A Note on Linear Independence and Bases

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Lemma 1: If $A \in \mathbb{R}^{m\times n}$ and $\mathrm{N}(A) = \{\mathbf{0}\}$, then $m \geq n$.

Proof. If $\mathrm{N}(A) = \{\mathbf{0}\}$, then by definition the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution $\mathbf{0}$. Equivalently, the system $A\mathbf{x} = \mathbf{0}$ has no free variables, so any echelon form of A has a pivot in each of its n columns. Since there are m rows, there can be at most m pivots, so $m \geq n$. $\square$

Lemma 2: If V is a subspace of $\mathbb{R}^n$ with basis $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, then any linearly independent subset of V has at most k vectors.

Proof. Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ be a linearly independent subset of V. We will prove $m \leq k$. Each $\mathbf{x}_i$ is a linear combination of the basis vectors in $\mathcal{V}$. That is, there exist scalars $a_{ij} \in \mathbb{R}$ such that

$$
\begin{aligned}
\mathbf{x}_1 &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \cdots + a_{k1}\mathbf{v}_k \\
\mathbf{x}_2 &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \cdots + a_{k2}\mathbf{v}_k \\
&\vdots \\
\mathbf{x}_m &= a_{1m}\mathbf{v}_1 + a_{2m}\mathbf{v}_2 + \cdots + a_{km}\mathbf{v}_k
\end{aligned} \tag{1}
$$

Note that, in the first equation above, the right-hand side is a matrix-vector product, so we can write this equation as follows:

$$
\mathbf{x}_1 = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{bmatrix}.
$$

Of course, the same is true of the other equations in (1), so we can write them all simultaneously as a the following system of equations:

$$
\begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{km} \end{bmatrix} \tag{2}
$$

Let X denote the matrix on the left-hand side of (2), and let A denote the second matrix on the right-hand side of (2). Since $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ is linearly independent, we have $\mathrm{N}(X) = \{\mathbf{0}\}$, and this implies $\mathrm{N}(A) = \{\mathbf{0}\}$. (Reason: if $A\mathbf{y} = \mathbf{0}$, then $X\mathbf{y} = \mathbf{0}$ by (2).) Finally, if $\mathrm{N}(A) = \{\mathbf{0}\}$, we know that A must have at least as many rows as columns, by Lemma 1. Therefore $m \leq k$, as desired. □

Proposition: Let $W \subseteq V$ be subspaces of $\mathbb{R}^n$. Suppose $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for V and $\mathcal{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ is a basis for W. Then $W = V$ if and only if $\ell = k$.

Proof.

($\Rightarrow$) If $W = V$, then $\mathcal{W}$ is a basis for V so $\ell = k$, since (as we proved earlier) all bases for a given subspace have the same number of vectors.

($\Leftarrow$) We want to prove $\ell = k$ implies $W = V$. We will prove the (equivalent) contrapositive statement, which is $W \neq V$ implies $\ell \neq k$.

If $W \neq V$, then there is a vector $\mathbf{v} \in V$ that does not belong to W. Therefore, $\mathbf{v} \notin \mathrm{Span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_\ell\}$, so $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_\ell, \mathbf{v}\}$ is a set of $\ell + 1$ linearly independent vectors in V. By Lemma 2 then, $\ell + 1 \leq k$, so $\ell < k$, as desired.

□

Corollary: If $W \subseteq V$ are subspaces of $\mathbb{R}^n$ and $\dim W = \dim V$, then $W = V$.

Proof. This follows directly from the proposition above and the definition of dimension. □

The following exercise is not really meant to illustrate the foregoing results, though it is related to them. It is an elaboration of an argument that a student came up with last semester, and I wanted to bring it to your attention so you can avoid making the same mistake.

Exercise: Let $\mathbf{v}_1 = (1,0,0)$, $\mathbf{v}_2 = (1,1,0)$, $\mathbf{w}_1 = (0,1,1)$, $\mathbf{w}_2 = (0,0,1)$, and let $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2\}$. Decide whether the following claim and its proof are valid. If not, what goes wrong?

Claim: The set $\mathcal{V} \cup \mathcal{W} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent.

Proof. Consider, for each $i \in \{1,2\}$, the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_i\}$. Both of the matrices

$$\begin{bmatrix} | & | & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{v}_1 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} | & | & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{v}_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

have reduced echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Therefore, we make the following observations:

- the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_1\}$ is linearly independent
- the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_2\}$ is linearly independent
- the set $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent (by assumption).

It follows that the set $\mathcal{V} \cup \mathcal{W} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent. $\square$