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INTERVAL ENFORCEABLE PROPERTIES OF FINITE GROUPS

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ABSTRACT. We propose a classification of group properties according to whether they can be deduced from the assumption that a group's subgroup lattice contains an interval isomorphic to some lattice. Suppose $\mathcal{P}$ is a group property and suppose there exists a lattice L such that if G is a group with L isomorphic to an interval $[\![H, G]\!]$ in $\mathrm{Sub}(G)$, with H core-free, then G has property $\mathcal{P}$. We call such $\mathcal{P}$ *core-free interval enforceable.* Among other things we show that if both a property and its negation could be proved core-free interval enforceable, this would settle an important open question in universal algebra.

1. INTRODUCTION

The study of subgroup lattices has a long history, starting with Richard Dedekind [**?**] and Ada Rottlaender [**?**], and later a number of important contributions by Reinhold Baer, Øystein Ore, Michio Suzuki, Roland Schmidt, and many others (see Schmidt [**?**]). Much of this work focuses on the problem of inferring properties of a group G based on the structure of its lattice of subgroups, $\mathrm{Sub}(G)$, or, conversely, inferring lattice theoretical properties of $\mathrm{Sub}(G)$ from properties of G.

Historically, less attention was paid to the local structure of the subgroup lattice of a finite group, perhaps because it seemed that very little about G could be inferred from knowledge of, say, an *upper interval*, $[\![H, G]\!] = \{K \mid H \leqslant K \leqslant G\}$, in the subgroup lattice of G. Recently, however, this topic has attracted more attention (see, e.g., [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**]), mostly owing to its connections with one of the most important open problems in universal algebra, the *Finite Lattice Representation Problem* (FLRP). This is the problem of characterizing the lattices that are (isomorphic to) congruence lattices of finite algebras (see, e.g., [**?**, **?**, **?**, **?**]). There is a remarkable theorem relating this problem to intervals in subgroup lattices of finite groups.

Theorem 1.1 (Pálfy and Pudlák [**?**]). *The following statements are equivalent:*

(A) Every finite lattice is isomorphic to the congruence lattice of a finite algebra.
(B) Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.

If these statements are true (resp., false), then we say the FLRP has a positive (resp., negative) answer. Thus, if we can find a finite lattice L for which it can be

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proved that there is no finite group G with $L \cong [\![H, G]\!]$ for some $H < G$, then the FLRP has a negative answer.

In this paper we propose a new classification of group properties according to whether or not they can be deduced from the assumption that $\mathrm{Sub}(G)$ has an upper interval isomorphic to some finite lattice. We believe that discovering which group properties can (or cannot) be connected to the local structure of a subgroup lattice is itself a worthwhile endeavor, but we will also describe how such a classification could be used to solve the FLRP.

Suppose $\mathcal{P}$ is a *group theoretical property*[1] and suppose there exists a finite lattice L such that if G is a finite group with $L \cong [\![H, G]\!]$ for some $H \leqslant G$, then G has property $\mathcal{P}$. We call such a property $\mathcal{P}$ *interval enforceable* (IE) *by* L. An *interval enforceable class of groups* is a class of all groups having property $\mathcal{P}$, for some interval enforceable property $\mathcal{P}$.

Although it depends on the lattice L, generally speaking it is difficult to deduce very much about a group G from the assumption that an upper interval in $\mathrm{Sub}(G)$ is isomorphic to L. It becomes easier easier if, in addition to the hypothesis $L \cong [\![H, G]\!]$, we assume that the subgroup H is *core-free* in G; that is, H contains no nontrivial normal subgroup of G. Properties of G that can be deduced from these assumptions are what we call *core-free interval enforceable* (cf-IE).

Extending this idea, we consider finite collections $\mathscr{L}$ of finite lattices and ask what can be proved about a group G if one assumes that each $L_i \in \mathscr{L}$ is isomorphic to an upper interval $[\![H_i, G]\!] \leqslant \mathrm{Sub}(G)$, with each H_i core-free in G. Clearly, if $\mathrm{Sub}(G)$ has such upper intervals, and if corresponding to each $L_i \in \mathscr{L}$ there is a property $\mathcal{P}_i$ that is cf-IE by L_i, then G must have all of the properties $\mathcal{P}_i$. A related question is the following: Given a set $\mathscr{P}$ of cf-IE properties, is the conjunction $\bigwedge \mathscr{P}$ cf-IE? Corollary **??** answers this question affirmatively.

In this paper, we will identify some group properties that are cf-IE, and others that are not. We will see that the cf-IE properties found thus far are negations of common group properties (for example, "not solvable," "not almost simple," "not alternating," "not symmetric"). Moreover, we prove that in these special cases the corresponding group properties ("solvable," "almost simple," "alternating," "symmetric") that are not cf-IE. This and other considerations suggest that a group property and its negation cannot both be cf-IE. As yet, we are unable to prove this. A related question is whether, for every group property $\mathcal{P}$, either $\mathcal{P}$ is cf-IE or $\neg\mathcal{P}$ is cf-IE.

One of our main results (Theorem **??**) connects the foregoing ideas with the FLRP, as follows:

Statement (B) of Theorem 1.1 is equivalent to each of the following statements:

- (C) *For every finite lattice L, for every finite collection $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ of cf-IE classes of groups, there exists a finite group $G \in \bigcap_{i=1}^{n} \mathfrak{X}_i$ such that $L \cong [\![H, G]\!]$ for some subgroup H that is core-free in G.*
- (D) *For every finite collection $\mathscr{L}$ of finite lattices, there exists a finite group G such that for each $L \in \mathscr{L}$ we have $L \cong [\![H, G]\!]$ for some subgroup H that is core-free in G.*

[1]This and other italicized terms in the introduction will be defined more formally in Section 2.

In fact, the arguments proving the equivalence of these statements are easily combined to show that the following is also equivalent to statement (B):

(E) For every finite collection $\mathscr{L}$ of finite lattices, for every finite collection $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ of cf-IE classes of groups, there exists a finite group $G \in \bigcap_{i=1}^{n} \mathfrak{X}_i$ such that each $L_i \in \mathscr{L}$ is isomorphic to an upper interval $[\![H_i, G]\!]$ with H core-free in G.

Core-free interval enforceable properties are related to permutation representations of groups in a manner that we now describe. If H is a core-free subgroup of G, then G has a faithful permutation representation $\varphi : G \hookrightarrow \mathrm{Sym}(G/H)$. Let $\langle G/H, \varphi(G)\rangle$ denote the algebra comprised of the right cosets G/H acted upon by right multiplication by elements of G; that is, $\varphi(g) : Hx \mapsto Hxg$. It is well known that the congruence lattice of this algebra (i.e., the lattice of systems of imprimitivity) is isomorphic to the interval $[\![H, G]\!]$ in the subgroup lattice of G.[2] This puts statement (E) into perspective. If the FLRP has a positive answer, then no matter what we take as our finite collection $\mathscr{L}$—for example, we might take $\mathscr{L}$ to be *all* finite lattices with at most N elements for some large $N < \omega$—we can always find a *single* finite group G such that every lattice in $\mathscr{L}$ is isomorphic to the interval in $\mathrm{Sub}(G)$ above a core-free subgroup. As a result, this group G must have so many faithful representations $G \hookrightarrow \mathrm{Sym}(G/H_i)$ with systems of imprimitivity isomorphic to L_i, one such representation for each distinct $L_i \in \mathscr{L}$. Moreover, the group G having this property can be chosen from the class $\bigcap_{i=1}^{n} \mathfrak{X}_i$, where $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ is an arbitrary collection of cf-IE classes of groups.

2. Notation and definitions

In this paper, *all groups and lattices are finite.* We use $\mathfrak{G}$ to denote the class of all finite groups. Given a group G, we denote the set of subgroups of G by $\mathrm{Sub}(G)$. The algebra $\langle \mathrm{Sub}(G), \wedge, \vee\rangle$ is a lattice where the $\wedge$ ("meet") and $\vee$ ("join") operations are defined for all H and K in $\mathrm{Sub}(G)$ by $H \wedge K = H \cap K$ and $H \vee K = \langle H, K\rangle =$ the smallest subgroup of G containing both H and K. We will refer to the set $\mathrm{Sub}(G)$ as a lattice, without explicitly mentioning the $\wedge$ and $\vee$ operations.

By $H \leqslant G$ (resp., $H < G$) we mean H is a subgroup (resp., proper subgroup) of G. For $H \leqslant G$, the *core of H in G*, denoted by $\mathrm{core}_G(H)$, is the largest normal subgroup of G contained in H. If $\mathrm{core}_G(H) = 1$, then we say that H is *core-free in* G. For $H \leqslant G$, by the *interval* $[\![H, G]\!]$ we mean the set $\{K \mid H \leqslant K \leqslant G\}$, which is a sublattice of $\mathrm{Sub}(G)$. With this notation, $\mathrm{Sub}(G) = [\![1, G]\!]$. When viewing $[\![H, G]\!]$ as a sublattice of $\mathrm{Sub}(G)$, we sometimes refer to it as an *upper interval.* Given a lattice L, if there is no mention of specific groups H and G, then the expression $L \cong [\![H, G]\!]$ means "there exist (finite) groups $H \leqslant G$ such that L is isomorphic to the interval $\{K \mid H \leqslant K \leqslant G\}$ in the subgroup lattice of G."

By a *group theoretical class*, or *class of groups*, we mean a collection $\mathfrak{X}$ of groups that is closed under isomorphism: if $G_0 \in \mathfrak{X}$ and $G_1 \cong G_0$, then $G_1 \in \mathfrak{X}$. A *group theoretical property*, or simply *property of groups*, is a property $\mathcal{P}$ such that if a group G_0 has property $\mathcal{P}$ and $G_1 \cong G_0$, then G_1 has property $\mathcal{P}$.[3] Thus if $\mathfrak{X}_{\mathcal{P}}$

[2] See [?, Lemma 4.20] or [?, Theorem 1.5A].

[3] It seems there is no single standard definition of *group theoretical class.* While some authors (e.g., [?], [?]) use the same definition we use here, others (e.g. [?], [?]) require that every group

denotes the collection of all groups having the group property $\mathcal{P}$, then $\mathfrak{X}_{\mathcal{P}}$ is a class of groups, and belonging to a particular class of groups is a group theoretical property. Therefore, we need not distinguish between a property of groups and the class of groups that possess this property.

If $\mathscr{K}$ is a class of algebras (e.g., a class of groups), then we say that $\mathscr{K}$ is *closed under homomorphic images* and we write $\mathbf{H}(\mathscr{K}) = \mathscr{K}$ provided $\varphi(G) \in \mathscr{K}$ whenever $G \in \mathscr{K}$ and φ is a homomorphism of G. By the first isomorphism theorem for groups, this is equivalent to: $G/N \in \mathscr{K}$ whenever $G \in \mathscr{K}$ and $N \trianglelefteqslant G$. For algebras, $\mathbf{H}(\mathscr{K}) = \mathscr{K}$ holds if and only if $\mathbf{A}/\theta \in \mathscr{K}$ for all $\mathbf{A} \in \mathscr{K}$ and all $\theta \in \operatorname{Con}\mathbf{A}$, where $\operatorname{Con}\mathbf{A}$ denotes the lattice of congruence relations of $\mathbf{A}$. Apart from possible notational differences, the foregoing terminology is standard.

We now introduce some new terminology that we find useful.[4] Let $\mathfrak{L}$ denote the class of all finite lattices, and $\mathfrak{G}$ the class of all finite groups. Let $\mathcal{P}$ be a given group theoretical property and $\mathfrak{X}_{\mathcal{P}}$ the associated class of all groups with property $\mathcal{P}$. We call $\mathcal{P}$ (and $\mathfrak{X}_{\mathcal{P}}$)

- *interval enforceable* (IE) provided
$$(\exists L \in \mathfrak{L})\ (\forall G \in \mathfrak{G})\ \big(L \cong [\![H, G]\!] \longrightarrow G \in \mathfrak{X}_{\mathcal{P}}\big)$$
- *core-free interval enforceable* (cf-IE) provided
$$(\exists L \in \mathfrak{L})\ (\forall G \in \mathfrak{G})\ \Big(\big(L \cong [\![H, G]\!] \bigwedge \operatorname{core}_G(H) = 1\big) \longrightarrow G \in \mathfrak{X}_{\mathcal{P}}\Big)$$
- *minimal interval enforceable* (min-IE) provided there exists $L \in \mathfrak{L}$ such that if $L \cong [\![H, G]\!]$ for some group $G \in \mathfrak{G}$ of minimal order (with respect to $L \cong [\![H, G]\!]$), then $G \in \mathfrak{X}_{\mathcal{P}}$.

In this paper we will have little to say about min-IE properties. Nonetheless, we include this class in our list of new definitions because properties of this type arise often (see, e.g., [**?**]), and a primary aim of this paper is to formalize various notions of interval enforceability that we believe are useful in applications.

3. Results

Clearly, if $\mathcal{P}$ is an interval enforceable property, then it is also core-free interval enforceable. There is an easy sufficient condition under which the converse holds. Suppose $\mathcal{P}$ is a group property, let $\mathfrak{X}_{\mathcal{P}}$ denote the class of all groups with property $\mathcal{P}$, and let $\mathfrak{X}^{c}_{\mathcal{P}}$ denote the class of all groups that do not have property $\mathcal{P}$.

Lemma 3.1. *Suppose $\mathcal{P}$ is a core-free interval enforceable property. If $\mathbf{H}(\mathfrak{X}^{c}_{\mathcal{P}}) = \mathfrak{X}^{c}_{\mathcal{P}}$, then $\mathcal{P}$ is an interval enforceable property.*

Proof. Since $\mathcal{P}$ is cf-IE there is a lattice L such that

$$\big(L \cong [\![H, G]\!] \bigwedge \operatorname{core}_G(H) = 1\big) \longrightarrow G \in \mathfrak{X}_{\mathcal{P}}. \tag{3.1}$$

Under the assumption $\mathbf{H}(\mathfrak{X}^{c}_{\mathcal{P}}) = \mathfrak{X}^{c}_{\mathcal{P}}$ we prove

$$L \cong [\![H, G]\!] \longrightarrow G \in \mathfrak{X}_{\mathcal{P}}. \tag{3.2}$$

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theoretical class contains the one element group. In the sequel we consider negations of group properties, and we would like these to qualify as group properties. Therefore, we don't require that every group theoretical class contains the one element group.

[4]The author thanks Bjørn Kjos-Hanssen and David Ross for suggesting improvements to the wording of these definitions.

If (3.2) fails, then there is a group $G \in \mathfrak{X}_{\mathcal{P}}^c$ with $L \cong [\![H, G]\!]$. Let $N = \operatorname{core}_G(H)$. Then $L \cong [\![H/N, G/N]\!]$ and H/N is core-free in G/N so, by hypothesis (3.1), $G/N \in \mathfrak{X}_{\mathcal{P}}$. But $G/N \in \mathfrak{X}_{\mathcal{P}}^c$, since $\mathfrak{X}_{\mathcal{P}}^c$ is closed under homomorphic images. □

In [?], Péter Pálfy gives an example of a lattice that cannot occur as an upper interval in the subgroup lattice finite solvable group. (We give other examples in Section ??.) In his Ph.D. thesis [?], Alberto Basile proves that if G is an alternating or symmetric group, then there are certain lattices that cannot occur as upper intervals in $\mathrm{Sub}(G)$. Another class of lattices with this property is described by Aschbacher and Shareshian in [?]. Thus, two classes of groups that are known to be at least cf-IE are the following:

- $\mathfrak{X}_0 = \mathfrak{S}^c =$ nonsolvable finite groups;
- $\mathfrak{X}_1 = \{G \in \mathfrak{G} \mid (\forall n < \omega)\ (G \neq A_n \bigwedge G \neq S_n)\}$,

where A_n and S_n denote, respectively, the alternating and symmetric groups on n letters. Note that both classes $\mathfrak{X}_0$ and $\mathfrak{X}_1$ satisfy the hypothesis of 3.1. Explicitly, $\mathfrak{X}_0^c = \mathfrak{S}$, the class of solvable groups, is closed under homomorphic images, as is the class $\mathfrak{X}_1^c$ of alternating and symmetric groups. Therefore, by Lemma 3.1, $\mathfrak{X}_0$ and $\mathfrak{X}_1$ are IE classes. By contrast, suppose there exists a finite lattice L such that

$$\big(L \cong [\![H, G]\!] \bigwedge \operatorname{core}_G(H) = 1\big) \longrightarrow G \text{ is subdirectly irreducible.}$$

Lemma 3.1 does not apply in this case since the class of subdirectly reducible groups is obviously not closed under homomorphic images.[5] In Section ?? below we describe lattices with which we can prove that the following classes are at least cf-IE:

- $\mathfrak{X}_2 =$ the subdirectly irreducible groups;
- $\mathfrak{X}_3 =$ the groups having no nontrivial abelian normal subgroups;
- $\mathfrak{X}_4 = \{G \in \mathfrak{G} \mid C_G(M) = 1 \text{ for all } 1 \neq M \trianglelefteqslant G\}$.

We noted above that $\mathfrak{X}_2$ fails to satisfy the hypothesis of 3.1. The same can be said of $\mathfrak{X}_3$ and $\mathfrak{X}_4$. That is, $\mathbf{H}(\mathfrak{X}_i^c) \neq \mathfrak{X}_i^c$ for $i = 2, 3, 4$. To verify this take $H \in \mathfrak{X}_i$, $K \in \mathfrak{X}_i^c$, and consider $H \times K$. In each case ($i = 2, 3, 4$) we see that $H \times K$ belongs to $\mathfrak{X}_i^c$, but the homomorphic image $(H \times K)/(1 \times K) \cong H$ does not.

3.1. Negations of interval enforceable properties.

If a lattice L is isomorphic to an interval in the subgroup lattice of a finite group, then we call L *group representable*. Recall, Theorem 1.1 says that the FLRP has a negative answer if we can find a finite lattice that is not group representable.

Suppose there exists a property $\mathcal{P}$ such that both $\mathcal{P}$ and its negation $\neg\mathcal{P}$ are interval enforceable by the lattices L and L_c, respectively. That is $L \cong [\![H, G]\!]$ implies G has property $\mathcal{P}$, and $L_c \cong [\![H_c, G_c]\!]$ implies G_c does not have property $\mathcal{P}$. Then clearly the lattice in Figure ?? could not be group representable. As the next result shows, however, if a group property and its negation are interval enforceable by the lattices L and L_c, then already at least one of these lattices is not group representable.

[5] Recall, for groups *subdirectly irreducible* is equivalent to having a unique minimal normal subgroup. Every algebra, in particular every group G, has a subdirect decomposition into subdirectly irreducibles, say, $G \hookrightarrow G/N_1 \times \cdots \times G/N_n$, so there are always subdirectly irreducible homomorphic images.