## Homework 4

## Michael Laughlin and Taylor Short

The label "Problem" is used for required problems. "Exercise" is for suggested exercises.

Problem 1 (Golan 199). Let V be a vector space of finite dimension n > 0 over  $\mathbb{R}$  and, for each positive integer i, let  $U_i$  be a proper subspace of V. Show that  $V \neq \bigcup_{i=1}^{\infty} U_i$ .

Problem 2 (Golan 210). Let V be a vector space over a field F and assume V is not finitely generated. Show that there exists an infinite sequence  $W_1, W_2, \ldots$  of proper subspaces of V satisfying  $\bigcup_{i=1}^{\infty} W_i = V$ .

Exercise (Golan 239). Let V and W be a vector space over  $\mathbb{Q}$  and let  $\alpha: V \to W$  be a function satisfying  $\alpha(x+y) = \alpha(x) + \alpha(y)$  for all  $x, y \in V$ . Is  $\alpha$  necessarily a linear transformation?

Exercise (Golan 240). Let  $\alpha : \mathbb{R} \to \mathbb{R}$  be a continuous function satisfying  $\alpha(x+y) = \alpha(x) + \alpha(y)$  for all  $a, b \in \mathbb{R}$ . Show that  $\alpha$  is a linear transformation.

Problem 3 (Golan 241). Let  $W_1$  and  $W_2$  be subspaces of a vector space V over a field F and assume we have linear transformations  $\alpha_1: W_1 \to V$  and  $\alpha_2: W_2 \to V$  satisfying the condition that  $\alpha_1(v) = \alpha_2(v)$  for all  $v \in W_1 \cap W_2$ . Find a linear transformation  $\theta: W_1 + W_2 \to V$  such that the restriction of  $\theta$  to  $W_i$  equals  $\alpha_i$  (i = 1, 2), or show why no such linear transformation exists.

Problem 4 (Golan 251). Let V, W and Y be vector spaces finitely generated over a field F and let  $\alpha \in \text{Hom}(V, W)$ . Let  $\text{ann}(\alpha)$  denote the set of those  $\beta \in \text{Hom}(W, Y)$  satisfying the condition that  $\beta \alpha$  is the 0-transformation. That is,

$$\operatorname{ann}(\alpha) = \{ \beta \in \operatorname{Hom}(W, Y) \mid \forall v \in V \ \beta \alpha(v) = 0_Y \}.$$

Prove that  $ann(\alpha)$  is a subspace of Hom(W,Y) and compute its dimension.

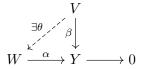
Exercise (Golan 253). Let V and W be vector spaces over a field F and assume that there are subspaces  $V_1$  and  $V_2$  of V, both of positive dimension, satisfying  $V = V_1 \bigoplus V_2$ . For i = 1, 2, let

 $U_i = \{\alpha \in \text{Hom}(V, W) \mid V_i \subseteq \text{ker}(\alpha)\}$ . Show that  $\{U_1, U_2\}$  is an independent set of subspaces of Hom(V, W). Is it necessarily true that  $\text{Hom}(V, W) = U_1 \bigoplus U_2$ ?

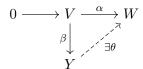
Problem 5 (Golan 256). Let V and W be vector spaces over a field F. Define a function  $\varphi: \operatorname{Hom}(V,W) \to \operatorname{Hom}(V\times W,V\times W)$  by setting  $\varphi(\alpha): \begin{bmatrix} v \\ w \end{bmatrix} \mapsto \begin{bmatrix} 0_V \\ \alpha(v) \end{bmatrix}$ . Is  $\varphi$  a linear transformation of vector spaces over F? Is it a monomorphism?

Problem 6 (Golan 293 & 294). Let V, W and Y be vector spaces over a field F, and let  $\beta \in \operatorname{Hom}(V,Y)$ . Prove the following:

1. If  $\alpha \in \text{Hom}(W,Y)$  is an epimorphism, then there exists  $\theta \in \text{Hom}(V,W)$  such that  $\beta = \alpha\theta$ .

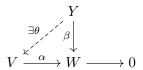


2. If  $\alpha \in \text{Hom}(V, W)$  is a monomorphism, then there exists  $\theta \in \text{Hom}(W, Y)$  such that  $\beta = \theta \alpha$ .

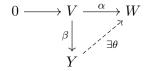


Note to students: Here is an alternative statement of the problem, with naming conventions that agree with Golan. You may solve whichever version you prefer.

1. If  $\alpha \in \text{Hom}(V, W)$  is an epimorphism, then for every  $\beta \in \text{Hom}(Y, W)$  there exists  $\theta \in \text{Hom}(Y, V)$  such that  $\beta = \alpha \theta$ .



2. If  $\alpha \in \text{Hom}(V, W)$  is a monomorphism, then for every  $\beta \in \text{Hom}(V, Y)$  there exists  $\theta \in \text{Hom}(W, Y)$  such that  $\beta = \theta \alpha$ .



Problem 7 (Golan 296).<sup>1</sup> Let V, W be vector spaces over a field F, let  $\alpha \in \text{Hom}(V, W)$ , and let D be a nonempty linearly independent subset of  $\text{im}(\alpha)$ . Show that there exists a basis B of V satisfying  $\{\alpha(v) \mid v \in B\} = D$ .

Problem 8 (Golan 306). Let V, W and Y be vector spaces over a field F. Let  $\{\alpha_1, \ldots, \alpha_n\}$  be a finite subset of  $\operatorname{Hom}(V, W)$  and let  $\beta \in \operatorname{Hom}(V, Y)$  be a linear transformation satisfying  $\bigcap_{i=1}^n \ker(\alpha_i) \subseteq \ker(\beta)$ . Show that there exist linear transformations  $\gamma_1, \ldots, \gamma_n$  in  $\operatorname{Hom}(W, Y)$  satisfying  $\beta = \sum_{i=1}^n \gamma_i \alpha_i$ .

Problem 9 (Golan 266). Let A and B be nonempty sets. Let V be the collection of all subsets of A and let W be the collection of all subsets of B, both of which are vector spaces over GF(2). Any function  $f: A \to B$  defines a function  $\alpha_f: W \to V$  by setting  $\alpha_f: D \mapsto \{a \in A: f(a) \in D\}$ . Show that each such function  $\alpha_f$  defines a linear transformation, and find its kernel.

The claim in this problem seems incorrect to me. If you agree, give a counter-example, then modify the claim so it is correct and prove it. If you disagree, and you believe the claim is correct, then prove it as given.