Homework 2

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Problem 1 (Golan 56). Is it possible to define on $\mathbb{Z}/(4)$ the structure of a vector space over GF(2) in such a way that the vector addition is the usual addition in $\mathbb{Z}/(4)$?

[*Hints*: Recall that (n) denotes the set $\{\dots, -2n, -n, 0, n, 2n, 3n, \dots\}$, which we denoted in lecture by $n\mathbb{Z}$. This is the ideal generated by n in the ring $\mathbb{Z}$, but don't worry about that for now. Just take $\mathbb{Z}/(n)$ to be the abelian group of integers $\{0, 1, 2, \dots, n-1\}$ with addition modulo n. In lecture, we used $\mathbb{Z}/n\mathbb{Z}$ to denote $\mathbb{Z}/(n)$. Use whichever notation you prefer.]

Solution. Assume toward a contradiction that $\mathbb{Z}/(4)$ is a vector space over GF(2), with vector addition defined as the usual modular addition. Then for any $v \in \mathbb{Z}/(4)$,

$$\begin{aligned} 0 &= (0)v \\ &= (1+1)v \\ &= 1 \cdot v + 1 \cdot v \\ &= v + v \end{aligned}$$

Now, let $v = 3$ to see that $3 + 3 = 2 \mod 4 \neq 0$. Thus we have a contradiction. The answer is no, it is not possible with the usual modular vector addition.

Problem 2 (Golan 60).[1] Let $V = C(0,1)$. Define the relation $\vee$ on V by setting $(f \vee g)(x) = \max\{f(x), g(x)\}$. If we think of $\vee$ as a "vector addition," does this, together with the usual scalar multiplication, make V into a vector space over $\mathbb{R}$?

Solution. It is true that $C(0,1)$ is closed uder $\vee$ (see the Appendix for a proof), and we can easily verify that $\vee$ is a commutative associative binary operation on the set $C(0,1)$, so $\langle C(0,1), \vee\rangle$ is a commutative semigroup. In fact, letting $f \wedge g = \min\{f, g\}$, we can check that $\langle C(0,1), \vee, \wedge\rangle$ is a lattice.[2]

However, recall that a vector space is built up from an additive abelian group. Is it possible for $\vee$ to serve as vector addition? If so, what would be the additive identity? We need a function $e \in C(0,1)$ such that for all $f \in C(0,1)$ we have $f \vee e = f$. It is clear that no such e exists.

[1] In the original problem, the notation $f \boxplus g$ was used. We use $f \vee g$ instead, since this is fairly standard notation for the function $\max\{f, g\}$.

[2] See [1, Sec. 30] for a discussion of vector lattices, such as $\langle C(0,1), \vee, \wedge\rangle$.

Problem 3 (Golan 63). Let $V = \{i \in \mathbb{Z} \mid 0 \leq i < 2^n\}$ for some fixed positive integer n. Define operations of vector addition and scalar multiplication on V in such a way as to turn it into a vector space over GF(2).

[*Hints*: Recall that GF(2) denotes the Galois field with two elements, $\{0, 1\}$, with addition mod 2 and the usual multiplication. Other than this field, the only restriction given in the problem is that V must have 2^n elements. Do you know of any sets of this size?]

Solution. Let W be the set of binary strings of length n, that is, length n sequences of 0's and 1's. We can also view these as maps from the set $n := \{0, 1, \ldots, n-1\}$ to the set $2 := \{0, 1\}$. So, in this sense, W *is* the set 2^n of maps from n to 2. So, it is not really an abuse of notation to write $W = 2^n$.

Since $|V| = 2^n$ (here 2^n is a number!), there is a bijection between V and W, and we will identify each $i \in V$ with its string representation in W using the notation $i = (i_0, i_1, \ldots, i_{n-1})$, where $i_k \in \{0, 1\}$. Define vector addition in V by adding strings "bitwise" modulo 2. That is

$$\begin{aligned} i + j &= (i_0, i_1, \ldots, i_{n-1}) + (j_1, \ldots, j_{n-1}) \\ &= (i_0, i_1, \ldots, i_{n-1}) + (j_1, \ldots, j_{n-1}) \\ &= (i_0 + j_0, i_1 + j_1, \ldots, i_{n-1} + j_{n-1}) \end{aligned}$$

where for each $0 \leq k < n$, the k-th element of $i + j$ is

$$i_k + j_k = \begin{cases} 0, & i_k = j_k \\ 1, & i_k \neq j_k. \end{cases}$$

Clearly the latter addition is commutative, and therefore, the vector addition is commutative: $i + j = j + i$. The zero vector $\mathbf{0} = (0, \ldots, 0)$ is the additive identity, and each vector is its own additive inverse, that is, $-v = v$. Thus, we have an abelian group $\langle 2^n, +, -, \mathbf{0} \rangle$. To make this into a vector space over GF(2), take the set of scalars $\{0, 1\}$ and define scalar multiplication as follows: $0i = 0$ and $1i = i$. It is easily verified that $\langle 2^n, +, \mathbf{0}, \{0, 1\} \rangle$ has the remaining (GF(2)-module) properties that make it a vector space over GF(2).

Problem 4 (Golan 70). Show that $\mathbb{Z}$ is not a vector space over any field.

Solution. Let $\mathbb{F} = \mathrm{GF}(2)$. Assume $\mathbb{Z}$ is a vector space over $\mathbb{F}$. Then

$$\begin{aligned} 0 &= 1_Z(1_F + 1_F) \\ &= 1_Z(1_F) +_Z 1_Z(1_F) \\ &= 2 \end{aligned}$$

But $0 \neq 2$, so $\mathbb{Z}$ is not a vector space over GF(2). Now let $\mathbb{F}$ be a field with characteristic greater than 2. Assume $\mathbb{Z}$ a vector space over $\mathbb{F}$. Then we know

$$1_F + 1_F = 2_F \implies 1_F = \frac{1}{2_F} + \frac{1}{2_F}$$

Therefore

$$\begin{aligned} 1_Z &= 1_F(1_Z) \\ &= (\frac{1}{2_F} + \frac{1}{2_F})1_Z \\ &= \frac{1}{2_F}(1_Z) +_Z \frac{1}{2_F}(1_Z) \end{aligned}$$

Now let $\frac{1}{2_F}(1_Z) = n \in \mathbb{Z}$. But there is no element in $n \in \mathbb{Z}$ that satisfies $n + n = 1$. So $\mathbb{Z}$ is not a vector space over any field with characteristic greater than 2. Thus $\mathbb{Z}$ is not a vector space over any field.

Problem 5 (Golan 76). Let $V = \mathbb{R}^{\mathbb{R}}$ and let W be the subset of V containing the constant function $x \mapsto 0$ and all of those functions $f \in V$ satisfying the following condition: $f(a) = 0$ for at most finitely many real numbers a. Is W a subspace of V.

[*Hint:* It's easy.]

Solution. Let us assume toward a contradiction that W is a subspace of V. Let $p(x), g(x), h(x)$ be distinct functions in W, such that $p(x) = g(x) + h(x)$ and $g(x) = 0$ for $x = b_0, \ldots, b_l \in \mathbb{R}$ and $h(x) = 0$ for $x = c_0, \ldots, c_m \in \mathbb{R}$. Then $p(x) = 0$ when $g(x) = h(x) = 0$ or when $g(x) = -h(x)$. The former happens when $b_i = c_j$ for $1 \leq i \leq l, 1 \leq j \leq m$. We can see this is a finite set of points. The latter, however, could happen for an infinite number of points (e.g. define $g(x) = -h(x)$ for $x > \max(b_l, c_m) \in \mathbb{R}$). In that case, $p(a) = 0$ for infinitely many real numbers a, but it is not the constant function $x \mapsto 0$. So vector addition is not closed and therefore W is not a subspace in V.

Problem 6 (Golan 79). A function $f \in \mathbb{R}^{\mathbb{R}}$ is *piecewise constant* if and only if it is a constant function $x \mapsto c$ or there exist $a_1 < a_2 < \cdots < a_n$ and $c_0, c_1, \cdots, c_n$ in $\mathbb{R}$ such that

$$f : x \mapsto \begin{cases} c_0 & \text{if } x < a_1, \\ c_i & \text{if } a_i \leq x < a_i \text{ for } 1 \leq i < n, \\ c_n & \text{if } a_n \leq x. \end{cases}$$

Does the set of all piecewise constant functions form a subspace of the vector space $\mathbb{R}^{\mathbb{R}}$ over $\mathbb{R}$?

Solution. Let W denote the set of piecewise constant functions. Then W is clearly a subset of $\mathbb{R}^{\mathbb{R}}$. Let $r \in \mathbb{R}$ and $k_i = rc_i$. Then

$$rf : x \mapsto \begin{cases} k_0 & \text{if } x < a_1, \\ k_i & \text{if } a_i \leq x < a_i \text{ for } 1 \leq i < n, \\ k_n & \text{if } a_n \leq x. \end{cases}$$

So W is closed under scalar multiplication.

Let us describe $f, g \in W$ using the characteristic function χ:

$$f(x) = \sum_{i=1}^{n} f(a_i)\chi_{[a_i, a_{i+1})}(x),$$

$$g(x) = \sum_{i=1}^{m} g(b_i)\chi_{[b_i, b_{i+1})}(x),$$

where $\chi_{[c_i, c_{i+1})}(x)$ is 1 if $c_i \leq x < c_{i+1}$ and 0 otherwise.

Now let the set $\{z_0, z_1, \ldots, z_N\}$ be the union $\{a_0, a_1, \ldots, a_n\} \cup \{b_0, b_1, \ldots, b_m\}$ such that $z_0 < z_1 < \cdots < z_N$.[3] Then we can describe the sum $f + g$ as follows:

$$f + g = \sum_{i=1}^{N} (f(z_i) + g(z_i))\chi_{[z_i, z_{i+1})}.$$

Thus $f + g$ is a piecewise constant function and so W is closed under vector addition. Therefore W is a subspace of $\mathbb{R}^{\mathbb{R}}$.

Problem 7 (Golan 81). Let W be the subset of $V = \mathrm{GF}(2)^5$ consisting of all vectors $(a_1, \ldots, a_5)$ satisfying $\sum_{i=1}^{5} a_i = 0$. Is W a subspace of V?

Solution. We know W is a subset of V and $O_V \in V$. Let $x \in V, x = (a_1, a_2, a_3, a_4, a_5)$ and $\sum_{i=1}^{5} a_i = 0$. Let $b \in F$. Then

$$bx = (b(a_1), b(a_2), b(a_3), b(a_4), b(a_5))$$

[3] Note to students: we use the word "union" explicitly to emphasize that the resulting $\{z_0, z_1, \ldots, z_N\}$ will be a set—i.e., there will be no repetitions.

And

$$\sum_{i=1}^{5}(ba_i) = b\left(\sum_{i=1}^{5} a_i\right) = b(0) = 0$$

So W is closed under scalar multiplication.

Let $y \in V$, $y = (c_1, c_2, c_3, c_4, c_5)$ such that $\sum_{i=1}^{5} c_i = 0$. Then

$$x + y = (a_1 + c_1, a_2 + c_2, a_3 + c_3, a_4 + c_4, a_5 + c_5)$$

Now let $d_i = a_i + c_i$.

$$\begin{aligned}\sum_{i=1}^{5} d_i &= \sum_{i=1}^{5}(a_i + c_i) \\ &= \sum_{i=1}^{5} a_i + \sum_{i=1}^{5} c_i \\ &= 0 + 0 = 0\end{aligned}$$

Therefore W is closed under vector addition and scalar multiplication so it is a subspace.

Problem 8 (Golan 85). Let $V = \mathbb{R}^{\mathbb{R}}$ and let W be the subset of V consisting of all functions f satisfying the following condition: there exists $c \in \mathbb{R}$ (that depends on f) such that $|f(a)| \leq c|a|$ for all $a \in \mathbb{R}$. Is W a subspace of V?

Solution. We know W is a subset of V. Let $g \in W$. There exists a $c_1 \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ $|g(x)| \leq c_1|x|$. Let $b \in \mathbb{R}$. Let $h = bg$. Then

$$\begin{aligned}|h(x)| &= |b(g(x))| \\ &= |b||g(x)| \\ &\leq |b|(c_1|x|) \\ &= c|x|,\end{aligned}$$

where $c = |b|c_1$. So $\exists c \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ we have $|h(x)| \leq c|x|$. Therefore W is closed under scalar multiplication.

Let $p, g \in W$. There exist $c_1, c_2 \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ $|p(x)| \leq c_1|x|$ and $|q(x)| \leq c_2|x|$. If $t(x) = p(x) + g(x)$, then

$$\begin{aligned}t(x) &= |p(x) + g(x)| \\ &\leq |p(x)| + |g(x)| \\ &\leq c_1|x| + c_2|x| \\ &= (c_1 + c_2)|x| \\ &= c_3|x|,\end{aligned}$$

where $c_3 = c_1 + c_2$. Therefore $\exists c \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ we have $t(x) \leq c|x|$. Thus W is closed under addition and scalar multiplication so W is a subspace of V.

Problem 9 (Golan 93)*.* Let V be a vector space over a field F and let P be the collection of all subsets of V, which we know is a vector space over GF(2). Is the collection of all subspaces of V a subspace of P?

Solution. Let U be the collection of all subspaces of V. As all subspaces of V must be subsets of V, we know that $U \subseteq P$. We will prove that $U \not\leq P$—that is, U is not a *subspace* of P—by showing that it is not closed under the scalar multiplication of P.

Indeed, fix a subspace $X \in U$, and recall that scalar multiplication $0 \in \mathrm{GF}(2)$ in P always results in the empty set. (See the example on page 24 of our textbook.) That is, $0X = \emptyset$, which is not a subspace. Since $0X \notin U$, we see that U is not closed under the scalar multiplication in P, so it is not a subspace of P.

Followup question: Could you fix the foregoing, perhaps by letting P be all subsets containing 0_V and defining $0X = \{0_V\}$ for all $X \in P$? It might not work if we take addition to be symmetric difference, but what if we use the binary string interpretation, where the string of all zeros corresponds to $\{0_V\}$? Note that P would then correspond to the set of all binary strings of length $|V| - 1$.

Problem 10 (Golan 105)*.* Let V be a vector space over a field F and let $0_V \neq w \in V$. Given a vector $v \in V \setminus Fw$, find the set G of all scalars $a \in F$ satisfying $F\{v, w\} = F\{v, aw\}$.

Solution.

$$F\{v,w\} = \{bv + cw | b, c \in \mathbb{F}\}$$

$$F\{v,aw\} = \{dv + e(aw) | d, e \in \mathbb{F}\}$$

F is a field, so it is closed under multiplication. v is not a scalar multiple of w. We know $1 \in G$. Thus $\mathbb{F}\{v, aw\} \leq \mathbb{F}\{v, w\} \forall a \in \mathbb{F}$. Let $x \in \mathbb{F}\{v, w\}$.. Then for $s, t \in \mathbb{F}$

$$\begin{aligned} x &= sv + tw \\ &= sv + \frac{t}{a}(aw) \end{aligned}$$

So $\frac{t}{a} = ta^-1$, and $t \in \mathbb{F}, a^-1 \in \mathbb{F}$ so $ta^-1 \in \mathbb{F}$ as long as $a \neq 0$. So $G = \mathbb{F} \setminus 0_V$.

Appendix

Notes on Problem 2. Here we show how one could prove that V is closed under $\boxplus$, that is, for all $f, g \in C(0, 1)$, we have $f \boxplus g \in C(0, 1)$. (Though, as we noted in the solution to Problem 2, closure only proves that $\boxplus$ is a binary operation on $C(0, 1)$; it does not prove that $\boxplus$ can serve as vector addition.)

Let $\epsilon > 0$. Assume f,g are continuous on $(0, 1)$. Then, there exist $\delta_f > 0$ and $\delta_g > 0$ such that

$$|x - x_0| < \delta_f \implies |f(x) - f(x_0)| < \frac{\epsilon}{2}$$
$$|x - x_0| < \delta_g \implies |g(x) - g(x_0)| < \frac{\epsilon}{2}$$

Let $h(x) = (f \vee g)(x) = \max\{f(x), g(x)\}$. Let $\delta_h = \min\{\delta_f, \delta_g\}$. Then

$$\begin{aligned}
|x - x_0| < \delta_h \quad \implies & \quad |h(x) - h(x_0)| \\
= & \quad \left| \frac{f(x) + g(x) + |f(x) - g(x)| - f(x_0) - g(x_0) - |f(x_0) - g(x_0)|}{2} \right| \\
\leq & \quad \left| \frac{f(x) - f(x_0)}{2} \right| + \left| \frac{g(x) - g(x_0)}{2} \right| + \left| \frac{f(x) - f(x_0) - (g(x) - g(x_0))}{2} \right| \\
< & \quad \frac{\epsilon}{2} + \left| \frac{f(x) - f(x_0) - (g(x) - g(x_0))}{2} \right| \\
< & \quad \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore $f \boxplus g$ is continuous for $x_0 \in (0, 1)$, so vector addition is closed in V. With the usual scalar multiplication, V is a vector space over $\mathbb{R}$.

Remarks. The proof above is correct. Alternatively, you could simply note that the sum (and difference) of two continuous functions is continuous, and the functions $x \mapsto |x|$ and $x \mapsto x/2$ are continuous. Therefore, since function composition preserves continuity, it is clear that $f \vee g = \frac{1}{2}(f + g + |f - g|)$ is continuous.

Yet another alternative is to use the fact that $h \in C(0, 1)$ if and only if for all $-\infty \leq a < b \leq \infty$ the set $h^{-1}(a, b) = \{x \in (0, 1) \mid a < h(x) < b\}$ is open in $(0, 1)$. Note that

$$(f \vee g)^{-1}(a, b) = (\{x : a < f(x)\} \cup \{x : a < g(x)\}) \cap \{x : f(x) < b\} \cap \{x : g(x) < b\}.$$

If f and g are continuous, all of the sets on the right are open, so $(f \vee g)^{-1}(a, b)$ is open.

Notes on on Problem 3. Consider the proposed solution:

> Let us define vector addition in a "bitwise xor" fashion such that $v + v = 0$ and $v + w = 1$ for all $v, w \in V, w \neq v$. Furthermore, let us define scalar multiplication in the natural way such that $1 \cdot v = v$ and $0 \cdot v = 0$. Then we can see that vector addition is closed as $0, 1 \in V$, as well as being associative and commutative. And every v has an additive inverse, namely v. Scalar multiplication is also closed in V, as the product is always $0 \in V$ or $v \in V$. So V is a vector space over GF(2).

Remarks. Having an addition that works as "xor" is the right idea. However, this proof is incorrect. In particular, you need an additive identity, that is, an $e \in V$ such that $v + e = v$ for all $v \in V$. In the proposed solution, 0 cannot serve as the additive identity because $v + w = 1$ for all $w \neq v$; in particular, $v + 0 = 1$ whenever $v \neq 0$. (See the correct solution given above.)

Notes on on Problem 9. The originally proposed began as follows:

> P is the collection of all subset of V. Let U be the collection of all subspaces of V. As all subspaces of V must be subsets of V, we know that $U \subseteq P$. We want to show that U is a subspace of P.
>
> Let $X \in U$. Then $X \in P$. Let $a \in F$. Let $y, z \in X$. Then $ay, az \in aX$. $ay + az = a(x + y)$. As X is a subspace, $(x + y) \in X$ so $a(x + y) \in aX$. Thus aX is closed under addition.

Here we have fallen into the trap of considering the wrong field. It is true that we should fix some $X \in U$, and then try to show that for each field element a, we have $aX \in U$. However, we must take a from GF(2), since that the field over which P is defined, and we are trying to prove $U \leq P$.

Now, you might then argue that now it is even easier because we only have to consider $0X$ and $1X$. The latter is simply X. However, what is $0X$? It would be nice if it were $\{0_V\}$, because then we would have $0X = \{0_V\} \in U$. Unfortunately, in P, scalar multiplication by 0 gives the empty set (or the binary string consisting of $|V|$ zeros, if you prefer to think of the elements of P as binary strings). The empty set is not a subspace.

References

[1] Charalambos D. Aliprantis and Owen Burkinshaw. *Principles of Real Analysis*. Academic Press, New York, 3rd edition, 1998.