MATH 317: SPRING 2016 **ANSWERS** **EXAM 2**

1. Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ real matrix. Given precise definitions of the *four fundamental subspaces* $\mathrm{R}(A)$, $\mathrm{C}(A)$, $\mathrm{N}(A)$, $\mathrm{N}(A^\top)$. Give both a symbolic definition as well as an **English sentence**. Also, in the blank space provided, insert the name of the space of which the given set is a subspace. The first part is done for you as an example.

(2pts) (a) The *row space* of A is the subspace of $\underline{\quad \mathbb{R}^n \quad}$ defined by...

(English) ...the set of all linear combinations of rows of A.

(symbols) $\mathrm{R}(A) = \{\mathbf{y}^\top A \mid \mathbf{y} \in \mathbb{R}^m\} = \{\mathbf{x} \in \mathbb{R}^n \mid \exists \mathbf{y} \in \mathrm{R}^m \,.\, \mathbf{x} = \mathbf{y}^\top A\}$.

(2pts) (b) The *column space* of A is the subspace of $\underline{\quad \mathbb{R}^m \quad}$ defined by...

(English) ...the set of all linear combinations of columns of A.

(symbols) $\mathrm{C}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathrm{R}^n\} = \{\mathbf{y} \in \mathbb{R}^m \mid \exists \mathbf{x} \in \mathbb{R}^n.\ A\mathbf{x} = \mathbf{y}\}$.

(2pts) (c) The *null space* of A is the subspace of $\underline{\quad \mathbb{R}^n \quad}$ defined by...

(English) ...the set of all solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

(symbols) $\mathrm{N}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$.

(2pts) (d) The *left null space* of A is the subspace of $\underline{\quad \mathbb{R}^m \quad}$ defined by...

(English) ...the set of all solutions to the homogeneous linear system $A^\top\mathbf{x} = \mathbf{0}$.

(symbols) $\mathrm{N}(A^\top) = \{\mathbf{x} \in \mathbb{R}^m \mid A^\top\mathbf{x} = \mathbf{0}\}$.

(3pts) (e) We learned a theorem that gives identities involving the four fundamental subspaces and their orthogonal complements. Two of these are $\mathrm{C}(A) = \mathrm{N}(A^\top)^\perp$ and $\mathrm{C}(A)^\perp = \mathrm{N}(A^\top)$. What are the other two?

Answer: $\mathrm{R}(A) = \mathrm{N}(A)^\perp$ and $\mathrm{R}(A)^\perp = \mathrm{N}(A)$.

2. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix}$. For the first two parts of this problem, circle the best answer. For the last part, write down the basis.

(3pts) (a) The rank of A is

(i) 1 (ii) 2 (iii) 3 (iv) 4

Answer: (ii)

(4pts) (b) A basis for the column space $\mathrm{C}(A)$ is given by which columns of A?

(i) columns 1 and 2
(ii) columns 1 and 3
(iii) columns 1 and 4
(iv) column 1 only
(v) column 4 only

Answer: (ii)

(5pts) (c) Find a basis for the nullspace, $\mathrm{N}(A)$.

Answer: Since $A \xrightarrow{\text{row red.}} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, a basis for $\mathrm{R}(A)$ is

$$\{(1, -2, 1, 0), (0, 0, 1, -1)\}.$$

(16pts) 3. Answer any **two** questions on this page. Clearly mark the answers you want graded.

(a) Prove that if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are vectors in $\mathbb{R}^n$, and if there is a matrix A such that the set $\{A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_m\}$ is linearly independent, then $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is linearly independent.

(b) Prove that if V is a vector space with inner product $\langle \cdot, \cdot \rangle$ and orthonormal basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then the $\mathcal{B}$-basis representation of the vector $\mathbf{x} \in V$ is given by $[\mathbf{x}]_{\mathcal{B}} = (\langle \mathbf{x}, \mathbf{v}_1 \rangle, \langle \mathbf{x}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{x}, \mathbf{v}_n \rangle)$.

(c) Either find a matrix A for which $\mathrm{N}(A)$ contains $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\mathrm{R}(A)$ contains $\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$, or explain why no such matrix exists.

Answer:

(a) Suppose $\{A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_m\}$ is linearly independent. To prove $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is linearly independent, we suppose $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m = \mathbf{0}$, and try to show that $c_1 = \cdots = c_m = 0$. Indeed, if $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m = \mathbf{0}$, then by applying A on the left of both sides of this equation we obtain, by linearity, $c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 + \cdots + c_mA\mathbf{x}_m = A\mathbf{0}$, which is $\mathbf{0}$. Thus, $c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 + \cdots + c_mA\mathbf{x}_m = \mathbf{0}$. So, by linear independences of $\{A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_m\}$, we have $c_1 = \cdots = c_m = 0$, as desired.

(b) Let $[\mathbf{x}]_{\mathcal{B}} = (c_1, c_2, \ldots, c_n)$. This means that $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$. Therefore, taking the inner product of $\mathbf{x}$ with $\mathbf{v}_i$, for any i, we have

$$\begin{aligned} \langle \mathbf{x}, \mathbf{v}_i \rangle &= \langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n, \mathbf{v}_i \rangle \\ &= c_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \cdots + c_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle \\ &= c_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle = c_i. \end{aligned}$$

The last two equalities hold since $\mathcal{B}$ is an orthonormal basis. We have thus proved for each i that $c_i = \langle \mathbf{x}, \mathbf{v}_i \rangle$. Thus, $[\mathbf{x}]_{\mathcal{B}} = (c_1, c_2, \ldots, c_n) = (\langle \mathbf{x}, \mathbf{v}_1 \rangle, \langle \mathbf{x}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{x}, \mathbf{v}_n \rangle)$, as desired.

4. Suppose $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, and $\mathbf{x} \in V$ is an arbitrary vector.

(3pts) (a) Denote the $\mathcal{B}$-*basis representation of* $\mathbf{x}$ by $[\mathbf{x}]_{\mathcal{B}}$. What does this mean? (i.e., interpret $[\mathbf{x}]_{\mathcal{B}} = (c_1, c_2, \ldots, c_n)$ in terms of the vector $\mathbf{x}$ and the vectors $\mathbf{v}_i$.)

Answer: This means that $\mathbf{x}$ in the standard basis is $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$.

(3pts) (b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (x_1 + 2x_2, 3x_2)$. Write down the standard matrix for T. (i.e., find the $\mathcal{E}$-basis representation of T).

Answer: $[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

(3pts) (c) Suppose $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Write down the change-of-basis matrix P that maps a $\mathcal{B}$-basis representation (e.g., $[\mathbf{x}]_{\mathcal{B}}$) to a standard basis representation (e.g., $\mathbf{x} = [\mathbf{x}]_{\mathcal{E}}$); then find the inverse P^{-1} (that goes from $\mathcal{E}$ to $\mathcal{B}$).

Answer:

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

(2pts) (d) Use the matrices from the previous part to find the $\mathcal{B}$-basis representation of T. That is, find $[T]_{\mathcal{B}}$, the matrix representation of T relative to the basis $\mathcal{B}$.

Answer:

$$[T]_{\mathcal{B}} = P^{-1}[T]_{\mathcal{E}}P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

(10pts) 5. Use Gram-Schmidt procedure to find an orthonormal basis for the subspace $V \leq \mathbb{R}^3$ spanned by $\mathbf{v}_1 = (3, 4, 0)$ and $\mathbf{v}_2 = (7, 1, 12)$. (*Hint:* You can simplify your answer using the identities $12^2 = 144$ and $13^2 = 169$.)

Answer: Let $\mathbf{w}_1' = \mathbf{v}_1$. Then

$$\|\mathbf{w}_1'\| = \sqrt{9+16} = 5 \quad \text{so} \quad \mathbf{w}_1 = \frac{1}{5}(3, 4, 0).$$

Let $\mathbf{w}_2' = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{w}_1)\mathbf{w}_1$. Then

$$\mathbf{w}_2' = (7, 1, 12) - \big[(7, 1, 12) \cdot (3/5, 4/5, 0)\big](3/5, 4/5, 0) = (4, -3, 12).$$

Therefore, $\|\mathbf{w}_2'\| = \sqrt{16 + 9 + 144} = \sqrt{169} = 13$, so $\mathbf{w}_2 = \frac{1}{13}(4, -3, 12)$.

Answer: $\mathbf{w}_1 = \frac{1}{5}(3, 4, 0)$ $\mathbf{w}_2 = \frac{1}{5}(4, -3, 12)$.