Math 317: Homework 11

NAME:

Section 4.4

The first exercise is recommended but not required. It will not be graded.

Exercise (SA 4.4.2). Let $V \le \mathcal{C}^\infty(\mathbb{R})$ be the given subspace. Let $D : V \to V$ be the *differentiation operator* $D(f) = f'$. Give the matrix for D with respect to the given basis.

a. $V = \mathrm{Span}\{1, e^x, e^{2x}, \ldots, e^{nx}\}$.

b. $V = \mathrm{Span}\{e^x, xe^x, x^2e^x, \ldots, x^ne^x\}$.

Hints: For part (a), if you have trouble, consult the solution in back of the textbook. For part (b), we have $D(e^x) = e^x$, $D(xe^x) = e^x + xe^x$, $D(x^2e^x) = 2xe^x + x^2e^x$, and, in general, $D(x^ke^x) = kx^{k-1}e^x + x^ke^x$. Now write down the matrix for D with respect to the given basis, i.e., the basis $\mathcal{B} = \{e^x, xe^x, x^2e^x, \ldots, x^ne^x\}$.

Problem 1 (SA 4.4.4). Recall from Example 5 (p. 229) (and from lecture), the matrix $[D]_{\mathcal{V},\mathcal{W}}$ representing the differentiation operator $D : \mathcal{P}_3 \to \mathcal{P}_2$ with respect to the bases $\mathcal{V} = \{1, t, t^2, t^3\}$ and $\mathcal{W} = \{1, t, t^2\}$ is given by

$$[D]_{\mathcal{V},\mathcal{W}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Use the *Change-of-Basis Formula* (Theorem 4.2) to find, in each case, the matrix $[D]_{\mathcal{V}',\mathcal{W}'}$ that represents the differential operator with respect to the given pair of bases.

a. $\mathcal{V}' = \{1, t-1, (t-1)^2, (t-1)^3\}$ and $\mathcal{W}' = \mathcal{W}$;

b. $\mathcal{V}' = \mathcal{V}$ and $\mathcal{W}' = \{1, t-1, (t-1)^2\}$;

c. $\mathcal{V}' = \{1, t-1, (t-1)^2, (t-1)^3\}$ and $\mathcal{W}' = \{1, t-1, (t-1)^2\}$.

Problem 2 (SA 4.4.16). Let V be a finite-dimensional vector space, let W be a vector space, and let $T : V \to W$ be a linear transformation. Prove the matrix-free analog of the "rank+nullity theorem" for linear transformations. That is, give a (matrix-free) proof that $\dim(\ker T) + \dim(\operatorname{im} T) = \dim V$ by following these steps.

a. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for $\ker(T)$, and (following Exercise 3.4.17) extend to obtain a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ for V. Show that $\{T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)\}$ gives a basis for $\operatorname{im}(T)$.

b. Conclude the desired result. Explain why this is a restatement of Corollary 4.7 of Chapter 3 when W is finite-dimensional.

Problem 3 (SA 4.4.18). Prove the following claims.

a. Claim 1: If $T : V \to W$ is a linear transformation and if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly dependent set of vectors in V, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ is a linearly dependent set in W.

b. Claim 2: Suppose $T : V \to V$ is a linear transformation and V is finite-dimensional. If $\mathrm{im}(T) = V$ and if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ is linearly independent. (*Hint:* Use Problem 2 (SA 4.4.16).)

Problem 4 (SA 4.4.21). Let V be a vector space.

a. Let V^* denote the set of all linear transformations from V to $\mathbb{R}$. Show that V^* is a vector space.[1]

b. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V. For $i = 1, 2, \ldots, n$, define $\mathbf{f}_i \in V^*$ by

$$\mathbf{f}_i(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) = a_i.$$

Prove that $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ gives a basis for V^*.

c. Deduce that whenever V is finite-dimensional, $\dim V^* = \dim V$.

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[1] The vectors in V^* are sometimes called the *linear functionals* of V.

Section 6.1

In the next exercise, parts f and k are required. Part p is recommended.

Problem 5 (SA 6.1.1fkp). Find eigenvalues and bases for the eigenspaces of the following matrices.

f. $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

k. $\begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & -1 \\ 0 & 2 & -1 \end{bmatrix}$

p. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Problem 6 (cf. SA 6.1.4)*.* Let $V \leq \mathbb{R}^n$ be a subspace and let P_V denote the projection onto V. What are the eigenvalues and eigenvectors of P_V? (*Hint:* Think about what P_V does to vectors $\mathbf{v} \in V$ and to vectors $\mathbf{w} \in V^\perp$. Are these the only vectors you need to consider?)

In the next exercise, part a is required. Part b and c are recommended.

Problem 7 (SA 6.1.13abc). In each of the following cases, find the eigenvalues and eigenvectors of the linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_3$.

a. $T(p)(t) = p'(t)$

b. $T(p)(t) = tp(t)$

c. $T(p)(t) = \int_0^t p'(u)\, du$