Homework 4

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The label "Problem" is used for required problems. "Exercise" is for suggested exercises.

Problem 1 (Golan 199). Let V be a vector space of finite dimension $n > 0$ over $\mathbb{R}$ and, for each positive integer i, let U_i be a proper subspace of V. Show that $V \neq \bigcup_{i=1}^{\infty} U_i$.

Solution. We proceed by induction on n. For the base case $n = 1$ the claim is obviously true, V cannot be written as a union of proper subspaces as the only subspaces are V and $\emptyset$. Assume the claim holds for vector spaces with dimension $n \geq 1$ and consider V with $\dim(V) = n + 1$. Now for $\bigcup_{i=1}^{\infty} U_i$ a union of proper subspaces of V, take $0 \neq u \in U_1$ and consider the span $\mathbb{R}\{u\}$. Since $\mathbb{R}$ is uncountable, the subspace $\mathbb{R}\{u\}$ is uncountable. Then by Proposition 5.15 in [1] there exists uncountably many, distinct complements $\{W_\alpha\}_{\alpha \in A}$ of $\mathbb{R}\{u\}$. Furthermore each W_α has dimension n, since by Grassman's Theorem

$$n + 1 = \dim(V) = \dim(\mathbb{R}\{u\}) + \dim(W_\alpha) - \dim(\mathbb{R}\{u\} \cap W_\alpha) = 1 + \dim(W_\alpha) - 0.$$

Now the collection $\{U_i\}_{i=1}^{\infty}$ of subspaces is countable and $\{W_\alpha\}_{\alpha \in A}$ is uncountable, so there exists some $W_{\alpha_0} \in \{W_\alpha\}_{\alpha \in A}$ that is distinct from all U_i. Moreover since W_{α_0} has dimension n, W_{α_0} is distinct from the U_i, and being proper the U_i can have dimension at most n, then we must have $\dim(W_{\alpha_0} \cap U_i) < n$ for all i, that is, $W_{\alpha_0} \cap U_i$ is a proper subspace of W_{α_0}. But now by the induction hypothesis,

$$W_{\alpha_0} \neq \bigcup_{i=1}^{\infty} (W_{\alpha_0} \cap U_i).$$

So there are elements of W_{α_0} a subspace of V that do not belong to $\bigcup_{i=1}^{\infty} U_i$ and hence we must have $V \neq \bigcup_{i=1}^{\infty} U_i$, finishing the proof by induction.

Problem 2 (Golan 210). Let V be a vector space over a field F and assume V is not finitely generated. Show that there exists an infinite sequence $W_1, W_2, \ldots$ of proper subspaces of V satisfying $\bigcup_{i=1}^{\infty} W_i = V$.

Solution. There exists a basis $B = \{b_1, b_2, b_3, \ldots\}$ for V. Let $W_i = F\{b_1, b_2, \ldots, b_i\}$. Then each subspace W_i is properly contained in V, and $\bigcup_{i=1}^{\infty} W_i = V$, since for every $v \in V$, v is representable as a finite linear combination of elements in B, and if b_n is the maximum indexed among these basis elements, $v \in W_n$.

Exercise (Golan 239). Let V and W be a vector space over $\mathbb{Q}$ and let $\alpha : V \to W$ be a function satisfying $\alpha(x + y) = \alpha(x) + \alpha(y)$ for all $x, y \in V$. Is α necessarily a linear transformation?

Solution. Let $c \in \mathbb{Q}$ be arbitrary. We need only verify that $\alpha(cv) = c\alpha(v)$. Suppose $c = \frac{p}{q}$ for $p, q \in \mathbb{Z}$. By supposition, for any $z \in \mathbb{Z}$, $\alpha(zv) = z\alpha(v)$. Then we have $q\alpha(cv) = q\alpha(\frac{p}{q}v) = \alpha(q\frac{p}{q}v) = \alpha(pv) = p\alpha(v)$. Hence, $\alpha(cv) = \frac{p}{q}\alpha(v) = c\alpha(v)$ and α is a linear transformation.

Exercise (Golan 240). Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $\alpha(x+y) = \alpha(x)+\alpha(y)$ for all $a, b \in \mathbb{R}$. Show that α is a linear transformation.

Solution. Let $c \in \mathbb{R}$ be arbitrary. We need only verify that $\alpha(cx) = c\alpha(x)$. Let $r \in \mathbb{Q}$. Since α is continuous, we have by the results of the previous exercise $c\alpha(x) = \lim_{r \to c} r\alpha(x) = \lim_{r \to c} \alpha(rx) = \alpha(cx)$.

Problem 3 (Golan 241). Let W_1 and W_2 be subspaces of a vector space V over a field F and assume we have linear transformations $\alpha_1 : W_1 \to V$ and $\alpha_2 : W_2 \to V$ satisfying the condition that $\alpha_1(v) = \alpha_2(v)$ for all $v \in W_1 \cap W_2$. Find a linear transformation $\theta : W_1 + W_2 \to V$ such that the restriction of θ to W_i equals α_i $(i = 1, 2)$, or show why no such linear transformation exists.

Solution. Let $\theta(v)$ be defined by $\theta(v) = \theta(w_1 + w_2) = \alpha_1(w_1) + \alpha_2(w_2)$ for all $v = w_1 + w_2 \in W_1 + W_2$. First we check that θ is well-defined. Suppose that $v = w_1 + w_2 = w_1' + w_2'$. Then we have that $w_1' - w_1 = w_2 - w_2'$ and for some $u \in V$, $u + \alpha_1(w_1) + \alpha_2(w_2) = \alpha_1(w_1') + \alpha_2(w_2')$. But then

$$
\begin{aligned}
u &= \alpha_1(w_1') - \alpha_1(w_1) + \alpha_2(w_2') - \alpha_2(w_2) \\
&= \alpha_1(w_1' - w_1) + \alpha_2(w_2' - w_2) \\
&= \alpha_2(w_2 - w_2') + \alpha_2(w_2' - w_2) \\
&= \alpha_2(0) \\
&= 0
\end{aligned}
$$

Since θ is well-defined, it suffices to check that θ is linear and satisfies the restriction property. Let $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$. Then we have, for $v_1 = w_1 + w_2$, $v_2 = w_1' + w_2'$, and $c \in F$

$$\begin{aligned}
\theta(v_1+v_2) &= \theta(w_1+w_2+w_1'+w_2') \\
&= \alpha_1(w_1+w_1')+\alpha_2(w_2+w_2') \\
&= \alpha_1(w_1)+\alpha_2(w_2)+\alpha_1(w_1')+\alpha_2(w_2') \\
&= \theta(w_1+w_2)+\theta(w_1'+w_2') \\
&= \theta(v_1)+\theta(v_2)
\end{aligned}$$

$$\begin{aligned}
\theta(cv_1) &= \theta(c(w_1+w_2)) \\
&= \theta(cw_1+cw_2) \\
&= \alpha_1(cw_1)+\alpha_2(cw_2) \\
&= c(\alpha_1(w_1)+\alpha_2(w_2)) \\
&= c\theta(v_1)
\end{aligned}$$

$$\begin{aligned}
\theta(w_1) &= \theta(w_1+0) \\
&= \alpha_1(w_1)+\alpha_2(0) \\
&= \alpha_1(w_1)
\end{aligned}$$

$$\begin{aligned}
\theta(w_2) &= \theta(0+w_2) \\
&= \alpha_1(0)+\alpha_2(w_2) \\
&= \alpha_2(w_2)
\end{aligned}$$

Problem 4 (Golan 251). Let V, W and Y be vector spaces finitely generated over a field F and let $\alpha \in \mathrm{Hom}(V,W)$. Let $\mathrm{ann}(\alpha)$ denote the set of those $\beta \in \mathrm{Hom}(W,Y)$ satisfying the condition that $\beta\alpha$ is the 0-transformation. That is,

$$\mathrm{ann}(\alpha) = \{\beta \in \mathrm{Hom}(W,Y) \mid \forall v \in V \ \beta\alpha(v) = 0_Y\}.$$

Prove that $\mathrm{ann}(\alpha)$ is a subspace of $\mathrm{Hom}(W,Y)$ and compute its dimension.

Solution. First we show that $\mathrm{ann}(\alpha)$ is a subspace of $\mathrm{Hom}(W,Y)$ by showing it is closed under addition and scalar multiplication. Letting $\beta,\gamma \in \mathrm{ann}(\alpha)$ and $a \in F$, then

$$(\beta+\gamma)\alpha = \beta\alpha+\gamma\alpha = 0+0$$

so $\mathrm{ann}(\alpha)$ is closed under addition. Also

$$(a\beta)\alpha = a(\beta\alpha) = a\cdot 0 = 0$$

so $\mathrm{ann}(\alpha)$ is closed under scalar multiplication and hence, a subspace of $\mathrm{Hom}(W,Y)$.

To compute the dimension of $\mathrm{ann}(\alpha)$, first recall by Proposition 8.1 $\dim(\mathrm{Hom}(W,Y)) = \dim(W)\cdot\dim(Y)$. Also since any $\beta \in \mathrm{ann}(\alpha)$ maps everything in the image of α to 0_Y, we must have $\dim(\mathrm{ann}(\alpha)) = \dim(\mathrm{Hom}(W/\mathrm{im}(\alpha),Y)$. It follows that

$$\begin{aligned}
\dim(\mathrm{ann}(\alpha)) &= \dim(\mathrm{Hom}(W/\mathrm{im}(\alpha),Y) \\
&= \dim(\mathrm{Hom}(W/\mathrm{im}(\alpha))\cdot\dim(Y) \\
&= (\dim(W)-\dim(\mathrm{im}(\alpha)))\cdot\dim(Y).
\end{aligned}$$

Exercise (Golan 253). Let V and W be vector spaces over a field F and assume that there are subspaces V_1 and V_2 of V, both of positive dimension, satisfying $V = V_1 \bigoplus V_2$. For $i = 1, 2$, let $U_i = \{\alpha \in \mathrm{Hom}(V,W) \mid V_i \subseteq \ker(\alpha)\}$. Show that $\{U_1, U_2\}$ is an independent set of subspaces of $\mathrm{Hom}(V,W)$. Is it necessarily true that $\mathrm{Hom}(V,W) = U_1 \bigoplus U_2$?

Solution. To show $\{U_1, U_2\}$ is an independent set of subspaces of $\mathrm{Hom}(V,W)$, we wish to show for any $\alpha_i \in U_i$ that $\alpha_1 + \alpha_2 = \sigma_0$ (σ_0 being the zero map) if, and only if, $\alpha_i = \sigma_0$. The direction $\alpha_i = \sigma_0$ implying $\alpha_1 + \alpha_2 = \sigma_0$ is obvious, so we only consider the remaining.

So suppose that $\alpha_1 + \alpha_2 = \sigma_0$. Then letting $v \in V$ be arbitrary, since $V = V_1 \bigoplus V_2$ we can write $v = v_1 + v_2$ uniquely where $v_i \in V_i$. Now by assumption

$$0 = (\alpha_1 + \alpha_2)(v) = \alpha_1(v_1 + v_2) + \alpha_2(v_1 + v_2) = \alpha_1(v_1) + \alpha_1(v_2) + \alpha_2(v_1) + \alpha_2(v_2)$$

where $\alpha_i(v_i) = 0$ since $V_i \subseteq \ker(\alpha_i)$. Thus we must have $0 = \alpha_1(v_2) + \alpha_2(v_1)$. As long as $\alpha_1(v_2) = -\alpha_2(v_1)$ is not possible (I don't see why this is), then we must have $0 = \alpha_1(v_2) = \alpha_2(v_1)$. Since v was arbitrary, then α_1 maps everything in V_2 to 0 and similarly for α_2 on V_1. Hence $\alpha_i = \sigma_0$ as desired.

Problem 5 (Golan 256). Let V and W be vector spaces over a field F. Define a function $\varphi : \mathrm{Hom}(V,W) \to \mathrm{Hom}(V \times W, V \times W)$ by setting $\varphi(\alpha) : \begin{bmatrix} v \\ w \end{bmatrix} \mapsto \begin{bmatrix} 0_V \\ \alpha(v) \end{bmatrix}$. Is φ a linear transformation of vector spaces over F? Is it a monomorphism?

Solution. We verify that φ is a monomorphism as follows: let $\alpha, \beta \in \mathrm{Hom}(V,W)$ and let $c \in F$. Then we have

$$\begin{aligned}
\varphi(\alpha+\beta)\begin{bmatrix} v \\ w \end{bmatrix} &= \begin{bmatrix} 0_V \\ (\alpha+\beta)(v) \end{bmatrix} \\
&= \begin{bmatrix} 0_V \\ \alpha(v) \end{bmatrix} + \begin{bmatrix} 0_V \\ \beta(v) \end{bmatrix} \\
&= \varphi(\alpha)\begin{bmatrix} v \\ w \end{bmatrix} + \varphi(\beta)\begin{bmatrix} v \\ w \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\varphi(c\alpha)\begin{bmatrix} v \\ w \end{bmatrix} &= \begin{bmatrix} 0_V \\ (c\alpha)(v) \end{bmatrix} \\
&= \begin{bmatrix} 0_V \\ c\alpha(v) \end{bmatrix} \\
&= c\begin{bmatrix} 0_V \\ \alpha(v) \end{bmatrix} \\
&= c\varphi(\alpha)\begin{bmatrix} v \\ w \end{bmatrix}
\end{aligned}$$

Now suppose that $\varphi(\alpha) = \varphi(\beta)$. Then we have $\begin{bmatrix} 0_V \\ \alpha(v) \end{bmatrix} = \varphi(\alpha)\begin{bmatrix} v \\ w \end{bmatrix} = \varphi(\beta)\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 0_V \\ \beta(v) \end{bmatrix}$. Hence, $\alpha(v) = \beta(v)$ for all $v \in V$, and $\alpha = \beta$. Therefore, φ is a monomorphism.

Problem 6 (Golan 293 & 294). Let V, W and Y be vector spaces over a field F. Prove the following:

1. If $\alpha \in \mathrm{Hom}(V, W)$ is an epimorphism, then for every $\beta \in \mathrm{Hom}(Y, W)$ there exists $\theta \in \mathrm{Hom}(Y, V)$ such that $\beta = \alpha\theta$.
2. If $\alpha \in \mathrm{Hom}(V, W)$ is a monomorphism, then for every $\beta \in \mathrm{Hom}(V, Y)$ there exists $\theta \in \mathrm{Hom}(W, Y)$ such that $\beta = \theta\alpha$.

Solution. (1) For v_1 and v_2 in V, consider v_1 and v_2 equivalent ($v_1 \sim v_2$) if $v_1 - v_2 \in \ker(\alpha)$. As the name implies, this is an equivalence relation, which is easily verified. For each equivalence class, pick exactly one representative, so that the equivalence class may be represented $[v]$ for some $v \in V$.

Take $\theta : y \mapsto v$ such that $\alpha(v) = \beta(y)$, and v is the representative of some equivalence class. This map is well-defined, for if $v_1 = \theta(y) = v_2$, then $\alpha(v_1) = \alpha(v_2)$, so $\alpha(v_1 - v_2) = 0$, $v_1 - v_2 \in \ker(\alpha)$, and v_1 and v_2 have the same equivalence class representative. Further, α is still surjective over the equivalence class representatives, for if $v_1 \sim v_2$, then $\alpha(v_1 - v_2) = 0$, and $\alpha(v_1) = \alpha(v_2)$.

We verify that θ is a homomorphism. Let $y_1, y_2 \in Y$, suppose that $\theta(y_1) = v_1$, $\theta(y_2) = v_2$, and let $c \in F$. Then we have $\beta(y_1 + y_2) = \beta(y_1) + \beta(y_2) = \alpha(v_1) + \alpha(v_2) = \alpha(v_1 + v_2)$. Hence, $\theta(y_1) + \theta(y_2) = \theta(y_1 + y_2)$ (allowing for representing vectors by their representatives). Also, $\alpha(cv_1) = c\alpha(v_1) = c\beta(y_1) = \beta(cy_1)$, so $c\theta(y_1) = cv_1 = \theta(cy_1)$. Hence, θ is the desired homomorphism.

(2) Take $\theta : \alpha(v) \mapsto \beta(v)$. This map is well-defined since α is injective. We verify that θ is a homomorphism. Let $v_1, v_2 \in V$ and let $c \in F$. Then we have $\theta(\alpha(v_1)+\alpha(v_2)) = \theta(\alpha(v_1+v_2)) = \beta(v_1 + v_2) = \beta(v_1) + \beta(v_2) = \theta(\alpha(v_1)) + \theta(\alpha(v_2))$. Also, $\theta(c\alpha(v_1)) = \theta(\alpha(cv_1)) = \beta(cv_1) = c\beta(v_1) = c\theta(\alpha(v_1))$. Hence, θ is the desired homomorphism.

Problem 7 (Golan 296).[1] Let V, W be vector spaces over a field F, let $\alpha \in \mathrm{Hom}(V, W)$, and let D be a nonempty linearly independent subset of $\mathrm{im}(\alpha)$. Show that there exists a basis B of V satisfying $\{\alpha(v) \mid v \in B\} = D$.

Solution. The claim is incorrect, consider $V = W = \mathbb{R}^2$ over $\mathbb{R}$, the identity homomorphism $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, and take $D = \{(1, 0)\}$. Then D is linearly independent, but the image of any basis for $\mathbb{R}^2$ will have at least two elements and D only has one, so we cannot have $\{\alpha(v) \mid v \in B\} = D$. We modify the claim as follows:

Claim: Let V, W be vector spaces over a field F, let $\alpha \in \mathrm{Hom}(V, W)$, and let D be a nonempty linearly independent subset of $\mathrm{im}(\alpha)$. Show that there exists a basis B of V satisfying $\{\alpha(v) \mid v \in B\} \supseteq D$.

To prove this, suppose $D = \{w_1, w_2, \ldots, w_n\}$ is the linearly independent subset in the claim. Since D is in the image of α, for each w_i there exists v_i such that $\alpha(v_i) = w_i$. We claim

[1] The claim in this problem seems incorrect to me. If you agree, give a counter-example, then modify the claim so it is correct and prove it. If you disagree, and you believe the claim is correct, then prove it as given.

the v_i are linearly independent and to show this, suppose they are not. Then there exists $c_i \in F$ such that $\sum_{i=1}^n c_i v_i = 0$. But then

$$0 = \alpha\left(\sum_{i=1}^n c_i v_i\right) = \sum_{i=1}^n c_i \alpha(v_i) = \sum_{i=1}^n c_i w_i$$

which is a contradiction to the fact that the w_i are linearly independent. Hence the v_i are linearly independent, so by Proposition 5.8 in [1] they are contained in some basis B and hence $\alpha(B) \supseteq D$.

Problem 8 (Golan 306). Let V, W and Y be vector spaces over a field F. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a finite subset of $\mathrm{Hom}(V, W)$ and let $\beta \in \mathrm{Hom}(V, Y)$ be a linear transformation satisfying $\bigcap_{i=1}^n \ker(\alpha_i) \subseteq \ker(\beta)$. Show that there exist linear transformations $\gamma_1, \ldots, \gamma_n$ in $\mathrm{Hom}(W, Y)$ satisfying $\beta = \sum_{i=1}^n \gamma_i \alpha_i$.

Solution. We shall use the facts that a homomorphism of spaces is uniquely determined by what it does to the basis elements and that a homomorphism of spaces always maps basis elements to basis elements (or to zero).

First note that since $\bigcap_{i=1}^n \ker(\alpha_i) \subseteq \ker(\beta)$, not all of the α_ks are zero on any basis element of V for which β is not also zero. Let $B = \{b_i\}_i$ and consider the collection of basis elements for which α_1 is nonzero. For each of these basis elements b, let $\gamma_1\alpha_1(b) = \beta(b)$. Now consider for all the remaining basis elements (B not including those basis elements for which $\gamma_1\alpha_1$ was nonzero) those on which α_2 is nonzero. For each of these basis elements b, let $\gamma_2\alpha_2(b) = \beta(b)$, and take $\gamma_2\alpha_2(b) = 0_Y$ for all other basis elements. Consider for all the remaining basis elements those on which α_3 is nonzero. For each of these basis elements b, let $\gamma_3\alpha_3(b) = \beta(b)$, and take $\gamma_3\alpha_3(b) = 0_Y$ for all other basis elements. Repeat this process to construct the collection of γ_is.

Each γ_i is a homomorphism, as they are being constructed from the basis elements $\alpha_j(b)$ in W. On the other hand, the sum of homomorphisms is a homomorphism, so $\sum_{i=1}^n \gamma_i\alpha_i$ is one, and it suffices to check that β and $\sum_{i=1}^n \gamma_i\alpha_i$ agree on all of B. This is trivial however, as for each $b \in B$, b is mapped to a nonzero element in Y for precisely one $\gamma_k\alpha_k$, and $\gamma_k\alpha_k(b) = \beta(b)$. Hence, the theorem holds.

Problem 9 (Golan 266). Let A and B be nonempty sets. Let V be the collection of all subsets of A and let W be the collection of all subsets of B, both of which are vector spaces over GF(2). Any function $f : A \to B$ defines a function $\alpha_f : W \to V$ by setting $\alpha_f : D \mapsto \{a \in A : f(a) \in D\}$. Show that each such function α_f defines a linear transformation, and find its kernel.

Solution. Recall V and W are vector spaces over GF(2) by taking vector addition to be symmetric difference and for a subset A we have $0 \cdot A = \emptyset$ and $1 \cdot A = A$. Then for each α_f

and any $D, E \in W$ we see that

$$\begin{aligned}
\alpha_f(D+E) &= \{a \in A : f(a) \in D+E\} \\
&= \{a \in A : f(a) \in (D \setminus E) \cup (E \setminus D)\} \\
&= \{a \in A : f(a) \in (D \setminus E)\} \cup \{a \in A : f(a) \in (E \setminus D)\} \\
&= \{a \in A : f(a) \in D\} \setminus \{a \in A : f(a) \in E\} \cup \{a \in A : f(a) \in E\} \setminus \{a \in A : f(a) \in D\} \\
&= \alpha_f(D) + \alpha_f(E).
\end{aligned}$$

Also we have

$$\alpha_f(0 \cdot D) = \emptyset_V = 0 \cdot \alpha_f(D)$$

and

$$\alpha_f(1 \cdot D) = 1 \cdot \alpha_f(D)$$

so indeed each α_f defines a linear transformation and we complete this problem by finding their kernel.

Now for some fixed α_f, we see that

$$\begin{aligned}
\ker(\alpha_f) &= \{D \in W : \alpha_f(D) = \emptyset_V\} \\
&= \{D \in W : f(a) \notin D \text{ for all } a \in A\} \\
&= \{D \subseteq B : D \cap \operatorname{im}(f) = \emptyset\}.
\end{aligned}$$

References

[1] Jonathan Golan. *The Linear Algebra a Beginning Graduate Student ought to know.* Springer, 3rd edition, 2012.