(DEMEO / POLLACK) MATH 207 - TEST 2 - SP 2015

1. (6 pts) Consider the set of all 3×3 nonsingular matrices with the standard operations. (check all that apply)

- ☐ The set is a vector space.
- ☒ The set is not a vector space because it is not closed under addition.
- ☐ The set is not a vector space because the associative property of addition is not satisfied.
- ☐ The set is not a vector space because a scalar identity does not exist.

(+6 pts)

(-2 for each mistake (min score: 0))

2. (6 pts) Determine whether the set $S = \{(-3, 6, 0), (6, 7, 1)\}$ spans $\mathbb{R}^3$. If not, then give a geometric description of the subspace that it does span.

- ☐ S spans $\mathbb{R}^3$. (+0)
- ☐ S does not span $\mathbb{R}^3$; S spans a point in $\mathbb{R}^3$. (+0)
- ☐ S does not span $\mathbb{R}^3$; S spans a line in $\mathbb{R}^3$. (+2)
- ☒ S does not span $\mathbb{R}^3$; S spans a plane in $\mathbb{R}^3$. (+6)

3. (6 pts) Is $S = \{(2, 6), (1, 0), (0, 1)\}$ a basis for $\mathbb{R}^2$? If not, then check the box next to the best explanation.

- ☐ S is a basis for $\mathbb{R}^2$. +0
- ☒ S is linearly dependent. (+6)
- ☐ S does not span $\mathbb{R}^2$. +0
- ☐ S is linearly dependent and does not span $\mathbb{R}^2$. (+3)

4. (12 pts) Consider the set $S = \{\mathbf{s}_1, \mathbf{s}_2\} = \{(1, 2, -2), (2, -1, 1)\}$ of vectors in $\mathbb{R}^3$. Write the vector $\mathbf{z} = (-7, 1, -1)$ as a linear combination of the vectors in S, if possible. In case it's not possible, write "impossible." (Show your work in the space provided, then use the symbols $\mathbf{s}_1$ and $\mathbf{s}_2$ in your final answer.)

$$\left[\begin{array}{cc|c} 1 & 2 & -7 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{array}\right] \xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{cc|c} 1 & 2 & -7 \\ 2 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & -7 \\ 0 & -5 & 15 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & -7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} c_1 = -1 \\ c_2 = -3 \end{array}$$

6 pts.

Answer: $\mathbf{z} = (-1)\mathbf{s}_1 + (-3)\mathbf{s}_2 = -\mathbf{s}_1 - 3\mathbf{s}_2$

$$= -\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - 3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

6 pts.

5. (9 pts) The set $S = \{(-1, -17, -4), (4, 0, -1), (1, -1, 4)\}$ of vectors in $\mathbb{R}^3$ is (check all that apply)

(a) orthogonal
(b) not orthogonal
(c) orthonormal
(d) not orthonormal
(e) a basis for $\mathbb{R}^3$
(f) not a basis for $\mathbb{R}^3$

+3 pts each (-2 for each mistake (min score: 0))

$$\left.\begin{array}{l} s_1 \cdot s_2 = -4 + 4 = 0 \\ s_1 \cdot s_3 = -1 + 17 - 16 = 0 \\ s_2 \cdot s_3 = 4 - 4 = 0 \end{array}\right\} \Rightarrow \text{orthogonal} \Rightarrow \text{a basis for } \mathbb{R}^3.$$

$\|s_2\|^2 = 16 + 1 = 17 \neq 1$ so not orthonormal

6. (16 pts) Consider the following **row equivalent** matrices:

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ -3 & -5 & 7 & 5 & -43 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Determine the *rank* and *nullity* of A.

4pts

rank(A) = 3 nullity(A) = 2

(b) Find a basis for the *row space* of A. (Use vectors appearing in matrices above!)

4pts

$\{(1,0,1,0,1),(0,1,-2,0,3),(0,0,0,1,-5)\}$.

Also acceptable: $\left\{ \begin{pmatrix}1\\0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-2\\0\\3\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-5\end{pmatrix} \right\}$

(c) Find a basis for the *column space* of A. (Use vectors appearing in matrices above!)

4pts

$\left\{ \begin{pmatrix}-2\\1\\-3\\1\end{pmatrix}, \begin{pmatrix}-5\\3\\-5\\7\end{pmatrix}, \begin{pmatrix}0\\1\\5\\5\end{pmatrix} \right\}$ (Also acceptable as row vectors.)

(d) Find a basis for the *nullspace* of A.

4pts

$A\underline{x} = 0 \iff B\underline{x} = 0$ so x in nullspace of A iff $B\underline{x} = 0$.

Let $x_3 = s$ $x_5 = t$ be free vars.

$x_1 + x_3 + x_5 = 0 \Rightarrow x_1 = -s - t$

$x_2 - 2x_3 + 3x_5 = 0 \Rightarrow x_2 = 2s - 3t$

$x_4 - 5x_5 = 0 \Rightarrow x_4 = 5t$

Basis: $\left\{ \begin{pmatrix}-1\\2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\-3\\0\\5\\1\end{pmatrix} \right\}$.

So $B\underline{x} = 0 \iff \underline{x} = \begin{pmatrix}-s-t\\2s-3t\\s\\5t\\t\end{pmatrix} = s\begin{pmatrix}-1\\2\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}-1\\-3\\0\\5\\1\end{pmatrix}$

7. (12 pts) Let $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (2, 4, 2)$, and $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 + u_3v_3$. Compute the following. *Show your work!* Caution: $\langle \mathbf{u}, \mathbf{v} \rangle$ is not the usual Euclidean inner (dot) product.

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = (1)(2) + (2)(1)(4) + (1)(2)$

$= 2 + 8 + 2 = 12$

Answer: $\langle \mathbf{u}, \mathbf{v} \rangle = \underline{12}$

(b) $\|\mathbf{v}\| = \sqrt{\langle v, v \rangle} = \sqrt{2^2 + 2 \cdot 4^2 + 2^2}$

$= \sqrt{4 + 32 + 4} = \sqrt{40} = 2\sqrt{10}$

Answer: $\|\mathbf{v}\| = \underline{\sqrt{40}} = \underline{2\sqrt{10}}$

(c) $d(\mathbf{u}, \mathbf{v}) = \|u - v\|$

$= \sqrt{\langle u-v, u-v \rangle} = \sqrt{(1-2)^2 + 2(1-4)^2 + (1-2)^2}$

$= \sqrt{1 + 18 + 1} = \sqrt{20} = 2\sqrt{5}$

Answer: $d(\mathbf{u}, \mathbf{v}) = \underline{\sqrt{20}} = \underline{2\sqrt{5}}$

8. (8 pts) Find the coordinate matrix of the vector $\mathbf{x} = (15, 20, 25)$ relative to the orthonormal basis

$$B = \{(\frac{3}{5}, \frac{4}{5}, 0), (-\frac{4}{5}, \frac{3}{5}, 0), (0, 0, 1)\} = \{\underline{b}_1, \underline{b}_2, \underline{b}_3\}$$

$$\underline{X} \cdot \underline{b}_1 = 15\left(\frac{3}{5}\right) + 20\left(\frac{4}{5}\right) = 9 + 16 = 25$$

$$\underline{X} \cdot \underline{b}_2 = 15\left(-\frac{4}{5}\right) + 20\left(\frac{3}{5}\right) = -12 + 12 = 0$$

$$\underline{X} \cdot \underline{b}_3 = 25$$

Check:

$$25\begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} + 0\begin{pmatrix} -4/5 \\ 3/5 \\ 0 \end{pmatrix} + 25\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \\ 25 \end{pmatrix} \checkmark$$

Answer: $[\mathbf{x}]_B = \begin{pmatrix} 25 \\ 0 \\ 25 \end{pmatrix} = (25, 0, 25).$

E.C. Apply the Gram-Schmidt orthonormalization process to transform the basis

$$B = \{\mathbf{v}_1, \mathbf{v}_2\} = \{\begin{bmatrix} 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\} \text{ for } \mathbb{R}^2 \text{ into an orthonormal basis.}$$

Use the usual Euclidean inner (dot) product on $\mathbb{R}^2$ and use the vectors in the order given. Simplify your answer, reducing any fractions that appear to lowest terms.

$$\underline{W}_1 = \frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{36+64}}\begin{pmatrix} 6 \\ 8 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$\underline{W}_2' = V_2 - \langle V_2, W_1 \rangle W_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 4/5\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} = \begin{pmatrix} -12/25 \\ 1 - 16/25 \end{pmatrix} = \begin{pmatrix} -12/25 \\ 9/25 \end{pmatrix}$$

$$W_2 = \frac{W_2'}{\|W_2\|}$$

Answer: $\mathbf{w}_1 =$ $\qquad\qquad$ $\mathbf{w}_2 =$