Midterm Exam

Math 700: Spring 2014

INSTRUCTIONS

- Solve all of the problems below and then write up your solutions (neatly!), giving complete proofs/justifications for all arguments. Since this is a take home exam, I expect the presentation quality to be higher quality than if this were an in-class exam.

- The questions are meant to test your understanding of basic concepts, so you should be more explicit than would if, say, you were writing an article for a journal. In particular, it will help if you write down definitions of any technical terms you use, even if these terms have already been mentioned in the statement of the problem. In the first problem, for instance, you should say what it means to be a projection, isomorphism, etc. Of course, you must use your best judgement about which definitions to state. (You probably don't want to provide definitions of the integers or real numbers, for example.)

- *Honor code.* You are expected to solve the exam problems on your own without outside help. You may consult the lecture notes and textbook for this course only. No other books or internet usage is allowed.[1] If you get stuck, please ask *me* for help, in which case I might be willing to provide hints on our wiki page.

- Finally, it will be helpful if you

  1. state what you are trying to prove,
  2. mention informally how you plan to prove it before giving the formal details, and
  3. if you believe your proof is complete, use an end-of-proof symbol (like QED or $\Box$) to indicate this; on the other hand, if you believe your proof is incomplete, please say so.

NOTATION

For the most part, we follow the notation used in the textbook. Recall that if V is a vector space over the field F and if $c \in F$, then $\sigma_c v = cv$ for all $v \in V$. In particular, σ_0 and σ_1 denote the zero and identity maps, respectively.

[1] There is one exception to this rule, since I've asked you to look at a specific Wikipedia page when solving Problem 2.

Problem 1. Let V be a finite dimensional vector space over a field F. Suppose $\alpha \in \operatorname{End}(V)$ is a projection with $X = \ker(\alpha)$ and $Y = \operatorname{im}(\alpha)$. Define an isomorphism $\varphi : V \cong X \oplus Y$ (i.e., define the appropriate map φ and then prove that it is an isomorphism from V onto $X \oplus Y$.

Problem 2. Suppose $V_0, V_1, V_2, \dots, V_n$ are vector spaces over the same field and suppose that $f_k \in \operatorname{Hom}(V_k, V_{k+1})$, for each $0 \le k < n$. We often use a diagram like the following to graphically depict such a sequence of maps:

$$V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} V_n$$

We call this an *exact sequence* if $\operatorname{im}(f_k) = \ker(f_{k+1})$ holds for all $0 \le k < n-1$.

(a) Suppose the following is an exact sequence:

$$0 \xrightarrow{\sigma_1} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \xrightarrow{\sigma_0} 0$$

Then, what kind of homomorphism is f_1? What about f_2?

(b) Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r, "\sigma_1"] & X \arrow[r, "p"] \arrow[d, "g"] & V \arrow[r, "q"] \arrow[d, "f"] & Y \arrow[r, "\sigma_0"] \arrow[d, "h"] & 0 \\
0 \arrow[r, "\sigma_1"] & X \arrow[r, "\texttt{inl}"] & X \oplus Y \arrow[r, "\texttt{snd}"] & Y \arrow[r, "\sigma_0"] & 0
\end{tikzcd}$$

Here `inl` denotes the left-inclusion map,

$$\texttt{inl} : X \hookrightarrow X \oplus Y, \quad \texttt{inl} : x \longmapsto x + 0,$$

and `snd` denotes the second-projection map,

$$\texttt{snd} : X \oplus Y \twoheadrightarrow Y, \quad \texttt{snd} : x + y \longmapsto y.$$

To complete this problem, first read the brief Wikipedia page on the *Short Five Lemma*: `http://en.wikipedia.org/wiki/Short_five_lemma`

Then, say how this lemma could be used in Problem 1 to prove that φ is an isomorphism. In particular, give appropriate definitions for each of the maps p, q, f, g, and h and say what else needs to be established about the diagram in order to apply the lemma. [*Hint:* When defining the maps, choose from among the following: σ_1, α, φ, σ_1.]

Problem 3.

Problem 4.