Math 317: Homework 6

NAME:

Problem 1 (SA 3.1.1aei). Which of the following are subspaces? Justify your answers.

a. $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1\}$

e. $\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 0\}$

i. $\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ for some } s, t \in R\}$

Problem 2 (SA 3.1.2cd). Decide whether each of the following collections of vectors spans $\mathbb{R}^3$.

c. $\{(1,0,1),(1,-1,1),(3,5,3),(2,3,2)\}$

d. $\{(1,0,-1),(2,1,1),(0,1,5)\}$

Problem 3 (SA 3.1.6). Let U and V be subspaces of $\mathbb{R}^n$. The *intersection* and *union* of U and V are defined, respectively, as follows:

$$U \cap V := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in U \text{ and } \mathbf{x} \in V\} \quad \text{and} \quad U \cup V := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in U \text{ or } \mathbf{x} \in V\},$$

a. Show that $U \cap V$ is a subspace of $\mathbb{R}^n$. Give two examples.

b. Is $U \cup V$ a subspace of $\mathbb{R}^n$? Give a proof or counterexample.

Problem 4 (SA 3.1.7). Let U and V be subspaces of $\mathbb{R}^n$. We define the *sum* of U and V to be

$$U + V := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{u} + \mathbf{v} \text{ for some } \mathbf{u} \in U \text{ and } \mathbf{v} \in V\}.$$

More simply, $U + V = \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in U \text{ and } \mathbf{v} \in V\}$.

Prove that if U and V are subspaces of $\mathbb{R}^n$ and W is a subspace of $\mathbb{R}^n$ containing all the vectors of U and all the vectors of V then $U + V \subseteq W$. That is, prove

$$U \leq W \text{ and } V \leq W \quad \text{implies} \quad U + V \leq W.$$

This means that $U + V$ is the smallest subspace containing both U and V.

Problem 5 (SA 3.1.9ad). Determine the intersection of the subspaces $\mathcal{P}_1$ and $\mathcal{P}_2$ in each case:

a. $\mathcal{P}_1 = \text{Span}\{(1,0,1),(2,1,2)\}$, $\mathcal{P}_2 = \text{Span}\{(1,-1,0),(1,3,2)\}$.

d. $\mathcal{P}_1 = \text{Span}\{(1,1,0,1),(0,1,1,0)\}$, $\mathcal{P}_2 = \text{Span}\{((0,0,1,1),(1,1,0,0)\}$.

The last required exercise of this assignment is Problem 6, which appears on the next page. The (unnumbered) problems on this page are recommended but optional. They merely test whether you know the definition of orthogonal complement (and thus prepare you for Problem 6). We will cover orthogonal complements in lecture, but in case we don't get to it before you come to this part of the homework, here's the definition. If V is a subspace of $\mathbb{R}^n$, then the *orthogonal complement* of V in $\mathbb{R}^n$ is denoted by $V^\perp$ and is defined as follows:

$$V^\perp := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{v} \in V\}.$$

Problem (SA 3.1.10). Let $V \leq \mathbb{R}^n$ be a subspace. Show that $V \cap V^\perp = \{\mathbf{0}\}$.

Problem (SA 3.1.11-2). Suppose V and W are *orthogonal subspaces* of $\mathbb{R}^n$, that is, $\mathbf{v} \cdot \mathbf{w} = 0$ for every $\mathbf{v} \in V$ and every $\mathbf{w} \in W$.

1. Prove that $V \subseteq W^\perp$.
2. Prove that $V \cap W = \{\mathbf{0}\}$.

Problem 6 (SA 3.1.14). Let V and W be subspaces of $\mathbb{R}^n$ with the property that $V \subseteq W$. Prove that $W^\perp \subseteq V^\perp$.