Homework 3

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Problem 1 (Golan 124). Let F be a field and let $(K, \bullet)$ be an associative unital F-algebra. If A and B are subsets of K, we let $A \bullet B$ be the set of all elements of K of the form $a \bullet b$, with $a \in A$ and $b \in B$ (in particular, $\emptyset \bullet B = A \bullet \emptyset = \emptyset$). We know that the set V of all subsets of K is a vector space over GF(2). Is $(V, \bullet)$ a GF(2)-algebra? If so, is it associative? Is it unital?

Solution. Let $u, v, w, x \in V$. The first two conditions we need to satisfy,
1. $u \bullet (v + w) = u \bullet v + u \bullet w$, and
2. $(u + v) \bullet w = u \bullet w + v \bullet w$
come from the fact that $(K, \bullet)$ is an associative F-algebra. The third condition,
3. $a(v \bullet w) = (av) \bullet w = v \bullet (aw)$,
is satisfied since $a \in \{0_F, 1_F\}$. (This works in either case). It is indeed associative from the definition of $\bullet$ given:
4. $v \bullet (w \bullet y) = (v \bullet w) \bullet y$
And it is also unital:
5. $v \bullet 1_F = v = 1_F \bullet v$

Problem 2 (Golan 132). Let F be a field and let L be the set of all polynomials $f(X) \in F[X]$ satisfying the condition that $f(-a) = -f(a)$ for all $a \in F$. Is L a subspace of $F[X]$?

Solution. L is the set of odd polynomials (i.e. only odd powers of x). To show that L is a subspace of $F[X]$, we need to show that L is a vector space in its own right with respect to the addition and scalar multiplication defined on F. Let $f, g \in L$, then $f(X) + g(X) = \sum_{i=0}^{\infty} a_{2i+1}X^{2i+1} + \sum_{i=0}^{\infty} b_{2i+1}X^{2i+1} = \sum_{i=0}^{\infty}(a_{2i+1} + b_{2i+1})X^{2i+1} = \sum_{i=0}^{\infty} c_{2i+1}X^{2i+1} \in L$. Let $c \in F$, then $cf(X) = c\sum_{i=0}^{\infty} a_{2i+1}X^{2i+1} = \sum_{i=0}^{\infty} ca_{2i+1}X^{2i+1} = \sum_{i=0}^{\infty} b_{2i+1}X^{2i+1} \in L$. Thus, L is a subspace of $F[X]$.

Problem 3 (Golan 133). Let F be a field and let L be the set of all polynomials $f(X) \in F[X]$ satisfying the condition that $\deg(f)$ is even. Is L a subspace of $F[X]$?

Solution. L is the set of polynomials with the highest power of even order. To show that L is a subspace of $F[X]$, we need to show that L is a vector space in its own right with respect to the addition and scalar multiplication defined on F. Let $f, g \in L$, then $f(X) + g(X) = \sum_{i=0}^{\infty} a_{2i}X^{2i} + \sum_{i=0}^{\infty} b_{2i}X^{2i} = \sum_{i=0}^{\infty}(a_{2i} + b_{2i})X^{2i} = \sum_{i=0}^{\infty} c_{2i}X^{2i} \in L$. Let $c \in F$, then $cf(X) = c\sum_{i=0}^{\infty} a_{2i}X^{2i} = \sum_{i=0}^{\infty} ca_{2i}X^{2i} = \sum_{i=0}^{\infty} b_{2i}X^{2i} \in L$. Thus, L is a subspace of $F[X]$.

Problem 4 (Golan 142). For a field F, compare the subsets $F[X^2]$ and $F[X^2 + 1]$ of $F[X]$.

Solution. $f(X) \in F[X]$ is defined as $\sum_{i=0}^{\infty} a_iX^i$, $f(X) \in F[X^2]$ is defined as $\sum_{i=0}^{\infty} a_{2i}X^{2i}$, and $f(X) \in F[X^2 + 1]$ is defined as $\sum_{i=0}^{\infty} a_{2i}(X^2 + 1)^i$. So, $F[X^2] = \{1, X^2, X^4, X^6, ...\}$ and $F[X^2 + 1] = \{1, X^2 + 1, (X^2 + 1)^2, (X^2 + 1)^3, ...\} = \{1, X^2 + 1, X^4 + 2X^2 + 1, X^6 + 3X^4 + 3X^2 + 1, ...\}$. $F[X^2] \subseteq F[X^2 + 1]$ since every element of $F[X^2 + 1]$ is of the form $\sum_{i=0}^{\infty} a_{2i}X^{2i}$ (e.g. $X^4 + 2X^2 + 1 = n_3 + 2n_2 + n_1$ for $n_i \in F[X^2]$). Similarly, $F[X^2 + 1] \subseteq F[X^2]$ (e.g. $X^6 = m_4 - 3m_3 + 3m_2 - m_1$ for $m_i \in F[X^2 + 1]$). Thus, $F[X^2] = F[X^2 + 1]$.

Problem 5 (Golan 154). Let F be a field and let $K = F^{\mathbb{N}}$. Define operations $+$ and $\bullet$ on K by setting $f + g : i \mapsto f(i) + g(i)$ and $f \bullet g : i \mapsto \sum_{j+k=i} f(j)g(k)$. Show that K is a an associative and commutative unital F-algebra. Is it entire?

Solution. To show that K is an associative commutative unital F-algebra, we want to show the following conditions hold: let $u, v, w, x \in K, a \in F$

1. $u \bullet (v + w) = u \bullet x = \sum\limits_{j+k=i} u(j)x(k) = \sum\limits_{j+k=i} u(j)[v(k) + w(k)] = \sum\limits_{j+k=i} u(j)v(k) + \sum\limits_{j+k=i} u(j)w(k) = u \bullet v + u \bullet w,$
2. $(u + v) \bullet w = x \bullet w = \sum\limits_{j+k=i} x(j)w(k) = \sum\limits_{j+k=i} [u(j) + v(j)]w(k) = \sum\limits_{j+k=i} u(j)w(k) + \sum\limits_{j+k=i} v(j)w(k) = u \bullet w + v \bullet w,$
3. $a(v\bullet w) = a \sum\limits_{j+k=i} v(j)w(k) = \sum\limits_{j+k=i} [av(j)]w(k) = (av)\bullet w = \sum\limits_{j+k=i} v(j)[aw(k)] = v\bullet(aw),$
4. $v \bullet (w \bullet y) = \sum\limits_{j+k+l=i} v(j)(w(k)y(l)) = \sum\limits_{j+k+l=i} (v(j)w(k))y(l) = (v \bullet w) \bullet y$
5. $v \bullet 1_F = \sum\limits_{j=i} v(j)1_F = v(i) = v = \sum\limits_{k=i} 1_F v(k) = 1_F \bullet v$
6. $v \bullet w = \sum\limits_{j+k=i} v(j)w(k) = \sum\limits_{k+j=i} w(k)v(j) = w \bullet v,$

If $v, w \neq 0$, then $v \bullet w = \sum\limits_{j+k=i} v(j)w(k)$ could be equal to 0 if the vectors v and w are orthogonal. Therefore, K is not entire.

Problem 6 (Golan 157). A *trigonometric polynomial* in $\mathbb{R}^{\mathbb{R}}$ is a function of the form $t \mapsto a_0 + \sum_{h=1}^{k}[a_h \cos(ht) + b_h \sin(ht)]$, where $a_0, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{R}$. Show that the subset, K, of $\mathbb{R}^{\mathbb{R}}$ consisting of all trigonometric polynomials is an entire $\mathbb{R}$-algebra.

Solution. Let $f, g, h, l \in K$ and $a \in \mathbb{R}^{\mathbb{R}}$. To show that K is an entire $\mathbb{R}$-algebra, we want to show the following:

1. $f \bullet (g + h) = (a_0 + \sum_{h=1}^{k} a_h \cos(ht) + \sum_{h=1}^{k} b_h \sin(ht))[(c_0 + \sum_{h=1}^{k} c_h \cos(ht) + \sum_{h=1}^{k} d_h \sin(ht)) + (e_0 + \sum_{h=1}^{k} e_h \cos(ht) + \sum_{h=1}^{k} f_h \sin(ht))] = (a_0 + \sum_{h=1}^{k} a_h \cos(ht) + \sum_{h=1}^{k} b_h \sin(ht))(c_0 + \sum_{h=1}^{k} c_h \cos(ht) + \sum_{h=1}^{k} d_h \sin(ht)) + (a_0 + \sum_{h=1}^{k} a_h \cos(ht) + \sum_{h=1}^{k} b_h \sin(ht))(e_0 + \sum_{h=1}^{k} e_h \cos(ht) + \sum_{h=1}^{k} f_h \sin(ht)) = f \bullet g + f \bullet h$,
2. $(f + g) \bullet h = [(a_0 + \sum_{h=1}^{k} a_h \cos(ht) + \sum_{h=1}^{k} b_h \sin(ht)) + (c_0 + \sum_{h=1}^{k} c_h \cos(ht) + \sum_{h=1}^{k} d_h \sin(ht))](e_0 + \sum_{h=1}^{k} e_h \cos(ht) + \sum_{h=1}^{k} f_h \sin(ht)) = (a_0 + \sum_{h=1}^{k} a_h \cos(ht) + \sum_{h=1}^{k} b_h \sin(ht))(e_0 + \sum_{h=1}^{k} e_h \cos(ht) + \sum_{h=1}^{k} f_h \sin(ht)) + (c_0 + \sum_{h=1}^{k} c_h \cos(ht) + \sum_{h=1}^{k} d_h \sin(ht))(e_0 + \sum_{h=1}^{k} e_h \cos(ht) + \sum_{h=1}^{k} f_h \sin(ht)) = f \bullet h + g \bullet h$,
3. $a(f \bullet g) = a(a_0 + \sum_{h=1}^{k} a_h \cos(ht) + \sum_{h=1}^{k} b_h \sin(ht))(c_0 + \sum_{h=1}^{k} c_h \cos(ht) + \sum_{h=1}^{k} d_h \sin(ht)) = (aa_0 + a\sum_{h=1}^{k} a_h \cos(ht) + \sum_{h=1}^{k} b_h \sin(ht)) = (af) \bullet g = (a_0 + \sum_{h=1}^{k} a_h \cos(ht) + \sum_{h=1}^{k} b_h \sin(ht))(ac_0 + a\sum_{h=1}^{k} c_h \cos(ht) + \sum_{h=1}^{k} d_h \sin(ht)) = f \bullet (ag)$.

Let $f = a_0 + \sum_{h=1}^{k}[a_h \cos(ht) + b_h \sin(ht)] = a_0 + b_0$ and $g = c_0 + \sum_{h=1}^{k}[c_h \cos(ht) + d_h \sin(ht)] = a_0 + d_0$. If $f, g \neq 0$, then $f \bullet g = (a_0 + b_0)(c_0 + d_0) = a_0c_0 + a_0d_0 + b_0c_0 + b_0d_0$. Here, we have 4 cases:

1. $a_0 = 0, b_0 \neq 0, c_0 = 0, d_0 \neq 0 \Rightarrow (a_0 + b_0)(c_0 + d_0) = b_0d_0 \neq 0$
2. $a_0 = 0, b_0 \neq 0, c_0 \neq 0, d_0 = 0 \Rightarrow (a_0 + b_0)(c_0 + d_0) = b_0c_0 \neq 0$
3. $a_0 \neq 0, b_0 = 0, c_0 = 0, d_0 \neq 0 \Rightarrow (a_0 + b_0)(c_0 + d_0) = a_0d_0 \neq 0$
4. $a_0 \neq 0, b_0 = 0, c_0 \neq 0, d_0 = 0 \Rightarrow (a_0 + b_0)(c_0 + d_0) = a_0c_0 \neq 0$

Thus, K is an entire $\mathbb{R}$-algebra.

Problem 7 (Golan 163). Let $F = \mathbb{Q}$. Is the subset

$$\left\{ \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$$

of F^3 linearly independent? What happens if $F = \mathrm{GF}(5)$?

Solution. To see if the given subset of F^3 is linearly independent, we can perform row operations to reach a RREF form of the matrix

$$\begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

.
We proceed as follows:

$$\begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 4 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -15 \\ 0 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -15 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the subset is linearly independent.

Problem 8 (Golan 177). Show that the subset

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a linearly independent subset of $\mathrm{GF}(p)^3$ if and only if $p \neq 3$.

Solution. Let $p = 3$. Perform row reductions as follows:

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, The set is linearly dependent if $p = 3$.

Let $p \neq 3$. Then if we set
$a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = c \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, we get the system of equations $a = 2c, 2a + b = 0, 2b = c$. Substituting, we then get $2(2c) + b = 0 \Rightarrow 4c + b = 0 \Rightarrow 4(2b) + b = 0 \Rightarrow 9b = 0$, which only happens if $p = 3$. Thus, if $p \neq 3$, then the set is linearly independent.

Clarification/amplification of Props 5.12 and 5.13. Let V be a vector space, and let $\{W_\omega : \omega \in \Omega\}$ be a collection of subspaces of V. Recall that $\sum_{\omega \in \Omega} W_\omega$ denotes the subspace of all (finite) linear combinations of vectors in $\{W_\omega : \omega \in \Omega\}$, which is equivalent to the subspace of vectors w of the form $w = \sum_{\lambda \in \Lambda} w_\lambda$, where $w_\lambda \in W_\lambda$ and Λ is a finite subset of Ω.
We call the set $\{W_\omega : \omega \in \Omega\}$ *independent* if and only if it satisfies the following condition: If Λ is a finite subset of Ω, if $w_\lambda \in W_\lambda$ for each $\lambda \in \Lambda$, and if $\sum_{\lambda \in \Lambda} w_\lambda = 0_V$, then $w_\lambda = 0_V$ for all $\lambda \in \Lambda$.

Problem 9. Let

$$W_1 = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\},\ W_2 = \left\{ \begin{bmatrix} b \\ b \end{bmatrix} : b \in \mathbb{R} \right\},\ W_3 = \left\{ \begin{bmatrix} 0 \\ c \end{bmatrix} : a \in \mathbb{R} \right\}.$$

1. Would our textbook author describe $\{W_1, W_2, W_3\}$ as "pairwise disjoint?" (explain)
2. Describe the space $W_1 + W_2 + W_3$.
3. Describe the space $W_i + W_j$ for each pair $i \neq j$ in $\{1, 2, 3\}$.
4. Is the set $\{W_1, W_2, W_3\}$ independent?

Solution. 1. Our book would not describe $\{W_1, W_2, W_3\}$ as "pairwise disjoint" since $W_1 \cap W_2 \neq 0_V$ and $W_2 \cap W_3 \neq 0_V$.

2. $W_1 + W_2 + W_3 = \left\{ \begin{bmatrix} d \\ e \end{bmatrix} : d, e \in \mathbb{R} \right\}$
3. $W_1 + W_2 = \left\{ \begin{bmatrix} d \\ b \end{bmatrix} : d, b \in \mathbb{R} \right\} = W_2 + W_3 = \left\{ \begin{bmatrix} b \\ d \end{bmatrix} : b, d \in \mathbb{R} \right\} = W_1 + W_3 = \left\{ \begin{bmatrix} a \\ c \end{bmatrix} : a, c \in \mathbb{R} \right\}$.
4. The set $\{W_1, W_2, W_3\}$ is independent since it obeys the definition of independence. If we choose elements $w_h \in \{W_h : h \in \{1, 2, 3\}\}$, then for every h in a finite sbuset of $\Lambda = \{1, 2, 3\}$ we get $\sum_{h \in \Lambda} w_h = 0_V$ when and only when $w_h = 0_V$ for each $h \in \Lambda$.

Problem 10. Below is an alleged theorem that attempts to combine Propositions 5.12 and 5.13 of the textbook. Prove it, or disprove it by providing a counterexample. If you find a counterexample, what additional hypothesis would fix the theorem?

Proposition. Suppose $\{W_\omega : \omega \in \Omega\}$ is a collection of subspaces of a vector space, and suppose, for each $\omega \in \Omega$, the set B_ω is a basis for W_ω. Then the following are equivalent:

1. The set $\{W_\omega : \omega \in \Omega\}$ is independent.
2. Every $w \in \sum_{\omega \in \Omega} W_\omega$ can be written as $w = \sum_{\lambda \in \Lambda} w_\lambda$ in exactly one way.
3. For every $\lambda \in \Omega$, $W_\lambda \cap \sum_{\omega \neq \lambda} W_\omega = \{0_V\}$.
4. The set $B = \bigcup_{\omega \in \Omega} B_\omega$ is a basis for $\sum_{\omega \in \Omega} W_\omega$.

The proof of Proposition 5.12 in the book covers $(1) \Rightarrow (2), (2) \Rightarrow (3)$, and $(3) \Rightarrow (1)$. As for showing that (4) is equivalent to any of $(1), (2)$, or (3), we will show that $(3) \Rightarrow (4)$. From (3), we see that the set $\{W_i : i \in \Omega\}$ is a pairwise disjoint collection of subspaces of V. For each $i \in \Omega$, let B_i be a basis of W_i. Then $V = \sum_{\omega \in \Omega} W_\omega \Rightarrow B = \bigcup_{\omega \in \Omega} B_\omega$ is a basis for $\sum_{\omega \in \Omega} W_\omega$, as can be seen in the book's proof of Proposition 5.13.

Final remark. We use $\bigoplus_{\omega \in \Omega} W_\omega$ to denote $\sum_{\omega \in \Omega} W_\omega$ *only when* the set $\{W_\omega : \omega \in \Omega\}$ is independent. When I mentioned this in class, instead of *only when*, I used the phrase *when and only when.* This is incorrect since the expression $\sum_{\omega \in \Omega} W_\omega$ does not necessarily imply that the set $\{W_\omega : \omega \in \Omega\}$ is dependent.