

# A Note on Linear Independence and Bases

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**Lemma 1:** If  $A \in \mathbb{R}^{m \times n}$  and  $N(A) = \{\mathbf{0}\}$ , then  $m \geq n$ .

*Proof.* If  $N(A) = \{\mathbf{0}\}$ , then by definition the only solution to  $A\mathbf{x} = \mathbf{0}$  is the trivial solution  $\mathbf{0}$ . It follows that any echelon form of  $A$  cannot have free variables, so there must be a pivot in each column of  $A$ . But there can be at most  $m$  pivots, so  $m \geq n$ .  $\square$

**Lemma 2:** If  $V$  is a subspace of  $\mathbb{R}^n$  with basis  $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ , then any linearly independent subset of  $V$  has at most  $k$  vectors.

*Proof.* Let  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$  be a linearly independent subset of  $V$ . We will prove  $m \leq k$ . Each  $\mathbf{x}_i$  is a linear combination of the basis vectors in  $\mathcal{V}$ . That is, there exist scalars  $a_{ij} \in \mathbb{R}$  such that

$$\begin{aligned} \mathbf{x}_1 &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \dots + a_{k1}\mathbf{v}_k \\ \mathbf{x}_2 &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \dots + a_{k2}\mathbf{v}_k \\ &\vdots \\ \mathbf{x}_m &= a_{1m}\mathbf{v}_1 + a_{2m}\mathbf{v}_2 + \dots + a_{km}\mathbf{v}_k \end{aligned} \tag{1}$$

Note that, in the first equation above, the right-hand side is a matrix-vector product, so we can write this equation as follows:

$$\mathbf{x}_1 = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{bmatrix}.$$

Of course, the same is true of the other equations in (1), so we can write them all simultaneously as a the following system of equations:

$$\begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{km} \end{bmatrix} \tag{2}$$

Let  $X$  denote the matrix on the left-hand side of (2), and let  $A$  denote the second matrix on the right-hand side of (2). Since  $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$  is linearly independent, we have  $N(X) = \{\mathbf{0}\}$ , and this implies  $N(A) = \{\mathbf{0}\}$ . (Reason: if  $A\mathbf{y} = \mathbf{0}$ , then  $X\mathbf{y} = \mathbf{0}$  by (2).) Finally, if  $N(A) = \{\mathbf{0}\}$ , we know that  $A$  must have at least as many rows as columns, by Lemma 1. Therefore  $m \leq k$ , as desired.  $\square$

**Proposition:** Let  $W \subseteq V$  be subspaces of  $\mathbb{R}^n$ . Suppose  $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is a basis for  $V$  and  $\mathcal{W} = \{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$  is a basis for  $W$ . Then  $W = V$  if and only if  $\ell = k$ .

*Proof.*

( $\Rightarrow$ ) If  $W = V$ , then  $\mathcal{W}$  is a basis for  $V$  so  $\ell = k$ , since (as we proved earlier) all bases for a given subspace have the same number of vectors.

( $\Leftarrow$ ) We want to prove  $\ell = k$  implies  $W = V$ . We will prove the (equivalent) contrapositive statement, which is  $W \neq V$  implies  $\ell \neq k$ .

If  $W \neq V$ , then there is a vector  $\mathbf{v} \in V$  that does not belong to  $W$ . Therefore,  $\mathbf{v} \notin \text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_\ell\}$ , so  $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_\ell, \mathbf{v}\}$  is a set of  $\ell + 1$  linearly independent vectors in  $V$ . By Lemma 2 then,  $\ell + 1 \leq k$ , so  $\ell < k$ , as desired.

$\square$