MATH 317: SPRING 2016 **EXAM 2**

RULES
• No books or notes or calculators allowed.
• No bathroom breaks until after you have completed and turned in your test.
• Out of consideration for your classmates, do not make disturbing noises during the exam.
• **Phones and other electronic devices must be off or in silent mode.**

Cheating will not be tolerated. If there is any indication that a student may have given or received unauthorized aid on this test, the case will be handed over to the ISU Office of Judicial Affairs. When you finish the exam, please sign the following statement acknowledging that you understand this policy:

"On my honor as a student I, ________________________________, have neither given nor received unauthorized aid on this exam." (print name clearly)

Signature: ____ Date: 2016-04-05

Page	Points	Score
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4	12	
5	16	
6	11	
7	10	
Total:	60	

1. Let $A \in \mathbb{R}^{m\times n}$ be an $m\times n$ real matrix. Given precise definitions of the *four fundamental subspaces* $\mathrm{R}(A)$, $\mathrm{C}(A)$, $\mathrm{N}(A)$, $\mathrm{N}(A^\top)$. Give both a symbolic definition as well as an **English sentence**. Also, in the blank space provided, insert the name of the space of which the given set is a subspace. The first part is done for you as an example.

(2pts) (a) The *row space* of A is the subspace of $\underline{\quad \mathbb{R}^n \quad}$ defined by...

(English) ...the set of all linear combinations of rows of A.

(symbols) $\mathrm{R}(A) = \big\{\mathbf{v} \in \mathbb{R}^n \mid \exists \mathbf{u} \in \mathrm{R}^m \text{ such that } \mathbf{v} = \mathbf{u}A\big\}$.

(2pts) (b) The *column space* of A is the subspace of ____________ defined by...

(English)

(symbols) $\mathrm{C}(A) = \{$

(2pts) (c) The *null space* of A is the subspace of ____________ defined by...

(English)

(symbols) $\mathrm{N}(A) = \{$

(2pts) (d) The *left null space* of A is the subspace of ____________ defined by...

(English)

(symbols) $\mathrm{N}(A^\top) = \{$

(3pts) (e) We learned a theorem that gives identities involving the four fundamental subspaces and their orthogonal complements. Two of these are $\mathrm{C}(A) = \mathrm{N}(A^\top)^\perp$ and $\mathrm{C}(A)^\perp = \mathrm{N}(A^\top)$. What are the other two?

2. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix}$. For the first two parts of this problem, circle the best answer. For the last part, write down the basis.

(3pts) (a) The rank of A is

(i) 1 (ii) 2 (iii) 3 (iv) 4

(4pts) (b) A basis for the column space $\mathrm{C}(A)$ is given by which columns of A?

(i) columns 1 and 2
(ii) columns 1 and 3
(iii) columns 1 and 4
(iv) column 1 only
(v) column 4 only

(5pts) (c) Find a basis for the nullspace, $\mathrm{N}(A)$.

Score for this page: __________ out of 12

(16pts) 3. Answer any **two** questions on this page. Clearly mark the answers you want graded.

(a) Prove that if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are vectors in $\mathbb{R}^n$, and if there is a matrix A such that the set $\{A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_m\}$ is linearly independent, then $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is linearly independent.

(b) Prove that if V is a vector space with inner product $\langle \cdot, \cdot \rangle$ and orthonormal basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then the $\mathcal{B}$-basis representation of the vector $\mathbf{x} \in V$ is given by $[\mathbf{x}]_{\mathcal{B}} = (\langle \mathbf{x}, \mathbf{v}_1 \rangle, \langle \mathbf{x}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{x}, \mathbf{v}_n \rangle)$.

(c) Either find a matrix A for which $\mathrm{N}(A)$ contains $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\mathrm{R}(A)$ contains $\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$, or explain why no such matrix exists.

Score for this page: ____________ out of 16

4. Suppose $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, and $\mathbf{x} \in V$ is an arbitrary vector.

(3pts) (a) Denote the *$\mathcal{B}$-basis representation of* $\mathbf{x}$ by $[\mathbf{x}]_{\mathcal{B}}$. What does this mean? (i.e., interpret $[\mathbf{x}]_{\mathcal{B}} = (c_1, c_2, \ldots, c_n)$ in terms of the vector $\mathbf{x}$ and the vectors $\mathbf{v}_i$.)

(3pts) (b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (x_1 + 2x_2, 3x_2)$. Write down the standard matrix for T. (i.e., find the $\mathcal{E}$-basis representation of T).

(3pts) (c) Suppose $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$. Write down the change-of-basis matrix P that maps a $\mathcal{B}$-basis representation (e.g., $[\mathbf{x}]_{\mathcal{B}}$) to a standard basis representation (e.g., $\mathbf{x} = [\mathbf{x}]_{\mathcal{E}}$); then find the inverse P^{-1} (that goes from $\mathcal{E}$ to $\mathcal{B}$).

(2pts) (d) Use the matrices from the previous part to find the $\mathcal{B}$-basis representation of T. That is, find $[T]_{\mathcal{B}}$, the matrix representation of T relative to the basis $\mathcal{B}$.

(10pts) 5. Use Gram-Schmidt procedure to find an orthonormal basis for the subspace $V \leq \mathbb{R}^3$ spanned by $\mathbf{v}_1 = (3,4,0)$ and $\mathbf{v}_2 = (7,1,12)$. (*Hint:* You can simplify your answer using the identities $12^2 = 144$ and $13^2 = 169$.)

Score for this page: ______________ out of 10

– scratch –

– scratch –