

## Homework 4

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The label “Problem” is used for required problems. “Exercise” is for suggested exercises.

*Problem 1* (Golan 199). Let  $V$  be a vector space of finite dimension  $n > 0$  over  $\mathbb{R}$  and, for each positive integer  $i$ , let  $U_i$  be a proper subspace of  $V$ . Show that  $V \neq \bigcup_{i=1}^{\infty} U_i$ .

*Problem 2* (Golan 210). Let  $V$  be a vector space over a field  $F$  and assume  $V$  is not finitely generated. Show that there exists an infinite sequence  $W_1, W_2, \dots$  of proper subspaces of  $V$  satisfying  $\bigcup_{i=1}^{\infty} W_i = V$ .

*Exercise* (Golan 239). Let  $V$  and  $W$  be a vector space over  $\mathbb{Q}$  and let  $\alpha : V \rightarrow W$  be a function satisfying  $\alpha(x + y) = \alpha(x) + \alpha(y)$  for all  $x, y \in V$ . Is  $\alpha$  necessarily a linear transformation?

*Exercise* (Golan 240). Let  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function satisfying  $\alpha(x+y) = \alpha(x) + \alpha(y)$  for all  $a, b \in \mathbb{R}$ . Show that  $\alpha$  is a linear transformation.

*Problem 3* (Golan 241). Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$  over a field  $F$  and assume we have linear transformations  $\alpha_1 : W_1 \rightarrow V$  and  $\alpha_2 : W_2 \rightarrow V$  satisfying the condition that  $\alpha_1(v) = \alpha_2(v)$  for all  $v \in W_1 \cap W_2$ . Find a linear transformation  $\theta : W_1 + W_2 \rightarrow V$  such that the restriction of  $\theta$  to  $W_i$  equals  $\alpha_i$  ( $i = 1, 2$ ), or show why no such linear transformation exists.

*Problem 4* (Golan 251). Let  $V$ ,  $W$  and  $Y$  be vector spaces finitely generated over a field  $F$  and let  $\alpha \in \text{Hom}(V, W)$ . Let  $\text{ann}(\alpha)$  denote the set of those  $\beta \in \text{Hom}(W, Y)$  satisfying the condition that  $\beta\alpha$  is the 0-transformation. That is,

$$\text{ann}(\alpha) = \{\beta \in \text{Hom}(W, Y) \mid \forall v \in V \beta\alpha(v) = 0_Y\}.$$

Prove that  $\text{ann}(\alpha)$  is a subspace of  $\text{Hom}(W, Y)$  and compute its dimension.

*Exercise* (Golan 253). Let  $V$  and  $W$  be vector spaces over a field  $F$  and assume that there are subspaces  $V_1$  and  $V_2$  of  $V$ , both of positive dimension, satisfying  $V = V_1 \oplus V_2$ . For  $i = 1, 2$ , let

$U_i = \{\alpha \in \text{Hom}(V, W) \mid V_i \subseteq \ker(\alpha)\}$ . Show that  $\{U_1, U_2\}$  is an independent set of subspaces of  $\text{Hom}(V, W)$ . Is it necessarily true that  $\text{Hom}(V, W) = U_1 \oplus U_2$ ?

*Problem 5* (Golan 256). Let  $V$  and  $W$  be vector spaces over a field  $F$ . Define a function  $\varphi : \text{Hom}(V, W) \rightarrow \text{Hom}(V \times W, V \times W)$  by setting  $\varphi(\alpha) : \begin{bmatrix} v \\ w \end{bmatrix} \mapsto \begin{bmatrix} 0_V \\ \alpha(v) \end{bmatrix}$ . Is  $\varphi$  a linear transformation of vector spaces over  $F$ ? Is it a monomorphism?

*Problem 6* (Golan 293 & 294). Let  $V$ ,  $W$  and  $Y$  be vector spaces over a field  $F$ , and let  $\beta \in \text{Hom}(V, Y)$ . Prove the following:

1. If  $\alpha \in \text{Hom}(W, Y)$  is an epimorphism, then there exists  $\theta \in \text{Hom}(V, W)$  such that  $\beta = \alpha\theta$ .

$$\begin{array}{ccccc} 0 & \longrightarrow & V & \xrightarrow{\alpha} & W \\ & & \downarrow \beta & \nearrow \exists \theta & \\ & & Y & & \end{array}$$

2. If  $\alpha \in \text{Hom}(V, W)$  is a monomorphism, then there exists  $\theta \in \text{Hom}(W, Y)$  such that  $\beta = \theta\alpha$ .

$$\begin{array}{ccccc} & & V & & \\ & \nwarrow \exists \theta & \downarrow \beta & \nearrow & \\ W & \xrightarrow{\alpha} & Y & \longrightarrow & 0 \end{array}$$

*Problem 7* (Golan 296).<sup>1</sup> Let  $V$ ,  $W$  be vector spaces over a field  $F$ , let  $\alpha \in \text{Hom}(V, W)$ , and let  $D$  be a nonempty linearly independent subset of  $\text{im}(\alpha)$ . Show that there exists a basis  $B$  of  $V$  satisfying  $\{\alpha(v) \mid v \in B\} = D$ .

*Problem 8* (Golan 306). Let  $V$ ,  $W$  and  $Y$  be vector spaces over a field  $F$ . Let  $\{\alpha_1, \dots, \alpha_n\}$  be a finite subset of  $\text{Hom}(V, W)$  and let  $\beta \in \text{Hom}(V, Y)$  be a linear transformation satisfying  $\bigcap_{i=1}^n \ker(\alpha_i) \subseteq \ker(\beta)$ . Show that there exist linear transformations  $\gamma_1, \dots, \gamma_n$  in  $\text{Hom}(W, Y)$  satisfying  $\beta = \sum_{i=1}^n \gamma_i \alpha_i$ .

*Problem 9* (Golan 266). Let  $A$  and  $B$  be nonempty sets. Let  $V$  be the collection of all subsets of  $A$  and let  $W$  be the collection of all subsets of  $B$ , both of which are vector spaces over  $\text{GF}(2)$ . Any function  $f : A \rightarrow B$  defines a function  $\alpha_f : W \rightarrow V$  by setting  $\alpha_f : D \mapsto \{a \in A : f(a) \in D\}$ . Show that each such function  $\alpha_f$  defines a linear transformation, and find its kernel.

<sup>1</sup> The claim in this problem seems incorrect to me. If you agree, give a counter-example, then modify the claim so it is correct and prove it. If you disagree, and you believe the claim is correct, then prove it as given.