Math 317: Homework 1

NAME:

General Hints: When you are asked to "show" or "prove" something, you should make it a point to write down clearly the information you are given and what it is you are to show. One word of warning regarding the second part of Problem 1.1.22: To say that $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is to say that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ for some scalars $c_1, \dots, c_k$. These scalars will surely be different when you express a different vector $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$, so be sure you give the scalars for $\mathbf{w}$ different names.

("SA 1.1.21" means exercise 21 in Section 1.1 of textbook by **S**hifrin and **A**dams.)

Problem 0 (SA 1.1.28). Carefully prove the following properties of vector arithmetic. Justify all steps. Optional: Give the geometric interpretation of each property in case $n = 2$ or $n = 3$. You are required to turn proofs for e. f. g. (but you are encouraged to try them all).

a. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

b. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.

c. For all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{0} + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

d. For each $\mathbf{x} \in \mathbb{R}^n$, there is a vector $-\mathbf{x}$ so that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.

e. For all $c, d \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, $c(d\mathbf{x}) = (cd)\mathbf{x}$.

f. For all $c \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$.

g. For all $c, d \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$.

h. For all $\mathbf{x} \in \mathbb{R}^n$, $1\mathbf{x} = \mathbf{x}$.

(You must either type your solutions using LaTeX, or print out a hard copy of the pdf file and write your solutions on the hardcopy. If hand-written, your solutions to Problem 0 should go on the next page.)

e. **Claim:** For all $c, d \in \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$, $c(d\mathbf{x}) = (cd)\mathbf{x}$.

Proof.

$\square$

f. For all $c \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$.

Proof.

$\square$

g. For all $c, d \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$.

Proof.

$\square$

Problem 1 (SA 1.1.21). Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and c is a scalar. Prove that $\mathrm{Span}(\mathbf{v} + c\mathbf{w}, \mathbf{w}) = \mathrm{Span}(\mathbf{v}, \mathbf{w})$. (See the blue box on p. 12 of the textbook.)

Proof.

$\square$

Problem 2 (SA 1.1.22). Suppose the vectors $\mathbf{v}$ and $\mathbf{w}$ are both linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

a. Prove for any scalar c that $c\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

b. Prove that $\mathbf{v} + \mathbf{w}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Proof.

$\square$

Problem 3 (SA 1.1.25). Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are nonparallel vectors. (Recall def, p.3 of text.)

a. Prove that if $s\mathbf{x} + t\mathbf{y} = 0$, then $s = t = 0$. (Hint: Show neither $s \neq 0$ nor $t \neq 0$ is possible.)

b. Prove that if $a\mathbf{x} + b\mathbf{y} = c\mathbf{x} + d\mathbf{y}$, then $a = c$ and $b = d$. (Hint: Use part a.)

Proof.

$\square$

Problem 4 (SA 1.2.11). Suppose $\mathbf{x}, \mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ and $\mathbf{x}$ is orthogonal to each of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Show that $\mathbf{x}$ is orthogonal to every linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$.

Proof.

$\square$

Problem 5 (SA 1.2.18). Prove the triangle inequality: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. (Hint: Use the dot product to calculate $\|\mathbf{x}+\mathbf{y}\|^2$.)

Proof.

$\square$

Problem 6 (SA 1.3.12). Suppose $\mathbf{a} \neq \mathbf{0}$ and $\mathscr{P} \subset \mathbb{R}^3$ is the plane through the origin with normal vector $\mathbf{a}$. Suppose $\mathscr{P}$ is spanned by $\mathbf{u}$ and $\mathbf{v}$.

a. Suppose $\mathbf{u} \cdot \mathbf{v} = 0$. Show that for every $\mathbf{x} \in \mathscr{P}$, we have $\mathbf{x} = \operatorname{proj}_{\mathbf{u}} \mathbf{x} + \operatorname{proj}_{\mathbf{v}} \mathbf{x}$.

b. Suppose $\mathbf{u} \cdot \mathbf{v} = 0$. Show that for every $\mathbf{x} \in \mathbb{R}^3$, we have $\mathbf{x} = \operatorname{proj}_{\mathbf{a}} \mathbf{x} + \operatorname{proj}_{\mathbf{u}} \mathbf{x} + \operatorname{proj}_{\mathbf{v}} \mathbf{x}$. (Hint: Apply part a. to the vector $\mathbf{x} - \operatorname{proj}_{\mathbf{a}} \mathbf{x}$.)

c. Give an example to show the result of part a is false when $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal.

Proof.

□