Homework 5

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The label "Problem" is used for required problems. "Exercise" is for suggested exercises.

Problem 1 (Golan 307). Let V be a vector space over a field F and let W be a subspace of V. For each $v \in V$, let $v + W = \{v + w \mid w \in W\}$. Let $V/W = \{v + W \mid v \in V\}$ be the collection of all sets of the form $v + W$, and define operations of addition and scalar multiplication on V/W by setting $(v + W) + (v' + W) = (v + v') + W$ and $c(v + W) = (cv) + W$ for all $v, v' \in V$ and $c \in F$. Show that

1. $v + W = v' + W$ if and only if $v - v' \in W$;
2. V/W, with the given operations, is a vector space over F;
3. The function $v \mapsto v + W$ is an epimorphism from V to V/W, the kernel of which equals W;
4. Every complement of W in V is isomorphic to V/W;
5. If $(v + W) \cap (v' + W) \neq \emptyset$, then $v + W = v' + W$.

The space V/W is called the *factor space* of V by W.

Solution. 1. Let $v + W = v' + W$ for some $v, v' \in V$. Then $v + 0_W = v' + w$ for some $w \in W$, so that $v - v' = w - 0_W \in W$.

Now let $v - v' \in W$. Then $v - v' = w'$ for some $w' \in W$, so that for any $w \in W$ we have $v + w = (v' + w') + w = v' + (w' + w) \in v' + W$, which shows $v + W \in v' + W$. Due to symmetry, we must also have $v' + W \in v + W$. Hence $v + W = v' + W$.

2. First, let's check that the given operations are well-defined, i.e. if $u + W = v + W$ for some $u, v \in W$ and $u' + W = v' + W$ for some $u', v' \in W$, then $(u + u') + W = (v + v') + W$ and $cu + W = cv + W$ for all $c \in F$.
If $u + W = v + W$ and $u' + W = v' + W$ for some $u, v, u', v' \in W$, then, by part 1, $u - v = w$ and $u' - v' = w'$ for some $w, w' \in W$. So $(u + u') - (v + v') = w + w' \in W$, and, using part 1 again, we get $(u + u') + W = (v + v') + W$. Also, $cu - cv = c(u - v) = cw \in W$, so that $cu + W = cv + W$, by part 1 as well. Thus, the given operations are indeed well-defined.

To show V/W, with the given operations, is a vector space, we first need to prove V/W is an abelian group.

Let $u + W, u_1 + W, u_2 + W, u_3 + W \in V/W$. Then
i) $((u_1 + W) + (u_2 + W)) + (u_3 + W) = ((u_1 + u_2) + W) + (u_3 + W) = ((u_1 + u_2) + u_3) + W = (u_1 + (u_2 + u_3)) + W = (u_1 + W) + ((u_2 + u_3) + W) = (u_1 + W) + ((u_2 + W) + (u_3 + W))$;
ii) $(0_V + W) + (u + W) = (0_V + u) + W = u + W = (u + 0_V) + W = (u + W) + (0_V + W)$;
iii) $(-u + W) + (u + W) = (-u + u) + W = 0_V + W$;
iv) $(u_1 + W) + (u_2 + W) = (u_1 + u_2) + W = (u_2 + u_1) + W = (u_2 + W) + (u_1 + W)$.
Hence $\underline{V/W} = < V/W, +, -, 0_V + W >$ is an abelian group.

For each $r \in F$ consider $f_r : \underline{V/W} \to \underline{V/W}$ by $f_r(v + W) = rv + W$.

Let $r, r_1, r_2 \in F$ and $v+W, v_1+W, v_2+W \in V/W$. Then
i) $f_r((v_1+W)+(v_2+W)) = f_r((v_1+v_2)+W) = r(v_1+v_2)+W = (rv_1+rv_2)+W = (rv_1+W)+(rv_2+W) = r(v_1+W)+r(v_2+W) = f_r(v_1+W)+f_r(v_2+W)$;
ii) $f_{r_1+r_2}(v+W) = (r_1+r_2)v+W = (r_1v+r_2v)+W = (r_1v+W)+(r_2v+W) = r_1(v+W)+r_2(v+W) = f_{r_1}(v+W)+f_{r_2}(v+W)$;
iii) $f_{r_1}(f_{r_2}(v+W)) = f_{r_1}(r_2v+W) = r_1r_2v+W = f_{r_1r_2}(v+W)$;
iv) $f_1(v+W) = 1v+W = v+W$.

Therefore V/W is a vector space over F.

3. The map $f: V \to V/W$ defined via $f(v) = v+W$ is obviously surjective and $f(v+u) = (v+u)+W = (v+W)+(u+W) = f(v)+f(u)$ for all $u, v \in V$, which shows it is a homomorphism. Using part 1, $f(v) = v+W = 0+W$ if and only if $v = v-0 \in W$, which shows $Ker(f) = W$.
4. Using the first homomorphism theorem and part 3, $W^c = V/Ker(f) \cong Im(f) = V/W$.
5. Let $(v+W)\cap(v'+W) \neq \emptyset$. This means $v+w = v'+w'$ for some $w, w' \in W$, which yields $v-v' = w'-w \in W$. By part 1, this implies $v+W = v'+W$.

Problem 2 (Golan 325). Let $\alpha \in \mathrm{Aut}(\mathbb{R}^2)$ be defined by $\alpha : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} -b \\ a \end{bmatrix}$. Show that $\mathbb{R}\{\alpha, \sigma_0\}$ is a unital subalgebra of $\mathrm{End}(\mathbb{R}^2)$. Show that it is proper by giving an example of an endomorphism of $\mathbb{R}^2$ not in this subalgebra.

Solution. Since $\alpha^4 = \sigma_1$, then $\mathbb{R}\{\alpha, \sigma_0\} = \{c_0\sigma_1 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 : c_0, c_1, c_2, c_3 \in \mathbb{R}\}$. Clearly this set is closed under addition, multiplication by scalars and compositions, and contains σ_1. Hence it is unital subalgebra of $\mathrm{End}(\mathbb{R}^2)$.

Define $\beta : \mathbb{R}^2 \to \mathbb{R}^2$ by $\beta : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a \\ 0 \end{bmatrix}$. Since β is a projection in $\mathbb{R}^2$, $\beta \in \mathrm{End}(\mathbb{R}^2)$. Suppose $\beta \in \mathbb{R}\{\alpha, \sigma_0\}$. Then there exist $c_0, c_1, c_2, c_3 \in \mathbb{R}$ such that $\begin{bmatrix} a \\ 0 \end{bmatrix} = \beta(\begin{bmatrix} a \\ b \end{bmatrix}) = c_0\sigma_1(\begin{bmatrix} a \\ b \end{bmatrix}) + c_1\alpha(\begin{bmatrix} a \\ b \end{bmatrix}) + c_2\alpha^2(\begin{bmatrix} a \\ b \end{bmatrix}) + c_3\alpha^3(\begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} (c_0-c_2)a + (c_3-c_1)b \\ (c_1-c_3)a + (c_0-c_2)b \end{bmatrix}$ for all $a, b \in \mathbb{R}$. Choosing $a = 1, b = 0$, we get $c_0 - c_2 = 1$. And, choosing $a = 0, b = 1$, we get $c_0 - c_2 = 0$. A contradiction. Hence $\beta \notin \mathbb{R}\{\alpha, \sigma_0\}$ and our subalgebra is proper.

Problem 3 (Golan 326). Let V be the space of all real-valued functions on the interval $[-1, 1]$ which are infinitely differentiable, and let δ be the endomorphism of V which assigns to each function f its derivative. Find the kernel and image of δ.

Solution. We have $Ker\delta = \{v \in V : \delta(v) = 0\} = \{f \in C^\infty[-1,1] : f' = 0\} = \{f \in C^\infty[-1,1] : f(x) = c \text{ for some c} \in R\}$, i.e., a set of all constant functions defined on $[-1, 1]$. We have $Im\delta = \{v \in V : \delta(u) = v \text{ for some } u \in V\} = \{f \in C^\infty[-1,1] : g' = f \text{ for some } g \in C^\infty[-1,1]\} = \{f \in C^\infty[-1,1] : \text{f is integrable on } [-1,1]\}$, i.e., a set of all integrable functions from $C^\infty[-1,1]$.

Problem 4 (Golan 338). Let V be a vector space over a field F which is not finitely generated, and let $\sigma_0 \neq \alpha \in \mathrm{End}(V)$. Set $A = \{\beta \in \mathrm{End}(V) \mid \alpha\beta = \sigma_1\}$. Show that if A has more than one element then it is infinite.

Solution. Suppose A has two elements, β_1 and β_2. Then there exists a basis vector v of V such that $\beta_1(v) \neq \beta_2(v)$. For $n \geq 3$, define $\beta_n \in \mathrm{End}(V)$ via $\beta_n(v) = (n-1)\beta_1(v) - (n-2)\beta_2(v)$ and $\beta_n(u) = \beta_1(u)$, where u is a basis vector of V such that $u \neq v$. Then $\alpha\beta_n(v) = (n-1)\alpha\beta_1(v) - (n-2)\alpha\beta_2(v) = (n-1)v - (n-2)v = v$ and $\alpha\beta_n(u) = \alpha\beta_1(u) = u$ for a basis vector u of V such that $u \neq v$. Thus, $\beta_n \in A$ for all n. For $n \neq k$, $\beta_n(v) - \beta_k(v) = (n-k)(\beta_1(v) - \beta_2(v)) \neq 0$, which shows $\beta_n \neq \beta_k$ for $n \neq k$. Hence A contains infinitely many elements.

Problem 5 (Golan 340). Let V be a vector space over a field F satisfying the condition that $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathrm{End}(V)$. Show that $\dim(V) = 1$.

Solution. Suppose $\dim(V) > 1$. Then there exist two linearly independent vectors e_1 and e_2 in V. Define $\alpha \in \mathrm{End}(V)$ via $\alpha(e_1) = e_2$ and $\alpha(v) = 0$ if $v \notin \mathit{span}(e_1)$. Define $\beta \in \mathrm{End}(V)$ via $\beta(e_2) = e_1$ and $\beta(v) = 0$ if $v \notin \mathit{span}(e_2)$. Then $\alpha\beta(e_1) = \alpha(0) = 0$, which does not equal $\beta\alpha(e_1) = \beta(e_2) = e_1$, a contradiction.

Problem 6 (Golan 354). Let V be a vector space over a field F and let $\alpha \in \mathrm{Aut}(V)$. Let $W_1, \ldots, W_k$ be subspaces of V satisfying $V = \bigoplus_{i=1}^k W_i$. For each $1 \leq i \leq k$, let $Y_i = \{\alpha(w) \mid w \in W_i\}$. Is $V = \bigoplus_{i=1}^k Y_i$?

Solution. Let $y \in V$. Since $\alpha \in Aut(V)$, then α is surjective and hence there exists $x \in V$ such that $\alpha(x) = y$. Since $x \in V$ and $V = \bigoplus_{i=1}^k W_i$, then $x = w_1 + w_2 + ... w_k$ for some $w_i \in W_i, 1 \leq i \leq k$. Then $y = \alpha(x) = \alpha(w_1 + w_2 + ... + w_k) = \alpha(w_1) + \alpha(w_2) + ... + \alpha(w_k) \in Y_1 + Y_2 + ... + Y_k$.

Let $v \in Y_i \cap Y_j$ for some $i \neq j$. Then $v = \alpha(w_i) = \alpha(w_j)$ for some $w_i \in W_i$ and $w_j \in W_j$. Since $\alpha \in Aut(V)$, then α is injective, and so $w_i = w_j \in W_i \cap W_j$. From $V = \bigoplus_{i=1}^k W_i$, it follows that $W_i \cap W_j = \{0\}$, and hence $w_i = w_j = 0$. So $v = \alpha(w_i) = \alpha(0) = 0$.

Exercise (Golan 415). Let V be the subspace of $\mathbb{R}[X]$ consisting of all polynomials of degree less than 3 and choose the basis $B = \{1, X, X^2\}$ for V. Let $\alpha \in \mathrm{End}(V)$ satisfy

$$\Phi_{BB}(\alpha) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Let D be the basis $\{1, X+1, 2X^2+4X+3\}$ for V. What is $\Phi_{DD}(\alpha)$?

Exercise (Golan 467). Let n be a positive integer and let F be a field. Let $A, B \in \mathcal{M}_{n\times n}(F)$ satisfy $A + B = I$. Show that $AB = \mathbf{0}$ if and only if A and B are idempotent.

Solution. Since $A+B=I$, then $B=I-A$ and $A=I-B$. So $AB=A(I-A)=AI-A^2=A-A^2$ and $AB=(I-B)B=IB-B^2=B-B^2$. Hence $AB=\mathbf{0}$ if and only if $A=A^2$ and $B=B^2$, i.e., if and only if A and B are idempotent.

Exercise (Golan 530). Let n be a positive integer and let F be a field. If $A \in \mathcal{M}_{n\times n}(F)$ is nonsingular, is the same necessarily true of $A+A^T$?

Solution. If $n=1$, then $A=(a)$ for some $a \in F$. If A is nonsingular, then $a \neq 0_F$, and so $A+A^T=(a)+(a)=(2a)$ is nonsingular, since $2a \neq 0_F$. Now, let $n>1$. Define $A=(a_{ij})$ via $a_{1n}=-1_F$, $a_{i,(n+1)-i}=1_F$ for $2 \leq i \leq n$, and $a_{ij}=0_F$ elsewhere. Then $det(A)=-1_F \neq 0_F$, so that A is nonsingular. However, $A+A^T$ has a zero first row, and hence is singular.