Math 317: Homework 8

NAME:

Section 3.4

Problem 1. Let $\mathcal{V}$ and $\mathcal{W}$ be subsets of $\mathbb{R}^4$ given by

$$\mathcal{V} := \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1,0,0,0),(1,1,0,0)\} \quad \text{and} \quad \mathcal{W} := \{\mathbf{w}_1, \mathbf{w}_2\} = \{(0,1,1,0),(0,0,1,0)\}.$$

It is not hard to see that each of these sets is linearly independent. What about their union? Decide whether the following claim and its proof are valid. If not, where is the error?

<u>Claim</u>: The set $\mathcal{V} \cup \mathcal{W} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent.

Proof Attempt: Consider, for each $i \in \{1,2\}$, the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_i\}$. Both of the matrices

$$\begin{bmatrix} | & | & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{v}_1 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} | & | & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{v}_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

have reduced echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. We deduce the following facts:

- the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_1\}$ is linearly independent;
- the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_2\}$ is linearly independent;
- the set $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

It follows that the set $\mathcal{V} \cup \mathcal{W} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent.

Problem 2. Recall,

Proposition 3.6. If $V \leq \mathbb{R}^n$, then $(V^\perp)^\perp = V$.

Proof. Choose a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for V, and consider the $k \times n$ matrix A with rows $\mathbf{v}_1, \ldots, \mathbf{v}_k$. By construction, $V = \mathrm{R}(A)$. By Theorem 2.5, $V^\perp = R(A)^\perp = \mathrm{N}(A)$, and $\mathrm{N}(A)^\perp = R(A)$, so $(V^\perp)^\perp = V$. $\square$

As mentioned in lecture, the inclusion $V \subseteq (V^\perp)^\perp$ follows directly from the definitions. It is the reverse inclusion $(V^\perp)^\perp \subseteq V$ that requires proof. Use Theorem 4.9 to show that $\mathbf{x} \in (V^\perp)^\perp$ implies $\mathbf{x} \in V$ (thus giving an alternative proof of Prop 3.6).

The next exercise is recommended but will not be graded.

Exercise (SA 3.4.15). Suppose $V \leq \mathbb{R}^n$. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for V, and let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_\ell \in V$ be vectors such that $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_\ell\} = V$. Prove that $\ell \geq k$.

Problem 3 (SA 3.4.16). Prove the Proposition 4.4. (*Hint:* 3.4.15 and Prop 4.3 are helpful.)

Proposition 4.4. Let $V \leq \mathbb{R}^n$ be a k-dimensional subspace. Then any k vectors that span V must be linearly independent, and any k linearly independent vectors in V must span V.

Proof.

$\square$

Problem 4 (SA 3.4.17). Let $V \leq \mathbb{R}^n$ be a subspace, and suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq V$ is a linearly independent set of vectors. Show that if $\dim V > k$, then there are vectors $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_\ell \in V$ such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ forms a basis for V.

The next exercise is recommended but will not be graded.

Exercise (SA 3.4.18). Suppose V and W are subspaces of $\mathbb{R}^n$ and $W \leq V$. Prove that $\dim W \leq \dim V$.

Problem 5 (SA 3.4.19). Suppose A is an $n \times n$ matrix, and let $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$. Suppose $\{A\mathbf{v}_1, \dots, A\mathbf{v}_n\}$ is linearly independent. Prove that A is nonsingular.

The exercises on this page are recommended but will not be graded.

Exercise (SA 3.3.23, 3.4.20). Let U and V be subspaces of $\mathbb{R}^n$ with $U \cap V = \{\mathbf{0}\}$.

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis for U and $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ is a basis for V, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ is a basis for $U + V$.

(b) $\dim(U + V) = \dim U + \dim V$.

Exercise (SA 3.4.21). Let U and V be subspaces of $\mathbb{R}^n$. Prove that

$$\dim(U + V) = \dim U + \dim V - \dim(U \cap V).$$

(Hint: Start with a basis for $U \cap V$, and use Exercise 3.4.17.)

Section 3.6

Problem 6 (SA 3.6.2abc). Decide whether the following sets of vectors are linearly independent. (Justify your answers.)

a. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\} \subseteq \mathbb{R}^{2\times 2}$

b. $\{f_1, f_2, f_3\} \subseteq \mathcal{P}_1$, where $f_1(t) = t,\ f_2(t) = t+1,\ f_3(t) = t+2$

c. $\{f_1, f_2, f_3\} \subseteq \mathcal{C}^\infty(\mathbb{R})$, where $f_1(t) = 1,\ f_2(t) = \cos t,\ f_3(t) = \sin t$

Problem 7 (SA 3.6.15a). Let $g_1(t) = 1$ and $g_2(t) = t$. Using the inner product defined in Example 10(c), find the orthogonal complement of $\text{Span}\{g_1, g_2\}$ in $\mathcal{P}_2 \subset \mathcal{C}^0([-1, 1])$.

The next exercise is recommended but will not be graded.

Exercise. Let V be the set of all functions mapping $\mathbb{R}$ to itself. That is, V is the set of all real-valued functions with domain the set of real numbers. (We often denote such functions by $f : \mathbb{R} \to \mathbb{R}$.) Let $W \subseteq V$ and suppose that W contains the constant function $x \mapsto 0$ (taking every x to 0) as well as all of those functions $f \in V$ satisfying the following condition: $f(x) = 0$ for at most finitely many real numbers x. Prove or disprove: W is a subspace of V.

The next exercise is just for fun. Think about it over dinner. You need not submit a solution.

Exercise. Fix an integer n. Let $V = \{0, 1, 2, \ldots, 2^n - 1\}$. (Notice that the size of the set V is 2^n.) Each number in V can be represented as a "binary vector" of length n as shown in Table 1. Let $F = \{0, 1\}$ be the scalar field with two elements, where multiplication on F is defined as usual, and addition is defined modulo 2. That is,

$$\begin{aligned} 0 \cdot 0 = 0, \quad 0 \cdot 1 = 0 = 1 \cdot 0, \quad 1 \cdot 1 = 1, \\ 0 + 0 = 0 = 1 + 1, \quad \text{and} \quad 1 + 0 = 1 = 0 + 1. \end{aligned}$$

Using the binary representation given above, each number in V is a vector (of length n) with entries 0 and 1. Show that V is a vector space over the scalar field F. (*Hints:* Say how vector addition and scalar multiplication should be defined, say what plays the role of the zero vector, and check that, with these definitions, V satisfies all the properties of a vector space. In particular, for each vector $\mathbf{v} \in V$, identify the vector $\mathbf{w}$ that satisfies $\mathbf{v} + \mathbf{w} = \mathbf{0}$.)

decimal	binary
0	$\begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$
1	$\begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$
2	$\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$
3	$\begin{bmatrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$
4	$\begin{bmatrix} 0 & 0 & 1 & 0 & \ldots & 0 & 0 \end{bmatrix}$
$\vdots$	$\vdots$
$2^n - 2$	$\begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$
$2^n - 1$	$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$

Tab. 1: representation of integers as binary vectors

The next exercise is just for fun. Think about it over the break. You need not submit a solution.

Exercise. Let X be a nonempty set. Consider the so called power set $\mathscr{P}(X)$ of all subsets of X. Can you find a way to turn $V = \mathscr{P}(X)$ into a vector space? (*Hints:* If each subset of X is a vector, how could you define vector addition? ...and scalar multiplication? What would you take as the zero vector? Find appropriate definitions for these operations so that the defining vector space properties are satisfied. Thinking about the problem on the previous page might help.)