

## Homework 2

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*Problem 1* (Golan 56). Is it possible to define on  $\mathbb{Z}/(4)$  the structure of a vector space over  $\text{GF}(2)$  in such a way that the vector addition is the usual addition in  $\mathbb{Z}/(4)$ ?

[*Hints:* Recall that  $(n)$  denotes the set  $\{\dots, -2n, -n, 0, n, 2n, 3n, \dots\}$ , which we denoted in lecture by  $n\mathbb{Z}$ . This is the ideal generated by  $n$  in the ring  $\mathbb{Z}$ , but don't worry about that for now. Just take  $\mathbb{Z}/(n)$  to be the abelian group of integers  $\{0, 1, 2, \dots, n-1\}$  with addition modulo  $n$ . In lecture, we used  $\mathbb{Z}/n\mathbb{Z}$  to denote  $\mathbb{Z}/(n)$ . Use whichever notation you prefer.]

**Solution.** Assume toward a contradiction that  $\mathbb{Z}/(4)$  is a vector space over  $\text{GF}(2)$ , with vector addition defined as the usual modular addition. Then for any  $v \in \mathbb{Z}/(4)$ ,

$$\begin{aligned} 0 &= (0)v \\ &= (1+1)v \\ &= 1 \cdot v + 1 \cdot v \\ &= v + v \end{aligned}$$

Now, let  $v = 3$  to see that  $3 + 3 = 2 \pmod{4} \neq 0$ . Thus we have a contradiction. The answer is no, it is not possible with the usual modular vector addition.

*Problem 2* (Golan 60).<sup>1</sup> Let  $V = C(0, 1)$ . Define the relation  $\vee$  on  $V$  by setting  $(f \vee g)(x) = \max\{f(x), g(x)\}$ . If we think of  $\vee$  as a “vector addition,” does this, together with the usual scalar multiplication, make  $V$  into a vector space over  $\mathbb{R}$ ?

**Solution.** It is true that  $C(0, 1)$  is closed under  $\vee$  (see the [Appendix](#) for a proof), and we can easily verify that  $\vee$  is a commutative associative binary operation on the set  $C(0, 1)$ , so  $\langle C(0, 1), \vee \rangle$  is a commutative semigroup. In fact, letting  $f \wedge g = \min\{f, g\}$ , we can check that  $\langle C(0, 1), \vee, \wedge \rangle$  is a lattice.<sup>2</sup>

However, recall that a vector space is built up from an additive abelian group. Is it possible for  $\vee$  to serve as vector addition? If so, what would be the additive identity? We need a function  $e \in C(0, 1)$  such that for all  $f \in C(0, 1)$  we have  $f \vee e = f$ . It is clear that no such  $e$  exists.

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<sup>1</sup> In the original problem, the notation  $f \boxplus g$  was used. We use  $f \vee g$  instead, since this is fairly standard notation for the function  $\max\{f, g\}$ .

<sup>2</sup> See [1, Sec. 30] for a discussion of vector lattices, such as  $\langle C(0, 1), \vee, \wedge \rangle$ .

*Problem 3* (Golan 63). Let  $V = \{i \in \mathbb{Z} \mid 0 \leq i < 2^n\}$  for some fixed positive integer  $n$ . Define operations of vector addition and scalar multiplication on  $V$  in such a way as to turn it into a vector space over  $\text{GF}(2)$ .

[*Hints:* Recall that  $\text{GF}(2)$  denotes the Galois field with two elements,  $\{0, 1\}$ , with addition mod 2 and the usual multiplication. Other than this field, the only restriction given in the problem is that  $V$  must have  $2^n$  elements. Do you know of any sets of this size?]

**Solution.** Let  $W$  be the set of binary strings of length  $n$ , that is, length  $n$  sequences of 0's and 1's. We can also view these as maps from the set  $n := \{0, 1, \dots, n-1\}$  to the set  $2 := \{0, 1\}$ . So, in this sense,  $W$  is the set  $2^n$  of maps from  $n$  to 2. So, it is not really an abuse of notation to write  $W = 2^n$ .

Since  $|V| = 2^n$  (here  $2^n$  is a number!), there is a bijection between  $V$  and  $W$ , and we will identify each  $i \in V$  with its string representation in  $W$  using the notation  $i = (i_0, i_1, \dots, i_{n-1})$ , where  $i_k \in \{0, 1\}$ . Define vector addition in  $V$  by adding strings “bitwise” modulo 2. That is

$$\begin{aligned} i + j &= (i_0, i_1, \dots, i_{n-1}) + (j_1, \dots, j_{n-1}) \\ &= (i_0, i_1, \dots, i_{n-1}) + (j_1, \dots, j_{n-1}) \\ &= (i_0 + j_0, i_1 + j_1, \dots, i_{n-1} + j_{n-1}) \end{aligned}$$

where for each  $0 \leq k < n$ , the  $k$ -th element of  $i + j$  is

$$i_k + j_k = \begin{cases} 0, & i_k = j_k \\ 1, & i_k \neq j_k. \end{cases}$$

Clearly the latter addition is commutative, and therefore, the vector addition is commutative:  $i + j = j + i$ . The zero vector  $\mathbf{0} = (0, \dots, 0)$  is the additive identity, and each vector is its own additive inverse, that is,  $-v = v$ . Thus, we have an abelian group  $\langle 2^n, +, -, \mathbf{0} \rangle$ . To make this into a vector space over  $\text{GF}(2)$ , take the set of scalars  $\{0, 1\}$  and define scalar multiplication as follows:  $0i = \mathbf{0}$  and  $1i = i$ . It is easily verified that  $\langle 2^n, +, \mathbf{0}, \{0, 1\} \rangle$  has the remaining ( $\text{GF}(2)$ -module) properties that make it a vector space over  $\text{GF}(2)$ .

*Problem 4* (Golan 70). Show that  $\mathbb{Z}$  is not a vector space over any field.

**Solution.** Let  $\mathbb{F} = \text{GF}(2)$ . Assume  $\mathbb{Z}$  is a vector space over  $\mathbb{F}$ . Then

$$\begin{aligned} 0 &= 1_Z(1_F + 1_F) \\ &= 1_Z(1_F) +_Z 1_Z(1_F) \\ &= 2 \end{aligned}$$

But  $0 \neq 2$ , so  $\mathbb{Z}$  is not a vector space over  $\text{GF}(2)$ . Now let  $\mathbb{F}$  be a field with characteristic greater than 2. Assume  $\mathbb{Z}$  a vector space over  $\mathbb{F}$ . Then we know

$$1_F + 1_F = 2_F \implies 1_F = \frac{1}{2_F} + \frac{1}{2_F}$$

Therefore

$$\begin{aligned} 1_Z &= 1_F(1_Z) \\ &= \left(\frac{1}{2_F} + \frac{1}{2_F}\right)1_Z \\ &= \frac{1}{2_F}(1_Z) +_Z \frac{1}{2_F}(1_Z) \end{aligned}$$

Now let  $\frac{1}{2_F}(1_Z) = n \in \mathbb{Z}$ . But there is no element in  $n \in \mathbb{Z}$  that satisfies  $n + n = 1$ . So  $\mathbb{Z}$  is not a vector space over any field with characteristic greater than 2. Thus  $\mathbb{Z}$  is not a vector space over any field.

*Problem 5* (Golan 76). Let  $V = \mathbb{R}^{\mathbb{R}}$  and let  $W$  be the subset of  $V$  containing the constant function  $x \mapsto 0$  and all of those functions  $f \in V$  satisfying the following condition:  $f(a) = 0$  for at most finitely many real numbers  $a$ . Is  $W$  a subspace of  $V$ .

[Hint: It's easy.]

**Solution.** Let us assume toward a contradiction that  $W$  is a subspace of  $V$ . Let  $p(x), g(x), h(x)$  be distinct functions in  $W$ , such that  $p(x) = g(x) + h(x)$  and  $g(x) = 0$  for  $x = b_0, \dots, b_l \in \mathbb{R}$  and  $h(x) = 0$  for  $x = c_0, \dots, c_m \in \mathbb{R}$ . Then  $p(x) = 0$  when  $g(x) = h(x) = 0$  or when  $g(x) = -h(x)$ . The former happens when  $b_i = c_j$  for  $1 \leq i \leq l, 1 \leq j \leq m$ . We can see this is a finite set of points. The latter, however, could happen for an infinite number of points (e.g. define  $g(x) = -h(x)$  for  $x > \max(b_l, c_m) \in \mathbb{R}$ ). In that case,  $p(a) = 0$  for infinitely many real numbers  $a$ , but it is not the constant function  $x \mapsto 0$ . So vector addition is not closed and therefore  $W$  is not a subspace in  $V$ .

*Problem 6* (Golan 79). A function  $f \in \mathbb{R}^{\mathbb{R}}$  is *piecewise constant* if and only if it is a constant function  $x \mapsto c$  or there exist  $a_1 < a_2 < \dots < a_n$  and  $c_0, c_1, \dots, c_n$  in  $\mathbb{R}$  such that

$$f : x \mapsto \begin{cases} c_0 & \text{if } x < a_1, \\ c_i & \text{if } a_i \leq x < a_{i+1} \text{ for } 1 \leq i < n, \\ c_n & \text{if } a_n \leq x. \end{cases}$$

Does the set of all piecewise constant functions form a subspace of the vector space  $\mathbb{R}^{\mathbb{R}}$  over  $\mathbb{R}$ ?

**Solution.** Let  $W$  denote the set of piecewise constant functions. Then  $W$  is clearly a subset of  $\mathbb{R}^{\mathbb{R}}$ . Let  $r \in \mathbb{R}$  and  $k_i = rc_i$ . Then

$$rf : x \mapsto \begin{cases} k_0 & \text{if } x < a_1, \\ k_i & \text{if } a_i \leq x < a_{i+1} \text{ for } 1 \leq i < n, \\ k_n & \text{if } a_n \leq x. \end{cases}$$

So  $W$  is closed under scalar multiplication.

Let us describe  $f, g \in W$  using the characteristic function  $\chi$ :

$$f(x) = \sum_{i=1}^n f(a_i) \chi_{[a_i, a_{i+1})}(x),$$

$$g(x) = \sum_{i=1}^m g(b_i) \chi_{[b_i, b_{i+1})}(x),$$

where  $\chi_{[c_i, c_{i+1})}(x)$  is 1 if  $c_i \leq x < c_{i+1}$  and 0 otherwise.

Now let the set  $\{z_0, z_1, \dots, z_N\}$  be the union  $\{a_0, a_1, \dots, a_n\} \cup \{b_0, b_1, \dots, b_m\}$  such that  $z_0 < z_1 < \dots < z_N$ .<sup>3</sup> Then we can describe the sum  $f + g$  as follows:

$$f + g = \sum_{i=1}^N (f(z_i) + g(z_i)) \chi_{[z_i, z_{i+1})}.$$

Thus  $f + g$  is a piecewise constant function and so  $W$  is closed under vector addition. Therefore  $W$  is a subspace of  $\mathbb{R}^{\mathbb{R}}$ .

*Problem 7* (Golan 81). Let  $W$  be the subset of  $V = \text{GF}(2)^5$  consisting of all vectors  $(a_1, \dots, a_5)$  satisfying  $\sum_{i=1}^5 a_i = 0$ . Is  $W$  a subspace of  $V$ ?

**Solution.** We know  $W$  is a subset of  $V$  and  $O_V \in W$ . Let  $x \in W, x = (a_1, a_2, a_3, a_4, a_5)$  and  $\sum_{i=1}^5 a_i = 0$ . Let  $b \in F$ . Then

$$bx = (b(a_1), b(a_2), b(a_3), b(a_4), b(a_5))$$

<sup>3</sup> Note to students: we use the word “union” explicitly to emphasize that the resulting  $\{z_0, z_1, \dots, z_N\}$  will be a set—i.e., there will be no repetitions.

And

$$\sum_{i=1}^5 (ba_i) = b \left( \sum_{i=1}^5 a_i \right) = b(0) = 0$$

So  $W$  is closed under scalar multiplication.

Let  $y \in V$ ,  $y = (c_1, c_2, c_3, c_4, c_5)$  such that  $\sum_{i=1}^5 c_i = 0$ . Then

$$x + y = (a_1 + c_1, a_2 + c_2, a_3 + c_3, a_4 + c_4, a_5 + c_5)$$

Now let  $d_i = a_i + c_i$ .

$$\begin{aligned} \sum_{i=1}^5 d_i &= \sum_{i=1}^5 (a_i + c_i) \\ &= \sum_{i=1}^5 a_i + \sum_{i=1}^5 c_i \\ &= 0 + 0 = 0 \end{aligned}$$

Therefore  $W$  is closed under vector addition and scalar multiplication so it is a subspace.

*Problem 8 (Golan 85).* Let  $V = \mathbb{R}^{\mathbb{R}}$  and let  $W$  be the subset of  $V$  consisting of all functions  $f$  satisfying the following condition: there exists  $c \in \mathbb{R}$  (that depends on  $f$ ) such that  $|f(a)| \leq c|a|$  for all  $a \in \mathbb{R}$ . Is  $W$  a subspace of  $V$ ?

**Solution.** We know  $W$  is a subset of  $V$ . Let  $g \in W$ . There exists a  $c_1 \in \mathbb{R}$  such that  $\forall x \in \mathbb{R}$   $|g(x)| \leq c_1|x|$ . Let  $b \in \mathbb{R}$ . Let  $h = bg$ . Then

$$\begin{aligned} |h(x)| &= |b(g(x))| \\ &= |b||g(x)| \\ &\leq |b|(c_1|x|) \\ &= c|x|, \end{aligned}$$

where  $c = |b|c_1$ . So  $\exists c \in \mathbb{R}$  such that  $\forall x \in \mathbb{R}$  we have  $|h(x)| \leq c|x|$ . Therefore  $W$  is closed under scalar multiplication.

Let  $p, g \in W$ . There exist  $c_1, c_2 \in \mathbb{R}$  such that  $\forall x \in \mathbb{R}$   $|p(x)| \leq c_1|x|$  and  $|q(x)| \leq c_2|x|$ . If  $t(x) = p(x) + g(x)$ , then

$$\begin{aligned} t(x) &= |p(x) + g(x)| \\ &\leq |p(x)| + |g(x)| \\ &\leq c_1|x| + c_2|x| \\ &= (c_1 + c_2)|x| \\ &= c_3|x|, \end{aligned}$$

where  $c_3 = c_1 + c_2$ . Therefore  $\exists c \in \mathbb{R}$  such that  $\forall x \in \mathbb{R}$  we have  $t(x) \leq c|x|$ . Thus  $W$  is closed under addition and scalar multiplication so  $W$  is a subspace of  $V$ .

*Problem 9* (Golan 93). Let  $V$  be a vector space over a field  $F$  and let  $P$  be the collection of all subsets of  $V$ , which we know is a vector space over  $\text{GF}(2)$ . Is the collection of all subspaces of  $V$  a subspace of  $P$ ?

**Solution.** Let  $U$  be the collection of all subspaces of  $V$ . As all subspaces of  $V$  must be subsets of  $V$ , we know that  $U \subseteq P$ . We will prove that  $U \not\subseteq P$ —that is,  $U$  is not a subspace of  $P$ —by showing that it is not closed under the scalar multiplication of  $P$ .

Indeed, fix a subspace  $X \in U$ , and recall that scalar multiplication  $0 \in \text{GF}(2)$  in  $P$  always results in the empty set. (See the example in our textbook [2, p. 24].) That is,  $0X = \emptyset$ , which is not a subspace. Since  $0X \notin U$ , we see that  $U$  is not closed under the scalar multiplication in  $P$ , so it is not a subspace of  $P$ .

*Followup question:* Could you fix the foregoing, perhaps by letting  $P$  be all subsets containing  $0_V$  and defining  $0X = \{0_V\}$  for all  $X \in P$ ? It might not work if we take addition to be symmetric difference, but what if we use the binary string interpretation, where the string of all zeros corresponds to  $\{0_V\}$ ? Note that  $P$  would then correspond to the set of all binary strings of length  $|V| - 1$ .

*Problem 10* (Golan 105). Let  $V$  be a vector space over a field  $F$  and let  $0_V \neq w \in V$ . Given a vector  $v \in V \setminus Fw$ , find the set  $G$  of all scalars  $a \in F$  satisfying  $F\{v, w\} = F\{v, aw\}$ .

**Solution.**

$$\begin{aligned} F\{v, w\} &= \{bv + cw \mid b, c \in F\} \\ F\{v, aw\} &= \{dv + e(aw) \mid d, e \in F\} \end{aligned}$$

Since  $F$  is a field, it is closed under multiplication. We know  $1 \in G$ . Thus  $F\{v, aw\} \subseteq F\{v, w\} \forall a \in F$ . Let  $x \in F\{v, w\}$ . Then for some  $s, t \in F$ ,

$$\begin{aligned} x &= sv + tw \\ &= sv + \frac{t}{a}(aw), \end{aligned}$$

where  $\frac{1}{a} = a^{-1} \in F$  so  $ta^{-1} \in F$  as long as  $a \neq 0$ . So  $G = F \setminus \{0\}$ .

## Appendix

*Notes on Problem 2.* Here we show how one could prove that  $V$  is closed under  $\boxplus$ , that is, for all  $f, g \in C(0, 1)$ , we have  $f \boxplus g \in C(0, 1)$ . (Though, as we noted in the solution to Problem 2, closure only proves that  $\boxplus$  is a binary operation on  $C(0, 1)$ ; it does not prove that  $\boxplus$  can serve as vector addition.)

Let  $\epsilon > 0$ . Assume  $f, g$  are continuous on  $(0, 1)$ . Then, there exist  $\delta_f > 0$  and  $\delta_g > 0$  such that

$$|x - x_0| < \delta_f \implies |f(x) - f(x_0)| < \frac{\epsilon}{2}$$

$$|x - x_0| < \delta_g \implies |g(x) - g(x_0)| < \frac{\epsilon}{2}$$

Let  $h(x) = (f \vee g)(x) = \max\{f(x), g(x)\}$ . Let  $\delta_h = \min\{\delta_f, \delta_g\}$ . Then

$$\begin{aligned} |x - x_0| < \delta_h &\implies |h(x) - h(x_0)| \\ &= \left| \frac{f(x) + g(x) + |f(x) - g(x)| - f(x_0) - g(x_0) - |f(x_0) - g(x_0)|}{2} \right| \\ &\leq \left| \frac{f(x) - f(x_0)}{2} \right| + \left| \frac{g(x) - g(x_0)}{2} \right| + \left| \frac{f(x) - f(x_0) - (g(x) - g(x_0))}{2} \right| \\ &< \frac{\epsilon}{2} + \left| \frac{f(x) - f(x_0) - (g(x) - g(x_0))}{2} \right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore  $f \boxplus g$  is continuous for  $x_0 \in (0, 1)$ , so vector addition is closed in  $V$ . With the usual scalar multiplication,  $V$  is a vector space over  $\mathbb{R}$ .

*Remarks.* The proof above is correct. Alternatively, you could simply note that the sum (and difference) of two continuous functions is continuous, and the functions  $x \mapsto |x|$  and  $x \mapsto x/2$  are continuous. Therefore, since function composition preserves continuity, it is clear that  $f \vee g = \frac{1}{2}(f + g + |f - g|)$  is continuous.

Yet another alternative is to use the fact that  $h \in C(0, 1)$  if and only if for all  $-\infty \leq a < b \leq \infty$  the set  $h^{-1}(a, b) = \{x \in (0, 1) \mid a < h(x) < b\}$  is open in  $(0, 1)$ . Note that

$$(f \vee g)^{-1}(a, b) = (\{x : a < f(x)\} \cup \{x : a < g(x)\}) \cap \{x : f(x) < b\} \cap \{x : g(x) < b\}.$$

If  $f$  and  $g$  are continuous, all of the sets on the right are open, so  $(f \vee g)^{-1}(a, b)$  is open.

*Notes on Problem 3.* Consider the proposed solution:

Let us define vector addition in a “bitwise xor” fashion such that  $v + v = 0$  and  $v + w = 1$  for all  $v, w \in V, w \neq v$ . Furthermore, let us define scalar multiplication in the natural way such that  $1 \cdot v = v$  and  $0 \cdot v = 0$ . Then we can see that vector addition is closed as  $0, 1 \in V$ , as well as being associative and commutative. And every  $v$  has an additive inverse, namely  $v$ . Scalar multiplication is also closed in  $V$ , as the product is always  $0 \in V$  or  $v \in V$ . So  $V$  is a vector space over  $\text{GF}(2)$ .

*Remarks.* Having an addition that works as “xor” is the right idea. However, this proof is incorrect. In particular, you need an additive identity, that is, an  $e \in V$  such that  $v + e = v$  for all  $v \in V$ . In the proposed solution, 0 cannot serve as the additive identity because  $v + w = 1$  for all  $w \neq v$ ; in particular,  $v + 0 = 1$  whenever  $v \neq 0$ . (See the correct solution given above.)

*Notes on Problem 9.* The originally proposed solution began as follows:

$P$  is the collection of all subset of  $V$ . Let  $U$  be the collection of all subspaces of  $V$ . As all subspaces of  $V$  must be subsets of  $V$ , we know that  $U \subseteq P$ . We want to show that  $U$  is a subspace of  $P$ .

Let  $X \in U$ . Then  $X \in P$ . Let  $a \in F$ . Let  $y, z \in X$ . Then  $ay, az \in aX$ .  $ay + az = a(x + y)$ . As  $X$  is a subspace,  $(x + y) \in X$  so  $a(x + y) \in aX$ . Thus  $aX$  is closed under addition.

Here we have fallen into the trap of considering the wrong field. It is true that we should fix some  $X \in U$ , and then try to show that for each field element  $a$ , we have  $aX \in U$ . However, we must take  $a$  from  $\text{GF}(2)$ , since that is the field over which  $P$  is defined, and we are trying to prove  $U \leq P$ .

Now, you might then argue that now it is even easier because we only have to consider  $0X$  and  $1X$ . The latter is simply  $X$ , but what is  $0X$ ? It would be nice if it were  $\{0_V\}$ , because then we would have  $0X = \{0_V\} \in U$ . Unfortunately, in  $P$ , scalar multiplication by 0 gives the empty set (or the length- $|V|$  string of zeros, if you prefer to think of the elements of  $P$  as binary strings). The empty set is not a subspace, so  $U$  is not closed under the scalar multiplication of  $P$ .

## References

- [1] Charalambos D. Aliprantis and Owen Burkinshaw. *Principles of Real Analysis*. Academic Press, New York, 3rd edition, 1998.
- [2] Jonathan Golan. *The Linear Algebra a Beginning Graduate Student ought to know*. Springer, 3rd edition, 2012.