

Homework 2

Matthew Corley and Melissa Murphy

Problem 1 (Golan 56). Is it possible to define on $\mathbb{Z}/(4)$ the structure of a vector space over $\text{GF}(2)$ in such a way that the vector addition is the usual addition in $\mathbb{Z}/(4)$?

[*Hints:* Recall that (n) denotes the set $\{\dots, -2n, -n, 0, n, 2n, 3n, \dots\}$, which we denoted in lecture by $n\mathbb{Z}$. This is the ideal generated by n in the ring \mathbb{Z} , but don't worry about that for now. Just take $\mathbb{Z}/(n)$ to be the abelian group of integers $\{0, 1, 2, \dots, n-1\}$ with addition modulo n . In lecture, we used $\mathbb{Z}/n\mathbb{Z}$ to denote $\mathbb{Z}/(n)$. Use whichever notation you prefer.]

Solution. Assume toward a contradiction that $\mathbb{Z}/(4)$ is a vector space over $\text{GF}(2)$, with vector addition defined as the usual modular addition. Then for any $v \in \mathbb{Z}/(4)$,

$$\begin{aligned} 0 &= (0)v \\ &= (1+1)v \\ &= 1 \cdot v + 1 \cdot v \\ &= v + v \end{aligned}$$

Now, let $v = 3$ to see that $3 + 3 = 2 \pmod{4} \neq 0$. Thus we have a contradiction. The answer is no, it is not possible with the usual modular vector addition.

Problem 2 (Golan 60).¹ Let $V = C(0, 1)$. Define the relation \vee on V by setting $(f \vee g)(x) = \max\{f(x), g(x)\}$. If we think of \vee as a “vector addition,” does this, together with the usual scalar multiplication, make V into a vector space over \mathbb{R} ?

Solution. It is true that $C(0, 1)$ is closed under \vee (see the [Appendix](#) for a proof), and we can easily verify that \vee is a commutative associative binary operation on the set $C(0, 1)$, so $\langle C(0, 1), \vee \rangle$ is a commutative semigroup. In fact, letting $f \wedge g = \min\{f, g\}$, we can check that $\langle C(0, 1), \vee, \wedge \rangle$ is a lattice.²

However, recall that a vector space is built up from an additive abelian group. Is it possible for \vee to serve as vector addition? If so, what would be the additive identity? We need a function $e \in C(0, 1)$ such that for all $f \in C(0, 1)$ we have $f \vee e = f$. It is clear that no such e exists.

¹ In the original problem, the notation $f \boxplus g$ was used. We use $f \vee g$ instead, since this is fairly standard notation for the function $\max\{f, g\}$.

² See [1, Sec. 30] for a discussion of vector lattices, such as $\langle C(0, 1), \vee, \wedge \rangle$.

Problem 3 (Golan 63). Let $V = \{i \in \mathbb{Z} \mid 0 \leq i < 2^n\}$ for some fixed positive integer n . Define operations of vector addition and scalar multiplication on V in such a way as to turn it into a vector space over $\text{GF}(2)$.

[*Hints:* Recall that $\text{GF}(2)$ denotes the Galois field with two elements, $\{0, 1\}$, with addition mod 2 and the usual multiplication. Other than this field, the only restriction given in the problem is that V must have 2^n elements. Do you know of any sets of this size?]

Solution. Let W be the set of binary strings of length n , that is, length n sequences of 0's and 1's. We can also view these as maps from the set $n := \{0, 1, \dots, n-1\}$ to the set $2 := \{0, 1\}$. So, in this sense, W is the set 2^n of maps from n to 2. So, it is not really an abuse of notation to write $W = 2^n$.

Since $|V| = 2^n$ (here 2^n is a number!), there is a bijection between V and W , and we will identify each $i \in V$ with its string representation in W using the notation $i = (i_0, i_1, \dots, i_{n-1})$, where $i_k \in \{0, 1\}$. Define vector addition in V by adding strings “bitwise” modulo 2. That is

$$\begin{aligned} i + j &= (i_0, i_1, \dots, i_{n-1}) + (j_1, \dots, j_{n-1}) \\ &= (i_0, i_1, \dots, i_{n-1}) + (j_1, \dots, j_{n-1}) \\ &= (i_0 + j_0, i_1 + j_1, \dots, i_{n-1} + j_{n-1}) \end{aligned}$$

where for each $0 \leq k < n$, the k -th element of $i + j$ is

$$i_k + j_k = \begin{cases} 0, & i_k = j_k \\ 1, & i_k \neq j_k. \end{cases}$$

Clearly the latter addition is commutative, and therefore, the vector addition is commutative: $i + j = j + i$. The zero vector $\mathbf{0} = (0, \dots, 0)$ is the additive identity, and each vector is its own additive inverse, that is, $-v = v$. Thus, we have an abelian group $\langle 2^n, +, -, \mathbf{0} \rangle$. To make this into a vector space over $\text{GF}(2)$, take the set of scalars $\{0, 1\}$ and define scalar multiplication as follows: $0i = \mathbf{0}$ and $1i = i$. It is easily verified that $\langle 2^n, +, \mathbf{0}, \{0, 1\} \rangle$ has the remaining ($\text{GF}(2)$ -module) properties that make it a vector space over $\text{GF}(2)$.

Problem 4 (Golan 70). Show that \mathbb{Z} is not a vector space over any field.

Solution. Let $\mathbb{F} = \text{GF}(2)$. Assume \mathbb{Z} is a vector space over \mathbb{F} . Then

$$\begin{aligned} 0 &= 1_Z(1_F + 1_F) \\ &= 1_Z(1_F) +_Z 1_Z(1_F) \\ &= 2 \end{aligned}$$

But $0 \neq 2$, so \mathbb{Z} is not a vector space over $\text{GF}(2)$. Now let \mathbb{F} be a field with characteristic greater than 2. Assume \mathbb{Z} a vector space over \mathbb{F} . Then we know

$$1_F + 1_F = 2_F \implies 1_F = \frac{1}{2_F} + \frac{1}{2_F}$$

Therefore

$$\begin{aligned} 1_Z &= 1_F(1_Z) \\ &= \left(\frac{1}{2_F} + \frac{1}{2_F}\right)1_Z \\ &= \frac{1}{2_F}(1_Z) +_Z \frac{1}{2_F}(1_Z) \end{aligned}$$

Now let $\frac{1}{2_F}(1_Z) = n \in \mathbb{Z}$. But there is no element in $n \in \mathbb{Z}$ that satisfies $n + n = 1$. So \mathbb{Z} is not a vector space over any field with characteristic greater than 2. Thus \mathbb{Z} is not a vector space over any field.

Problem 5 (Golan 76). Let $V = \mathbb{R}^{\mathbb{R}}$ and let W be the subset of V containing the constant function $x \mapsto 0$ and all of those functions $f \in V$ satisfying the following condition: $f(a) = 0$ for at most finitely many real numbers a . Is W a subspace of V .

[Hint: It's easy.]

Solution. Let us assume toward a contradiction that W is a subspace of V . Let $p(x), g(x), h(x)$ be distinct functions in W , such that $p(x) = g(x) + h(x)$ and $g(x) = 0$ for $x = b_0, \dots, b_l \in \mathbb{R}$ and $h(x) = 0$ for $x = c_0, \dots, c_m \in \mathbb{R}$. Then $p(x) = 0$ when $g(x) = h(x) = 0$ or when $g(x) = -h(x)$. The former happens when $b_i = c_j$ for $1 \leq i \leq l, 1 \leq j \leq m$. We can see this is a finite set of points. The latter, however, could happen for an infinite number of points (e.g. define $g(x) = -h(x)$ for $x > \max(b_l, c_m) \in \mathbb{R}$). In that case, $p(a) = 0$ for infinitely many real numbers a , but it is not the constant function $x \mapsto 0$. So vector addition is not closed and therefore W is not a subspace in V .

Problem 6 (Golan 79). A function $f \in \mathbb{R}^{\mathbb{R}}$ is *piecewise constant* if and only if it is a constant function $x \mapsto c$ or there exist $a_1 < a_2 < \cdots < a_n$ and c_0, c_1, \dots, c_n in \mathbb{R} such that

$$f : x \mapsto \begin{cases} c_0 & \text{if } x < a_1, \\ c_i & \text{if } a_i \leq x < a_{i+1} \text{ for } 1 \leq i < n, \\ c_n & \text{if } a_n \leq x. \end{cases}$$

Does the set of all piecewise constant functions form a subspace of the vector space $\mathbb{R}^{\mathbb{R}}$ over \mathbb{R} ?

Solution. Let W denote the set of piecewise constant functions. Then W is clearly a subset of $\mathbb{R}^{\mathbb{R}}$. Let $r \in \mathbb{R}$ and $k_i = rc_i$. Then

$$rf : x \mapsto \begin{cases} k_0 & \text{if } x < a_1, \\ k_i & \text{if } a_i \leq x < a_{i+1} \text{ for } 1 \leq i < n, \\ k_n & \text{if } a_n \leq x. \end{cases}$$

So W is closed under scalar multiplication.

Let us describe $f, g \in W$ using the characteristic function χ :

$$f(x) = \sum_{i=1}^n f(a_i) \chi_{[a_i, a_{i+1})}(x),$$

$$g(x) = \sum_{i=1}^m g(b_i) \chi_{[b_i, b_{i+1})}(x),$$

where $\chi_{[c_i, c_{i+1})}(x)$ is 1 if $c_i \leq x < c_{i+1}$ and 0 otherwise.

Now let the set $\{z_0, z_1, \dots, z_N\}$ be the union $\{a_0, a_1, \dots, a_n\} \cup \{b_0, b_1, \dots, b_m\}$ such that $z_0 < z_1 < \cdots < z_N$.³ Then we can describe the sum $f + g$ as follows:

$$f + g = \sum_{i=1}^N (f(z_i) + g(z_i)) \chi_{[z_i, z_{i+1})}.$$

Thus $f + g$ is a piecewise constant function and so W is closed under vector addition. Therefore W is a subspace of $\mathbb{R}^{\mathbb{R}}$.

Problem 7 (Golan 81). Let W be the subset of $V = \text{GF}(2)^5$ consisting of all vectors (a_1, \dots, a_5) satisfying $\sum_{i=1}^5 a_i = 0$. Is W a subspace of V ?

Solution. We know W is a subset of V and $O_V \in W$. Let $x \in W, x = (a_1, a_2, a_3, a_4, a_5)$ and $\sum_{i=1}^5 a_i = 0$. Let $b \in F$. Then

$$bx = (b(a_1), b(a_2), b(a_3), b(a_4), b(a_5))$$

³ Note to students: we use the word “union” explicitly to emphasize that the resulting $\{z_0, z_1, \dots, z_N\}$ will be a set—i.e., there will be no repetitions.

And

$$\sum_{i=1}^5 (ba_i) = b \left(\sum_{i=1}^5 a_i \right) = b(0) = 0$$

So W is closed under scalar multiplication.

Let $y \in V$, $y = (c_1, c_2, c_3, c_4, c_5)$ such that $\sum_{i=1}^5 c_i = 0$. Then

$$x + y = (a_1 + c_1, a_2 + c_2, a_3 + c_3, a_4 + c_4, a_5 + c_5)$$

Now let $d_i = a_i + c_i$.

$$\begin{aligned} \sum_{i=1}^5 d_i &= \sum_{i=1}^5 (a_i + c_i) \\ &= \sum_{i=1}^5 a_i + \sum_{i=1}^5 c_i \\ &= 0 + 0 = 0 \end{aligned}$$

Therefore W is closed under vector addition and scalar multiplication so it is a subspace.

Problem 8 (Golan 85). Let $V = \mathbb{R}^{\mathbb{R}}$ and let W be the subset of V consisting of all functions f satisfying the following condition: there exists $c \in \mathbb{R}$ (that depends on f) such that $|f(a)| \leq c|a|$ for all $a \in \mathbb{R}$. Is W a subspace of V ?

Solution. We know W is a subset of V . Let $g \in W$. There exists a $c_1 \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ $|g(x)| \leq c_1|x|$. Let $b \in \mathbb{R}$. Let $h = bg$. Then

$$\begin{aligned} |h(x)| &= |b(g(x))| \\ &= |b||g(x)| \\ &\leq |b|(c_1|x|) \\ &= c|x|, \end{aligned}$$

where $c = |b|c_1$. So $\exists c \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ we have $|h(x)| \leq c|x|$. Therefore W is closed under scalar multiplication.

Let $p, g \in W$. There exist $c_1, c_2 \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ $|p(x)| \leq c_1|x|$ and $|q(x)| \leq c_2|x|$. If $t(x) = p(x) + g(x)$, then

$$\begin{aligned} t(x) &= |p(x) + g(x)| \\ &\leq |p(x)| + |g(x)| \\ &\leq c_1|x| + c_2|x| \\ &= (c_1 + c_2)|x| \\ &= c_3|x|, \end{aligned}$$

where $c_3 = c_1 + c_2$. Therefore $\exists c \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ we have $t(x) \leq c|x|$. Thus W is closed under addition and scalar multiplication so W is a subspace of V .

Problem 9 (Golan 93). Let V be a vector space over a field F and let P be the collection of all subsets of V , which we know is a vector space over $\text{GF}(2)$. Is the collection of all subspaces of V a subspace of P ?

Solution. Let U be the collection of all subspaces of V . As all subspaces of V must be subsets of V , we know that $U \subseteq P$. We will prove that $U \not\subseteq P$ —that is, U is not a subspace of P —by showing that it is not closed under the scalar multiplication of P .

Indeed, fix a subspace $X \in U$, and recall that scalar multiplication $0 \in \text{GF}(2)$ in P always results in the empty set. (See the example in our textbook [2, p. 24].) That is, $0X = \emptyset$, which is not a subspace. Since $0X \notin U$, we see that U is not closed under the scalar multiplication in P , so it is not a subspace of P .

Followup question: Could you fix the foregoing, perhaps by letting P be all subsets containing 0_V and defining $0X = \{0_V\}$ for all $X \in P$? It might not work if we take addition to be symmetric difference, but what if we use the binary string interpretation, where the string of all zeros corresponds to $\{0_V\}$? Note that P would then correspond to the set of all binary strings of length $|V| - 1$.

Problem 10 (Golan 105). Let V be a vector space over a field F and let $0_V \neq w \in V$. Given a vector $v \in V \setminus Fw$, find the set G of all scalars $a \in F$ satisfying $F\{v, w\} = F\{v, aw\}$.

Solution.

$$F\{v, w\} = \{bv + cw \mid b, c \in F\}$$

$$F\{v, aw\} = \{dv + e(aw) \mid d, e \in F\}$$

F is a field, so it is closed under multiplication. v is not a scalar multiple of w . We know $1 \in G$. Thus $F\{v, aw\} \subseteq F\{v, w\} \forall a \in F$. Let $x \in F\{v, w\}$. Then for $s, t \in F$

$$\begin{aligned} x &= sv + tw \\ &= sv + \frac{t}{a}(aw) \end{aligned}$$

So $\frac{t}{a} = ta^{-1}$, and $t \in F, a^{-1} \in F$ so $ta^{-1} \in F$ as long as $a \neq 0$. So $G = F \setminus 0_V$.

Appendix

Notes on Problem 2. Here we show how one could prove that V is closed under \boxplus , that is, for all $f, g \in C(0, 1)$, we have $f \boxplus g \in C(0, 1)$. (Though, as we noted in the solution to Problem 2, closure only proves that \boxplus is a binary operation on $C(0, 1)$; it does not prove that \boxplus can serve as vector addition.)

Let $\epsilon > 0$. Assume f, g are continuous on $(0, 1)$. Then, there exist $\delta_f > 0$ and $\delta_g > 0$ such that

$$|x - x_0| < \delta_f \implies |f(x) - f(x_0)| < \frac{\epsilon}{2}$$

$$|x - x_0| < \delta_g \implies |g(x) - g(x_0)| < \frac{\epsilon}{2}$$

Let $h(x) = (f \vee g)(x) = \max\{f(x), g(x)\}$. Let $\delta_h = \min\{\delta_f, \delta_g\}$. Then

$$\begin{aligned} |x - x_0| < \delta_h &\implies |h(x) - h(x_0)| \\ &= \left| \frac{f(x) + g(x) + |f(x) - g(x)| - f(x_0) - g(x_0) - |f(x_0) - g(x_0)|}{2} \right| \\ &\leq \left| \frac{f(x) - f(x_0)}{2} \right| + \left| \frac{g(x) - g(x_0)}{2} \right| + \left| \frac{f(x) - f(x_0) - (g(x) - g(x_0))}{2} \right| \\ &< \frac{\epsilon}{2} + \left| \frac{f(x) - f(x_0) - (g(x) - g(x_0))}{2} \right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore $f \boxplus g$ is continuous for $x_0 \in (0, 1)$, so vector addition is closed in V . With the usual scalar multiplication, V is a vector space over \mathbb{R} .

Remarks. The proof above is correct. Alternatively, you could simply note that the sum (and difference) of two continuous functions is continuous, and the functions $x \mapsto |x|$ and $x \mapsto x/2$ are continuous. Therefore, since function composition preserves continuity, it is clear that $f \vee g = \frac{1}{2}(f + g + |f - g|)$ is continuous.

Yet another alternative is to use the fact that $h \in C(0, 1)$ if and only if for all $-\infty \leq a < b \leq \infty$ the set $h^{-1}(a, b) = \{x \in (0, 1) \mid a < h(x) < b\}$ is open in $(0, 1)$. Note that

$$(f \vee g)^{-1}(a, b) = (\{x : a < f(x)\} \cup \{x : a < g(x)\}) \cap \{x : f(x) < b\} \cap \{x : g(x) < b\}.$$

If f and g are continuous, all of the sets on the right are open, so $(f \vee g)^{-1}(a, b)$ is open.

Notes on Problem 3. Consider the proposed solution:

Let us define vector addition in a “bitwise xor” fashion such that $v + v = 0$ and $v + w = 1$ for all $v, w \in V, w \neq v$. Furthermore, let us define scalar multiplication in the natural way such that $1 \cdot v = v$ and $0 \cdot v = 0$. Then we can see that vector addition is closed as $0, 1 \in V$, as well as being associative and commutative. And every v has an additive inverse, namely v . Scalar multiplication is also closed in V , as the product is always $0 \in V$ or $v \in V$. So V is a vector space over $\text{GF}(2)$.

Remarks. Having an addition that works as “xor” is the right idea. However, this proof is incorrect. In particular, you need an additive identity, that is, an $e \in V$ such that $v + e = v$ for all $v \in V$. In the proposed solution, 0 cannot serve as the additive identity because $v + w = 1$ for all $w \neq v$; in particular, $v + 0 = 1$ whenever $v \neq 0$. (See the correct solution given above.)

Notes on Problem 9. The originally proposed solution began as follows:

P is the collection of all subset of V . Let U be the collection of all subspaces of V . As all subspaces of V must be subsets of V , we know that $U \subseteq P$. We want to show that U is a subspace of P .

Let $X \in U$. Then $X \in P$. Let $a \in F$. Let $y, z \in X$. Then $ay, az \in aX$. $ay + az = a(x + y)$. As X is a subspace, $(x + y) \in X$ so $a(x + y) \in aX$. Thus aX is closed under addition.

Here we have fallen into the trap of considering the wrong field. It is true that we should fix some $X \in U$, and then try to show that for each field element a , we have $aX \in U$. However, we must take a from $\text{GF}(2)$, since that is the field over which P is defined, and we are trying to prove $U \leq P$.

Now, you might then argue that now it is even easier because we only have to consider $0X$ and $1X$. The latter is simply X , but what is $0X$? It would be nice if it were $\{0_V\}$, because then we would have $0X = \{0_V\} \in U$. Unfortunately, in P , scalar multiplication by 0 gives the empty set (or the length- $|V|$ string of zeros, if you prefer to think of the elements of P as binary strings). The empty set is not a subspace, so U is not closed under the scalar multiplication of P .

References

- [1] Charalambos D. Aliprantis and Owen Burkinshaw. *Principles of Real Analysis*. Academic Press, New York, 3rd edition, 1998.
- [2] Jonathan Golan. *The Linear Algebra a Beginning Graduate Student ought to know*. Springer, 3rd edition, 2012.