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INTERVAL ENFORCEABLE PROPERTIES OF FINITE GROUPS

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ABSTRACT. We propose a classification of group properties according to whether they can be deduced from the assumption that a group's subgroup lattice contains an interval isomorphic to some lattice. We are able to classify a few group properties as being "interval enforceable" in this sense, and we establish that other properties satisfy a weaker notion of "core-free interval enforceable." We also show that if there exists a group property and its negation that are both core-free interval enforceable, this would settle an important open question in universal algebra.

1. INTRODUCTION

The study of subgroup lattices has a long history that began with Richard Dedekind [7] and Ada Rottlaender [21], and continued with important contributions by Reinhold Baer, Øystein Ore, Michio Suzuki, Roland Schmidt, and many others (see Schmidt [22]). Much of this work focuses on the problem of deducing properties of a group G from assumptions about the structure of its lattice of subgroups, $\mathrm{Sub}(G)$, or, conversely, deducing lattice theoretical properties of $\mathrm{Sub}(G)$ from assumptions about G.

Historically, less attention was paid to the local structure of the subgroup lattice of a finite group, perhaps because it seemed that very little about G could be inferred from knowledge of, say, an *upper interval*, $[\![H, G]\!] = \{K \mid H \leqslant K \leqslant G\}$, in the subgroup lattice of G. Recently, however, this topic has attracted more attention (see, e.g., [1, 2, 4, 6, 11, 13, 15, 16]), mostly owing to its connection with one of the most important open problems in universal algebra, the *Finite Lattice Representation Problem* (FLRP). This is the problem of characterizing the lattices that are (isomorphic to) congruence lattices of finite algebras (see, e.g., [5, 8, 16, 17]). There is a remarkable theorem relating this problem to intervals in subgroup lattices of finite groups.

Theorem 1.1 (Pálfy and Pudlák [18]). *The following statements are equivalent:*
(A) Every finite lattice is isomorphic to the congruence lattice of a finite algebra.
(B) Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.

If these statements are true (resp., false), then we say the FLRP has a positive (resp., negative) answer. Thus, if we can find a finite lattice L for which it can be

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proved that there is no finite group G with $L \cong [\![H,G]\!]$ for some $H < G$, then the FLRP has a negative answer.

In this paper we propose a new classification of group properties according to whether or not they can be deduced from the assumption that $\mathrm{Sub}(G)$ has an upper interval isomorphic to some finite lattice. We believe that discovering which group properties can (or cannot) be connected to the local structure of a subgroup lattice is itself a worthwhile endeavor, but we will also describe how this classification could provide a solution of the FLRP.

Suppose $\mathcal{P}$ is a *group theoretical property*[1] and suppose there exists a finite lattice L such that if G is a finite group with $L \cong [\![H,G]\!]$ for some $H \leqslant G$, then G has property $\mathcal{P}$. We call such a property $\mathcal{P}$ *interval enforceable* (IE). If the lattice involved is germaine to the discussion, we say that $\mathcal{P}$ is *interval enforceable by L*. An *interval enforceable class of groups* is a class of groups all of which have a common interval enforceable property.

Although it depends on the lattice L, generally speaking it is difficult to deduce very much about a group G from the assumption that an upper interval in $\mathrm{Sub}(G)$ is isomorphic to L. It becomes easier easier if, in addition to the hypothesis $L \cong [\![H,G]\!]$, we assume that the subgroup H is *core-free* in G; that is, H contains no nontrivial normal subgroup of G. Properties of G that can be deduced from these assumptions are what we call *core-free interval enforceable* (cf-IE).

Extending this idea, we consider finite collections $\mathscr{L}$ of finite lattices and ask what can be proved about a group G if one assumes that each $L_i \in \mathscr{L}$ is isomorphic to an upper interval $[\![H_i,G]\!] \leqslant \mathrm{Sub}(G)$, with each H_i core-free in G. Clearly, if $\mathrm{Sub}(G)$ has such upper intervals, and if corresponding to each $L_i \in \mathscr{L}$ there is a property $\mathcal{P}_i$ that is cf-IE by L_i, then G must have all of the properties $\mathcal{P}_i$. A related question is the following: Given a set $\mathscr{P}$ of cf-IE properties, is the conjunction $\bigwedge \mathscr{P}$ cf-IE? Corollary 3.9 answers this question affirmatively.

In this paper, we will identify some group properties that are cf-IE, and others that are not. We will see that the cf-IE properties found thus far are negations of common group properties (for example, "not solvable," "not almost simple," "not alternating," "not symmetric"). Moreover, we prove that in these special cases the corresponding group properties ("solvable," "almost simple," "alternating," "symmetric") that are not cf-IE. This and other considerations suggest that a group property and its negation cannot both be cf-IE. As yet, we are unable to prove this. A related question is whether, for every group property $\mathcal{P}$, either $\mathcal{P}$ is cf-IE or $\neg\mathcal{P}$ is cf-IE.

One of our main results (Theorem 3.7) connects the foregoing ideas with the FLRP, as follows:

Statement (B) of Theorem 1.1 is equivalent to each of the following statements:

- *(C) For every finite lattice L, for every finite collection $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ of cf-IE classes of groups, there exists a finite group $G \in \bigcap_{i=1}^{n} \mathfrak{X}_i$ such that $L \cong [\![H,G]\!]$ for some subgroup H that is core-free in G.*
- *(D) For every finite collection $\mathscr{L}$ of finite lattices, there exists a finite group G such that for each $L \in \mathscr{L}$ we have $L \cong [\![H,G]\!]$ for some subgroup H that is core-free in G.*

[1] This and other italicized terms in the introduction will be defined more formally in Section 2.

In fact, the arguments proving the equivalence of these statements are easily combined to show that the following is also equivalent to statement (B):

(E) For every finite collection $\mathscr{L}$ of finite lattices, for every finite collection $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ of cf-IE classes of groups, there exists a finite group $G \in \bigcap_{i=1}^{n} \mathfrak{X}_i$ such that each $L_i \in \mathscr{L}$ is isomorphic to an upper interval $[\![H_i, G]\!]$ with H core-free in G.

Core-free interval enforceable properties are related to permutation representations of groups in a manner that we now describe. If H is a core-free subgroup of G, then G has a faithful permutation representation $\varphi : G \hookrightarrow \mathrm{Sym}(G/H)$. Let $\langle G/H, \varphi(G)\rangle$ denote the algebra comprised of the right cosets G/H acted upon by right multiplication by elements of G; that is, $\varphi(g) : Hx \mapsto Hxg$. It is well known that the congruence lattice of this algebra (i.e., the lattice of systems of imprimitivity) is isomorphic to the interval $[\![H, G]\!]$ in the subgroup lattice of G.[2] This puts statement (E) into perspective. If the FLRP has a positive answer, then no matter what we take as our finite collection $\mathscr{L}$—for example, we might take $\mathscr{L}$ to be *all* finite lattices with at most N elements for some large $N < \omega$—we can always find a *single* finite group G such that every lattice in $\mathscr{L}$ is isomorphic to the interval in $\mathrm{Sub}(G)$ above a core-free subgroup. As a result, this group G must have so many faithful representations $G \hookrightarrow \mathrm{Sym}(G/H_i)$ with systems of imprimitivity isomorphic to L_i, one such representation for each distinct $L_i \in \mathscr{L}$. Moreover, the group G having this property can be chosen from the class $\bigcap_{i=1}^{n} \mathfrak{X}_i$, where $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ is an arbitrary collection of cf-IE classes of groups.

2. Notation and definitions

In this paper, *all groups and lattices are finite*. We use $\mathfrak{G}$ to denote the class of all finite groups. Given a group G, we denote the set of subgroups of G by $\mathrm{Sub}(G)$. The algebra $\langle \mathrm{Sub}(G), \wedge, \vee\rangle$ is a lattice where the $\wedge$ ("meet") and $\vee$ ("join") operations are defined for all H and K in $\mathrm{Sub}(G)$ by $H \wedge K = H \cap K$ and $H \vee K = \langle H, K\rangle =$ the smallest subgroup of G containing both H and K. We will refer to the set $\mathrm{Sub}(G)$ as a lattice, without explicitly mentioning the $\wedge$ and $\vee$ operations.

By $H \leqslant G$ (resp., $H < G$) we mean H is a subgroup (resp., proper subgroup) of G. For $H \leqslant G$, the *core of H in G*, denoted by $\mathrm{core}_G(H)$, is the largest normal subgroup of G contained in H. If $\mathrm{core}_G(H) = 1$, then we say that H is *core-free in* G. For $H \leqslant G$, by the *interval* $[\![H, G]\!]$ we mean the set $\{K \mid H \leqslant K \leqslant G\}$, which is a sublattice of $\mathrm{Sub}(G)$. With this notation, $\mathrm{Sub}(G) = [\![1, G]\!]$. When viewing $[\![H, G]\!]$ as a sublattice of $\mathrm{Sub}(G)$, we sometimes refer to it as an *upper interval.* Given a lattice L, if there is no mention of specific groups H and G, then the expression $L \cong [\![H, G]\!]$ means "there exist (finite) groups $H \leqslant G$ such that L is isomorphic to the interval $\{K \mid H \leqslant K \leqslant G\}$ in the subgroup lattice of G."

By a *group theoretical class*, or *class of groups*, we mean a collection $\mathfrak{X}$ of groups that is closed under isomorphism: if $G_0 \in \mathfrak{X}$ and $G_1 \cong G_0$, then $G_1 \in \mathfrak{X}$. A *group theoretical property*, or simply *property of groups*, is a property $\mathcal{P}$ such that if a group G_0 has property $\mathcal{P}$ and $G_1 \cong G_0$, then G_1 has property $\mathcal{P}$.[3] Thus if $\mathfrak{X}_{\mathcal{P}}$

[2] See [14, Lemma 4.20] or [9, Theorem 1.5A].

[3] It seems there is no single standard definition of *group theoretical class.* While some authors (e.g., [10], [3]) use the same definition we use here, others (e.g. [19], [20]) require that every group

denotes the collection of all groups having the group property $\mathcal{P}$, then $\mathfrak{X}_{\mathcal{P}}$ is a class of groups, and belonging to a particular class of groups is a group theoretical property. Therefore, we need not distinguish between a property of groups and the class of groups that possess this property.

If $\mathscr{K}$ is a class of algebras (e.g., a class of groups), then we say that $\mathscr{K}$ is *closed under homomorphic images* and we write $\mathbf{H}(\mathscr{K}) = \mathscr{K}$ provided $\varphi(G) \in \mathscr{K}$ whenever $G \in \mathscr{K}$ and φ is a homomorphism of G. By the first isomorphism theorem for groups, this is equivalent to: $G/N \in \mathscr{K}$ whenever $G \in \mathscr{K}$ and $N \trianglelefteq G$. For algebras, $\mathbf{H}(\mathscr{K}) = \mathscr{K}$ holds if and only if $\mathbf{A}/\theta \in \mathscr{K}$ for all $\mathbf{A} \in \mathscr{K}$ and all $\theta \in \operatorname{Con} \mathbf{A}$, where $\operatorname{Con} \mathbf{A}$ denotes the lattice of congruence relations of $\mathbf{A}$. Apart from possible notational differences, the foregoing terminology is standard.

We now introduce some new terminology that we find useful.[4] Let $\mathfrak{L}$ denote the class of all finite lattices, and $\mathfrak{G}$ the class of all finite groups. Let $\mathcal{P}$ be a given group theoretical property and $\mathfrak{X}_{\mathcal{P}}$ the associated class of all groups with property $\mathcal{P}$. We call $\mathcal{P}$ (and $\mathfrak{X}_{\mathcal{P}}$)

- *interval enforceable* (IE) provided
$$(\exists L \in \mathfrak{L})\ (\forall G \in \mathfrak{G})\ \big(L \cong [\![H, G]\!] \longrightarrow G \in \mathfrak{X}_{\mathcal{P}}\big)$$
- *core-free interval enforceable* (cf-IE) provided
$$(\exists L \in \mathfrak{L})\ (\forall G \in \mathfrak{G})\ \Big(\big(L \cong [\![H, G]\!] \bigwedge \operatorname{core}_G(H) = 1\big) \longrightarrow G \in \mathfrak{X}_{\mathcal{P}}\Big)$$
- *minimal interval enforceable* (min-IE) provided there exists $L \in \mathfrak{L}$ such that if $L \cong [\![H, G]\!]$ for some group $G \in \mathfrak{G}$ of minimal order (with respect to $L \cong [\![H, G]\!]$), then $G \in \mathfrak{X}_{\mathcal{P}}$.

In this paper we will have little to say about min-IE properties. Nonetheless, we include this class in our list of new definitions because properties of this type arise often (see, e.g., [13]), and a primary aim of this paper is to formalize various notions of interval enforceability that we believe are useful in applications.

3. Results

Clearly, if $\mathcal{P}$ is an interval enforceable property, then it is also core-free interval enforceable. There is an easy sufficient condition under which the converse holds. Suppose $\mathcal{P}$ is a group property, let $\mathfrak{X}_{\mathcal{P}}$ denote the class of all groups with property $\mathcal{P}$, and let $\mathfrak{X}^c_{\mathcal{P}}$ denote the class of all groups that do not have property $\mathcal{P}$.

Lemma 3.1. *Suppose $\mathcal{P}$ is a core-free interval enforceable property. If $\mathbf{H}(\mathfrak{X}^c_{\mathcal{P}}) = \mathfrak{X}^c_{\mathcal{P}}$, then $\mathcal{P}$ is an interval enforceable property.*

Proof. Since $\mathcal{P}$ is cf-IE there is a lattice L such that

$$\big(L \cong [\![H, G]\!] \bigwedge \operatorname{core}_G(H) = 1\big) \longrightarrow G \in \mathfrak{X}_{\mathcal{P}}. \tag{3.1}$$

Under the assumption $\mathbf{H}(\mathfrak{X}^c_{\mathcal{P}}) = \mathfrak{X}^c_{\mathcal{P}}$ we prove

$$L \cong [\![H, G]\!] \longrightarrow G \in \mathfrak{X}_{\mathcal{P}}. \tag{3.2}$$

theoretical class contains the one element group. In the sequel we consider negations of group properties, and we would like these to qualify as group properties. Therefore, we don't require that every group theoretical class contains the one element group.

[4]The author thanks Bjørn Kjos-Hanssen and David Ross for suggesting improvements to the wording of these definitions.

If (3.2) fails, then there is a group $G \in \mathfrak{X}_{\mathcal{P}}^c$ with $L \cong [\![H, G]\!]$. Let $N = \mathrm{core}_G(H)$. Then $L \cong [\![H/N, G/N]\!]$ and H/N is core-free in G/N so, by hypothesis (3.1), $G/N \in \mathfrak{X}_{\mathcal{P}}$. But $G/N \in \mathfrak{X}_{\mathcal{P}}^c$, since $\mathfrak{X}_{\mathcal{P}}^c$ is closed under homomorphic images. □

In [16], Péter Pálfy gives an example of a lattice that cannot occur as an upper interval in the subgroup lattice finite solvable group. (We give other examples in Section 3.3.) In his Ph.D. thesis [4], Alberto Basile proves that if G is an alternating or symmetric group, then there are certain lattices that cannot occur as upper intervals in $\mathrm{Sub}(G)$. Another class of lattices with this property is described by Aschbacher and Shareshian in [1]. Thus, two classes of groups that are known to be at least cf-IE are the following:

- $\mathfrak{X}_0 = \mathfrak{S}^c =$ nonsolvable finite groups;
- $\mathfrak{X}_1 = \{G \in \mathfrak{G} \mid (\forall n < \omega)\ (G \neq A_n \bigwedge G \neq S_n)\}$,

where A_n and S_n denote, respectively, the alternating and symmetric groups on n letters. Note that both classes $\mathfrak{X}_0$ and $\mathfrak{X}_1$ satisfy the hypothesis of 3.1. Explicitly, $\mathfrak{X}_0^c = \mathfrak{S}$, the class of solvable groups, is closed under homomorphic images, as is the class $\mathfrak{X}_1^c$ of alternating and symmetric groups. Therefore, by Lemma 3.1, $\mathfrak{X}_0$ and $\mathfrak{X}_1$ are IE classes. By contrast, suppose there exists a finite lattice L such that

$$\big(L \cong [\![H, G]\!] \bigwedge \mathrm{core}_G(H) = 1\big) \longrightarrow G \text{ is subdirectly irreducible.}$$

Lemma 3.1 does not apply in this case since the class of subdirectly reducible groups is obviously not closed under homomorphic images.[5] In Section 3.3 below we describe lattices with which we can prove that the following classes are at least cf-IE:

- $\mathfrak{X}_2 =$ the subdirectly irreducible groups;
- $\mathfrak{X}_3 =$ the groups having no nontrivial abelian normal subgroups;
- $\mathfrak{X}_4 = \{G \in \mathfrak{G} \mid C_G(M) = 1 \text{ for all } 1 \neq M \trianglelefteq G\}$.

We noted above that $\mathfrak{X}_2$ fails to satisfy the hypothesis of 3.1. The same can be said of $\mathfrak{X}_3$ and $\mathfrak{X}_4$. That is, $\mathbf{H}(\mathfrak{X}_i^c) \neq \mathfrak{X}_i^c$ for $i = 2, 3, 4$. To verify this take $H \in \mathfrak{X}_i$, $K \in \mathfrak{X}_i^c$, and consider $H \times K$. In each case ($i = 2, 3, 4$) we see that $H \times K$ belongs to $\mathfrak{X}_i^c$, but the homomorphic image $(H \times K)/(1 \times K) \cong H$ does not.

3.1. **Negations of interval enforceable properties.** If a lattice L is isomorphic to an interval in the subgroup lattice of a finite group, then we call L *group representable*. Recall, Theorem 1.1 says that the FLRP has a negative answer if we can find a finite lattice that is not group representable.

Suppose there exists a property $\mathcal{P}$ such that both $\mathcal{P}$ and its negation $\neg\mathcal{P}$ are interval enforceable by the lattices L and L_c, respectively. That is $L \cong [\![H, G]\!]$ implies G has property $\mathcal{P}$, and $L_c \cong [\![H_c, G_c]\!]$ implies G_c does not have property $\mathcal{P}$. Then clearly the lattice in Figure 1 could not be group representable. As the next result shows, however, if a group property and its negation are interval enforceable by the lattices L and L_c, then already at least one of these lattices is not group representable.

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[5] Recall, for groups *subdirectly irreducible* is equivalent to having a unique minimal normal subgroup. Every algebra, in particular every group G, has a subdirect decomposition into subdirectly irreducibles, say, $G \hookrightarrow G/N_1 \times \cdots \times G/N_n$, so there are always subdirectly irreducible homomorphic images.

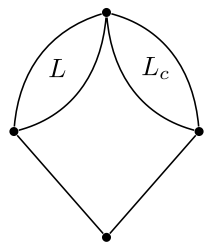

FIGURE 1.

Lemma 3.2. *If $\mathcal{P}$ is a group property that is interval enforceable by a group representable lattice, then it is not the case that $\neg\mathcal{P}$ is interval enforceable by a group representable lattice.*

Proof. Assume $\mathcal{P}$ is interval enforceable by the group representable lattice L, and let $H \leqslant G$ be groups for which $L \cong [\![H, G]\!]$. If $\neg\mathcal{P}$ is interval enforceable by the group representable lattice L_c, then there exist $H_c \leqslant G_c$ satisfying $L_c \cong [\![H_c, G_c]\!]$. Consider the group $G \times G_c$. This has upper intervals $L \cong [\![H \times G_c, G \times G_c]\!]$ and $L_c \cong [\![G \times H_c, G \times G_c]\!]$ and therefore, by the interval enforceability assumptions, the group $G \times G_c$ has the properties $\mathcal{P}$ and $\neg\mathcal{P}$ simultaneously, which is a contradiction. $\square$

To take a concrete example, nonsolvability is IE. However, solvability is obviously not IE. For, if $L \cong [\![H, G]\!]$ then for any nonsolvable group K we have $L \cong [\![H \times K, G \times K]\!]$, and of course $G \times K$ is nonsolvable. Note that here (and in the proof of Lemma 3.2) the group $H \times K$ at the bottom of the interval is not core-free. So a more interesting question is whether a property and its negation can both be cf-IE. Again, if such a property were found, a lattice of the form in Figure 1 would give a negative answer to the FLRP, though this requires additional justification to address the core-free aspect (see Section 3.3). We suspect the answer is no, as suggested by

We will answer this question affirmatively for a few special cases—namely, when $\mathcal{P}$ means "not solvable" or "not almost simple." Indeed, Lemma 3.3 below implies that the class of solvable groups, and more generally any class of groups that omits certain wreath products, cannot be core-free interval enforceable by a group representable lattice.

Lemma 3.3. *Suppose $\mathcal{P}$ is core-free interval enforceable by a group representable lattice. Then, for any finite nonabelian simple group S, there exists a wreath product group of the form $W = S \wr \bar{U}$ that has property $\mathcal{P}$.*

Proof. Let L be a group representable lattice such that if $L \cong [\![H, G]\!]$ and $\operatorname{core}_G(H) = 1$ then $G \in \mathfrak{X}_{\mathcal{P}}$. Since L is group representable, there exists a $\mathcal{P}$-group G with $L \cong [\![H, G]\!]$. We apply an idea of Hans Kurzweil (see [12]) twice. Fix a finite nonabelian simple group S. Suppose the index of H in G is $|G : H| = n$. Then the action of G on the cosets of H induces an automorphism of the group S^n by permutation of coordinates. Denote this representation by $\varphi : G \to \operatorname{Aut}(S^n)$, and let the image of G be $\varphi(G) = \bar{G} \leqslant \operatorname{Aut}(S^n)$. The wreath product under this action is the group

$$U := S \wr_{\varphi} G = S^n \rtimes_{\varphi} G = S^n \rtimes \bar{G},$$

with multiplication given by

$$(s_1, \ldots, s_n, x)(t_1, \ldots, t_n, y) = (s_1 t_{x(1)}, \ldots, s_n t_{x(n)}, xy),$$

for $s_i, t_i \in S$ and $x, y \in \bar{G}$. (For the remainder of the proof, we suppress the semidirect product symbol and write, for example, $S^n\bar{G}$ instead of $S^n \rtimes \bar{G}$.)

An illustration of the subgroup lattice of such a wreath product appears in Figure 2. Note that the interval $[\![D, S^n]\!]$, where D denotes the diagonal subgroup of S^n, is isomorphic to $\mathrm{Eq}(n)'$, the dual of the lattice of partitions of an n-element set. The dual lattice L' is an upper interval of $\mathrm{Sub}(U)$, namely, $L' \cong [\![D\bar{G}, U]\!]$.[6]

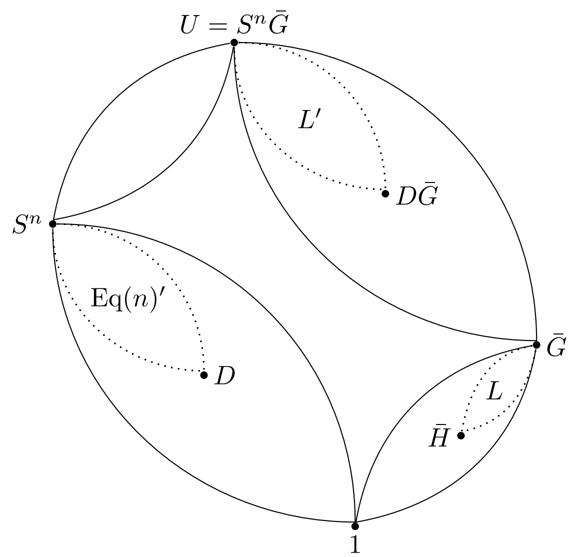

FIGURE 2. Hasse diagram illustrating some features of the subgroup lattice of the wreath product U.

It is important to note (and we prove below) that if H is core-free in G – equivalently, if $\ker\varphi = 1$ – then the foregoing construction results in the subgroup $D\bar{G}$ being core-free in U. Therefore, by repeating the foregoing procedure, with $H_1 = D\bar{G}$ denoting the (core-free) subgroup of U such that $L' \cong [\![H_1, U]\!]$, we find that $L = L'' \cong [\![D_1\bar{U}, S^m\bar{U}]\!]$, where $m = |U : H_1|$, and D_1 denotes the diagonal subgroup of S^m. Since $D_1\bar{U}$ will be core-free in $S^m\bar{U}$ then, it follows by the original hypothesis that $S^m\bar{U} = S \wr \bar{U}$ must have property $\mathcal{P}$.

To complete the proof, we check that starting with a core-free subgroup $H \leqslant G$ in the Kurzweil construction just described results in a core-free subgroup $D\bar{G} \leqslant U$. Let $N = \mathrm{core}_U(D\bar{G})$. Then, for all $w = (d, \ldots, d, x) \in N$ and for all $u = (t_1, \ldots, t_n, g) \in U$, we have $uwu^{-1} \in N$. Fix $w = (d, \ldots, d, x) \in N$. We will choose $u \in U$ so that the condition $uwu^{-1} \in N$ implies x acts trivially on $\{1, \ldots, n\}$. First

[6]These facts, which were proved by Kurzweil in [12], are discussed in greater detail in [8, Section 2.2].

note that if $u = (t_1, \dots, t_n, 1)$, then

$$\begin{aligned} uwu^{-1} &= (t_1, \dots, t_n, 1)(d, \dots, d, x)(t_1^{-1}, \dots, t_n^{-1}, 1) \\ &= (t_1 d\, t_{x(1)}^{-1}, \dots, t_n d\, t_{x(n)}^{-1}, 1) \in N, \end{aligned}$$

and this implies that $t_1 d\, t_{x(1)}^{-1} = t_2 d\, t_{x(2)}^{-1} = \cdots = t_n d\, t_{x(n)}^{-1}$. Suppose by way of contradiction that $x(1) = j \neq 1$. Then, since x is a permutation (hence, one-to-one), $x(k) \neq j$ for each $k \in \{2, 3, \dots, n\}$. Pick one such k other than j. (This is possible since $n = |G : H| > 2$; for otherwise $H \trianglelefteqslant G$ contradicting $\operatorname{core}_G(H) = 1$.) Since $u \in U$ is arbitrary, we may assume $t_1 = t_k$ and $t_{x(1)} = t_j \neq t_{x(k)}$. But this contradicts $t_1 d\, t_{x(1)}^{-1} = t_k d\, t_{x(k)}^{-1}$. Therefore, $x(1) = 1$. The same argument shows that $x(i) = i$ for each $1 \leqslant i \leqslant n$, and we see that $w = (d, \dots, d, x) \in N$ implies $x \in \ker \varphi = 1$. This puts N below D, and the only normal subgroup of U that lies below D is the trivial group. □

By the foregoing result we conclude that a class of groups that does not include wreath products of the form $S \wr G$, where S is an arbitrary finite nonabelian simple group, is not a core-free interval enforceable class. The class of solvable groups is an example.

3.2. **Dedekind's rule and its consequences.** When A and B are subgroups of a group G, by AB we mean the set $\{ab \mid a \in A, b \in B\}$, and we write $A \vee B$ or $\langle A, B\rangle$ to denote the subgroup of G generated by A and B. Clearly $AB \subseteq \langle A, B\rangle$; equality holds if and only if A and B *permute*, by which we mean $AB = BA$.

We will need the following well known result:[7]

Theorem 3.4 (Dedekind's rule)**.** *Let G be a group and let A, B and C be subgroups of G with $A \leqslant B$. Then,*

$$A(C \cap B) = AC \cap B, \qquad \text{and} \tag{3.3}$$

$$(C \cap B)A = CA \cap B. \tag{3.4}$$

For $A \in [\![H, G]\!]$, let $A^{\perp(H,G)}$ denote the set of complements of A in the interval $[\![H, G]\!]$. That is,

$$A^{\perp(H,G)} := \{B \in [\![H, G]\!] \mid A \cap B = H,\ \langle A, B\rangle = G\}.$$

Clearly $H^{\perp(H,G)} = \{G\}$ and $G^{\perp(H,G)} = \{H\}$. Recall that an *antichain* of a partially ordered set is a subset of pairwise incomparable elements.

Corollary 3.5. *If $A \in [\![H, G]\!]$ and if A permutes with each subgroup in $A^{\perp(H,G)}$, then $A^{\perp(H,G)}$ is either an antichain or the empty set.*

Proof. If $A^{\perp(H,G)}$ contains fewer that two elements, the result holds trivially. Let B_1 and B_2 denote two distinct elements of $A^{\perp(H,G)}$. We will prove $\neg(B_1 \leqslant B_2)$. Indeed, if $B_1 \leqslant B_2$, then Theorem 3.4 implies

$$B_1 = B_1 H = B_1(A \cap B_2) = B_1 A \cap B_2 = G \cap B_2 = B_2,$$

which is a contradiction. The penultimate equality holds by the hypothesis that A permutes with B_1. □

For the present work, the most important consequence of Dedekind's rule is the following:

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[7] See [20, p. 122], for example.

Corollary 3.6. *If $H \leqslant G$ and $N \trianglelefteq G$, then $(HN)^{\perp(H,G)}$ is an antichain.*

Proof. Note that HN permutes with each subgroup in $[\![H,G]\!]$. Indeed, if $H \leqslant A \leqslant G$, then $HNA = HAN = HN = AHN$. Therefore, by Corollary 3.5, the result holds. □

3.3. **Parachute lattices.** We now prove the equivalence of statements (B), (C), and (D) mentioned in Section 1. (That statement (E) of Section 1 is also equivalent to (B) follows easily from the arguments given below, so we omit the details.)

Theorem 3.7. *The following statements are equivalent:*

- *(B) Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*
- *(C) For every finite lattice L, for every finite collection $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ of cf-IE classes of groups, there exists a finite group $G \in \bigcap_{i=1}^{n} \mathfrak{X}_i$ such that $L \cong [\![H,G]\!]$ for some core-free subgroup $H \leqslant G$.*
- *(D) For every finite collection $\mathscr{L}$ of finite lattices, there exists a finite group G such that each $L_i \in \mathscr{L}$ is isomorphic to $[\![H_i,G]\!]$ for some core-free subgroup $H_i \leqslant G$.*
- *(E) For every finite collection $\mathscr{L}$ of finite lattices, for every finite collection $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ of cf-IE classes of groups, there exists a finite group $G \in \bigcap_{i=1}^{n} \mathfrak{X}_i$ such that for each $L_i \in \mathscr{L}$ we have $L_i \cong [\![H_i,G]\!]$ for some subgroup H_i that is core-free in G.*

Remark. By (C), the FLRP would have a negative answer if we could find a collection $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ of cf-IE classes such that $\bigcap_{i=1}^{n} \mathfrak{X}_i$ is empty.

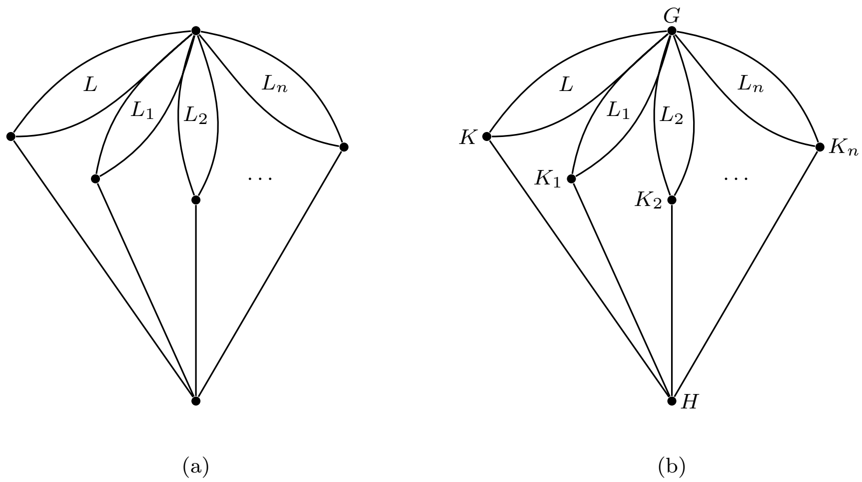
Proof. We prove the equivalence of (B) and (C). Obviously (C) implies (B). Assume (B) holds and let L be any finite lattice. Suppose $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ is a collection of cf-IE enforceable classes of groups. Then there exist finite lattices $L_1, \ldots, L_n$ such that $L_i \cong [\![H_i, G_i]\!]$ with $\operatorname{core}_G(H_i) = 1$ implies $G_i \in \mathfrak{X}_i$. Construct a new lattice, denoted $\mathscr{P} = \mathscr{P}(L, L_1, \ldots, L_n)$, as shown in the Hasse diagram of Figure 3 (a). Note that the bottoms of the L_i sublattices are atoms in $\mathscr{P}$. By (B), there exist groups $H < G$ with $\mathscr{P} \cong [\![H,G]\!]$. We can assume H is a core-free subgroup of G. (If not, replace G and H with G/N and H/N, where $N = \operatorname{core}_G(H)$, and note that $\mathscr{P} \cong [\![H,G]\!] \cong [\![H/N, G/N]\!]$.) Let $K, K_1, \ldots, K_n$ be the subgroups of G in which H is maximal and for which $L \cong [\![K,G]\!]$ and $L_i \cong [\![K_i,G]\!]$, $1 \leqslant i \leqslant n$; see Figure 3 (b). If we can prove for each $1 \leqslant i \leqslant n$ that $\operatorname{core}_G(K_i) = 1$, then $G \in \mathfrak{X}_i$ by hypothesis, and it will follow that $G \in \bigcap_{i=1}^{n} \mathfrak{X}_i$, proving that (B) implies (C).

If $L \cong \mathbf{2}$, the two element lattice, or if $L_j \cong \mathbf{2}$ for all $1 \leqslant j \leqslant n$, then the consequent of the implication in (C) holds trivially. So we can assume without loss of generality that $L \ncong \mathbf{2}$ and that there is at least one $1 \leqslant j \leqslant n$ for which $L_j \ncong \mathbf{2}$. Let N be a minimal normal subgroup of G and suppose $N \leqslant K_i$. Since H is core-free, $K_i = NH$. Suppose $K_i \neq K$. Note that the groups K and K_i permute:

$$KK_i = KNH = NKH = NHK = K_iK.$$

Therefore, by Lemma **??** (i) we see that the interval $[\![K,G]\!] \cong L$ must be isomorphic to a sublattice of the interval $[\![H,K_i]\!] \cong \mathbf{2}$, but this contradicts $L \ncong \mathbf{2}$. Suppose

FIGURE 3. The parachute construction.



instead that $K = K_i$. Note that $K \neq K_j$, and since $K = NH$, we see that K and K_j permute. Therefore, by Lemma **??** (i) again, the lattice $L_j \ncong \mathbf{2}$ is isomorphic to a sublattice of $[\![H, K]\!] \cong \mathbf{2}$, which is impossible. This proves that $NH = G$ for all $N \trianglelefteq G$, so each K_i is core-free.

The proof that statements (B) and (D) are equivalent follows by a similar construction. Roughly, if $\mathscr{L} = \{L_1, \ldots, L_n\}$, we form the lattice $\mathscr{P} = \mathscr{P}(L_1, \ldots, L_n)$. If (B) holds, then there exists a group G with $\mathscr{P} \cong [\![H, G]\!]$ and $\mathrm{core}_G(H) = 1$. The proof that each K_i is core-free, where $L_i \cong [\![K_i, G]\!]$, is similar to the argument above. □

By a *parachute lattice*, denoted $\mathscr{P}(L_1, \ldots, L_m)$, we mean a lattice just like the one illustrated in Figure 3 (a), but with the lattices $L_1, \ldots, L_m$ appearing as the upper intervals.

Next we prove that any group that has a nontrivial parachute lattice as an upper interval in its subgroup lattice must have some rather special properties.

Lemma 3.8. *Let $\mathscr{P} = \mathscr{P}(L_1, \ldots, L_n)$ with $n \geqslant 2$ and $|L_i| > 2$ for all i, and suppose $\mathscr{P} \cong [\![H, G]\!]$, with H core-free in G.*

(i) If $1 \neq N \trianglelefteq G$, then $NH = G$ and $C_G(N) = 1$.
(ii) G is subdirectly irreducible and nonsolvable.

Remark. If a subgroup $N \leqslant G$ is abelian, then $N \leqslant C_G(N)$, so (i) implies that every nontrivial normal subgroup of G is nonabelian.

Proof. (i) Let $1 \neq N \trianglelefteq G$. Then $N \nleqslant H$, since H is core-free in G. Therefore, $H < NH$. As in Section 3.3, we let K_i denote the subgroups of G corresponding to the atoms of $\mathscr{P}$. Then H is covered by each K_i, so $K_j \leqslant NH$ for some $1 \leqslant j \leqslant n$. Suppose, by way of contradiction, that $NH < G$. By assumption, $n \geqslant 2$ and $|L_i| > 2$. Thus for any $i \neq j$ we have $K_i \leqslant Y < Z < G$ for some subgroups Y and Z which satisfy $(NH) \cap Z = H$ and $(NH) \vee Y = G$. Also, $(NH)Y = NY$ is a group, so $(NH)Y = NH \vee Y = G$. But then, by Dedekind's rule, we have

$$Y = HY = ((NH) \cap Z)Y = (NH)Y \cap Z = G \cap Z = Z,$$

contrary to $Y < Z$. This contradiction proves that $NH = G$.

To prove that $C_G(N) = 1$, let M be a minimal normal subgroup of G contained in N. It suffices to prove $C_G(M) = 1$. Assume the contrary. Then, since $C_G(M) \trianglelefteq N_G(M) = G$, it follows by what we just proved that $C_G(M)H = G$. Consider any $H < K < G$. Then $1 < M \cap K < M$ (strictly, by Lemma **??**). Now $M \cap K$ is normalized by H and centralized (hence normalized) by $C_G(M)$. Therefore, $M \cap K \trianglelefteq C_G(M)H = G$, contradicting the minimality of M.

To prove (ii) we first show that G has a unique minimal normal subgroup. Let M be a minimal normal subgroup of G and let $N \trianglelefteq G$ be any normal subgroup not containing M. We show that $N = 1$. Since both subgroups are normal, the commutator subgroup $[M, N]$ lies in the intersection $M \cap N$, which is trivial by the minimality of M. Thus, M and N centralize each other. In particular, $N \leqslant C_G(M) = 1$, by (i).

Finally, since G has a unique minimal normal subgroup that is nonabelian (see the remark preceding the proof), G is nonsolvable. $\square$

To summarize what we have thus far, the lemmas above imply that (B) holds if and only if every finite lattice is an interval $[\![H, G]\!]$, with H core-free in G, where

(i) G is nonsolvable, not alternating, and not symmetric;
(ii) G has a unique minimal normal subgroup M which satisfies $MH = G$ and $C_G(M) = 1$; in particular, M is nonabelian and $\operatorname{core}_G(X) = 1$ for all $H \leqslant X < G$.

We conclude this section by formalizing the remarks of the previous sentence. Given two group theoretical properties $\mathcal{P}_1, \mathcal{P}_2$, we write $\mathcal{P}_1 \to \mathcal{P}_2$ to denote that property $\mathcal{P}_1$ implies property $\mathcal{P}_2$. In other words, G has property $\mathcal{P}_1$ only if G has property $\mathcal{P}_2$. Thus $\to$ provides a natural partial order on any given set of properties, as follows:

$$\mathcal{P}_1 \leqslant \mathcal{P}_2 \quad \Longleftrightarrow \quad \mathcal{P}_1 \to \mathcal{P}_2 \quad \Longleftrightarrow \quad \mathfrak{X}_{\mathcal{P}_1} \subseteq \mathfrak{X}_{\mathcal{P}_2},$$

where $\mathfrak{X}_{\mathcal{P}_i} = \{G \in \mathfrak{G} \mid G \text{ has property } \mathcal{P}_i\}$. The following is an immediate corollary of the parachute construction described above.

Corollary 3.9. *If $\mathscr{P} = \{\mathcal{P}_i \mid i \in \mathscr{I}\}$ is a collection of (cf-)IE properties, then $\bigwedge \mathscr{P}$ is (cf-)IE.*

The conjunction $\bigwedge \mathscr{P}$ corresponds to the class $\bigcap_{i \in \mathscr{I}} \mathfrak{X}_{\mathcal{P}_i} = \{G \in \mathfrak{G} \mid (\forall i \in \mathscr{I})\ G \text{ has property } \mathcal{P}_i\}$.

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