Math 317: Homework 12

NAME:

Section 6.2

Problem 1 (cf. SA 6.2.1fkp). Recall, the matrices in Problem 6.1.1 of Homework 11 were,

$$\text{f. } \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}, \qquad \text{k. } \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & -1 \\ 0 & 2 & -1 \end{bmatrix}, \qquad \text{and} \qquad \text{p. } \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Decide which of these matrices is diagonalizable. Give your reasoning.

Problem 2 (cf. SA 6.2.2). Three of the six claims below are valid, three are not. Prove the valid claims and find a counterexample refuting the invalid claims.

a. If A is an $n \times n$ matrix with n distinct (real) eigenvalues, then A is diagonalizable.

b. If A is diagonalizable and $AB = BA$, then B is diagonalizable.

c. If there is an invertible matrix P so that $A = P^{-1}BP$, then A and B have the same eigenvalues.

d. If A and B have the same eigenvalues, then there is an invertible matrix P so that $A = P^{-1}BP$.

e. There is no real 2×2 matrix A satisfying $A^2 = -I_2$.

f. If A and B are diagonalizable and have the same eigenvalues (with the same algebraic multiplicities), then there is an invertible matrix P so that $A = P^{-1}BP$.

Hints: You may find Lemma 1.4 and Corollary 2.2 helpful.

The following exercise is recommended but not required.

Exercise (SA 6.2.11). Suppose A is an $n \times n$ matrix with the property that $A^2 = A$.

a. Show that if λ is an eigenvalue of A, then $\lambda = 0$ or $\lambda = 1$.

b. Prove that A is diagonalizable. (*Hint:* See Exercise 3.2.13.)

Problem 3 (SA 6.2.20a). Let λ and μ be distinct eigenvalues of a linear transformation. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbf{E}(\lambda)$ is linearly independent and $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\} \subset \mathbf{E}(\mu)$ is linearly independent. Prove that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ is linearly independent. (This is essentially the result needed to complete the proof of Theorem 2.4.)

Section 6.4

Problem 4 (SA 6.4.9).

a. Suppose A is a symmetric $n \times n$ matrix. Using the Spectral Theorem, prove that if $A\mathbf{x} \cdot \mathbf{x} = 0$ for every vector $\mathbf{x} \in \mathbb{R}^n$, then $A = O$.

b. Give a counterexample to show that if we drop the symmetry hypothesis, then the statement in part a is false.

The next exercise is recommended but not required.

Exercise (cf. SA 6.4.3). Find a matrix A with the following properties:

- A is symmetric;
- A has eigenvalues 1 and 2;
- $\mathbf{E}(2) = \operatorname{Span}\{(1,1,1)\}$.

(*Hint:* the Spectral Theorem implies $\mathbf{E}(1) = \mathbf{E}(2)^\top$. Find a basis for $\mathbf{E}(1)$, and then compute $A = P\Lambda P^{-1}$ for an appropriate choice of P and Λ.)