximal subgroup, all proper subgroups in the interval $[\![H, G]\!]$ are core-free.

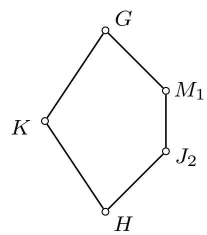
Remark. It is obvious that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), and (ii) $\Rightarrow$ (v), but we include these easy consequences in the statement of the result for emphasis; for, although the hard work will be in proving (ii) and (vi), our main goal is the pair of restrictions (iii) and (v), which allow us to rule out a number of the O'Nan-Scott types describing primitive permutation groups.

Assume the hypotheses of the proposition above. In particular, throughout this section *all groups are finite, H is a core-free subgroup of G, and* $[\![H, G]\!] \cong L_7$. Label the seven subgroups of G in the interval $[\![H, G]\!]$ as in the following diagram:

[9]The author thanks John Shareshian for this suggestion. See the comments at MathOverflow [9].



We now prove the foregoing proposition through a series of claims. The first thing to notice about the interval $[\![H, G]\!]$ is that K is a *non-modular element* of the interval. This means that there is a pentagonal (N_5) sublattice of the interval with K as the incomparable proper element. (See the diagram below, for example.)



Using this non-modularity property of K, it is easy to prove the following

Claim 4.1. *K is a core-free subgroup of G.*

Proof. Let $N = \mathrm{core}_G(K)$. If $N \leqslant X$ for some $X \in \{M_1, M_2, J_1, J_2\}$, then $N < X \cap K = H$, so $N = 1$ (since H is core-free). If $N \nleqslant X$ for all $X \in \{M_1, M_2, J_1, J_2\}$, then $NJ_2 = G$. But then Dedekind's rule leads to the following contradiction:

$$J_2 \leqslant M_1 \implies J_2 = J_2(N \cap M_1) = J_2N \cap M_1 = G \cap M_1 = M_1.$$

Therefore, $N = 1$. □

Note that (i) of the proposition follows from Claim 4.1. Since K is core-free, G acts faithfully on the cosets G/K by right multiplication. Since K is a maximal subgroup, the action is primitive.

The next claim is slightly harder than the previous one as it requires the more general consequence of Dedekind's rule that we established above in Lemma 3.8 (i).

Claim 4.2. *J_1 and J_2 are core-free subgroups of G.*

Proof. First note that if $N \trianglelefteqslant G$ then the subgroup NH permutes[10] with any subgroup containing H. To see this, let $H \leqslant X \leqslant G$ and note that

$$NHX = NX = XN = XHN = XNH,$$

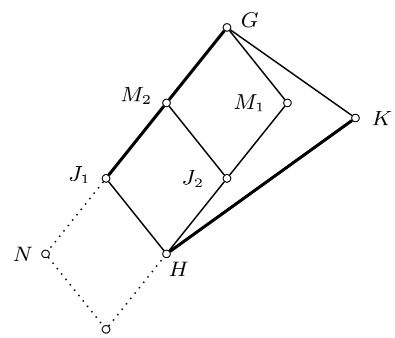
since $H \leqslant X$ and $N \trianglelefteqslant G$.

Suppose $1 \neq N \leqslant J_1$ for some $N \lhd G$. Then $NH = J_1$, so J_1 and K are permuting subgroups. Since $J_1K = G$ and $J_1 \cap K = H$, Lemma 3.8 yields

$$[\![J_1, G]\!] \cong [\![H, K]\!]^{J_1} := \{X \in [\![H, K]\!] \mid J_1X = XJ_1\}.$$

[10]Recall, for subgroups X and Y of a group G, we define the *sets* $XY = \{xy \mid x \in X, y \in Y\}$, and $YX = \{yx \mid x \in X, y \in Y\}$, and we say that X and Y are *permuting subgroups* (or that X and Y permute, or that X permutes with Y) provided the two sets XY and YX coincide, in which case the set forms a group: $XY = \langle X, Y\rangle = YX$.

But this is impossible since $[\![H,K]\!]^{J_1} \leqslant [\![H,K]\!] \cong \mathbf{2}$, while $[\![J_1,G]\!] \cong \mathbf{3}$. This proves that $\mathrm{core}_G(J_1) = 1$. The intervals involved in the argument are drawn with bold lines in the following diagram.



The proof that J_2 is core-free is similar. Suppose $1 \neq N \leqslant J_2$ where $N \lhd G$. Then $NH = J_2$ and the subgroups J_2 and K permute. Therefore, $[\![H,K]\!]^{J_2} \cong [\![J_2,G]\!]$, by Lemma 3.8, which is a contradiction since $[\![H,K]\!]^{J_2} \leqslant [\![H,K]\!] \cong \mathbf{2}$, while $[\![J_2,G]\!] \cong \mathbf{2} \times \mathbf{2}$. □

Now that we know K, J_1, J_2 are each core-free in G, we use this information to prove that at least one of the other maximal subgroups, M_1 or M_2, is core-free in G, thereby establishing (vi) of the proposition. We will also see that G is subdirectly irreducible, proving (v). The proof of (ii) will then follow from the same argument used to prove Lemma 3.8 (ii), which we repeat below.

Claim 4.3. Either M_1 or M_2 is core-free in G. If M_2 has non-trivial core and $N \lhd G$ is contained in M_2, then $C_G(N) = 1$ and G is subdirectly irreducible.

Proof. Suppose M_2 has non-trivial core. Then there is a minimal normal subgroup $1 \neq N \lhd G$ contained in M_2. Since H, J_1, J_2 are core-free, $NH = M_2$. Consider the centralizer, $C_G(N)$, of N in G. Of course, this is a normal subgroup of G. If $C_G(N) = 1$, then, since minimal normal subgroups centralize each other, N must be the unique minimal normal subgroup of G. Furthermore, M_1 must be core-free in this case. Otherwise $N \leqslant M_1 \cap M_2 = J_2$, contradicting $\mathrm{core}_G(J_2) = 1$. Therefore, in case $C_G(N) = 1$ we conclude that G is subdirectly irreducible and M_1 is core-free.

We now prove that the alternative, $C_G(N) \neq 1$, does not occur. This case is a bit more challenging and must be split up into further subcases, each of which leads to a contradiction. Throughout, the assumption $1 \neq N \leqslant M_2$ is in force, and it helps to keep in mind the diagram in Figure 6.

Suppose $C_G(N) \neq 1$. Then, since $C_G(N) \trianglelefteq G$, and since H, J_1, J_2, K are core-free, it's clear that $C_G(N)H \in \{G, M_1, M_2\}$. We consider each case separately.

Case 1: Suppose $C_G(N)H = G$. Note that $N \cap H < N \cap J_1 < N$ (strictly). The subgroup $N \cap J_1$ is normalized by J_1 and by $C_G(N)$, and so it is normal in $C_G(N)J_1 \geqslant C_G(N)H = G$, contradicting the minimality of N. Thus, the case $C_G(N)H = G$ does not occur.

Case 2: Suppose $C_G(N)H = M_1$. The subgroup $N \cap J_1$ is normalized by both H and $C_G(N)$. For, $C_G(N)$ centralizes, hence normalizes, every subgroup of

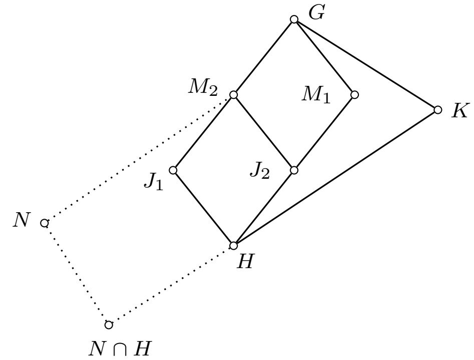


FIGURE 6. Hasse diagram illustrating the cases in which M_2 has non-trivial core: $1 \neq N \leqslant M_2$ for some $N \lhd G$.

N. Therefore, $N \cap J_1$ is normalized by $C_G(N)H = M_1$. Of course, it's also normalized by J_1, so $N \cap J_1$ is normalized by the set M_1J_1, so it's normalized by the group generated by that set, which is $\langle M_1, J_1\rangle = G$.[11] The conclusion is that $N \cap J_1 \lhd G$. Since J_1 is core-free, $N \cap J_1 = 1$. But this contradicts the (by now familiar) consequence of Dedekind's rule:

$$H < J_1 < M_2 \implies N \cap H < N \cap J_1 < N \cap M_2.$$

Therefore, $C_G(N)H = M_1$ does not occur.

Case 3: Suppose $C_G(N)H = M_2$. The subgroup $N \cap M_1$ is normalized by both H and $C_G(N)$. Therefore, $N\cap M_1$ is normalized by $C_G(N)H = M_2$. Of course, it's also normalized by M_1, so $N\cap M_1$ is normalized by $\langle M_1, M_2\rangle = G$. The conclusion is that $N \cap M_1 \lhd G$. By minimality of the normal subgroup N, we must have either $N \cap M_1 = 1$ or $N \cap M_1 = N$. The former equality implies $N \cap J_2 = 1$, which contradicts the strict inequalities of Dedekind's rule,

$$H < J_2 < M_2 \implies N \cap H < N \cap J_2 < N \cap M_2, \tag{4.1}$$

while the latter equality ($N \cap M_1 = N$) implies that $N \leqslant M_1 \cap M_2 = J_2$ which contradicts $\operatorname{core}_G(J_2) = 1$.

□

We have proved that either M_1 or M_2 is core-free in G, and we have shown that, if M_2 has non-trivial core, then G is subdirectly irreducible. In fact, we proved that $C_G(N) = 1$ for the unique minimal normal subgroup N in this case. It remains to prove that G is subdirectly irreducible in case M_1 has non-trivial core. The argument is similar to the foregoing, and we omit some of the details that can be checked exactly as above.

Claim 4.4. If M_1 has non-trivial core and $N \lhd G$ is contained in M_1, then $C_G(N) = 1$ and G is subdirectly irreducible.

---

[11] Actually, the set is already a group in this case since $M_1J_1 = C_G(N)HJ_1 = J_1C_G(N)H = J_1M_1$.

Proof. If M_1 has non-trivial core, then there is a minimal normal subgroup $N \lhd G$ contained in M_1. We proved above that M_2 must be core-free in this case, so either $C_G(N)H = G$, $C_G(N)H = M_1$, or $C_G(N) = 1$. The first case is easily ruled out exactly as in Case 1 above. The second case is handled by the argument we used in Case 3. Indeed, if we suppose $C_G(N)H = M_1$, then $N \cap M_2$ is normalized by both H and $C_G(N)$, hence by M_1. It is also normalized by M_2, so $N \cap M_2 \lhd G$. Thus, by minimality of N, and since M_2 is core-free, $N \cap M_2 = 1$. But then $N \cap J_2 = 1$, leading to a contradiction similar to (4.1) but with M_1 replacing M_2. Therefore, the case $C_G(N)H = M_1$ does not occur, and we have proved $C_G(N) = 1$. □

So far we have proved that all intermediate proper subgroups in the interval $[\![H, G]\!]$ are core-free except possibly at most one of M_1 or M_2. Moreover, we proved that if one of the maximal subgroups has non-trivial core, then there is a unique minimal normal subgroup $N \lhd G$ with trivial centralizer, $C_G(N) = 1$. As explained above, G is subdirectly irreducible in this case, since minimal normal subgroups centralize each other.

In order to prove (ii), there remains only one case left to check, and the argument is by now very familiar.

Claim 4.5. If each $H \leqslant X < G$ is core-free and N is a minimal normal subgroup of G, then $C_G(N) = 1$.

Proof. Let N be a minimal normal subgroup of G. Then, by the core-free hypothesis we have $NH = G$. Fix a subgroup $H < X < G$. Then $N \cap H < N \cap X < N$. The subgroup $N \cap X$ is normalized by H and by $C_G(N)$. If $C_G(N) \neq 1$, then $C_G(N)H = G$, by the core-free hypothesis, so $N \cap X \lhd G$, contradicting the minimality of N. Therefore, $C_G(N) \neq 1$. □

Finally, we note that the claims above taken together prove (ii), and thereby complete the proof of the proposition. For if G is subdirectly irreducible with unique minimal normal subgroup N, and if $C_G(N) = 1$, then all normal subgroups (which necessarily lie above N) must have trivial centralizers.

We conclude with a final observation which helps us describe the O'Nan-Scott type of a group that has L_7 as an interval in its subgroup lattice. By what we have proved, G acts primitively on the cosets of K, and it also acts primitively on the cosets of at least one of M_1 or M_2. Suppose M_1 is core-free so that G is a primitive permutation group in its action on cosets of M_1, and let N be a minimal normal subgroup of G. As we have seen, N has trivial centralizer, so it is nonabelian and is the unique minimal normal subgroup of G. Now, we have seen that $NH \geqslant M_2$ in this case, so $H < J_2 < NH$ implies that $N \cap M_1 \neq 1$. Similarly, if we had started out by assuming that M_2 is core-free, then $NH \geqslant M_1$, and $H < J_2 < NH$ would imply that $N \cap M_2 \neq 1$.

By an elementary result (see [15, Lemma 8.5]), if G acts transitively on a set Ω with stabilizer G_ω, then a subgroup $N \leqslant G$ acts transitively on Ω if and only if $NG_\omega = G$, and N is *regular*[12] if and only if in addition $N \cap G_\omega = 1$. Thus, in the present application, we see that the action of N on the cosets of the core-free maximal subgroup M_i is not regular. Consequently, G is characterized by the

[12]Recall, a transitive permutation group N is *acts regularly* on a set Ω provided the stabilizer subgroup of N is trivial. Equivalently, every non-identity element of N is fixed-point-free. Equivalently, N is regular on Ω if and only if for each $\omega_1, \omega_2 \in \Omega$ there is a unique $n \in N$ such that $n\omega_1 = \omega_2$. In particular, $|N| = |\Omega|$.

Aschbacher-O'Nan-Scott Theorem as being either almost simple, of product action type, or of diagonal type (see [13, Theorem 4.6A]).

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