Homework 4

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The label "Problem" is used for required problems. "Exercise" is for suggested exercises.

Problem 1 (Golan 199). Let V be a vector space of finite dimension $n > 0$ over $\mathbb{R}$ and, for each positive integer i, let U_i be a proper subspace of V. Show that $V \neq \bigcup_{i=1}^{\infty} U_i$.

Problem 2 (Golan 210). Let V be a vector space over a field F and assume V is not finitely generated. Show that there exists an infinite sequence $W_1, W_2, \ldots$ of proper subspaces of V satisfying $\bigcup_{i=1}^{\infty} W_i = V$.

Exercise (Golan 239). Let V and W be a vector space over $\mathbb{Q}$ and let $\alpha : V \to W$ be a function satisfying $\alpha(x + y) = \alpha(x) + \alpha(y)$ for all $x, y \in V$. Is α necessarily a linear transformation?

Exercise (Golan 240). Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $\alpha(x+y) = \alpha(x)+\alpha(y)$ for all $a, b \in \mathbb{R}$. Show that α is a linear transformation.

Problem 3 (Golan 241). Let W_1 and W_2 be subspaces of a vector space V over a field F and assume we have linear transformations $\alpha_1 : W_1 \to V$ and $\alpha_2 : W_2 \to V$ satisfying the condition that $\alpha_1(v) = \alpha_2(v)$ for all $v \in W_1 \cap W_2$. Find a linear transformation $\theta : W_1 + W_2 \to V$ such that the restriction of θ to W_i equals α_i $(i = 1, 2)$, or show why no such linear transformation exists.

Problem 4 (Golan 251). Let V, W and Y be vector spaces finitely generated over a field F and let $\alpha \in \mathrm{Hom}(V, W)$. Let $\mathrm{ann}(\alpha)$ denote the set of those $\beta \in \mathrm{Hom}(W, Y)$ satisfying the condition that $\beta\alpha$ is the 0-transformation. That is,

$$\mathrm{ann}(\alpha) = \{\beta \in \mathrm{Hom}(W, Y) \mid \forall v \in V \;\; \beta\alpha(v) = 0_Y\}.$$

Prove that $\mathrm{ann}(\alpha)$ is a subspace of $\mathrm{Hom}(W, Y)$ and compute its dimension.

Exercise (Golan 253). Let V and W be vector spaces over a field F and assume that there are subspaces V_1 and V_2 of V, both of positive dimension, satisfying $V = V_1 \bigoplus V_2$. For $i = 1, 2$, let

$U_i = \{\alpha \in \mathrm{Hom}(V, W) \mid V_i \subseteq \ker(\alpha)\}$. Show that $\{U_1, U_2\}$ is an independent set of subspaces of $\mathrm{Hom}(V, W)$. Is it necessarily true that $\mathrm{Hom}(V, W) = U_1 \bigoplus U_2$?

Problem 5 (Golan 256). Let V and W be vector spaces over a field F. Define a function $\varphi : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V \times W, V \times W)$ by setting $\varphi(\alpha) : \begin{bmatrix} v \\ w \end{bmatrix} \mapsto \begin{bmatrix} 0_V \\ \alpha(v) \end{bmatrix}$. Is φ a linear transformation of vector spaces over F? Is it a monomorphism?

Problem 6 (Golan 293 & 294). Let V, W and Y be vector spaces over a field F, and let $\beta \in \mathrm{Hom}(V, Y)$. Prove the following:

1. If $\alpha \in \mathrm{Hom}(W, Y)$ is an epimorphism, then there exists $\theta \in \mathrm{Hom}(V, W)$ such that $\beta = \alpha\theta$.

$$\begin{tikzcd} & V \arrow[dl, dashed, "\exists\theta"'] \arrow[d, "\beta"] & \\ W \arrow[r, "\alpha"] & Y \arrow[r] & 0 \end{tikzcd}$$

2. If $\alpha \in \mathrm{Hom}(V, W)$ is a monomorphism, then there exists $\theta \in \mathrm{Hom}(W, Y)$ such that $\beta = \theta\alpha$.

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "\alpha"] \arrow[d, "\beta"'] & W \arrow[dl, dashed, "\exists\theta"] \\ & Y & \end{tikzcd}$$

Note to students: Here is an alternative statement of the problem, with naming conventions that agree with Golan. You may solve whichever version you prefer.

1. If $\alpha \in \mathrm{Hom}(V, W)$ is an epimorphism, then for every $\beta \in \mathrm{Hom}(Y, W)$ there exists $\theta \in \mathrm{Hom}(Y, V)$ such that $\beta = \alpha\theta$.

$$\begin{tikzcd} & Y \arrow[dl, dashed, "\exists\theta"'] \arrow[d, "\beta"] & \\ V \arrow[r, "\alpha"] & W \arrow[r] & 0 \end{tikzcd}$$

2. If $\alpha \in \mathrm{Hom}(V, W)$ is a monomorphism, then for every $\beta \in \mathrm{Hom}(V, Y)$ there exists $\theta \in \mathrm{Hom}(W, Y)$ such that $\beta = \theta\alpha$.

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "\alpha"] \arrow[d, "\beta"'] & W \arrow[dl, dashed, "\exists\theta"] \\ & Y & \end{tikzcd}$$

Problem 7 (Golan 296).[1] Let V, W be vector spaces over a field F, let $\alpha \in \mathrm{Hom}(V, W)$, and let D be a nonempty linearly independent subset of $\mathrm{im}(\alpha)$. Show that there exists a basis B of V satisfying $\{\alpha(v) \mid v \in B\} = D$.

Problem 8 (Golan 306). Let V, W and Y be vector spaces over a field F. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a finite subset of $\mathrm{Hom}(V, W)$ and let $\beta \in \mathrm{Hom}(V, Y)$ be a linear transformation satisfying $\bigcap_{i=1}^n \ker(\alpha_i) \subseteq \ker(\beta)$. Show that there exist linear transformations $\gamma_1, \ldots, \gamma_n$ in $\mathrm{Hom}(W, Y)$ satisfying $\beta = \sum_{i=1}^n \gamma_i \alpha_i$.

Problem 9 (Golan 266). Let A and B be nonempty sets. Let V be the collection of all subsets of A and let W be the collection of all subsets of B, both of which are vector spaces over $\mathrm{GF}(2)$. Any function $f : A \to B$ defines a function $\alpha_f : W \to V$ by setting $\alpha_f : D \mapsto \{a \in A : f(a) \in D\}$. Show that each such function α_f defines a linear transformation, and find its kernel.

[1] The claim in this problem seems incorrect to me. If you agree, give a counter-example, then modify the claim so it is correct and prove it. If you disagree, and you believe the claim is correct, then prove it as given.