Math 317: Homework 11

NAME:

Section 4.4

The first exercise is recommended but not required. It will not be graded.

Exercise (SA 4.4.2). Let $V \leq \mathcal{C}^{\infty}(\mathbb{R})$ be the given subspace. Let $D: V \to V$ be the differentiation operator D(f) = f'. Give the matrix for D with respect to the given basis.

a.
$$V = \text{Span}\{1, e^x, e^{2x}, \dots, e^{nx}\}.$$

b.
$$V = \text{Span}\{e^x, xe^x, x^2e^x, \dots, x^ne^x\}.$$

Hints: For part (a), if you have trouble, consult the solution in back of the textbook. For part (b), we have $D(e^x) = e^x$, $D(xe^x) = e^x + xe^x$, $D(x^2e^x) = 2xe^x + x^2e^x$, and, in general, $D(x^ke^x) = kx^{k-1}e^x + x^ke^x$. Now write down the matrix for D with respect to the given basis, i.e., the basis $\mathcal{B} = \{e^x, xe^x, x^2e^x, \dots, x^ne^x\}$.

Problem 1 (SA 4.4.4). Recall from Example 5 (p. 229) (and from lecture), the matrix $[D]_{\mathcal{V},\mathcal{W}}$ representing the differentiation operator $D: \mathcal{P}_3 \to \mathcal{P}_2$ with respect to the bases $\mathcal{V} = \{1, t, t^2, t^3\}$ and $\mathcal{W} = \{1, t, t^2\}$ is given by

$$[D]_{\mathcal{V},\mathcal{W}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Use the *Change-of-Basis Formula* (Theorem 4.2) to find, in each case, the matrix $[D]_{V',W'}$ that represents the differential operator with respect to the given pair of bases.

a.
$$\mathcal{V}' = \{1, t - 1, (t - 1)^2, (t - 1)^3\}$$
 and $\mathcal{W}' = \mathcal{W}$;

b.
$$V' = V$$
 and $W' = \{1, t - 1, (t - 1)^2\};$

c.
$$\mathcal{V}' = \{1, t - 1, (t - 1)^2, (t - 1)^3\}$$
 and $\mathcal{W}' = \{1, t - 1, (t - 1)^2\}.$

Problem 2 (SA 4.4.16). Let V be a finite-dimensional vector space, let W be a vector space, and let $T:V\to W$ be a linear transformation. Prove the matrix-free analog of the "rank+nullity theorem" for linear transformations. That is, give a (matrix-free) proof that $\dim(\ker T) + \dim(\operatorname{im} T) = \dim V$ by following these steps.

- **a.** Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a basis for $\ker(T)$, and (following Exercise 3.4.17) extend to obtain a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ for V. Show that $\{T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\}$ gives a basis for $\operatorname{im}(T)$.
- **b.** Conclude the desired result. Explain why this is a restatement of Corollary 4.7 of Chapter 3 when W is finite-dimensional.

Problem 3 (SA 4.4.18). Prove the following claims.

- **a. Claim 1:** If $T: V \to W$ is a linear transformation and if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a linearly dependent set of vectors in V, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$ is a linearly dependent set in W.
- **b. Claim 2:** Suppose $T: V \to V$ is a linear transformation and V is finite-dimensional. If im(T) = V and if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly independent, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$ is linearly independent. (*Hint:* Use Problem 2 (SA 4.4.16).)

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Problem 4 (SA 4.4.21). Let V be a vector space.

- **a.** Let V^* denote the set of all linear transformations from V to \mathbb{R} . Show that V^* is a vector space.
- **b.** Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V. For $i = 1, 2, \dots, n$, define $\mathbf{f}_i \in V^*$ by

$$\mathbf{f}_i(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n) = a_i.$$

Prove that $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n\}$ gives a basis for V^* .

c. Deduce that whenever V is finite-dimensional, dim $V^* = \dim V$.

HW 11 due 2016/04/18

¹ The vectors in V^* are sometimes called the *linear functionals* of V.

Section 6.1

In the next exercise, parts f and k are required. Part p is recommended.

 $Problem\ 5$ (SA 6.1.1fkp). Find eigenvalues and bases for the eigenspaces of the following matrices.

$$\mathbf{f.} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\mathbf{k.} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\mathbf{p.} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Problem 6 (cf. SA 6.1.4). Let $V \leq \mathbb{R}^n$ be a subspace and let P_V denote the projection onto V. What are the eigenvalues and eigenvectors of P_V ? (Hint: Think about what P_V does to vectors $\mathbf{v} \in V$ and to vectors $\mathbf{w} \in V^{\perp}$. Are these the only vectors you need to consider?)

In the next exercise, part a is required. Part b and c are recommended.

Problem 7 (SA 6.1.13abc). In each of the following cases, find the eigenvalues and eigenvectors of the linear transformation $T: \mathcal{P}_3 \to \mathcal{P}_3$.

a.
$$T(p)(t) = p'(t)$$

b.
$$T(p)(t) = tp(t)$$

c.
$$T(p)(t) = \int_0^t p'(u) \, du$$