MATH 317: FALL 2015 **FINAL EXAM** **NAME:** ______________________

(print clearly)

RULES:

- No books or notes or calculators allowed.
- No bathroom breaks until after you have completed and turned in your test.
- Out of consideration for your classmates, do not make disturbing noises during the exam. If you need a tissue, please ask for one.
- **Phones and other electronic devices must be off or in silent mode.**

Cheating will not be tolerated. If there is any indication that a student may have given or received unauthorized aid on this test, the case will be handed over to the ISU Office of Judicial Affairs. When you finish the exam, please sign the following statement acknowledging that you understand this policy:

"On my honor as a student, I have neither given nor received unauthorized aid on this exam."

Signature: ______________________________ Date: 2015-12-14

Prob#	max	score
1–7.	49	
8.	17	
9.	17	
10.	17	
Tot.	100	

NOTATION: For the most part, we follow the notation used in the textbook.

- Recall that if V and W are vector spaces, we use the notation $W \leq V$ to mean that W is a subspace of V, whereas $W \subseteq V$, means that W is a subset of V (which may or may not be a subspace). The dimension of V is denoted by $\dim(V)$.

- If $A \in \mathbb{R}^{n \times m}$ is a matrix, then $\mathrm{N}(A)$, $\mathrm{C}(A)$, and $\mathrm{R}(A)$ denote the nullspace, column space, and row space of A, respectively. Denote the entry in row i and column j of a matrix A by a_{ij}, unless referring to a matrix named B or C, in which cases we use b_{ij} or c_{ij}, etc.

- If $T : V \to W$ is a linear transformation, then $\ker(T) := \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}_W\}$ is the kernel of T, the dimension of which is called the nullity of T, which is denoted $\mathrm{nullity}(T)$. The image of V under T is sometimes denoted by $T(V)$, and sometimes by $\mathrm{im}(T)$. The dimension of $\mathrm{im}(T)$ is called the rank of T, denoted by $\mathrm{rank}(T)$.

Part 1: Multiple Choice. For the first seven problems, mark your answers by filling in the ovals in the answer box on this page. Each problem is worth 7 points.

problem	answer choices				
1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)

1. Does the set $S = \{(-3, 6, 0), (6, 7, 3)\}$ span $\mathbb{R}^3$? If the set does not span $\mathbb{R}^3$, then give a geometric description of the subspace that it does span.

(a) S spans $\mathbb{R}^3$.

(b) S does not span $\mathbb{R}^3$. S spans a plane in $\mathbb{R}^3$.

(c) S does not span $\mathbb{R}^3$. S spans a line in $\mathbb{R}^3$.

(d) S does not span $\mathbb{R}^3$. S spans a point in $\mathbb{R}^3$.

2. The set $S = \{(0, -1, 4), (-17, -4, -1), (-1, 4, 1)\}$ of vectors in $\mathbb{R}^3$ has which of the following properties (select all that apply):

(a) S is orthogonal.

(b) S is orthonormal.

(c) S is a basis for $\mathbb{R}^3$.

In Problems **3** and **4**, V and W are finite dimensional vector spaces, and $T : V \to W$ is a linear transformation. Select the letters next to each true statement. There may be more than one. Only select statements you are certain are true. If you are unsure whether or not a statement is true, or you are certain it is false, then do not mark the corresponding letter.

3(a) Suppose $T(\mathbf{v}) = \mathbf{0}_W$ if and only if $\mathbf{v} = \mathbf{0}_V$. Then $\dim(V) = \dim(W)$.

(b) Suppose $\dim(V) = \operatorname{rank}(T)$. Then $\ker(T) = \{\mathbf{0}_V\}$.

(c) $\ker(T) \leq \ker(T^2)$ and $\operatorname{im}(T) \geq \operatorname{im} T^2$

(d) $\operatorname{nullity}(T) \leq \operatorname{rank}(T)$

(e) $\operatorname{nullity}(T) \leq \dim(V)$

4(a) T is one-to-one if and only if $\ker(T) = \{\mathbf{0}_V\}$.

(b) T is one-to-one if and only if $\dim(V) \leq \dim(W)$.

(c) T is one-to-one if and only if $\operatorname{nullity}(T) = 0$.

(d) T is onto if and only if $\dim(V) \geq \dim(W)$.

(e) T is onto if and only if $\operatorname{rank}(T) = \dim(W)$.

5. The system of linear equations

$$\begin{array}{rcrcrcrcr} 2x & - & y & + & 4z & & & = & 5 \\ -x & + & 2y & - & 5z & - & w & = & -1 \\ & & y & - & 2z & - & w & = & 1 \\ x & + & y & - & z & + & w & = & 3 \end{array}$$

has

(a) no solution

(b) a unique solution

(c) infinitely many solutions with 1 free variable

(d) infinitely many solutions with 2 free variables

The next two problems refer to the matrix $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix}$.

6. The rank of A is

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

7. A basis for the column space $\mathrm{C}(A)$ is given by which columns of A?

(a) columns 1 and 2

(b) columns 1 and 3

(c) columns 1 and 4

(d) column 1 only

(e) column 4 only

8. (17pts) Either find a matrix A with the following properties or explain why no such matrix exists:

(a) The nullspace contains $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}$ and the row space contains $\begin{bmatrix}1 & 1 & -1\end{bmatrix}$.

(b) The column space and nullspace both have basis $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}$.

9. (17pts) Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be vectors in $\mathbb{R}^n$. Suppose there is a matrix A for which the set $\{A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_m\}$ is linearly independent. Prove that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is linearly independent.

10. (17pts) Complete ONE of the following problems, either **10.1.abc** OR **10.2.abc**. Clearly mark which problem you want graded. If you work on more than one, cross out what you don't want graded.

10.1 Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose matrix representation relative to the standard basis is $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$.

(a) One of the eigenvalues of A is $\lambda_0 = 2$. Find the other eigenvalue of A.

$$(\lambda_0, \lambda_1) = (2, \qquad)$$

(b) A basis for the eigenspace $\mathbf{E}(2)$ is $B_0 = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$. Find a basis for the eigenspace $\mathbf{E}(\lambda_1)$. (Please use 1 as the first coordinate of the basis vector.)

$$B_0 = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}, \qquad B_1 =$$

(c) Let $\mathcal{B} = B_0 \cup B_1$. Find $[T]_{\mathcal{B}}$, the matrix representation of T with respect to the basis $\mathcal{B}$. **Using this** compute A^5. (Do not compute A^5 directly, that is, with 5 "dense" matrix-matrix multiplications. Instead, use $[T]_{\mathcal{B}}$ and its relation to A. Show your work!)

$$[T]_{\mathcal{B}} = \qquad\qquad A^5 =$$

10.2 Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(x_1, x_2) = (x_1 + 2x_2, 3x_2)$, which has standard matrix representation $[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

(a) Write down the change-of-basis matrix P that maps $\mathcal{B}$-basis representations of vectors (e.g., $[\mathbf{x}]_{\mathcal{B}}$) to standard basis representations of vectors (e.g., $\mathbf{x} = [\mathbf{x}]_{\mathcal{E}}$); then find the inverse change-of-basis matrix P^{-1} (going from $\mathcal{E}$ to $\mathcal{B}$).

$P =$

$P^{-1} =$

(b) Using the matrices in part (a), find the $\mathcal{B}$-basis representation of the vector $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$[\mathbf{x}]_{\mathcal{B}} =$

(c) Find $[T]_{\mathcal{B}}$, the matrix representation of T relative to the basis $\mathcal{B}$, and then find the vector $[T(\mathbf{x})]_{\mathcal{B}}$; that is, find the $\mathcal{B}$-basis representation of the result of applying T to $\mathbf{x}$.

$[T]_{\mathcal{B}} =$

$[T(\mathbf{x})]_{\mathcal{B}} =$

- scratch -

- scratch -