Final Exam

Math 700: Spring 2014

Deadline: **5pm Monday, May 5**

INSTRUCTIONS:

- Solve the problems below. Write up your solutions, giving complete justifications for all arguments. When you have finished,

  turn in a hard copy of your solutions to my office by **5pm Monday, May 5**.

  If I am not in my office when you are ready to turn in your exam, please slide it under my office door.

- The questions are meant to test your understanding of elementary concepts, and you should write down definitions of any technical terms you use, even if these terms are mentioned in the statement of the problem. Of course, you must use your best judgment about which definitions to state. (You probably don't want to define the integers or real numbers, for example.)

- It will help me (and probably your grade) if you do the following:

  1. State what you are trying to prove.
  2. Mention informally how you plan to prove it before giving the details.
  3. If you believe your proof is complete, use an end-of-proof symbol (like QED or $\square$); on the other hand, if you believe your proof is incomplete, say so.

HONOR CODE: You are expected to solve the exam problems on your own with no outside help. You may consult the lecture notes and textbook for this course only. No other books or internet usage is allowed. If you get stuck, please ask *me* for help, and I may post hints on our wiki page.

When you finish the exam, please sign the following pledge:

"On my honor as a student I, ________________________, have neither given nor received unauthorized aid on this exam." (Print Name)

Signature: ______________________________ Date: _______________

NOTATION: For the most part, we follow the notation used in the textbook. Recall that if V is a vector space over the field F, then $W \leqslant V$ denotes that W is a subspace of V, whereas $W \subseteq V$ means that W is a subset of V (which may or may not be a subspace). If $\varphi : V \to W$, then $\operatorname{im}(\varphi) := \varphi(V)$, $\ker(\varphi) := \{v \in V : \varphi(v) = 0_W\}$. Finally, if α belongs to $\operatorname{End}(V)$ or $\mathcal{M}_{n\times n}(F)$, then $\operatorname{spec}(\alpha)$ denotes the set of eigenvalues of α.

Problem 1. Let $F = \mathrm{GF}(q)$ be the finite field of order q, and let $V = F^n$.

(a) How many nonzero vectors are there in V?

(b) Given a nonzero vector $v_1 \in V$, how many vectors $v_2 \in V$ are such that $\{v_1, v_2\}$ is a linearly independent set?

(c) Let $S = \{v_1, v_2, \ldots, v_k\} \subset V$ be a subset of k linearly independent vectors. How many such subsets are there in V? (Hint: use the previous question to count the number of ways to build such sets.)

(d) Fix a subspace $W \leqslant V$ with $\dim(W) = k$. How many distinct bases does W have?

(e) How many k-dimensional subspaces of V are there?

Problem 2. Let F be a field and let $(K, \bullet)$ be an associative unital F-algebra. If $\alpha \in K$ and if $p(X) = \sum_{i=0}^{k} c_i X^i \in F[X]$, then $p(\alpha) = \sum_{i=0}^{k} c_i \alpha^i \in F[X]$. Recall that

$$\operatorname{Ann}(\alpha) := \{p(X) \in F[X] \mid p(\alpha) = 0_K\}.$$

(a) State the definition of the *minimal polynomial*, $m_\alpha(X)$, of α, and explain why $m_\alpha(X)$ is unique, if it exists.

(b) Give a condition on $(K, \bullet)$ that guarantees existence of $m_\alpha(X)$. A detailed proof is not required, but you should give some justification for your claim; e.g., by citing a result you learned in lecture or from the text.

(c) Suppose $A \in K = \mathcal{M}_{n\times n}(F)$. Is $m_A(X)$ guaranteed to exist in this case?

Problem 3. Let $\alpha \in \operatorname{End}(\mathbb{R}^3)$ be given by

$$\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ y+z \\ x+y \end{pmatrix}.$$

(a) Find the matrix $\Phi_{BB}(\alpha)$ that represents α with respect to the basis

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(b) Find the characteristic and minimal polynomials of α?

(c) Compute the eigenvalues of α?

Problem 4. Let F be a field. Recall that a linear transformation is *diagonalizable* means there exists a basis of eigenvectors for the transformation. A pair of linear transformations is *simultaneously diagonalizable* means that there exists a basis consisting of vectors that are eigenvectors of both transformations. Said another way, there exists a change of basis matrix that diagonalizes both transformations.

(a) Let $A \in \mathcal{M}_{n\times n}(F)$, let $W \subseteq F^n$ be a nontrivial A-invariant subspace, and let $A|_W$ denote the restriction of A to W. Show that if A is diagonalizable, then so is $A|_W$.
(Hint: consider the minimum polynomials of A and $A|_W$.)

(b) Let $\lambda \in F$ and let $W_\lambda = \{x \in F^n \mid Ax = \lambda x\}$ be the eigenspace of A associated with λ. Let A be diagonalizable and let $\lambda_1, \ldots, \lambda_k$ be its distinct eigenvalues. Show that $F^n = W_{\lambda_1} \oplus \cdots \oplus W_{\lambda_k}$.

(c) Let $B \in \mathcal{M}_{n\times n}(F)$. Show that if B commutes with A (i.e., $AB = BA$), then $BW_\lambda \subseteq W_\lambda$.

(d) Let $A, B \in \mathcal{M}_{n\times n}(F)$ and assume that both matrices are individually diagonalizable. Use the previous parts to show that if $AB = BA$, then A and B are simultaneously diagonalizable.

(e) Prove the converse: If A and B are simultaneously diagonalizable, then $AB = BA$.

Problem 5. Let $V = \mathbb{C}_3[X]$ be the vector space of polynomials with complex coefficients and degree at most three. Define $\alpha \in \mathrm{End}(V)$ by $\alpha p(X) = p(X+1)$. Find the eigenvalues of α and, for each eigenvalue, describe the corresponding eigenspace.
(Hint: begin by finding $\Phi_{BB}(\alpha)$ for the basis $B = \{1, X, X^2, X^3\}$.)

Problem 6. Let V be a vector space over a field F. The *dual space* of V is defined by $V^* := \mathrm{Hom}(V, F)$. For a subspace $U \leqslant V$, the *annihilator* of U is

$$\mathrm{ann}(U) := \{\theta \in V^* \mid \theta(u) = 0 \text{ for all } u \in U\} = \{\theta \in V^* \mid U \subseteq \ker(\theta)\}.$$

As we have seen, $\mathrm{ann}(U) \leqslant V^* \leqslant F^V$, a chain of subspaces.

(a) For this part, assume that V is finitely generated over F. Show that if $\theta_1, \ldots, \theta_r$ is a basis for $\mathrm{ann}(U)$, then $U = \bigcap_{i=1}^{r} \ker(\theta_i)$.

(b) If W is another vector space over F, and if $\varphi \in \mathrm{Hom}(V, W)$, then we define the *dual* of φ to be the linear transformation $\varphi^* \in \mathrm{Hom}(W^*, V^*)$ that takes each $\beta \in W^*$ to the composition $\beta \circ \varphi$. That is, $\varphi^*(\beta) = \beta \circ \varphi$.

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "\beta\circ\varphi"'] & & W \arrow[dl, "\beta"] \\ & F & \end{tikzcd}$$

Prove that $\ker(\varphi^*) = \operatorname{ann}(\operatorname{im}(\varphi))$ and that $\operatorname{im}(\varphi^*) = \operatorname{ann}(\ker(\varphi))$.