Math 317: Homework 9

NAME:

Section 4.1

Recall the following

Definition. If $V \leq \mathbb{R}^m$ is a subspace, and $\mathbf{b} \in \mathbb{R}^m$, then we define the *projection of* $\mathbf{b}$ *onto* V to be the unique vector $\mathbf{p} \in V$ with the property that $\mathbf{b} - \mathbf{p} \in V^{\perp}$, and we write $\mathbf{p} = \operatorname{proj}_V \mathbf{b}$ in this case.

The following exercise is recommended; it will not be graded.

Exercise (SA 4.1.13). Use the definition of projection given above to show that for any subspace $V \leq \mathbb{R}^m$, the function $\operatorname{proj}_V : \mathbb{R}^m \to \mathbb{R}^m$ is a linear transformation.

Problem 1 (SA 4.1.14). Prove *directly from the definition above* that if we let P denote the matrix projection onto V—that is, $P\mathbf{b} = \mathrm{proj}_V\,\mathbf{b}$—then $P = P^2$ and $P = P^\top$. [*Hints:* For the latter, show that $P\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot P\mathbf{y}$ for all $\mathbf{x}, \mathbf{y}$. It may be helpful to write $\mathbf{x}$ and $\mathbf{y}$ as the sum of vectors in V and $V^\perp$. Then use Exercise 2.5.24.]

The next exercise is recommended but will not be graded.

Exercise (SA 4.1.15). Prove the converse of the fact in the last exercise. That is, if A is a matrix and $A^2 = A$ and $A^{\perp} = A$, then A is a projection matrix. [*Hints:* First decide onto which subspace V it should be projecting. Then Show that for any $\mathbf{b}$, the vector $\mathbf{p} = A\mathbf{b}$ satisfies the definition above of the projection of $\mathbf{b}$ on the subspace V.]

As we have seen in lecture, if $V \leq \mathbb{R}^m$ is a subspace and $\mathbf{b} \in \mathbb{R}^m$, then

$$\mathbf{b} = \operatorname{proj}_V \mathbf{b} + \operatorname{proj}_{V^\perp} \mathbf{b}.$$

Therefore, if we know $\mathbf{b}$ and $\operatorname{proj}_{V^\perp} \mathbf{b}$, then we easily compute $\operatorname{proj}_V \mathbf{b}$ as follows: $\operatorname{proj}_V \mathbf{b} = \mathbf{b} - \operatorname{proj}_{V^\perp} \mathbf{b}$. It's sometimes the case that $\operatorname{proj}_{V^\perp} \mathbf{b}$ is very easy to compute, as in this first exercise below, where the subspace V is a plane and $V^\perp$ is equal to the span of a "normal" vector (i.e., a vector orthogonal to the plane V). For example, in Part (b), a normal vector to V is $\mathbf{n} = (1, 1, 1, 0)$, so the projection of $\mathbf{b}$ onto $V^\perp$ is $\frac{\mathbf{b}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n} = (1, 1, 1, 0)$. Therefore, $\operatorname{proj}_V \mathbf{b} = (-1, 0, 1, 3)$.

Problem 2 (SA 4.1.1). Find the projection of the given vector $\mathbf{b} \in \mathbb{R}^m$ onto the given hyperplane $V \leq \mathbb{R}^m$ by first finding the projection onto $V^\perp$, as suggested above.

(a) $V = \{x_1 + x_2 + x_3 = 0\} \leq \mathbb{R}^3$, $\mathbf{b} = (2, 1, 1)$.

(b) $V = \{x_1 + x_2 + x_3 = 0\} \leq \mathbb{R}^4$, $\mathbf{b} = (0, 1, 2, 3)$.

(c) $V = \{x_1 - x_2 + x_3 + 2x_4 = 0\} \leq \mathbb{R}^4$, $\mathbf{b} = (1, 1, 1, 1)$.

Problem 3 (SA 4.1.2). Use the formula $P_V = A(A^\top A)^{-1}A^\top$ for the projection matrix to check that $P_V = P_V^\top$ and $P_V^2 = P_V$. Show that $I - P_V$ has the same properties, and explain why.

Problem 4 (SA 4.1.4). Let $V = \text{Span}\{(1,0,1),(0,1,-2)\} \leq R^3$. Construct the matrix P_V representing proj_V in two ways:

(a) by finding $P_{V^\perp}$;

(b) by using the formula $P_V = A(A^\top A)^{-1}A^\top$.

Problem 5 (SA 4.1.7). Find the least squares solution of

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - 3x_2 &= 4 \\ 2x_1 + x_2 &= 3. \end{aligned}$$

Use your answer to find the point on the plane spanned by $(1, 1, 2)$ and $(1, -3, 1)$ that is closest to $(1, 4, 3)$.

Problem 6 (SA 4.2.3). Let $V = \text{Span}\{(2,1,0,-2),(3,3,1,0)\} \leq \mathbb{R}^4$.

(a) Find an orthogonal basis for V.

(b) Use your answer to part (a) to find the projection of $\mathbf{b} = (0,4,-4,-7)$ onto V.

(c) Use your answer to part (a) to find the projection matrix P_V.

Problem 7. Let $\mathcal{C}^0([a,b])$ denote the vector space of continuous real-valued functions defined on $[a,b]$. Recall that an inner product on $\mathcal{C}^0([a,b])$ can be defined as follows: for $f,g \in \mathcal{C}^0([a,b])$,

$$\langle f, g\rangle = \int_a^b f(t)g(t)\,dt.$$

Using this inner product, find an orthogonal basis for the subspace $\mathcal{P}_1 \leq C^0([0,1])$, and use your answer to find the projection of $f(t) = t^2 + t - 1$ onto $\mathcal{P}_1$.