Math 317: Homework 1

NAME:

General Hints: When you are asked to "show" or "prove" something, you should make it a point to write down clearly the information you are given and what it is you are to show. One word of warning regarding the second part of Problem 1.1.22: To say that $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is to say that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ for some scalars $c_1, \ldots, c_k$. These scalars will surely be different when you express a different vector $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, so be sure you give the scalars for $\mathbf{w}$ different names.

("SA 1.1.21" means exercise 21 in Section 1.1 of textbook by **S**hifrin and **A**dams.)

Problem 0 (SA 1.1.28). Carefully prove the following properties of vector arithmetic. Justify all steps. Optional: Give the geometric interpretation of each property in case $n = 2$ or $n = 3$. You are required to turn proofs for e. f. g. (but you are encouraged to try them all).

a. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

b. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.

c. For all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{0} + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

d. For each $\mathbf{x} \in \mathbb{R}^n$, there is a vector $-\mathbf{x}$ so that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.

e. For all $c, d \in \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$, $c(d\mathbf{x}) = (cd)\mathbf{x}$.

f. For all $c \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$.

g. For all $c, d \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$.

h. For all $\mathbf{x} \in \mathbb{R}^n$, $1\mathbf{x} = \mathbf{x}$.

(You must either type your homework using LaTeX, or write them by hand on this prinout. If you will use this printout to complete your homework, write your solutions to Problem 0 on the next page.)

e. **Claim:** For all $c, d \in \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$, $c(d\mathbf{x}) = (cd)\mathbf{x}$.

Proof.

$\square$

f. For all $c \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$.

Proof.

$\square$

g. For all $c, d \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$.

Proof.

$\square$

Problem 1 (SA 1.1.21). Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and c is a scalar. Prove that $\text{Span}(\mathbf{v} + c\mathbf{w}, \mathbf{w}) = \text{Span}(\mathbf{v}, \mathbf{w})$. (See the blue box on p. 12 of the textbook.)

Proof.

$\square$

Problem 2 (SA 1.1.22). Suppose the vectors $\mathbf{v}$ and $\mathbf{w}$ are both linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

a. Prove for any scalar c that $c\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

b. Prove that $\mathbf{v} + \mathbf{w}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

Proof.

$\square$

Problem 3 (SA 1.1.25). Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are nonparallel vectors. (Recall def, p.3 of text.)

a. Prove that if $s\mathbf{x} + t\mathbf{y} = 0$, then $s = t = 0$. (Hint: Show neither $s \neq 0$ nor $t \neq 0$ is possible.)

b. Prove that if $a\mathbf{x} + b\mathbf{y} = c\mathbf{x} + d\mathbf{y}$, then $a = c$ and $b = d$. (Hint: Use part a.)

Proof.

$\square$

Problem 4 (SA 1.2.11). Suppose $\mathbf{x}, \mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ and $\mathbf{x}$ is orthogonal to each of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Show that $\mathbf{x}$ is orthogonal to every linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$.

Proof.

$\square$

Problem 5 (SA 1.2.18). Prove the triangle inequality: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. (Hint: Use the dot product to calculate $\|\mathbf{x}+\mathbf{y}\|^2$.)

Proof.

$\square$

Problem 6 (SA 1.3.12). Suppose $\mathbf{a} \neq \mathbf{0}$ and $\mathscr{P} \subset \mathbb{R}^3$ is the plane through the origin with normal vector $\mathbf{a}$. Suppose $\mathscr{P}$ is spanned by $\mathbf{u}$ and $\mathbf{v}$.

a. Suppose $\mathbf{u} \cdot \mathbf{v} = 0$. Show that for every $\mathbf{x} \in \mathscr{P}$, we have $\mathbf{x} = \operatorname{proj}_{\mathbf{u}} \mathbf{x} + \operatorname{proj}_{\mathbf{v}} \mathbf{x}$.

b. Suppose $\mathbf{u} \cdot \mathbf{v} = 0$. Show that for every $\mathbf{x} \in \mathbb{R}^3$, we have $\mathbf{x} = \operatorname{proj}_{\mathbf{a}} \mathbf{x} + \operatorname{proj}_{\mathbf{u}} \mathbf{x} + \operatorname{proj}_{\mathbf{v}} \mathbf{x}$. (Hint: Apply part a. to the vector $\mathbf{x} - \operatorname{proj}_{\mathbf{a}} \mathbf{x}$.)

c. Give an example to show the result of part a is false when $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal.

Proof.

$\square$