Homework 1

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Problem 1 (Golan 12). For a field $\mathbb{F} = \langle F, +, \cdot, -, 0, 1\rangle$, show that the function $a \mapsto a^{-1}$ is a permutation of the set $F \setminus \{0_F\}$.

Solution. In order to show that $f : F \setminus \{0_F\} \to F \setminus \{0_F\}$, $f(a) = a^{-1}$ is a permutation, we prove that f is a well-defined bijection.

Well-defined: For each $a \in F\setminus\{0_F\}$, there is an element $x \in F\setminus\{0_F\}$ such that $ax = xa = 1$ because F is a field. In other words, $x = a^{-1}$. Now suppose there exists $y \in F \setminus \{0_F\}$ such that $ay = ya = 1$. Then $y = 1y = xay = x1 = xax = 1x = x$. Therefore inverses in F are unique.

Surjective: For any $b \in F \setminus \{0_F\}$, we have $f(b) = b^{-1} \in F \setminus \{0_F\}$ because F is a field. Observe that $bb^{-1} = b^{-1}b = 1$. Therefore, b is the unique inverse of b^{-1}. So $f(b^{-1}) = b$.

Injective: For any $a, b \in F \setminus \{0_F\}$, suppose $f(a) = c = f(b)$ for some $c \in F \setminus \{0_F\}$. Then $ac = 1 = bc$, so $a = a1 = acc^{-1} = 1c^{-1} = bcc^{-1} = b1 = b$.

[Note: "If $c \neq 0$, then $ac = bc$ implies $a = b$" is a general property of integral domains. You can alternatively use the fact that every field is an integral domain to prove that f is well-defined and injective.]

Problem 2 (Golan 16). Let z_1, z_2, and z_3 be complex numbers satisfying $|z_i| = 1$ for $i = 1, 2, 3$. Show that $|z_1z_2 + z_1z_3 + z_2z_3| = |z_1 + z_2 + z_3|$.

Solution. For each $j \in [3]$, since $|z_i| = 1$, note that for some real θ_j,

$$\frac{1}{z_j} = \frac{1}{e^{i\theta_j}} = e^{-i\theta_j} = \overline{z_j}.$$

Therefore,

$$\begin{aligned} |z_1z_2 + z_1z_3 + z_2z_3| &= |z_1| \cdot |z_2| \cdot |z_3| \cdot \left|\frac{1}{z_3} + \frac{1}{z_2} + \frac{1}{z_1}\right| \\ &= 1 \cdot 1 \cdot 1 \cdot |\overline{z_3} + \overline{z_2} + \overline{z_1}| \\ &= |\overline{z_1 + z_2 + z_3}| \\ &= |z_1 + z_2 + z_3|. \end{aligned}$$

Problem 3 (Golan 22 *Abel's inequality*). Let $z_1, \ldots, z_n$ be a list of complex numbers and, for each $1 \leq k \leq n$, let $s_k = \sum_{i=1}^{k} z_i$. For real numbers $a_1, \ldots, a_n$ satisfying $a_1 \geq a_2 \geq \cdots \geq$

$a_n \geq 0$, show that

$$\left|\sum_{i=1}^{n} a_i z_i\right| \leq a_1 \left(\max_{1\leq k\leq n} |s_k|\right). \tag{1}$$

Solution. Define $s_0 := 0$. Observe $a_i z_i = a_i s_i - a_i s_{i-1}$ for $1 \leq i \leq n$. Therefore

$$\begin{aligned}
\left|\sum_{i=1}^{n} a_i z_i\right| &= \left|\sum_{i=1}^{n} a_i s_i - a_i s_{i-1}\right| \\
&= \left|\sum_{i=1}^{n-1} s_i(a_i - a_{i+1}) + s_n a_n\right| \\
&\leq \sum_{i=1}^{n-1} |s_i(a_i - a_{i+1})| + |s_n a_n| && \text{by the Triangle Inequality} \\
&= \sum_{i=1}^{n-1} (a_i - a_{i+1})|s_i| + a_n\,|s_n| \\
&\leq \sum_{i=1}^{n-1}(a_i - a_{i+1}) \max_{1\leq k\leq n} |s_k| + a_n \max_{1\leq k\leq n} |s_k| \\
&= \left(\sum_{i=1}^{n-1}(a_i - a_{i+1}) + a_n\right) \max_{1\leq k\leq n} |s_k| \\
&= a_1 \max_{1\leq k\leq n} |s_k|. && \text{by telescoping sums}
\end{aligned}$$

Problem 4 (Golan 24). If p is a prime positive integer, find all subfields of $GF(p)$.

Solution. Since every field is a unital ring, every subfield of $GF(p)$ contains the unit 1. The order of 1 is p, since $\underbrace{1+1+\cdots+1}_{p} = 0$, but $\underbrace{1+1+\cdots+1}_{k} \neq 0$ for positive $k < p$.

Suppose $\underbrace{1+1+\cdots+1}_{a} = \underbrace{1+1+\cdots+1}_{b}$ for some positive $a, b < p$. Then $a = b$. Thus, any subfield with 1 contains at least p distinct elements. Since $|GF(p)| = p$, it is itself the only subfield.

Problem 5. Write down the definition of a *module* as a (universal) algebra, $\mathbf{M} = \langle M, F\rangle$. That is, describe the set F of operations and give the conditions that they should satisfy in order for $\mathbf{M}$ to agree with the classical definition of a module over a ring.

[*Hint:* Let $\mathbf{R} = \langle R, +, \cdot, -, 0, 1\rangle$ be a ring and, for each $r \in R$, define a scalar multiply operation $f_r \in F$.]

Solution. Let $\mathbf{A} = \langle V, +, -, 0\rangle$ be an Abelian group. Let $\mathbf{R}$ be the unital ring

$$\mathbf{R} = \langle R, +, \cdot, -, 0, 1\rangle$$

Define module $\mathbf{V}$ as follows:

$$\mathbf{V} = \langle V, +, -, 0, \{f_r : r \in R\}\rangle$$

where the addition, additive inverse, and zero are defined on V as they are in the Abelian group $\mathbf{A}$. Each f_r is a function $f_r : V \to V$ so that for any $r, r_1, r_2 \in R$ and any $v, v_1, v_2 \in V$ each of the following is satisfied:

- $f_r(v_1 + v_2) = f_r(v_1) + f_r(v_2)$
- $f_{r_1+r_2}(v) = f_{r_1}(v) + f_{r_2}(v)$
- $f_{r_1}(f_{r_2}(v)) = f_{r_1 r_2}(v)$
- $f_1(v) = v$

Problem 6. Let $\mathbf{R} = \langle R, +, -, \cdot, 0, 1\rangle$ be a ring.

1. Define *left ideal* of $\mathbf{R}$.
2. Let $\mathscr{A} = \{A_i : i \in \mathscr{I}\}$ be a family of left ideals of $\mathbf{R}$. Prove that $\bigcap \mathscr{A}$ is a left ideal.

Solution.

1. A left ideal I of $\mathbf{R}$ is a subset of R which forms a subgroup under addition and for any $a \in I$ and $r \in R$, we have $ra \in I$.
2. Clearly $\bigcap \mathscr{A}$ is a subset of R since each $A_i \in \mathscr{A}$ is a subset of R. Let $a, b \in \bigcap \mathscr{A}$ and $r \in R$. By definition of intersection, $a, b \in A_i$ for all $A_i \in \mathscr{A}$. Therefore $ra, a + b \in A_i$ for all $A_i \in \mathscr{A}$ because each A_i is a left ideal. But this implies that $ra, a + b \in \bigcap \mathscr{A}$, proving that $\bigcap \mathscr{A}$ is a left ideal.

Problem 7. Let $\mathbf{R}$ be a ring and fix $a, b \in R$. Prove that if $1 - ba$ is left invertible, then $1 - ab$ is also left invertible. What is the inverse?

[*Hint:* Consider the left ideal $R(1 - ab)$. It contains the left ideal $Rb(1 - ab) = Rb$ and therefore contains 1. Verify these statements, then try to compute the inverse of $1 - ab$. (Ask for more hints as needed.)]

Solution. We can show existence as follows: Observe $Rb(1 - ab) \subseteq R(1 - ab)$ since $Rb \subseteq R$. Because $(1 - ba)$ is left invertible, $R = R(1 - ba)$ and

$$Rb = R(1 - ba)b = R(b - bab) = Rb(1 - ab).$$

Therefore $Rb \subseteq R(1 - ab)$. Observe $ab \in Rb$ and $(1 - ab) \in R(1 - ab)$ so

$$1 = (1 - ab) + ab \in R(1 - ab),$$

which implies that $(1 - ab)$ is invertible.

To explicitly find the left inverse of $(1 - ab)$, let $c \in R$ be such that $c(1 - ba) = 1$. Then

$$\begin{aligned} 1 &= 1 - ab + ab \\ &= 1 - ab + a1b \\ &= 1 - ab + ac(1 - ba)b \\ &= (1 - ab) + ac(b - bab) \\ &= 1(1 - ab) + acb(1 - ab) \\ &= (1 + acb)(1 - ab) \end{aligned}$$

Therefore, the left inverse of $(1 - ab)$ is $(1 + acb)$, where c is the left inverse of $(1 - ba)$.