Midterm Exam

Math 700: Spring 2014

INSTRUCTIONS:

- Solve the problems below. Write up your solutions (neatly!), giving complete justifications for all arguments, and turn in a hard copy of your solutions in class on the

  Due Date: **Wednesday, March 19**

- The questions are meant to test your understanding of elementary concepts, and you should write down definitions of any technical terms you use, even if these terms are mentioned in the statement of the problem. Of course, you must use your best judgment about which definitions to state. (You probably don't want to define the integers or real numbers, for example.)

- It will help me (and probably your grade) if you do the following:

  1. State what you are trying to prove.
  2. Mention informally how you plan to prove it before giving the details.
  3. If you believe your proof is complete, use an end-of-proof symbol (like QED or $\square$); on the other hand, if you believe your proof is incomplete, say so.

HONOR CODE: You are expected to solve the exam problems on your own with no outside help. You may consult the lecture notes and textbook for this course only. No other books or internet usage is allowed. If you get stuck, please ask *me* for help, and I may post hints on our wiki page.

When you finish the exam, please sign the following pledge:

"On my honor as a student I, ________________________, have neither given nor received unauthorized aid on this exam." (Print Name)

Signature: ______________________________ Date: _______________

NOTATION: For the most part, we follow the notation used in the textbook. Recall that if V is a vector space over the field F and if $c \in F$, then $\sigma_c v = cv$ for all $v \in V$. In particular, σ_0 and σ_1 denote the zero and identity maps, respectively. However, when V and W are vector spaces over the same field, it is clearer to denote their identity maps by id_V and id_W, resp. We use $W \leqslant V$ to denote that W is a subspace of V, whereas $W \subseteq V$, means that W is a subset of V (which may or may not be a subspace). By an "F-vector space" we mean a vector space over the field F. If $\varphi : V \to W$, then $\mathrm{im}(\varphi) := \varphi(V)$, $\ker(\varphi) := \{v \in V : \varphi(v) = 0_W\}$, $\mathrm{rank}(\alpha) := \dim(\mathrm{im}(\alpha))$, and $\mathrm{null}(\alpha) := \dim(\ker(\alpha))$.

Problem 1. Let V and W be finite dimensional vector spaces over the field F, and suppose $\alpha \in \mathrm{Hom}(V, W)$. Circle true or false (no proof required).

(a) If $\alpha(v) = 0_W$ only when $v = 0_V$, then $\dim(V) = \dim(W)$. true false

(b) If $\mathrm{im}(\alpha) = \{0_W\}$, then $\alpha = \sigma_0$. true false

(c) If $\dim(V) = \mathrm{rank}(\alpha)$, then $\ker(\alpha) = \{0_V\}$. true false

(d) $\ker(\alpha) \leqslant \ker(\alpha^2)$ true false

(e) $\mathrm{im}(\alpha) \geqslant \mathrm{im}(\alpha^2)$ true false

(f) $\mathrm{null}(\alpha) \leqslant \mathrm{rank}(\alpha)$ true false

(g) $\mathrm{null}(\alpha) \leqslant \dim(V)$ true false

(h) α is a one-to-one if and only if $\ker(\alpha) = \{0_V\}$. true false

(i) α is a one-to-one if and only if $\dim(V) \leqslant \dim(W)$. true false

(j) α is a one-to-one if and only if $\mathrm{null}(\alpha) = 0$. true false

(k) α is a onto if and only if $\dim(V) \geqslant \dim(W)$. true false

(l) α is a onto if and only if $\mathrm{rank}(A) = \dim(W)$. true false

Problem 2. Prove that the lattice of subspaces of a vector space is modular but not necessarily distributive, as follows: Let U, Y, and W be subspaces of a vector space V.

(a) Show that $U \cap (Y + (U \cap W)) = (U \cap Y) + (U \cap W)$.

(b) Show that $U \cap (Y + W) = (U \cap Y) + (U \cap W)$ is not always valid.

Problem 3. Prove that every nontrivial vector space V has a basis.

[*Hint:* First prove that every linearly independent subset of V is contained in a basis. As we did in class, let S be a linearly independent subset and let $\mathcal{S}$ be the set of all linearly independent subsets that contain S. Partially order $\mathcal{S}$ by inclusion $\subseteq$ and apply Zorn's Lemma.[1] Finally, say why if follows from this that every vector space has a basis.]

Problem 4. Let V be a vector space over the field F, and suppose the subset $S \subseteq V$ satisfies $FS = V$. Prove that S contains a basis.

Problem 5. Let V be a vector space over the field F and let Ω be a (possibly uncountable) set. Describe V^Ω. Can you make V^Ω into an F-vector space? An F-algebra? Explain.

[1] **Zorn's Lemma:** If a partially ordered set $(\mathcal{S}, \subseteq)$ has the property that every chain $S_1 \subseteq S_2 \subseteq \cdots$ has an upper bound in $\mathcal{S}$, then $\mathcal{S}$ contains a maximal element.

Problem 6. Let V and W be vector spaces over the field F, and let $\varphi : V \to W$ be a linear transformation. Give details and proof of the following: φ is injective (respectively, surjective) if and only if there is a linear transformation $\psi : W \to V$ such that the composition of φ with ψ is the identity map.

[*Hint:* There are two claims to prove, (a) "φ is injective iff...", and (b) "φ is surjective iff..." There are two ways to form the composition, $\varphi\psi = \mathrm{id}_W$ and $\psi\varphi = \mathrm{id}_V$. Figure out which composition you need to prove each claim.]

Problem 7. Let V and W be vector spaces over the field F. Recall that W^V denotes the set of maps $\{f : V \to W\}$. For fixed $\alpha \in \mathrm{Hom}(V, W)$ and $w \in W$, define the *affine transformation* $\zeta_{\alpha,w} : V \to W$ to be the map $v \mapsto \alpha(v) + w$. Denote by $\mathrm{Aff}(V, W)$ the set of all such affine transformations. That is, $\mathrm{Aff}(V, W) := \{\zeta_{\alpha,w} : \alpha \in \mathrm{Hom}(V, W) \text{ and } w \in W\}$.

(a) Can you make W^V into an F-vector space? An F-algebra? Explain.

(b) Prove or disprove: $\mathrm{Hom}(V, W) \leqslant \mathrm{Aff}(V, W) \leqslant W^V$. (Interpret $\leqslant$ here as you see fit.)

(c) Note that ζ (without subscripts) may be viewed as a map from $\mathrm{Hom}(V, W)$ to $\mathrm{Aff}(V, W)^W$. Is ζ a vector space homomorphism? An F-algebra homomorphism? Prove.

Problem 8.

Let V be a finite dimensional vector space over the field F and suppose $T \in \mathrm{End}(V)$. Prove the following:

(a) $0 \leqslant \ker(T) \leqslant \ker(T^2) \leqslant \cdots$

(b) $V \geqslant \mathrm{im}(T) \geqslant \mathrm{im}(T^2) \geqslant \cdots$

(c) $\dim(V) = \mathrm{rank}(T^k) + \mathrm{null}(T^k)$, for each $k = 0, 1, \ldots$

(d) The sets $V_1 := \bigcap_{k=1}^{\infty} \mathrm{im}(T^k)$ and $V_2 := \bigcup_{k=1}^{\infty} \ker(T^k)$ are T-invariant subspaces.

(e) $V = V_1 \oplus V_2$.

(f) If T_i is the restriction of T to V_i, then T_1 is an isomorphism and T_2 is *nilpotent*.[2]

[2] $\alpha \in \mathrm{End}(V)$ is called *nilpotent* if there is a positive integer k such that $\alpha^k = \sigma_0$.