Homework 4

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Problem 1 (Golan 199). Let V be a vector space of finite dimension $n > 0$ over $\mathbb{R}$ and, for each positive integer i, let U_i be a proper subspace of V. Show that $V \neq \bigcup_{i=1}^{\infty} U_i$.

Problem 2 (Golan 210). Let V be a vector space over a field F and assume V is not finitely generated. Show that there exists an infinite sequence $W_1, W_2, \ldots$ of proper subspaces of V satisfying $\bigcup_{i=1}^{\infty} W_i = V$.

Problem 3 (Golan 239). Let V and W be a vector space over $\mathbb{Q}$ and let $\alpha : V \to W$ be a function satisfying $\alpha(x + y) = \alpha(x) + \alpha(y)$ for all $x, y \in V$. Is α necessarily a linear transformation?

Problem 4 (Golan 240). Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $\alpha(x + y) = \alpha(x) + \alpha(y)$ for all $a, b \in \mathbb{R}$. Show that α is a linear transformation.

Problem 5 (Golan 241). Let W_1 and W_2 be subspaces of a vector space V over a field F and assume we have linear transformations $\alpha_1 : W_1 \to V$ and $\alpha_2 : W_2 \to V$ satisfying the coondition that $\alpha_1(v) = \alpha_2(v)$ for all $v \in W_1 \cap W_2$. Find a linear transformation $\theta : W_1 + W_2 \to V$ such that the restriction of θ to W_i equals α_i $(i = 1, 2)$, or show why no such linear transformation exists.

Problem 6 (Golan 251). Let V, W and Y be vector spaces finitely generated over a field F and let $\alpha \in \mathrm{Hom}(V, W)$. Let $\mathrm{ann}(\alpha)$ denote the set of those $\beta \in \mathrm{Hom}(W, Y)$ satisfying the condition that $\beta\alpha$ is the 0-transformation. That is,

$$\mathrm{ann}(\alpha) = \{\beta \in \mathrm{Hom}(W, Y) \mid \forall v \in V \ \beta\alpha(v) = 0_Y\}.$$

Prove that $\mathrm{ann}(\alpha)$ is a subspace of $\mathrm{Hom}(W, Y)$ and compute its dimension.

Problem 7 (Golan 253). Let V and W be vector spaces over a field F and assume that there are subspaces V_1 and V_2 of V, both of positive dimension, satisfying $V = V_1 \bigoplus V_2$. For

$i = 1, 2$, let $U_i = \{\alpha \in \mathrm{Hom}(V,W) \mid V_i \subseteq \ker(\alpha)\}$. Show that $\{U_1, U_2\}$ is an independed set of subspaces of $\mathrm{Hom}(V,W)$. Is it necessarily true that $\mathrm{Hom}(V,W) = U_1 \bigoplus U_2$?

Problem 8 (Golan 256). Let V and W be vector spaces over a field F. Define a function $\varphi : \mathrm{Hom}(V,W) \to \mathrm{Hom}(V \times W, V \times W)$ by setting $\varphi(\alpha) : \begin{bmatrix} v \\ w \end{bmatrix} \mapsto \begin{bmatrix} 0_V \\ \alpha(v) \end{bmatrix}$. Is φ a linear transformation of vector spaces over F? Is it a monomorphism?

Problem 9 (Golan 293 & 294). Let V, W and Y be vector spaces over a field F, let $\alpha \in \mathrm{Hom}(V,W)$, and let $\beta \in \mathrm{Hom}(V,Y)$. Prove the following:
1. If α is an epimorphism, then there exists $\theta \in \mathrm{Hom}(W,Y)$ such that $\beta = \alpha\theta$.
2. If α is a monomorphism, then there exists $\theta \in \mathrm{Hom}(W,Y)$ such that $\beta = \theta\alpha$.

Problem 10 (Golan 296).[1] Let V, W be vector spaces over a field F, let $\alpha \in \mathrm{Hom}(V,W)$, and let D be a nonempty linearly independent subset of $\mathrm{im}(\alpha)$. Show that there exists a basis B of V satisfying $\{\alpha(v) \mid v \in B\} = D$.

Problem 11 (Golan 306). Let V, W and Y be vector spaces over a field F. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a finite subset of $\mathrm{Hom}(V,W)$ and let $\beta \in \mathrm{Hom}(V,Y)$ be a linear transformation satisfying $\bigcap_{i=1}^n \ker(\alpha_i) \subseteq \ker(\beta)$. Show that there exist linear transformations $\gamma_1, \ldots, \gamma_n$ in $\mathrm{Hom}(W,Y)$ satisfying $\beta = \sum_{i=1}^n \gamma_i \alpha_i$.

Problem 12 (Golan 307). Let V be a vector space over a field F and let W be a subspace of V. For each $v \in V$, let $v + W = \{v + w \mid w \in W\}$. Let $V/W = \{v + W \mid v \in V\}$ be the collection of all sets of the form $v + W$, and define operations of addition and scalar multiplication on V/W by setting $(v + W) + (v' + W) = (v + v') + W$ and $c(v + W) = (cv) + W$ for all $v, v' \in V$ and $c \in F$. Show that
1. $v + W = v' + W$ if and only if $v - v' \in W$;
2. V/W, with the given operations, is a vector space over F;
3. The function $v \mapsto v + W$ is an epimorphism from V to W, the kernel of which equals W;
4. Every complement of W in V is isomorphic to V/W;
5. If $(v + W) \cap (v' + W) \neq \emptyset$, then $v + W = v' + W$.

The space V/W is called the *factor space* of V by W.

[1] I believe the statement of this problem is incorrect. If you agree, give a counter-example and explain how to fix the statement so that it is correct. If you disagree, and you believe the statement is correct, then prove it as stated.

Problem 13 (Golan 325). Let $\alpha \in \mathrm{Aut}(\mathbb{R}^2)$ be defined by $\alpha : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} -b \\ a \end{bmatrix}$. Show that $\mathbb{R}\{\alpha, \sigma_1\}$ is a unital subalgebra of $\mathrm{End}(\mathbb{R}^2)$. Show that it is proper by giving an example of an endomorphism of $\mathbb{R}^2$ not in this subalgebra.

Problem 14 (Golan 326). Let V be the space of all real-valued functions on the interval $[-1, 1]$ which are infinitely differentiable, and let δ be the endomorphism of V which assigns to each function f its derivative. Find the kernel and image of δ.

Problem 15 (Golan 338). Let V be a vector space over a field F which is not finitely generated, and let $\sigma_0 \neq \alpha \in \mathrm{End}(V)$. Set $A = \{\beta \in \mathrm{End}(V) \mid \alpha\beta = \sigma_1\}$. Show that if A has more than one element then it is infinite.

Problem 16 (Golan 340). Let V be a vector space over a field F satisfying the condition that $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathrm{End}(V)$. Show that $\dim(V) = 1$.

Problem 17 (Golan 354). Let V be a vector space over a field F and let $\alpha \in \mathrm{Aut}(V)$. Let $W_1, \ldots, W_k$ be subspaces of V satisfying $V = \bigoplus_{i=1}^k W_i$. For each $1 \leq i \leq k$, let $Y_i = \{\alpha(w) \mid w \in W_i\}$. Is $V = \bigoplus_{i=1}^k Y_i$?