Math 317: Homework 3

NAME:

Problem 1 (SA 1.5.2b). Is the vector

$$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \text{a linear combination of the vectors} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}?$$

If the answer is "no," prove it. If the answer is "yes," write down the *general solution* to the system $A\mathbf{x} = \mathbf{b}$, where A is the matrix with the vector $\mathbf{v}_i$ in column i. (You must do this problem by hand and show your work to get credit. You may use Sage to check your answer.)

Problem 2 (SA 1.5.3cd). Find constraint equations (if any) that $\mathbf{b}$ must satisfy in order for $A\mathbf{x} = \mathbf{b}$ to be consistent.

c.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

d.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

(*Hint:* See Examples 3, 4, and 5 of Section 1.5.)

Problem 3 (SA 1.5.10). Let A be an $m \times n$ matrix. Prove, or disprove with counterexample, the following claim: If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, then for each $\mathbf{b} \in \mathbb{R}^m$ the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

Problem 4 (SA 1.5.12cd). In each case, give positive integers m and n and an example of an $m \times n$ matrix A with the stated property, or explain why none can exist.

c. $A\mathbf{x} = \mathbf{b}$ has no solutions for some $\mathbf{b} \in \mathbb{R}^m$ and one solution for every other $\mathbf{b} \in \mathbb{R}^m$.

d. $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for every $\mathbf{b} \in \mathbb{R}^m$.

Problem 5 (cf. SA 2.1.8). Consider the following matrix

$$A = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix}.$$

a. Find a formula for A^k that holds for positive integers k. In other words, express the product $AA \cdots A$ (of k factors) as a function of the entries of the matrix A.
(*Hint:* if you're having trouble starting this one, try computing higher powers of a small example, like $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, then notice the pattern and guess the formula.)

b. Use the principle of induction to prove that your formula is correct.

Problem 6 (SA 2.1.11a). Suppose $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$, and $BA = I_n$. Prove that if for some $\mathbf{b} \in \mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has a solution, then that solution is unique.

> *Hint:* To prove the statement "if a solution exists, then it is unique," one approach (which works well here) is to suppose that $\mathbf{x}$ satisfies the equation and find a formula that determines it. Another approach is to assume that $\mathbf{x}$ and $\mathbf{y}$ are both solutions and then use the equations to prove that $\mathbf{x} = \mathbf{y}$.