Math 317: Homework 4

NAME:

Note: Only numbered exercises have potential to be graded. Unnumbered exercises (like the first problem in this assignment) are recommended but not required. Solutions to unnumbered problems will not be graded.

Problem (Block multiplication). We can think of an $(m+n) \times (m+n)$ matrix as being decomposed into "blocks," and thinking of these blocks as matrices themselves, we can form products and sums appropriately. Suppose A and A' are $m \times m$ matrices, B and B' are $m \times n$ matrices, C and C' are $n \times m$ matrices, and D and D' are $n \times n$ matrices. Verify the following formula for the product of "block" matrices:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right] = \left[\begin{array}{c|c} AA' + BC' & AB' + BD' \\ \hline CA' + DC' & CB' + DD' \end{array}\right]. \tag{1}$$

Problem 1 (SA 2.1.10). Suppose A and B are nonsingular $n \times n$ matrices. Prove that AB is nonsingular.

> *Hint:* Although it is tempting to try to show that the reduced echelon form of AB is the identity matrix, there is no direct way to do this. As is the case in most non-numerical problems regarding nonsingularity, you should remember that AB is nonsingular precisely when the only solution of $(AB)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

Problem 2 (SA 2.2.1). Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation and that

$$T \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ and } T \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}. \quad \text{Compute } T\left(2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right), T \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}, \text{ and } T \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}.$$

Hint: begin by writing each vector as a linear combination of the vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Problem 3 (SA 2.2.3b). Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with $T\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}5\\3\end{bmatrix}$, and $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\-3\end{bmatrix}$. Find the standard matrix representation of T.

Problem 4 (SA 2.2.4bef). Determine whether each of the following functions is a linear transformation. If so, provide a proof; if not, explain why.

b. $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 0 \end{bmatrix}$.

e. $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ -x_1 + 3x_2 \end{bmatrix}$.

f. $T : \mathbb{R}^n \to \mathbb{R}$ given by $T\mathbf{x} = \|\mathbf{x}\|$.

Problem 5 (SA 2.2.11). Solving each part is recommended, but only part **c** will be graded.

a. Prove that if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and c is any scalar, then the function $cT : \mathbb{R}^n \to \mathbb{R}^m$ defined by $(cT)(\mathbf{x}) = cT(\mathbf{x})$ (i.e., the scalar c times the vector $T(\mathbf{x})$) is also a linear transformation.

b. Prove that if $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations, then the function $S + T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $(S + T)(\mathbf{x}) = S(\mathbf{x}) + T(\mathbf{x})$ is also a linear transformation.

c. Prove that if $S : \mathbb{R}^m \to \mathbb{R}^p$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations, then the function $S \circ T : \mathbb{R}^n \to \mathbb{R}^p$ defined by $(S \circ T)(\mathbf{x}) = S(T(\mathbf{x}))$ is also a linear transformation.

Problem 6 (SA 2.3.1d). Use Gaussian elimination to find A^{-1} (if it exists) where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

Problem 7 (SA 2.3.2c). Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

(i) Find A^{-1}.

(ii) Use your answer to (i) to solve $A\mathbf{x} = \mathbf{b}$.

(iii) Use your answer to (ii) to express $\mathbf{b}$ as a linear combination of the columns of A.

Problem 8 (SA 2.3.12). Suppose A is an invertible $m \times m$ matrix and B is an invertible $n \times n$ matrix. Let O denote a matrix of all zeros (with the appropriate dimensions).

a. Show that the matrix $\left[\begin{array}{c|c} A & O \\ \hline O & B \end{array}\right]$ is invertible and give a formula for its inverse.

b. Suppose C is an arbitrary $m \times n$ matrix. Is the matrix $\left[\begin{array}{c|c} A & C \\ \hline O & B \end{array}\right]$ invertible? If so, find a formula for the inverse. If not, explain why not.

[*Hint:* to see how "block matrix multiplication" works, look at Equation (1) on page 1 above.]

The next problem is recommended, but solutions will not be graded.

Problem (SA 2.3.3). Suppose A is an $n \times n$ matrix and B is an $n \times n$ invertible matrix. Simplify the following.

a. $(BAB^{-1})^2$

b. $(BAB^{-1})^n$ (n a positive integer)

c. $(BAB^{-1})^{-1}$ (What additional assumption is required here?)