Math 317: Homework 7

Due: 11 March 2016

NAME:

Section 3.3

Problem 1 (SA 3.3.3). Decide whether the following sets of vectors give a basis for the indicated space. (Show your work and/or justify your answer.)

a. $\{(1,2,1),(2,4,5),(1,2,3)\}$; $\mathbb{R}^3$.

b. $\{(1,0,1),(1,2,4),(2,2,5),(2,2,-1)\}$; $\mathbb{R}^3$.

c. $\{(1,0,2,3),(0,1,1,1),(1,1,4,4)\}$; $\mathbb{R}^4$.

d. $\{(1,0,2,3),(0,1,1,1),(1,1,4,4),(2,-2,1,2)\}$; $\mathbb{R}^4$.

Problem 2 (SA 3.3.4ac). In each case, check that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ and give the coordinates of the given vector $\mathbf{b} \in \mathbb{R}^n$ with respect to that basis.

a. $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

c. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

Problem 3 (SA 3.3.5b). Give a basis for each of the subspaces $\mathbf{R}(A)$, $\mathbf{C}(A)$, $\mathbf{N}(A)$ where
$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

Problem 4 (SA 3.3.11). Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are nonzero, mutually orthogonal vectors in $\mathbb{R}^n$.

a. Prove that they form a basis for $\mathbb{R}^n$.

b. Given any $\mathbf{x} \in \mathbb{R}^n$, give an explicit formula for the coordinates of $\mathbf{x}$ with respect to the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

c. Deduce from your answer to part b that $\mathbf{x} = \sum_{i=1}^{n} \operatorname{proj}_{\mathbf{v}_i} \mathbf{x}$.

Problem 5 (SA 3.4.1bcd). For each subspace V, find a basis and determine $\dim V$.

b. $V = \{\mathbf{x} \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0,\ x_2 + x_4 = 0\} \subseteq \mathbb{R}^4$.

c. $V = (\mathrm{Span}\{(1,2,3)\})^\perp \subseteq \mathbb{R}^3$.

d. $V = \{\mathbf{x} \in \mathbb{R}^5 : x_1 = x_2,\ x_3 = x_4\} \subseteq \mathbb{R}^5$.

Problem 6. Let $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix}$.

(a) Give bases for $\mathbf{R}(A)$, $\mathbf{N}(A)$, $\mathbf{C}(A)$, and $\mathbf{N}(A^\top)$.

(b) Use your answer to Part (a) to determine the dimension of each of these subspaces; confirm your answer using the "Nullity-Rank Theorem" (Corollary 4.7 of the text).

(c) Using your answer to Part (a), verify the orthogonality conditions given in Theorem 3.2.5. (That is, check $\mathbf{N}(A)^\perp = \mathbf{R}(A)$ and $\mathbf{N}(A^\top)^\perp = \mathbf{C}(A)$.)

Problem 7 (SA 3.4.24)*.* Continuing Exercise 3.2.10 from last week's homework...
Let A be an $m \times n$ matrix.

a. Use Theorem 2.5 to prove that $\mathbf{N}(A^\top A) = \mathbf{N}(A)$.
   (*Hint:* if $\mathbf{x} \in \mathbf{N}(A^\top A)$, then $A\mathbf{x} \in \mathbf{C}(A) \cap \mathbf{N}(A^\top)$.)

b. Prove that $\operatorname{rank}(A) = \operatorname{rank}(A^\top A)$.

c. Prove that $\mathbf{C}(A^\top A) = \mathbf{C}(A^\top)$.

The following problem is recommended but not required; it will not be graded.

Problem (SA 3.4.17). Let $V \leq R^n$ be a subspace. Prove that any linearly independent set of vectors in V can be extended to a basis for V. In other words, prove the following: if $\dim V > k$ and if we are given a linearly independent set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subseteq V$, then there exist vectors $\mathbf{v}_{k+1}, \dots, \mathbf{v}_\ell \in V$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_\ell\}$ is a basis for V.