Homework 5

Alexander Brylev

The label "Problem" is used for required problems. "Exercise" is for suggested exercises.

Problem 1 (Golan 307). Let V be a vector space over a field F and let W be a subspace of V. For each $v \in V$, let $v + W = \{v + w \mid w \in W\}$. Let $V/W = \{v + W \mid v \in V\}$ be the collection of all sets of the form $v + W$, and define operations of addition and scalar multiplication on V/W by setting $(v + W) + (v' + W) = (v + v') + W$ and $c(v + W) = (cv) + W$ for all $v, v' \in V$ and $c \in F$. Show that
1. $v + W = v' + W$ if and only if $v - v' \in W$;
2. V/W, with the given operations, is a vector space over F;
3. The function $v \mapsto v + W$ is an epimorphism from V to W, the kernel of which equals W;
4. Every complement of W in V is isomorphic to V/W;
5. If $(v + W) \cap (v' + W) \neq \emptyset$, then $v + W = v' + W$.

The space V/W is called the *factor space* of V by W.

Problem 2 (Golan 325). Let $\alpha \in \mathrm{Aut}(\mathbb{R}^2)$ be defined by $\alpha : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} -b \\ a \end{bmatrix}$. Show that $\mathbb{R}\{\alpha, \sigma_1\}$ is a unital subalgebra of $\mathrm{End}(\mathbb{R}^2)$. Show that it is proper by giving an example of an endomorphism of $\mathbb{R}^2$ not in this subalgebra.

Problem 3 (Golan 326). Let V be the space of all real-valued functions on the interval $[-1, 1]$ which are infinitely differentiable, and let δ be the endomorphism of V which assigns to each function f its derivative. Find the kernel and image of δ.

Problem 4 (Golan 338). Let V be a vector space over a field F which is not finitely generated, and let $\sigma_0 \neq \alpha \in \mathrm{End}(V)$. Set $A = \{\beta \in \mathrm{End}(V) \mid \alpha\beta = \sigma_1\}$. Show that if A has more than one element then it is infinite.

Problem 5 (Golan 340). Let V be a vector space over a field F satisfying the condition that $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathrm{End}(V)$. Show that $\dim(V) = 1$.

Problem 6 (Golan 354). Let V be a vector space over a field F and let $\alpha \in \mathrm{Aut}(V)$. Let $W_1, \ldots, W_k$ be subspaces of V satisfying $V = \bigoplus_{i=1}^{k} W_i$. For each $1 \leq i \leq k$, let $Y_i = \{\alpha(w) \mid$

$w \in W_i\}$. Is $V = \bigoplus_{i=1}^{k} Y_i$?

Exercise (Golan 415). Let V be the subspace of $\mathbb{R}[X]$ consisting of all polynomials of degree less than 3 and choose the basis $B = \{1, X, X^2\}$ for V. Let $\alpha \in \operatorname{End}(V)$ satisfy

$$\Phi_{BB}(\alpha) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Let D be the basis $\{1, X+1, 2X^2+4X+3\}$ for V. What is $\Phi_{DD}(\alpha)$?

Exercise (Golan 467). Let n be a positive integer and let F be a field. Let $A, B \in \mathcal{M}_{n\times n}(F)$ satisfy $A + B = I$. Show that $AB = \mathbf{0}$ if and only if A and B are idempotent.

Exercise (Golan 530). Let n be a positive integer and let F be a field. If $A \in \mathcal{M}_{n\times n}(F)$ is nonsingular, is the same necessarily true of $A + A^T$?